hybr i2d Theory of Hybrid Systems Informatik 2

RWTH AACHEN UNIVERSITY

Satisfiability Checking - WS 2023/2024

Series 10

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Exercise 1: Interval arithmetic

Apply basic interval arithmetic as presented in the lecture.

$$x \in I_x = [-1; 3], y \in I_y = [2; 6]$$

Calculate:

1. $2 \cdot x + y$
2. $x^2 - 4 \cdot x + 7$
3. $x \cdot x \cdot y$
4. $\frac{2 \cdot x}{y}$
5. $z \in (3; 4]$, calculate: $x + z - y$

Solution:

1. $2 \cdot x + y \in [2; 2] \cdot [-1; 3] + [2; 6] = [-2; 6] + [2; 6] = [0; 12]$
2. $x^2 - 4 \cdot x + 7 \in [-1; 3]^2 - 4 \cdot [-1; 3] + 7 = [0; 9] - [4; 4] \cdot [-1; 3] + [7; 7] = [0; 9] - [-4; 12] + [7; 7] = [-12; 13] + [7; 7] = [-5; 20]$
   此处的x・x不是x^2, 不需要 U [0;+∞)
3. $x \cdot x \cdot y \in [-1; 3] \cdot [-1; 3] \cdot [2; 6] = [-3; 9] \cdot [2; 6] = [-18; 54]$
4. $\frac{2 \cdot x}{y} \in \frac{[2;2] \cdot [-1;3]}{[2;6]} = \frac{[-2;6]}{[2;6]} = [-2; 6] \cdot [\frac{1}{6}; \frac{1}{2}] = [-1; 3]$
5. $z \in (3; 4]$, calculate: $x + z - y \in [-1; 3] + (3; 4] - [2; 6] = (2; 7] - [2; 6] = (-4; 5]$

Exercise 2: Propagation

a) Given the following constraints

$$c_1 : 2 \cdot x - 3 \cdot y = 0, \qquad c_2 : x^2 - 2 \cdot y = 0,$$

perform two interval propagation steps with the first contraction method from the lecture. In each step choose the contraction candidate with the highest relative contraction (which you have to compute first). The initial intervals of x and y are $x, y \in [1; 10]$.

Solution:

Possible contractions in the first step:

- $(c_1, x) : x = \frac{3 \cdot y}{2} \in \frac{[3;30]}{[2;2]} = [\frac{3}{2}; 15] \rightarrow x \in [1; 10] \cap [\frac{3}{2}; 15] = [\frac{3}{2}; 10]$
  (relative contraction: $\sim$ 0.056) relative contraction= (D_old-D_new)/D_old
- $(c_1, y) : y = \frac{2 \cdot x}{3} \in \frac{[2;20]}{[3;3]} = [\frac{2}{3}; \frac{20}{3}] \rightarrow y \in [1; 10] \cap [\frac{2}{3}; \frac{20}{3}] = [1; \frac{20}{3}]$
  (relative contraction: $\sim$ 0.370)

- $(c_2, x) : x = \sqrt{2 \cdot y} \in [\sqrt{2}; \sqrt{20}] \to x \in [1; 10] \cap [\underbrace{\sqrt{2}}_{\sim 1.41}; \underbrace{\sqrt{20}}_{\sim 4.47}] = [\sqrt{2}; \sqrt{20}]$
  (relative contraction: ~ 0.660)
  Note that we skipped the case $-\sqrt{2 \cdot y}$ because $x \in [1; 10]$ and thus we know that negative values would be cut away anyway.
- $(c_2, y) : y = \frac{x^2}{2} \in \frac{[1;100]}{2} = [\frac{1}{2}; 50] \to y \in [1; 10] \cap [\frac{1}{2}; 50] = [1; 10]$
  (relative contraction: 0)

Choose (c_2, x) as the first contraction candidate. Updated intervals: $x \in [\sqrt{2}; \sqrt{20}]$, $y \in [1; 10]$.

Possible contractions in the second step:

- $(c_1, x) : x = \frac{3 \cdot y}{2} \in \frac{[3;30]}{[2;2]} = [\frac{3}{2}; 15] \to x \in [\sqrt{2}; \sqrt{20}] \cap [\frac{3}{2}; 15] = [\frac{3}{2}; \sqrt{20}]$
  (relative contraction: ~ 0.028)
- $(c_1, y) : y = \frac{2 \cdot x}{3} \in \frac{2 \cdot [\sqrt{2}; \sqrt{20}]}{[3;3]} = [\frac{2 \cdot \sqrt{2}}{3}; \frac{2 \cdot \sqrt{20}}{3}] \to y \in [1; 10] \cap \left[\underbrace{\frac{2 \cdot \sqrt{2}}{3}}_{\sim 0.94}; \underbrace{\frac{2 \cdot \sqrt{20}}{3}}_{\sim 2.98}\right] = [1; \frac{2 \cdot \sqrt{20}}{3}]$
  (relative contraction: ~ 0.780)
- $(c_2, x) : x = \sqrt{2 \cdot y} \in [\sqrt{2}; \sqrt{20}] \to x \in [\sqrt{2}; \sqrt{20}] \cap [\sqrt{2}; \sqrt{20}] = [\sqrt{2}; \sqrt{20}]$
  (relative contraction: 0)
- $(c_2, y) : y = \frac{x^2}{2} \in \frac{[2;20]}{2} = [1; 10] \to y \in [1; 10] \cap [1; 10] = [1; 10]$
  (relative contraction: 0)

Intervals after two optimal contractions: $x \in [\sqrt{2}; \sqrt{20}], y \in [1; \frac{2 \cdot \sqrt{20}}{3}]$.

b) Given the constraints

$$a^2 + b^2 \leq 1 \text{ and } a \cdot b \geq 1$$

preprocess: 对多项式中的每一项用变量h进行替换

preprocessing yields the following equations and initial bounds:

$$\begin{array}{lll}
 & & a \in [-\infty; \infty] \\
 & & b \in [-\infty; \infty] \\
c_1 : & h_1 + h_2 \leq 1 & \\
c_2 : & h_1 = a^2 & h_1 \in [0; \infty] \\
c_3 : & h_2 = b^2 & h_2 \in [0; \infty] \\
c_4 : & h_3 \geq 1 & \\
c_5 : & h_3 = a \cdot b & h_3 \in [-\infty; \infty]
\end{array}$$

Use the first contraction method from the lecture and propagate using these equations until one of the variable domains gets empty.

Solution:

In the following we mean by $\rightsquigarrow_{i,c_j}$ that we used the constraint c_j in the i-th step for contracting the variable in the corresponding row.

$$\begin{array}{llllll}
a \in [-\infty; \infty] & & \rightsquigarrow_{3,c_2} [-1; 1] & & \rightsquigarrow_{7,c_5} [-1; -1] \cup [1; 1] & \\
b \in [-\infty; \infty] & & \rightsquigarrow_{4,c_3} [-1; 1] & & \rightsquigarrow_{8,c_5} [-1; -1] \cup [1; 1] & \\
h_1 \in [0; \infty] & \rightsquigarrow_{1,c_1} [0; 1] & & & \rightsquigarrow_{9,c_2} [1; 1] & \\
h_2 \in [0; \infty] & \rightsquigarrow_{2,c_1} [0; 1] & & & \rightsquigarrow_{10,c_3} [1; 1] & \rightsquigarrow_{11,c_1} \emptyset \\
h_3 \in [-\infty; \infty] & & \rightsquigarrow_{5,c_5} [-1; 1] & \rightsquigarrow_{6,c_4} [1; 1] & &
\end{array}$$

Exercise 3: Questions

Give a short answer to the following questions:

1. The ICP algorithm from the lecture maintains two threshold values as parameters. Describe the purpose of these values.
2. Which are the two events causing a split in the ICP algorithm presented in the lecture?
3. ICP is not a complete method. Why does it still make sense to use it as a preprocessing to a complete method, such as CAD or VS?

Solution:

1. ICP maintains a threshold for the weight of the contraction candidates as well as a threshold for the target diameter of the box. The first one is required to stop contraction: If the weight of a contraction candidate is below the threshold, the candidate is not longer considered for contraction, which means that the contraction gain of this candidate has been too low during the last runs and the candidate does not seem to be promising for contraction. The second parameter is used to stop the overall loop. The loop is stopped as soon as the box is smaller than this threshold in every dimension. This parameter specifies the precision of the algorithm.
2. On the one hand a split is performed whenever there is no promising contraction candidate available - the contraction gain has been too low and the algorithm does not make enough progress and the target diameter is not yet reached. On the other hand a split can occur, when during contraction (e.g. via the componentwise Newton operator) a division by an interval containing 0 occurs.
3. Complete methods such as CAD or VS profit from a reduced search space as they are able to drop samples/test candidates not contained in the search space.