Satisfiability Checking

25 The cylindrical algebraic decomposition method II

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1. What is a cylindrical algebraic decomposition?
2. Computing cylindrical algebraic decompositions for $\mathbb{R}$
3. Computing cylindrical algebraic decompositions for $\mathbb{R}^n$

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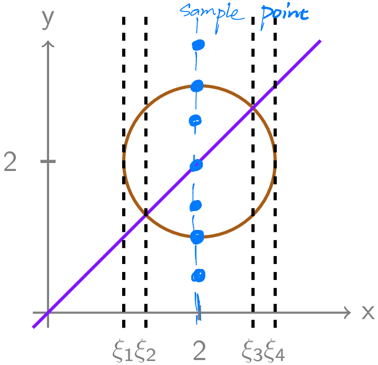
Reminder: CAD definition

$$P = \begin{pmatrix} (x-2)^2 + \\ (y-2)^2 - 1, \\ x - y \end{pmatrix}$$



The projected CAD cells in $\mathbb{R}$ are:

$(-\infty, \xi_1)$, $\{\xi_1\}$, (ξ_1, ξ_2), $\{\xi_2\}$, (ξ_2, ξ_3), $\{\xi_3\}$, (ξ_3, ξ_4), $\{\xi_4\}$, (ξ_4, ∞)

Reminder

A CAD for P is a

- decomposition of $\mathbb{R}^n$
- which is cylindrical,
- semi-algebraic,
- and its cells are P-sign-invariant.

Variable ordering

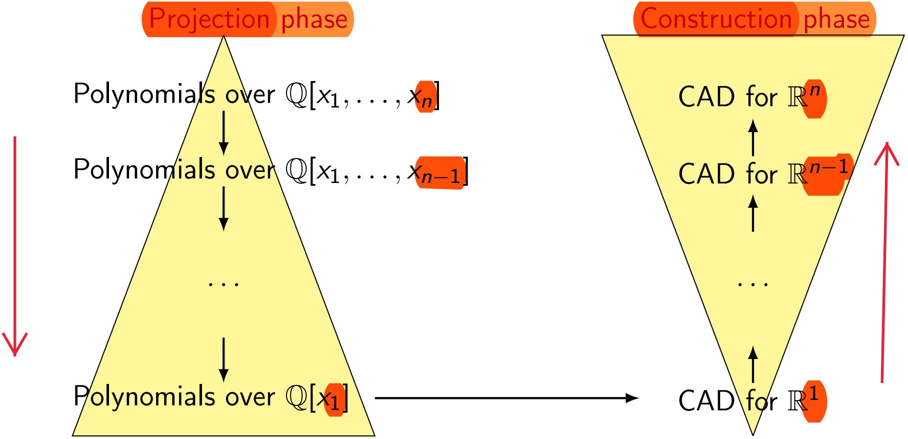
- Let $x_1, \ldots, x_n$ be variables.
- CAD assumes a static variable order.
- We will use $x_1 < x_2 < \ldots < x_n$.

- When we use other variables (e.g. x, y), we will explicitly fix the order.

先投射到x轴上

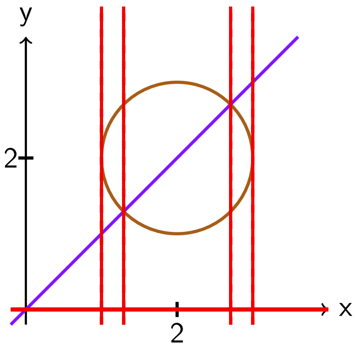
A CAD for a set P of polynomials from $\mathbb{Q}[x_1, \ldots, x_n]$ splits $\mathbb{R}^n$ into finitely many P-sign-invariant regions.

construction即是构建sample point的过程



Variable order: $x < y$

$$P = \begin{pmatrix} (x-2)^2 + \\ (y-2)^2 - 1, \\ \\ x - y \end{pmatrix}$$



P-delineable regions:

- $(2 - \frac{\sqrt{2}}{2}, 2 + \frac{\sqrt{2}}{2})$
- $\{2 - \frac{\sqrt{2}}{2}\}$, $\{2 + \frac{\sqrt{2}}{2}\}$
- $(1, 2 - \frac{\sqrt{2}}{2})$, $(2 + \frac{\sqrt{2}}{2}, 3)$
- $\{1\}$, $\{3\}$
- $(-\infty, 1)$, $(3, \infty)$

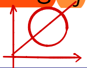
Delineability

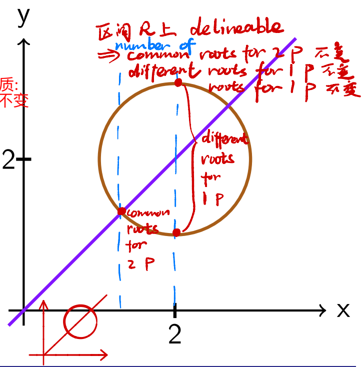
Let $R \subseteq \mathbb{R}^{n-1}$ be a region and
$P = \{p_1, \ldots, p_m\} \subset \mathbb{Q}[x_1, \ldots, x_n]$, where
$m \geq 1$ and $n \geq 2$.

即之前通过画线所划分的sign-invariant所对应的性质:
real roots在区间内连续变化, 根的数量和order保持不变

Intuition: If P is delineable on R then the real roots of P vary continuously over R, while maintaining their number and order.

For $p_i \in P$ and $a = (a_1, \ldots, a_{n-1}) \in R$, let $p_i(a, x_n)$ denote the result of substituting a_j for x_j for each $j \in \{1, \ldots, n-1\}$.





Definition

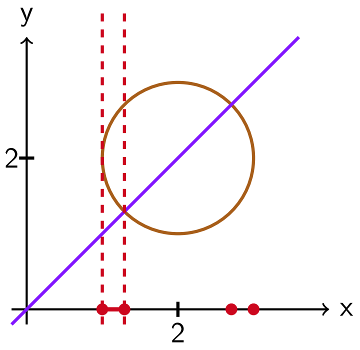
p(a, Xn)为用sample point a来代表变量的P-sign-invariant

P is delineable on R if for each $1 \leq i, j \leq m$ with $i \neq j$, for all $a \in R$

1. the number of roots of $p_i(a, x_n)$ is constant,
2. the number of different roots of $p_i(a, x_n)$ is constant, and
   对于 turnaround. 为2个相同root, different roots 为1
3. the number of common roots of $p_i(a, x_n)$ and $p_j(a, x_n)$ is constant.

CAD projection

Let $P = \{p_1, \ldots, p_m\} \subset \mathbb{Q}[x_1, \ldots, x_n]$ where $n \geq 2$ and $m \geq 1$.



Definition

A mapping

$$\mathrm{proj} : 2^{\mathbb{Q}[x_1,\ldots,x_n]} \longrightarrow 2^{\mathbb{Q}[x_1,\ldots,x_{n-1}]}$$

is called a CAD-Projection if any $\mathrm{proj}(P)$-sign-invariant region $R \subseteq \mathbb{R}^{n-1}$ is P-delineable.

Remarks

- Usually, $|\mathrm{proj}(P)| = |P|^2$. Thus, projecting recursively up to the univariate case is in $\mathcal{O}(|P|^{2^{n-1}})$.

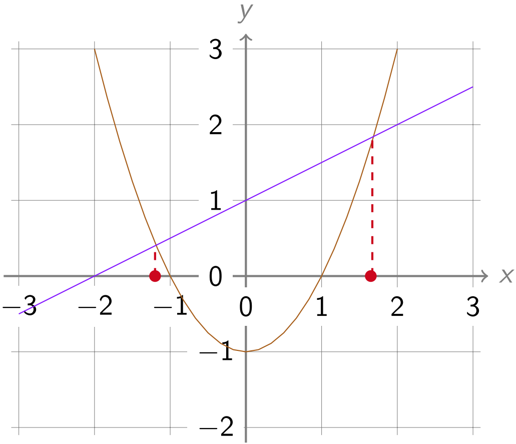
Where must be cylinder boundaries?

There are three types of projections, with the following informal role:

1. resultant of two polynomials: its roots cover (projections of) common roots of the different polynomials ("crossing points")
2. discriminant of a polynomial: whose roots cover (projections of) changes in the number and order of roots of a single polynomial ("turn arounds")
3. coefficients: whose roots cover (projections of) divergence points (at singularities of the polynomials)

CAD: Example

Variable oder: $x < y$



$-x^2 + y + 1 = 0$, $-x + 2y - 2 = 0$

The resultant of $-x^2 + y + 1$ and $-x + 2y - 2$ in y is $2x^2 - x - 4$, which has roots at $\frac{1}{4} \pm \frac{\sqrt{33}}{4}$.

$y = x^2 - 1$

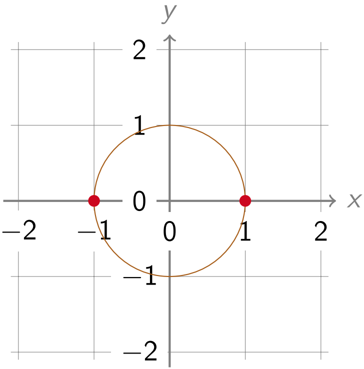
$y = \frac{x+2}{2}$

$x^2 - 1 = \frac{x+2}{2}$

$2x^2 - x - 4 = 0$

CAD: Example

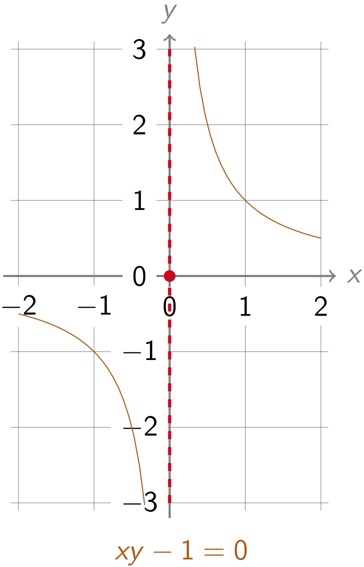
Variable oder: $x < y$



$x^2 + y^2 - 1 = 0$

The discriminant of $x^2 + y^2 - 1$ in y is $-4x^2 + 4$, which has two roots at -1 and $+1$.

Variable oder: $x < y$

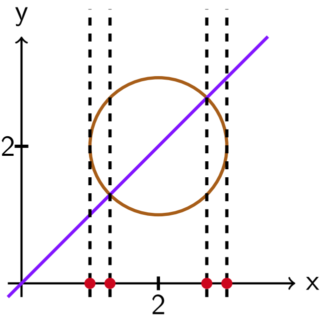


The leading coefficient of $xy - 1$ in y is x, which has a single root at 0.

CAD projection: Example

Variable order: $x < y$

$$P = \begin{pmatrix} (x-2)^2 + \\ (y-2)^2 - 1, \\ \\ x - y \end{pmatrix}$$



Projection ... and its real roots

$\mathrm{proj}(P) = \{x^2 - 4x + 3,$	discriminant of 1st poly
$1,$	discriminant of 2nd poly
$-4x + x^2 + \frac{7}{2},$	resultant of polys
$1,$	leading coeff of 1st poly
$-1\}$	leading coeff of 2nd poly

The CAD sample construction in a nutshell

$$
\begin{tikzcd}[row sep=large]
\to P_n \subseteq \mathbb{Q}[x_1,\ldots,x_n] \arrow[d, "\text{eliminate } x_n"'] & Z_n \subseteq \mathbb{R}^n \\
P_{n-1} \subseteq \mathbb{Q}[x_1,\ldots,x_{n-1}] \arrow[d, "\text{eliminate } x_{n-1}"'] & \vdots \arrow[u, "\text{extend } z \in Z_{n-1} \text{ using roots of } \{p(z) \mid p \in P_n\}"'] \\
\vdots \arrow[d, "\text{eliminate } x_2"'] & Z_2 \subseteq \mathbb{R}^2 \arrow[u, "\text{extend } z \in Z_2 \text{ using roots of } \{p(z) \mid p \in P_3\}"'] \\
P_1 \subseteq \mathbb{Q}[x_1] \arrow[r, bend right, "\text{roots of } P_1"'] & Z_1 \subseteq \mathbb{R} \arrow[u, "\text{extend } z \in Z_1 \text{ using roots of } \{p(z) \mid p \in P_2\}"']
\end{tikzcd}
$$

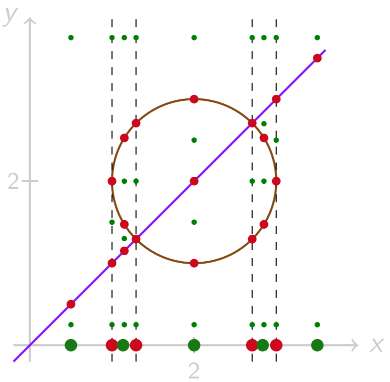
Projection | Construction

Samples are: 1. all roots, a sample 2. between each two neighboured roots, one sample 3. below the smallest and one 4. above the largest root

CAD sample construction ("lifting"): Example

Variable order: $x < y$

$$P = \begin{pmatrix} (x-2)^2 + (y-2)^2 - 1, \\ x - y \end{pmatrix}$$



One-dimensional samples for $\mathrm{proj}(P)$

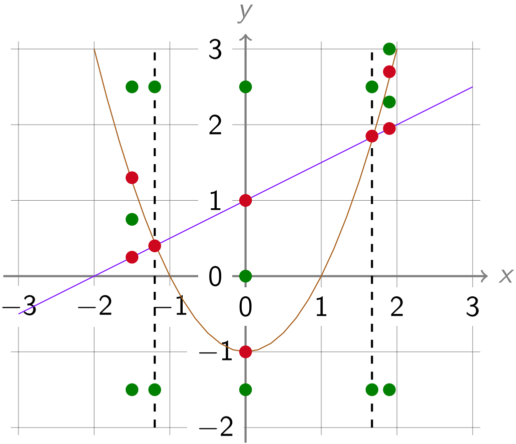
$\{1, 2 - \frac{\sqrt{2}}{2},\ 2 + \frac{\sqrt{2}}{2},\ 3\}$
$\{0.5,\ 1.135, 2,\ 2.835,\ 3.5\}$

Extending samples to $\mathbb{R}^2$ (example at $x = 2$)

- $(2-2)^2 + (y-2)^2 - 1 = (y-2)^2 - 1$ is now univariate and has roots at $y = 1$ and $y = 3$, yields samples $(2, 1)$ and $(2, 3)$,
- $2 - y$ has a root at $y = 2$, yields sample $(2, 2)$.

CAD: Example

Variable order: $x < y$

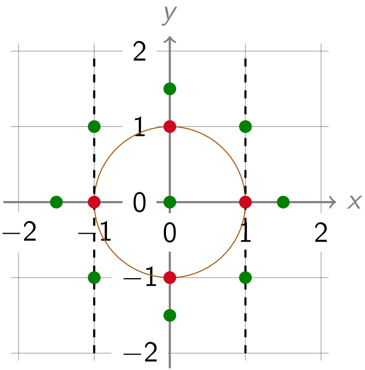


$-x^2 + y + 1 = 0$, $-x + 2y - 2 = 0$

21 cells

CAD: Example

Variable order: $x < y$

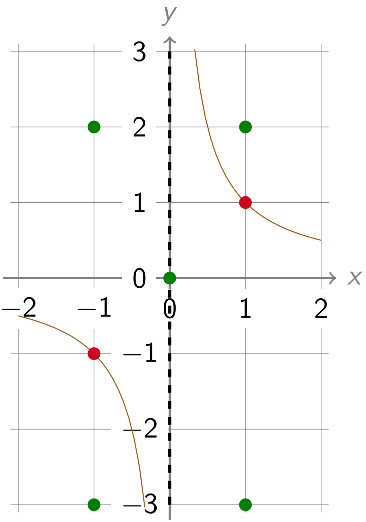


$x^2 + y^2 - 1 = 0$

13 cells

CAD: Example

Variable order: $x < y$



$xy - 1 = 0$

7 cells

Learning target

- What is a cylindrical algebraic decomposition for a set of polynomials?
- How to compute it for the univariate case?
- What is delineability?
- Given a graphical representation of the real roots of some polynomials, how to illustrate their CAD and the generated samples graphically?