hybr i 2 d Theory of Hybrid Systems Informatik 2

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Satisfiability Checking - WS 2023/2024

Series 13

Exercise 1

The Cylindrical Algebraic Decomposition aims at decomposing the whole solution space into *sign-invariant regions*. Each such region is represented by a single sample point.

Why can you decide satisfiability using only a few sample points, although the solution space is infinitely large?

Solution: The inequalities are in a canonical form and thus the polynomials only compare to zero. Within a region, no polynomial changes its sign and thus, all points within a region are equivalent with respect to the input inequalities – they fulfill and conflict with the same inequalities. Hence, it suffices to select (at least) one sample point per region. As there are only finitely many polynomials of finite degree, the number of regions is finite.

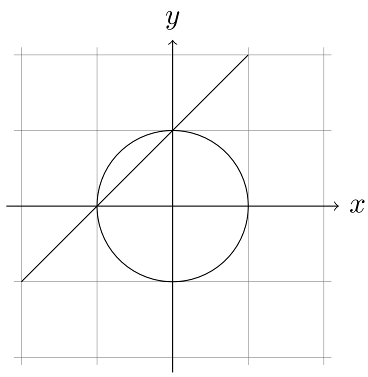
Exercise 2

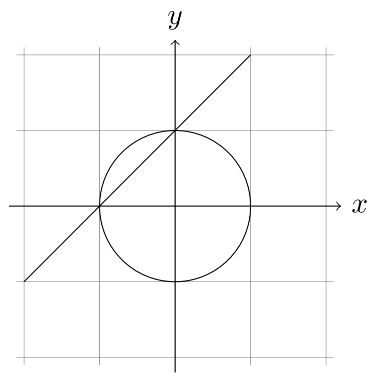
1. For the polynomials $p_1(x, y) = x^2 + y^2 - 1$ and $p_2(x, y) = x - y + 1$, whose real zeros are depicted below, please give a minimal selection $S \subset \mathbb{R}^2$ of sample points such that for all real-arithmetic satisfiability problems φ that put sign conditions on these polynomials only, for example $(p_1 \leq 0 \land p_2 \leq 0) \lor (p_1 \geq 0 \land p_2 \geq 0)$, it holds that

   因为题目加上了限定条件: constraint只为 p≥0, p≤0, 所以不需要全部的sample point, 只需要(大于0, 等于0, 小于0)3×3=9种组合各取一个就行

$$\exists x. \exists y. \varphi \iff \bigvee_{(v,u) \in S} \varphi[v/x][u/y] \; .$$

   You can draw the sample points as dots in the diagram.



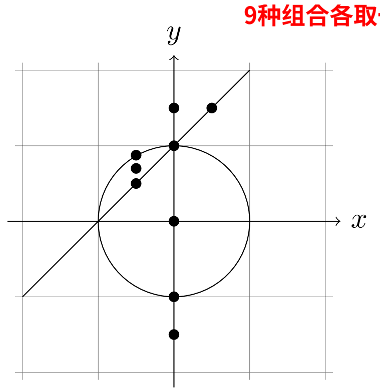


2. Due to the way how the CAD algorithm determines the sample points, the set of sample points that will actually be used is much larger. Give a minimal set of sample points that the CAD method could generate when projecting y first for the above example, and argue why the additional sample points are included.

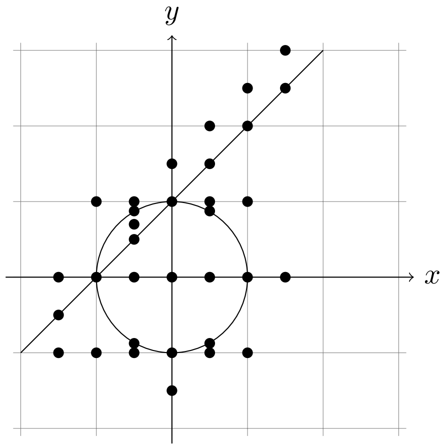
Solution:

1. Minimum selection of sample points:

9种组合各取一个, 9个sample point



2. An actual selection of sample points:



The CAD algorithm first selects a set of sample points in one dimension only. If we project y first then this will be the x dimension. For this, every x-value at which *the number or order of the zeros changes* is selected, i.e. at least $-1, 0, 1$ as well as points smaller, inbetween and greater, for example $-1.5, -0.5, 0.5, 1.5$. Then, each of these sample points is substituted for x into each polynomial and for the next dimension, once again all points at which *the number or oder of the zeros changes* are selected. However, the lifting is done independently for different values of x and thus multiple samples that describe the same maximal sign-invariant region are constructed.

Exercise 3

How many cells are in the coarsest CAD, i.e. the one with the minimal number of cells, for

$$P = \{\underbrace{x - y}_{p_1}, \underbrace{x + y}_{p_2}\}$$

将包含y的数据点投射到x轴上

when projecting y first (i.e. selecting samples for x first)?

Solution: Note that for each fixed x-value, both polynomials have a single root. The projection will contain a polynomial with a root at $x = 0$. This is the only x-value at which the number/order of the zeros of the two polynomials changes:

- for $x < 0$ the only zero of p_1 is smaller than the only zero of p_2,
- at $x = 0$ the two zeros are identical,
- and for $x > 0$ the only zero of p_2 is smaller than the only zero of p_1.

Thus there are 3 one-dimensional samples: (i) one below 0, (ii) 0 and (iii) one above zero.

Each of these 3 one-dimensional samples are substituted into the polynomials and their zeros are computed. The two-dimensional samples extend the one-dimensional samples by these zeros, one value below the smallest zero, one between each two consequtive zeros and one above the largest zero. This gives us 13 two-dimensional samples: the left and right cylinders have 5 samples each, whereas the cylinder in the middle has 3 samples. Such a set of samples could be e.g.: