Northwestern University
Department of Electrical and Computer Engineering

ELEC ENG 422 Winter 2020

Problem set 4: Date due: Feb. 26, 2020.

Reading: Supplemental Notes on Markov chains. Sections 4.1-4.3 and 6.1 and 6.2 in Gallager cover Markov Chains and stationary distributions.

1. A fair die is rolled repeatedly. Consider the following sequences of random variables, where the nth variable corresponds to the time of the nth die roll. Which of these are Markov chains? For those that are, give the transition diagram.

   a.) The largest number X_n shown up to the nth roll.

   b.) The number N_n of sixes in n rolls.

   c.) At time n, the time C_n since the most recent six.

   d.) At time n, the time B_n until the next six.

   e.) The sum M_n of the largest two rolls out of the first n rolls.

2. Consider a variation of the gambler's ruin problem where the gambler has a rich uncle who covers his losses when his wealth is zero allowing the gambler to continue to play. (i.e., his wealth is never negative and if he losses when it is zero, it simply stays at zero). Suppose that the gambler has a dollars and plays until the first time T at which he has $c > a$ dollars. All other parameters are the same as in the notes on Markov chains.

   a) Draw the state transition diagram for the gambler's wealth after each game.

   b) Using first step analysis to write down a system of equations whose solution would give you the expected time until the gambler stops playing.

   c) Suppose instead you were interested in the probability that the gambler has to borrow any money from his uncle before he stops. Use first step analysis to write down a system of equations to solve for this.

3. Exercise 4.3 in Gallager.

4. Exercise 4.8 in Gallager.

5. Exercise 4.10 in Gallager.

6. Suppose an absent minded professor has two umbrellas that she uses to commute from work to home. Each time she commutes, it rains with probability p (independent from trip to trip). If it is raining and she has an umbrella at her current location, she takes one with her. If it is not raining she does not take an umbrella.

   a) Construct a Markov chain model for the number of umbrellas at the professor's current location.

b) Determine the stead-state probability that it is raining and the professor does not have an umbrella.

7. Let $\{X_n\}_{n=0}^{\infty}$, be a discrete-time Markov chain such that for each time n, X_n is equal to 0 or 1 and over time the process evolves according to the following rule:

$$X_{n+1} = \begin{cases} X_n, & \text{with probability } p, \\ \bar{X}_n, & \text{with probability } 1-p, \end{cases}$$

where $\bar{X}$ denotes the "opposite" value of X. At time 0, X_0 is equal to 0 with probability q and equal to 1, otherwise.

a) Give the transition matrix for this Markov chain.

b) Find a value of q which gives the stationary distribution of this Markov chain (i.e. so that the chain starts already in its stationary distribution).

c) Suppose that the Markov chain starts with $X_0 = 0$ and determine the expected time until the state is zero again, i.e. the expected value of $T = \min\{n \geq 1 : X_n = 0\}$. This quantity is called the *mean recurrence time* of state 0. (*hint: you can use an idea similar to first step analysis to do this.*)

d) What is the relation between your answers to (b) and (c)?

8. Consider a Markov chain with state space S and transitions probabilities P_{ij} Let A be a subset of the states. Show that any stationary distribution, $\{\pi_i\}$ for this Markov chain must satisfy

$$\sum_{j \in A} \pi_j \sum_{i \notin A} P_{ji} = \sum_{i \notin A} \pi_i \sum_{j \in A} P_{ij}.$$

This is a type of "flow-balance" equation as discussed in class, where the left-hand side of this expression is the stationary probability of transitioning out of the set A and the right-hand side is the stationary probability of transitioning into this set.

9. Consider the Markov chain shown in Figure 6.2 in the textbook (this is sometimes referred to as a random walk with a barrier at 0).

a) Use the "flow-balance" approach form the previous problem (similar to the example in lecture) to solve for the stationary distribution of this Markov Chain assuming $p < q$.

b) If $p \geq q$ argue that there is no pmf that satisfies the flow-balance equations in this case. (when $p = q$ this Markov chain is null recurrent, when $p > q$ it is transient.)