# 02 Propositional logic :

**Assignments**: special interpretation ↗ Assigns Boolean values to the variables : $\alpha : AP \to \{0,1\}$ and use Assign to donate the Set of all assignments.

- See assignment $\alpha$ as a set of variables ( $\alpha \in 2^{AP}$ as being of type $\alpha \in \{0,1\}^{AP}$

**Satisfaction relation**: $\alpha \models \varphi$ and say $\alpha$ satisfies $\varphi$ or $\alpha$ is a model of $\varphi$

formula $\varphi$ is **valid** or **tautology** if $sat(\varphi) = Assign$
formula $\varphi$ is **satisfiable** if $sat(\varphi) \neq \emptyset$
formula $\varphi$ is **unsatisfiable** or **contradiction** if $sat(\varphi) = \emptyset$
→ formula $\varphi$ is valid iff $\neg\varphi$ is unsatisfiable

**Literal**: is either a variable or the negation of a variable
**Term** : conjunction of literals ( $a \land \neg b \land c$ )
**clause** : disjunction of literals ( $a \lor \neg b \lor c$ )

**Negation Normal Form (NNF)**:
1. Contain only $\neg, \lor, \land$
2. only variables are negated

(time and)

- Transformation to NNF done with linear effort (place)
- Idea: Nr. of transf. steps ≤ Nr. of operands in the formula

## Disjunctive Normal Form (DNF):

- $\bigvee_i (\bigwedge_j l_{i,j})$
- formula to DNF in exponential time and space:
    1. convert to NNF
    2. Distribute disjunctions following the rule:

       $$\models \varphi_1 \wedge (\varphi_2 \vee \varphi_3) \Leftrightarrow (\varphi_1 \wedge \varphi_2) \vee (\varphi_1 \wedge \varphi_3)$$
- $2^n$ clauses
- satisfiability check of DNF formula in linear (time and space)

## Conjunctive Normal Form (CNF):

- $\bigwedge_i (\bigvee_j l_{i,j})$
- formula to CNF in exponential time and space:
    1. convert to NNF
    2. Distribute disjunctions following the rule:

       $$\models \varphi_1 \vee (\varphi_2 \wedge \varphi_3) \Leftrightarrow (\varphi_1 \vee \varphi_2) \wedge (\varphi_1 \vee \varphi_3)$$
- $2^n$ clauses

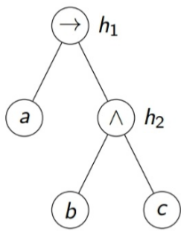
## Tseitin's encoding:

- formula to CNF in linear Time and speace if new variables are added

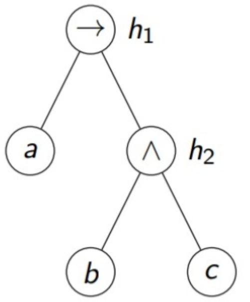
- not equivalent but equi-satisfiable
- 3n+1 clauses, 3n variables

- Consider the formula $\varphi = (a \rightarrow (b \wedge c))$
- Associate a new auxiliary variable with each inner (non-leaf) node.
- Add constraints that define these new variables.
- Finally, enforce the truth of the root node.

Parse tree:




- Tseitin's encoding:
  $(h_1 \leftrightarrow (a \rightarrow h_2)) \wedge$
  $(h_2 \leftrightarrow (b \wedge c)) \wedge$
  $(h_1)$




- Each node's encoding has a CNF representation with 3 or 4 clauses.
  $h_1 \leftrightarrow (a \rightarrow h_2)$ in CNF: $(h_1 \vee a) \wedge (h_1 \vee \neg h_2) \wedge (\neg h_1 \vee \neg a \vee h_2)$
  $h_2 \leftrightarrow (b \wedge c)$ in CNF: $(\neg h_2 \vee b) \wedge (\neg h_2 \vee c) \wedge (h_2 \vee \neg b \vee \neg c)$

Is $\varphi$ satisfiable? finding out: → Enumeration
→ Deduction

## Enumeration

## Deduction:

A deductive Proof system consists of a set of axioms and inference rules

**Inference rules**: $\frac{\text{Antecedents}}{\text{Consequents}}$

**Axioms**: are inference rules with no antecedents

Provability relation
↑
$\Gamma \vdash_{\mathcal{H}} \varphi$ : there is a proof of $\varphi$ in system $\mathcal{H}$ whose

premises are included in $\Gamma$

**Soudness**: Does $\vdash$ conclude "correct" conclusions from premises?

- if $\Gamma \vdash_{\mathcal{H}} \varphi$ then $\Gamma \models \varphi$

**Completeness**: Can we conclude all true statements with $\mathcal{H}$?

- if $\Gamma \models \varphi$ then $\Gamma \vdash_{\mathcal{H}} \varphi$

**Resolution**: is a sound and complete proof system of CNF

- if Input unsatisfiable, there exists a proof of the empty clause.
- $$\frac{(\ell \vee \ell_1 \vee \ell_2 \vee \ldots \vee \ell_n) \quad (\neg \ell \vee \ell'_1 \vee \ldots \vee \ell'_m)}{(\ell_1 \vee \ldots \vee \ell_n \vee \ell'_1 \vee \ldots \vee \ell'_m)}$$