Solitons and the KdV Equation

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PHYS305 - Computational Physics

December 2025

Overview

What is a Soliton?

A soliton is a nonlinear wave with [1]:

- permanent shape,
- spatial localization,
- elastic collisions (only a phase shift).

Meaning that it is a wave packet that keeps its shape, speed, and size as it travels.

One popular solution for solitons is given by the Korteweg-de Vries (KdV) equation:

$$\phi_t - 6\phi\phi_x + \phi_{xxx} = 0. \tag{1}$$

$$\frac{\partial \phi}{\partial t} - 6\phi\frac{\partial \phi}{\partial x} + \frac{\partial^3 \phi}{\partial x^3} = 0 \tag{2}$$

Traveling-Wave Ansatz and the reduction to an ODE

We can start by looking for solutions of the form:

$$\phi(x,t) = f(x-ct) = f(X), \qquad X = x - ct. \tag{3}$$

Then:

$$\frac{\partial X}{\partial x} = 1, \qquad \frac{\partial X}{\partial t} = -c. \tag{4}$$

Finally by using the chain rule:

$$\phi_x = f'(X), \qquad \phi_t = -c\, f'(X), \qquad \phi_{xxx} = f'''(X). \tag{5}$$

Plugging this equation into the KdV equation:

$$\phi_t - 6\phi\,\phi_x + \phi_{xxx} = 0, \implies -cf'(X) - 6f(X)f'(X) + f'''(X) = 0. \tag{6}$$

Thus we get the third order ODE:

$$f'''(X) = (c + 6f(X))f'(X). \tag{7}$$

Constant of Motion

Notice that by integrating the ODE, we get

$$f''(X) = cf(X) + 3f(X)^2 + C, \tag{8}$$

which can be rewritten as the constant of motion:

$$C = f''(X) - (c + 3f(X))f(X). \tag{9}$$

Using the initial conditions that we are assuming, alongside the simplification that $c = 1$ throughout this problem, we get

$$f(0) = -\frac{1}{2}, \qquad f'(0) = 0, \qquad f''(0) = \frac{1}{4}, \implies C = 0 \tag{10}$$

This is great, as the constant of motion can be used to determine the accuracy of our numerical solutions later on.

Reduction to a First-Order System

To numerically integrate this, we can rewrite this third-order ODE as a system of three first order ODEs: Define:

$$u_1 = f, \qquad u_2 = f', \qquad u_3 = f''. \tag{11}$$

Then the system becomes:

$$\begin{cases} u_1' = u_2, \longrightarrow u_1(0) = -\frac{1}{2}, \\ u_2' = u_3, \longrightarrow u_2(0) = 0, \\ u_3' = (1 + 6u_1)\, u_2. \longrightarrow u_3(0) = \frac{1}{4}. \end{cases} \tag{12}$$

Now we can use the 4th Order Runge-Kutta(RK4) method to numerically integrate this system over a given interval.

Fourth-Order Runge–Kutta (RK4) Method

We have the system

$$\vec{u}'(X) = \begin{bmatrix} u_2 \\ u_3 \\ (1+6u_1)u_2 \end{bmatrix}, \qquad \vec{u}(0) = \begin{bmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{4} \end{bmatrix}, \tag{13}$$

we use RK4 with step size h for a system of ODEs, where we have:

$$\vec{k}_1 = \vec{F}(X_n,\ \vec{u}_n), \qquad \vec{k}_2 = \vec{F}\left(X_n + \tfrac{h}{2},\ \vec{u}_n + \tfrac{h}{2}\vec{k}_1\right), \tag{14}$$

$$\vec{k}_3 = \vec{F}\left(X_n + \tfrac{h}{2},\ \vec{u}_n + \tfrac{h}{2}\vec{k}_2\right), \qquad \vec{k}_4 = \vec{F}\left(X_n + h,\ \vec{u}_n + h\vec{k}_3\right). \tag{15}$$

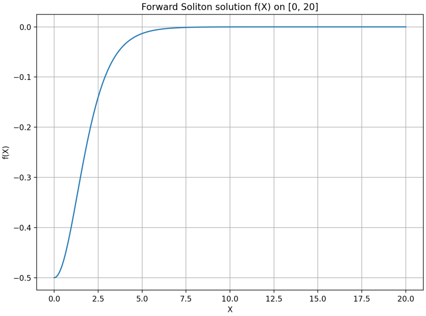
Where we can get $\vec{u}_{n+1}$ by:

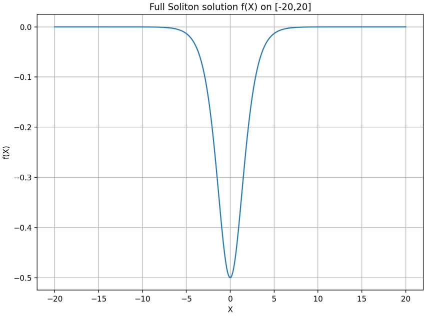
$$\vec{u}_{n+1} = \vec{u}_n + \frac{h}{6}\left(\vec{k}_1 + 2\vec{k}_2 + 2\vec{k}_3 + \vec{k}_4\right). \tag{16}$$

RK4 has global error $\mathcal{O}(h^4)$, which we can verify through the convergence test later on.

Numerical Integration of the ODE

Using the described RK4 scheme, we can integrate the system on the interval $X \in [-20, 20]$ with the previously mentioned initial conditions. Observe that the ODE is symmetric, meaning $f(X) = f(-X)$, so we only have to compute $X > 0$ and then reflect the results to get the full interval of solutions.





Convergence Test: Constant of Motion

To test the accuracy of these numerical solutions, use the constant of motion C(X):

$$C(X) = u_3 - (1 + 3u_1)\, u_1. \tag{17}$$

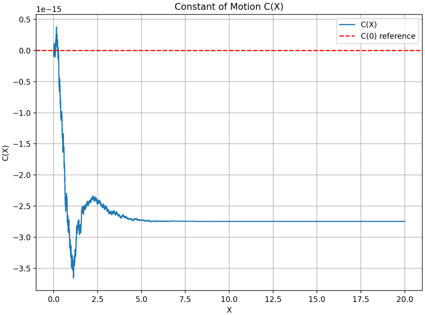
where analytically we know that $C(X) = 0 \quad \forall X$. Calculating the absolute error (Since $C(0) = 0$, we can't do relative error since we'd get a division by 0):

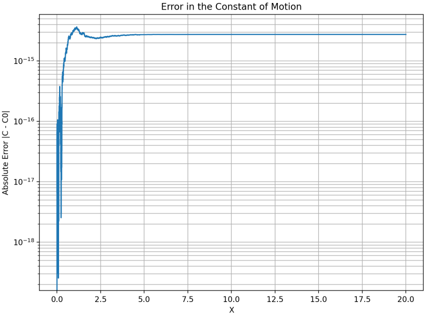
$$\delta C = |C(5) - C(0)| = |C(5)|, \tag{18}$$

for decreasing step sizes h.

$C(X)$ and error in the Constant of Motion

We can create the plots:





So overall, pretty good!

Creating the Traveling Wave

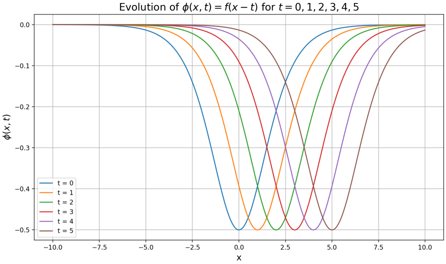
Recall that

$$\phi(x, t) = f(x - t) \tag{19}$$

Using the numerical solution $f(X)$ and a third-order Lagrange interpolation, we evaluate

$$\phi(x, t_k) = f(x - t_k) \tag{20}$$

for $x \in [-10, 10]$ and $t_k = 0, 1, 2, 3, 4, 5$. This is exactly what a soliton should look like as it travels to the right (+x direction)!



Convergence Test: Constant of Motion

To test the order of accuracy, we compute $C(X)$ at $X = 5$ for different step sizes h:

$$C(5; h) = u_3(5; h) - \big(1 + 3u_1(5; h)\big)\, u_1(5; h). \tag{21}$$

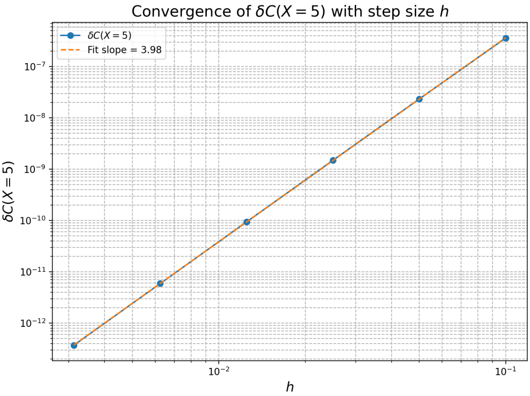
Since analytically $C(X) = C(5) = 0$, we can define the absolute error as

$$\delta C(h) = |C(5; h)|. \tag{22}$$

And so we can see how δC decreases as $h \to 0$.

Convergence of $\delta C(X = 5)$

Drawing a log-log plot:



Where δC approximately is $\propto h^4$ from the slope of the fitted line. Matches the expected error order of RK4.

Self Convergence of $f(5)$

Since in this case the exact solution is unknown, we can test self convergence. Let $f(h)$ be the numerical value of $f(5)$ at step size h, and let h_2 and h_3 be the two finest resolutions (N is the largest for these cases). Then similar to how it was done in the last homework, we form the ratio

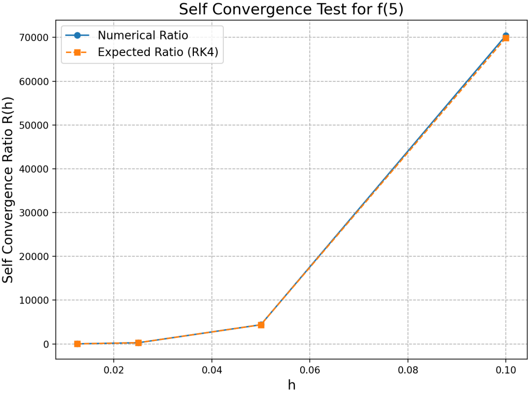
$$R_{\mathrm{num}}(h) = \frac{f(h) - f(h_3)}{f(h_2) - f(h_3)} \tag{23}$$

And

$$R_{\mathrm{expected}}(h) = \frac{(h/h_3)^n - 1}{2^n - 1}. \tag{24}$$

Where for the RK4 method, n is 4.

Self Convergence of $f(5)$ - Continued



The numerical ratio $R_{\text{num}}(h)$ and the theoretical ratio $R_{\text{expected}}(h)$ are very close here!

Richardson Extrapolation for $f(5)$

Finally by using the Richardson extrapolation, we can can determine the error at any time (say f(5)). So

$$f_{h_2} \text{ for } N = 800, \qquad f_{h_3} \text{ for } N = 1600. \tag{25}$$

The 4th-order Richardson estimate is

$$f_R = \frac{2^4 f_{h_3} - f_{h_2}}{2^4 - 1}. \tag{26}$$

Numerically:

$$f_{h_2} \approx -1.3296113376 \times 10^{-2}, \tag{27}$$
$$f_{h_3} \approx -1.3296113344 \times 10^{-2}, \tag{28}$$
$$f_R \approx -1.3296113342 \times 10^{-2}. \tag{29}$$

The estimated error in the finest-resolution value is $|f_R - f_{h_3}| \sim 10^{-12}$.

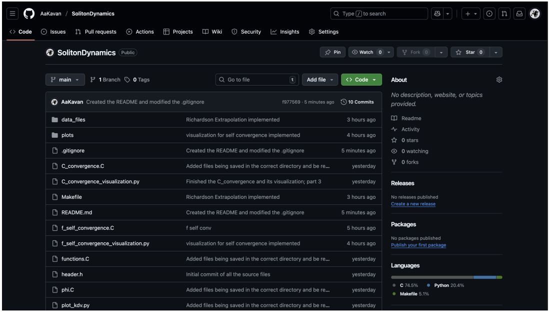
Conclusions

- Successfully solved the KdV ODE using the RK4 method
- Recreated the traveling wave solution for t $=$ 0,1,2,3,4,5
- Convergence test of the constant of motion showed $\delta C \propto h^4$, which matches the expected RK4 accuracy.
- Self-convergence test for $f(5)$ also agreed with this 4-th order convergence
- Found the error to the solutions for $f(X = 5)$ using Richardson extrapolation; the solutions were very accurate!

Conclusions - continued

The code worked extremely accurately!

All of the code for this final project is publicly available on GitHub at https://github.com/AaKavan/SolitonDynamics



References

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