

Satisfiability Checking - WS 2023/2024

Series 1

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Exercise 1

Let $AP = \{a, b\}$ be a set of propositions and let

$$\varphi_1 := ((a \oplus \neg b) \rightarrow b) \vee (\neg a \leftrightarrow \neg b)$$

$$\varphi_2 := (((b \rightarrow \neg a) \oplus \neg b)$$

$$\varphi_3 := (\varphi_2 \wedge (a \vee \neg b))$$

be formulas over AP .

- What are the truth tables for the above formulas?
- What are $\text{sat}(\varphi_1)$, $\text{sat}(\varphi_2)$ and $\text{sat}(\varphi_3)$?
- Which of the above formulas are satisfiable, which are unsatisfiable, and which are tautologies?

Solution:

a)

a	b	$a \oplus \neg b$	$(a \oplus \neg b) \rightarrow b$	$\neg a \leftrightarrow \neg b$	φ_1
0	0	1	0	1	1
0	1	0	1	0	1
1	0	0	1	0	1
1	1	1	1	1	1

a	b	$b \rightarrow \neg a$	$\neg b$	φ_2	$a \vee \neg b$	φ_3
0	0	1	1	0	1	0
0	1	1	0	1	0	0
1	0	1	1	0	1	0
1	1	0	0	0	1	0

- $\text{sat}(\varphi_1) = \text{Assign}$
 - $\text{sat}(\varphi_2) = \{\alpha\}$, with $\alpha(a) = 0$ and $\alpha(b) = 1$ and
 - $\text{sat}(\varphi_3) = \emptyset$
- Satisfiable: φ_1, φ_2 tautology 也为 satisfiable!!!
 - Unsatisfiable: φ_3
 - Tautology: φ_1

Exercise 2

Let $AP = \{a, b\}$ be a set of propositions and let $\alpha, \beta \in \text{Assign}$ with $\alpha(a) = 1$, $\alpha(b) = 1$ and $\beta(a) = 0$, $\beta(b) = 1$. Do the following hold?

- $\alpha \models a \vee \neg b$
- $\beta \not\models \neg a \wedge \neg b$

3. $\{\alpha, \beta\} \models a \wedge b$ 此处题目意思为: , 都能使 $a \wedge b$ 为真
4. $\{\alpha, \beta\} \models a \rightarrow b$ 此处题目意思为: , 都能使 $a \rightarrow b$ 为真
5. $a \vee b \models a \oplus b$
6. $\text{sat}(a \leftrightarrow b) \subseteq \text{sat}(a \rightarrow b)$

Solution:

1. $\alpha \models a \vee \neg b$ is true
2. $\beta \not\models \neg a \wedge \neg b$ is true
3. $\{\alpha, \beta\} \models a \wedge b$ is false
4. $\{\alpha, \beta\} \models a \rightarrow b$ is true
5. $a \vee b \models a \oplus b$ is false satisfiable, 但是不是前者为真后者就一定为真
6. $\text{sat}(a \leftrightarrow b) \subseteq \text{sat}(a \rightarrow b)$ is true

Exercise 3

Let $AP := \{a, b\}$ be a set of propositions and let $\varphi := (a \leftrightarrow b)$ be a formula over AP . Give a formula equivalent to φ that contains only propositions from AP and

1. the operators \neg and \wedge ,
2. the operators \neg and \vee ,
3. or the operator \uparrow (called NAND).

(The binary operator \uparrow has the following semantics: $\alpha \models (a \uparrow b) \leftrightarrow \alpha \models (\neg(a \wedge b))$ for all $a, b \in AP$ and $\alpha \in \text{Assigns.}$)

Solution:

1. Operators \neg and \wedge :

$$\begin{aligned}
 & (a \leftrightarrow b) \\
 \stackrel{1.}{\equiv} & (a \rightarrow b) \wedge (b \rightarrow a) \\
 \stackrel{2.}{\equiv} & (\neg a \vee b) \wedge (\neg b \vee a) \\
 \stackrel{3.}{\equiv} & \neg(a \wedge \neg b) \wedge \neg(b \wedge \neg a)
 \end{aligned}$$

2. Operators \neg and \vee :

$$\begin{aligned}
 & (a \leftrightarrow b) \\
 \stackrel{1., -2.}{\equiv} & (\neg a \vee b) \wedge (\neg b \vee a) \\
 \equiv & \neg(\neg(\neg a \vee b) \vee \neg(\neg b \vee a))
 \end{aligned}$$

3. Operator \uparrow : We show that the operators \neg and \wedge can be expressed by \uparrow .

$$\begin{aligned}
 \neg a & \equiv (a \uparrow a) \\
 (a \wedge b) & \equiv (a \uparrow b) \uparrow (a \uparrow b)
 \end{aligned}$$

Then:

$$\begin{aligned}
 & (a \leftrightarrow b) \\
 & \stackrel{1.-3.}{\equiv} \neg(a \wedge \neg b) \wedge \neg(b \wedge \neg a) \\
 & \equiv \neg(a \wedge (b \uparrow b)) \wedge \neg(b \wedge (a \uparrow a)) \\
 & \equiv (a \uparrow (b \uparrow b)) \wedge (b \uparrow (a \uparrow a)) \\
 & \equiv ((a \uparrow (b \uparrow b)) \uparrow (b \uparrow (a \uparrow a))) \uparrow ((a \uparrow (b \uparrow b)) \uparrow (b \uparrow (a \uparrow a)))
 \end{aligned}$$