Satisfiability Checking - WS 2019/2020

Written Exam II

Tuesday, May 19, 2020

Forename and surname:	**Matriculation number:**
Sign here:	

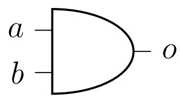
- Do not open the exam until we give the start signal.
- Put your name and matriculation number on the front page.
- The duration of the exam is 120 minutes.
- Use a blue or black (permanent) pen only.
- Please write your name and matriculation number on each page of this exam.
- Please write clear and legible answers. Clearly cross out parts you do *not* wish to be evaluated.
- Please use the space below each task to solve it. Additional paper is provided at the back of the exam.
- If you have problems understanding a task, indicate this by a hand signal. Please cover your mouth and nose when the minimum distance to others is not guaranteed.
- You are not allowed to use auxiliary material except for a pen. In particular, switch off your electronic devices! Cheating disqualifies from the exam.
- When the exam is over stop writing, turn around your sheets and leave them at your seat.

Task:	1.)	2.)	3.)	4.)	5.)	6.)	7.)	8.)	9.)	**Total**
Maximum score:	20	18	14	15	10	14	12	5	12	120
Reached score:										

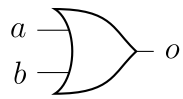
Good luck!

1.) SAT Checking **6+3+11 Points**

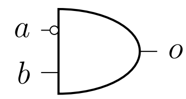
i) Logical circuits are often depicted graphically using the symbols defined below.



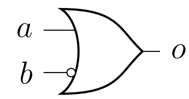
(a) $o = a \wedge b$



(b) $o = a \vee b$

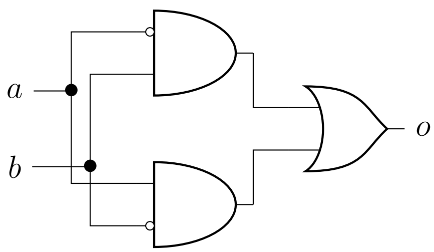


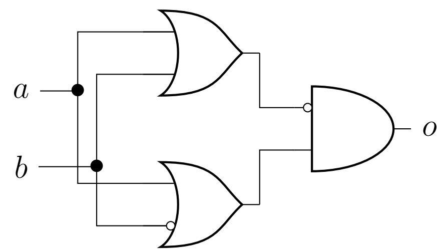
(c) $o = \neg a \wedge b$



(d) $o = a \vee \neg b$

An engineer designed the following two circuits and claims they are equivalent, that is for all possible inputs a and b, their outputs are the same. Formulate the two circuits as propositional logic formulas $\varphi_1(a, b)$ and $\varphi_2(a, b)$ and give a formula φ that allows a SAT solver to check whether they are in fact equivalent. Black dots represent junctions of wires and not negations.





(a) $\varphi_1(a, b) =$

(b) $\varphi_2(a, b) =$

(c) $\varphi =$

ii) Are the circuits in fact equivalent? If they are, argue why. If not, give inputs a and b such that the outputs differ.

iii) Now consider the following propositional formula in conjunctive normal form, which is satisfiability-equivalent to φ from i). Solve this formula using the CDCL algorithm as given in the lecture. You may select any decision heuristic. Explicitly state all decisions and propagations (including the clauses used for these propagations), as well as all conflict resolutions with the learnt clauses.

$$
\begin{array}{llll}
(a \vee \neg h_2) & \wedge(a \vee \neg b \vee \neg h_4) & \wedge(\neg a \vee \neg h_1) & \wedge(\neg a \vee \neg h_3) \\
\wedge(b \vee \neg h_1) & \wedge(\neg b \vee \neg h_2) & \wedge(\neg b \vee \neg h_3) & \wedge(h_1 \vee h_2 \vee \neg h_5) \\
\wedge(\neg h_1 \vee \neg h_2 \vee \neg h_7) & \wedge(h_3 \vee \neg h_8) & \wedge(\neg h_3 \vee \neg h_4 \vee \neg h_6) & \wedge(h_4 \vee \neg h_8) \\
\wedge(h_5 \vee h_7 \vee \neg h_9) & \wedge(h_5 \vee h_8 \vee \neg h_9) & \wedge(h_6 \vee h_7 \vee \neg h_9) & \wedge(h_6 \vee h_8 \vee \neg h_9) \\
\wedge(h_9) & & &
\end{array}
$$

2.) SMT Solving

3+2+13 Points

i) Name the three main requirements on a theory solver that are desirable for SMT solving.

ii) Describe eager SMT solving.

iii) Consider the following equality logic formula:

$$\begin{array}{rrlcccccl}
\varphi^{EQ} & := & & (& x_3 = x_4 & \vee & x_2 = x_4 & & &) \\
& & \wedge & (& x_3 = x_4 & \vee & x_1 = x_2 & \vee & x_1 = x_4 &) \\
& & \wedge & (& \neg(x_1 = x_2) & \vee & x_1 = x_4 & & &) \\
& & \wedge & (& \neg(x_3 = x_4) & \vee & \neg(x_2 = x_4) & \vee & \neg(x_2 = x_3) &)
\end{array}$$

The Boolean abstraction of this formula is

$$(a \vee b) \wedge (a \vee c \vee d) \wedge (\neg c \vee d) \wedge (\neg a \vee \neg b \vee \neg e)$$

Explain how a *less-lazy* SMT solver solves φ^{EQ} for satisfiability as presented in the lecture. Describe

- the *SAT solver's assignments with decision levels* before each theory solver invocation,
- the *equivalence classes* determined by the theory solver in each invocation and whether a *conflict* occurs,
- for each theory conflict, how a *minimal infeasible subset* is generated and
- the subsequent *conflict resolution* in the SAT solver.

Assume that

- if the SAT solver makes a decision it chooses the lexicographically smallest unassigned variable and assigns it to *false*, and that
- all equalities (or disequalities) are given to the theory solver (i.e. no constraints are dropped because only one polarity is present).

3.) Fourier-Motzkin Variable Elimination **9+5 Points**

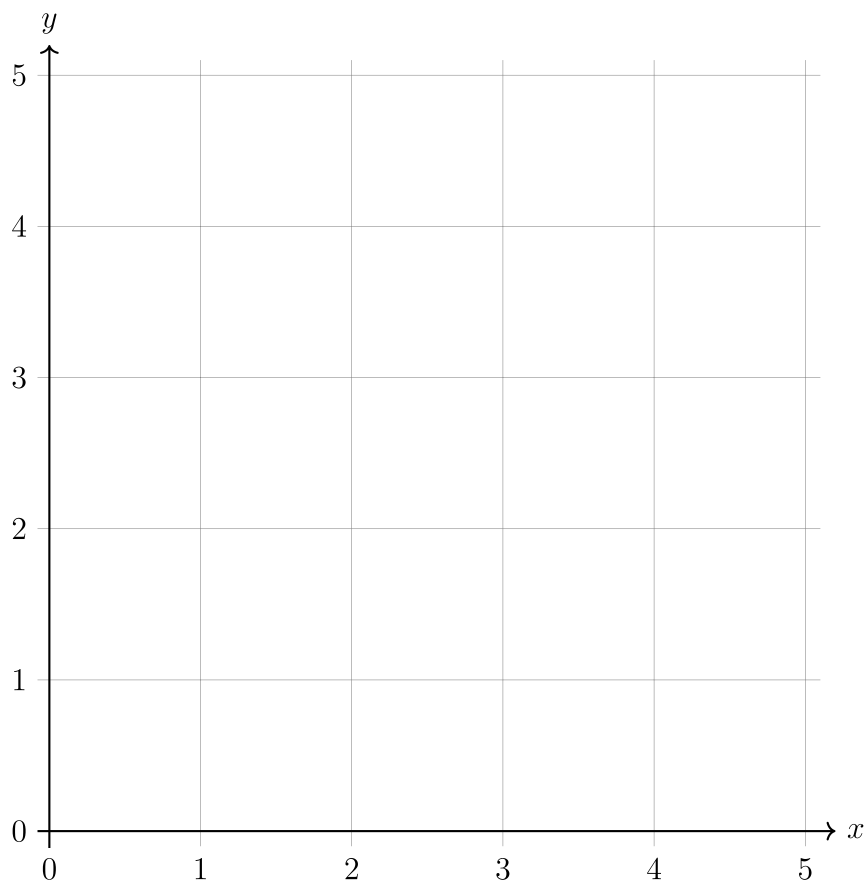
i) *Eliminate* x by applying the Fourier-Motzkin variable elimination method to the following set of linear real-arithmetic constraints:

$$
\begin{array}{lrcrcrcr}
c_1: & -x & + & 2y & - & 2z & \leq & 0 \\
c_2: & x & - & y & + & z & \leq & -2 \\
c_3: & x & - & y & - & z & \geq & 2 \\
c_4: & x & - & 2y & & & \leq & 1
\end{array}
$$

Answer without further applying Fourier-Motzkin: Is this constraint set satisfiable? Why?

ii) Sketch the *solution set* of the following set of linear real-arithmetic constraints graphically:

$$\begin{array}{lrcrcl} c_1: & -x & + & y & \leq & 2 \\ c_2: & x & & & \leq & 3 \\ c_3: & x & - & 2y & \leq & 1 \\ c_4: & 2x & + & y & \geq & 2 \end{array}$$



4.) Simplex **12+3 Points**

i) Apply one *pivoting step* and update the current assignment, where the variable order is $x_1 < x_2 < s_1 < s_2 < s_3 < s_4$ and where the current tableau, the bounds and the current assignment are given by:

	s_4	s_3
s_2	1	3
x_1	-2	1
s_1	-1	2
x_2	2	1

$$\begin{aligned} s_1 &\leq -2 \\ s_2 &\geq 8 \\ s_3 &\geq 2 \\ s_4 &\geq 1 \end{aligned}$$

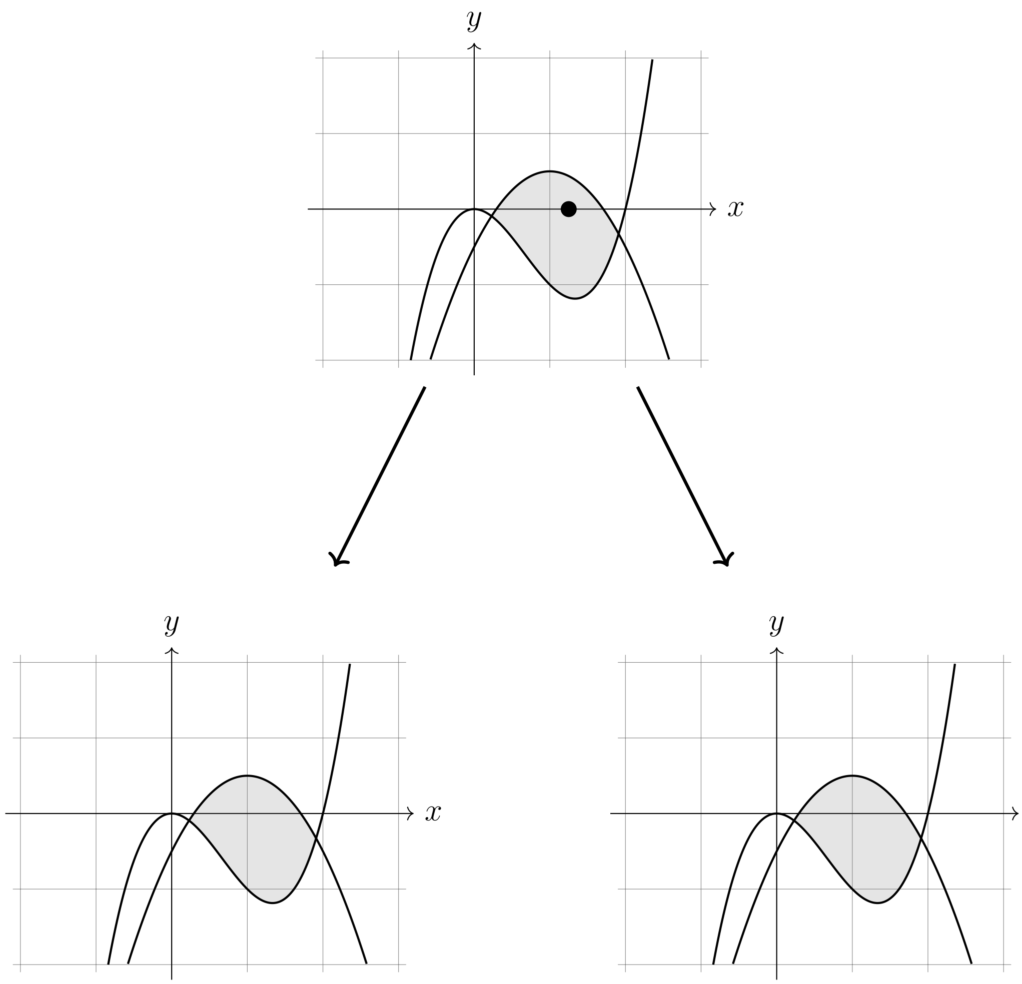
$$\begin{aligned} \alpha(s_1) &= 3 \\ \alpha(s_2) &= 7 \\ \alpha(s_3) &= 2 \\ \alpha(s_4) &= 1 \\ \alpha(x_1) &= 0 \\ \alpha(x_2) &= 4 \end{aligned}$$

ii) The simplex method terminates on the resulting tableau. Is the tableau satisfied or conflicting? Why? Give the satisfying assignment for the original variables or the infeasible subset.

5.) Integer Arithmetic **7+3 Points**

i) Branch&Bound employs a decision procedure for real arithmetic to solve integer arithmetic problems. Though we only showed linear examples in the lecture it can also be applied to non-linear examples, given an algorithm for non-linear real arithmetic. Consider the following example and *execute one branching* of the Branch&Bound method.

- Give the *variable used for branching.*
- Give the *new bounds* for each sub-problem.
- Sketch the *branch* for each sub-problem graphically.
- Sketch the *real solution space* for each sub-problem.



ii) In some applications, problems contain both integer-valued and real-valued variables. Can we apply Branch&Bound (as presented in the lecture) to such problems? How?

6.) Interval Constraint Propagation **6+8 Points**

i) Apply *basic interval arithmetic* as presented in the lecture to compute the following:

a) $[1; 2] + [-3; -3] + [-1; 3] =$

b) $[2; 4] \cdot [-2; 2] =$

c) $[1; \infty) \cdot [-2; -1] =$

d) $[1; 2]/[-2; 1] =$

ii) Assume an initial box $\mathcal{B}_0 = [2; 5] \times [1; 6]$ for the variables x and y and a set of constraints as follows:

$$C_1 : -x + 2y - 4 = 0$$
$$C_2 : -x - 2y + 10 \leq 0$$

Please contract each constraint C_i once individually for each variable. Do not reuse previous contractions, i.e., always use $\mathcal{B}_0$ as given above for your initial interval values for the variables. Please give all intermediate computations.

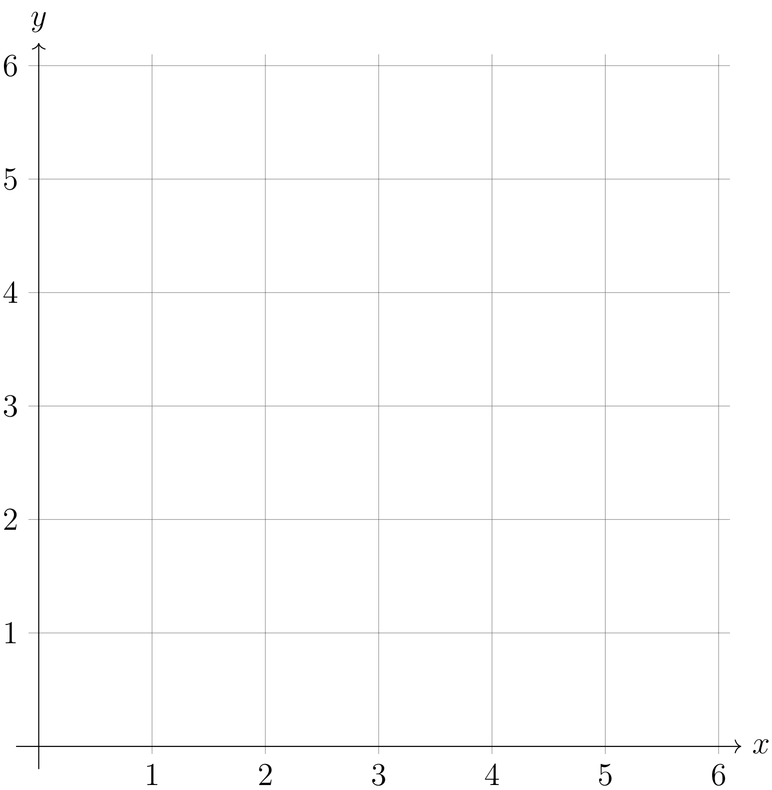
7.) Subtropical Satisfiability **6+6 Points**

i) Consider the polynomial $f(x, y) = x^2 - y - \frac{3}{4}$. We have already evaluated $f(x, y)$ at the two points $f(1, 1) = -\frac{3}{4}$ and $f(2, 2) = \frac{5}{4}$.

Use this information to compute a real root of $f(x, y)$ as shown in the lecture:

- Construct a univariate $f^*(t)$ for $f(x, y)$.
- Identify a real root of $f^*(t)$ with $t \in [0, 1]$.
- Use this real root to compute a real root of $f(x, y)$.

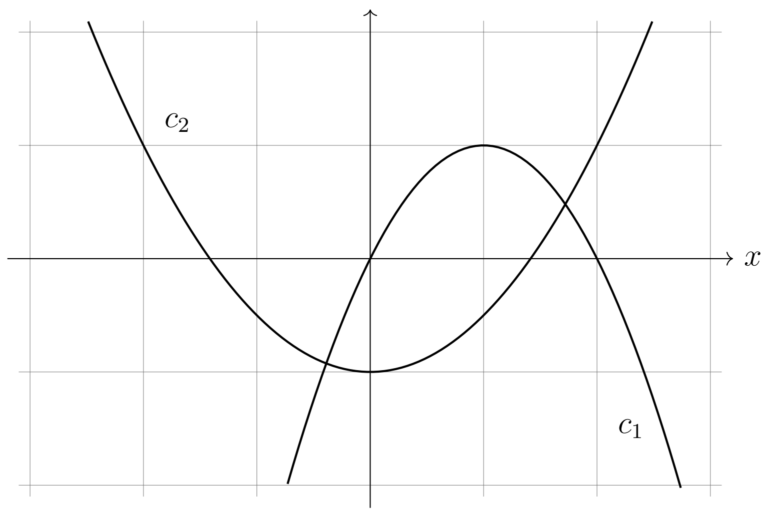
ii) Visualize $newton(g)$ for the polynomial $g(x, y) = 2x^2y^3 - 5yx^2 + 7x^3y^4 + x^4y^2 - 2x^4y^5 - 4y^4x - x^5y^2$.
Is the subtropical method as presented in the lecture suitable to determine the satisfiability of the constraint $g = 0$? Explain why not or describe how to apply the method to this example.



8.) Virtual Substitution **5 Points**

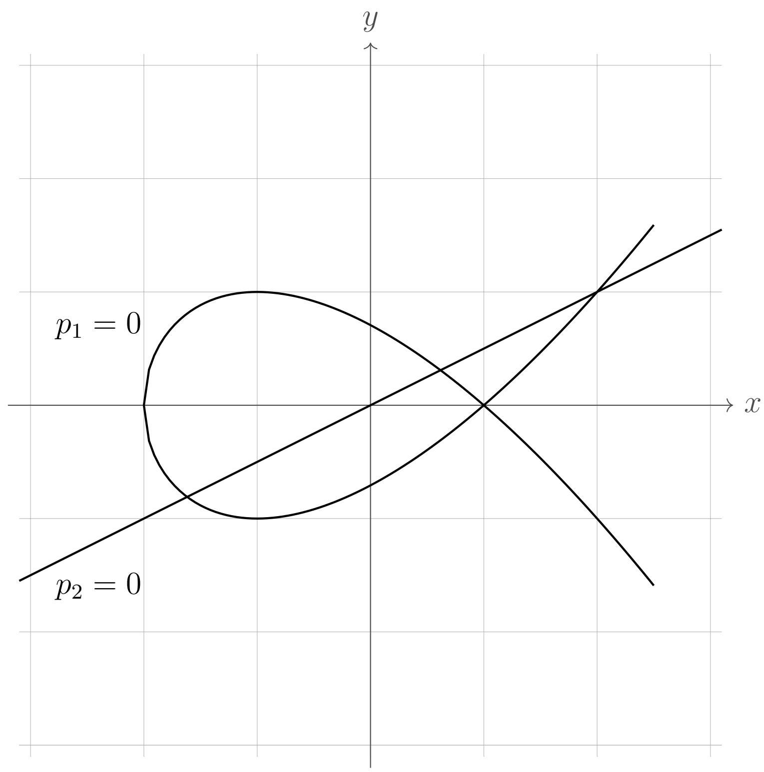
Consider the following constraints and their graphical depiction below. Give the test candidates for x resulting from these constraints *graphically* and label them appropriately.

$$c_1 : -x^2 + 2x > 0 \qquad c_2 : \frac{1}{2}x^2 - 1 \leq 0$$



9.) Cylindrical Algebraic Decomposition **8+4 Points**

i) Consider two polynomials p_1 and p_2 whose varieties are depicted in the picture below. Assume that we project onto the x axis. Identify every point that yields a cylinder boundary and state why it does so in the context of the definition of *delineability* from the lecture.



ii) One requirement of SMT compliancy (as defined in the lecture) is the generation of *infeasible subsets*. Consider a one-dimensional CAD. What is the basic idea to compute infeasible subsets once the CAD method has determined infeasibility? How could we attempt to get *small* infeasible subsets?