Northwestern University
Department of Electrical and Computer Engineering

ELEC ENG 422 Winter 2020

Take Home Final Exam Date Issued: March 11, 2019

- Submit your completed exam by 5:00pm on Tuesday. Mar. 17. You may scan your solution and submit it electronically or give your completed exam to me in L352 (or slide it under my door if I am not there).
- You may use the course book, notes and reference books when completing the exam, but *not* any other references including the Internet or your classmates.
- Show all your work and clearly indicate your answers.
- **Be sure to justify your answers.** A correct answer without justification may be considered wrong.
- Write legibly - if I can't read it, I can't grade it.

Good luck and enjoy spring break!

1. A Centered Poisson Process. Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate $\lambda > 0$. Define a new process $\{M(t) : t \geq 0\}$ by $M(t) = N(t) - \lambda t$, i.e., $M(t)$ indicates whether the number of arrivals is larger or smaller than the expected number from $N(t)$ at any time t.

a.) What are the mean and covariance functions for $M(t)$?

b.) The mean and covariance functions of this process should be the same as those for another process we studied - which process is this? (this shows that process with very different sample paths can have the same mean and covariance functions).

c.) Is $M(t)$ a stationary process? Explain.

d.) Let T_1 indicate the time of the first arrival for $N(t)$. Given that you observe that $M(1) = -\lambda$, what is the MMSE estimate of T_1?

e.) Given you observe that $M(3) = 1 - 3\lambda$ and $M(1) = -\lambda$, what is the MMSE estimate of $M(6)$?

2. Batch processing. Consider a system that can process up to K jobs in a batch. Jobs arrive over time and if there are fewer than K jobs waiting, a newly arriving job is accepted and queued until the system can process it. If there are K jobs already queued, then a newly arriving job is rejected. Whenever the system finishes processing a batch of jobs, it takes all waiting jobs and processes them.

Suppose we adopt the following discrete-time model for the number of jobs that are queued for processing. In each time interval, *exactly one* of the following events occurs: a job arrives with probability p, all jobs depart with probability q (where $p + q \leq 1$) or nothing occurs. The events in each time period are independent.

a.) Let $\{X_n\}_{n=0}^{\infty}$ be the number of jobs in the system at time n, with $X_0 = 0$. Explain why this is a Markov chain.

b.) Draw the transition graph for this Markov chain when $K = 3$.

c.) What is the expected time until the system first contains 2 jobs? (Your answer should be an expression that may depend on K, p and q).

d.) For what values of $K \geq 1$, p and q is this an ergodic Markov chain? Explain.

d.) Assuming K, p and q satisfy the conditions in part (c.) determine the steady-state distribution π (again your answer should be expressions that may depend on K, p and q).

e.) Let $\pi = (\pi_0, \ldots, \pi_K)$ be the steady-state distribution found in part (d.). In terms of this, what is the steady-state probability that when the system begins processing a batch, the batch contains K jobs.

f.) Suppose now that instead whenever the system begins processing a batch of jobs it takes at least three time-units to complete processing the batch (during this time new jobs can arrive but no jobs will be processed). Is $\{X_n\}_{n=0}^{\infty}$ still a Markov chain? Explain.

3. Periodicity and Covariances. Let $K_X(t)$ be the covariance function of a zero-mean stationary process $\{X(t)\}$ which satisfies $K_X(\tau) = K_X(0)$ for some given $\tau > 0$.

a.) For such a process show that $K_X(\tau)$ must be periodic with period τ.

b.) Show that $E[(X(t) - X(t+\tau))^2] = 0$ for all t. A process satisfying this is said to be *mean square periodic.*

c.) Does the converse to the previous part hold? In other words, if $\{Y(t)\}$ is a stationary zero-mean process which is mean-square periodic with period τ, then must its covariance function also be periodic with period τ?

For the remaining parts of this question assume additionally that $\{X(t)\}$ is a zero-mean Gaussian process.

d.) Give the MMSE estimate $\hat{X}$ of $X(s+2\tau)$ given $X(s) = x$. Also characterize the estimation error $X(s+2\tau) - \hat{X}$.

e.) Suppose that $0 < K_X(u) < K_X(0)$ for some $u > 0$. For a given t, which of the following sets of random variables will have a well defined joint probability density. Explain your answers (you do not need to write down the density, just explain if it exists or not).

i.) $X(t), X(t+2\tau)$

ii.) $X(t), X(t+u)$

iii.) $X(t+u), X(t+\tau)$

iv.) $X(t), X(t+u), X(t+\tau)$

4. Functions of a Wiener Process. Let $\{W(t) : 0 \leq t \leq \infty\}$ be a Wiener process with $W(0) = 0$ and the parameter $\sigma^2 = 1$ (such a process is called a standard Wiener process). When answering the following questions be sure to justify your answers.

a.) Let $X(t) = W(t + \alpha) - W(t)$ for some $\alpha > 0$. Is $\{X(t)\}$ a Gaussian Process? Explain.

b.) Is $\{X(t)\}$ a stationary process? Explain.

c.) Determine the probability $X(t) > 0$ at time $t = 10$.

d.) Suppose that $\alpha < 1$, does $\frac{1}{N}\sum_{n=1}^{N} X(n)$ converge to a limit in probability as $N \to \infty$? Explain.

e.) Let $B(t) = W(t) - tW(1)$ for $0 \leq t \leq 1$. Is $\{B(t)\}$ a Gaussian Process?

f.) Derive the covariance function for $\{B(t)\}$.

g.) Show that for any t, $B(t)$ is independent of $W(1)$.