Satisfiability Checking

14 The simplex algorithm

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Satisfiability with general simplex

- The simplex method was originally designed for solving linear programming problems, i.e., to find a solution for a set of linear real-arithmetic constraints that is optimal with respect to an objective function.
- We are only interested in the feasibility problem (=satisfiability problem), which is solved in the first phase of the simplex method (we learn a variant called general simplex). In this lecture we do not handle the second phase for optimization.
- Assumptions:

  不等式 weak inequality

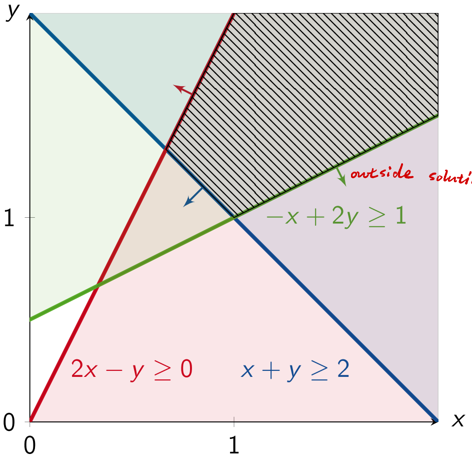
  1. no equalities $t_1 = t_2$ (transform into $t_1 \leq t_2 \wedge t_1 \geq t_2$)
  2. no strict inequalities (simplex can be exteded to strict inequalities but it is a 无<, > bit involved and we do not handle that case in this lecture)
  3. no disequalities $t_1 \neq t_2$ (needs strict inequalities: case split on $t_1 < t_2 \vee t_1 > t_2$)

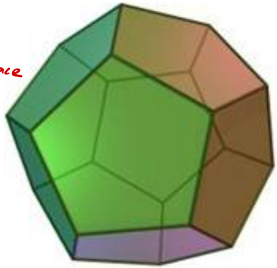
inequality: 即不等号, weak inequality">", "<"   strict inequality "≤", "≥"

disequality: 即等式不相等, "≠"

Geometric view

Geometrically, the solution set of
a conjunction of non-strict linear real-arithmetic constraints
is a (possibly empty) convex polyhedron. 凸多面体





Problem statement

- Input: m linear real-arithmetic constraints $c_1, \ldots, c_m$ of the form

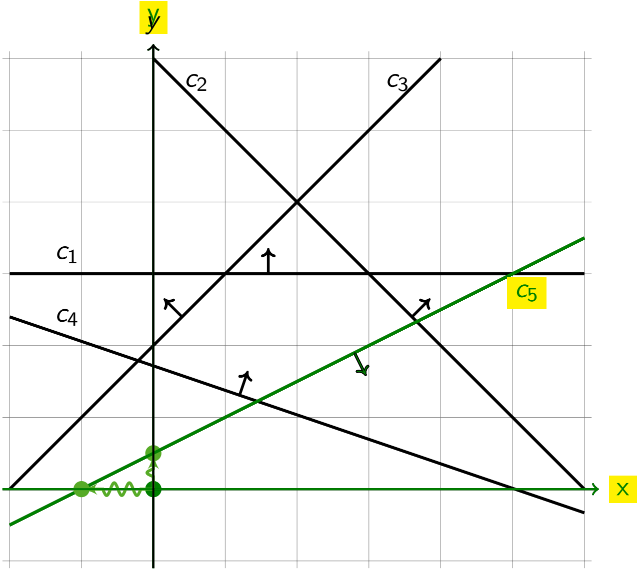
$$c_i : \qquad \sum_{j=1}^{n} a_{ij} x_j \ \sim_i \ b_i$$

  using

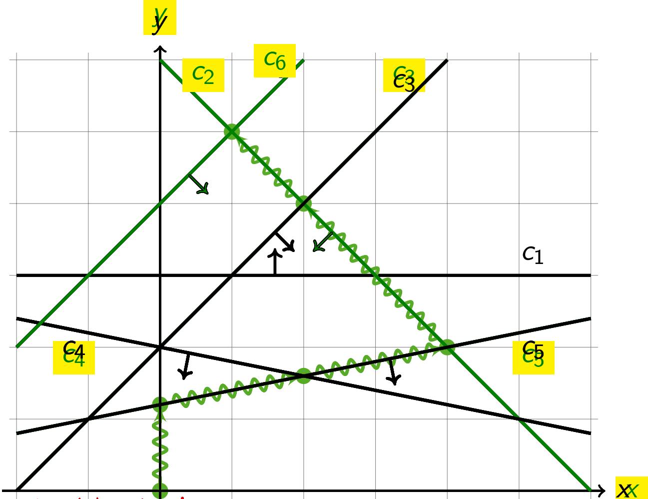
  - n real-valued variables $x_1, \ldots, x_n$,
  - rational constants $a_{ij}, b_i \in \mathbb{Q}$ and
  - a comparison predicate $\sim_i \in \{\leq, \geq\}$.

- Problem: Decide whether the conjunction $\bigwedge_{i=1}^{m} c_i$ of the contraints is satisfiable, i.e., whether there exist real values for $x_1, \ldots, x_n$ which evaluate all constraints to true.
- Note: no $=, >, <, \neq$!

Geometric view (SAT example)



SAT



UNSAT

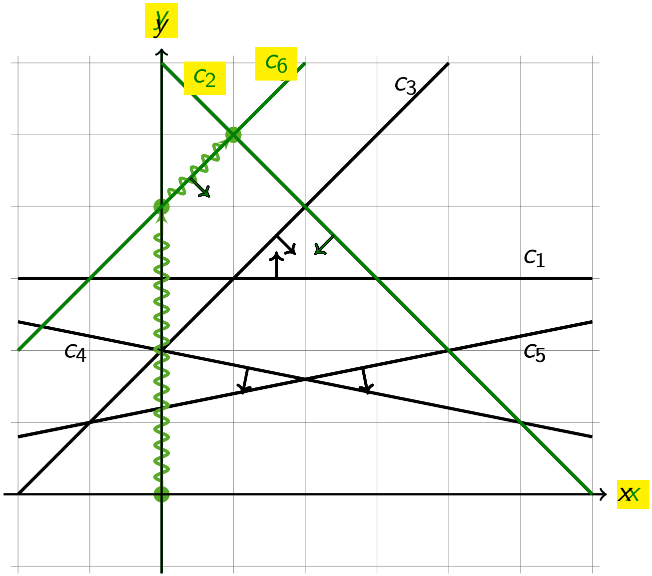
移动规则：保持在一个 hyperplane 上移动（在线上）

一个 hyperplane 不变

$\boxed{\sum a_{ij} x_j = b}$ 是一个 hyperplane

$\sum a_{ij} b_j \le b$ 是 half plane

Geometric view (UNSAT example)



UNSAT

Transformation to general form

- We introduce fresh additional or slack variables $s_1, \ldots, s_m$ and transform

$$\sum_j a_{ij} x_j \quad \sim_i \quad b_i \qquad \sim_i \in \{\geq, \leq\}$$

to

$$\sum_j a_{ij} x_j \quad - \quad s_i \quad = \quad 0 \quad \text{and} \quad s_i \quad \sim_i \quad b_i$$

Definition (General form)

$$A \cdot \begin{pmatrix} \vec{x} \\ \vec{s} \end{pmatrix} = 0 \quad \text{and} \quad \bigwedge_{i=1}^{m} \ell_i \leq s_i \leq u_i \quad \ell_i \in \mathbb{Q} \cup \{-\infty\},\ u_i \in \mathbb{Q} \cup \{+\infty\}$$

Note: A is now an $m \times (n+m)$ matrix due to the additional variables.

Example 1

Convert $x + y \geq 2$!

Result:
$x + y - s_1 = 0$
$s_1 \geq 2$

It is common to keep the conjunctions implicit

Example 2

Convert:

$$\begin{array}{rrl} x & +y & \geq 2 \\ 2x & -y & \geq 0 \\ -x & +2y & \geq 1 \end{array}$$

Result:

$$\begin{array}{rrrl} x & +y & -s_1 & = 0 \\ 2x & -y & -s_2 & = 0 \\ -x & +2y & -s_3 & = 0 \\ & & s_1 & \geq 2 \\ & & s_2 & \geq 0 \\ & & s_3 & \geq 1 \end{array}$$

Recall the general form: $A \cdot \begin{pmatrix} \vec{x} \\ \vec{s} \end{pmatrix} = 0$ and $\bigwedge_{i=1}^{m} \ell_i \leq s_i \leq u_i$

Matrix A:

$$\begin{array}{c} \begin{array}{ccccc} x & y & s_1 & s_2 & s_3 \end{array} \\ \begin{pmatrix} 1 & 1 & -1 & 0 & 0 \\ 2 & -1 & 0 & -1 & 0 \\ -1 & 2 & 0 & 0 & -1 \end{pmatrix} \end{array}$$

The diagonal part is inherent to the general form.

Simplex tableau:

	x	y
s_1	1	1
s_2	2	-1
s_3	-1	2

The tableau

- The tableaux changes throughout the algorithm, but maintains its $m \times n$ structure
- Distinguish basic (also called dependent) and non-basic variables

$$\begin{array}{cc} & \begin{array}{cc} x & y \end{array} \\ \begin{array}{c} s_1 \\ s_2 \\ s_3 \end{array} & \begin{pmatrix} 1 & 1 \\ 2 & -1 \\ -1 & 2 \end{pmatrix} \end{array}$$

Non-basic variables

Basic variables

Notation:

$\mathcal{B}$	the set of basic variables
$\mathcal{N}$	the set of non-basic variables

- Initially, basic variables = the additional variables.
- The initial tableaux is simply a different notation for the system

$$\bigwedge_{s_i \in \mathcal{B}} \left(s_i = \sum_{x_j \in \mathcal{N}} a_{ij} \cdot x_j \right)$$

Data structures

- Simplex maintains
  - the tableau and
  - an assignment α to all (problem and additional) variables.
- For simplicity, let us name the variables $s_1, \ldots, s_m$ as $x_{n+1}, \ldots, x_{n+m}$.
- Initially, $\alpha(x_i) = 0$ for $i \in \{1, ..., n+m\}$.

- Two invariants are maintained throughout:
  1. $A \cdot \vec{x} = 0$
  2. All non-basic variable values satisfy the respective bounds

  Note: The basic variables might violate their bounds.

basic variable also satisfied $\Rightarrow$ solution

- Can you see why these invariants are maintained initially?

Invariants

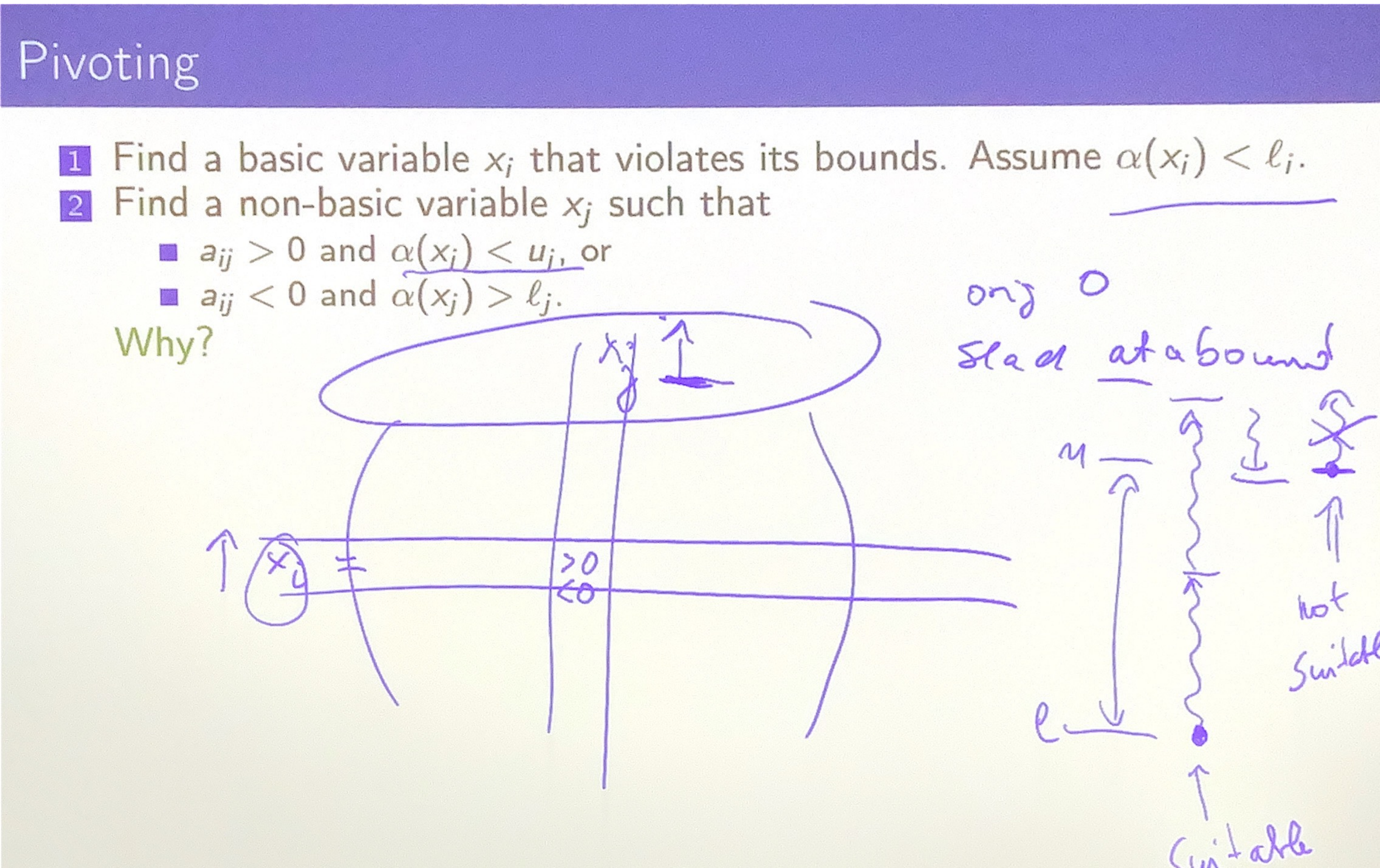
- The initial assignment α satisfies $A \cdot \vec{x} = 0$
- If the bounds of all basic variables are satisfied by α, return "satisfiable".
- Otherwise... *pivot*.

Pivoting

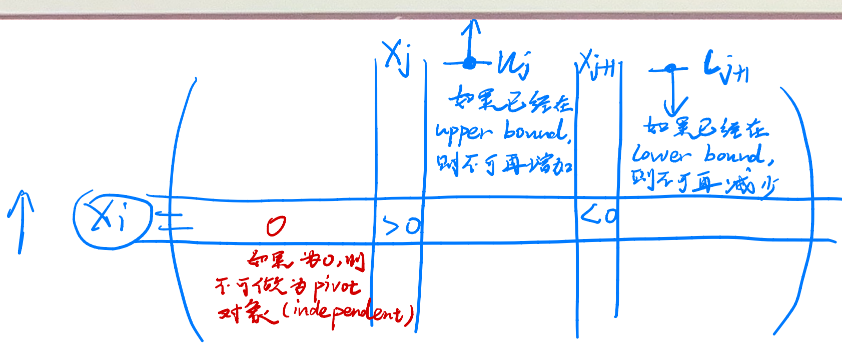
1. Find a basic variable x_i that violates its bounds. Assume $\alpha(x_i) < \ell_i$.
2. Find a non-basic variable x_j such that
   - $a_{ij} > 0$ and $\alpha(x_j) < u_j$, or
   - $a_{ij} < 0$ and $\alpha(x_j) > \ell_j$.

   Why? If there is such an x_j then we can increase the value of x_i to its lower bound, and in compensation we can change the value of x_j to maintain the equation in the row of x_i. Such a variable is called suitable. However, if a_{ij} would be 0 then we cannot compensate the value change of x_i by changing the independent value of x_j. If $a_{ij} \geq 0$ and $\alpha(x_j) \geq u_j$ then x_j is on its upper bound (remember that non-basic variables satisfy their bounds) and we need to further increase its value in order to compensate the value increment for x_i, however, then we would need to decrement it later with at least the same value, thus it would bring no progress. The case for $a_{ij} < 0$ and $\alpha(x_j) \leq \ell_j$ is similar.
3. If there is no suitable variable then return "unsatisfiable".

   Why? The maximal value of the linear term to which x_i should be equal to is smaller than the lower bound on x_i.



尚未到达upper bound, 则可以再增加, 即使会超过upper bound (超过upper bound说明此时为了使得basic variable增加, 单靠一个变量无法完全满足, 还需要靠其他变量. 此后因为超过upper bound, 还会通过pivot使得其他变量增加或减小来完成调节)



对于一行 (一个 basic variable)
如果所有 non-basic variable 都处于 bounds
则此时为最值 ⟹ unsatisfable

Pivoting x_i and x_j (1)

1. Transform $B_i = a_{i,j} NB_j + \sum_{\ell \neq j} a_{i,\ell} NB_\ell$ to
$$NB_j = \boxed{\frac{1}{a_{i,j}}} B_i + \sum_{\ell \neq j} \left(-\frac{a_{i,\ell}}{a_{i,j}}\right) NB_\ell$$
2. Swap B_i and NB_j, and update the i-th row accordingly
3. Update all other rows, replacing NB_j with its equivalent from 1.

	$\dots$	NB_j	$\dots$	NB_ℓ	$\dots$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
B_i	$\dots$	$\boxed{a_{i,j}}$	$\dots$	$\boxed{a_{i,\ell}}$	$\dots$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
B_k	$\dots$	$\boxed{a_{k,j}}$	$\dots$	$\boxed{a_{k,\ell}}$	$\dots$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$

$\rightsquigarrow$

	$\dots$	B_i	$\dots$	NB_ℓ	$\dots$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
NB_j	$\dots$	$\boxed{\frac{1}{a_{i,j}}}$	$\dots$	$\boxed{-\frac{a_{i,\ell}}{a_{i,j}}}$	$\dots$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
B_k	$\dots$	$\boxed{\frac{a_{k,j}}{a_{i,j}}}$	$\dots$	$\boxed{a_{k,\ell} - \frac{a_{k,j} a_{i,\ell}}{a_{i,j}}}$	$\dots$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$

Bonus exercise 20

Denoting original variables as x_i and slack variables as s_i, assume the following simplex tableau and slack variable bounds, with the current values of the variables given in square brackets:

此时都在 upper bound，不可再增加

如果不在 upper bound，则可以再增加，即使为了使 basic variable 到达 bound 会超过 upper bound

	s_4 $[-2]$	s_0 $[0]$	x_2 $[0]$
x_1 $[-\frac{6}{5}]$	$\frac{3}{5}$	$\frac{1}{5}$	2
s_1 $[\frac{12}{5}]$	$-\frac{6}{5}$	$-\frac{7}{5}$	-8
s_2 $[\frac{4}{5}]$	$-\frac{2}{5}$	$\frac{1}{5}$	-1
s_3 $[\frac{4}{5}]$	$-\frac{2}{5}$	$-\frac{4}{5}$	0
x_0 $[\frac{2}{5}]$	$-\frac{1}{5}$	$-\frac{2}{5}$	-1

pivot 的目的是使 non-basic variable 处于 bound 最值

$s_0 \leq 0$
$s_1 \leq 0$
$s_2 \leq 0$
$s_3 \leq 1$
$s_4 \leq -2$

The basic variable s_1 violates its bound.
Which of the non-basic variables are suitable for pivoting with s_1?

- Option 1: s_4
- Option 2: s_0
- Option 3: x_2
- Option 4: None of the non-basic variables

Pivoting x_i and x_j (2)

$$
\begin{array}{c|ccccc}
 & \dots & NB_j & \dots & NB_\ell & \dots \\
\hline
\dots & \dots & \dots & \dots & \dots & \dots \\
B_i & \dots & \boxed{a_{i,j}} & \dots & \boxed{a_{i,\ell}} & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots \\
B_k & \dots & \boxed{a_{k,j}} & \dots & \boxed{a_{k,\ell}} & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots
\end{array}
\quad \rightsquigarrow \quad
\begin{array}{c|ccccc}
 & \dots & B_i & \dots & NB_\ell & \dots \\
\hline
\dots & \dots & \dots & \dots & \dots & \dots \\
NB_j & \dots & \boxed{\frac{1}{a_{i,j}}} & \dots & \boxed{-\frac{a_{i,\ell}}{a_{i,j}}} & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots \\
B_k & \dots & \boxed{\frac{a_{k,j}}{a_{i,j}}} & \dots & \boxed{a_{k,\ell} - \frac{a_{k,j}a_{i,\ell}}{a_{i,j}}} & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots
\end{array}
$$

To increase the value of B_i to its lower bound LB_i, update α as follows:

- $\alpha(NB_j) := \alpha(NB_j) + \underbrace{\frac{LB_i - \alpha(B_i)}{a_{ij}}}_{\theta}$

  Note: Now NB_j is a basic variable and it may violate its bound.
- $\alpha(B_i) := LB_i$
- The values of the other non-basic variables do not change.
- Update α for all other basic (dependent) variables.

Pivoting: Example (1)

- Recall the tableau and constraints in our example:

	x	y
s_1	1	1
s_2	2	-1
s_3	-1	2

$$\begin{array}{ccc} 2 & \leq & s_1 \\ 0 & \leq & s_2 \\ 1 & \leq & s_3 \end{array}$$

- Initially, α assigns 0 to all variables

$\Longrightarrow$ Violated are the bounds of s_1 and s_3

- We will fix s_1.
- x is a *suitable* non-basic variable for pivoting.
  It has no upper bound!
- So now we pivot s_1 with x

Pivoting: Example (2)

	x	y
s_1	1	1
s_2	2	-1
s_3	-1	2

$$\begin{array}{ccc} 2 & \leq & s_1 \\ 0 & \leq & s_2 \\ 1 & \leq & s_3 \end{array}$$

- Solve 1st row for x:

$$s_1 = x + y \quad \leftrightarrow \quad x = s_1 - y$$

- Replace x in other rows:

$$\begin{array}{ccccccc} s_2 & = & 2x - y & = & 2(s_1 - y) - y & = & 2s_1 - 3y \\ s_3 & = & -x + 2y & = & -(s_1 - y) + 2y & = & -s_1 + 3y \end{array}$$

$$\begin{array}{rcl} x & = & s_1 - y \\ s_2 & = & 2s_1 - 3y \\ s_3 & = & -s_1 + 3y \end{array}$$

This results in the following new tableau:

	s_1	y
x	1	−1
s_2	2	−3
s_3	−1	3

$$\begin{array}{rcl} 2 & \leq & s_1 \\ 0 & \leq & s_2 \\ 1 & \leq & s_3 \end{array}$$

What about the assignment?

- Keep the values of all non-basic variables but the pivoted s_1.
- We set s_1 to its lower bound $\alpha(s_1) = 2$.
- Update all basic variables according to the tableau.
- Especially, we increase the value of x to $\alpha(x) = 2$.

Pivoting: Example (4)

The new state:

	s_1	y
x	1	-1
s_2	2	-3
s_3	-1	3

$$\begin{array}{rcl} \alpha(x) & = & 2 \\ \alpha(y) & = & 0 \\ \alpha(s_1) & = & 2 \\ \alpha(s_2) & = & 4 \\ \alpha(s_3) & = & -2 \end{array}$$

$$\begin{array}{rcl} 2 & \leq & s_1 \\ 0 & \leq & s_2 \\ 1 & \leq & s_3 \end{array}$$

- Now s_3 violates its lower bound
- Which non-basic variable is suitable for pivoting?
  That's right... y
- We should increase y by $\theta = \frac{1-(-2)}{3} = 1$.

Pivoting: Example (5)

The final state:

	s_1	s_3
x	$2/3$	$-1/3$
s_2	1	-1
y	$1/3$	$1/3$

$$\begin{aligned} \alpha(x) &= 1 \\ \alpha(y) &= 1 \\ \alpha(s_1) &= 2 \\ \alpha(s_2) &= 1 \\ \alpha(s_3) &= 1 \end{aligned}$$

$$\begin{aligned} 2 &\leq s_1 \\ 0 &\leq s_2 \\ 1 &\leq s_3 \end{aligned}$$

All constraints are satisfied.

Observations I

The additional variables:

- Only additional variables have bounds.
- These bounds are static.
- Additional variables enter the base only on extreme points (their lower or upper bounds).
- When entering the base, they shift towards the other bound and possibly cross it (violate it).

Observations II

Q: Can it be that we pivot x_i, x_j and then pivot x_j, x_i and thus enter a (local) cycle?

A: No.

再pivot xj, xi说明为了增大xi, xj超过了upper bound

- For example, suppose that $a_{ij} > 0$.
- We increased $\alpha(x_j)$ so now $\alpha(x_i) = \ell_i$.
- After pivoting, possibly $\alpha(x_j) > u_j$, but $a'_{ij} = 1/a_{ij} > 0$, hence the coefficient of x_i is not suitable

假设xi为basic variable, xj为non-basic variable, xi< lowerbound, 需要增加xi
经过pivot后xi可能已经处于lowerbound, 而xi和xj前系数符号一样, 为了减少xj, 需要xi也减少, 而处于Lowerbound的变量不能再减少了

Termination

Is termination guaranteed?

- No, there might be bigger cycles.

- In order to avoid circles, we use Bland's rule:
  1. Determine a total order on the variables
  2. Choose the first basic variable that violates its bound, 即选择下标最小
     and the first non-basic suitable variable for pivoting.
  3. It can be shown that this guarantees that no base is repeated, which implies termination.

General simplex with Bland's rule

1. Transform the system into the general form

$$A \cdot \begin{pmatrix} \vec{x} \\ \vec{s} \end{pmatrix} = 0 \quad \text{and} \quad \bigwedge_{i=1}^{m} \ell_i \leq s_i \leq u_i \ .$$

2. Set $\mathcal{B}$ to be the set of additional variables $s_1, \ldots, s_m$.
3. Construct the tableau for A.
4. Determine a fixed order on the variables.
5. If there is no basic variable that violates its bounds, return "satisfiable". Otherwise, let x_i be the first basic variable in the order that violates its bounds.
6. Search for the first suitable non-basic variable x_j in the order for pivoting with x_i. If there is no such variable, return "unsatisfiable".
7. Perform the pivot operation on x_i and x_j.
8. Go to step 5.

Geometric view revisited

- The solution of each constraint is a halfspace, whose facet is a hyperplane.
- The simplex method iterates over intersections of n of these hyperplanes (and the initial constraints $x_i = 0$ for original variables x_i) until it finds a vertex of the solution space or detects infeasibility.
- The n hyperplanes are defined by the n non-basic variables, if they are all additional variables.
- Pivoting exchanges one of the hyperplanes.
- Bland's rule assures that we visit each intersection of n hyperplanes at most once.

Learning target

- What is the input for the simplex method?
- How to bring a set of constraints into general form?
- How to construct the initial simplex tableau and the initial assignment?
- How to modify the initial tableau and assignment by pivoting?
- When iterating pivoting, when does the simplex algorithm terminate?
- Which condition assures termination (i.e. completeness)?