Satisfiability Checking - WS 2023/2024

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Series 7

Exercise 1

Consider the propositional logical formula with equalities:

$$
\begin{array}{rlccccc}
\varphi^{EQ} := & & x_3 = x_5 & \wedge & (\neg x_1 = x_4 & \vee & \neg x_4 = x_5) \\
& \wedge & x_4 = x_6 & \wedge & (x_4 = x_5 & \vee & x_3 = x_4) \\
& \wedge & x_1 = x_2 & \wedge & (x_4 = x_5 & \vee & x_3 = x_6)
\end{array}
$$

The Boolean abstraction of this formula is

$$a_1 \wedge (\neg a_2 \vee \neg a_3) \wedge a_4 \wedge (a_3 \vee a_5) \wedge a_6 \wedge (a_3 \vee a_7).$$

Simulate how a less-lazy SMT solver solves φ^{EQ} for satisfiability as presented in the lecture. Show the progress in the SAT solver and the theory solver implementing an incremental and infeasible subset generating procedure for solving a conjunction of equalities for satisfiability. If the SAT solver makes a decision, it chooses the unassigned variable a_i with the lowest index and assigns it to false. Show how the theory solver benefits from its incrementality support, both when adding and removing constraints, and show how the infeasible subset(s) are computed.

Solution:

SAT solver	**Theory solver**
We apply Boolean constraint propagation. $DL0: \quad a_1 : 1, a_4 : 1, a_6 : 1$	Received (in)equalities: $x_3 = x_5, \quad x_4 = x_6, \quad x_1 = x_2$ We create for each variable an equivalence class: $\langle x_1 \rangle = \{x_1\}$ $\langle x_2 \rangle = \{x_2\}$ $\langle x_3 \rangle = \{x_3\}$ $\langle x_4 \rangle = \{x_4\}$ $\langle x_5 \rangle = \{x_5\}$ $\langle x_6 \rangle = \{x_6\}$ For each equality we merge the equivalence classes of the variables at its left- and right-hand side resulting in: $\langle x_1 \rangle = \{x_1, x_2\}$ $\langle x_3 \rangle = \{x_3, x_5\}$ $\langle x_4 \rangle = \{x_4, x_6\}$ As there is no inequality, no conflict can occur.
We decide that a_2 is assigned to 0 and applyBoolean constraint propagation (which implies no assignments) resulting in: $DL0: \quad a_1 : 1, a_4 : 1, a_6 : 1$ $DL1: \quad a_2 : 0$	Received (in)equalities: $x_3 = x_5, \quad x_4 = x_6, \quad x_1 = x_2$ $x_1 \neq x_4$ No new equation, hence the equivalence classes remain untouched. The left and right-hand side of the only inequality $x_1 \neq x_4$ are in different equivalent classes, hence there is no conflict.

SAT solver

We decide that a_3 is assigned to 0 and apply-Boolean constraint propagation resulting in:

$DL0:\quad a_1:1, a_4:1, a_6:1$

$DL1:\quad a_2:0$

$DL2:\quad a_3:0, a_5:1, a_7:1$

Theory solver

Received (in)equalities:

$x_3 = x_5,\quad x_4 = x_6,\quad x_1 = x_2$

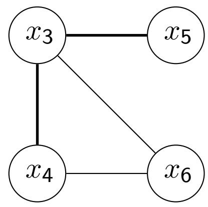
$x_1 \neq x_4,$

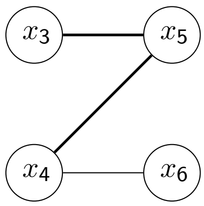
$x_4 \neq x_5,\quad x_3 = x_4,\quad x_3 = x_6$

The equivalence classes are now:

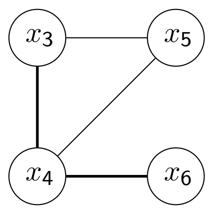
$$\begin{array}{lcl} \langle x_1 \rangle & = & \{x_1,\ x_2\} \\ \langle x_3 \rangle & = & \{x_3,\ x_4,\ x_5,\ x_6\} \end{array}$$

The inequality $x_1 \neq x_4$ is still not conflicting, but the inequality $x_4 \neq x_5$ has its left- and right-hand side in the same equivalence class $\langle x_3 \rangle$ and is therefore conflicting. We consider the equality graph (without polarity as only equations are relevant) of the equations containing the variables in $\langle x_3 \rangle$. In the graph we search for the shortest path between the nodes of the variables of the conflicting inequality which are x_4 and x_5. The equations corresponding to this path together with the conflicting inequality form a minimal infeasible subset $\{x_4 \neq x_5,\ x_3 = x_4,\ x_3 = x_5\}$.



<table>
<tr><th>SAT solver</th><th>Theory solver</th></tr>
<tr><td>We exclude the assignment corresponding to the infeasible subset by considering the conflicting clause ($\neg a_1 \vee a_3 \vee \neg a_5$). The SAT solver applies resolution in order to achieve an asserting clause:
$$\frac{(a_3 \vee a_5)\ (\neg a_1 \vee a_3 \vee \neg a_5)}{(\neg a_1 \vee a_3)}$$
Note that we skipped the clause $(a_3 \vee a_7)$ as the implied literal a_7 is not part of the conflict clause. We add the clause $(\neg a_1 \vee a_3)$ to the SAT solver's set of clauses, backtrack to decision level 0, assign a_3 to 1 and apply Boolean constraint propagation leading to:
$DL0: \quad a_1:1, a_4:1, a_6:1, a_3:1, a_2:0$</td><td>Received (in)equalities:
$x_3 = x_5, \quad x_4 = x_6, \quad x_1 = x_2$
$x_4 = x_5, \quad x_1 \neq x_4,$
The equivalence classes are still:
$\langle x_1 \rangle \quad = \quad \{x_1,\ x_2\}$
$\langle x_3 \rangle \quad = \quad \{x_3,\ x_4,\ x_5,\ x_6\}$
As x_1 and x_4 are in different equivalence classes we have no conflict.</td></tr>
<tr><td>We decide that a_5 is assigned to 0 (BCP leads to no assignments):
$DL0: \quad a_1:1, a_4:1, a_6:1, a_3:1, a_2:0$
$DL1: \quad a_5:0$</td><td>Received (in)equalities:
$x_3 = x_5, \quad x_4 = x_6, \quad x_1 = x_2$
$x_4 = x_5, \quad x_1 \neq x_4,$
$x_3 \neq x_4$
The equivalence classes are unchanged, but x_3 and x_4 are already in the same equivalence class. We consider the equality graph:


Infeasible subset:
$\{x_3 = x_5,\ x_4 = x_5,\ x_3 \neq x_4\}$</td></tr>
</table>

SAT solver	**Theory solver**
The conflicting clause $(\neg a_1 \vee \neg a_3 \vee a_5)$ is already asserting as it only contains one literal of the current decision level. We add the clause to our set of clauses backtrack to decision level 0 and propagate. $DL0: \quad a_1:1, a_4:1, a_6:1, a_3:1, a_2:0,$ $a_5:1$	Received (in)equalities: $x_3 = x_5, \quad x_4 = x_6, \quad x_1 = x_2$ $x_4 = x_5, \quad x_1 \neq x_4, \quad x_3 = x_4$ No change in the equivalence classes and no conflict.
We decide that a_7 is assigned to 0 (BCP leads to no assignments): $DL0: \quad a_1:1, a_4:1, a_6:1, a_3:1, a_2:0,$ $a_5:1$ $DL1: \quad a_7:0$	Received (in)equalities: $x_3 = x_5, \quad x_4 = x_6, \quad x_1 = x_2$ $x_4 = x_5, \quad x_1 \neq x_4, \quad x_3 = x_4$ $x_3 \neq x_6$ The equivalence classes are unchanged. The variables x_3 and x_6 are in the same equivalence class, therefore we have a conflict. We consider the equality graph:  Infeasible subset: $\{x_4 = x_6,\ x_3 = x_4,\ x_3 \neq x_6\}$
The conflicting clause $(\neg a_4 \vee \neg a_5 \vee a_7)$ is already asserting as it only contains one literal of the current decision level. We add the clause to our set of clauses backtrack to decision level 0 and propagate. $DL0: \quad a_1:1, a_4:1, a_6:1, a_3:1, a_2:0,$ $a_5:1, a_7:1$	Received (in)equalities: $x_3 = x_5, \quad x_4 = x_6, \quad x_1 = x_2$ $x_4 = x_5, \quad x_1 \neq x_4, \quad x_3 = x_4$ $x_3 = x_6$ The equivalence classes are still: $\langle x_1 \rangle \quad = \quad \{x_1,\ x_2\}$ $\langle x_3 \rangle \quad = \quad \{x_3,\ x_4,\ x_5,\ x_6\}$ The only inequality $x_1 \neq x_4$ is not conflicting.

The SMT solver's SAT solver found a full satisfying assignment of the Boolean skeleton and it's corresponding theory constraints are consistent, therefore the SMT solver returns that φ^{EQ} is satisfiable.