Satisfiability Checking - WS 2023/2024

Series 11

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Exercise 1: Subtropical satisfiability

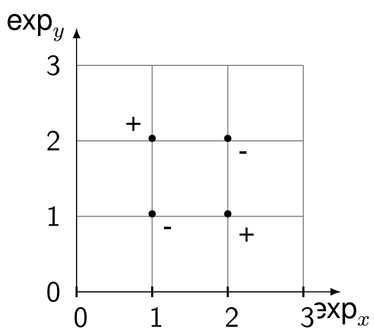
Find a satisfying assignment for $2xy^2 + x^2y - 2xy - 2x^2y^2 = 0$ using the subtropical satisfiability method as shown in the lecture. If you need to identify the real roots of a univariate polynomial of degree larger than two, please use WolframAlpha (`https://www.wolframalpha.com`).

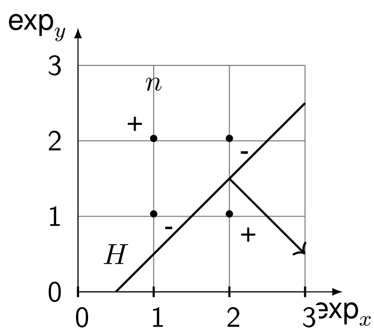
Solution:

I. We observe that for $(1, 1)$ we have $2 + 1 - 2 - 2 = -1 < 0$.

II. We proceed to find a solution for $\underbrace{2xy^2 + x^2y - 2xy - 2x^2y^2}_{p(x,y)} > 0$.

The frame of p is *frame*$(p) = \{(1, 2), (2, 1), (1, 1), (2, 2)\}$ and we have the positive frame *frame*$^+(p) = \{(1, 2), (2, 1)\}$. We identify a direction $n = (1, -1)$ and an offset $c = 0.5$ so that $n \cdot (2, 1)^T > 0.5$ and $nu^T < 0.5$ for all $u \in \{(1, 2), (1, 1), (2, 2)\}$.





(Note that separating the other positive frame point $(1, 2)$ is also a valid option.)

We thus have that $p(a^1, a^{-1}) > 0$ for some sufficiently large a. We start with $a = 2$ and get $p(2, 2^{-1}) = -1 < 0$. We increase to $a = 4$ and get $p(4, 4^{-1}) = 0.5 > 0$.

From $a = 4$ we obtain the satisfying assignment $x = 4, y = \frac{1}{4}$.

III. We identify a real root for $2xy^2 + x^2y - 2xy - 2x^2y^2$ on the line segment between $(1, 1)$ and

$(4, \frac{1}{4})$. That means, we search for a $t \in [0,1] \subset \mathbb{R}$ with

$$\begin{aligned}
x &= 1 + t(4-1) = 3t+1 \\
y &= 1 + t(\frac{1}{4} - 1) = -\frac{3}{4}t + 1 \\
p(x,y) &= 2xy^2 + x^2y - 2xy - 2x^2y^2 \\
&= xy \cdot (2y + x - 2 - 2xy) \\
&= (3t+1)(-\frac{3}{4}t+1) \cdot (2(-\frac{3}{4}t+1) + (3t+1) - 2 - 2(3t+1)(-\frac{3}{4}t+1)) \\
&= (-\frac{9}{4}t^2 + 3t - \frac{3}{4}t + 1) \cdot (-\frac{6}{4}t + 2 + 3t + 1 - 2 - 2(-\frac{9}{4}t^2 + 3t - \frac{3}{4}t + 1)) \\
&= (-\frac{9}{4}t^2 + 3t - \frac{3}{4}t + 1) \cdot (\frac{9}{2}t^2 - 3t - 1) \\
&= -\frac{81}{8}t^4 + \frac{27}{4}t^3 + \frac{9}{4}t^2 + \frac{27}{2}t^3 - 9t^2 - 3t - \frac{27}{8}t^3 + \frac{9}{4}t^2 + \frac{3}{4}t + \frac{9}{2}t^2 - 3t - 1 \\
&= -\frac{81}{8}t^4 + \frac{135}{8}t^3 - \frac{21}{4}t - 1 \\
&= \frac{1}{8}(-81t^4 + 135t^3 - 42t - 8)
\end{aligned}$$

WolframAlpha tell us that this polynomial has 4 real roots, but only one real root between 0 and 1 at $t = \frac{1}{3} + \frac{1}{\sqrt{3}}$. We get:

$$\begin{aligned}
x &= 3t + 1 = 3 \cdot (\frac{1}{3} + \frac{1}{\sqrt{3}}) + 1 = \sqrt{3} + 2 \\
y &= -\frac{3}{4}t + 1 = -\frac{3}{4} \cdot (\frac{1}{3} + \frac{1}{\sqrt{3}}) + 1 = \frac{3 - \sqrt{3}}{4}
\end{aligned}$$

We control the result by substituting these values into the polynomial $p(x, y)$, and get 0.

Exercise 2: Subtropical satisfiability

Assume that the subtropical satisfiability method found a solution $s = (s_1, \ldots, s_d)$ for a constraint $p(x_1, \ldots, x_d) > 0$ based on a separating hyperplane with normal vector $n = (n_1, \ldots, n_d) \in \mathbb{R}^d$. Further, we assume that the corresponding monomial dominates p at s. Let $i \in \{1, \ldots, d\}$.

Prove the following statements:

1. If $n_i > 0$ then for all $s'_i \in \mathbb{R}_{>0}$ with $s'_i > s_i$ there exist $s'_1, \ldots, s'_{i-1}, s'_{i+1}, \ldots, s'_d \in \mathbb{R}_{>0}$ such that $p(s'_1, \ldots, s'_d) > 0$.
2. If $n_i = 0$ then for all $s'_i \in \mathbb{R}_{>0}$ there exist $s'_1, \ldots, s'_{i-1}, s'_{i+1}, \ldots, s'_d \in \mathbb{R}_{>0}$ such that $p(s'_1, \ldots, s'_d) > 0$.
3. If $n_i < 0$ then for all $s'_i \in \mathbb{R}_{>0}$ with $s'_i < s_i$ there exist $s'_1, \ldots, s'_{i-1}, s'_{i+1}, \ldots, s'_d \in \mathbb{R}_{>0}$ such that $p(s'_1, \ldots, s'_d) > 0$.

Solution:

1. Assume $n_i > 0$. Then the solution $s = (s_1, \ldots, s_d) \in \mathbb{R}^d_{>0}$ will have a "sufficiently large" value in dimension i. Let $s'_i \in \mathbb{R}_{>0}$ with $s'_i > s_i$. We show that there exist $s'_1, \ldots, s'_{i-1}, s'_{i+1}, \ldots, s'_d \in \mathbb{R}_{>0}$ such that $s' = (s'_1, \ldots, s'_d)$ is also a solution.

Assume that the solution was found for some $a \in \mathbb{R}_{>0}$ with $p(a^{n_1}, \ldots, a^{n_d}) > 0$, i.e. we have $s_i = a^{n_i}$, or equivalently $a = \sqrt[n_i]{s_i}$. By the construction of n we know that $p(b^{n_1}, \ldots, b^{n_d}) > 0$ for all $b \in \mathbb{R}$ with $b > a$. We set $b = \sqrt[n_i]{s'_i}$. Note that $s_i < s'_i$ assures that $a = \sqrt[n_i]{s_i} < \sqrt[n_i]{s'_i} = b$.

2. Assume $n_i = 0$. We show that for all $s'_i \in \mathbb{R}_{>0}$ there exist $s'_1, \ldots, s'_{i-1}, s'_{i+1}, \ldots, s'_d \in \mathbb{R}_{>0}$ such that $s' = (s'_1, \ldots, s'_d)$ is a solution.

   To find such $s'_1, \ldots, s'_{i-1}, s'_{i+1}, \ldots, s'_d \in \mathbb{R}_{>0}$, assume that the method gives us some $a \in \mathbb{R}_{>0}$ with $p(a^{n_1}, \ldots, a^{n_d}) > 0$. The vector n is constructed such a way that there exists a $b \in \mathbb{R}^d$ $v \cdot n^T > b$ for a positive frame point $v = (v_1, \ldots, v_d)$, and $u \cdot n^T < b$ for all other frame points.

   We substitute s'_i for x_i in $p(x_1, \ldots, x_d)$, resulting in a polynomial $q(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d)$. Note that since $s'_i > 0$, this substitution does not change the signs of the coefficients in the terms. Note furthermore that each solution of q can be extended to a solution of p by assigning the value s'_i to x_i.

   Let $n' = (n_1, \ldots, n_{i-1}, n_{i+1}, \ldots, n_d)$ result from n by removing the entry $n_i = 0$ at position i. Note that the frame points u' of q are the frame points u of p after removing their ith entries. Then, since $n_i = 0$, we have $v \cdot n^T = v' \cdot n'^T > b$ and $u \cdot n^T = u' \cdot n'^T < b$ for all other frame points u of p that are different from v. That means, n' is suited to separate v' from all other frame points in q, thus the subtropical satisfiability method gives us a solution for $q > 0$, which we can extend to a solution for $p > 0$ by assigning the value s'_i to x_i.

3. Assume $n_i < 0$. Then the solution $s = (s_1, \ldots, s_d) \in \mathbb{R}^d_{>0}$ will have a "sufficiently small" value in dimension i. Let $s'_i \in \mathbb{R}_{>0}$ with $s'_i < s_i$. We show that there exist $s'_1, \ldots, s'_{i-1}, s'_{i+1}, \ldots, s'_d \in \mathbb{R}_{>0}$ such that $s' = (s'_1, \ldots, s'_d)$ is also a solution.

   Assume that the solution was found for some $a \in \mathbb{R}_{>0}$ with $p(a^{n_1}, \ldots, a^{n_d}) > 0$, i.e. we have $s_i = a^{n_i}$, or equivalently $a = \frac{1}{\sqrt[-n_i]{s_i}}$. By the construction of n we know that $p(b^{n_1}, \ldots, b^{n_d}) > 0$ for all $b \in \mathbb{R}$ with $b > a$. We set $b = \frac{1}{\sqrt[-n_i]{s'_i}}$. Note that $s_i > s'_i$ assures that $a = \frac{1}{\sqrt[-n_i]{s_i}} < \frac{1}{\sqrt[-n_i]{s'_i}} = b$.