Satisfiability Checking - WS 2020/2021

Written Exam I

Saturday, March 06, 2021

Sample solution

Task	1.)	2.)	3.)	4.)	5.)	6.)	7.)	8.)	9.)	**Total**
Maximum score	20	13	15	15	10	13	10	12	12	120
Reached score										

Good luck!

1.) SAT Checking **(6+4)+(5+3)+2 Points**

i) Consider the following propositional logic formula:

$$\begin{array}{llll} & c_1 : (b \vee \neg c \vee \neg d) & \wedge \quad c_2 : (a \vee b \vee d \vee f) & \wedge \quad c_3 : (\neg c \vee e) \\ \wedge & c_4 : (\neg a) & \wedge \quad c_5 : (a \vee \neg b) & \wedge \quad c_6 : (a \vee b \vee \neg c \vee \neg f) \end{array}$$

(a) **Apply the CDCL SAT-checking algorithm** to the given formula until the first conflict appears. When making a decision, take the lexicographically smallest unassigned variable first and assign **true** to it. Show the (partial) assignments after each decision level.

(b) Apply **conflict resolution** as presented in the lecture.

Solution:

(a) Propagate $a = 0@0$ using c_4
Propagate $b = 0@0$ using c_5
Updated assignment: $(a = 0@0, b = 0@0)$
Decide $c = 1@1$
Propagate $d = 0@1$ using c_1
Propagate $e = 1@1$ using c_3
Propagate $f = 1@1$ using c_2
Updated assignment: $(a = 0@0, b = 0@0, c = 1@1, d = 0@1, e = 1@1, f = 1@1)$
Clause $c_6 : (a \vee b \vee \neg c \vee \neg f)$ is conflicting

(b) Resolve using variable f:

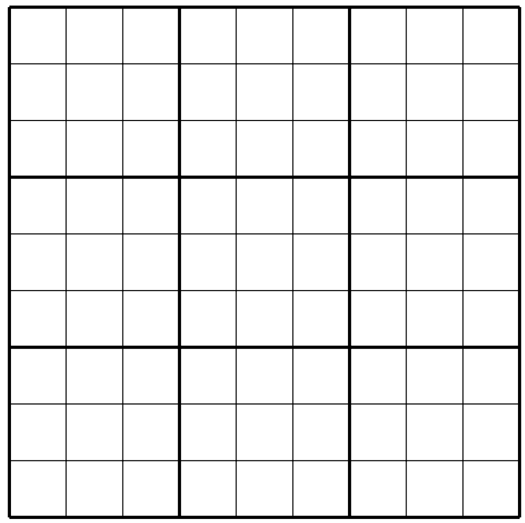
$$\frac{c_6 : (a \vee b \vee \neg c \vee \neg f) \quad c_2 : (a \vee b \vee d \vee f)}{c_7 : (a \vee b \vee \neg c \vee d)}$$

Resolve using variable d:

$$\frac{c_7 : (a \vee b \vee \neg c \vee d) \quad c_1 : (b \vee \neg c \vee \neg d)}{c_8 : (a \vee b \vee \neg c)}$$

Resulting clause $c_8 : (a \vee b \vee \neg c)$ is asserting, backtrack to decision level 0
Propagate $c = 0@0$ using c_8

ii) The objective of a Sudoku puzzle is to fill a 9×9 grid with digits in $\{1, 2, \ldots, 9\}$ such that no digit appears twice in a row, a column, or in a 3×3 subgrid (see figure below).



Note: The set of feasible solutions to a Sudoku is usually constrained by some digits filled into the grid. For this task, no such constraints are given.

Using the Boolean variables x_{ijk}, $i, j, k \in \{1, \dots, 9\}$ to encode that digit k is assigned to the cell in row i and column j, define the below-listed parts of the following formula to encode the set of feasible solutions for the Sudoku puzzle:

$$\varphi_{\text{digit}} \wedge \bigwedge_{i=1,\dots,k} \varphi_{\text{row}}(i) \wedge \bigwedge_{j=1,\dots,k} \varphi_{\text{column}}(j) \wedge \varphi_{\text{subgrid}}$$

a) φ_{digit}: Exactly one digit is assigned to every cell.

b) $\varphi_{\text{row}}(i)$: No digit appears twice in row i.

Note: The subformulas $\varphi_{\text{column}}(j)$ encoding that no digit appears twice in column j and φ_{subgrid} encoding that no digit appears twice in a subgrid do not need to be specified in this task.

Solution:

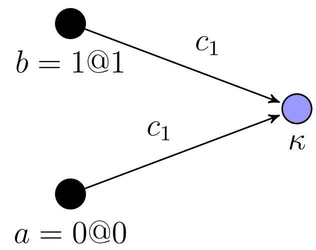
a)

$$\varphi_{\text{digit}} = \bigwedge_{i=1,\dots,9} \bigwedge_{j=1,\dots,9} \left(\bigvee_{k=1,\dots,9} \left(x_{ijk} \wedge \bigwedge_{\substack{k' = 1, \dots, 9 \\ k' \neq k}} \neg x_{ijk'} \right) \right)$$

b)

$$\varphi_{\text{row}}(i) = \bigwedge_{k=1,\dots,9} \bigwedge_{j=1,\dots,8} \left(\neg x_{ijk} \vee \bigwedge_{j'=j+1,\dots,9} \neg x_{ij'k} \right)$$

iii) Can the following conflict occur in DPLL? Why? (Note: A reason for the answer is neccessary to obtain any points for this subtask.)



Solution: No. After deciding $a = 0@0$, $c_1 = a \vee \neg b$ is unit, thus $b = 0@0$ is propagated.

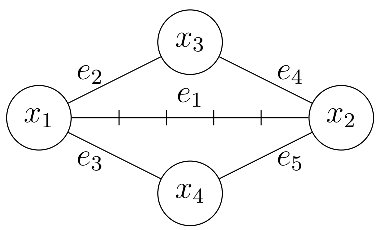
2.) Equality Logic and Uninterpreted Functions 6+7 Points

i) Consider the equality logic formula

$$\varphi^{EQ} \quad := \quad \neg(\underbrace{x_1 = x_2}_{e_1}) \quad \vee \quad \Big(\underbrace{x_1 = x_3}_{e_2} \quad \wedge \quad \underbrace{x_1 = x_4}_{e_3}\Big) \quad \vee \quad \Big(\underbrace{x_2 = x_3}_{e_4} \quad \wedge \quad \underbrace{x_2 = x_4}_{e_5}\Big)$$

Draw the E-graph **with polarity** and use it (without any modification) to transform φ^{EQ} to a satisfiability-equivalent propositional logic formula φ as presented in the lecture for **eager** SMT solving.

Solution:



$$\begin{aligned}
\varphi_{sk} &:= \neg e_1 \vee (e_2 \wedge e_3) \vee (e_4 \wedge e_5) \\
\varphi_{trans} &:= ((e_2 \wedge e_4) \to e_1) \wedge (e_3 \wedge e_5) \to e_1) \\
\varphi &:= \varphi_{sk} \wedge \varphi_{trans}
\end{aligned}$$

ii) Consider the following conjunction of constraints in equality logic with uninterpreted functions:

$$a = b \quad \wedge \quad f(f(a)) = a \quad \wedge \quad f(f(b)) \neq b$$

Check the satisfiability of the above conjunction using the sparse method from the lecture. Please describe the partition

- initially,
- after considering transitivity and
- after considering congruence,

and explain how we can read off the answer to the satisfiability question.

Solution:

Initial partition:

$$\boxed{a} \quad \boxed{b} \quad \boxed{f(f(a))} \quad \boxed{f(f(b))} \quad \boxed{f(a)} \quad \boxed{f(b)}$$

After considering transitivity:

$$\boxed{a \quad b \quad f(f(a))} \quad \boxed{f(f(b))} \quad \boxed{f(a)} \quad \boxed{f(b)}$$

After considering congruence:

$$\boxed{a \quad b \quad f(f(a)) \quad f(f(b))} \quad \boxed{f(a) \quad f(b)}$$

The input conjunction is **unsatisfiable**, because the two sides of the inequality $f(f(b) \neq b$ are in the same equivalence class.

3.) Fourier-Motzkin Variable Elimination **5+6+2+2 Points**

Consider the following set of linear real-arithmetic constraints:

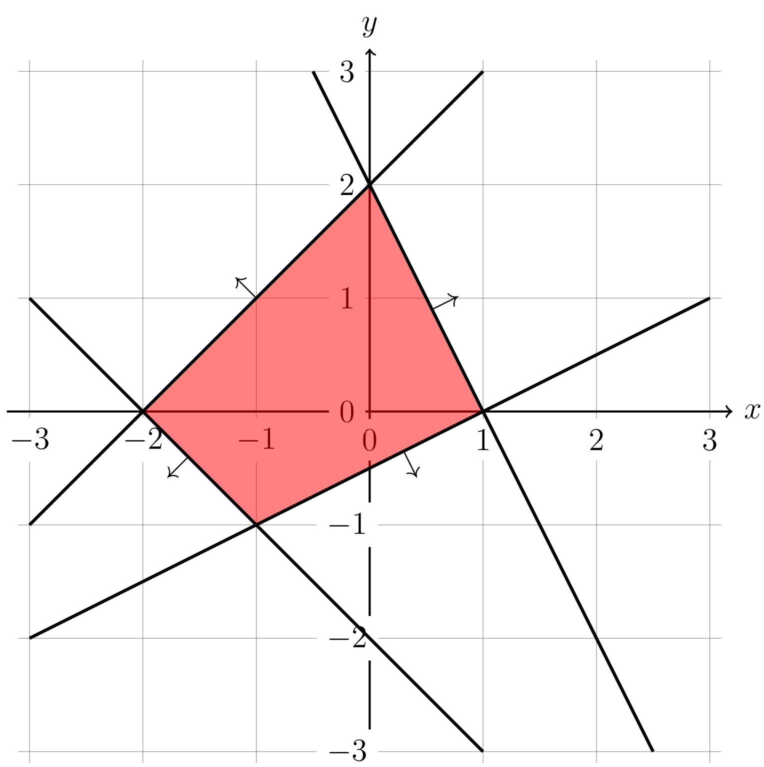
$$\begin{array}{rrcrcr} c_1: & 2x & + & y & \leq & 2 \\ c_2: & x & - & y & \geq & -2 \\ c_3: & x & - & 2y & \leq & 1 \\ c_4: & -x & - & y & \leq & 2 \end{array}$$

i) Sketch the **solution set** of this problem graphically in a coordinate system.

Solution: The following equalities describe the surfaces of the solution sets (halfspaces) of the constraints:

$$\begin{array}{rcrcr} y & = & -2x & + & 2 \\ y & = & x & + & 2 \\ y & = & {}^1\!/\!_2 x & - & {}^1\!/\!_2 \\ y & = & -x & - & 2 \end{array}$$

The common solution set of all constraints is the intersection of the halfspaces.



ii) **Eliminate** y from this constraint set by applying the Fourier-Motzkin variable elimination. Simplify the resulting constraints such that they define constant bounds on x.

Solution: We first identify the lower and the upper bounds on y:

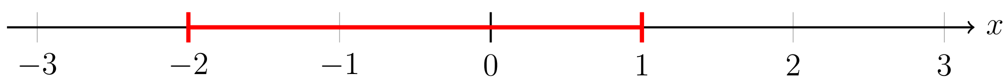
$$\begin{array}{lrcl} c_1: & & y & \leq -2x + 2 \\ c_2: & & y & \leq x + 2 \\ c_3: & {}^1\!/\!_2 x - {}^1\!/\!_2 \leq & y & \\ c_4: & -x - 2 \leq & y & \end{array}$$

We now state that each lower bound is less or equal each upper bound:

$$
\begin{array}{lrcl}
(c_3, c_1): & {}^1\!/\!_2 x - {}^1\!/\!_2 & \leq & -2x + 2 \\
\Rightarrow & x & \leq & 1 \\
\hline
(c_3, c_2): & {}^1\!/\!_2 x - {}^1\!/\!_2 & \leq & x + 2 \\
\Rightarrow & -5 & \leq & x \\
\hline
(c_4, c_1): & -x - 2 & \leq & -2x + 2 \\
\Rightarrow & x & \leq & 4 \\
\hline
(c_4, c_2): & -x - 2 & \leq & x + 2 \\
\Rightarrow & -2 & \leq & x
\end{array}
$$

iii) Sketch the **solution set** of the resulting constraints in x graphically.

Solution:



iv) What is the relation between the two solution sets, that you depicted in i) and iii)?

Solution: The one-dimensional set is the projection of the two-dimensional set to the x axis.

4.) Simplex **12+3 Points**

i) Apply one **pivoting step** and update the current assignment, where the variable order is $x_1 < x_2 < s_1 < s_2 < s_3$ and where the current tableau, the bounds and the current assignment are given by:

	s_1	s_2
x_1	$1/2$	$1/2$
x_2	$1/2$	$-1/2$
s_3	$1/2$	-1

$$\begin{array}{rcr} s_1 & \geq & 2 \\ s_2 & \leq & -2 \\ s_3 & \geq & 5 \end{array}$$

$$\begin{aligned} \alpha(x_1) &= 0 \\ \alpha(x_2) &= 2 \\ \alpha(s_1) &= 2 \\ \alpha(s_2) &= -2 \\ \alpha(s_3) &= 3 \end{aligned}$$

Solution: We pivot the third row and first column with

$$s_3 = 1/2 \cdot s_1 + -1 \cdot s_2 \Leftrightarrow s_1 = 2 \cdot s_3 + 2 \cdot s_2$$

and obtain

	s_3	s_2
x_1	1	$3/2$
x_2	1	$1/2$
s_1	2	2

$$\begin{array}{rcr} s_1 & \geq & 2 \\ s_2 & \leq & -2 \\ s_3 & \geq & 5 \end{array}$$

$$\begin{aligned} \alpha(x_1) &= 2 \\ \alpha(x_2) &= 4 \\ \alpha(s_1) &= 6 \\ \alpha(s_2) &= -2 \\ \alpha(s_3) &= 5 \end{aligned}$$

ii) The simplex method terminates on the resulting tableau. Is the tableau satisfied or conflicting? Why? Give the satisfying assignment for the original variables or the slack variables corresponding to the infeasible subset.

Solution: The tableau is satisfied. All auxiliary variables $s_1, \ldots s_4$ are within their bounds. The assignment is simply taken from α and we have $\alpha(x_1) = 2$ and $\alpha(x_2) = 4$.

5.) Integer Arithmetic **10 Points**

In the lecture, we presented a Branch and Bound algorithm in pseudocode, which used recursive calls to itself. The following pseudocode describes an alternative algorithm, which uses **iteration** instead of recursion. In the code,

- each value of type `ConstraintSet` stores a set of linear constraints over variables V,
- `n` is an integer,
- `A` is an infinite array of constraint sets and
- the method's return value as well as the variable `relaxed` are of type `Result`, storing either the value `UNSAT` or a variable assignment $f : V \to \mathbb{Q}$.

Please specify the **five missing details**, such that the recursive and the iterative algorithms always give the same result to each input. *Note: when branching, the recursive method processes the lower (i.e. upper-bounded) branch first.*

```
Result Branch&Bound(ConstraintSet P) {

  int n;
  ConstraintSet[] A;
  Result relaxed;

  n := 1; A[n] := P;
  while (n>0) do {
    P := A[n]; n := n-1;
    relaxed = LRA(Relaxed(P));
    if (          1          ){
      if (isInteger(relaxed))          2          ;
      let v be a variable such that relaxed(v) ∉ ℤ;
      n := n+1; A[n] :=          3          ;
      n := n+1; A[n] :=          4          ;
    }
  }
  return          5          ;
}
```

Solution:

```
Result Branch&Bound(ConstraintSet P) {

  int n;
  ConstraintSet[] A;
  Result relaxed;

  n := 1; A[n] := P;
  while (n>0) do {
    P := A[n]; n := n-1;
    relaxed = LRA(Relaxed(P));
    if (relaxed != UNSAT){
      if (isInteger(relaxed)) return relaxed;
      let v be a variable such that relaxed(v) ∉ ℤ;
```

```
        n := n+1; A[n] := P ∪ {v ≥ ⌈relaxed(v)⌉};
        n := n+1; A[n] := P ∪ {v ≤ ⌊relaxed(v)⌋};
      }
    }
    return UNSAT;
}
```

6.) Interval Constraint Propagation **4+9 Points**

i) Specify the formal rule for **squaring** an arbitrary interval as presented in the lecture, without using arithmetic operators for intervals.

Solution: If the interval is empty then $[\underline{A}, \overline{A}]^2 = [1; 0]$, otherwise

$$[\underline{A}, \overline{A}]^2 = ([\underline{A}, \overline{A}] \cdot [\underline{A}, \overline{A}]) \cap [0; \infty) = [\min(\underline{A} \cdot \underline{A}, \underline{A} \cdot \overline{A}, \overline{A} \cdot \overline{A}); \max(\underline{A} \cdot \underline{A}, \underline{A} \cdot \overline{A}, \overline{A} \cdot \overline{A})] \cap [0; \infty)$$

ii) As an advanced contraction method, the **interval Newton** method was presented in the lecture. Consider the polynomial $f(x) = {}^1\!/_3 \cdot x^3 - x^2 + 1$ and the starting interval $A = [1; 2]$ for x with $f'(A) = [-3; 2]$. Let $s = 1$ be a sample point from A. Perform one Newton contraction step.

Solution: $f(s) = {}^1\!/_3 \cdot 1^3 - 1^2 + 1 = {}^1\!/_3$. Compute Newton step:

$$s - \frac{f(s)}{f'(A)} = 1 - \frac{{}^1\!/_3}{[-3; 2]} = 1 - {}^1\!/_3 \cdot \frac{1}{[-3; 2]} = 1 - {}^1\!/_3 \cdot ((-\infty; -{}^1\!/_3] \cup [{}^1\!/_2; +\infty))$$

$$= 1 - ((-\infty; -{}^1\!/_9] \cup [{}^1\!/_6; +\infty)) = (-\infty; {}^5\!/_6] \cup [{}^{10}\!/_9; \infty)$$

And we get the new interval A' by

$$A' := A \cap \left(s - \frac{f(s)}{f'(A)}\right) = [1; 2] \cap ((-\infty; {}^5\!/_6] \cup [{}^{10}\!/_9; \infty)) = [{}^{10}\!/_9; 2]$$

7.) Subtropical Satisfiability **2+4+4 Points**

i) Please specify the **frame** *frame*(p) of the polynomial

$$p(a,b,c) = -5a^4 + 3ab^4 - abc - 3 \ .$$

Solution:

$\mathit{frame}(p) = \{(4,0,0),(1,4,0),(1,1,1),(0,0,0)\}$

ii) Specify a formula using the variables n_x, n_y, n_u and b that encodes the **separating hyperplanes** for the only positive frame element of

$$p(x,y,u) = -x^5 + 2x^2y^4 - 5u - xyu \ .$$

Solution:

The separating hyperplanes are encoded by

$$2n_x + 4n_y > b \quad \wedge \quad 5n_x < b \quad \wedge \quad n_u < b \quad \wedge \quad n_x + n_y + n_u < b \ .$$

iii) Specify conditions on the coefficients to describe the exact set of all *univariate* polynomials $p \in \mathbb{Z}[x]$, for which the subtropical satisfiability method as presented in the lecture *cannot* find a solution for $p(x) > 0$.

Solution:

The constraint $p(x) > 0$ can be solved by the method iff there exists $v \in \mathit{frame}^+(p) \subseteq \mathbb{N}$ such that

$$n_x \cdot v > b \wedge \bigwedge_{u \in \mathit{frame}(p) \setminus \{v\}} n_x \cdot u < b \ .$$

Thus the method finds *no* solution for $p(x) > 0$ iff one of the following two conditions hold:

- The polynomial $p(x)$ has no positive frame point, e.g. it has only negative coefficients. Example: $p(x) = -x$.
- There are positive frame points but these are not vertices of the Newton polytope. For the one-dimensional case it means that the monomials with the largest and smallest exponents need both to have a negative coefficient. Example: $p(x) = -x^3 + x^2 - x$.

8.) Virtual Substitution **(6+4)+2 Points**

i) Assume we want to virtually substitute the test candidate $t = y^2 + \epsilon$ **for the variable** $\boldsymbol{x}$ in the constraint $x \cdot y - y^3 + 2 \cdot x \cdot y^2 \leq 0$ using one of the following rules:

- Substitute $e + \varepsilon$ for x in $b \cdot x + c \leq 0$:

$$\begin{array}{lllll} & (& (bx + c < 0)[e//x] & &) \\ \vee & (& (bx + c = 0)[e//x] & \wedge \ \ b < 0 &) \\ \vee & (& b = 0 & \wedge \ \ c = 0 &) \end{array}$$

- Substitute $e + \varepsilon$ for x in $a \cdot x^2 + b \cdot x + c \leq 0$:

$$\begin{array}{lllllll} & (& (ax^2 + bx + c < 0)[e//x] & & & &) \\ \vee & (& (ax^2 + bx + c = 0)[e//x] & \wedge & (2ax + b < 0)[e//x] & &) \\ \vee & (& (ax^2 + bx + c = 0)[e//x] & \wedge & (2ax + b = 0)[e//x] & \wedge \ \ 2a < 0 &) \\ \vee & (& a = 0 & \wedge & b = 0 & \wedge \ \ c = 0 & \end{array}$$

a) Identify the appropriate substitution rule from above and apply virtual substitution. Normalize all polynomials by transforming them to sums of products.

b) Further simplify the result as far as possible. Use the result to state whether the formula is satisfiable.

Solution:

a) We need to compute $(b \cdot x + c \leq 0)[e + \epsilon//x]$ with $e = y^2$, $b = (y + 2 \cdot y^2)$ and $c = -y^3$. I.e., we need to instantiate the first rule.

Note: the last rule is also applicable with $a = 0$ (actually it is then the same).

Substitute $e + \varepsilon$ into $b \cdot x + c \leq 0$:

$$\begin{array}{lllll} & (& (bx + c < 0)[e//x] & &) \\ \vee & (& (bx + c = 0)[e//x] & \wedge \ \ b < 0 &) \\ \vee & (& b = 0 & \wedge \ \ c = 0 &) \end{array}$$

$$\begin{array}{lllll} & (& ((y + 2 \cdot y^2) \cdot x - y^3 < 0)[y^2//x] & &) \\ \vee & (& ((y + 2 \cdot y^2) \cdot x - y^3 = 0)[y^2//x] & \wedge \ \ y + 2 \cdot y^2 < 0 &) \\ \vee & (& (y + 2 \cdot y^2) = 0 & \wedge \ \ -y^3 = 0 &) \end{array}$$

$$\begin{array}{lllll} & (& ((y + 2 \cdot y^2) \cdot y^2 - y^3 < 0) & &) \\ \vee & (& ((y + 2 \cdot y^2) \cdot y^2 - y^3 = 0) & \wedge \ \ y + 2 \cdot y^2 < 0 &) \\ \vee & (& (y + 2 \cdot y^2) = 0 & \wedge \ \ -y^3 = 0 &) \end{array}$$

Normalization:

$$\begin{array}{lllll} & (& (y^3 + 2 \cdot y^4 - y^3 < 0) & &) \\ \vee & (& (y^3 + 2 \cdot y^4 - y^3 = 0) & \wedge \ \ y + 2 \cdot y^2 < 0 &) \\ \vee & (& (y + 2 \cdot y^2) = 0 & \wedge \ \ -y^3 = 0 &) \end{array}$$

$$\begin{array}{lllll} & (& (2 \cdot y^4 < 0) & &) \\ \vee & (& (2 \cdot y^4 = 0) & \wedge \ \ (y + 2 \cdot y^2 < 0) &) \\ \vee & (& (y + 2 \cdot y^2) = 0 & \wedge \ \ -y^3 = 0 &) \end{array}$$

b) Simplification:

$$\begin{array}{lcccl} & (& false & &) \\ \vee & (& y = 0 & \wedge \ (y + 2 \cdot y^2 < 0) &) \\ \vee & (& (y + 2 \cdot y^2) = 0 & \wedge \ y = 0 &) \end{array}$$

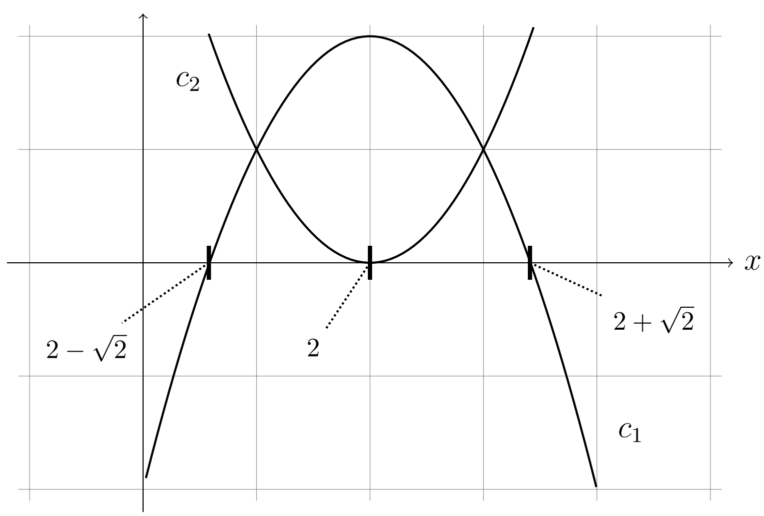
$$\begin{array}{lcccl} & (& false & &) \\ \vee & (& false & &) \\ \vee & (& (y + 2 \cdot y^2) = 0 & \wedge \ y = 0 &) \end{array}$$

$$\begin{array}{lccl} & (& false &) \\ \vee & (& false &) \\ \vee & (& \quad y = 0 &) \end{array}$$

Yes, the formula is satisfiable.

ii) Consider the following constraints and their graphical depiction below. Give the test candidates for x resulting from these constraints with **true** side conditions as symbolic expressions.

$$c_1 : -(x-2)^2 + 2 > 0 \qquad c_2 : x^2 - 4x + 4 \leq 0$$



Solution:

- $-\infty$
- $2 - \sqrt{2} + \varepsilon$
- $2 + \sqrt{2} + \varepsilon$
- 2

9.) Cylindrical Algebraic Decomposition **5+7 Points**

Consider the polynomial $p = x^3 + 2 \cdot x^2 - 3 \cdot x$.

i) Use the Cauchy bound $C = 4$ and the Sturm sequence of p:

$$\begin{aligned} p_0 &= x^3 + 2 \cdot x^2 - 3 \cdot x \\ p_1 &= 3 \cdot x^2 + 4 \cdot x - 3 \\ p_2 &= {}^{26}\!/\!_{9} \cdot x - {}^{2}\!/\!_{3} \\ p_3 &= {}^{324}\!/\!_{169} \end{aligned}$$

to determine the number of real roots of p as presented in the lecture.

Solution: All real roots of p are in the interval $[-4; 4]$ according to the lecture. Since $p(-4) \neq 0$ all real roots are in the interval $(-4; 4]$.
We use the Sturm sequence to compute the number of real roots in $(-4; 4]$:

Sturm sequence	values at		
	-4	0	4
$p_0 = x^3 + 2 \cdot x^2 - 3 \cdot x$	-	0	$+$
$p_1 = 3 \cdot x^2 + 4 \cdot x - 3$	$+$	-	$+$
$p_2 = {}^{26}\!/\!_{9} \cdot x - {}^{2}\!/\!_{3}$	-	-	$+$
$p_3 = {}^{324}\!/\!_{169}$	$+$	$+$	$+$
# sign changes $\sigma(\cdot)$	3	1	0

There are $\sigma(-4) - \sigma(4) = 3 - 0 = 3$ real roots in $(-4; 4]$.

ii) Isolate all real roots of p with the method presented in the lecture, using interval midpoints for splitting. Give the resulting interval representations of the real roots.

Solution: In the interval $(-4; 4)$ there are three real roots of p (note that $p(4) \neq 0$). To isolate the real roots, we split the interval at its midpoint 0 into $(-4; 0)$, $[0; 0]$ and $(0; 4)$.
Number of real roots of p in $(-4; 0)$: 1; in $[0; 0]$: 1 ; in $(0; 4)$: 1
In every interval there is at most one real root, therefore, all real roots are isolated. The interval representations of the real roots are $(p, (-4; 0))$, $(p, [0; 0])$ and $(p, (0; 4))$.