Prob1 (a) Since $N(t)$ is a Poisson process with rate $\lambda > 0$, $E[N(t)] = \lambda t$.

Thus, $E[M(t)] = E[N(t) - \lambda t] = E[N(t)] - \lambda t = \lambda t - \lambda t = 0$

For $0 \le s \le t < \infty$,

$$Cov(M(t), M(s)) = E[(M(t) - E(M(t)))(M(s) - E(M(s)))]$$

$$= E[M(t) \cdot M(s)]$$

$$= E[(N(t) - \lambda t)(N(s) - \lambda s)]$$

$$= E[N(t)N(s)] - \lambda t E[N(s)] - \lambda s E[N(t)] + \lambda^2 st$$

Since $E[M(s)] = \lambda s$, $E[N(t)] = \lambda t$

$$= E[N(t)N(s)] - \lambda^2 ts$$

Since $E[N(t)N(s)] = E[(N(t) - N(s))(N(s) - N(0))] + E[N^2(s)]$, $N(0) = 0$,

and $(0, s)$, (s, t) are disjoint and therefore independent

$$E[N(t)N(s)] = E[N(t) - N(s)] \cdot E[N(s) - N(0)] + E[N^2(s)]$$

$$= \lambda^2 (t-s) s + E[N^2(s)] = \lambda^2 ts - \lambda^2 s^2 + E[N^2(s)]$$

since $E(X^2) = Var(X) + E^2(X)$,

$$Cov(M(t), M(s)) = \lambda^2 ts - \lambda^2 s^2 + E(N^2(s)) - \lambda^2 ts$$

$$= -\lambda^2 s^2 + E(N^2(s))$$

$$= \lambda^2 s^2 + Var(N(s)) + [E(N(s))]^2$$

$$= -\lambda^2 s^2 + s\lambda + (\lambda s)^2 = \lambda s, \text{ where } 0 \le s \le t < \infty$$

Thus, in general, $Cov(N(s), N(t)) = \lambda \min\{s, t\}$.

(b) Wiener Process.

For a Wiener Process, we have $E[W(t)] = 0$, $Cov_w(s,t) = \sigma^2 \min\{s, t\}$.

if we let $\sigma^2 = \lambda$, then we have a Wiener Process which is equivalent to $M(t)$

(c). $M(t)$ is not a stationary process.

since the covariance function for $M(t)$, $Cov(M(s), M(t)) = \lambda \min\{s, t\}$, depends on s or t,

not on $|s-t|$.

(d) since $M(1) = -\lambda$, $N(1) = 0$.

Thus, the MMSE estimate of T_1 is that $E(T_1 \mid N(1) = 0)$

Since $\{N(1)=0\} = \{T_1 > 1\}$, $E(T_1 \mid T_1 > 1) = 1 + E(T_1)$ by memorylessness

$= 1 + \frac{1}{\lambda}$ $\quad T_1 \sim e^{\lambda}$.

Thus, $E(T_1 \mid M(1) = -\lambda) = 1 + \frac{1}{\lambda}$.

(e) $M(3) = 1 - 3\lambda \Rightarrow N(3) = 1$

$M(1) = -\lambda \Rightarrow N(1) = 0$

$E(N_6 \mid N_3, N_1) = E(N_6 \mid N_3 = 1) = E(N_3 \mid N_3 = 1) + E(N_6 - N_3 \mid N_3 = 1) = N_3 + E(N_6 - N_3)$.

since $N_6 - N_3 \sim \text{Poisson}(\lambda(6-3) = 3\lambda)$, $E[N_6 - N_3] = \lambda(3) = 3\lambda$.

Thus, $E[N_6 \mid N_3, N_1] = N_3 + E(N_6 - N_3) = 1 + 3\lambda$.

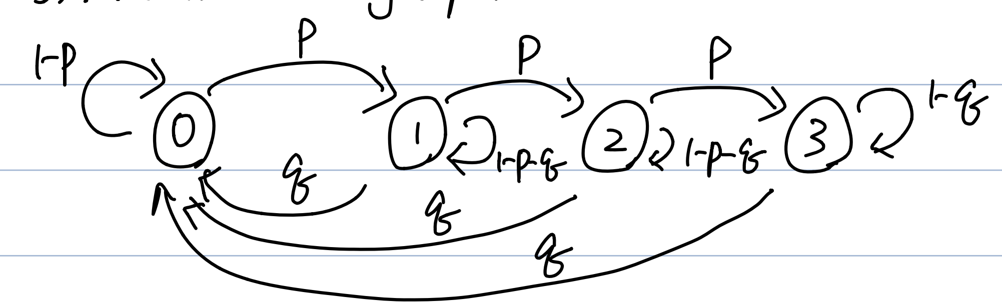
$\therefore E(M_6 \mid M_3, M_1) = E[N_6 \mid N_3, N_1) - 6\lambda = 1 + 3\lambda - 6\lambda = 1 - 3\lambda$.

Prob 2. (a). when $0 < X_i < K$, $X_{i+1} = \begin{cases} X_i + 1 & \text{with probability } p \\ 0 & \text{with probability } q \\ X_i & \text{with probability } 1-p-q \end{cases}$

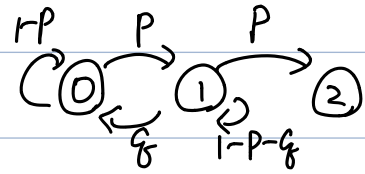
when $X_i = 0$, $X_{i+1} = \begin{cases} X_i + 1 & \text{with probability } p \\ 0 & \text{with probability } 1-p \end{cases}$

when $X_i = K$, $X_{i+1} = \begin{cases} K & \text{with probability } 1-q \\ 0 & \text{with probability } q \end{cases}$

(b). transition graph:



(c). Let $T(i)$ be the expected time to first arrive at state 2, starting from i. Then, we will have



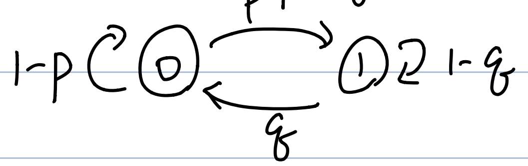
$T(2) = 0$

$T(1) = 1 + pT(2) + (1-p-q)T(1) + qT(0)$

$T(0) = 1 + (1-p)\, T(0) + pT(1)$

Thus $T(0) = \frac{p+q}{p^2} + \frac{1}{p} \quad (k \geq 2)$.

(d) $k=1$, $p=q=\frac{1}{2}$



if $k>1$, for some states, it will be transient, since $p < p+q$

(e) $P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ Thus, $\pi = [\frac{1}{2} \ \ \frac{1}{2}]$

(f). No.

the state of X_{n+1} depends on the state of X_n, X_{n-1} and X_{n-2}.

(d)(e) $\pi \cdot P = \pi$

since $P = \begin{bmatrix} 1-p & p & 0 & \cdots & 0 \\ q & 1-p-q & p & \cdots & 0 \\ q & 0 & 1-p-q & p \cdots & 0 \\ q & \vdots & \vdots & \ddots & p \\ q & \cdots & 0 & & 1-q \end{bmatrix}$ $\quad \Sigma \pi_i = 1 \Rightarrow \pi_0 = \frac{q}{p-q}$

$\pi_i = \frac{q\, p^i}{(p-q)^{i+1}} \quad i<k$

$\pi_k = \frac{p^k}{(p+q)^{k+1}}$

$\boxed{p>0,\ q>0 \\ p+q \leq 1}$.

Prob 3. (a). $\{X(t)\}$ is a zero-mean stationary process.

Thus $K_X(s,t) = E[X(s)X(t)] = E[X(0)X(t-s)] = K_X(t-s)$.

let $\tau = t-s$, then $K_X(\tau) = E[X(t)X(t+\tau)]$

From Gauchy-Schwarz inequality, if $K_X(\tau) = K_X(0)$, for som τ,

then $[E((X(\tau+t)-X(t))X(0))]^2 = (K_X(\tau+t) - K_X(t))^2$

$$\leq E[(X(t+\tau)-X(t))^2]\,E[(X(0))^2]$$

$$= (E[(X(t+\tau))^2] + E[(X(t))^2] - 2E[(X(t+\tau))(X(t))]) \cdot K_X(0)$$

$$= (2K_X(0) - 2K_X(\tau))K_X(0) = 0$$

Thus $K_X(t+\tau) = K_X(t)$, for some τ

Thus $K_X(\tau)$ must be periodic with period τ.

(b). $E[(X(t+\tau)-X(t))^2]$

$= E[(X(t+\tau))^2] + E[(X(t))^2] - 2E[X(t+\tau)X(t)]$

$= 2K_X(0) - 2K_X(\tau) = 0$.

(c) Yes.

(d). $E[X|Y] = E[X] + K_{XY}K_Y^{-1}(y - E(Y))$

Thus $E[X(S+2\tau) \mid X(S) = x]$

$= E[X(S+2\tau)] + K_{X,Y}K_Y^{-1}(y - E(Y))$, where $Y = X(S) = x$.

$E[X(S+2\tau)] = 0$,

$K_{X,Y} = K_X(S+2\tau, S) = K_X(2\tau) = K_X(0)$

$K_Y = K_X(S,S) = K_X(0)$

$\therefore E[X(S+2\tau) \mid X(S) = x] = 0 + K_X(0)\cdot K_X^{-1}(S)(x-0) = \dfrac{K_X(0)}{K_X(0)}x = x$

$E[X(S+2\tau) - \hat{X}] = E[X(S+2\tau)] - E[\hat{X}]$

$= 0 - E[E[X(S+2\tau) \mid X(S) = x]] = 0$.

$Var[X(S+2\tau) - \hat{X}] = [K_X] - [K_{XY}\cdot K_Y^{-1}K_{XY}^T]$

$= K_X(0) - \dfrac{K_X^2(0)}{K_X(0)} = 0$.

$K_x(0)$

$\begin{bmatrix} K_x & K_{xy} \\ K_{xy} & K_y \end{bmatrix}$ — if Covariance is non-singular their joint pdf doesn't exist.

(e). (i) $K_x(t, t+2\tau) = K_x(2\tau) = K_x(0)$ not exist. (non-singular)

(ii). $K_x(t, t+u) = K_x(u) \neq K_x(0)$ exist.

(iii) $K_x(t+u, t+\tau) = K_x(u-\tau) = K_x(u)$ exist.

(iv). $K_x(t, t+u) = K_x(u)$, $K_x(t+u, t+\tau) = K_x(u)$

$K_x(t, t+\tau) = K_x(0)$ not exist. (non-singular).

prob4. (a). Yes.

let $\Delta = \frac{t+\alpha-t}{n} = \frac{\alpha}{n}$

Since $X(t+\alpha) - W(t) = [X(t+\alpha) - X(t+\alpha-\Delta)] + [X(t+\alpha-\Delta) - X(t+\alpha-2\Delta)]$

$+ \cdots + [X(t+\alpha-(n-1)\Delta) - X(t)]$

$= \sum_{i=1}^{n} Y_i.$

when $n \to \infty$, $\Delta \to 0$, then $Y_i = X[s-(i-1)\Delta] - X[s-i\Delta] \to 0$.

Thus, according to CLT, $X(t+\alpha) - X(t)$ follows gaussian distribution

$\{X(t)\}$ is a gaussian process.

(b). Yes.

let $0 < t_1 < t_2$, $X(t_2) - X(t_1) = [W(t_2+\alpha) - W(t_2)] - [W(t_1+\alpha) - W(t_1)]$

$= [W(t_2+\alpha) - W(t_1+\alpha)] - [W(t_2) - W(t_1)]$

Since Wiener process satisfies that the distribution of $W(t_2) - W(t_1)$ is only related to $|t_2 - t_1|$, $X(t_2) - X(t_1)$ is only related to $t_2 - t_1$. Thus, $X(t)$ is a stationary process.

(c). $X(10) = W(10+\alpha) - W(10)$

mean=0

$P(X(10) > 0) = P[W(10+\alpha) - W(10) > 0] = \frac{1}{2}$

(d). $X(n) \sim N(0, \alpha)$

$\therefore \frac{1}{N} \sum_{n=1}^{N} X_n \sim N(0, \frac{N\alpha}{N^2}) = N(0, \frac{\alpha}{N})$

Thus, $N \to \infty$, $\frac{\alpha}{N} \to 0$. $\therefore$ converge to 0 in probability.

(f). $E[B(t)] = 0$

if $s<t$, $Cov[B(s), B(t)] = Cov(W(s) - sW(1), W(t) - tW(1))$

$$= Cov(W(s), W(t)) - t\,Cov(W(s), W(1))$$

$$- s\,Cov(W(1), W(t)) + st\,Var(W(1))$$

$$= s - ts - st + st$$

$$= s(1-t)$$

in which case, $Var(B(t)) = t(1-t)$

In general, $Cov[B(s), B(t)] = \min(s,t) - st$.

(e) Yes. $B(t) = W(t) - tW(1)$ $0 \le t \le 1$ It is the linear combination of Gaussian.

(g). For $0 \le t \le 1$

$$Cov(W(t) - tW(1), W(1)) = E[(W(t) - tW(1))W(1)]$$

$$= \min\{t, 1\} - t = 0$$

As their covariance is zero, and their joint distribution Gaussian, $B(t) = W(t) - tW(1)$ and $W(1)$ are independent.