Satisfiability Checking - WS 2022/2023

Written Exam I

Monday, February 27, 2023

Forename and surname:	**Matriculation number:**
Sign here:	

- Do not open the exam until we give the start signal.
- The duration of the exam is 120 minutes.
- Use a blue or black (permanent) pen only. Everything else will be ignored.
- Please write your name and matriculation number on each page of this exam.
- Please write clear and legible answers. Clearly cross out parts you do *not* wish to be evaluated.
- Please use the space below each task to solve it. Additional paper is provided at the back of the exam.
- If you have problems understanding a task, indicate this by a hand signal.
- You are not allowed to use auxiliary material except for a pen and a ruler. In particular, switch off your electronic devices! Cheating disqualifies from the exam.
- When the exam is over stop writing, turn around your sheets and leave them at your seat.

Task	1.)	2.)	3.)	4.)	5.)	6.)	7.)	8.)	9.)	Bonus	**Total**
Maximum score	17	14	10	17	7	16	13	12	14	-	120
Reached score											

Good luck!

1.) SAT Checking

3 + 4 + 3 + (4+1+2) points

i) In the clause $(a \vee \neg b \vee \neg c \vee d)$, which literal pairs are suited to be watched under the assigment $a = 0$, $b = 1$, $c = 0$ and d unassigned?

ii) Assume the following propositional logic formula in CNF:

$$(A \vee B \vee D) \wedge (C \vee \neg D) \wedge (\neg A \vee \neg C \vee \neg D) \wedge (A \vee \neg B \vee \neg C \vee D)$$

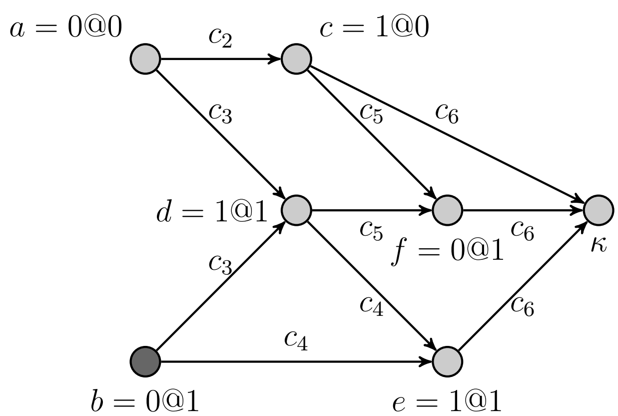
Apply the DPLL+CDCL algorithm until it detects either a conflict or a complete solution. Please specify each step and the intermediate assignment after its execution. For a decision, always take the smallest unassigned variable in the order $A < B < C < D$ and assign *false* to it.

iii) Prove or disprove by providing a counterexample: For every propositional logic formula φ holds: φ is valid if and only if its Tseitin encoding is valid.

iv) Consider the given trail and implication graph for the propositional logic formula

$$\begin{array}{llll} & c_1 : (\neg a) & \wedge \quad c_2 : (a \vee c) & \wedge \quad c_3 : (a \vee b \vee d) \\ \wedge & c_4 : (b \vee \neg d \vee e) & \wedge \quad c_5 : (\neg c \vee \neg d \vee \neg f) & \wedge \quad c_6 : (\neg c \vee \neg e \vee f) \end{array}$$

DL0: $\neg a : c_1, \quad c : c_2$ DL1: $\neg b : nil, \quad d : c_3, \quad e : c_4, \quad \neg f : c_5$



a) The clause c_6 is conflicting. Perform two resolution steps of the Boolean conflict resolution as presented in the lecture. For each step give the current conflicting clause, the used antecedent clause and the new conflicting clause.

b) Depict the cut corresponding to the resulting clause graphically in the above implication graph.

c) Is this conflict resolution finished after the two steps? Please explain why.

2.) Equality Logic and Uninterpreted Functions 4 + 5 + (2 + 3) points

i) Apply *lazy* SMT solving for equality logic as presented in the lecture to the following conjunction of equation and disequations, considering equations from left to right.

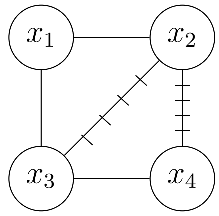
$$v = y \land u \neq v \land y = z \land y = v \land v = z \land x = v$$

Please specify the initial partition, each execution step and the partition after the step.

ii) Consider the following formula in *equality logic with uninterpreted functions*. Apply *Ackermann's reduction* to eliminate the function symbols and to obtain a satisfiability-equivalent formula in *equality logic*.

$$\varphi : (F(a) = G(a, b)) \land (F(F(a)) \neq G(b, F(b)))$$

iii) Consider the following *E-graph with polarity*.



a) Give all simple contradictory cycles contained in the depicted graph.

b) Give a satisfiable formula φ_1 and an unsatisfiable formula φ_2 in equality logic which both produce the depicted E-graph.

3.) Fourier-Motzkin Variable Elimination **6+(2+2) points**

i) Assume the following constraint set:

$$\{2x - y \leq -8, \quad 2x + y \leq 8, \quad -2x - y \leq -8, \quad -2x + y \leq 8\}$$

a) Please apply the method of Fourier-Motzkin to eliminate first y and then x. Specify the result after each elimination step.

b) Please specify a solution using the elimination result. Is this solution unique?

ii) For $n \in \{1, 2, \ldots, 9, 10\}$, consider the linear integer systems

$$S_n = \{-n \cdot x \leq -1,\ n \cdot x \leq 3\}.$$

(a) For which values of $n \in \{1, 2, \ldots, 9, 10\}$ would the Fourier-Motzkin method answer "satisfiable" on the input S_n?

(b) The Fourier-Motzkin variable elimination is *not* correct on problems of linear *integer* arithmetic (LIA). Give all values of $n \in \{1, 2, \ldots, 9, 10\}$, for which the Fourier-Motzkin method would give the correct result (i.e. sat, when S_n is satisfiable with an *integer solution* or unsat, when S_n has no such solution).

4.) Simplex **2+4+5+4+2 points**

i) Denoting original variables as x_i and slack variables as s_i, assume the following simplex tableau and slack variable bounds, with the current values of the variables given in square brackets:

	s_2 [−1]	s_0 [1]	x_2 [0]	
x_1 [1]	-3	-2	-2	$s_0 \leq 1$
s_1 [1]	0	1	-1	$s_1 \leq 0$
x_0 [0]	-1	-1	-1	$s_2 \leq -1$

The basic variable s_1 violates its bound. Please state for each non-basic variable whether it is suitable for pivoting with s_1 or not (no explanation required).

ii) Apply the simplex method to the following constraint set until termination:

$$\begin{aligned} s_0 &= -1x_1 & \qquad s_0 &\leq -2 \\ s_1 &= -3x_0 - 2x_1 & \qquad s_1 &\leq 3 \end{aligned}$$

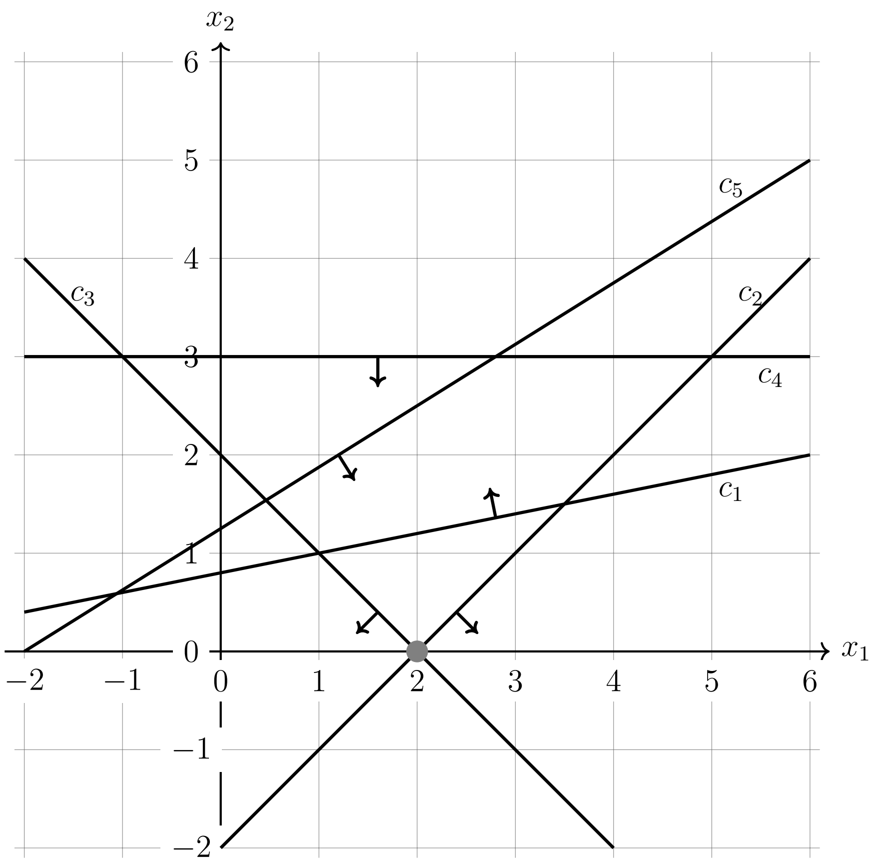
Please specify the simplex tableau and the assignment initially and after each pivot step. When choosing pivot variables, use the order $x_0 \prec x_1 \prec s_0 \prec s_1$ and take the smallest possible variable.

iii) Consider the following tableau (the current values of the variables are given in square brackets) with some information missing. Fill out the missing fields, assuming that (1) both basic slack variables violate their bounds and (2) s_3 is the only variable suitable for pivoting with the basic variable chosen by Bland's rule. Assume the variable ordering $x_1 \prec x_2 \prec s_1 \prec s_2 \prec s_3 \prec s_4$ preferring the smallest variable for Bland's rule.

	s_1 [3]	s_3 [-1]
s_2 [1]	1	2
x_2 [-6]	-2	0
s_4 [10]	3	-1
x_1 [7]	2	-1

s_1	$\geq$		
s_2	$\leq$		-1
s_3			
s_4			3

iv) Consider the following system of linear inequations $\{c_1, \ldots, c_5\}$. The point marked at at $(2, 0)$ corresponds to a state of the simplex algorithm on the system.



By s_i we denote the slack variable that stems from the linear inequation c_i.

Mark the assignment that is obtained after one pivot step according to Bland's rule. Assume the variable ordering $x_1 \prec x_2 \prec s_1 \prec s_2 \prec s_3 \prec s_4$ preferring the smallest variable for Bland's rule.

v) Give all possible infeasible subsets of the following conflicting tableau (the current values of the variables are given in square brackets). Denote the constraints corresponding to $s_1, \ldots, s_4$ by $c_1, \ldots, c_4$.

	s_1 [0]	s_2 [0]
x_1 [0]	-1	0
x_2 [0]	0	-1
s_3 [0]	0	1
s_4 [0]	1	1

$$\begin{aligned} s_1 &\geq 0 \\ s_2 &\leq 0 \\ s_3 &\geq 1 \\ s_4 &\leq -1 \end{aligned}$$

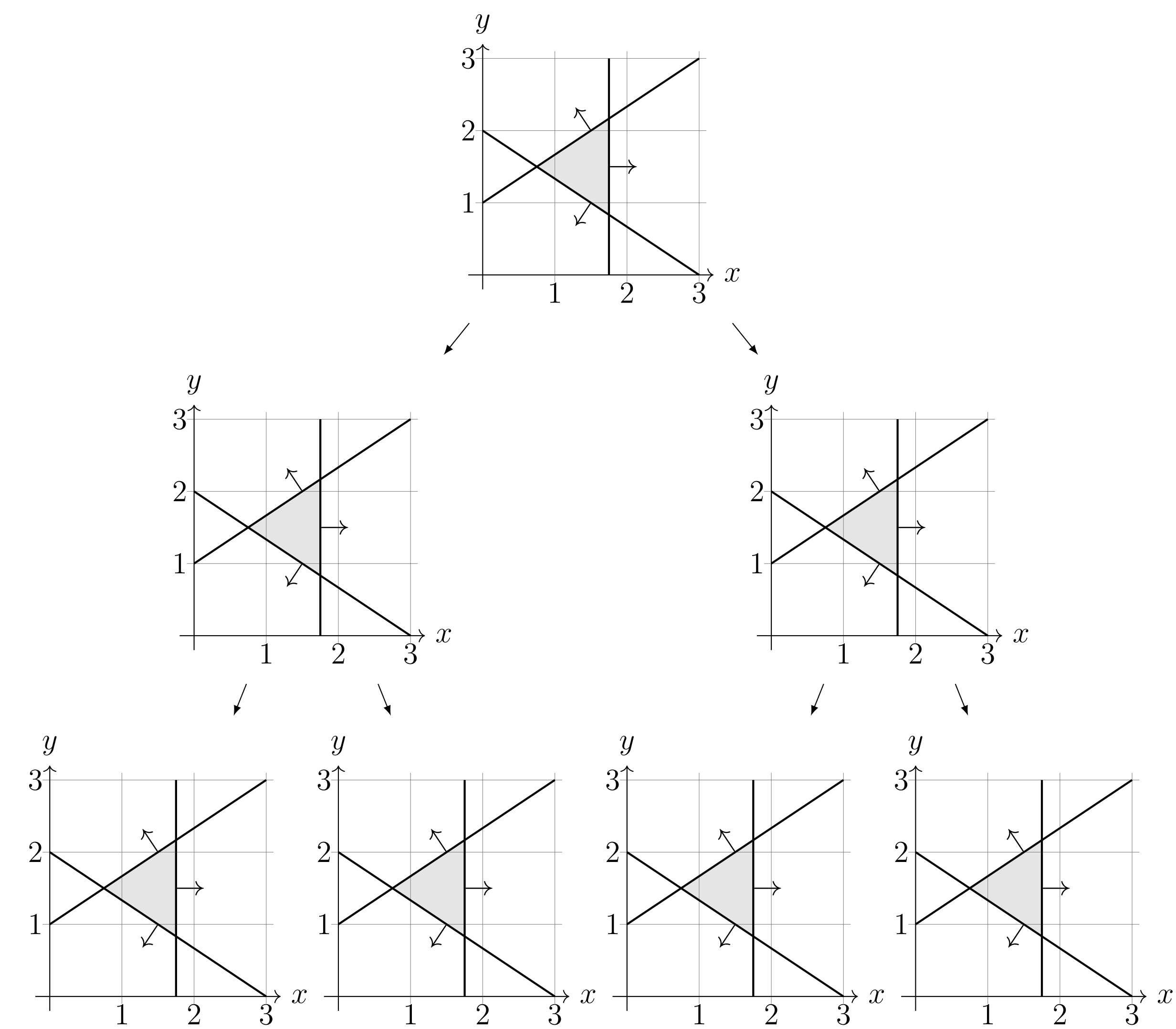
5.) Linear Integer Arithmetic **7 points**

Consider the example depicted below. Execute the *Branch&Bound* algorithm until its termination.

Use the following assumptions:

- The real solution (x^*, y^*) found by simplex will always be the *lexicographically largest* among all solutions (i.e. for all solutions (x, y): $x^* \geq x \vee (x^* = x \wedge y^* \geq y)$).
- x is preferred over y for branching.

Depict each call of the recursive Branch&Bound method in a new plot (there might be more plots than necessary). Mark the real solution and sketch the additional constraints used for branching. Indicate if a call has no real solution by writing "UNSAT".



6.) Interval Constraint Propagation **6 + 10 points**

i) Contract the domain $x \in A = [1;6]$ with the help of the univariate interval Newton method from the lecture using the constraint $-x^2 - 2x - 4 = 0$ and sample point $s = 4$. Please write down the computations and the resulting contracted domain for x.

ii) Consider the constraints $c_1 : x - y = 0$ and $c_2 : x - \frac{1}{2}y^2 = 0$ and the initial intervals $x \in [-1, 1]$, $y \in [-1, 1]$.

a) For each contraction candidate, give the relative contraction when applying contraction method I once.

b) Give a sequence of contraction candidates which can be repeated infinitely so that each contraction has a gain strictly greater than zero.

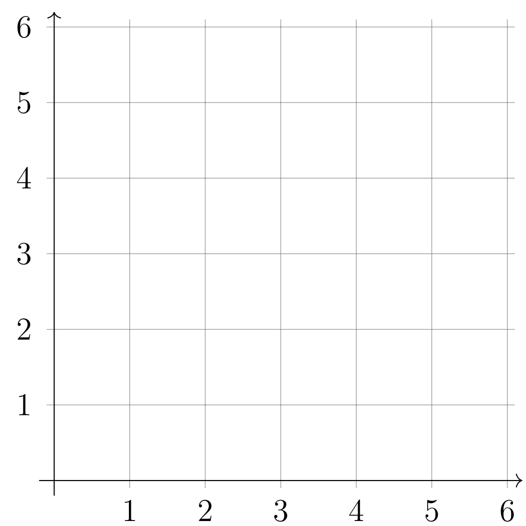
7.) Subtropical Satisfiability

4+3+2+4 points

i) Please specify a real value n_x such that the direction $(n_x, 1) \in \mathbb{R}^2$ is suited to separate a positive frame point from all the other frame points for the constraint

$$2x^1y^3 - 2x^5y^4 - 5x^4y^1 - 2x^5y^5 > 0 \ .$$

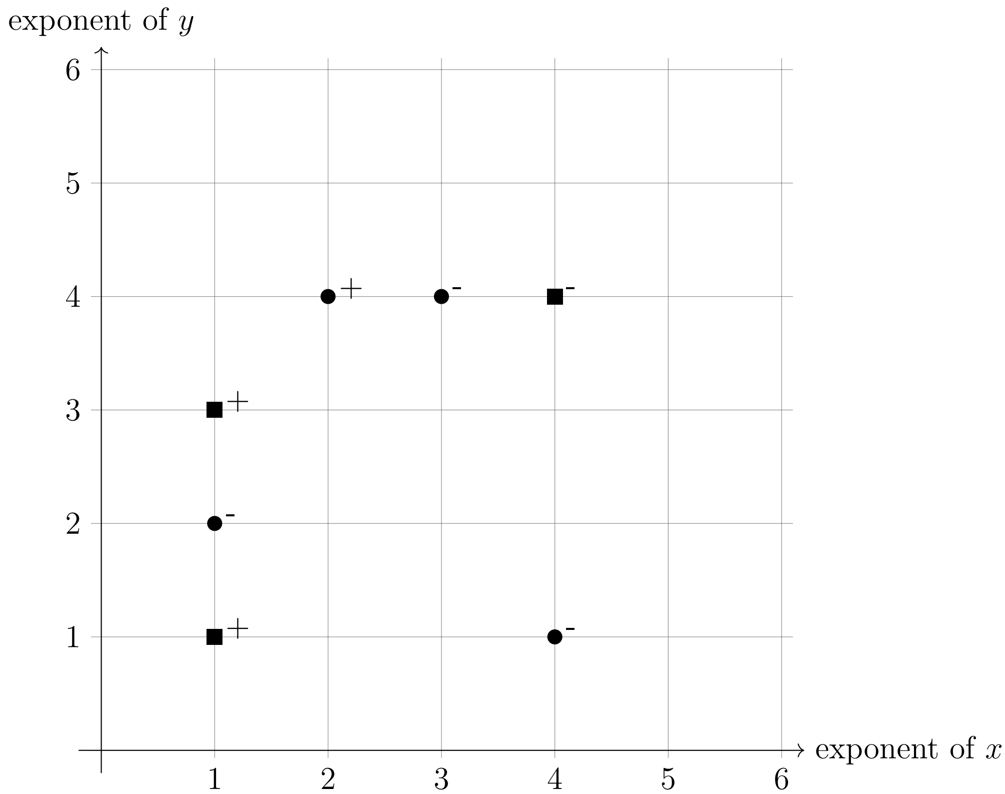
You may use the following coordinate system for auxiliary calculations (this is not evaluated):



ii) Give the subtropical encoding for the satisfiability of $x^2y - 2xy > 0$ according to the lecture.

iii) The following image depicts the frame points of two polynomials $p_1(x, y)$ (circles) and $p_2(x, y)$ (squares).

Depict a suitable separating hyperplane and its normal vector $n = (n_x, n_y) \in \mathbb{Z}^2$ such that there exists an $a \in \mathbb{R}_{>0}$ with $p_1(a^{n_x}, a^{n_y}) > 0$ and $p_2(a^{n_x}, a^{n_y}) > 0$.



iv) Consider the polynomial $p_d(x, y) := -xy + 3x^{2d}y - 5x^2y^d$ for $d \in \mathbb{N}$.

Give the set of all values for d such that there exists a hyperplane with normal vector $n = (1, 1)$ that separates a positive frame point of p_d from all other frame points of p_d.

8.) Virtual Substitution **4+3+5 points**

i) Please specify a test candidate different from $-\infty$ that is generated for x by at least two of the following constraints:

$$x + 2 = 0 \quad x + 2 < 0 \quad x + 2 > 0$$

ii) Let $\exists x.\ \exists y.\ \varphi$ be a formula over the variables x, y. Assume that for the elimination of y via virtual substitution, the set of test candidates according to the lecture is given as

Test candidate		Side condition
$-\infty$	, if	true
$\frac{1}{x}$	, if	$x \neq 0$

Use the virtual substitution method from the lecture to eliminate y from the above formula, without applying the substitution rules. That means you may use $\varphi[t//y]$ to denote the sub-formula after substituting a test candidate t for y in φ.

iii) Design a virtual substitution rule for $(b \cdot x + c \leq 0)[e - \varepsilon//x]$ (where b and c are coefficients not containing x) for some test candidate $e - \varepsilon$.

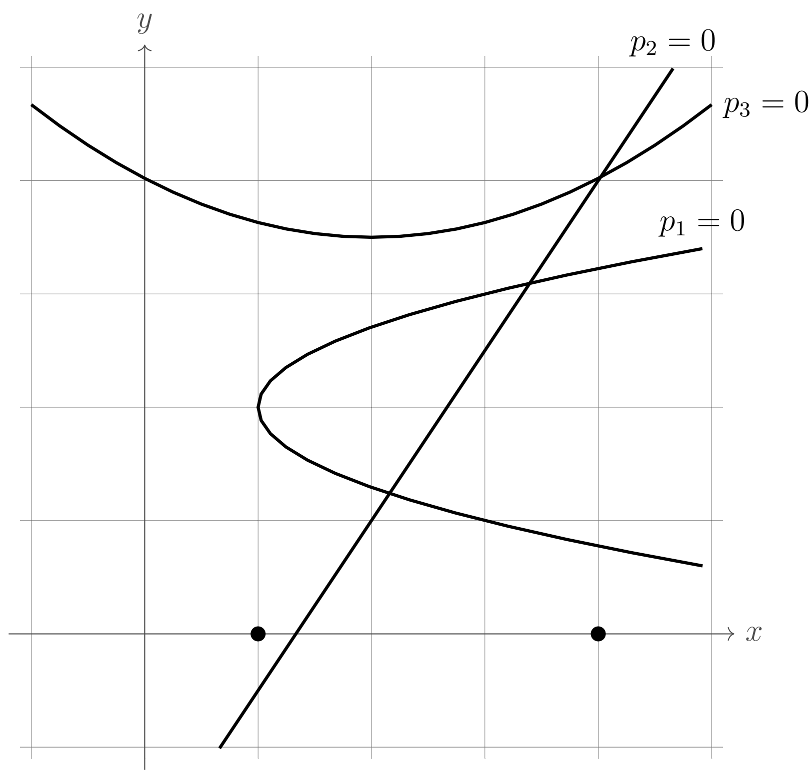
You may use $C[e//x]$ for any constraint C in your answer without applying the virtual substitution rule for it.

Note that the test candidate contains $-\varepsilon$, *not* $+\varepsilon$.

9.) Cylindrical Algebraic Decomposition **4+4+6 points**

i) Assume the variable order $x < y$. Give the maximal interval in the x-dimension that contains 0 and on which $P = \{-9x + y - 1,\ -8x + y + 4\} \subset \mathbb{Z}[y][x]$ is delineable.

ii) Consider the following varieties (sets of zeros) of some polynomials p_1, p_2 and p_3:



The dots on the x-axis denote roots of some univariate polynomials in x generated by the CAD method during the projection of y with p_1, p_2, p_3 as input. Which are they?

Write $res(p_a, p_b)$ for the resultant of polynomials p_a and p_b, $disc(p)$ for the discriminant of p and $ldcf(p)$ for the leading coefficient of p.

iii) Isolate all real zeros of the univariate polynomial plotted below in the interval $(-2, 2)$ with the method presented in the lecture, using interval midpoints for splitting. You can read off all needed information from the plots below. Depict all resulting *isolating* intervals in the picture below.