Satisfiability Checking - WS 2020/2021

Written Exam II

Tuesday, April 06, 2021

Forename and surname:	**Matriculation number:**
Forename Surename	000000
Your identifier: abcd **Your room number:** 2	

- Do not start the exam until we give the start signal.
- The duration of the exam is 120 minutes.
- Write your solution on empty sheets. You may use blank, lined or squared paper.
- Use a blue or black (permanent) pen only.
- Please write your name, matriculation number and a page number on every page.
- Please use a new page for every of the nine tasks and indicate which task is solved on this page.
- Please write clear and legible answers.
- Please clearly cross out parts you do *not* wish to be evaluated.
- If you have problems understanding a task, indicate this by a hand signal.
- You are not allowed to use auxiliary material except for a pen and a ruler. Cheating disqualifies from the exam.
- In case the upload via Moodle fails, please upload your solution to `https://gigamove.rz.rwth-aachen.de/` and send us an E-Mail to exam@ths.rwth-aachen.de with `2 000000` as subject and only the GigaMove link as body.
- In case there are any troubles during the exam, call us at `0241/123456`.

Task:	1.)	2.)	3.)	4.)	5.)	6.)	7.)	8.)	9.)	**Total**
Maximum score:	20	15	14	12	8	14	12	13	12	120
Reached score:										

Good luck!

1.) SAT Checking

(6+4)+4+6 Points

i) Consider the following propositional logic formula:

$$\begin{array}{llllll} & c_1 : (\neg a \vee e) & \wedge & c_2 : (\neg a \vee d \vee \neg e) & \wedge & c_3 : (\neg b \vee \neg c) \\ \wedge & c_4 : (\neg a \vee c \vee f) & \wedge & c_5 : (\neg a \vee \neg b \vee \neg f) & & \end{array}$$

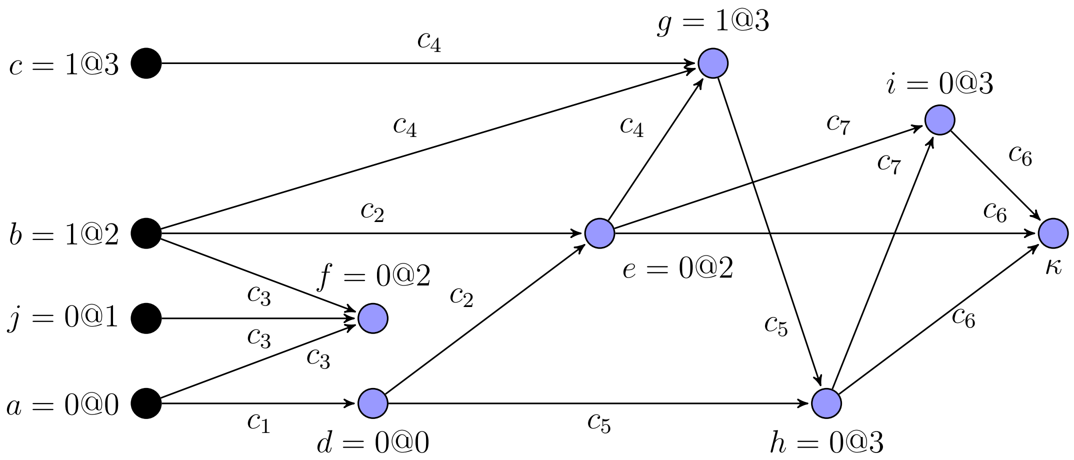
(a) **Apply the CDCL SAT-checking algorithm** to the given formula until the first conflict appears. When making a decision, take the lexicographically smallest unassigned variable first and assign **true** to it. Show the (partial) assignments after each decision level.

(b) Apply **conflict resolution** as presented in the lecture.

ii) Apply Tseitin transformation to the following formula:

$$a \vee (b \wedge (c \vee d))$$

iii) Consider the following implication graph:



a) What is the conflicting clause? Give its label **and** its literals.

b) Which clause is generated as result of conflict resoltion in CDCL? Which is the first UIP?

c) Give the node labels of all further unique implication points.

d) Which decisions from this implication graph were sufficient to cause this conflict? Give a **minimal** set of decisions.

2.) Equality Logic and Uninterpreted Functions 5+10

Points

i) Assume the following formula of equality logic with uninterpreted functions:

$$\varphi^{\mathbf{UF}} \quad := \quad x_1 = x_2 \land (F(x_1, x_3) \neq F(x_1, x_2) \lor F(x_1, x_3) = F(x_1, x_1))$$

Use Ackermann's reduction to construct an equality logic formula φ^E **without** uninterpreted functions that is satisfiability-equivalent to $\varphi^{\mathbf{UF}}$.

ii) Consider the following formula in equality logic:

$$\varphi^E \quad := \quad \big(\neg(\underbrace{x_1 = x_2}_{e_1}) \lor \underbrace{x_1 = x_3}_{e_2}\big) \land \big(\underbrace{x_3 = x_4}_{e_3} \lor \underbrace{x_4 = x_2}_{e_4}\big) \land \big(\neg(\underbrace{x_4 = x_5}_{e_5}) \lor \underbrace{x_1 = x_4}_{e_6}\big)$$

a) Construct the equality graph **with** polarity for φ^E.

b) Simplify the constructed equality graph and the formula using the method presented in the lecture.

c) Make the simplified equality graph without polarity chordal.

d) Construct the satisfiability-equivalent propositional logic formula for φ using the previous results from c) (without considering polarities).

3.) Fourier-Motzkin Variable Elimination **5+9 Points**

i) Assume that we apply Fourier-Motzkin variable elimination to eliminate **all** variables $x_1, ..., x_n$ from a set of non-strict inequalities from linear real arithmetic. Assume that the variables are eliminated in the order $x_n, ..., x_1$.

How can we generate a **satisfying assignment** for the variables if the constraint set turned out to be satisfiable? Start with the base case and continue by extending a partial assignment for a single variable at a time.

ii) Consider the following set of linear real-arithmetic constraints:

$$\begin{array}{lrcrcr} c_1: & x & - & 2y & \leq & -5 \\ c_2: & x & + & y & \leq & 2 \\ c_3: & -x & + & y & \leq & 2 \\ c_4: & -2x & - & 3y & \leq & 3 \end{array}$$

a) **Eliminate** y by applying the Fourier-Motzkin variable elimination.

b) Is the constraint set satisfiable? Either give a **satisfying assignment** or determine a **minimal infeasible subset**.

4.) Simplex **9+3 Points**

i) Apply one **pivoting step** and update the current assignment, where the variable order is $x_1 < x_2 < s_1 < s_2 < s_3 < s_4$ and where the current tableau, the bounds and the current assignment are given by:

	s_1	x_2
x_1	1	-1
s_2	1	-2
s_3	1	1
s_4	0	1

$$\begin{array}{lcc} s_1 & \geq & 3 \\ s_2 & \leq & -1 \\ s_3 & \geq & 2 \\ s_4 & \leq & 3/2 \end{array}$$

$$\begin{array}{ll} \alpha(x_1) & = 3 \\ \alpha(x_2) & = 0 \\ \alpha(s_1) & = 3 \\ \alpha(s_2) & = 3 \\ \alpha(s_3) & = 3 \\ \alpha(s_4) & = 0 \end{array}$$

ii) The simplex method terminates on the resulting tableau. Is the tableau satisfied or conflicting? Why? Give the satisfying assignment for the original variables or the slack variables corresponding to the infeasible subset.

5.) Integer Arithmetic **2+6 Points**

i) Is the Branch and Bound method complete on input problems, whose relaxation has a bounded solution set? Please argue why.

ii) We want to check the satisfiability of the linear integer arithmetic constraint set

$$P = \{y \leq x + 0.5, \quad y \leq -x + 1.5, \quad y \geq x - 0.5, \quad y \geq -x + 0.5\}$$

using the Branch and Bound method as presented in the lecture, with the heuristics to prefer x for splitting (if its value is non-integer) and handle lower branches first. Starting with the initial call with parameter P, please define the parameter sequence of all recursive calls until termination.

6.) Interval Constraint Propagation **4+10 Points**

i) Specify the formal rule for **dividing** an interval $A = [\underline{A}, \overline{A}] \subseteq \mathbb{R}^{>0}$, i.e. with $0 < \underline{A}$, by an arbitrary interval B with $0 \in B$ as presented in the lecture.

ii) Consider the constraints $c_1 : x^2 - 2 = y$ and $c_2 : 2 \cdot x \geq y + 2$ and initial intervals $x \in [-1, 1], y \in [-2, 2]$. Give **all contraction candidates**, **perform** contractions with all of them and argue which caused the largest relative contraction.

7.) Subtropical Satisfiability **6+6 Points**

i) Let $p(x, y)$ be the polynomial $-x^2 - xy + 1$. Which solutions can be found for the equation $p(x, y) = 0$ by the subtropical satisfiability method using the sample points $p(1, 1) < 0$ and $p(0, 0) > 0$?

ii) For the polynomial $p(x, y) = -x^2y - 3x + y$ and the separating hyperplane specified by $(n_x, n_y) = (-1, 1)$ and $b = 0$, use the subtropical satisfiability method to compute a solution for $p(x, y) > 0$. As in the lecture, please start with $a = 2$.

8.) Virtual Substitution **(5+3)+5 Points**

i) Assume we want to virtually substitute the test candidate $t = -\infty$ **for the variable** y in the constraint $x \cdot y^2 + 2 \cdot y^2 + x^2 + y + z^2 \cdot y < 0$ using one of the following rules:

- Substitute $-\infty$ for x in $b \cdot x + c < 0$:

$$\begin{array}{llllll} & (& b > 0 & & &) \\ \vee & (& b = 0 & \wedge & (c < 0 &) \end{array}$$

- Substitute $-\infty$ for x in $a \cdot x^2 + b \cdot x + c < 0$:

$$\begin{array}{llllllll} & (& a < 0 & & & & &) \\ \vee & (& a = 0 & \wedge & b > 0 & & &) \\ \vee & (& a = 0 & \wedge & b = 0 & \wedge & c < 0 &) \end{array}$$

a) Identify the appropriate substitution rule from above and apply virtual substitution. Normalize all polynomials by transforming them to sums of products.

b) Further simplify the result as far as possible. Use the result to state whether the formula is satisfiable.

ii) Give all test candidates of the following polynomial in variable x and their side conditions:

$$zx^2 + x^2y^2 + 2xy + z < 0$$

Simplify all expressions as far as possible by multiplying out all brackets.

9.) Cylindrical Algebraic Decomposition **12 Points**

Consider the polynomial $p = x^3 - 4x^2 + 3 \cdot x$.

Use the Cauchy bound $C = 5$ and the Sturm sequence of p:

$$\begin{aligned} p_0 &= x^3 - 4x^2 + 3 \cdot x \\ p_1 &= 3 \cdot x^2 - 8 \cdot x + 3 \\ p_2 &= {}^{14}\!/_{9} \cdot x - {}^{4}\!/_{3} \\ p_3 &= {}^{81}\!/_{49} \end{aligned}$$

to isolate all real roots of p with the method presented in the lecture, using interval midpoints for splitting. Give the table of sign changes for all relevant interval bounds. Describe and explain every split and give the resulting interval representations of the real roots.