E-Test 2

Question 1

Apply the Fourier-Motzkin method, eliminating first x then y then z, to determine the satisfiability of the following set of constraints over the reals:
$\{4x + 4y - 5z \leq 0, \quad -4x + 3y + 5z \leq -2, \quad 2x + 2y - 4z \leq -2, \quad -5y - 5z \leq -3, \quad 4x - 2y - 4z \leq 4, \quad -1x - 4z \leq 5\}$
Please collect all constraints that are generated in during the whole procedure. Then remove all non-trivial constraints (i.e. constraints where at least one of the variables has a non-zero coefficient), and iteratively remove any constraint that is a multiple of another constraint in the set or a multiple of an input constraint. How many constraints remain? Please answer by entering the number with digits, without whitespaces.

Antwort: 15 ✔

Remember: When applying Fourier-Motzkin we must first isolate the variable (i.e., $x < y + z$), and then pair all lower with all upper bounds.

<u>Eliminate x:</u>

$4x + 4y - 5z \leq 0$	$x \leq -y + \frac{5}{4}z$	Original Constraint
$-4x + 3y + 5z \leq -2$	$\frac{3}{4}y + \frac{5}{4}z + \frac{1}{2} \leq x$	Original Constraint
$2x + 2y - 4z \leq -2$	$x \leq -y + 2z - 1$	Original Constraint
$-5y - 5z \leq -3$	Not relevant here, keep it for later use when eliminating y/z.	
$4x - 2y - 4z \leq 4$	$x \leq \frac{1}{2}y + z + 1$	Original Constraint
$-1x - 4z \leq 5$	$-4z - 5 \leq x$	Original Constraint

<u>Pair all lower with all upper bounds, and then eliminate y:</u>

$\frac{3}{4}y + \frac{5}{4}z + \frac{1}{2} \leq -y + \frac{5}{4}z$	$y \leq -\frac{2}{7}$	(1)
$\frac{3}{4}y + \frac{5}{4}z + \frac{1}{2} \leq -y + 2z - 1$	$y \leq \frac{3}{7}z - \frac{6}{7}$	(2)
$\frac{3}{4}y + \frac{5}{4}z + \frac{1}{2} \leq \frac{1}{2}y + z + 1$	$y \leq 2 - z$	(3)
$-4z - 5 \leq -y + \frac{5}{4}z$	$y \leq \frac{21}{4}z + 5$	(4)
$-4z - 5 \leq -y + 2z - 1$	$y \leq 6z + 4$	(5)
$-4z - 5 \leq \frac{1}{2}y + z + 1$	$-10z - 12 \leq y$	(6)
$-5y - 5z \leq -3$	$-z + \frac{3}{5} \leq y$	Original Constraint

<u>Pair all lower with all upper bounds, and eliminate z:</u>

$-10z - 12 \leq -\frac{2}{7}$	$z \geq -\frac{41}{35}$	(7)
$-10z - 12 \leq \frac{3}{7}z - \frac{6}{7}$	$z \geq -\frac{78}{73}$	(8)
$-10z - 12 \leq 2 - z$	$z \geq -\frac{14}{9}$	(9)
$-10z - 12 \leq \frac{21}{4}z + 5$	$z \geq -\frac{68}{61}$	(10)
$-10z - 12 \leq 6z + 4$	$z \geq -1$	(11)
$-z + \frac{3}{5} \leq -\frac{2}{7}$	$z \geq \frac{31}{35}$	(12)
$-z + \frac{3}{5} \leq \frac{3}{7}z - \frac{6}{7}$	$z \geq \frac{31}{50}$	(13)
$-z + \frac{3}{5} \leq 2 - z$	$\frac{3}{5} \leq 2$	Trivial!
$-z + \frac{3}{5} \leq \frac{21}{4}z + 5$	$z \geq -\frac{88}{125}$	(14)
$-z + \frac{3}{5} \leq 6z + 4$	$z \geq -\frac{17}{35}$	(15)

Thus, we have generated 15 new, non-trivial constraints in total.

Question 2

Construct the simplex tableau and apply the simplex method to the following constraint set until termination:

$s_0 = 1x_0 - 1x_1 - 1x_2 \quad s_0 \leq 2$
$s_1 = 1x_0 + 1x_1 + 1x_2 \quad s_1 \leq -1$
$s_2 = 1x_0 + 3x_1 + 2x_2 \quad s_2 \leq 1$
$s_3 = -3x_0 - 3x_1 - 1x_2 \quad s_3 \leq -1$
$s_4 = -1x_0 + 1x_1 + 2x_2 \quad s_4 \leq -2$

When chosing pivot variables, use the order $x_0 < \ldots < x_2 < s_0 < \ldots < s_4$ and take the smallest possible variable.
How many of the (original and slack) variables have integer values in the final assignment? Please answer by entering the number with digits, without whitespaces.

Antwort: 6

Sidenote:
You can calculate the table as we've learned in the lecture by simply constructing all the formulas, replacing them, compute it, etc. However, you can also just learn the following 4 rules by heart for the four cases (orange, green, blue, white) and apply them much quicker:

Pivoting B_i with NB_j

	$\cdots$	NB_j	$\cdots$	NB_l	$\cdots$
$\cdots$					
B_i		a_{ij}		a_{il}	
$\cdots$					
B_k		a_{kj}		a_{kj}	
$\cdots$					

$\rightarrow$

	$\cdots$	NB_j	$\cdots$	NB_l	$\cdots$
$\cdots$					
B_i		$\frac{1}{a_{ij}}$		$-\frac{a_{il}}{a_{ij}}$	
$\cdots$					
B_k		$\frac{a_{kj}}{a_{ij}}$		$a_{kj} - \frac{a_{kj}a_{il}}{a_{ij}}$	
$\cdots$					

<u>Construct the Simplex Tableau</u>

$s_0 \leq 2 \qquad s_1 \leq -1 \qquad s_2 \leq 1 \qquad s_3 \leq -1 \qquad s_4 \leq -2$

	$x_0[0]$	$x_1[0]$	$x_2[0]$
$s_0[0]$	1	-1	-1
$s_1[0]$	1	1	1
$s_2[0]$	1	3	2
$s_3[0]$	-3	-3	-1
$s_4[0]$	-1	1	2

s_1 violates its bounds. The first Non-Basic Variable suitable for Pivoting is x_0.

	$s_1[-1]$	$x_1[0]$	$x_2[0]$
$s_0[-2]$	1	-2	-2
$x_0[-1]$	1	-1	-1
$s_2[-1]$	1	2	1
$s_3[3]$	-3	0	2
$s_4[1]$	-1	2	3

s_3 violates its bounds. The first Non-Basic Variable suitable for Pivoting is x_2.

	$s_1[-1]$	$x_1[0]$	$s_3[-1]$
$s_0[3]$	-2	-2	-1
$x_0[1]$	$-\frac{1}{2}$	-1	$-\frac{1}{2}$
$s_2[-3]$	$\frac{5}{2}$	2	$\frac{1}{2}$
$x_2[-2]$	$\frac{3}{2}$	0	$\frac{1}{2}$
$s_4[-5]$	$\frac{7}{2}$	2	$\frac{3}{2}$

s_0 violates its bounds. The first Non-Basic Variable suitable for Pivoting is x_1.

	$s_1[-1]$	$s_0[2]$	$s_3[-1]$
x_1[-1.5]	-1	$-\frac{1}{2}$	$-\frac{1}{2}$
x_0[2.5]	$-\frac{3}{2}$	$\frac{1}{2}$	0
$s_2[-6]$	$\frac{9}{2}$	-1	$-\frac{1}{2}$
$x_2[-2]$	$\frac{3}{2}$	0	$\frac{1}{2}$
$s_4[-8]$	$\frac{3}{2}$	-1	$\frac{1}{2}$

All Side-Conditions are fulfilled. Subsequently, we have 6 Integer Values in our final Tableau.

Question 3

How many final equivalence classes are defined by Algorithm 2 in Lecture 13 for the following conjunction (set) of equalities and disequalities?
$x \neq F(F(u)) \wedge x = F(z) \wedge y = z \wedge v = x \wedge y = v \wedge x = z$
Please answer by entering the digit without white spaces.

Antwort: 4

1. Start by adding a fresh equivalence class for each variable and UF (also Sub-UFs!):
   $\langle x\rangle, \langle y\rangle, \langle z\rangle, \langle u\rangle, \langle v\rangle, \langle \underbrace{f_1}_{=F(u)} \rangle, \langle \underbrace{f_2}_{=F(F(u))} \rangle, \langle \underbrace{f_3}_{F(z)} \rangle$
2. Go over each equality and join their corresponding Equivalence-Classes:
   a. $x = F(z)$: $\langle x, f_3\rangle, \langle y\rangle, \langle z\rangle, \langle u\rangle, \langle v\rangle, \langle f_1\rangle, \langle f_2\rangle$
   b. $y = z$: $\langle x, f_3\rangle, \langle y, z\rangle, \langle u\rangle, \langle v\rangle, \langle f_1\rangle, \langle f_2\rangle$
   c. $v = x$: $\langle x, f_3, v\rangle, \langle y, z\rangle, \langle u\rangle, \langle f_1\rangle, \langle f_2\rangle$
   d. $y = v$: $\langle x, f_3, v, y, z\rangle, \langle u\rangle, \langle f_1\rangle, \langle f_2\rangle$
   e. $x = z$: $\langle x, f_3, v, y, z\rangle, \langle u\rangle, \langle f_1\rangle, \langle f_2\rangle$
3. Join Equivalence-Classes of UFs to account for Functional Congruence. That is, if either explicitly $F(x) = F(y)$ or if x, y are in the same Equivalence-Class.
   In our case, we don't have to do anything, because the arguments u (from f_1), f_1 (from f_2), and z (from f_3) are all in distinct equivalence classes.

So, we end up with 4 different equivalence classes.

Question 4

Please match the following interval arithmetic terms with their meaning.

$([7;18] + [-4;8]) - [-2,3]$	[0;28]
$([2;14] - [-10;4]) * [-7;3]$	[-168;72]
$([9;17] * [-3;8]) + [-8;3]$	[-59;139]
$[10;20]^2 - [5;16]$	[84;395]
$[14;28] \div [2;7]$	[2;14]

When applying interval arithmetic, a rule of thumb is that our lower bound is the "lowest possible result" and the upper bound our "biggest possible result". So, i.e.,when subtracting two intervals $A - B$, the lower bound is the lowest value of A minus the biggest value of B.

1. $([7;18] + [-4;8]) - [-2;3] = [3;26] - [-2;3] = [0;28]$
2. $([2;14] - [-10;4]) \cdot [-7;3] = [-2;24] \cdot [-7;3] = [-168;72]$
3. $([9;17] \cdot [-3;8]) + [-8;3] = [-51;136] + [-8;3] = [-59;139]$
4. $[10;20]^2 - [5;16] = ([10^2;20^2] \cap [0;\infty]) - [5;16] = [100;400] - [5;16] = [84;395]$
5. $[14;28] \div [2;7] = [14;28] \cdot \left[\frac{1}{7};\frac{1}{2}\right] = [2;14]$

Question 5

Interval Newton Method.

1. Compute $f(s_0)$
   $f(s_0) = 2^3 + 2^2 - 1 = 11$
2. Derive $f'(x)$ and compute $f'(A)$
   $f'(x) = 3x^2 + 2x$
   $f'(A) = f'([1;3]) = 3[1;3]^2 + 2[1;3] = 3[1;9] + [2;6] = [5;33]$
3. Compute the Newton-Interval
   $$N = s_0 - \frac{f(s_0)}{f'(A)} = 2 - \frac{11}{[5;33]} = 2 - 11 \cdot \left[\frac{1}{33};\frac{1}{5}\right] = [2;2] - \left[\frac{1}{3};\frac{11}{5}\right] = \left[-\frac{1}{5};\frac{5}{3}\right]$$
4. Compute the new contracted interval by intersecting A and N:
   $$A_{new} = A \cap N = [1;3] \cap \left[-\frac{1}{5};\frac{5}{3}\right] = \left[1;\frac{5}{3}\right] = [1;1.667]$$

As the integer part of the upper bound of the resulting interval A_{new} (rounded downwards) we get 1.