Satisfiability Checking

12 (Full/Less) lazy SMT solving for equality logic

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Reminder: Equality logic with uninterpreted functions

We extend the propositional logic with

- equalities and
- uninterpreted functions (UFs).

Syntax:

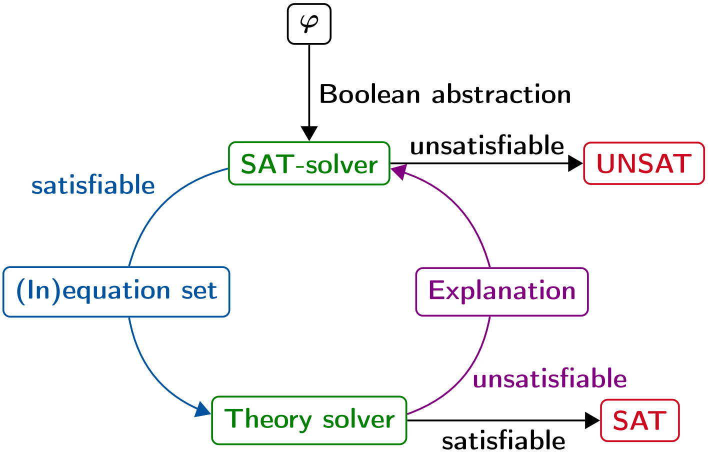
- variables x over an arbitrary domain D,
- constants c from the same domain D,
- function symbols F for functions of the type $D^n \to D$, and
- equality as predicate symbol.

$$
\begin{array}{lllllll}
\textit{Terms:} & t & ::= & c & \mid & x & \mid & F(t, \ldots, t) \\
\textit{Formulas:} & \varphi & ::= & t = t & \mid & (\varphi \land \varphi) & \mid & (\neg \varphi)
\end{array}
$$

Semantics: standard FO semantics

Again, we assume constants to be uninterpreted and treat them as variables.

Full lazy SMT solving



We need a theory solver for conjunctions/sets of equations and disequations over terms (variables and uninterpreted functions, see page 2).

Basic idea for the theory solver (EQ+UF, full lazy)

Input: set E of equations and disequations; set T of terms in E
Output: whether E is satisfiable (with a "sufficiently large" theory domain)

1. Compute a partition $\mathcal{C}$ of T such that for each $t, t' \in T$,

   t and t' are in the same equivalence class (written: $[t] = [t']$)
   if and only if
   all models that satisfy all equations in E assign equal values to t and t'.

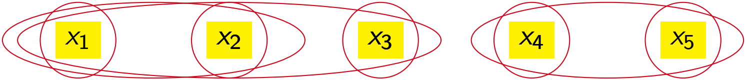
2. Check for each disequation in E whether the two operands (sides) are in different equivalence classes (yes: E is SAT, no: E is UNSAT).

How to compute such a partition?

- Initial partition: Each subterm from T has its own equivalence class.
- Transitive closure for equality: For each equation in E, merge the equivalence classes for the two sides.
- Congruence for uninterpreted functions: Iteratively merge the equivance classes of each $F(t), F(t') \in T$ with $[t] = [t']$.

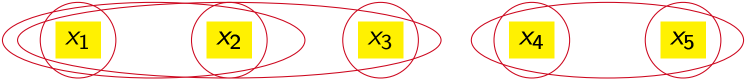
Example: EQ (no UF, full lazy)

$$\varphi^E : \quad x_1 = x_2 \wedge x_2 = x_3 \wedge x_4 = x_5 \wedge x_1 \neq x_5$$



SAT

$$\varphi^E : \quad x_1 = x_2 \land x_2 = x_3 \land x_4 = x_5 \land x_1 \neq x_3$$



UNSAT

Algorithm 1: satCheck (EQ, no UF, full lazy)

Input: Set V of variables, set E of equations and disequations over V.
Output: Satisfiability of $\bigwedge_{e \in E} e$.

1 Initial partition has an own equivalence class for each variable in V:
$\mathcal{C} := \{\{x\} \mid x \in V\}$;

2 Assure transitivity for equality: For each input equation $x = x'$, if the equivalence classes $[x]$ and $[x']$ of the two sides differ then merge them:
$\texttt{for each } (x = x') \in E$
$\quad \texttt{if } ([x] \neq [x']) \texttt{ then } \mathcal{C} := (\mathcal{C} \setminus \{[x], [x']\}) \cup \{[x] \cup [x']\}$;

3 For each disequation $(x \neq x') \in E$, if the equivalence classes of the two sides coincide then return unsatisfiability:
$\texttt{for each } (x \neq x') \in E \texttt{ if } ([x] = [x']) \texttt{ then return } \texttt{UNSAT}$;

4 Else return satisfiability:
$\texttt{return } \texttt{SAT}$.

Next: Add uninterpreted functions (EQ+UF, full lazy)

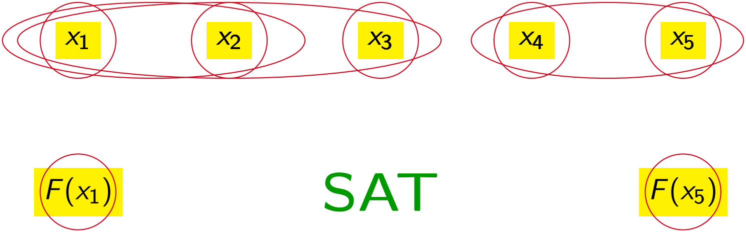
Are these conjunctions satisfiable?

- $x = y \wedge F(x) = F(y)$: satisfiable
- $x = y \wedge F(x) \neq F(y)$: unsatisfiable 相同的定义域取值必须有相同的值域取值
- $x \neq y \wedge F(x) = F(y)$: satisfiable 不同的定义域取值可以有不用的值域取值
- $x \neq y \wedge F(x) \neq F(y)$: satisfiable

- $x = y \wedge F(G(x)) \neq F(G(y))$: unsatisfiable 多重函数也是如此

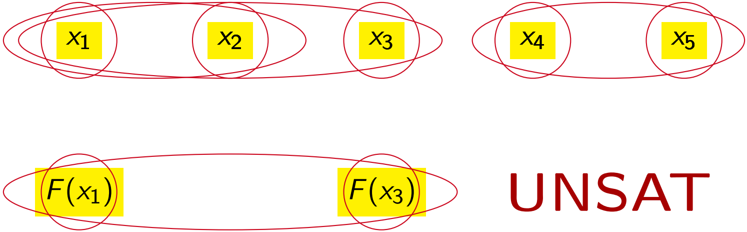
Next: Add uninterpreted functions (EQ+UF, full lazy)

$\varphi^E : \quad x_1 = x_2 \wedge x_2 = x_3 \wedge x_4 = x_5 \wedge F(x_1) \neq F(x_5)$



Next: Add uninterpreted functions (EQ+UF, full lazy)

$$\varphi^E : \quad x_1 = x_2 \wedge x_2 = x_3 \wedge x_4 = x_5 \wedge F(x_1) \neq F(x_3)$$



UNSAT

Algorithm 2: satCheck (EQ+UF, full lazy)

Input: Set $\mathcal{T}$ of terms (definition see page 3) with sub-terms T,
set E of equalities and disequalities over $\mathcal{T}$.

Output: Satisfiability of $\bigwedge_{e \in E} e$.

1. Initial partition has an own equivalence class for each term in T:
   $\mathcal{C} := \{\{t\} \mid t \in T\}$;
2. Assure transitivity for equality: For each input equation $(t = t') \in E$, if the equivalence classes $[t]$ and $[t']$ of the two sides differ then merge them:
   $\texttt{for each } (t = t') \in E$
   $\quad \texttt{if } ([t] \neq [t']) \texttt{ then } \mathcal{C} := (\mathcal{C} \setminus \{[t], [t']\}) \cup \{[t] \cup [t']\}$;
3. Assure functional congruence for uninterpreted functions:
   $\texttt{while exist subterms } F(t), F(t') \texttt{ in } T \texttt{ with } [t] = [t'] \texttt{ and } [F(t)] \neq [F(t')]$
   $\quad \mathcal{C} := (\mathcal{C} \setminus \{[F(t)], [F(t')]\}) \cup \{[F(t)] \cup [F(t')]\}$;
4. For each disequation $(t \neq t') \in E$, if the equivalence classes of the two sides coincide then return unsatisfiability:
   $\texttt{for each } (t \neq t') \in E \texttt{ if } ([t] = [t']) \texttt{ then return UNSAT}$;
5. Else return satisfiability:
   $\texttt{return SAT}$.

Full lazy EQ+UF theory solver

Algorithm 3: getExplanation (EQ+UF, full lazy)

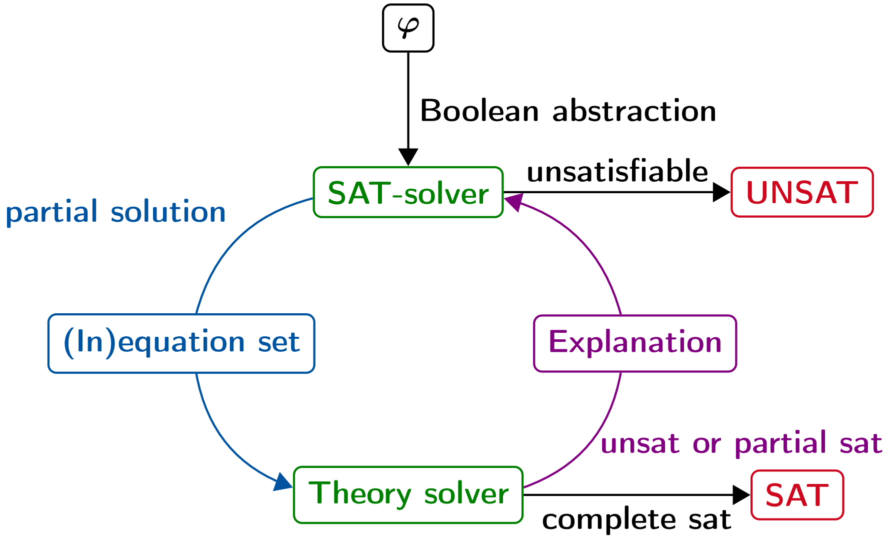
Input: Final state of Algorithm 2.

Output: IF the current state is UNSAT THEN an unsatisfiable subset of the previously received equations and disequations ELSE the empty set.

1. $\varphi := \emptyset$;
2. $E_= = \{(t = t') \in E\}$;
3. **for each** $(t \neq t') \in E$
   - **if** $\left(t \neq t' \wedge \bigwedge_{e \in E_=} e\right)$ **is UNSAT (use Alg.2) then**
     - $\varphi := \{t \neq t'\}$;
     - **break**;
   - **fi**
4. **if** $(\varphi = \emptyset)$ **return** $\emptyset$;
5. **while** $E_= \neq \emptyset$
   - **let** $(t = t') \in E_=$; $E_= := E_= \setminus \{(t = t')\}$;
   - **if** $(\bigwedge_{e \in \varphi \cup E_=} e)$ **is SAT then** $\varphi := \varphi \cup \{t = t'\}$;

   od
6. **return** φ;

Less lazy SMT solving



Requirements on the theory solver

Needed for less lazy SMT solving:

1. Incrementality: In less lazy solving we extend the set of constraints. The solver should make use of the previous satisfiability check for the check of the extended set.
2. (Preferably minimal) infeasible subsets: Compute a reason for unsatisfaction
3. Backtracking: The theory solver should be able to remove constraints in inverse chronological order.

Requirements on the theory solver

Solution:

1. Incrementality:
   - When a new equation is added, update the partition and check the previously added disequations for satisfiability.
   - When a new disequation is added, check the satisfiability of the new disequation.
2. (Preferably minimal) infeasible subsets: A conflict appears when a disequation $t \neq t'$ cannot be true together with the current equations; build the set of this disequation $t \neq t'$ and (a minimal number of) equations that imply $t = t'$ by transitivity and congruence.
3. Backtracking: Remember computation history.

Less lazy EQ+UF theory solver

Algorithm 4: Init (EQ+UF, less lazy)

Input: Set T of all subterms which can be used in (dis)equations

Output: none

1. No equations or disequations received yet:
   $E := \emptyset$;
2. Initial partition over T:
   $\mathcal{C} := \{\{t\} \mid t \in T\}$;
3. Remember current state of satisfiability:
   $state := SAT$;

Less lazy EQ+UF theory solver

Algorithm 5: addEquation (EQ+UF, less lazy)

Input: Equation $t_1 = t_2$ with subterms from T

Output: Satisfiability of the conjunction of all received (and not yet removed) equations and disequations

1. Remember the new equation: $E := E \cup \{t_1 = t_2\}$;
2. If the problem was already unsatisfiable then it is still unsatisfiable:
   $\texttt{if } (\mathit{state} = \mathit{UNSAT}) \texttt{ then return UNSAT;}$
3. Update the partition using the new equation and re-check satisfiability of the disequations:

```
if ([t1] ≠ [t2]) then
  C := (C \ {[t1],[t2]}) ∪ {[t1] ∪ [t2]};
  while exist subterms F(t), F(t') ∈ T with [t]=[t'] and [F(t)]≠[F(t')]
    C := (C \ {[F(t)],[F(t')]}) ∪ {[F(t)] ∪ [F(t')]};
  for each (t ≠ t') ∈ E
    if ([t] = [t']) then
      state := UNSAT; return UNSAT;
    fi
fi
return SAT.
```

Algorithm 6: addDisequation (EQ+UF, less lazy)

Input: Disequation $t_1 \neq t_2$ with subterms from T

Output: Satisfiability of the conjunction of all received (and not yet removed) equations and disequations

1. Remember the new disequation:
   $E := E \cup \{t_1 \neq t_2\}$;
2. If the problem was already unsatisfiable then it is still unsatisfiable:
   `if` $(state = UNSAT)$ `then return UNSAT`;
3. Check satisfiability of the new disequation:

```
if ([t1] = [t2]) then
  state := UNSAT;
  return UNSAT;
fi
return SAT.
```

Less lazy EQ+UF theory solver

Algorithm 7: getExplanation (EQ+UF, less lazy)

Same as for full lazy.

Less lazy EQ+UF theory solver

This is just a rather naive (and informal) solution for backtracking... more optimal solutions need smart datastructures for efficient book-keeping.

Algorithm 8: backtrack (EQ+UF, less lazy)

Input: Equation and disequation set S

Output: none

1. Remove the equations and disequations:
   $E := E \setminus S;$
2. Apply `Algorithm 2` (page 11) to compute satisfiability for T and the conjunction of all equations and disequations from E.
3. Remember the satisfiability result in `status`.
4. Return the satisfiability result.

Learning target

- How to encode transitivity for equality by defining an equivalence relation over involved terms?
- How to use this equivalence relation to check the satisfiability of EQ-formulas?
- How to encode congruence for uninterpreted functions by extending the equivalence relation for equality?
- How to use this extended equivalence relation to check the satisfiability of EQ+UF-formulas?
- How to generate explanations (infeasible subsets) for the full-lazy EQ+UF-theory solver?
- How to achieve incrementality, explanation (infeasible subset) generation and backtracking for the less-lazy EQ+UF-theory solver?