Satisfiability Checking

23 The virtual substitution method II

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Substitution of a variable by a test candidate in a constraint

- Standard substitution $\varphi[t/x]$ could lead to formulas containing ϵ, ∞, $\sqrt{\ }$ or division.
- Virtual substitution $\varphi[t /\!\!/ x]$ generates real algebraic formulas that are semantically equivalent to the application of standard substitution but these formulas do not contain ϵ, ∞, $\sqrt{\ }$ or division.
- There are rules that define how to virtually substitute a test candidate into a constraint. These rules distinguish between
  - the constraint's relation symbol, and
  - the test candidate's type (whether it contains $-\infty$, ϵ, $\sqrt{\ }$ or division).

  We look at all rules for substituting $-\infty$ and fractions of polynomials, and a few rules for the other cases.

Substitution of a variable by a test candidate in a constraint

	$p(x) \sim 0$	$(p(x) \sim 0)[-\infty /\!\!/ x]$
	$bx + c = 0$	$b = 0 \wedge c = 0$
	$bx + c \neq 0$	$b \neq 0 \vee c \neq 0$ ☆ 此条件间为 ∨
	$bx + c < 0$	$b > 0 \vee (b = 0 \wedge c < 0)$
无 a	$bx + c > 0$	$b < 0 \vee (b = 0 \wedge c > 0)$
	$bx + c \leq 0$	$b > 0 \vee (b = 0 \wedge c \leq 0)$
	$bx + c \geq 0$	$b < 0 \vee (b = 0 \wedge c \geq 0)$
	$ax^2 + bx + c = 0$ 否定	$a = 0 \wedge b = 0 \wedge c = 0$ 否定，∧ → ∨
	$ax^2 + bx + c \neq 0$	$a \neq 0 \vee b \neq 0 \vee c \neq 0$ ☆ 此条件间为 ∨
	$ax^2 + bx + c < 0$	$a < 0 \vee (a = 0 \wedge b > 0) \vee (a = 0 \wedge b = 0 \wedge c < 0)$
有 a	$ax^2 + bx + c > 0$	$a > 0 \vee (a = 0 \wedge b < 0) \vee (a = 0 \wedge b = 0 \wedge c > 0)$
	$ax^2 + bx + c \leq 0$	$a < 0 \vee (a = 0 \wedge b > 0) \vee (a = 0 \wedge b = 0 \wedge c \leq 0)$
	$ax^2 + bx + c \geq 0$	$a > 0 \vee (a = 0 \wedge b < 0) \vee (a = 0 \wedge b = 0 \wedge c \geq 0)$

$\exists x.\ [(x > 0 \land 1 - x^2 \geq 0) \lor (x < 0 \land 1 - x^2 \leq 0)] \land 1 - x^2 < 0 \land x \neq 0$

Eliminate x. 1. test candidate: $-\infty$

因为 $x \to -\infty$, 所以 b 要 >0

$= (a<0) \lor (a=0 \land b>0) \lor (a=0 \land b=0 \land c \leq 0)$

前为 False, 后面 $\land$ 的项则无需代入检验

c a b $+0 \cdot x$

$([(x > 0 \mathrm{False} \land 1 - x^2 \geq 0) \lor (x < 0 \mathrm{True} \land 1 - x^2 \leq 0 \mathrm{True})] \land 1 - x^2 < 0 \mathrm{True} \land x \neq 0 \mathrm{True})\ [-\infty /\!/ x]$

$(1 - x^2 \leq 0)\ [-\infty /\!/ x]$

$= (-1 < 0) \lor (-1 = 0 \land 0 > 0) \lor (-1 = 0 \land 0 = 0 \land 1 \leq 0)$

$(1 \neq 0 \lor 0 \neq 0)$

$$
\begin{aligned}
& (x > 0)[-\infty /\!/ x] \\
= \quad & (1 < 0 \lor (1 = 0 \land 0 > 0)) \\
= \quad & \text{False} \\
& (x < 0)[-\infty /\!/ x] \\
= \quad & (1 > 0 \lor (1 = 0 \land 0 < 0)) \\
= \quad & \text{True}
\end{aligned}
$$

$\exists x.\exists y.\ ((\ xy - 1 = 0 \quad \vee \quad y - x \geq 0\ ) \quad \wedge \quad y^2 - 1 < 0\ )$, eliminate $\boxed{y}$

1. test candidate: $-\infty$

$$
\begin{array}{lll}
\exists x.(& (& (xy - 1 = 0)[-\infty /\!/ y] \\
 & & \vee\ (y - x \geq 0)[-\infty /\!/ y] \\
 & \wedge & (y^2 - 1 < 0)[-\infty /\!/ y] \\
\Leftrightarrow \exists x.(& (& (x = 0 \wedge -1 = 0) \\
 & & \vee\ (1 < 0 \vee (1 = 0 \wedge -x \geq 0)) \\
 & \wedge & (1 < 0 \vee (1 = 0 \wedge 0 > 0) \vee (1 = 0 \wedge 0 = 0 \wedge -1 < 0)) \\
\Leftrightarrow \exists x.(\mathit{false}) & &
\end{array}
$$

Substitution of a variable by a test candidate in a constraint

e.g. $x^2+2x+1 \sim 0 \Rightarrow p(e) = (\frac{q}{r})^2 + 2(\frac{q}{r}) + 1 \sim 0$ 此时 max degree $= 2 = k$

$\delta = 0$

$p(x) \sim 0$	$(p(x) \sim 0)[e /\!/ x]$ *for* $e = \frac{q}{r}$ *with* q *and* r *polynomials* *let* k *be the maximum degree of* x *in* p *let* $\delta = 1$ *if* k *is odd and* $\delta = 0$ *else*
$p(x) = 0$	$p(e) \cdot r^k = 0$
$p(x) \neq 0$	$p(e) \cdot r^k \neq 0$
$p(x) < 0$	$(r^\delta > 0 \wedge p(e) \cdot r^k < 0) \vee (r^\delta < 0 \wedge p(e) \cdot r^k > 0)$
$p(x) > 0$	$(r^\delta > 0 \wedge p(e) \cdot r^k > 0) \vee (r^\delta < 0 \wedge p(e) \cdot r^k < 0)$
$p(x) \leq 0$	$(r^\delta > 0 \wedge p(e) \cdot r^k \leq 0) \vee (r^\delta < 0 \wedge p(e) \cdot r^k \geq 0)$
$p(x) \geq 0$	$(r^\delta > 0 \wedge p(e) \cdot r^k \geq 0) \vee (r^\delta < 0 \wedge p(e) \cdot r^k \leq 0)$

$$\begin{cases} (\frac{q}{r})^2 \cdot r^2 + 2(\frac{q}{r}) \cdot r^2 + r^2 \sim 0 & r^\delta > 0 \\ (\frac{q}{r})^2 \cdot r^2 + 2(\frac{q}{r}) \cdot r^2 + r^2 \underbrace{\neg\sim}_{\text{取反}} 0 & r^\delta < 0 \end{cases}$$

Note that for the quadratic case $k = 2$, $\delta = 0$,

$p(e) \cdot r^k = (ax^2 + bx + c)[\frac{q}{r}/x] \cdot r^2 = (a\frac{q^2}{r^2} + b\frac{q}{r} + c) \cdot r^2 = aq^2 + bqr + cr^2$

always has the same sign as $p(e)$, and in this case $r^\delta = r^0 = 1 > 0$.

However, for the linear case $k = 1$, $\delta = 1$,

$p(e) \cdot r^k = (bx + c)[\frac{q}{r}/x] \cdot r^1 = (b\frac{q}{r} + c) \cdot r^1 = b \cdot q + c \cdot r$

the sign might change if $r < 0$.

Virtual substitution: Example

$\exists x.\exists y.\ ((\ xy - 1 = 0 \ \lor\ y - x \geq 0\ )\ \land\ y^2 - 1 < 0\ )$, eliminate y

2. test candidate: $\frac{1}{x}$, if $x \neq 0$

p中x的最大的degree为1

此时k=1,δ =1

q=1, r=x

$$
\begin{array}{lllll}
 & \exists x.(& (& (xy - 1 = 0)[\frac{1}{x} /\!/ y] & k=1 \quad \delta=1 \\
 & & \lor & (y - x \geq 0)[\frac{1}{x} /\!/ y] & k=1 \quad \delta=1 \quad) \\
 & & \land & (y^2 - 1 < 0)[\frac{1}{x} /\!/ y] & k=2 \quad \delta=0 \\
 & & \land & x \neq 0 &)
\end{array}
$$

每一个式子单独判断!!

r^δ =x^1=x p(e)*r^k=p(e)*x

r^δ =x^0=1 p(e)*r^k=p(e)*x^2

$$
\begin{array}{lllll}
\Leftrightarrow & \exists x.(& (& (0 = 0) & \\
 & & \lor & ((x > 0 \ \land\ 1 - x^2 \geq 0)\ \lor\ (x < 0 \ \land\ 1 - x^2 \leq 0)) &) \\
 & & \land & ((1 > 0 \ \land\ 1 - x^2 < 0)\ \lor\ (1 < 0 \ \land\ 1 - x^2 > 0)) & \\
 & & \land & x \neq 0 &) \\
\Leftrightarrow & \exists x.(& & true & \\
 & & \land & 1 - x^2 < 0 & \\
 & & \land & x \neq 0 &)
\end{array}
$$

Substitution of a variable by a test candidate in a constraint

$(p(x) = 0)[\frac{q+r\sqrt{t}}{s} /\!/ x]$

1. Substitute x by $\frac{q+r\sqrt{t}}{s}$ in $p(x) = 0$ in the common way.
   先 normal substitution
2. Transform the result to $\frac{\hat{q}+\hat{r}\sqrt{t}}{\hat{s}} = 0$ where $\hat{q}$, $\hat{r}$, and $\hat{s}$ are polynomials
   然后.转化为此形式
   (always possible, proof exercise)
3. $(p(x) = 0)[\frac{q+r\sqrt{t}}{s} /\!/ x] := (\hat{q}\hat{r} \leq 0 \wedge \hat{q}^2 - \hat{r}^2 t = 0)$
4. Explanation:

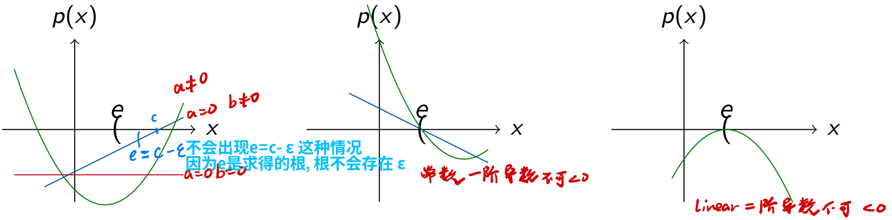
$$\begin{aligned} \frac{\hat{q}+\hat{r}\sqrt{t}}{\hat{s}} = 0 \quad &\Leftrightarrow \quad \hat{q} + \hat{r}\sqrt{t} = 0 \\ &\Leftrightarrow \quad \hat{q}\hat{r} \leq 0 \ \wedge \ \|\hat{q}\| = \|\hat{r}\sqrt{t}\| \\ &\Leftrightarrow \quad \hat{q}\hat{r} \leq 0 \wedge \hat{q}^2 - \hat{r}^2 t = 0 \end{aligned}$$

^q ^r异号,且^q和^r√ t大小相等

$$(p(x) < 0)[e + \epsilon /\!\!/ x]$$

$$\underbrace{p[e /\!\!/ x] < 0}_{\text{Case 1}} \vee \underbrace{p[e /\!\!/ x] = 0 \wedge p'[e /\!\!/ x] < 0}_{\text{Case 2}} \vee \underbrace{p[e /\!\!/ x] = 0 \wedge p'[e /\!\!/ x] = 0 \wedge p''[e /\!\!/ x] < 0}_{\text{Case 3}}$$

Explanation:



Case 1 Case 2 Case 3

Complexity

We consider in the following the elimination of one existential quantifier (existentially quantified variable):

$$\exists x_1 \ldots \exists x_n.\ \varphi \quad \equiv \quad \exists x_1 \ldots \exists x_{n-1}.\ \bigvee_{t \in T} \varphi[t /\!/ x_n].$$

- Degree $D(x_i, \cdot)$ of a remaining variable x_i, $1 \leq i < n$:

$$D(x_i, \bigvee_{t \in T} \varphi[t /\!/ x_n]) \in \mathcal{O}(6D(x_i, \varphi) - 8)$$

- Number of atoms $at(\cdot)$:

$$at(\bigvee_{t \in T} \varphi[t /\!/ x_n]) \in \mathcal{O}(8at(\varphi) + at(\varphi)(8 + 63at(\varphi)))$$

Learning target

- What is the basic idea of the virtual substitution?
- How to compute the test candidates?
- How to apply virtual substitution?
- Is the virtual substitution method complete?