E-Test 3

Question 1

Construct a real solution for the constraint
$2x^2 - 7y^2 - 4 = 0$
on the line segment between $(1, 1)$ (where the polynomial has a negative sign) and $(19, 7)$ (where the polynomial has a positive sign). What is the value of x in your solution? Please answer using digits without whitespaces.

Antwort: 4 ✔

1. Constructing the line equations for the coordinates $(x_1, y_1) = (1,1)$ and $(x_2, y_2) = (19, 7)$:
   $\binom{x}{y} = \binom{1}{1} + t\left(\binom{19}{7} - \binom{1}{1}\right) = \binom{1}{1} + t\binom{18}{6}$
   $\Rightarrow x = 1 + 18t$
   $\Rightarrow y = 1 + 6t$
2. Plugging those values into the original polynomial and compute the real roots:
   $2(1 + 18t)^2 - 7(1 + 6t)^2 - 4 = 0$
   $\Leftrightarrow 2 + 72t + 648t^2 - 7 - 84t - 252t^2 - 4 = 0$
   $\Leftrightarrow 396t^2 - 12t - 9 = 0$

   Compute Roots:
   $$t_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{12 \pm \sqrt{12^2 + 4 \cdot 396 \cdot 9}}{2 \cdot 396} = \frac{1 \pm 10}{66}$$

   By convention, we only look at the root contained in the interval $[0,1]$, that is:
   $$t_0 = \frac{11}{66} = \frac{1}{6}$$

3. Plug t_0 back into x and y computed in step 1:
   $\binom{x}{y} = \binom{1}{1} + \frac{1}{6}\binom{18}{6} = \binom{4}{2}$

The value of x is: 4

Question 2

Besides $-\infty$, which of the following expressions are generated as test candidates to eliminate x from $y \leq 0 \wedge 2x \geq 0 \wedge 5x + 5y > 0$?
(Multiple choice: please select all generated test candidates.)

Wählen Sie eine oder mehrere Antworten:

- [ ] a. -1
- [ ] b. $-1 + \epsilon$
- [x] c. 0 ✔
- [ ] d. $0 + \epsilon$
- [ ] e. 1
- [ ] f. $1 + \epsilon$
- [ ] g. $-y$
- [x] h. $-y + \epsilon$ ✔
- [ ] i. y
- [ ] j. $y + \epsilon$

Recall from the lecture. To eliminate x in a set of constraints, we need to test:

- $-\infty$ (always)

- For all strict inequalities ($<, >, \neq$) containing x:
  Test $\xi_i + \epsilon$ with ξ_i being the roots of the inequalities and ϵ an infinitesimal that "slightly moves" our root to the left so we fulfill the strict inequality.
- For all "normal" inequalities ($\leq, \geq, =$) containing x:
  Test ξ_i with ξ_i being the roots of the inequalities

In our case:

1. $y \leq 0$
   As we're only interested in eliminating x, we can ignore this constraint.
2. $2x \geq 0$
   Root: $\xi_0 = 0$, thus we can add 0 as Test Candidate
3. $5x + 5y > 0$
   Root: $\xi_1 = -y$, Thus we can add $-y + \epsilon$ as Test Candidate (because we have strict inequality)

In total, the test candidates are:
$\{-\infty, -y + \epsilon, 0\}$

Question 3

Let $p = 4zx^2 + 5y^3$ be a polynomial. Please pair the following virtual substitution results with the substitution that they implement.

$4z < 0 \vee (4z = 0 \wedge 0 > 0) \vee (4z = 0 \wedge 0 = 0 \wedge 5y^3 \leq 0)$	(p< =0)[-oo//x] ✔
$4z \neq 0 \vee 0 \neq 0 \vee 5y^3 \neq 0$	(p! =0)[-oo//x] ✔
$4z = 0 \wedge 0 = 0 \wedge 5y^3 = 0$	(p=0)[-oo//x] ✔
$4z < 0 \vee (4z = 0 \wedge 0 > 0) \vee (4z = 0 \wedge 0 = 0 \wedge 5y^3 < 0)$	(p<0)[-oo//x] ✔
$4z > 0 \vee (4z = 0 \wedge 0 < 0) \vee (4z = 0 \wedge 0 = 0 \wedge 5y^3 > 0)$	(p>0)[-oo//x] ✔
$4z > 0 \vee (4z = 0 \wedge 0 < 0) \vee (4z = 0 \wedge 0 = 0 \wedge 5y^3 \geq 0)$	(p> =0)[-oo//x] ✔

Quick Answer:

- Just look at the last sign…

More formally:
Look into the following table to determine the case and thus concluding the condition – however, this is exactly the same and takes much longer; so don't do it :D

$p(x) \sim 0$	$(p(x) \sim 0)[-\infty /\!/ x]$
$bx + c = 0$	$b = 0 \wedge c = 0$
$bx + c \neq 0$	$b \neq 0 \vee c \neq 0$
$bx + c < 0$	$b > 0 \vee (b = 0 \wedge c < 0)$
$bx + c > 0$	$b < 0 \vee (b = 0 \wedge c > 0)$
$bx + c \leq 0$	$b > 0 \vee (b = 0 \wedge c \leq 0)$
$bx + c \geq 0$	$b < 0 \vee (b = 0 \wedge c \geq 0)$
$ax^2 + bx + c = 0$	$a = 0 \wedge b = 0 \wedge c = 0$
$ax^2 + bx + c \neq 0$	$a \neq 0 \vee b \neq 0 \vee c \neq 0$
$ax^2 + bx + c < 0$	$a < 0 \vee (a = 0 \wedge b > 0) \vee (a = 0 \wedge b = 0 \wedge c < 0)$
$ax^2 + bx + c > 0$	$a > 0 \vee (a = 0 \wedge b < 0) \vee (a = 0 \wedge b = 0 \wedge c > 0)$
$ax^2 + bx + c \leq 0$	$a < 0 \vee (a = 0 \wedge b > 0) \vee (a = 0 \wedge b = 0 \wedge c \leq 0)$
$ax^2 + bx + c \geq 0$	$a > 0 \vee (a = 0 \wedge b < 0) \vee (a = 0 \wedge b = 0 \wedge c \geq 0)$

Question 4

Which of the following are interval representations of real roots? (Multiple choice: please select all interval representations.)

Wählen Sie eine oder mehrere Antworten:

- a. $(-x^2 - 5x - 4, (2, 10))$
- b. $(-x^2 - 5x - 4, [-3, 8])$
- c. $(-x^2 - 5x - 4, (-8, -3))$ ✔
- d. $(-x^2 - 5x - 4, (-13, 5))$
- e. $(-x^2 - 5x - 4, (-3, -1))$
- f. None of the above

Recall from the lecture, that the interval representation is just another way to represent one single root of a polynomial without having to deal with things like $\sqrt{\ }$ or similar.

1. For this, we first need to compute the real roots of the polynomial:
   $-x^2 - 5x - 4$
   $$x_{1,2} = \frac{5 \pm \sqrt{5^2 - 4 \cdot 4}}{-2} = -\frac{5 \pm 3}{2} \Rightarrow x_1 = -4, x_2 = -1$$
2. The interval representation must contain exactly one of the roots in the interval. Additionally, we only consider OPEN intervals (thus, NOT [ ])
   a. No, 0 roots are included.
   b. No, closed interval.
   c. Yes, only x_1 is included.
   d. No, 2 roots are included.
   e. No, 0 roots are included.

Question 5

Assume a polynomial $p = x^3 + 6x^2 - 9x - 6$ and its Sturm sequence

$p_0 = x^3 + 6x^2 - 9x - 6$
$p_1 = 3x^2 + 12x - 9$
$p_2 = 14x$
$p_3 = 9$

Compute the Cauchy bound for p and isolate p's real roots using the algorithm presented in the lecture, choosing always the mid points of intervals to split.

Let $(p, I_1), \ldots, (p, I_k)$ be the resulting interval representations of p's real roots. Compute the sum of all upper bounds in $I_1, \ldots, I_k$ and round it down to the next integer (i.e. compute the floor of the sum). What is the result?

Please answer using digits without whitespaces.

Antwort: 5 ✔

Remember, that for the interval representation, we must do the following:

1. Compute the Cauchy-Bound $C \Rightarrow$ all roots must be in the interall $[-C, C]$
2. Check $-C$ and $(-C, C]$ (= Sturm-Sequence can only be applied on $I = (a, b]$).
   If the interval has more than one real root $\Rightarrow$ Split into 3 intervals $((-C, a), [a, a], (a, C))$.
   Usually, we split the interval in half.
3. Re-do step 2 until each interval only contains at most one root.

Cauchy Bound

$|\xi| \leq 1 + \max_{i=0,\ldots,k-1} \frac{|a_i|}{|a_k|} = 1 + \max\left(\frac{|6|}{|1|}, \frac{|-9|}{|1|}, \frac{|0|}{|1|}\right) = 10$

1. Check Interval $[-10, 10]$

Use Sturm Sequence to determine the number of Sign Changes:

$p(-C) \neq 0$, thus, all real roots are in $I_1 \coloneqq (-10, 10]$ and we can use the Sturm-Sequence

Polynomial	**Value at -10**	**Value at 10**
p_0	-316	1504
p_1	171	411
p_2	-140	140
p_3	9	9
Sign Changes $\sigma(\cdot)$	3	0

Number of Real Roots: $\sigma(-10) - \sigma(10) = 3$

Split the Interval

Split $(-10,10)$ into $I_2 \coloneqq (-10, 0)$, $[0, 0]$, $I_3 \coloneqq (0, 10)$

2. Check Interval $(-10, 0)$

Use Sturm Sequence to determine the number of Sign Changes:

Polynomial	**Value at -10**	**Value at 0**
p_0	-316	-6
p_1	171	-9
p_2	-140	0
p_3	9	9
Sign Changes $\sigma(\cdot)$	3	1

Number of Real Roots: $\sigma(-10) - \sigma(0) = 2$

Split the Interval:

Split $(-10, 0)$ into $I_4 \coloneqq (-10, -5)$, $[-5, -5]$, $I_5 \coloneqq (-5, 0)$

3. Check Interval $(-10, -5)$

Use Sturm Sequence to determine the number of Sign Changes:

Polynomial	**Value at -10**	**Value at -5**
p_0	-316	64
p_1	171	6
p_2	-140	-70
p_3	9	9
Sign Changes $\sigma(\cdot)$	3	2

Number of Real Roots: $\sigma(-10) - \sigma(-5) = 1 \Rightarrow$ **No more split necessary**

4. Check Interval $(-5, 0)$

Use Sturm Sequence to determine the number of Sign Changes:

Polynomial	**Value at -5**	**Value at 0**
p_0	64	-6
p_1	6	-9
p_2	-70	0

p_3	9	9
Sign Changes $\sigma(\cdot)$	2	1

Number of Real Roots: $\sigma(-5) - \sigma(0) = 1 \Rightarrow$ **No more split necessary**

5. Check Interval $(\mathbf{0}, \mathbf{10})$

Use Sturm Sequence to determine the number of Sign Changes:

Polynomial	**Value at 0**	**Value at 10**
p_0	-6	1504
p_1	-9	411
p_2	0	140
p_3	9	9
Sign Changes $\sigma(\cdot)$	1	0

Number of Real Roots: $\sigma(0) - \sigma(10) = 1 \Rightarrow$ **No more split necessary**

So in total, we have computed the following representation of real roots:

1) $(p, I_3) = \left(x^3 + 6x^2 - 9x - 6, (\mathbf{0}, \mathbf{10})\right)$
2) $(p, I_4) = \left(x^3 + 6x^2 - 9x - 6, (-10, -5)\right)$
3) $(p, I_5) = \left(x^3 + 6x^2 - 9x - 6, (-5, 0)\right)$

Finally, to answer the question, we need to sum up the upper bounds, which gives us:

$\boldsymbol{a = 10 - 5 + 0 = 5}$