Satisfiability Checking - WS 2022/2023

Written Exam I

Monday, February 27, 2023

Sample solution

1.) SAT Checking

3 + 4 + 3 + (4+1+2) points

i) In the clause $(a \vee \neg b \vee \neg c \vee d)$, which literal pairs are suited to be watched under the assigment $a = 0$, $b = 1$, $c = 0$ and d unassigned?

Solution: $\{a, \neg c\}$, $\{\neg b, \neg c\}$, $\{\neg c, d\}$

ii) Assume the following propositional logic formula in CNF:

$$(A \vee B \vee D) \wedge (C \vee \neg D) \wedge (\neg A \vee \neg C \vee \neg D) \wedge (A \vee \neg B \vee \neg C \vee D)$$

Apply the DPLL+CDCL algorithm until it detects either a conflict or a complete solution. Please specify each step and the intermediate assignment after its execution. For a decision, always take the smallest unassigned variable in the order $A < B < C < D$ and assign *false* to it.

Solution:

- There are no clauses with a single literal, thus propagating at decision level 0 does not yield any assignments.
- Decide $A = 0$.
- Propagating $A = 0$ yields no further assignments.
- Decide $B = 0$.
- Propagating $B = 0$ yields $D = 1$ from the first clause.
- Propagating $D = 1$ yields $C = 1$ from the second clause.

  We have a complete solution.

iii) Prove or disprove by providing a counterexample: For every propositional logic formula φ holds: φ is valid if and only if its Tseitin encoding is valid.

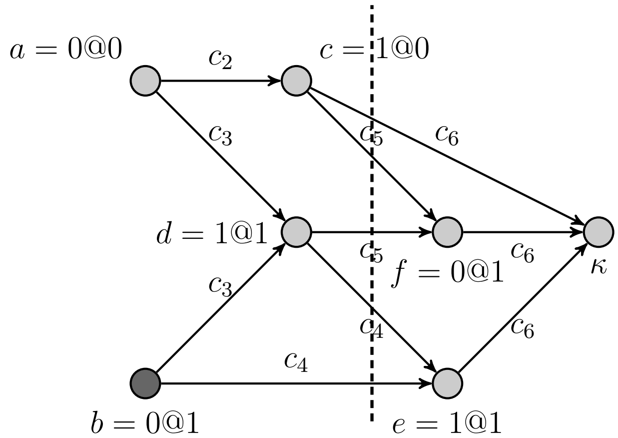
Solution:

This statement is false. Example: $\neg(a \wedge \neg a)$ is valid, but its Tseitin encoding is equivalent to $h_1 \wedge (h_1 \leftrightarrow \neg h_2) \wedge (h_2 \leftrightarrow (a \wedge \neg a))$, which is not satisfied for any assignment with $h_1 = 0$.

iv) Consider the given trail and implication graph for the propositional logic formula

$$\begin{array}{llllll} & c_1 : (\neg a) & \wedge & c_2 : (a \vee c) & \wedge & c_3 : (a \vee b \vee d) \\ \wedge & c_4 : (b \vee \neg d \vee e) & \wedge & c_5 : (\neg c \vee \neg d \vee \neg f) & \wedge & c_6 : (\neg c \vee \neg e \vee f) \end{array}$$

DL0: $\neg a : c_1$, $c : c_2$ DL1: $\neg b : nil$, $d : c_3$, $e : c_4$, $\neg f : c_5$



a) The clause c_6 is conflicting. Perform two resolution steps of the Boolean conflict resolution as presented in the lecture. For each step give the current conflicting clause, the used antecedent clause and the new conflicting clause.

b) Depict the cut corresponding to the resulting clause graphically in the above implication graph.

c) Is this conflict resolution finished after the two steps? Please explain why.

Solution:

The initial conflicting clause is $c_6 : (\neg c \vee \neg e \vee f)$. It is not asserting as both e and f are from $DL1$. The most recently assigned literal in c_6 is $\neg f$. The antecedent of c_6 is c_5. We resolve c_6 and c_5 with respect to f.

$$\frac{c_6 : (\neg c \vee \neg e \vee f) \quad c_5 : (\neg c \vee \neg d \vee \neg f)}{c_7 : (\neg c \vee \neg d \vee \neg e)}$$

The resulting conflicting clause c_7 is not asserting as both d and e are from $DL1$. The most recently assigned literal in c_7 is $\neg e$ with antecedent c_4. We resolve with c_7 and c_4 with respect to e.

$$\frac{c_7 : (\neg c \vee \neg d \vee \neg e) \quad c_4 : (b \vee \neg d \vee e)}{c_8 (b \vee \neg c \vee \neg d)}$$

The corresponding cut for the resulting conflicting clause c_8 is depicted in the implication graph above. The conflict resolution is not yet finished as the clause is not asserting.

2.) Equality Logic and Uninterpreted Functions **4 + 5 + (2 + 3) points**

i) Apply *lazy* SMT solving for equality logic as presented in the lecture to the following conjunction of equation and disequations, considering equations from left to right.

$$v = y \wedge u \neq v \wedge y = z \wedge y = v \wedge v = z \wedge x = v$$

Please specify the initial partition, each execution step and the partition after the step.

Solution:

Initial partition: $\{\{x\}, \{y\}, \{z\}, \{v\}, \{u\}\}$,
Merging for $v = y$: $\{\{x\}, \{y, v\}, \{z\}, \{u\}\}$,
Merging for $y = z$: $\{\{x\}, \{y, z, v\}, \{u\}\}$,
Merging for $y = v$: $\{\{x\}, \{y, z, v\}, \{u\}\}$, (no change)
Merging for $v = z$: $\{\{x\}, \{y, z, v\}, \{u\}\}$, (no change)
Merging for $x = v$: $\{\{x, y, z, v\}, \{u\}\}$.

ii) Consider the following formula in *equality logic with uninterpreted functions.* Apply *Ackermann's reduction* to eliminate the function symbols and to obtain a satisfiability-equivalent formula in *equality logic.*

$$\varphi : (F(a) = G(a, b)) \wedge (F(F(a)) \neq G(b, F(b)))$$

Solution:

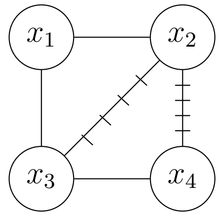
We replace each occurence of $F(a)$ by a new variable f_a, of $F(b)$ by f_b. Then we replace $F(f_a)$ by f_{f_a}, $G(a, b)$ by g_1 and $G(b, f_b)$ by g_2. We get

$$\varphi : (f_a = g_1) \wedge (f_{f_a} \neq g_2) \wedge \varphi_{cong}$$

with

$$\begin{aligned} \varphi_{cong} : &((a = b) \rightarrow (f_a = f_b))\wedge \\ &((a = f_a) \rightarrow (f_a = f_{f_a}))\wedge \\ &((b = f_a) \rightarrow (f_b = f_{f_a}))\wedge \\ &((a = b \wedge b = f_b) \rightarrow (g_1 = g_2)) \end{aligned}$$

iii) Consider the following *E-graph with polarity*.



a) Give all simple contradictory cycles contained in the depicted graph.

b) Give a satisfiable formula φ_1 and an unsatisfiable formula φ_2 in equality logic which both produce the depicted E-graph.

Solution:

a) x_1, x_2, x_3 and x_1, x_2, x_4, x_3
b) E.g. $\varphi_1 : (x_1 = x_2) \vee (x_1 = x_3) \vee (x_3 = x_4) \vee (x_2 \neq x_3) \vee (x_2 \neq x_4)$ and φ_2 the same with conjunctions instead of disjunctions.

3.) Fourier-Motzkin Variable Elimination 6+(2+2) points

i) Assume the following constraint set:

$$\{2x - y \leq -8, \quad 2x + y \leq 8, \quad -2x - y \leq -8, \quad -2x + y \leq 8\}$$

a) Please apply the method of Fourier-Motzkin to eliminate first y and then x. Specify the result after each elimination step.

b) Please specify a solution using the elimination result. Is this solution unique?

Solution:

a) Eliminate y:
Bring y to one-hand side:
$\{c_1 : 2x + 8 \leq y, \quad c_2 : y \leq -2x + 8, \quad c_3 : -2x + 8 \leq y, \quad c_4 : y \leq 2x + 8\}$
Combining lower-upper bounds on y:

- (c_1, c_2): $2x + 8 \leq -2x + 8$, i.e. $4x \leq 0$
- (c_1, c_4): $2x + 8 \leq 2x + 8$, i.e. $0 \leq 0$
- (c_3, c_2): $-2x + 8 \leq -2x + 8$, i.e. $0 \leq 0$
- (c_3, c_4): $-2x + 8 \leq 2x + 8$, i.e. $0 \leq 4x$

Result after eliminating y: $\{0 \leq 0, \quad 4x \leq 0, \quad 0 \leq 4x\}$

Eliminate x:
Bring x to one-hand side: $\{c_5 : x \leq 0, \quad c_6 : 0 \leq x\}$
Combining lower-upper bounds on x:

- (c_6, c_5): $0 \leq 0$

Result after eliminating x: $\{0 \leq 0\}$

b) After eliminating y, from $x \leq 0 \leq x$ we derive $x = 0$. Substituting $x = 0$ into the bounds on y we get $8 \leq y \leq 8$, yielding $y = 8$ Thus $x = 0$ and $y = 8$ is the only solution.

ii) For $n \in \{1, 2, \ldots, 9, 10\}$, consider the linear integer systems

$$S_n = \{-n \cdot x \leq -1, \ n \cdot x \leq 3\}.$$

(a) For which values of $n \in \{1, 2, \ldots, 9, 10\}$ would the Fourier-Motzkin method answer "satisfiable" on the input S_n?

(b) The Fourier-Motzkin variable elimination is *not* correct on problems of linear *integer* arithmetic (LIA). Give all values of $n \in \{1, 2, \ldots, 9, 10\}$, for which the Fourier-Motzkin method would give the correct result (i.e. sat, when S_n is satisfiable with an *integer solution* or unsat, when S_n has no such solution).

Solution:

a) All

b) $n \in \{1, 2, 3\}$

4.) Simplex

2+4+5+4+2 points

i) Denoting original variables as x_i and slack variables as s_i, assume the following simplex tableau and slack variable bounds, with the current values of the variables given in square brackets:

	s_2 $[-1]$	s_0 $[1]$	x_2 $[0]$	
x_1 $[1]$	-3	-2	-2	$s_0 \leq 1$
s_1 $[1]$	0	1	-1	$s_1 \leq 0$
x_0 $[0]$	-1	-1	-1	$s_2 \leq -1$

The basic variable s_1 violates its bound. Please state for each non-basic variable whether it is suitable for pivoting with s_1 or not (no explanation required).

Solution: s_2: no, s_0: yes, x_2: yes

Note that s_1 needs to be decreased.

- s_2 is not suitable: it has a coefficient zero in the row of s_1.
- s_0 is suitable: it has a positive coefficient in the row of s_1, thus suitability requires that it is not on its lower bound, which is the case (it has no lower bound).
- x_2 is suitable: its coefficient is non-zero and it has no bounds (because it is an original variable).

ii) Apply the simplex method to the following constraint set until termination:

$$
\begin{array}{ll}
s_0 = -1x_1 & s_0 \leq -2 \\
s_1 = -3x_0 - 2x_1 & s_1 \leq 3
\end{array}
$$

Please specify the simplex tableau and the assignment initially and after each pivot step. When choosing pivot variables, use the order $x_0 \prec x_1 \prec s_0 \prec s_1$ and take the smallest possible variable.

Solution:

Initially the tableau is as follows:

	$x_0[0]$	$x_1[0]$
$s_0[0]$	0	-1
$s_1[0]$	-3	-2

We pivot s_0 with x_1, yielding the tableau:

	$x_0[0]$	$s_0[-2]$
x_1 $[\ \ 2]$	0	-1
s_1 $[-4]$	-3	2

All non-basic variables satisfy their bounds, thus simplex terminates with reporting satisfiability.

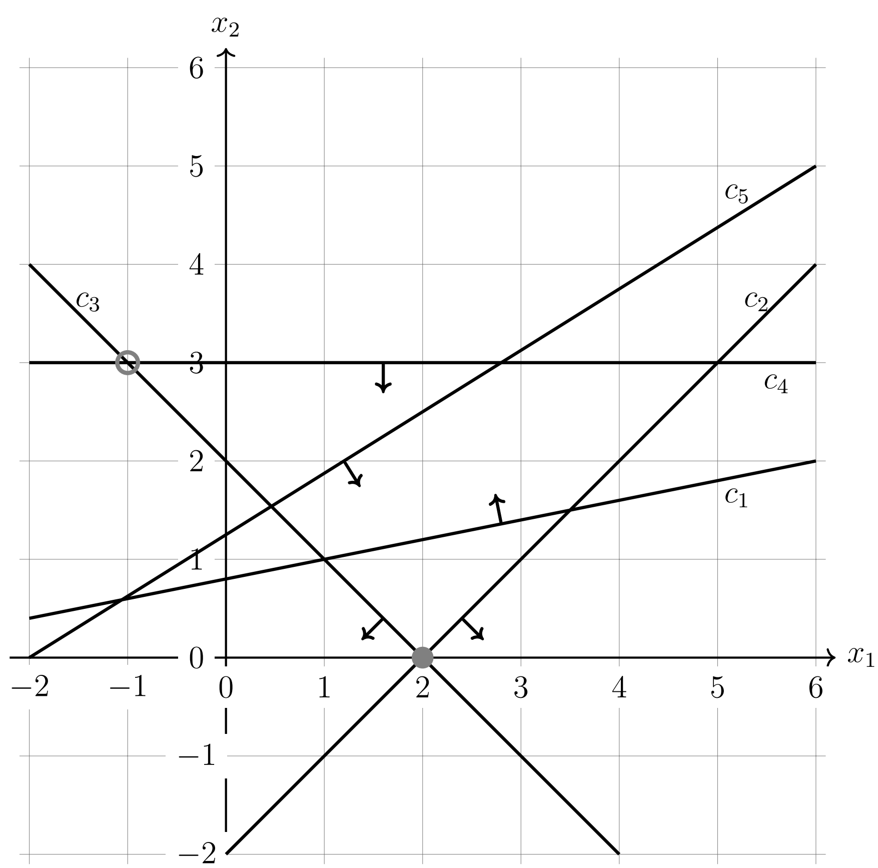
iii) Consider the following tableau (the current values of the variables are given in square brackets) with some information missing. Fill out the missing fields, assuming that (1) both basic slack variables violate their bounds and (2) s_3 is the only variable suitable for pivoting with the basic variable chosen by Bland's rule. Assume the variable ordering $x_1 \prec x_2 \prec s_1 \prec s_2 \prec s_3 \prec s_4$ preferring the smallest variable for Bland's rule.

	s_1 [3]	s_3 [-1]
s_2 [1]	1	2
x_2 [-6]	-2	0
s_4 [10]	3	-1
x_1 [7]	2	-1

s_1	$\geq$	$\boxed{3}$
s_2	$\leq$	-1
s_3	$\boxed{\leq}$	$\boxed{-1}$
s_4	$\boxed{\leq}$	3

Solution: see above

iv) Consider the following system of linear inequations $\{c_1, \ldots, c_5\}$. The point marked at at $(2, 0)$ corresponds to a state of the simplex algorithm on the system.



By s_i we denote the slack variable that stems from the linear inequation c_i.

Mark the assignment that is obtained after one pivot step according to Bland's rule. Assume the variable ordering $x_1 \prec x_2 \prec s_1 \prec s_2 \prec s_3 \prec s_4$ preferring the smallest variable for Bland's rule.

Solution: See non-filled circle above.

v) Give all possible infeasible subsets of the following conflicting tableau (the current values of the variables are given in square brackets). Denote the constraints corresponding to $s_1, \ldots, s_4$ by $c_1, \ldots, c_4$.

	s_1 [0]	s_2 [0]
x_1 [0]	-1	0
x_2 [0]	0	-1
s_3 [0]	0	1
s_4 [0]	1	1

$$\begin{aligned} s_1 &\geq 0 \\ s_2 &\leq 0 \\ s_3 &\geq 1 \\ s_4 &\leq -1 \end{aligned}$$

Solution: $\{c_2, c_3\}$

5.) Linear Integer Arithmetic **7 points**

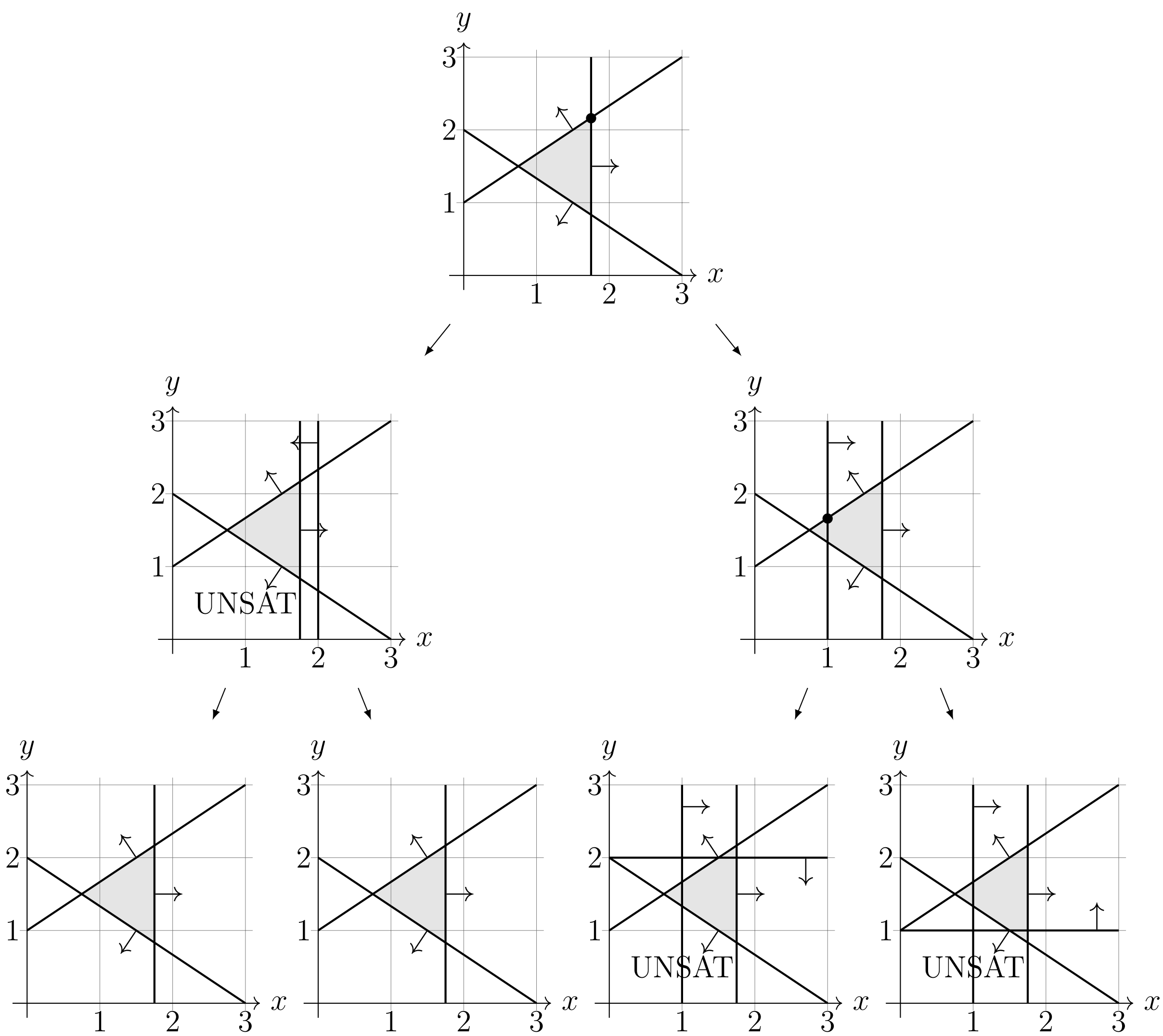
Consider the example depicted below. Execute the *Branch&Bound* algorithm until its termination.

Use the following assumptions:

- The real solution (x^*, y^*) found by simplex will always be the *lexicographically largest* among all solutions (i.e. for all solutions (x, y): $x^* \geq x \vee (x^* = x \wedge y^* \geq y)$).
- x is preferred over y for branching.

Depict each call of the recursive Branch&Bound method in a new plot (there might be more plots than necessary). Mark the real solution and sketch the additional constraints used for branching. Indicate if a call has no real solution by writing "UNSAT".

Solution:



6.) Interval Constraint Propagation **6 + 10 points**

i) Contract the domain $x \in A = [1;6]$ with the help of the univariate interval Newton method from the lecture using the constraint $-x^2 - 2x - 4 = 0$ and sample point $s = 4$. Please write down the computations and the resulting contracted domain for x.

Solution:

- $f(s) = -1 \cdot 4^2 - 2 \cdot 4 - 4 = -28$
- $f'(A) = [2;2] \cdot [-1;-1] \cdot [1;6] + [-2;-2] = [-14;-4]$
- $A_{new} = s - \frac{f(s)}{f'(A)} = [-3;2]$
- Intersecting the previous interval domain A of x with this new interval A_{new} yields $[1;6] \cap [-3;2] = [1;2]$.

ii) Consider the constraints $c_1 : x - y = 0$ and $c_2 : x - \frac{1}{2}y^2 = 0$ and the initial intervals $x \in [-1,1]$, $y \in [-1,1]$.

a) For each contraction candidate, give the relative contraction when applying contraction method I once.

b) Give a sequence of contraction candidates which can be repeated infinitely so that each contraction has a gain strictly greater than zero.

Solution: The CCs are $(c_1, x), (c_1, y), (c_2, x), (c_2, y)$.

(c_1, x): $x \in I_x \cap I_y = I_x$. Gain: 0

(c_1, y): $y \in I_y \cap I_x = I_y$. Gain: 0

(c_2, x): $x \in I_x \cap (0.5 I_y^2) = [0; 0.5]$. Gain: 0.75

(c_2, y): $y \in I_y \cap \sqrt{2 \cdot I_x} = I_y$. Gain: 0

b) $((c_2, x), (c_1, y))$ is the only solution

7.) Subtropical Satisfiability **4+3+2+4 points**

i) Please specify a real value n_x such that the direction $(n_x, 1) \in \mathbb{R}^2$ is suited to separate a positive frame point from all the other frame points for the constraint

$$2x^1y^3 - 2x^5y^4 - 5x^4y^1 - 2x^5y^5 > 0 \; .$$

Solution: Any value $n_x < -0.5$ satisfies the requirement.

ii) Give the subtropical encoding for the satisfiability of $x^2y - 2xy > 0$ according to the lecture.

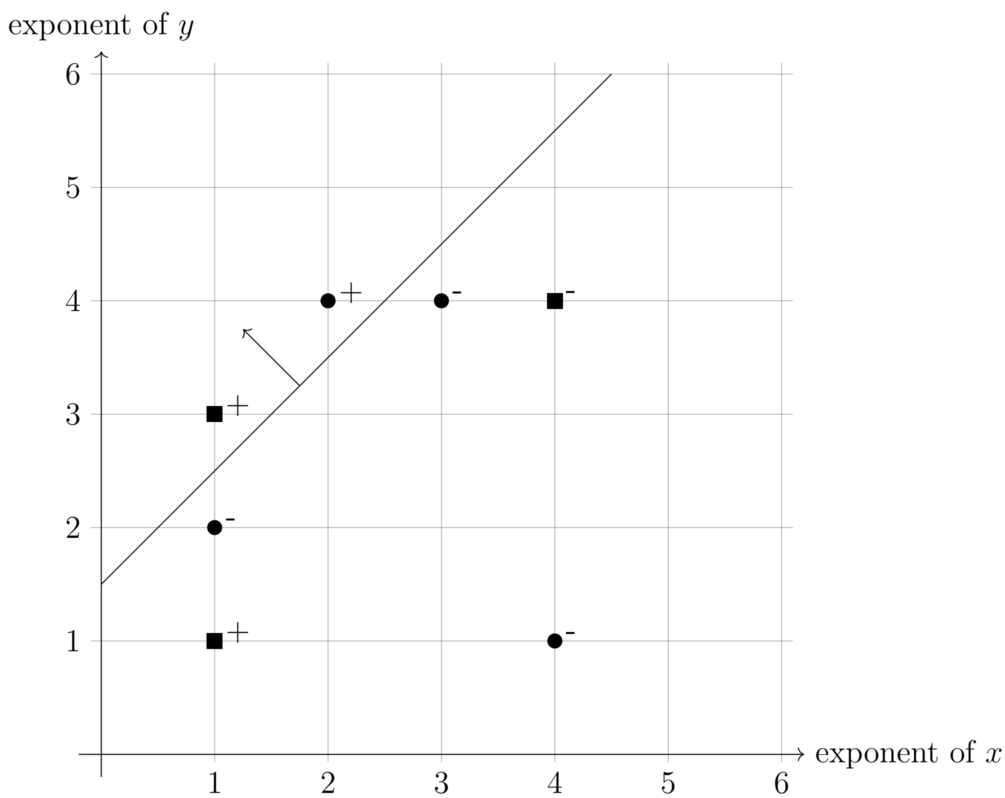
Solution: The only positive frame point is $(2, 1)$, the only negative one is $(1, 1)$. We encode that there is a hyperplane $H : nx^T = b$ separating a positive frame point from the others using variables n_x, n_y, b that encode this hyperplane:

$$(n_x \cdot 2 + n_y \cdot 1 > b \wedge n_x \cdot 1 + n_y \cdot 1 < b)$$

iii) The following image depicts the frame points of two polynomials $p_1(x, y)$ (circles) and $p_2(x, y)$ (squares).

Depict a suitable separating hyperplane and its normal vector $n = (n_x, n_y) \in \mathbb{Z}^2$ such that there exists an $a \in \mathbb{R}_{>0}$ with $p_1(a^{n_x}, a^{n_y}) > 0$ and $p_2(a^{n_x}, a^{n_y}) > 0$.

Solution:



iv) Consider the polynomial $p_d(x, y) := -xy + 3x^{2d}y - 5x^2y^d$ for $d \in \mathbb{N}$.

Give the set of all values for d such that there exists a hyperplane with normal vector $n = (1, 1)$ that separates a positive frame point of p_d from all other frame points of p_d.

Solution: $[2; \infty]$

8.) Virtual Substitution **4+3+5 points**

i) Please specify a test candidate different from $-\infty$ that is generated for x by at least two of the following constraints:

$$x + 2 = 0 \quad x + 2 < 0 \quad x + 2 > 0$$

Solution: $-2 + \epsilon$

Test candidates with not-trivially–false side conditions:

- $x + 2 = 0$: $-\infty$ and -2
- $x + 2 < 0$: $-\infty$ and $-2 + \epsilon$
- $x + 2 > 0$: $-\infty$ and $-2 + \epsilon$

Thus the only test candidate different from $-\infty$ that is generated by at least two of the constraints is $-2 + \epsilon$.

ii) Let $\exists x.\ \exists y.\ \varphi$ be a formula over the variables x, y. Assume that for the elimination of y via virtual substitution, the set of test candidates according to the lecture is given as

Test candidate		Side condition
$-\infty$	, if	true
$\frac{1}{x}$	, if	$x \neq 0$

Use the virtual substitution method from the lecture to eliminate y from the above formula, without applying the substitution rules. That means you may use $\varphi[t//y]$ to denote the sub-formula after substituting a test candidate t for y in φ.

Solution: $\varphi[-\infty//y] \vee (\varphi[\frac{1}{x}//y] \wedge x \neq 0)$

iii) Design a virtual substitution rule for $(b \cdot x + c \leq 0)[e - \varepsilon//x]$ (where b and c are coefficients not containing x) for some test candidate $e - \varepsilon$.

You may use $C[e//x]$ for any constraint C in your answer without applying the virtual substitution rule for it.

Note that the test candidate contains $-\varepsilon$, *not* $+\varepsilon$.

Solution:

$$\begin{array}{lllll} & (& (bx + c < 0)[e//x] & &) \\ \vee & (& (bx + c = 0)[e//x] & \wedge \quad (b > 0)[e//x] &) \\ \vee & (& b = 0 & \wedge \quad c = 0 &) \end{array}$$

9.) Cylindrical Algebraic Decomposition **4+4+6 points**

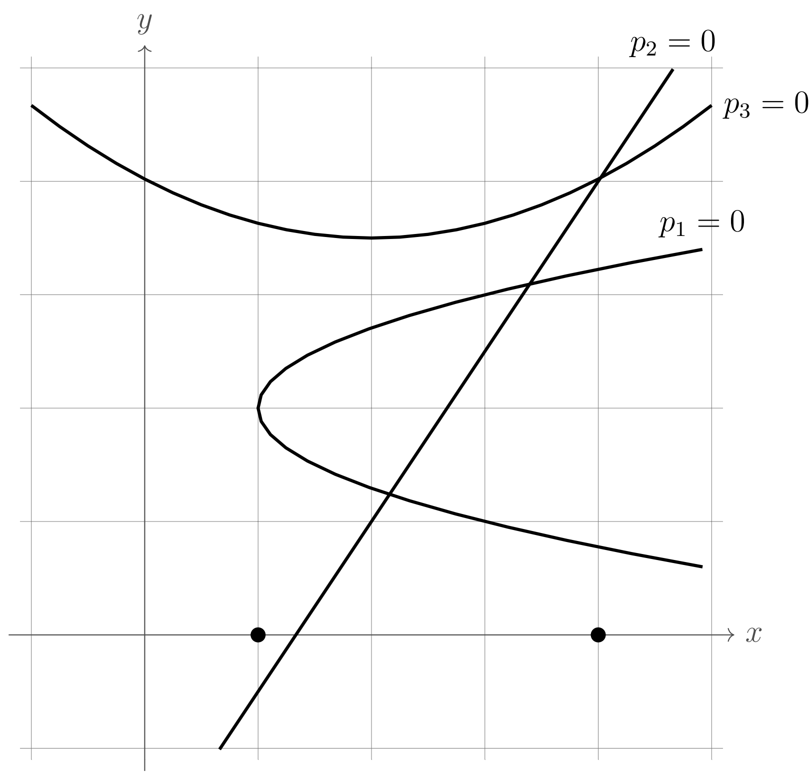
i) Assume the variable order $x < y$. Give the maximal interval in the x-dimension that contains 0 and on which $P = \{-9x + y - 1,\ -8x + y + 4\} \subset \mathbb{Z}[y][x]$ is delineable.

Solution: $(-5; \infty)$

We need to determine the point(s) in the x-dimension, over which the number or order of the real roots of the polynomials change. Note that the two polynomials are linearly independent and linear in the largest variable y. Thus both have exactly one real root for each fixed x-value. The number/order of the roots changes when both polynomials are zero at the same x-value. The two polynomials have a common root if $-9x + y - 1 = 0$ and $-8x + y + 4 = 0$. We use the method of Gauß to eliminate y: From $-9x+y-1 = 0$ we get $y = 9x+1$. Substituting this into $-8x+y+4 = 0$ results in $x = \frac{-1--4}{-8+9} = -5$. Thus the polynomials have exactly one common root at $x = -5$. The delineable regions are $[-5; -5]$ and all regions that do not contain -5. (Note that open intervals do not contain their bounds.)

Thus the answer is $(-5; \infty)$.

ii) Consider the following varieties (sets of zeros) of some polynomials p_1, p_2 and p_3:



The dots on the x-axis denote roots of some univariate polynomials in x generated by the CAD method during the projection of y with p_1, p_2, p_3 as input. Which are they?

Write $res(p_a, p_b)$ for the resultant of polynomials p_a and p_b, $disc(p)$ for the discriminant of p and $ldcf(p)$ for the leading coefficient of p.

Solution: $\{disc(p_1), res(p_2, p_3)\}$

iii) Isolate all real zeros of the univariate polynomial plotted below in the interval $(-2, 2)$ with the method presented in the lecture, using interval midpoints for splitting. You can read off all needed information from the plots below. Depict all resulting *isolating* intervals in the picture below.

Solution: