hybr2d Theory of Hybrid Systems Informatik 2

RWTH AACHEN UNIVERSITY

Satisfiability Checking - WS 2023/2024

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Series 4

Exercise 1

For each of the following theories, give their *signature* and *domain*, and state whether the theory is *decidable*.

Theory	Signature	Domain	Decidable?
Linear real arithmetic			
Linear integer arithmetic			
Nonlinear real arithmetic			
Nonlinear integer arithmetic			

Exercise 2

Assume a signature with the non-logical symbols: constants a, b; unary function f, binary function g; unary predicate p , binary predicate r, 3ary predicate q.

Say whether the following strings of symbols are well formed FOL Σ-formulas or terms:

1. $q(a)$
2. $p(y)$
3. $p(g(b))$
4. $\neg r(x, a)$
5. $q(x, p(a), b)$
6. $p(g(f(a), g(x, f(x))))$
7. $q(f(a), f(f(x)), f(g(f(z), g(a, b))))$
8. $r(a, r(a, a))$

Exercise 3

Assume a signature Σ with the non-logical symbols: constants a, b; unary function f, binary function g; unary predicate p , binary predicate r, 3ary predicate q.

Please specify all free variable occurrences in the following Σ-formulas:

1. $p(x) \land \neg r(y, a)$
2. $\exists x.\, r(x, y)$
3. $(\forall x.\, p(x)) \rightarrow (\exists y. \neg q(f(x), y, f(y)))$
4. $\forall x.\, \exists y.\, r(x, f(y))$
5. $\forall x.\, \exists y.\, (r(x, f(y)) \rightarrow r(x, y))$
6. $\forall x.\, (\exists y.\, (r(x, f(y))) \rightarrow r(x, y))$

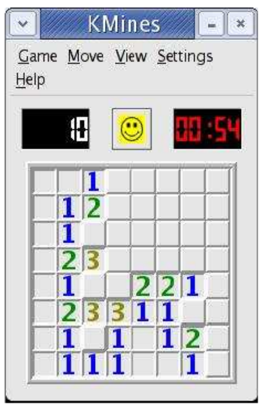
Exercise 4

Define an appropriate signature Σ and formalize the following sentences using Σ-formulas:

1. All students are smart.
2. There exists a student.
3. There exists a smart student.
4. Every student loves some student.
5. Every student loves some other student.
6. There is a student who is loved by every other student.
7. Bill is a student.
8. Bill takes either Analysis or Geometry, but not both.
9. Bill takes Analysis and Geometry.
10. Bill doesn't take Analysis.
11. No student takes Geometry.

Exercise 5

Minesweeper is a single-player computer game invented by Robert Donner in 1989. The game field is an $k \times k$ matrix of cells, out which $n \in [0, k^2]$ contain a mine. At the beginning, all cells are covered. Each covered cell can be uncovered by clicking on it. If a cell that contains a mine is clicked, the game is over. Otherwise, if the clicked cell does not contain a mine, one of two things happens:



i. A number between 1 and 8 appears indicating the amount of adjacent (including diagonally-adjacent) squares containing mines, or

ii. no number appears, in which case there are no mines in the adjacent cells.

The objective is to uncover each cell that does not contain a mine, without uncovering any cell with a mine in it.

Provide a signature for a first-order language that allows to formalize the knowledge of a player about a game state. In your language, formalize the following knowledge as axioms:

1. The minefield is a matrix of 8×8 cells.
2. For a given cell, its adjacent cells are its left, right, top, bottom and the four diagonal neightbours.
3. There are exactly n mines in the minefield.
4. If a cell contains the number 1, then there is exactly one mine in the adjacent cells.

Show by means of deduction that there must be a mine in the position $(3, 3)$ (3rd row and 3rd column, counting from 1) of the game state depicted on the right.
Suggestion: define the predicate *adj*(x, y) to formalize the fact that two cells x and y are adjacent.

Exercise 6*

As help for understanding the definition of a theory. Not relevant for the exam. In this exercise, we give some more details on the concept of *logical theory* and how it is related to axioms.

We fix an arbitrary signature Σ and an arbitrary structure $\mathcal{S}$ over Σ. In the following, all sentences are over Σ and Φ[1] is a set of sentences. We use the following notation:

- $\mathcal{S} \models \varphi$: $\mathcal{S}$ is a model of a sentence φ.
- $\mathcal{S} \models \Phi$: $\mathcal{S}$ is a model of all sentences φ from the set Φ.

Definitions:

- A sentence φ **is a consequence of** Φ ($\Phi \models \varphi$) iff $\mathcal{S} \models \varphi$ for each model $\mathcal{S} \models \Phi$.
- $\Phi^{\models} := \{\varphi \mid \Phi \models \varphi\}$ denotes the **set of consequences of** Φ.
- Φ is called **consistent** if there is no sentence φ with $\Phi \models \varphi$ and $\Phi \models \neg\varphi$.
- A satisfiable set of sentences T is called a **theory** if for all sentences φ

$$T \models \varphi \quad \Longleftrightarrow \quad \varphi \in T.$$

- A theory T is **complete** iff for all sentences φ

$$\text{either} \quad \varphi \in T \quad \text{or} \quad \neg\varphi \in T.$$

Prove the following five statements.

1. Each theory T is consistent.
2. Let Φ be a set of sentences. Φ is consistent iff $\Phi^{\models}$ is a theory.
3. The set $\mathtt{Th}(\mathcal{S}) := \{\varphi \mid \mathcal{S} \models \varphi\}$ is a theory. It is called the **theory of** $\mathcal{S}$.
4. $\mathtt{Th}(\mathcal{S})$ is complete.
5. Let $\Sigma = \{+, \cdot, \leq, =\}$. Give one example each:
   (a) a complete Σ-theory T_1,
   (b) an incomplete Σ-theory T_2.

   Hint: You can use different ways to define a theory.

[1] Imagine Φ to be a (finite) set of axioms.