F-Test 2

Question 1

Apply the Fourier-Motzkin method, eliminating first x then y then z, to determine the satisfiability of the following set of constraints over the reals: $\{4x+4y-5z \leq 0, \quad -4x+3y+5z \leq -2, \quad 2x+2y-4z \leq -2, \quad -5y-5z \leq -3, \quad 4x-2y-4z \leq 4, \quad -1x-4z \leq 5\}$ Please collect all constraints that are generated in during the whole procedure. Then remove all non-trivial constraints (i.e. constraints where at least one of the variables has a non-zero coefficient), and iteratively remove any constraint that is a multiple of another constraint in the set or a multiple of an input constraint. How many constraints remain? Please answer by entering the number with digits, without whitespaces.

Antwort: 15

Remember: When applying Fourier-Motzkin we must first isolate the variable (i.e., x < y + z), and then pair all lower with all upper bounds.

Eliminate *x*:

 $-5y - 5z \le -3$

 $4x + 4y - 5z \le 0 x \le -y + \frac{5}{4}z$ $-4x + 3y + 5z \le -2 \frac{3}{4}y + \frac{5}{4}z + \frac{1}{2} \le x$ **Original Constraint Original Constraint** $2x + 2y - 4z \le -2$ $x \leq -y + 2z - 1$ **Original Constraint** $-5y - 5z \le -3$ Not relevant here, keep it for later use when eliminating y/z. $x \le \frac{1}{2}y + z + 1$ $-4z - 5 \le x$ $4x - 2y - 4z \le 4$ **Original Constraint** $-1x - 4z \le 5$ **Original Constraint**

Pair all lower with all upper bounds, and then eliminate y:

Pair all lower with all upper bounds, and then eliminate
$$y$$
:
$$\frac{3}{4}y + \frac{5}{4}z + \frac{1}{2} \le -y + \frac{5}{4}z \qquad y \le -\frac{2}{7} \qquad (1)$$

$$\frac{3}{4}y + \frac{5}{4}z + \frac{1}{2} \le -y + 2z - 1 \qquad y \le \frac{3}{7}z - \frac{6}{7} \qquad (2)$$

$$\frac{3}{4}y + \frac{5}{4}z + \frac{1}{2} \le \frac{1}{2}y + z + 1 \qquad y \le 2 - z \qquad (3)$$

$$-4z - 5 \le -y + \frac{5}{4}z \qquad y \le \frac{21}{4}z + 5 \qquad (4)$$

$$-4z - 5 \le -y + 2z - 1 \qquad y \le 6z + 4 \qquad (5)$$

$$-4z - 5 \le \frac{1}{2}y + z + 1 \qquad -10z - 12 \le y \qquad (6)$$

Original Constraint

 $-z + \frac{3}{5} \le y$

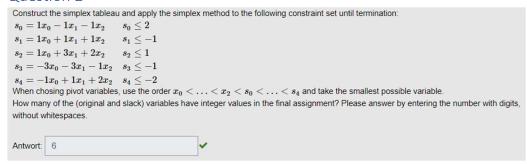
Pair all lower with all upper bounds, and eliminat

Pair all lower with all upper bounds,	and eliminate z:	
$-10z - 12 \le -\frac{2}{7}$	$z \ge -\frac{41}{35}$	(7)
$-10z - 12 \le \frac{3}{7}z - \frac{6}{7}$	$z \ge -\frac{78}{73}$	(8)
$-10z - 12 \le 2 - z$	$z \ge -\frac{14}{9}$	(9)
$-10z - 12 \le \frac{21}{4}z + 5$	$z \ge -\frac{68}{61}$	(10)
$-10z - 12 \le 6z + 4$	$z \ge -1$	(11)
$-z + \frac{3}{5} \le -\frac{2}{7}$	$Z \ge \frac{31}{35}$	(12)
$-z + \frac{3}{5} \le \frac{3}{7}z - \frac{6}{7}$	$z \ge \frac{31}{50}$	(13)
$-z + \frac{3}{5} \le 2 - z$	$\frac{3}{5} \le 2$	Trivial!
$-z + \frac{3}{5} \le \frac{21}{4}z + 5$	$z \ge -\frac{88}{125}$	(14)
$-z + \frac{3}{5} \le 6z + 4$	$z \ge -\frac{17}{35}$	(15)

Thus, we have generated 15 new, non-trivial constraints in total.

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Question 2



Sidenote:

You can calculate the table as we've learned in the lecture by simply constructing all the formulas, replacing them, compute it, etc. However, you can also just learn the following 4 rules by heart for the four cases (orange, green, blue, white) and apply them much quicker:



	 NB_j	 NB_l	
•••			
$\boldsymbol{B_i}$	a_{ij}	a_{il}	
B_k	a_{kj}	a_{kj}	

	•••	NB_j	 NB_l	
•••				
B_i		$\frac{1}{a_{ij}}$	$-\frac{a_{il}}{a_{ij}}$	
B_k		$\frac{a_{kj}}{a_{ij}}$	$a_{kj} - \frac{a_{kj}a_{il}}{a_{ij}}$	

Construct the Simplex Tableau

$s_0 \le 2$	$s_1 \le -1$		
	$x_0[0]$	$x_1[0]$	$x_2[0]$
$s_0[0]$	1	-1	-1
$s_1[0]$	1	1	1
$s_2[0]$	1	3	2
$s_3[0]$	-3	-3	-1
[N] o	_1	1	2

$$s_3 \le -1 \qquad s_4 \le -2$$

 s_1 violates its bounds. The first Non-Basic Variable suitable for Pivoting is x_0 .

 $s_2 \leq 1$

	$s_1[-1]$	$x_1[0]$	$x_2[0]$
$s_0[-2]$	1	-2	-2
$x_0[-1]$	1	-1	-1
$s_2[-1]$	1	2	1
$s_3[3]$	-3	0	2
$s_4[1]$	-1	2	3

 s_3 violates its bounds. The first Non-Basic Variable suitable for Pivoting is x_2 .

	$s_1[-1]$	$x_1[0]$	$s_3[-1]$
$s_0[3]$	-2	-2	-1
$x_0[1]$	$-\frac{1}{2}$	-1	$-\frac{1}{2}$
$s_2[-3]$	<u>5</u> 2	2	$\frac{1}{2}$
$x_2[-2]$	$\frac{3}{2}$	0	$\frac{1}{2}$
$s_4[-5]$	$\frac{7}{2}$	2	$\frac{3}{2}$

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 s_0 violates its bounds. The first Non-Basic Variable suitable for Pivoting is x_1 .

	$s_1[-1]$	$s_0[2]$	$s_3[-1]$
<i>x</i> ₁ [-1.5]	-1	$-\frac{1}{2}$	$-\frac{1}{2}$
$x_0[2.5]$	$-\frac{3}{2}$	$\frac{1}{2}$	0
$s_2[-6]$	9 2	-1	$-\frac{1}{2}$
$x_2[-2]$	$\frac{3}{2}$	0	$\frac{1}{2}$
s ₄ [-8]	$\frac{3}{2}$	-1	$\frac{1}{2}$

All Side-Conditions are fulfilled. Subsequently, we have 6 Integer Values in our final Tableau.

Question 3

1. Start by adding a fresh equivalence class for each variable and UF (also Sub-UFs!):

$$\begin{array}{c} \langle x \rangle, \langle y \rangle, \langle z \rangle, \langle u \rangle, \langle v \rangle, \langle \underbrace{f_1}_{=F(u)} \rangle, \langle \underbrace{f_2}_{=F(F(u))} \rangle, \langle \underbrace{f_3}_{=F(z)} \rangle \end{array}$$

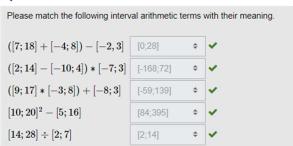
2. Go over each equality and join their corresponding Equivalence-Classes:

a.
$$x = F(z)$$
: $\langle x, f_3 \rangle, \langle y \rangle, \langle z \rangle, \langle u \rangle, \langle v \rangle, \langle f_1 \rangle, \langle f_2 \rangle$
b. $y = z$: $\langle x, f_3 \rangle, \langle y, z \rangle, \langle u \rangle, \langle v \rangle, \langle f_1 \rangle, \langle f_2 \rangle$
c. $v = x$: $\langle x, f_3, v \rangle, \langle y, z \rangle, \langle u \rangle, \langle f_1 \rangle, \langle f_2 \rangle$
d. $y = v$: $\langle x, f_3, v, y, z \rangle, \langle u \rangle, \langle f_1 \rangle, \langle f_2 \rangle$
e. $x = z$: $\langle x, f_3, v, y, z \rangle, \langle u \rangle, \langle f_1 \rangle, \langle f_2 \rangle$

3. Join Equivalence-Classes of UFs to account for Functional Congruence. That is, if either explicitly F(x) = F(y) or if x, y are in the same Equivalence-Class. In our case, we don't have to do anything, because the arguments u (from f_1), f_1 (from f_2), and f_3 0 are all in distinct equivalence classes.

So, we end up with 4 different equivalence classes.

Question 4



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When applying interval arithmetic, a rule of thumb is that our lower bound is the "lowest possible result" and the upper bound our "biggest possible result". So, i.e., when subtracting two intervals A - B, the lower bound is the lowest value of A minus the biggest value of B.

- 1. ([7;18] + [-4;8]) [-2;3] = [3;26] [-2;3] = [0;28]
- 2. $([2;14]-[-10;4])\cdot[-7;3]=[-2;24]\cdot[-7;3]=[-168;72]$
- 3. $([9;17] \cdot [-3;8]) + [-8;3] = [-51;136] + [-8;3] = [-59;139]$
- 4. $[10; 20]^2 [5; 16] = ([10^2; 20^2] \cap [0; \infty]) [5; 16] = [100; 400] [5; 16] = [84; 395]$
- 5. $[14; 28] \div [2; 7] = [14; 28] \cdot \left[\frac{1}{7}; \frac{1}{2}\right] = [2; 14]$

Question 5

Interval Newton Method.

- 1. Compute $f(s_0)$ $f(s_0) = 2^3 + 2^2 - 1 = 11$
- 2. Derive f'(x) and compute f'(A) $f'(x) = 3x^2 + 2x$
 - $f'(A) = f'([1;3]) = 3[1;3]^2 + 2[1;3] = 3[1;9] + [2;6] = [5;33]$
- 3. Compute the Newton-Interval

$$N = s_0 - \frac{f(s_0)}{f'(A)} = 2 - \frac{11}{[5;33]} = 2 - 11 \cdot \left[\frac{1}{33}; \frac{1}{5}\right] = [2;2] - \left[\frac{1}{3}; \frac{11}{5}\right] = \left[-\frac{1}{5}; \frac{5}{3}\right]$$

4. Compute the new contracted interval by intersecting A and N:

$$A_{new} = A \cap N = [1;3] \cap \left[-\frac{1}{5}; \frac{5}{3} \right] = \left[1; \frac{5}{3} \right] = [1;1.667]$$

As the integer part of the upper bound of the resulting interval A_{new} (rounded downwards) we get 1.

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