Mid Semester: MSO202M (2023-2024 I)

Date: 18 September 2023 Time: 13:00 pm-15:00 pm Maximum marks: 60

Name:

Instructions: (Read carefully)

- Please enter your NAME and ROLL NUMBER in the space provided on EACH page.
- Only those booklets with name and roll number on every page will be graded. All other booklets will NOT be graded.
- This answer booklet has 12 pages. Check to see if the print is either faulty or missing on any of the pages. In such a case, ask for a replacement immediately.
- Please answer each question ONLY in the space provided. Answers written outside the space provided for it WILL NOT be considered for grading. So remember to use space judiciously.
- For rough work, separate sheets will be provided to you. Write your name and roll number on rough sheets as well. However, they WILL NOT be collected back along with the answer booklet.
- No calculators, mobile phones, smart watches, or other electronic gadgets are permitted in the exam hall.
- Notations: All notations used are as discussed in class.
- All questions are compulsory.
- Do NOT remove any of the sheets in this booklet.

Q 1. (a) Find Möbius transformation T that maps the points $z = i, 2, -2$ to $w = i, 1, -1$. Also, find the fixed points of T.

(b) Is there a Möbius transformation that has exactly one fixed point? Justify.

4+2 marks

a) let w be mobius transformation T of z, such as $w = T(z)$

the cross ratio for points, z_1, z_2, z_3, z_4 can be written as

$$\frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$$ as we know by property of cross ratio that

replacing z with ~~w=T~~ $w \Rightarrow T(z)$ will also give same cross ratio

so $$\frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)} = \frac{(w_1 - w_3)(w_2 - w_4)}{(w_1 - w_4)(w_2 - w_3)}$$, where $z_1 = i, z_2 = 2$, $z_3 = -2, z_4 = z$

& $w_1 \Rightarrow i, w_2 = 1$, $w_3 = -1, w_4 = w$

$$\frac{(i+2)(2-z)}{(i-z)\,4} = \frac{(i+1)(1-w)}{(i-w)\,2}$$

$(2i - iz + 4 - 2z)(i - w) = 2(1-w)[-1 + i - iz - z]$

$\Rightarrow -\cancel{2} - 2iw + z + iwz + 4i - 4w - 2\cancel{i}z + 2z\cancel{i}w$

$\Rightarrow \cancel{2} - \cancel{2} + 2i - 2\cancel{i}z - 2z + 2w - 2\cancel{w}i + 2\cancel{w}iz + 2w\cancel{z}$

$\Rightarrow iwz + 6w = 2i + 3z$, so

$w \Rightarrow \dfrac{3z + 2i}{6 + iz}$ which is the mobius transformation, for fixed point $z(6 + iz) = 3z + 2i \Rightarrow 6z + iz^2 = 3z + 2i \Rightarrow iz^2 - 3z - 2i = 0$

$\Rightarrow$ fixed point are $z_1 \Rightarrow -i$, $z_2 \Rightarrow -2i$

For mobius ~~transfo~~ transfer to have fixed point

$z = \dfrac{az + b}{cz + d} \Rightarrow cz^2 + (d-a)z - b = 0$ so from

here we can see that mobius transfer has ~~to~~ 2 fixed points

for having one fixed point $\boxed{(d-a)^2 + 4bc = 0}$

so it can have fixed point

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Q 2. Evaluate the integral **6 marks**

$$\int_{-\infty}^{\infty} \frac{x \sin 2x}{x^3+3} dx.$$

$\int_{-\infty}^{\infty} \frac{x \sin 2x}{x^3+3}$ can be written as $\Rightarrow I \Rightarrow Im \int_{-\infty}^{\infty} \frac{x e^{i2x}}{x^3+3} \Rightarrow Im \int \frac{z e^{i2z}}{z^3+3}$

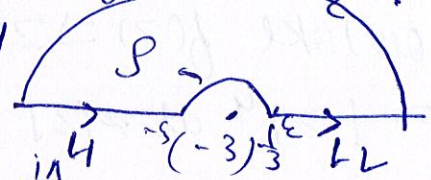
Now as we know by jordan's lemma that if $|f(R)| \to 0$ at $R \to \infty$

so $\int_{C_R} f(z) e^{i\alpha z} = 0$ along a curve C_R now as for

$\left|\frac{x}{x^3+3}\right| \leq \frac{R}{R^3-3}$ so at $R \to \infty$ $\frac{R}{R^3-3} \Rightarrow 0$ so $\int_{C_R} \frac{x e^{i2x}}{x^3+3} = 0$

so /we/can/say/that/ $\int_{-\infty}^{\infty} \frac{x e^{i2x}}{x^3+3} \Rightarrow 2\pi i \times$ Res/

but here there is a singularity at $x \Rightarrow (3)^{\frac{1}{3}}$ so Integral $\Rightarrow$ where $R \to \infty$, the function has singularity at $z \Rightarrow (-3)^{\frac{1}{3}}$



so Integral can be written as and $3^{\frac{1}{3}} e^{i\frac{\pi}{3}} \to$ above axis

$$\int_{C_R} \to 0 + \int_{L_1} + \int_{L_2} + \int_{\rho} = 2\pi i \times [\text{residue at } 3^{\frac{1}{3}} e^{i\frac{\pi}{3}}]$$

$\Rightarrow \cancel{2\pi i \times e^{i2(\pi/3)}}$

also $\int_{\rho}$ would be $\Rightarrow \pi i \times$ residue at $z = -1$

so ans $\Rightarrow 2\pi i$ [res at $3^{\frac{1}{3}} e^{i\pi/3}$] $- \pi i$ [res at -1]

Q 3 Find the number of roots of polynomial $p(z) = z^4 - 8z + 10$ in the unit disk $|z| < 1$ and in annulus $1 < |z| < 3$. **6 marks**

By rouche theorm we can say that if $|f(z)| > |g(z)|$ at $|z| = R$ so the noi of roots of polynomial $f(z)+g(z)$ is same as roots of polynomial $f(z)$, so for $|z| < 1$ let in $p(z)$, $f(z) = 10$, $g(z) \Rightarrow z^4 - 8z$, so $|g(z)| \leq |z|^4 + 8|z|$

~~g(z)~~ $|g(z)| \leq |z|^4 + 8|z| \Rightarrow 9$ for $|z| = 1$ (by triangle inequality), and $|f(z)| \Rightarrow |10| \Rightarrow 10$

so as $|f(z)| > |g(z)|$, so $f(z)$ & $f(z)+g(z) \Rightarrow p(z)$ have same roots, as $f(z) = 10$ has no root so $f(z)+g(z) \Rightarrow p(z)$ will also have no root

so $p(z)$ for $|z| < 1$ ~~has~~ have zero roots.

ii) Now take $f(z) \Rightarrow z^4$ & $g(z) \Rightarrow -8z + 10$, so $|f(z)|$ ~~at~~ $\Rightarrow |z|^4$ at $|z| = 3 \Rightarrow 81$, whereas $|g(z)| \leq 8|z| + 10$ so for $|z| = 3$, $|g(z)| \leq 34$, so $|f(z)| > |g(z)|$ so $f(z)$ & $p(z)$ have same roots for $|z| < 3 \Rightarrow$ roots for $f(z)$ as z^4 ~~#~~ has 4 roots $p(z)$ has 4 roots in $|z| < 3$ and as ~~f~~ $p(z)$ has no roots in $|z| < 1$, ~~f f(z)~~ so $p(z)$ has all 4 roots in $1 < |z| < 3$, so $p(z)$ has 4 roots between $1 < |z| < 3$

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Q 4. Write Cauchy Riemann equation in Polar coordinates. Let $u(r,\theta) = \sqrt{r}\cos(\theta/2)$. Find $f'(z)$ and $f(z)$. **6 marks**

Cauchy Reiman equation in polar coordinate is

$$r u_r = v_\theta \quad \& \quad r v_r = -u_\theta,$$

$u_r \Rightarrow \frac{1}{2\sqrt{r}} \cos(\theta/2)$, $r u_r \Rightarrow \frac{\sqrt{r}}{2}\cos(\theta/2) = \frac{\partial v}{\partial \theta}$

so $v \Rightarrow \int \frac{\sqrt{r}}{2}\cos\left(\frac{\theta}{2}\right) d\theta \Rightarrow \boxed{\sqrt{r}\sin\left(\frac{\theta}{2}\right) + f(r),}$

$v_r \Rightarrow \frac{1}{2\sqrt{r}}\sin\left(\frac{\theta}{2}\right) + f'(r)$, $r v_r \Rightarrow \frac{\sqrt{r}}{2}\sin\left(\frac{\theta}{2}\right) + r f'(r)$

$u_\theta \Rightarrow -\frac{\sqrt{r}}{2}\sin\left(\frac{\theta}{2}\right)$, $-u_\theta \Rightarrow \frac{\sqrt{r}}{2}\sin\left(\frac{\theta}{2}\right)$, as $r v_r = u_\theta$

we can write that, $\cancel{\frac{\sqrt{r}}{2}\sin\left(\frac{\theta}{2}\right)} + f'(r) r = \cancel{\frac{\sqrt{r}}{2}\sin\left(\frac{\theta}{2}\right)}$ so we can say that $f'(r) = 0$, $f(r) =$ constant, WLOG let constant be zero

so $v \Rightarrow \sqrt{r}\sin\left(\frac{\theta}{2}\right)$, $u \Rightarrow \sqrt{r}\cos\left(\frac{\theta}{2}\right)$

so $f(r,\theta) \Rightarrow \sqrt{r}\cos\left(\frac{\theta}{2}\right) + \sqrt{r}\sin\left(\frac{\theta}{2}\right)$, this could be written in $f(z)$ in terms of (x,y) as we know that $x = r\cos\theta$ & $y = r\sin\theta$

$\cos\frac{\theta}{2} \Rightarrow \sqrt{\frac{1+\cos\theta}{2}} \Rightarrow \sqrt{\frac{1+\frac{x}{r}}{2}} \Rightarrow \sqrt{\frac{x+r}{2r}}$, $\sqrt{r}\cos\frac{\theta}{2} \Rightarrow \sqrt{\frac{x+r}{2}}$

$\sin\frac{\theta}{2} \Rightarrow \sqrt{\frac{1-\cos\theta}{2}} \Rightarrow \sqrt{r}\sin\frac{\theta}{2} \Rightarrow \sqrt{\frac{\sqrt{x^2+y^2}-x}{2}}$

so $f(z) \Rightarrow \sqrt{\frac{x+\sqrt{x^2+y^2}}{2}} + i\sqrt{\frac{\sqrt{x^2+y^2}-x}{2}}$ $\Rightarrow \sqrt{\frac{x+\sqrt{x^2+y^2}}{2}}$

$f'(r,\theta) \Rightarrow \frac{1}{2\sqrt{r}}\cos\left(\frac{\theta}{2}\right) + \frac{\sqrt{r}}{2}\cos\left(\frac{\theta}{2}\right) \Rightarrow \cancel{\frac{1}{2\sqrt{2}}\frac{\sqrt{x+\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}}}$

$e^{i\theta}$ $\Rightarrow$ this could be expressed in terms of $f(z)$

Q 5. Find the annulus of convergence of the series **6 marks**

$$\sum_{n=0}^{\infty}\left(\frac{z}{5}-i\right)^{n^2}+\sum_{n=1}^{\infty}\frac{(-1)^n}{2^n(z-5i)^n}.$$

$$\sum_{n=0}^{\infty}\left(\frac{z}{5}-i\right)^{n^2} \Rightarrow \sum_{n=0}^{\infty}\left(\frac{1}{5}\right)^{n^2} \times [z-5i]^{n^2}$$

as we know that radius of convergence $\Rightarrow \frac{1}{R} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|^{\frac{1}{(2n+1)}}$

$$\frac{1}{R} = \lim_{n\to\infty}\left(\frac{\left(\frac{1}{5}\right)^{(n+1)^2}}{\left(\frac{1}{5}\right)^{n^2}}\right)^{\frac{1}{2n+1}}$$

$$\Rightarrow \lim_{n\to\infty}\left(\left(\frac{1}{5}\right)^{2n+1}\right)^{\frac{1}{2n+1}} \Rightarrow \frac{1}{5}$$

so $R_1 \Rightarrow 5$, domain of converge $\Rightarrow |z-5i| \le 5$

then $\sum_{n=1}^{\infty}\frac{(-1)^n}{2^n} \times (z-5i)^{-n}$

$$\Rightarrow \frac{1}{R} = \lim_{n\to\infty}\left|\left(\frac{\left(\frac{-1}{2}\right)^{n+1}}{\left(\frac{-1}{2}\right)^{n}}\right)\right|^{\frac{1}{n+1+n}} \Rightarrow \frac{1}{R} = \frac{1}{2}$$

so $R \Rightarrow 2$, domain of convergence is $|z-5i| \le 2$

so the annulus of converge would be written as

~~$2 < |z-5i| < 5$~~ $0 < |z-5i| < 2$

as 2nd function converges of $R < 2$ and 1st function converges of $R < 5$, ~~calculs~~

Q 6. (a) Write the statement of Schwarz lemma.

(b) Suppose $f(z)$ is analytic for $|z| \leqslant 1$ and $|f(z)| \geqslant 1$ for $|z| \leqslant 1$. If $f(0) = 1$, then show that f is constant.

3+3 marks

a) Schwarz lemma states that if a function is analytic in domain $D \Rightarrow |z| < R$ and we know that $f(z)$ is bounded, $|f(z)| < M$ so we can write that

$$\boxed{|f(z)| \leq \frac{M|z|}{R}}$$

b) as $f(z)$ is analytic in $|z| \leq 1$ so we can also state that $1/f(z)$ will be analytic in $|z| \leq 1$ as we know that $f(z) \neq 0$ as $|f(z)| \geq 1$

so $1/f(z)$ is also analytic and $\left|\frac{1}{f(z)}\right| \leq 1$

~~so by this we will get to know that the~~ so by schwarz lemma we get to know that $\left|\frac{1}{f(z)}\right| \leq \frac{1 \times |z|}{1}$, and as it is maximum at boundary so $\frac{1}{f(z)}$ is max at boundary for $|z| = 1$, so $\frac{1}{f(z)}$ has max value of 1, ~~also~~ as ~~$\frac{1}{f(z)}$~~ has max ~~value at~~ $z =$ also as $f(0) \Rightarrow 1$ we get to know that function is constant, so if $\frac{1}{f(z)}$ is constant, $f(z)$ is also constant. Also as it satifies Louiville's thm. we can say it is constant.

Name:...

Q 7. Find singularities and residues of the function $f(z)$ in extended complex plane, where

$$f(z) = \frac{z^3+3}{z(z^2-4)^2}.$$

Show that the sum of all residue is zero. **6 marks**

~~singularity~~ $f(z) \Rightarrow \dfrac{z^3+3}{z(z-2)^2(z+2)^2}$ so the function is

singular at $z=0$, $z=2$, $z=-2$ residue at $z=0$

$\Rightarrow f(z) \times z \big|_{z=0} \Rightarrow \dfrac{3}{16}$

residue at $z=2 \Rightarrow$ ~~$f(z)$~~ residue $\Rightarrow \dfrac{1}{1!}\dfrac{d}{dz} f(z) \times (z-2)^2$

as it is a pole of order 2

$\Rightarrow \dfrac{d}{dz}\dfrac{z^3+3}{z(z+2)^2}\Big|_{z=2} \Rightarrow \dfrac{3z^2(z(z+2)^2) - (z^3+3)[(z+2)^2 + 2z(z+2)]}{(z(z+2)^2)^2}$

$\Rightarrow \dfrac{384-352}{32\times 32} \Rightarrow \dfrac{1}{32}$

~~singularity~~ residue at $z=-2$

$\dfrac{d}{dz}\dfrac{z^3+3}{z(z-2)^2} \times 1 \Big|_{z=-2} \Rightarrow \dfrac{3z^2(z(z-2)^2) - (z^3+3)[(z-2)^2 + 2z(z-2)]}{z(z-2)^2}$

$\Rightarrow \dfrac{-384-160}{32\times 32} \Rightarrow \dfrac{-17}{32}$

residue at $z=\infty$ (as it is extended complex plane)

$\Rightarrow \dfrac{5}{16}$

so sum of residues

$\Rightarrow \dfrac{3}{16} + \dfrac{5}{16} + \dfrac{1}{32} - \dfrac{17}{32} = 0$ (sum of residue in extended plane $= 0$)

Name:..

Q 8. (a) Let C_R be the circle of radius R centered at origin and oriented clock-wise direction. Show that

$$\lim_{R\to\infty}\int_{C_R}\frac{Log(z^3)}{z^{3/2}}dz = 0. \quad \left(z^\alpha := \exp(\alpha \text{ Log } z)\right)$$

(b) The value of $(2^i)^i =$ **4 + 2 marks**

let $I = \lim_{R\to\infty} \int_{C_R} \frac{\log(z^3)}{z^{3/2}} dz$

a) so as we know that $\log(z^3) \Rightarrow \ln|z|^3 + i\,Arg(z^3)$

$\Rightarrow 3\ln|z| + i\,Arg(z^3)$, so max value of $|\log(z^3)|$

$\leq 3\ln|z| + \pi$ [by triangle inequality and as Arg's max value is π]

$$\left|\int_{C_R}\frac{\log(z^3)}{z^{3/2}}\right| \leq \int_{C_R}\left|\frac{\log(z^3)}{z^{3/2}}\right| \leq \left|\frac{3\ln|z| + \pi}{z^{3/2}}\right| \times \pi R$$

so at $R\to\infty$ $\cancel{\left|\frac{3\ln|R| + \pi}{R^{\frac{3}{2}}}\right| \to 0}$ $|I| \leq \left|\frac{3\ln|R| + \pi}{R^{\frac{1}{2}}}\right| \times 2\pi$

and clearly solving by L'hopital or by limits we can directly see that $\lim_{R\to\infty}\left|\frac{3\ln|R| + \pi}{R^{\frac{1}{2}}}\right| \times 2\pi = 0$, so as $|I| \leq 0$

$I = 0$, hence proved

[could solve limit by L'hopital $\Rightarrow \frac{3\times\frac{1}{R}}{R^{\frac{1}{2}}}$ at $R\to\infty = 0$]

b) $(2^i)^i$, let $z = (2^i)^i$

$\cancel{\log z \Rightarrow i\log(2^i)}$

$\log z \Rightarrow i\log(2^i)$, $2^i \Rightarrow e^{i\log 2}$ $\Rightarrow \cancel{i[i\ln|2| + i\log 2]}$ $\cancel{i\times\log(2)}$

$\log(2^i) \Rightarrow \log[\cos(\log 2) + i\sin(\log 2)]$ $\Rightarrow \cancel{i\log[i\times\ln 2]}$

$$\boxed{\text{so } z \Rightarrow e^{i\log[\cos(\log 2) + i\sin(\log 2)]}} \Rightarrow \text{Ans}$$

Q 9. (a) Discuss the continuity of $f(x, y)$, where

$$f(x,y) = \begin{cases} \frac{x^3-2y^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

(b) Let $f = u + iv$ be an analytic function on a domain D. Is uv a harmonic function on D? Justify. **3+3 marks**

a) For the given function to be continuous at $z=0$ the function should be 0 in any direction we approach 0

let $y = mx$ ~~for~~ be the direcn $f(x,y) \Rightarrow \frac{x^3-2m^3x^3}{x^2+m^2x^2}$

$f(x,y) \Rightarrow \frac{x^3(1-2m^3)}{x^2(1+m^2)} \Rightarrow \frac{x(1-2m^3)}{1+m^2}$, so we can say that

$f(x,y) \to 0$ as $x, y \to (0,0)$ here in this case $f(x,y) \to 0$ for any direcn so we can say that $f(x,y)$ is a continuous function

~~b) let $f \Rightarrow u+iv$, $\frac{\partial}{\partial x^2} uv \Rightarrow \frac{\partial}{\partial x}[u_x v + v_x u]$~~

b) Yes uv would be a harmonic function $\Rightarrow$ ~~$u_{xx} v + u_x v_x + v_{xx} u + v_x u_x$~~

Name:...

Q 10. (a) Suppose $f(z)$ is an entire function. Show that $f(\mathbb{R}) \subset (\mathbb{R})$ if and only if $f^{(n)}(0) \in \mathbb{R}$ $\forall\, n = 0, 1, 2 \cdots$

(b) Let $\omega \neq 1$ be an n-th root of unity Show that

$$1 + 2\omega + 3\omega^2 + \cdots + n\omega^{n-1} = \frac{-n}{1-\omega}. \qquad \textbf{4 + 2marks}$$

b) $S = 1 + 2\omega + 3\omega^2 \text{ ---- } n\omega^{n-1}$

$\omega S \Rightarrow \omega + 2\omega^2 + 3\omega^3 - (n-1)\omega^{n-1} + n\omega^n$

$S(1-\omega) \Rightarrow 1 + \omega + \omega^2 + \omega^3 \text{ ---- } \omega^{n-1} - n$

$(\omega^n = 1$

$\Rightarrow$ As sum of roots of unity $= 0$)

$S(1-\omega) = 0 - n, \; S \Rightarrow \dfrac{-n}{1-\omega}$ hence proved

a) $\dfrac{f^n(0)}{n!} = \dfrac{1}{2\pi i} \displaystyle\int \frac{f(z)}{z^{n+1}}$

~~Safe~~ So $f(z)$ is an entire function, so the ~~Integral will be~~ so the integral would be

$\dfrac{f^n(0)}{n!} \times 2\pi i$

or we can write

$\cancel{f^n(0)}$

$\cancel{I \Rightarrow \dfrac{1}{2\pi i} \times \dfrac{1}{n!}}$

Q 10 Use this space for the solution of question 10 only

c