Satisfiability Checking - WS 2023/2024

teaching@ths.rwth-aachen.de
https://ths.rwth-aachen.de/teaching/

Series 9

Exercise 1

Consider the first-order logical formula over the integers with addition:

$$\varphi^{LIA} \quad := \quad 2 \cdot x_1 \geq 1 \wedge 2 \cdot x_2 \geq 1 \wedge -2 \cdot x_1 - x_2 \geq -3$$

Show how a theory solver that employs a Simplex algorithm with branch and bound solves this formula.

For pivoting, choose the smallest variable and branch on the smallest variable x_k according to the following variable order:

$$s_1 < \cdots < s_7 < x_1 < x_2$$

Branch for $x_k \leq i$ first and for $x_k \geq i + 1$ afterwards.

Solution:
First build the initial tableau:

	x_1	x_2
s_1	2	0
s_2	0	2
s_3	-2	-1

$$\begin{array}{lcr} s_1 & \geq & 1 \\ s_2 & \geq & 1 \\ s_3 & \geq & -3 \end{array}$$

The initial assignment for all variables is

$$\alpha(s_1) = 0, \alpha(s_2) = 0, \alpha(s_3) = 0, \alpha(x_1) = 0, \alpha(x_2) = 0$$

We apply the Simplex method. The first variable violating its bound is s_1, only x_1 is suitable to fix this. We solve the equation for the row of s_1:

$$s_1 = 2 \cdot x_1 \Leftrightarrow x_1 = 0.5 \cdot s_1$$

We substitute and update the tableau and the assignments:

	s_1	x_2
x_1	0.5	0
s_2	0	2
s_3	-1	-1

$$\begin{array}{lcr} s_1 & \geq & 1 \\ s_2 & \geq & 1 \\ s_3 & \geq & -3 \end{array}$$

$$\alpha(s_1) = 1, \alpha(s_2) = 0, \alpha(s_3) = -1, \alpha(x_1) = 0.5, \alpha(x_2) = 0$$

The first variable violating its bound is s_2, only x_2 is suitable to fix this. We solve the equation for the row of s_2:

$$s_2 = 2 \cdot x_2 \Leftrightarrow x_2 = 0.5 \cdot s_2$$

We substitute and update the tableau and the assignments:

	s_1	s_2
x_1	0.5	0
x_2	0	0.5
s_3	-1	-0.5

$$\begin{array}{lcr} s_1 & \geq & 1 \\ s_2 & \geq & 1 \\ s_3 & \geq & -3 \end{array}$$

$$\alpha(s_1) = 1, \alpha(s_2) = 1, \alpha(s_3) = -1.5, \alpha(x_1) = 0.5, \alpha(x_2) = 0.5$$

Now, all variables satisfy their bounds and we obtain $x_1 = 0.5, x_2 = 0.5$ as first real assignment. We select x_1 for branching and try $x_1 \leq 0$ first. We add a new constraint with a new slack variable s_4.

	s_1	s_2
x_1	0.5	0
x_2	0	0.5
s_3	-1	-0.5
s_4	0.5	0

$$\begin{array}{lcl} s_1 & \geq & 1 \\ s_2 & \geq & 1 \\ s_3 & \geq & -3 \\ s_4 & \leq & 0 \end{array}$$

$$\alpha(s_1) = 1, \alpha(s_2) = 1, \alpha(s_3) = -1.5, \alpha(s_4) = 0.5, \alpha(x_1) = 0.5, \alpha(x_2) = 0.5$$

s_4 violates its bound, but s_1 is not suitable for pivoting. Hence, we continue with the other branch $x_1 \geq 1$: We backtrack and add a new slack variable s_5.

	s_1	s_2
x_1	0.5	0
x_2	0	0.5
s_3	-1	-0.5
s_5	0.5	0

$$\begin{array}{lcl} s_1 & \geq & 1 \\ s_2 & \geq & 1 \\ s_3 & \geq & -3 \\ s_5 & \geq & 1 \end{array}$$

$$\alpha(s_1) = 1, \alpha(s_2) = 1, \alpha(s_3) = -1.5, \alpha(s_5) = 0.5, \alpha(x_1) = 0.5, \alpha(x_2) = 0.5$$

s_5 violates its bound and s_1 is suitable for pivoting. We solve the equation for the row of s_5:

$$s_5 = 0.5s_1 \Leftrightarrow s_1 = 2 \cdot s_5$$

We substitute and update the tableau and the assignments:

	s_5	s_2
x_1	1	0
x_2	0	0.5
s_3	-2	-0.5
s_1	2	0

$$\begin{array}{lcl} s_1 & \geq & 1 \\ s_2 & \geq & 1 \\ s_3 & \geq & -3 \\ s_5 & \geq & 1 \end{array}$$

$$\alpha(s_1) = 2, \alpha(s_2) = 1, \alpha(s_3) = -2.5, \alpha(s_5) = 1, \alpha(x_1) = 1, \alpha(x_2) = 0.5$$

Now, all variables satisfy their bounds and we obtain $x_1 = 1, x_2 = 0.5$ as real assignment. We select x_2 for branching and try $x_2 \leq 0$ first. We add a new constraint with a new slack variable s_6.

	s_5	s_2
x_1	1	0
x_2	0	0.5
s_3	-2	-0.5
s_1	2	0
s_6	0	0.5

$$\begin{array}{lcl} s_1 & \geq & 1 \\ s_2 & \geq & 1 \\ s_3 & \geq & -3 \\ s_5 & \geq & 1 \\ s_6 & \leq & 0 \end{array}$$

$$\alpha(s_1) = 2, \alpha(s_2) = 1, \alpha(s_3) = -2.5, \alpha(s_5) = 1, \alpha(s_6) = 0.5, \alpha(x_1) = 1, \alpha(x_2) = 0.5$$

s_6 violates its bound, but s_2 is not suitable for pivoting. Hence, we continue with the other branch $x_2 \geq 1$: We backtrack and add a new slack variable s_7.

	s_5	s_2
x_1	1	0
x_2	0	0.5
s_3	-2	-0.5
s_1	2	0
s_7	0	0.5

$$\begin{aligned} s_1 &\geq 1 \\ s_2 &\geq 1 \\ s_3 &\geq -3 \\ s_5 &\geq 1 \\ s_7 &\geq 1 \end{aligned}$$

$$\alpha(s_1) = 2, \alpha(s_2) = 1, \alpha(s_3) = -2.5, \alpha(s_5) = 1, \alpha(s_7) = 0.5, \alpha(x_1) = 1, \alpha(x_2) = 0.5$$

s_7 violates its bound and s_2 is suitable for pivoting. We solve the equation for the row of s_7:

$$s_7 = 0.5 s_2 \Leftrightarrow s_2 = 2 \cdot s_7$$

We substitute and update the tableau and the assignments:

	s_5	s_7
x_1	1	0
x_2	0	1
s_3	-2	-1
s_1	2	0
s_2	0	2

$$\begin{aligned} s_1 &\geq 1 \\ s_2 &\geq 1 \\ s_3 &\geq -3 \\ s_5 &\geq 1 \\ s_7 &\geq 1 \end{aligned}$$

$$\alpha(s_1) = 2, \alpha(s_2) = 2, \alpha(s_3) = -3, \alpha(s_5) = 1, \alpha(s_7) = 1, \alpha(x_1) = 1, \alpha(x_2) = 1$$

Now, all variables satisfy their bounds and we obtain $x_1 = 1, x_2 = 1$ as real assignment. As this assignment is integral, we are done and obtain a satisfying assignment for the input formula.