Satisfiability Checking - WS 2022/2023

Written Exam II

Monday, March 30, 2023

Sample solution

1.) SAT Checking **7 + 5 + 5 points**

i) Assume the following propositional logic formula in CNF:

$$c_1 : (\neg x_1 \vee x_2) \quad \wedge \quad c_2 : (\neg x_2 \vee \neg y_1 \vee y_2) \quad \wedge \quad c_3 : (\neg y_2 \vee y_3) \quad \wedge$$
$$c_4 : (\neg y_3 \vee y_4) \quad \wedge \quad c_5 : (\neg y_3 \vee \neg y_4)$$

Assume furthermore the following trail:

$$\begin{array}{ll} DL0: & - \\ DL1: & x_1 : nil, \quad x_2 : c_1 \\ DL2: & y_1 : nil, \quad y_2 : c_2, \quad y_3 : c_3, \quad y_4 : c_4 \end{array}$$

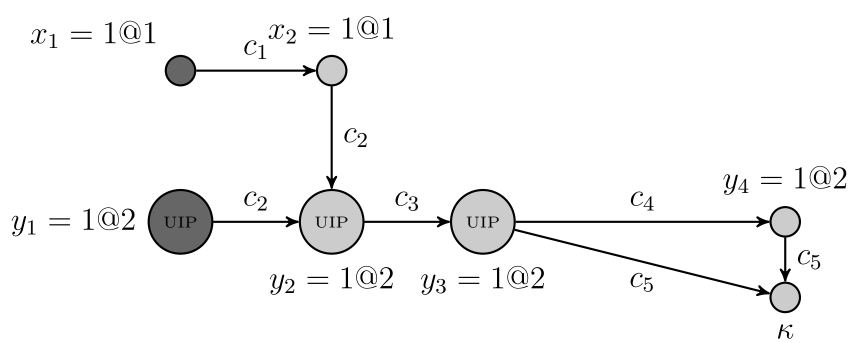
We detect a conflicting clause c_5.

a) Please draw the implication graph and mark all unique implication points in it with the label "UIP".

b) Apply conflict resolution to c_5 till the first unique implication point as presented in the lecture. Please specify all resolution steps and their results.

Solution:

- There are three unique implication points, which are the nodes for y_1, y_2 and y_3 in the following illustration:



- The conflicting clause is c_5, its most recently assigned literal is y_4 with antecedent c_4. We resolve c_5 with c_4 wrt. y_4, yielding $c_6 : (\neg y_3)$. This clause is already asserting.

ii) Please apply Tseitin's encoding to the following propositional logic formulas. Please specify

I. the intermediate formula that encodes the meaning of sub-formulas using auxiliary variables $h_1, h_2, \ldots$, and

II. the result after transforming the intermediate formula into CNF.

a) $\neg(a \vee b)$

b) $(a \to b) \wedge c$

Solution:

- I. $(h_1 \leftrightarrow (\neg h_2)) \wedge (h_2 \leftrightarrow (a \vee b)) \wedge h_1$
  II. $(\neg h_1 \vee \neg h_2) \wedge (h_2 \vee h_1) \wedge (\neg h_2 \vee a \vee b) \wedge (\neg a \vee h_2) \wedge (\neg b \vee h_2) \wedge h_1$
- I. $(h_1 \leftrightarrow (h_2 \wedge c)) \wedge (h_2 \leftrightarrow (a \rightarrow b)) \wedge h_1$
  II. $(\neg h_1 \vee h_2) \wedge (\neg h_1 \vee c) \wedge (\neg h_2 \vee \neg c \vee h_1) \wedge (\neg h_2 \vee \neg a \vee b) \wedge (a \vee h_2) \wedge (\neg b \vee h_2) \wedge h_1$

iii) In the termination proof of the DPLL+CDCL algorithm, we used a (partial) ordering on partial assignments, which decreases during the execution. Order the following assignments α_1, α_2, α_3, α_4 accordingly from *largest to smallest (descending).*

	α_1	α_2	α_3	α_4
DL0	a	a, b	a	a
DL1	$\neg b$	c, d	$\neg b, c$	$\neg b$
DL2	$\neg c, d$	-	$\neg d$	$\neg c$

Solution:

$$\alpha_4 > \alpha_1 > \alpha_3 > \alpha_2$$

Explanation: α_1 is an extension of α_4. α_3 disagrees with α_1 on DL1 and has more assignments on that DL. α_2 disagrees with all other assignments on b and has the most assignments on DL0.

2.) Equality Logic and Uninterpreted Functions 4 + 4 + 6 points

i) Apply *lazy* SMT solving for equality logic and uninterpreted functions as presented in the lecture to the following conjunction of equalities and disequalities, considering equations from left to right.

$$x = y \wedge u = F(x) \wedge F(F(u)) = y \wedge u \neq y$$

Please specify the initial partition, each execution step and the partition after the step, even if there is no change.

Solution: Initial partition: $\{\{x\}, \{y\}, \{u\}, \{F(x)\}, \{F(u)\}, \{F(F(u))\}\}$

Transitivity:
After merging for $x = y$: $\{\{x, y\}, \{u\}, \{F(x)\}, \{F(u)\}, \{F(F(u))\}\}$
After merging for $u = F(x)$: $\{\{x, y\}, \{u, F(x)\}, \{F(u)\}, \{F(F(u))\}\}$
After merging for $F(F(u)) = y$: $\{\{x, y, F(F(u))\}, \{u, F(x)\}, \{F(u)\}\}$

Congruence:
No congruence merging.

ii) Let φ be an arbitrary formula in equality logic with uninterpreted functions so that the algorithm from i) produces the final partition

$$\{\{a, b, F(a)\}, \{c, d\}, \{F(c)\}\}.$$

Give a disequation e only using a, b, c, d and/or F so that the algorithm from i) produces a different number of final equivalence classes for the inputs φ and $\varphi \wedge e$.

Solution:

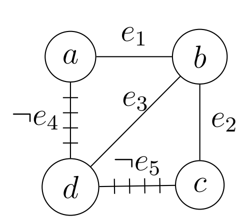
Any disequation containing $F(F(c))$ or $F(F(d))$ (or $F(F(F(...F(c)...)))$ or $F(F(F(...F(d)...))))$ works. E.g. $a \neq F(F(c))$.

iii) Consider the following formula in *equality logic*:

$$\varphi^{EQ} : (a = b \ \vee \ b = c \ \vee \ b = d) \ \wedge \ (a \neq d \ \vee \ c \neq d)$$

Draw the corresponding E-graph *with polarity* and use it to transform φ^{EQ} into a satisfiability-equivalent propositional logic formula as presented in the lecture for eager SMT-solving.

Solution:



$$\varphi = \varphi_{sk} \wedge \varphi_{trans} \ \textit{with} \quad \varphi_{sk} : \ (e_1 \vee e_2 \vee e_3) \wedge (\neg e_4 \vee \neg e_5) \quad \textit{and}$$
$$\varphi_{trans} : \ (e_1 \wedge e_3 \rightarrow e_4) \wedge (e_2 \wedge e_3 \rightarrow e_5)$$

3.) Fourier-Motzkin Variable Elimination **2 + 3 points**

i) For any integer $a \in \mathbb{Z}$, consider the following set of linear real arithmetic constraints:

$$S_a = \{a \cdot x + y \leq 2, \quad -2x - y \leq 0, \quad (a+1) \cdot x \leq 2, \quad -x - a \cdot y \leq -1\}$$

a) Assume $a = 1$. How many constraints does the *Fourier-Motzkin method* compute when eliminating only x from S_1 (including duplicates and trivial constraints)?

b) Give a value of $a \in \mathbb{Z}$ for which the number of constraints computed by the Fourier-Motzkin method when eliminating only x from S_a is *minimal*.

Solution:

a) 4

b) All values $a \leq -1$

4.) Simplex

4+2+4+4 points

i) Apply the simplex method to the following constraint set until termination:

$$\begin{array}{ll} s_0 = -1x_0 + 2x_1 & s_0 \leq -2 \\ s_1 = -1x_0 - 2x_1 & s_1 \leq 2 \end{array}$$

Please specify the simplex tableau and the assignment initially and after each pivot step. When choosing pivot variables, use the order $x_0 \prec x_1 \prec s_0 \prec s_1$ and take the smallest possible variable.

Solution: Initially the tableau is as follows:

	$x_0[0]$	$x_1[0]$
$s_0[0]$	-1	2
$s_1[0]$	-1	-2

We pivot s_0 with x_1, yielding the tableau:

	$s_0[-2]$	$x_1[0]$
$x_0\ [\ \ 2]$	-1	2
$s_1\ [-2]$	1	-4

All non-basic variables satisfy their bounds, thus simplex terminates with reporting satisfiability.

ii) Consider the following tableau and the bounds on the slack variables. Compute the corresponding assignment α for all variables.

	s_2	s_1
x_1	-2	1
x_2	2	1

$$\begin{array}{lcl} s_1 & \geq & 2 \\ s_2 & \geq & 1 \end{array}$$

$$\begin{array}{lcl} \alpha(s_1) & = & \boxed{2} \\ \alpha(s_2) & = & \boxed{1} \\ \alpha(x_1) & = & \boxed{0} \\ \alpha(x_2) & = & \boxed{4} \end{array}$$

Solution: see above

iii) Consider the following tableau (the current values of the variables are given in square brackets).

	$s_1\ [-1]$	$x_2\ [0]$	$s_2\ [1]$
$x_1\ [2]$	-1	1	1
$s_3\ [0]$	1	1	1
$s_4\ [0]$	-1	-1	-1
$x_3\ [-2]$	1	-1	-1

$$\begin{array}{lcl} s_1 & \leq & -1 \\ s_2 & \geq & 1 \\ s_3 & \leq & -1 \\ s_4 & \geq & 1 \end{array}$$

a) Give all suitable pairs suitable for pivoting.

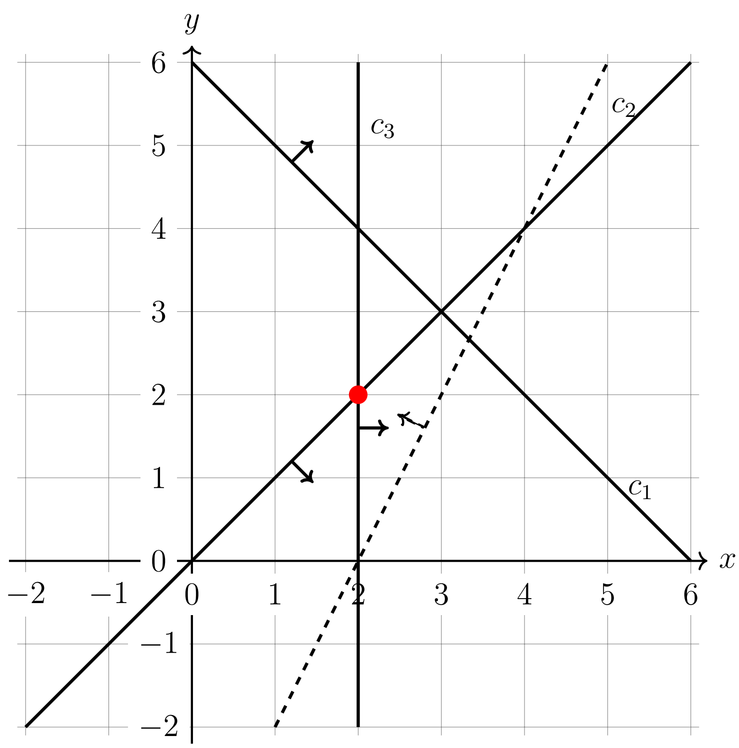
b) Which one would be chosen by Bland's rule? Assume the variable ordering $x_1 \prec x_2 \prec s_1 \prec s_2 \prec s_3 \prec s_4$ preferring the smallest variable for Bland's rule.

Solution:

a) $(s_3, x_2), (s_3, s_1), (s_4, x_2), (s_4, s_1)$

b) (s_3, x_2)

iv) Consider the following system of linear inequations $\{c_1, \ldots, c_3\}$. The point marked at at $(2, 2)$ corresponds to a state of the simplex algorithm on the system.



Add a linear inequality to the above coordinate system (i.e. draw a hyperplane and its normal vector) such that c_2 and c_3 are part of a conflict (i.e. the Simplex algorithm would terminate with an infeasible subset containing c_2 and c_3).

Solution: See the dashed line above. Any constraint with hyperplane crossing c_3 below $y = 2$ and crossing c_2 right of $x = 2$, and normal vector in direction to north-west.

5.) Linear Integer Arithmetic **5 points**

Let P be a set of linear integer arithmetic constraints containing only the variables x and y. Below, the initial call and *all* recursive calls of the *Branch & Bound* algorithm for the input P are given. Complete the missing information for each call under the assumption that the final result of the algorithm is *unsatisfiable*. In particular,

- complete the specification of the input to the simplex method,
- specify the answer of simplex for the relaxed version of each input by writing SAT (satisfiable) or UNSAT (unsatisfiable), and
- give a possible variable assignment only if simplex would find one and otherwise leave the gaps for the assignment empty.

Input: P
Answer of simplex: $\boxed{\text{SAT}}$ Assignment: $x = 0$, $y = \boxed{0.5}$

Input: $P \cup \{y \geq 1\}$
Answer of simplex: $\boxed{\text{UNSAT}}$ Assignment: $x = \boxed{-}$, $y = \boxed{-}$

Input: $P' := P \cup \{\boxed{y \leq 0}\}$
Answer of simplex: $\boxed{\text{SAT}}$ Assignment: $x = 1/2$, $y = 0$

Input: $P' \cup \{\boxed{x \geq 1}\}$
Answer of simplex: $\boxed{\text{UNSAT}}$ Assignment: $x = \boxed{-}$, $y = \boxed{-}$

Input: $P' \cup \{\boxed{x \leq 0}\}$
Answer of simplex: $\boxed{\text{UNSAT}}$ Assignment: $x = \boxed{-}$, $y = \boxed{-}$

Solution: See above.

6.) Interval Constraint Propagation **4 + 4 + 7 + 3 points**

i) Please compute the result of the interval division $\frac{[-11;-4]}{[-1;2]}$.

Solution: The result is $(-\infty;-2] \cup [4;+\infty)$.

ii) Assume $x \in [0;16]$ and $y \in [2;3]$. Please contract the domain for x using $x = 2y^2 - 3y + 4$ with the help of the contraction method I from the lecture.

Solution:

$$\begin{aligned} [2;2] \cdot [2;3]^2 - [3;3] \cdot [2;3] + [4;4] &= [2;2] \cdot [4,9] - [6;9] + [4;4] \\ &= [8;18] - [6;9] + [4;4] \\ &= [-1;12] + [4;4] \\ &= [3;16] \end{aligned}$$

Intersecting the previous interval domain of x with this new interval yields: $[0;16] \cap [3;16] = [3;16]$.

iii) Apply the necessary preprocessing for the ICP contraction method I to the following constraints:

$$c_1 : xyz + x^2y + 2z < 0 \quad \wedge \quad c_2 : x - 2y = 0$$

Solution:

$$h_1 = xyz \wedge h_2 = x^2y \wedge h_1 + h_2 + 2z < 0 \wedge x - 2y = 0$$

iv) Give a value $a \geq 0$ so that the relative contraction of the ICP contraction method II applied to the constraint $x^2 + a = 0$ with $x \in A = [0;2]$ and $s = 1$ is *minimal*. Specify your intermediate computations and argue why your value for a is correct.

Solution:

The formula for the realtive contraction is $gain_{rel} = 1 - \frac{D(A')}{D(A)}$, which is mininmal if $D(A')$ is maximal. The result of the contraction is $A' = A \cap \left(s - \frac{f(s)}{f'(A)}\right) = A \cap \left(1 - \frac{1+a}{[0;4]}\right) = A \cap [-\infty;(3-a)/4] = [0;(3-a)/4]$. This interval is the largest for $a = 0$ and thus its relative contraction is minimal.

7.) Subtropical Satisfiability **4+4+3+4 points**

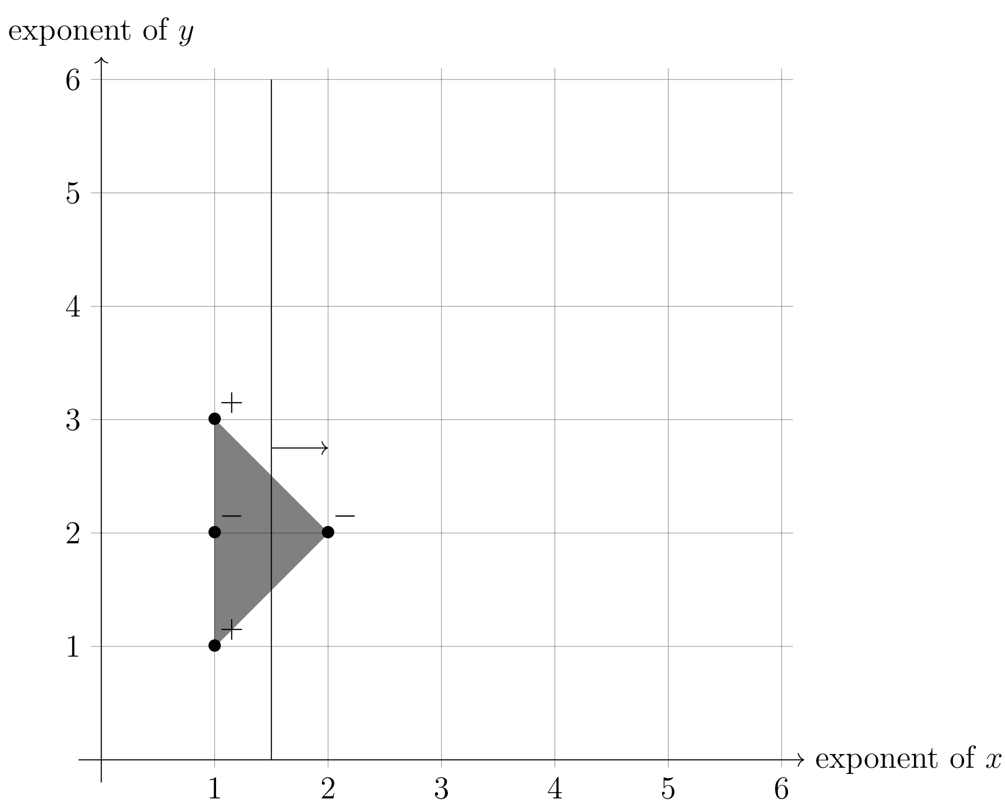
i) Construct a real solution for the constraint $-2x^2 + 4y^2 - 8 = 0$ on the line segment between $(1, 1)$ (where the polynomial has a negative sign) and $(3, 3)$ (where the polynomial has a positive sign). Please give the computations and the resulting values for x and y.

Solution:

- $x = 1 + t(3 - 1) = 2t + 1$
- $y = 1 + t(3 - 1) = 2t + 1$
- $-2x^2 + 4y^2 - 8 = -2(2t + 1)^2 + 4(2t + 1)^2 - 8 = 2(2t + 1)^2 - 8 = 0$
- $(2t + 1)^2 = 4$
- $2t + 1 = \pm 2$
- $t \in [0, 1] \rightarrow t = \frac{1}{2}$
- $x = 1 + t(3 - 1) = 2$, $y = 1 + t(3 - 1) = 2$

ii) Draw the Newton polytope for the polynomial $p(x, y) = 5xy - 2xy^2 + xy^3 - 3x^2y^2$. Label all frame points accordingly.
Is the subtropical method as presented in the lecture suitable to determine the satisfiability of the constraint $p(x, y) = 0$? If yes, draw a *suitable* separating hyperplane and its normal vector that could be generated during the process. If not, mark the frame points which need to be removed to make the method applicable.

Solution:



As $p(1,1) > 0$, we need to separate a negative frame point.

iii) Assume the subtropical method computes a separating hyperplane with normal vector $n = (-2, 3)$ for some polynomial $p(x, y)$. The method can construct a satisfying assignment (α_x, α_y) for $p(x, y) > 0$ using n. We assume that we initialize the value which is increased during that process with 2.

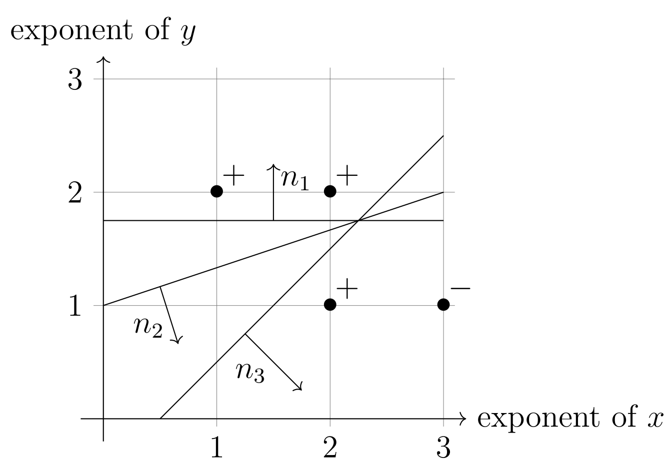
Give inclusion-minimal intervals I_x, I_y in which α_x and α_y must be contained for any polynomial $p \in \mathbb{Z}[x, y]$.

$I_x = \boxed{(0, 0.25]}$

$I_y = \boxed{[8, \infty)}$

Solution: see above

iv) The following image depicts the frame points of some polynomial $p(x, y)$ and three hyperplanes h_1, h_2 and h_3 with their normal vectors.



a) None of these normal vectors satisfies the sufficient condition from the subtropical method as given in the lecture. However, with one of them we can construct a solution for the constraint $p(x, y) > 0$ with the same method from the lecture. Which is it?

b) Based on this observation, adapt the encoding of a suitable normal vector n for a given polynomial p by filling in the two gaps in the following formula:

$$\bigvee_{v_+ \in \boxed{frame_+(p)}} \left(nv_+^T > b \wedge \bigwedge_{v_- \in \boxed{frame_-(p)}} (nv_-^T < b) \right)$$

Solution: a) n_1

8.) Virtual Substitution **4+5+5 points**

i) Please specify the constraint c such that the result of the virtual substitution $c[-\infty//x]$ is

$$-2 = 0 \wedge 0 = 0 \wedge 4z = 0\ .$$

Solution: $-2x^2 + 4z = 0$

ii) Give all test candidates for x with their side conditions that we get from the following constraint:

$$2xz^2 + x^2y + yz < 0$$

Simplify all expressions as far as possible by multiplying out all brackets.

Write your results in the following table. There are more rows than necessary.

Solution: $(y)x^2 + (2z^2)x + yz < 0$

Zeros of $ax^2 + bx + c$:

Real root	Side condition
$\xi_0 = -\frac{c}{b}$	, if $a = 0 \wedge b \neq 0$
$\xi_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}$	, if $a \neq 0 \wedge b^2 - 4ac \geq 0$
$\xi_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$	, if $a \neq 0 \wedge b^2 - 4ac > 0$

... with $a = y, b = 2z^2, c = yz$:

Real root	Side condition
$\xi_0 = -\frac{yz}{2z^2}$	, if $y = 0 \wedge 2z^2 \neq 0$
$\xi_1 = \frac{-z^2+\sqrt{z^4-y^2z}}{y}$	, if $y \neq 0 \wedge z^4 - y^2z \geq 0$
$\xi_2 = \frac{-z^2-\sqrt{z^4-y^2z}}{y}$	, if $y \neq 0 \wedge z^4 - y^2z > 0$

Resulting test candidates:

Test candidate	Side condition
$-\infty$	, if true
$-\frac{yz}{2z^2} + \varepsilon$	, if $y = 0 \wedge 2z^2 \neq 0$
$\frac{-z^2+\sqrt{z^4-y^2z}}{y} + \varepsilon$	, if $y \neq 0 \wedge z^4 - y^2z \geq 0$
$\frac{-z^2-\sqrt{z^4-y^2z}}{y} + \varepsilon$	, if $y \neq 0 \wedge z^4 - y^2z > 0$

iii) Let $tcs(\varphi, x)$ be the set of test candidates of φ for the variable in x and $sc(t)$ denote the side condition of a test candidate $t \in tcs(\varphi, x)$. Then by the virtual substitution method it holds

$$\exists x.\varphi \quad \leftrightarrow \quad \bigvee_{t \in tcs(\varphi,x)} (\varphi[t//x] \wedge sc(t))$$

I.e. we can eliminate existentially quantified variables this way.

Virtual substitution can also be applied to eliminate universally quantified variables. Give an analogous statement for $\forall x.\varphi$. Simplify the formula as far as possible.

Solution:

$$
\begin{aligned}
&\forall x.\varphi \equiv \neg\exists x.\neg\varphi \\
&\equiv \neg\left(\bigvee_{t \in tcs(\neg\varphi, x)} ((\neg\varphi)[t//x] \wedge sc(t))\right) \equiv \bigwedge_{t \in tcs(\neg\varphi, x)} (\neg(\neg\varphi)[t//x] \vee \neg sc(t)) \\
&\equiv \bigwedge_{t \in tcs(\varphi, x)} (\varphi[t//x] \vee \neg sc(t)) \equiv \bigwedge_{t \in tcs(\varphi, x)} (sc(t) \rightarrow \varphi[t//x])
\end{aligned}
$$

9.) Cylindrical Algebraic Decomposition **4+4+2+8 points**

i) What is the Cauchy bound for the polynomial $p = 2x^3 - 10x^2 - 10x - 8$? Please show your computations.

Solution: 6

The leading coefficient is $a_3 = 2$. The other coefficients are $a_2 = -10$, $a_1 = -10$ and $a_0 = -8$. Thus the Cauchy bound of p is

$$C = 1 + max\{\frac{|-10|}{|2|}, \frac{|-10|}{|2|}, \frac{|-8|}{|2|}\} = 6 \ .$$

ii) Assume the polynomial $p = x^3 - 18x^2 - 6x + 12$ and its Sturm sequence

$$\begin{aligned} p_0 &= x^3 - 18x^2 - 6x + 12 \\ p_1 &= 3x^2 - 36x - 6 \\ p_2 &= 76x \\ p_3 &= 6 \end{aligned}$$

Use the Sturm sequence to compute the number of real roots of p contained in the interval $(-1; 1]$. Also give your intermediate calculations.

Solution: 2

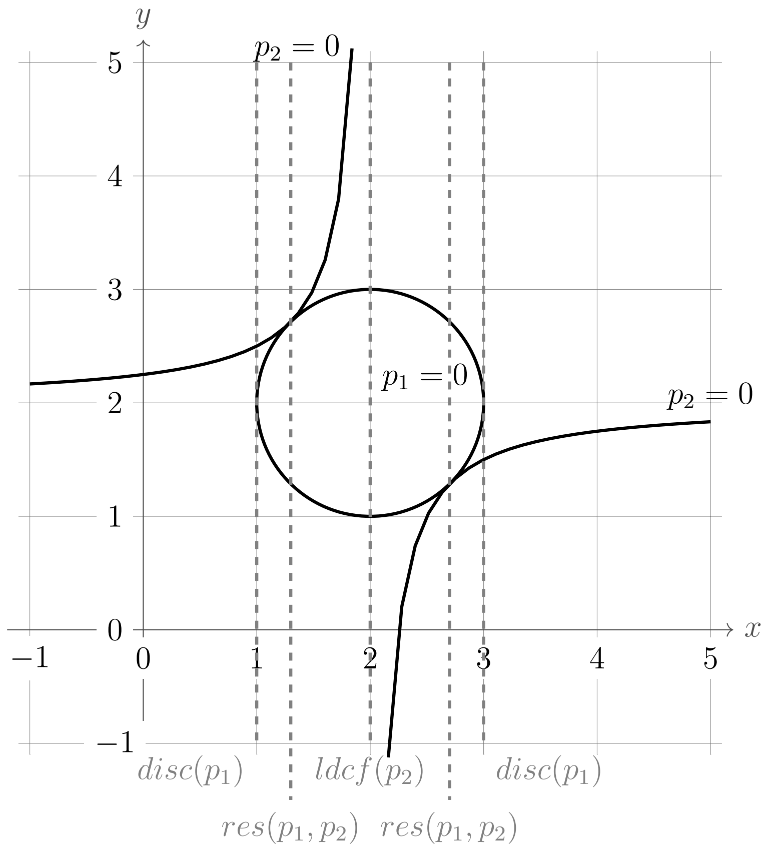
Sturm sequence	*value at* -1	*value at* 1
p_0	-1	-11
p_1	33	-39
p_2	-76	76
p_3	6	6
number of sign changes $\sigma(\cdot)$	3	1

Thus p has $3 - 1 = 2$ real roots in the interval $(-1; 1]$.

iii) During the lifting phase of the CAD, on each level, we select a sample at every root, a sample below all roots, a sample above all roots, and a sample between all pairs of neighbouring roots. In the cases where we can choose a sample,which property assures that we can choose *any* such point for lifting? Please *name* the property. (No explanation needed.)

Solution: Delineability

iv) Consider the following varieties of some polynomials p_1 and p_2:



Note that p_2 has a singularity at $x = 2$.

Draw the cylinder boundaries of the CAD computed for the input p_1, p_2 with the CAD method as presented in the lecture. Label each boundary with the corresponding projection polynomial that induced this boundary.

Write $res(p_a, p_b)$ for the resultant of polynomials p_a and p_b, $disc(p)$ for the discriminant of p and $ldcf(p)$ for the leading coefficient of p.

Solution: see above