Northwestern University
Department of Electrical and Computer Engineering

ELEC ENG 422 Winter 2020

Problem set 3: Date due: Feb. 5, 2020

Announcements:

- The mid-term will be on Feb. 12 during regular class time.

Reading: Sections 1.7-1.8 in the text;

Problems:

1. As we saw in the coupon collector example, often instead of directly evaluating Chernoff bounds, it is more convenient to use approximations (which are also typically referred to as Chernoff bounds). This problem will look at one such example to develop a Chernoff bound for the cumulative arrivals S_n for a Bernoulli Process. Specifically we will use the following bound on the moment generating function of the binomial random variable, S_n,
$$E[e^{rS_n}] \leq e^{np(e^r-1)}$$
where p is the probability of an arrival at each time instant.

   a.) Let $Y = S_n/n$ be the average number of arrivals per time-instant. Using the above bound, develop the following Chernoff bound
   $$\Pr(Y - p > a) \leq \left(\frac{e^a}{(1+\frac{a}{p})^{a+p}} \right)^n$$
   for some $a > 0$ (*Hint: to get this bound you don't need to optimize over* r*, but do need to carefully choose* r*.*)

   b.) For 100 independent and fair coin flips, use your result in the previous part to give a bound on the probability that more than 3/4 of the flips were heads.

2. Exercise 1.26 parts (a) -(c) in Gallager. This shows why the log-moment generating funciton is convex as was indicated in lecture.

3. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of random variables defined by
$$X_n = \begin{cases} n^2, & \text{with probability } n^{-2}, \\ 0, & \text{with probability } 1 - n^{-2}. \end{cases}$$

   a.) Show that X_n converges to a limit in probability and identify the limit.

   b.) Does X_n converge to a limit in the mean square sense? Justify your answer.

4. Given a sequence of random variables $\{X_n\}_{n=1}^{\infty}$, in class we discussed the following useful technique for showing that this sequence convergences almost surely to a random variable X:[1]

   For any $\epsilon > 0$, *and all* $n = 1, 2, 3, \ldots$, *let* $p_n(\epsilon) = \Pr(|X_n - X| > \epsilon)$. X_n *converges to* X *almost surely if* $\sum_{n=1}^{\infty} p_n(\epsilon) < \infty$ *for all* $\epsilon > 0$.

   In other words, if the probability of the events $\{|X_n - X| > \epsilon\}$ becomes small enough as n increases so that their sum converges, then X_n converges to X almost surely. Use this to show that X_n in the previous problem converges to a limit almost surely.

5. A sequence of random variables $\{X_n\}_{n=1}^{\infty}$ is said to *converge in the mean* to a random variable X, if $\lim_{n\to\infty} E(|X_n - X|) = 0$.

   a.) Use Markov's inequality to show that if a sequence of random variables converges in the mean, it converges in probability.

   b.) Does the sequence in Problem 3 converge to a limit in the mean?

6. This problem considers a slight generalization of the weak law of large numbers. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of independent but **not** identically distributed random variables and suppose that there is a constant A so that $\mathrm{var}(X_n) \leq A$ for all n. Show that for all $\epsilon > 0$,

$$\lim_{n\to\infty} \Pr\left\{\left|\frac{S_n}{n} - \frac{E(S_n)}{n}\right| \geq \epsilon\right\} = 0,$$

   where $S_n = X_1 + X_2 + \cdots + X_n$.

7. Exercise 1.34 parts (a) and (b) in Gallager.

8. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of i.i.d. random variables, with finite mean μ. Using a similar argument as in the proof of the central limit theorem sketched in class, show that the characteristic function of $\frac{S_n}{n}$ converges to the characteristic function of the constant μ, where $S_n = X_1 + \cdots + X_n$. (This essentially gives a proof of a law of large numbers using convergence in distribution).

[1] This is a special case of a result known as the Borel-Cantelli Lemma.