Theory of Hybrid Systems Informatik 2

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Satisfiability Checking - WS 2023/2024

Series 5

Exercise 1

a) Give a formula describing the Unequal game instance of Series 3 Exercise 2 in *equality logic with uninterpreted functions*. Remember, that the formula shall be satisfiable iff the game instance has a solution. You must not use propositional variables in your solution!

b) Compare the resulting formula to the propositional encoding. More precisely, compare the number of literals and clauses using the big $\mathcal{O}$ notation. Draw a conclusion.

Solution:

a) In the following, $g_{i,j}$ $(1 \leq i, j \leq n)$ is an uninterpreted variable and we introduce the set of constants $N := \{1, \dots, n\}$. Then $g_{i,j}$ represents the number in the grid at the coordinates $(i,\ j)$. We define the uninterpreted function $greater(r, s)$ on $N \times N$ mapping to the constants T and F.

$$\varphi_{grid} \quad := \quad \bigwedge_{i=1}^{n} \bigwedge_{j=1}^{n} \bigvee_{k=1}^{n} g_{i,j} = k$$

$$\varphi_{row} \quad := \quad \bigwedge_{i=1}^{n} \bigwedge_{j=1}^{n} \bigwedge_{k=j+1}^{n} g_{i,j} \neq g_{i,k}$$

$$\varphi_{column} \quad := \quad \bigwedge_{j=1}^{n} \bigwedge_{i=1}^{n} \bigwedge_{k=i+1}^{n} g_{i,j} \neq g_{k,j}$$

$$\varphi_{>} \quad := \quad \bigwedge_{\substack{r\,=\,1\\ s\,=\,1\\ r\,>\,s}}^{n} greater(r,s) = T \quad \wedge \quad \bigwedge_{\substack{r\,=\,1\\ s\,=\,1\\ r\,<=\,s}}^{n} greater(r,s) = F$$

Hence, we get the formula representing the game instance of Series 3 Exercise 2:

$$\begin{aligned}
\varphi^{UF} \quad := \quad & g_{1,1} = 3 \quad \wedge \quad g_{4,1} = 1 \quad \wedge \quad greater(g_{1,2},\ g_{1,3}) = T \\
& greater(g_{2,3},\ g_{2,2}) = T \quad \wedge \quad greater(g_{1,4},\ g_{2,4}) = T \\
& greater(g_{3,4},\ g_{2,4}) = T \quad \wedge \quad greater(g_{4,3},\ g_{4,4}) = T \\
& \varphi_{grid} \quad \wedge \quad \varphi_{row} \quad \wedge \quad \varphi_{column} \quad \wedge \quad \varphi_{>}
\end{aligned}$$

b) By comparing the literals/constraints and clauses of the result of part a) with the result of Series 3 Exercise 2 you see:

- As the number of variables in the formula using propositional logic of Series 3 Exercise 2 has been n^3, the number of literals is:

$$\mathcal{O}(n^3)$$

- The number of clauses in the propositional formula is:

$$\mathcal{O}(n^2) + 2 \cdot \mathcal{O}(n^4) + 5 \cdot \mathcal{O}(n^2) + 2 \;=\; \mathcal{O}(n^4)$$

- The number of literals (different equations) in φ^{UF} is:

$$3 \cdot \mathcal{O}(n^3) + \mathcal{O}(n^2) + 5 + 2 \;=\; \mathcal{O}(n^3)$$

- The number of clauses in φ^{UF} is:

$$\mathcal{O}(n^2) + 2 \cdot \mathcal{O}(n^3) + \mathcal{O}(n^2) + 5 + 2 \;=\; \mathcal{O}(n^3)$$

Conclusion: The formula using propositional logic needs less literals but $\mathcal{O}(n)$ times more clauses than the formula using equality logic with uninterpreted functions.

Exercise 2

Consider the following formula in equality logic:

$$\varphi \quad := \quad a = b \ \wedge \ (\ b = c \ \vee \ c = e\ ) \ \wedge \ (\ b = d \ \vee \ c = f\ ) \ \wedge \ a \neq e$$

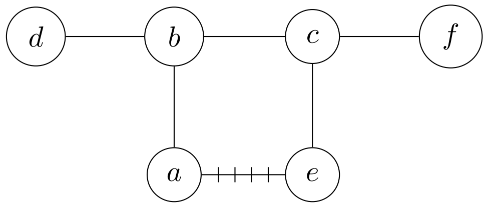
a) Simplify the formula φ using the method presented in the lecture, based on polar equality graphs (slides 37-38).

b) For the simplied formula, construct the equality graph without polarity and make it chordal. What are the chord-free simple cycles?

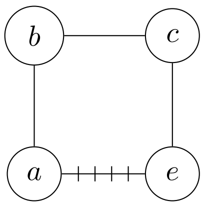
c) Construct the satisfiability-equivalent propositional logic formula for φ using the previous results from b).

Solution:

a) The equality graph with polarity for φ is given by:



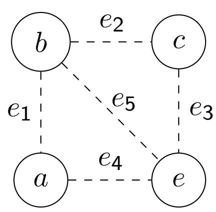
The edges (b, d) and (c, f) are not part of any contradictory cycle, thus we can remove them and replace in the formula both $b = d$ and $c = f$ by *true*. We get the following simplified equality graph:



The simplified formula is

$$a = b \wedge (b = c \vee c = e) \wedge a \neq e$$

b) A chordal completion of the simplified equality graph without polarity is:



The chord-free simple cycles are $(a,\ b,\ e,\ a)$ and $(b,\ c,\ e,\ b)$.

c) The satisfiability-equivalent propositional logic formula for φ using the previous results is:

$$
\begin{aligned}
\varphi^{prop} &:= \varphi_{sk} \quad \wedge \quad \varphi_{trans} \\
\varphi_{sk} &:= e_1 \wedge (e_2 \vee e_3) \wedge \neg e_4 \\
\varphi_{trans} &:= \quad ((e_1 \wedge e_4) \rightarrow e_5) \\
&\quad \wedge ((e_1 \wedge e_5) \rightarrow e_4) \\
&\quad \wedge ((e_4 \wedge e_5) \rightarrow e_1) \\
&\quad \wedge ((e_2 \wedge e_3) \rightarrow e_5) \\
&\quad \wedge ((e_2 \wedge e_5) \rightarrow e_3) \\
&\quad \wedge ((e_3 \wedge e_5) \rightarrow e_2)
\end{aligned}
$$