Theory of Hybrid Systems Informatik 2

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Satisfiability Checking - WS 2023/2024

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Series 12

Exercise 1

The solution domain of formulas in propositional logic is always finite, hence we can check the formula for satisfiability by testing all assignments. How about formulas in linear or non-linear real arithmetic? Would a procedure testing all assignments always terminate, if the formula is satisfiable?

Solution:

Linear and non-linear real arithmetic formulas have an infinite solution domain, therefore this procedure would not terminate if the formula is unsatisfiable. As the solution domain $\mathbb{R}$ is even not countable, such a procedure would in general neither terminate if the formula is satisfiable. However, checking the satisfiability of linear real arithmetic formulas is equivalent to checking the satisfiability of linear rational arithmetic formulas, that is the variables have the countably infinite domain $\mathbb{Q}$. Hence, there exists an enumerating procedure testing all assignments of the variables to $\mathbb{Q}$ which always terminates if the given formula is satisfiable. For instance you can count the rationals by

$$0,\ \pm\frac{1}{1},\ \pm\frac{1}{2},\ \pm\frac{2}{1},\ \pm\frac{1}{3},\ \pm\frac{2}{3},\ \pm\frac{3}{2},\ \pm\frac{3}{1},\ \ldots$$

eventually reaching any rational. For non-linear real arithmetic formulas we cannot use this property.

Exercise 2

For the polynomial

$$p(x, y, z) = x^2 + 2yx - z + 1$$

please list the symbolic descriptions of its real roots in x with side conditions in all those cases, where the side conditions are not trivially false.

Solution:

1. p cannot be constant nor linear in x, since p's quadratic term has the coefficient 1.
2. Quadratic in x, 1st solution: $\frac{-2y+\sqrt{(2y)^2-4(-z+1)}}{2}$
   with side condition $1 \neq 0 \land (2y)^2 - 4(-z+1) \geq 0$
3. Quadratic in x, 2nd solution: $\frac{-2y-\sqrt{(2y)^2-4(-z+1)}}{2}$
   with side condition $1 \neq 0 \land (2y)^2 - 4(-z+1) > 0$

Exercise 3

Consider the following non-linear real arithmetic formula:

$$\varphi \quad = \quad \exists x, y.((\ xy - 1 = 0 \lor y - x \geq 0\ ) \land (\ y^2 - 1 < 0 \lor x + y + 1 > 0\ ))$$

a) List the test candidates you obtain for y by the constraints of φ.

b) Apply the virtual substitution[1] of y by all test candidates of the constraint $y - x \geq 0$.

[1] You find the virtual substitution rules in the learning room besides the lecture slides.

c) List all test candidates you obtain for x by the constraints of the result of part b).

d) Choose one of these test candidates, not containing a square root but an infinitesimal, and apply it to one of the resulting constraints.

e) Why can the virtual substitution method as presented in the lecture not solve all non-linear real arithmetic formulas? Could this procedure check formulas for satisfiability, where each variable occurs at most quadratic?

Solution:

a) We get the following test candidates for y:

- The constraint $xy - 1 = 0$ provides for y the test candidates $-\infty$ and $\frac{1}{x}$ with the side condition $x \neq 0$. Note that $-\infty$ would fulfill a constraint $xy - c = 0$ if $x = 0$ and $c = 0$. That is why the general algorithm would provide $-\infty$ for each constraint containing y.
- The constraint $y - x \geq 0$ provides for y additionally the test candidate x.
- The constraint $y^2 - 1 < 0$ provides for y additionally the test candidates $-1 + \epsilon$ and $1 + \epsilon$.
- The constraint $x + y + 1 > 0$ provides for y additionally the test candidate $-x - 1 + \epsilon$.

Note that a test candidate provided by a constraint does not necessarily fulfill this constraint.

b) The constraint $y - x \geq 0$ provides for y the test candidates $-\infty$ and x. We substitute each constraint containing y by $-\infty$ using the substitution rules of the virtual substitution method.

$$
\begin{aligned}
& (xy - 1 = 0)[-\infty//y] \\
\equiv\ & x = 0 \ \wedge\ -1 = 0 \\
\equiv\ & \textit{False}
\end{aligned}
$$

$$
\begin{aligned}
& (y - x \geq 0)[-\infty//y] \\
\equiv\ & 1 < 0 \ \vee\ (\ 1 = 0 \wedge\ -x \geq 0\ ) \\
\equiv\ & \textit{False}
\end{aligned}
$$

$$
\begin{aligned}
& (y^2 - 1 < 0)[-\infty//y] \\
\equiv\ & 1 < 0 \ \vee\ (\ 1 = 0 \wedge\ 0 > 0\ ) \ \vee\ (\ 1 = 0 \wedge\ 0 = 0 \ \wedge\ -1 < 0) \\
\equiv\ & \textit{False}
\end{aligned}
$$

$$
\begin{aligned}
& (x + y + 1 > 0)[-\infty//y] \\
\equiv\ & 1 < 0 \ \vee\ (\ 1 = 0 \wedge\ x + 1 > 0\ ) \\
\equiv\ & \textit{False}
\end{aligned}
$$

Hence, the resulting formula is

$$(\ \textit{False} \vee \textit{False}\ ) \wedge (\ \textit{False} \vee \textit{False}\ ) \quad \equiv \quad \textit{False}.$$

Now, we substitute each constraint containing y by x using the substitution rules of the virtual substitution method.

$$\begin{aligned} & (xy - 1 = 0)[x//y] \\ \equiv\ & x^2 - 1 = 0 \end{aligned}$$

$$\begin{aligned} & (y - x \geq 0)[x//y] \\ \equiv\ & (\ 1 > 0\ \wedge\ 0 \geq 0\ )\ \vee\ (\ 1 < 0\ \wedge\ 0 \leq 0\ ) \\ \equiv\ & \textit{True} \end{aligned}$$

$$\begin{aligned} & (y^2 - 1 < 0)[x//y] \\ \equiv\ & (\ 1 > 0\ \wedge\ x^2 - 1 < 0\ )\ \vee\ (\ 1 < 0\ \wedge\ x^2 - 1 > 0\ ) \\ \equiv\ & x^2 - 1 < 0 \end{aligned}$$

$$\begin{aligned} & (x + y + 1 > 0)[x//y] \\ \equiv\ & (\ 1 > 0\ \wedge\ 2x + 1 > 0\ )\ \vee\ (\ 1 < 0\ \wedge\ 2x + 1 > 0\ ) \\ \equiv\ & 2x + 1 > 0 \end{aligned}$$

Hence, the resulting formula is

$$(\ x^2 - 1 < 0\ \vee\ 2x + 1 > 0\ ).$$

c) We get the following test candidates for x:

- The constraint $x^2 - 1 < 0$ provides for x the test candidates $-\infty$, $-1 + \epsilon$ and $1 + \epsilon$.
- The constraint $2x + 1 > 0$ provides for x additionally the test candidate $-\frac{1}{2} + \epsilon$.

d) We choose the test candidate $1 + \epsilon$ and the constraint $x^2 - 1 < 0$.

$$\begin{aligned} & (x^2 - 1 < 0)[1 + \epsilon//x] \\ \equiv\ & \quad (x^2 - 1 < 0)[1//x] \\ & \vee (\ (x^2 - 1 = 0)[1//x]\ \wedge\ (2x < 0)[1//x]\ ) \\ & \vee (\ (x^2 - 1 = 0)[1//x]\ \wedge\ (2x = 0)[1//x]\ \wedge\ 2 < 0\ ) \\ \equiv\ & \quad 0 < 0 \\ & \vee (\ 0 = 0\ \wedge\ 2 < 0\ ) \\ & \vee (\ 0 = 0\ \wedge\ 2 = 0\ \wedge\ 2 < 0\ ) \\ \equiv\ & \textit{False} \end{aligned}$$

e) In general there is no solution formula to calculate the zeros of polynomials with arbitrary degrees. This would be necessary to apply this kind of the virtual substitution to general real arithmetic formulas. Furthermore, we gave only substitution rules, which can deal with expressions possibly containing $\sqrt{}$, $-\infty$ and $+\epsilon$. For instance, they do not consider $\sqrt[3]{}$. that a single quantifier resp. variable elimation step can increase the degree of the remaining variables. Hence, it can even happen that virtual substitution cannot check formulas for satisfiability, where each variable occurs at most quadratic.