5. Brownian Motion (3/30/06, cf. Ross + DG)

1. Introduction.
2. Gaussian Processes.

5.1. Introduction

Consider a symmetric random walk: Suppose that $X_1, X_2, \ldots$ are i.i.d. ± 1, each with probability 1/2, i.e., $P_{i,i+1} = P_{i,i-1} = 1/2$.

Take smaller steps (Δx) in smaller time intervals (Δt), and let

$$X(t) = \Delta x \, (X_1 + X_2 + \cdots + X_{[t/\Delta t]}),$$

where $[y]$ is the greatest integer $\leq y$.

Notice that $\mathsf{E}[X_i] = 0$ and $\mathsf{Var}(X_i) = \mathsf{E}[X_i^2] = 1$, so that

$$\mathsf{E}[X(t)] = 0 \quad \text{and} \quad \mathsf{Var}(X(t)) = (\Delta x)^2 [t/\Delta t].$$

Now let $\Delta x = \sigma\sqrt{\Delta t}$ for some constant $\sigma > 0$, and see what happens when we take $\Delta t \to 0\ldots$

$$\mathsf{E}[X(t)] = 0 \quad \text{and} \quad \mathsf{Var}(X(t)) \to \sigma^2 t.$$

Reasonable things to expect:

(i) Since $X(t)$ is the sum of a bunch of i.i.d. X_i's, $X(t) \sim \text{Nor}(0, \sigma^2 t)$.

(ii) Since changes in the value of the r.w. in disjoint time intervals are indep, we have *indep increments*, i.e., for $t_1 < t_2 < \cdots < t_n$, $X(t_n) - X(t_{n-1}), X(t_{n-1}) - X(t_{n-2}), \ldots, X(t_2) - X(t_1), X(t_1)$ are all indep.

(iii) Since changes depend only on the length of the interval, we have *stationary increments*, i.e., the distribution of $X(t+s) - X(t)$ doesn't depend on t.

Definition: The stochastic process $\{X(t), t \geq 0\}$ is a *Brownian motion* process with parameter σ if:

(a) $X(0) = 0$.

(b) $X(t) \sim \text{Nor}(0, \sigma^2 t)$.

(c) $\{X(t), t \geq 0\}$ has stationary and indep increments.

$\sigma = 1$ corresponds to *standard* BM.

Discovered by Brown; first analyzed rigorously by Einstein; mathematical rigor established by Wiener (also called *Wiener* process).

Remark: Here's another way to construct BM:

Suppose $Y_1, Y_2, \ldots$ is any sequence of identically distributed RV's with mean zero and finite variance. (To some extent, the Y_i's don't even have to be indep!) *Donsker's* CLT says that

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} Y_i \xrightarrow{\mathcal{D}} \sigma \mathcal{W}(t) \quad \text{as } n \to \infty,$$

where, henceforth, $\mathcal{W}(t)$ denotes standard BM, and

$$\sigma^2 = \lim_{n \to \infty} n \mathrm{Var}(\bar{Y}_n) \quad \text{with } \bar{Y}_n \equiv \textstyle\sum_{i=1}^{n} Y_i / n.$$

Facts (which we won't prove):

$\mathcal{W}(t)$ produces continuous sample paths!

$\mathcal{W}(t)$ is nowhere differentiable!

Now let's get the joint p.d.f. of $\mathcal{W}(t_1), \mathcal{W}(t_2), \ldots, \mathcal{W}(t_n)$, where we assume $t_1 < t_2 < \cdots < t_n$ and $\sigma^2 = 1$. . . .

First of all,

$$\mathcal{W}(t_1) = w_1, \mathcal{W}(t_2) = w_2, \ldots, \mathcal{W}(t_n) = w_n$$

$$\Longleftrightarrow$$

$$\mathcal{W}(t_1) = w_1, \mathcal{W}(t_2) - \mathcal{W}(t_1) = w_2 - w_1, \ldots, \mathcal{W}(t_n) - \mathcal{W}(t_{n-1}) = w_n - w_{n-1},$$

where we note that these increments are indep.

Further, by stationary increments,

$$\mathcal{W}(t_k) - \mathcal{W}(t_{k-1}) \sim \text{Nor}(0, t_k - t_{k-1}), \quad k = 1, 2, \ldots, n.$$

Thus, the joint p.d.f. of $\mathcal{W}(t_1), \mathcal{W}(t_2), \ldots, \mathcal{W}(t_n)$ is

$$\begin{aligned}
& f(w_1, \ldots, w_n) \\
& \quad = \prod_{i=1}^{n} f_{\mathcal{W}_i - \mathcal{W}_{i-1}}(w_i - w_{i-1}) \quad \text{with } \mathcal{W}_0 \equiv 0 = w_0 \\
& \quad = \frac{\exp\left\{-\frac{1}{2}\sum_{i=1}^{n} \frac{(w_i - w_{i-1})^2}{t_i - t_{i-1}}\right\}}{(2\pi)^{n/2}\left[\prod_{i=1}^{n}(t_i - t_{i-1})\right]^{1/2}} \quad \text{with } t_0 \equiv 0.
\end{aligned}$$

We can get many properties from this p.d.f.

Theorem: Conditional distribution of BM. For $s < t$,

$$[\mathcal{W}(s)|\mathcal{W}(t) = b] \;\sim\; \text{Nor}\left(\frac{bs}{t}, \; \frac{s(t-s)}{t}\right).$$

Proof: Using some algebra, we have

$$\begin{aligned} f_{\mathcal{W}(s)|\mathcal{W}(t)}(x|b) &= \frac{f_{\mathcal{W}(s)}(x) f_{\mathcal{W}(t)-\mathcal{W}(s)}(b-x)}{f_{\mathcal{W}(t)}(b)} \\ &= C \exp\left\{\frac{-x^2}{2s} - \frac{(b-x)^2}{2(t-s)}\right\} \\ &= C' \exp\left\{\frac{-(x - bs/t)^2}{2s(t-s)/t}\right\}. \quad \diamondsuit \end{aligned}$$

Example: Suppose we can model the difference $Y(t)$ in two stock prices as a BM with variance parameter σ^2. Suppose stock 1 is ahead of stock 2 by σ at the 6-month mark. What's the prob that it'll also be ahead at $t = 1$ year?

$$
\begin{aligned}
&\Pr(Y(1) > 0 \,|\, Y(1/2) = \sigma) \\
&\quad = \Pr(Y(1) - Y(1/2) > -\sigma \,|\, Y(1/2) = \sigma) \\
&\quad = \Pr(Y(1) - Y(1/2) > -\sigma) \quad \text{(indep increments)} \\
&\quad = \Pr(Y(1/2) > -\sigma) \quad \text{(stationary increments)} \\
&\quad = \Phi(\sqrt{2}) \doteq 0.92. \quad \diamondsuit
\end{aligned}
$$

Now suppose that stock 1 is ahead by σ at time $t = 1$. What's the prob that it was ahead at the 6-month mark?

$$\begin{aligned}
&\Pr(Y(1/2) > 0 \,|\, Y(1) = \sigma) \\
&= \Pr(\mathcal{W}(1/2) > 0 \,|\, \mathcal{W}(1) = 1) \quad (\mathcal{W}(t) \equiv Y(t)/\sigma) \\
&= \Pr\left(\text{Nor}\left(\frac{1}{2}, \frac{1}{4}\right) > 0\right) \\
&\qquad \text{(condl theorem with } s = 1/2,\ t = 1,\ b = 1) \\
&= \Phi(1) \doteq 0.84. \quad \diamondsuit
\end{aligned}$$

Theorem: $\mathrm{Cov}(\mathcal{W}(s), \mathcal{W}(t)) = \min(s, t)$.

Proof: Suppose $s < t$. Then

$$\begin{aligned}
&\mathrm{Cov}(\mathcal{W}(s), \mathcal{W}(t)) \\
&\quad = \mathrm{Cov}(\mathcal{W}(s), \mathcal{W}(t) - \mathcal{W}(s) + \mathcal{W}(s)) \\
&\quad = \mathrm{Cov}(\mathcal{W}(s), \mathcal{W}(t) - \mathcal{W}(s)) + \mathrm{Var}(\mathcal{W}(s)) \\
&\quad = \mathrm{Var}(\mathcal{W}(s)) \quad \text{(indep increments)} \\
&\quad = s. \quad \diamondsuit
\end{aligned}$$

5.2 Gaussian Processes

Definition: A SP $\{X(t), t \geq 0\}$ is a *Gaussian process* if $X(t_1), \ldots, X(t_n)$ is jointly normal for all $t_1, \ldots, t_n$.

Example: BM $\mathcal{W}(t)$ is Gaussian.

Definition: A *Brownian bridge* process is Gaussian. Two equiv definitions:

$$\begin{aligned} 1. \quad \mathcal{B}(t) &\equiv \mathcal{W}(t) \,|\, \mathcal{W}(1) = 0 \\ 2. \quad \mathcal{B}(t) &\equiv \mathcal{W}(t) - t\mathcal{W}(1). \end{aligned}$$

Note $\mathsf{E}[\mathcal{B}(t)] = \mathsf{E}[\mathcal{W}(t) - t\mathcal{W}(1)] = 0$. Further, if $s < t$,

$$\begin{aligned}
&\mathsf{Cov}(\mathcal{B}(s), \mathcal{B}(t)) \\
&\quad = \mathsf{Cov}(\mathcal{W}(s) - s\mathcal{W}(1), \mathcal{W}(t) - t\mathcal{W}(1)) \\
&\quad = \mathsf{Cov}(\mathcal{W}(s), \mathcal{W}(t)) - t\mathsf{Cov}(\mathcal{W}(s), \mathcal{W}(1)) \\
&\qquad\quad - s\mathsf{Cov}(\mathcal{W}(1), \mathcal{W}(t)) + st\mathsf{Var}(\mathcal{W}(1)) \\
&\quad = s - ts - st + st \\
&\quad = s(1-t),
\end{aligned}$$

in which case $\mathsf{Var}(\mathcal{B}(t)) = t(1-t)$. $\diamondsuit$

Example: Area under a Brownian bridge, $A \equiv \int_0^1 \mathcal{B}(t)\,dt$.

Obviously, $\mathsf{E}[A] = 0$. Further,

$$\begin{aligned}
\mathsf{Var}(A) &= \mathsf{Cov}\left(\int_0^1 \mathcal{B}(t)\,dt,\ \int_0^1 \mathcal{B}(s)\,ds\right) \\
&= \int_0^1 \int_0^1 \mathsf{Cov}(\mathcal{B}(t), \mathcal{B}(s))\,ds\,dt \\
&= 2\int_0^1 \int_0^t s(1-t)\,ds\,dt \;=\; 1/12.
\end{aligned}$$

So $A \sim \mathsf{Nor}(0, 1/12)$. $\quad\diamondsuit$

Definition: *Integrated BM*, $\mathcal{Z}(t) \equiv \int_0^t \mathcal{W}(s)\,ds$. Note that $\mathsf{E}[\mathcal{Z}(t)] = \int_0^t \mathsf{E}[\mathcal{W}(s)]\,ds = 0$. Further, if $s < t$,

$$
\begin{aligned}
\mathsf{Cov}(\mathcal{Z}(s), \mathcal{Z}(t)) &= \mathsf{Cov}\left(\int_0^s \mathcal{W}(u)\,du,\ \int_0^t \mathcal{W}(v)\,dv\right) \\
&= \int_0^s \int_0^t \mathsf{Cov}(\mathcal{W}(u), \mathcal{W}(v))\,du\,dv \\
&= \int_0^s \left[\int_0^s + \int_s^t\right] \mathsf{Cov}(\mathcal{W}(u), \mathcal{W}(v))\,du\,dv \\
&= 2\int_0^s \int_0^v \mathsf{Cov}\,du\,dv + \int_0^s \int_s^t \mathsf{Cov}\,du\,dv \\
&= 2\int_0^s \int_0^v u\,du\,dv + \int_0^s \int_s^t v\,du\,dv \;=\; \frac{ts^2}{2} - \frac{s^3}{6},
\end{aligned}
$$

in which case $\mathsf{Var}(\mathcal{Z}(t)) = t^3/3$. $\diamondsuit$

Definition: The SP $\{X(t), t \leq 0\}$ is BM *with drift* coefficient μ and variance parameter σ^2 if:

(a) $X(0) = 0$.

(b) $\{X(t)\}$ has stationary and indep increments.

(c) $X(t) \sim \text{Nor}(\mu t, \sigma^2 t)$.

$X(t) = \sigma \mathcal{W}(t) + \mu t$.

Definition: *Geometric BM* is a non-Gaussian process:

$$G(t) \;=\; e^{X(t)} \;=\; \mathsf{exp}\{\sigma\mathcal{W}(t) + \mu t\}.$$

For $s < t$,

$$\begin{aligned}
&\mathsf{E}[G(t)\,|\,G(u), 0 \le u \le s] \\
&\quad = \; \mathsf{E}[e^{X(t)}\,|\,X(u), 0 \le u \le s] \\
&\quad = \; \mathsf{E}[e^{X(t)-X(s)+X(s)}\,|\,X(u), 0 \le u \le s] \\
&\quad = \; e^{X(s)}\mathsf{E}[e^{X(t)-X(s)}\,|\,X(u), 0 \le u \le s] \\
&\qquad\; \text{(since } X(s) \text{ is given)} \\
&\quad = \; G(s)\mathsf{E}[e^{X(t)-X(s)}] \quad \text{(indep increments)} \\
&\quad = \; G(s)\mathsf{E}[e^{X(t-s)}].
\end{aligned}$$

Now, note that the m.g.f. of a normal RV N is

$$M_N(a) \;=\; \mathsf{E}[e^{aN}] \;=\; \exp\left\{a\mathsf{E}[N] + \frac{a^2}{2}\mathsf{Var}(N)\right\}.$$

Taking $N = X(t-s) \sim \mathsf{Nor}(\mu(t-s), \sigma^2(t-s))$, we have

$$\mathsf{E}[e^{X(t-s)}] \;=\; M_{X(t-s)}(1) \;=\; e^{\mu(t-s)+\sigma^2(t-s)/2}.$$

So, mopping up from the previous page, we have

$$\mathsf{E}[G(t)\,|\,G(u), 0 \le u \le s] \;=\; G(s)e^{\mu(t-s)+\sigma^2(t-s)/2}. \quad \diamondsuit$$

Geom BM can model stock prices if you're willing to assume that % changes are i.i.d. I.e., if X_n is the stock price at time n, then we'll assume that the sequence formed by $Y_n \equiv X_n / X_{n-1}$ is $\approx$ i.i.d. Further,

$$X_n \;=\; Y_n X_{n-1} \;=\; \cdots \;=\; Y_n Y_{n-1} \cdots Y_1 X_0,$$

so that

$$\ell\mathrm{n}(X_n) \;=\; \sum_{i=1}^{n} \ell\mathrm{n}(Y_i) + \ell\mathrm{n}(X_0).$$

Since the Y_i's are $\approx$ i.i.d., the CLT implies that $\ell\mathrm{n}(X_n) \approx$ normal.

Definition: Suppose $f(\cdot)$ is a function with a continuous derivative in $[a, b]$. Consider the stochastic integral

$$\int_a^b f(t)\, d\mathcal{W}(t)$$
$$\equiv \lim_{\substack{n \to \infty \\ |t_i - t_{i-1}| \to 0, \forall i}} \sum_{i=1}^{n} f(t_{i-1})(\mathcal{W}(t_i) - \mathcal{W}(t_{i-1})). \quad (*)$$

The term $d\mathcal{W}(t)$ is known as *white noise*. It's sort of the "derivative" of BM. Now pretend you can use integration by parts in the usual way. . . .

Then

$$\int_a^b f(t)\, d\mathcal{W}(t) \;=\; f(b)\mathcal{W}(b) - f(a)\mathcal{W}(a) - \int_a^b \mathcal{W}(t)\, df(t).$$

This is usually regarded as the definition of the left-hand side.

Assuming you can bring the expectation inside, we immediately have $\mathsf{E}[\int_a^b f(t)\, d\mathcal{W}(t)] = 0$.

How about the variance?

By indep increments, we have

$$\begin{aligned}
&\mathsf{Var}\left(\sum_{i=1}^{n} f(t_{i-1})(\mathcal{W}(t_i) - \mathcal{W}(t_{i-1}))\right) \\
&\quad = \sum_{i=1}^{n} f^2(t_{i-1})\mathsf{Var}(\mathcal{W}(t_i) - \mathcal{W}(t_{i-1})) \\
&\quad = \sum_{i=1}^{n} f^2(t_{i-1})(t_i - t_{i-1}).
\end{aligned}$$

Taking the limit as in (*) (which you have to be careful about), we get

$$\mathsf{Var}\left(\int_a^b f(t)\, d\mathcal{W}(t)\right) = \int_a^b f^2(t)\, dt. \quad \Diamond$$

Example: Suppose a particle moves in a liquid. At time t, it has velocity $V(t)$, but has to move against a viscous force that slows it at a rate proportional to $V(t)$. Further suppose the velocity changes instantly by a multiple of white noise. Then

$$V'(t) = -\beta V(t) + \alpha \mathcal{W}'(t)$$

$$\iff e^{\beta t}(V'(t) + \beta V(t)) = \alpha e^{\beta t} \mathcal{W}'(t)$$

$$\iff \frac{d}{dt}\Big(e^{\beta t} V(t)\Big) = \alpha e^{\beta t} \mathcal{W}'(t)$$

$$\iff e^{\beta t}V(t) \;=\; V(0) + \alpha \int_0^t e^{\beta s}\mathcal{W}'(s)\,ds$$

$$\iff V(t) \;=\; V(0)e^{-\beta t} + \alpha \int_0^t e^{-\beta(t-s)}\,d\mathcal{W}(s).$$

Applying the integration by parts formula for the stochastic integral $\int_a^b f(t)\,d\mathcal{W}(t)$, we finally obtain

$$V(t) \;=\; V(0)e^{-\beta t} + \alpha\left(\mathcal{W}(t) - \int_0^t \mathcal{W}(s)\beta e^{-\beta(t-s)}\,ds\right). \quad \diamondsuit$$

Definition: The *hitting time* $T_a \equiv \text{argmin}_t\{\mathcal{W}(t) = a\}$ is the first time that $\mathcal{W}(t)$ "hits" the value $a > 0$.

Before deriving an expression for $\Pr(T_a \leq t)$, note the *reflection principle*: If $s < t$, then we can reflect $\mathcal{W}(t)$ around the horizontal line $y = a$ to obtain the "equally likely" path

$$\widetilde{\mathcal{W}}(t) \;=\; \begin{cases} \mathcal{W}(t), & \text{if } t < T_a \\ a - (\mathcal{W}(t) - a), & \text{if } t > T_a \end{cases}.$$

This implies that

$$\Pr(\mathcal{W}(t) \geq a \,|\, T_a \leq t) \;=\; 1/2.$$

Then by the law of total prob,

$$\begin{aligned}
&\Pr(\mathcal{W}(t) \geq a) \\
&= \Pr(\mathcal{W}(t) \geq a \,|\, T_a \leq t)\Pr(T_a \leq t) \\
&\qquad \Pr(\mathcal{W}(t) \geq a \,|\, T_a > t)\Pr(T_a > t) \\
&= \frac{1}{2}\Pr(T_a \leq t) + 0,
\end{aligned}$$

so that

$$\begin{aligned}
\Pr(T_a \leq t) &= 2\Pr(\mathcal{W}(t) \geq a) \\
&= 2\Pr(\mathrm{Nor}(0,1) \geq a/\sqrt{t}) \\
&= 2(1 - \Phi(a/\sqrt{t})).
\end{aligned}$$

Similarly, for $a < 0$, $\Pr(T_a \leq t) = 2(1 - \Phi(-a/\sqrt{t}))$.

Thus, for any a, $\Pr(T_a \leq t) = 2(1 - \Phi(|a|/\sqrt{t}))$. $\diamondsuit$

Can use symmetry to show that

$$\Pr(\max_{0 \leq s \leq t} \mathcal{W}(s) \geq a) \;=\; \Pr(T_a \leq t) \;=\; 2(1 - \Phi(a/\sqrt{t})),$$

where a must be > 0, and

$$\Pr(T_a < T_b) \;=\; \frac{b}{b-a}, \quad \text{for } a < 0,\ b > 0. \quad \diamondsuit$$