Satisfiability Checking - WS 2022/2023

Written Exam II

Monday, March 30, 2023

Forename and surname:	**Matriculation number:**
Sign here:	

- Do not open the exam until we give the start signal.
- The duration of the exam is 120 minutes.
- Use a blue or black (permanent) pen only. Everything else will be ignored.
- Please write your name and matriculation number on each page of this exam.
- Please write clear and legible answers. Clearly cross out parts you do *not* wish to be evaluated.
- Please use the space below each task to solve it. Additional paper is provided at the back of the exam.
- If you have problems understanding a task, indicate this by a hand signal.
- You are not allowed to use auxiliary material except for a pen and a ruler. In particular, switch off your electronic devices! Cheating disqualifies from the exam.
- When the exam is over stop writing, turn around your sheets and leave them at your seat.

Task	1.)	2.)	3.)	4.)	5.)	6.)	7.)	8.)	9.)	Bonus	**Total**
Maximum score	17	14	5	14	5	18	15	14	18	-	120
Reached score											

Good luck!

1.) SAT Checking **7 + 5 + 5 points**

i) Assume the following propositional logic formula in CNF:

$$\begin{array}{lll} c_1 : (\neg x_1 \vee x_2) \quad \wedge & c_2 : (\neg x_2 \vee \neg y_1 \vee y_2) \quad \wedge & c_3 : (\neg y_2 \vee y_3) \quad \wedge \\ c_4 : (\neg y_3 \vee y_4) \quad \wedge & c_5 : (\neg y_3 \vee \neg y_4) & \end{array}$$

Assume furthermore the following trail:

$$\begin{array}{ll} DL0: & - \\ DL1: & x_1 : nil, \quad x_2 : c_1 \\ DL2: & y_1 : nil, \quad y_2 : c_2, \quad y_3 : c_3, \quad y_4 : c_4 \end{array}$$

We detect a conflicting clause c_5.

a) Please draw the implication graph and mark all unique implication points in it with the label "UIP".

b) Apply conflict resolution to c_5 till the first unique implication point as presented in the lecture. Please specify all resolution steps and their results.

ii) Please apply Tseitin's encoding to the following propositional logic formulas. Please specify

I. the intermediate formula that encodes the meaning of sub-formulas using auxiliary variables $h_1, h_2, \ldots$, and

II. the result after transforming the intermediate formula into CNF.

a) $\neg(a \vee b)$

b) $(a \rightarrow b) \wedge c$

iii) In the termination proof of the DPLL+CDCL algorithm, we used a (partial) ordering on partial assignments, which decreases during the execution. Order the following assignments α_1, α_2, α_3, α_4 accordingly from *largest to smallest (descending).*

	α_1	α_2	α_3	α_4
DL0	a	a, b	a	a
DL1	$\neg b$	c, d	$\neg b, c$	$\neg b$
DL2	$\neg c, d$	-	$\neg d$	$\neg c$

2.) Equality Logic and Uninterpreted Functions **4 + 4 + 6 points**

i) Apply *lazy* SMT solving for equality logic and uninterpreted functions as presented in the lecture to the following conjunction of equalities and disequalities, considering equations from left to right.

$$x = y \wedge u = F(x) \wedge F(F(u)) = y \wedge u \neq y$$

Please specify the initial partition, each execution step and the partition after the step, even if there is no change.

ii) Let φ be an arbitrary formula in equality logic with uninterpreted functions so that the algorithm from i) produces the final partition

$$\{\{a, b, F(a)\}, \{c, d\}, \{F(c)\}\}.$$

Give a disequation e only using a, b, c, d and/or F so that the algorithm from i) produces a different number of final equivalence classes for the inputs φ and $\varphi \wedge e$.

iii) Consider the following formula in *equality logic*:

$$\varphi^{EQ} : (a = b \;\vee\; b = c \;\vee\; b = d) \;\wedge\; (a \neq d \;\vee\; c \neq d)$$

Draw the corresponding E-graph *with polarity* and use it to transform φ^{EQ} into a satisfiability-equivalent propositional logic formula as presented in the lecture for eager SMT-solving.

3.) Fourier-Motzkin Variable Elimination **2 + 3 points**

i) For any integer $a \in \mathbb{Z}$, consider the following set of linear real arithmetic constraints:

$$S_a = \{a \cdot x + y \leq 2, \quad -2x - y \leq 0, \quad (a+1) \cdot x \leq 2, \quad -x - a \cdot y \leq -1\}$$

a) Assume $a = 1$. How many constraints does the *Fourier-Motzkin method* compute when eliminating only x from S_1 (including duplicates and trivial constraints)?

b) Give a value of $a \in \mathbb{Z}$ for which the number of constraints computed by the Fourier-Motzkin method when eliminating only x from S_a is *minimal*.

4.) Simplex

4+2+4+4 points

i) Apply the simplex method to the following constraint set until termination:

$$\begin{array}{ll} s_0 = -1x_0 + 2x_1 & s_0 \leq -2 \\ s_1 = -1x_0 - 2x_1 & s_1 \leq 2 \end{array}$$

Please specify the simplex tableau and the assignment initially and after each pivot step. When choosing pivot variables, use the order $x_0 \prec x_1 \prec s_0 \prec s_1$ and take the smallest possible variable.

ii) Consider the following tableau and the bounds on the slack variables. Compute the corresponding assignment α for all variables.

	s_2	s_1
x_1	-2	1
x_2	2	1

$$\begin{array}{lll} s_1 & \geq & 2 \\ s_2 & \geq & 1 \end{array}$$

$\alpha(s_1) =$ ☐

$\alpha(s_2) =$ ☐

$\alpha(x_1) =$ ☐

$\alpha(x_2) =$ ☐

iii) Consider the following tableau (the current values of the variables are given in square brackets).

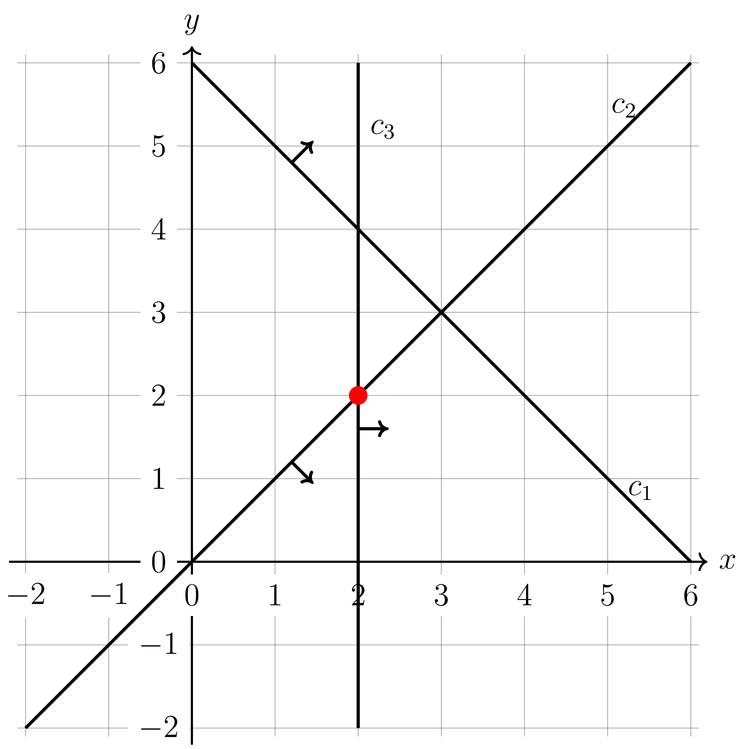
	s_1 [−1]	x_2 [0]	s_2 [1]
x_1 [2]	−1	1	1
s_3 [0]	1	1	1
s_4 [0]	−1	−1	−1
x_3 [−2]	1	−1	−1

$$\begin{array}{rcr} s_1 & \leq & -1 \\ s_2 & \geq & 1 \\ s_3 & \leq & -1 \\ s_4 & \geq & 1 \end{array}$$

a) Give all suitable pairs suitable for pivoting.

b) Which one would be chosen by Bland's rule? Assume the variable ordering $x_1 \prec x_2 \prec s_1 \prec s_2 \prec s_3 \prec s_4$ preferring the smallest variable for Bland's rule.

iv) Consider the following system of linear inequations $\{c_1, \ldots, c_3\}$. The point marked at at $(2, 2)$ corresponds to a state of the simplex algorithm on the system.



Add a linear inequality to the above coordinate system (i.e. draw a hyperplane and its normal vector) such that c_2 and c_3 are part of a conflict (i.e. the Simplex algorithm would terminate with an infeasible subset containing c_2 and c_3).

5.) Linear Integer Arithmetic **5 points**

Let P be a set of linear integer arithmetic constraints containing only the variables x and y. Below, the initial call and *all* recursive calls of the *Branch & Bound* algorithm for the input P are given. Complete the missing information for each call under the assumption that the final result of the algorithm is *unsatisfiable.* In particular,

- complete the specification of the input to the simplex method,
- specify the answer of simplex for the relaxed version of each input by writing SAT (satisfiable) or UNSAT (unsatisfiable), and
- give a possible variable assignment only if simplex would find one and otherwise leave the gaps for the assignment empty.

Input: P
Answer of simplex: $\boxed{}$ Assignment: $x = 0$, $y = \boxed{}$

Input: $P \cup \{y \geq 1\}$
Answer of simplex: $\boxed{}$ Assignment: $x = \boxed{}$, $y = \boxed{}$

Input: $P' := P \cup \{\boxed{}\}$
Answer of simplex: $\boxed{}$ Assignment: $x = 1/2$, $y = 0$

Input: $P' \cup \{\boxed{}\}$
Answer of simplex: $\boxed{}$ Assignment: $x = \boxed{}$, $y = \boxed{}$

Input: $P' \cup \{\boxed{}\}$
Answer of simplex: $\boxed{}$ Assignment: $x = \boxed{}$, $y = \boxed{}$

6.) Interval Constraint Propagation **4 + 4 + 7 + 3 points**

i) Please compute the result of the interval division $\frac{[-11;-4]}{[-1;2]}$.

ii) Assume $x \in [0;16]$ and $y \in [2;3]$. Please contract the domain for x using $x = 2y^2 - 3y + 4$ with the help of the contraction method I from the lecture.

iii) Apply the necessary preprocessing for the ICP contraction method I to the following constraints:

$$c_1 : xyz + x^2y + 2z < 0 \quad \wedge \quad c_2 : x - 2y = 0$$

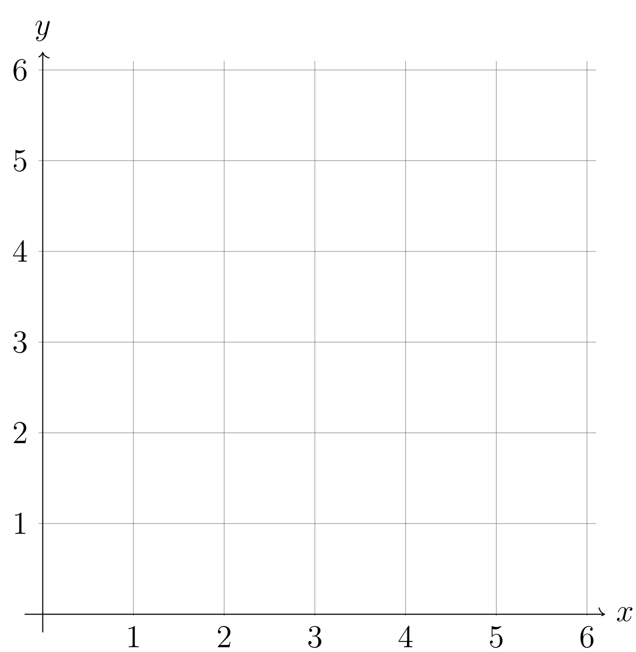
iv) Give a value $a \geq 0$ so that the relative contraction of the ICP contraction method II applied to the constraint $x^2 + a = 0$ with $x \in A = [0; 2]$ and $s = 1$ is *minimal*. Specify your intermediate computations and argue why your value for a is correct.

7.) Subtropical Satisfiability

4+4+3+4 points

i) Construct a real solution for the constraint $-2x^2 + 4y^2 - 8 = 0$ on the line segment between $(1, 1)$ (where the polynomial has a negative sign) and $(3, 3)$ (where the polynomial has a positive sign). Please give the computations and the resulting values for x and y.

ii) Draw the Newton polytope for the polynomial $p(x, y) = 5xy - 2xy^2 + xy^3 - 3x^2y^2$. Label all frame points accordingly.
Is the subtropical method as presented in the lecture suitable to determine the satisfiability of the constraint $p(x, y) = 0$? If yes, draw a *suitable* separating hyperplane and its normal vector that could be generated during the process. If not, mark the frame points which need to be removed to make the method applicable.



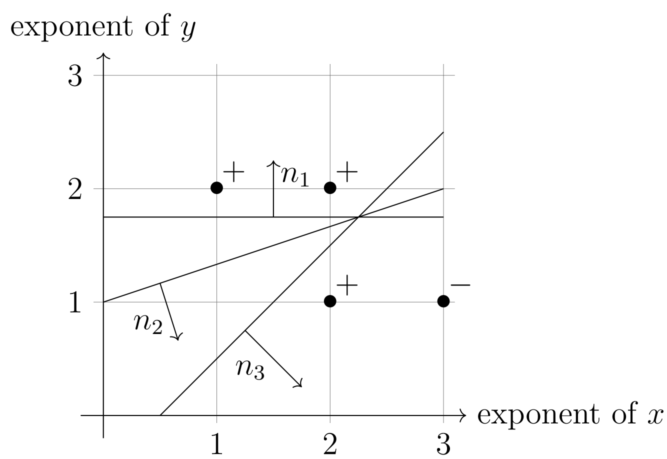
iii) Assume the subtropical method computes a separating hyperplane with normal vector $n = (-2, 3)$ for some polynomial $p(x, y)$. The method can construct a satisfying assignment (α_x, α_y) for $p(x, y) > 0$ using n. We assume that we initialize the value which is increased during that process with 2.

Give inclusion-minimal intervals I_x, I_y in which α_x and α_y must be contained for any polynomial $p \in \mathbb{Z}[x, y]$.

$I_x =$ $\boxed{}$

$I_y =$ $\boxed{}$

iv) The following image depicts the frame points of some polynomial $p(x, y)$ and three hyperplanes h_1, h_2 and h_3 with their normal vectors.



a) None of these normal vectors satisfies the sufficient condition from the subtropical method as given in the lecture. However, with one of them we can construct a solution for the constraint $p(x, y) > 0$ with the same method from the lecture. Which is it?

b) Based on this observation, adapt the encoding of a suitable normal vector n for a given polynomial p by filling in the two gaps in the following formula:

$$\bigvee_{v_+ \in \boxed{}} \left(nv_+^T > b \wedge \bigwedge_{v_- \in \boxed{}} (nv_-^T < b) \right)$$

8.) Virtual Substitution

4+5+5 points

i) Please specify the constraint c such that the result of the virtual substitution $c[-\infty//x]$ is

$$-2 = 0 \land 0 = 0 \land 4z = 0 \ .$$

ii) Give all test candidates for x with their side conditions that we get from the following constraint:

$$2xz^2 + x^2y + yz < 0$$

Simplify all expressions as far as possible by multiplying out all brackets.

Write your results in the following table. There are more rows than necessary.

Test candidate		Side condition
	, if	
	, if	
	, if	
	, if	
	, if	
	, if	

iii) Let $tcs(\varphi, x)$ be the set of test candidates of φ for the variable in x and $sc(t)$ denote the side condition of a test candidate $t \in tcs(\varphi, x)$. Then by the virtual substitution method it holds

$$\exists x.\varphi \quad \leftrightarrow \quad \bigvee_{t \in tcs(\varphi,x)} (\varphi[t//x] \wedge sc(t))$$

I.e. we can eliminate existentially quantified variables this way.

Virtual substitution can also be applied to eliminate universally quantified variables. Give an analogous statement for $\forall x.\varphi$. Simplify the formula as far as possible.

9.) Cylindrical Algebraic Decomposition **4+4+2+8 points**

i) What is the Cauchy bound for the polynomial $p = 2x^3 - 10x^2 - 10x - 8$? Please show your computations.

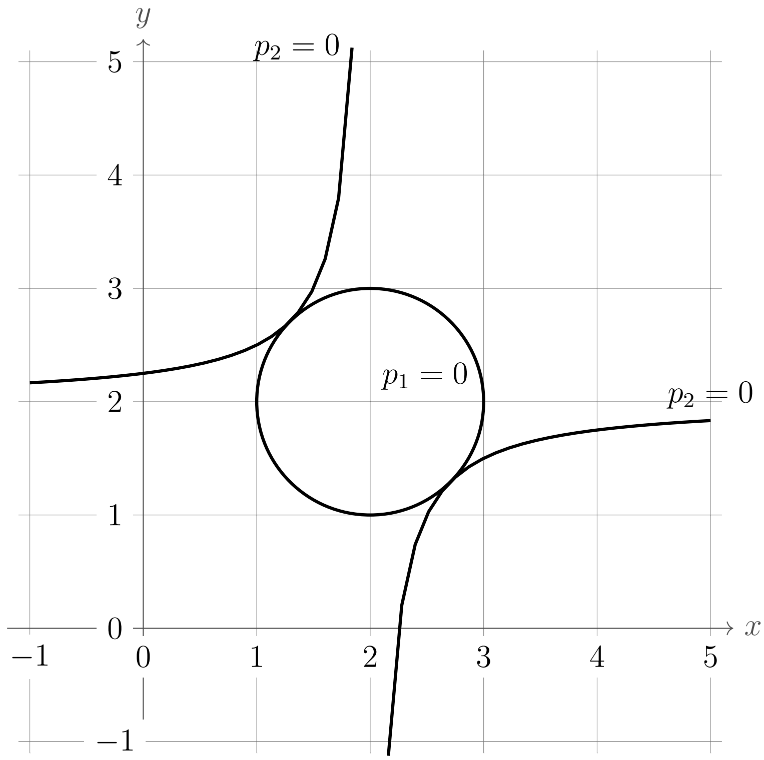
ii) Assume the polynomial $p = x^3 - 18x^2 - 6x + 12$ and its Sturm sequence

$$\begin{aligned} p_0 &= x^3 - 18x^2 - 6x + 12 \\ p_1 &= 3x^2 - 36x - 6 \\ p_2 &= 76x \\ p_3 &= 6 \end{aligned}$$

Use the Sturm sequence to compute the number of real roots of p contained in the interval $(-1; 1]$. Also give your intermediate calculations.

iii) During the lifting phase of the CAD, on each level, we select a sample at every root, a sample below all roots, a sample above all roots, and a sample between all pairs of neighbouring roots. In the cases where we can choose a sample,which property assures that we can choose *any* such point for lifting? Please *name* the property. (No explanation needed.)

iv) Consider the following varieties of some polynomials p_1 and p_2:



Note that p_2 has a singularity at $x = 2$.

Draw the cylinder boundaries of the CAD computed for the input p_1, p_2 with the CAD method as presented in the lecture. Label each boundary with the corresponding projection polynomial that induced this boundary.

Write $res(p_a, p_b)$ for the resultant of polynomials p_a and p_b, $disc(p)$ for the discriminant of p and $ldcf(p)$ for the leading coefficient of p.