Worksheet 4

Problem 1. Sample from Cauchy [Bi] Ex.11.3

Given a random variable z uniformly distributed over $(0, 1)$, find a transformation y $= f(\mathrm{z})$ such that y has Cauchy distribution

$$p_{\mathrm{y}}(y) = \frac{1}{\pi}\frac{1}{1+y^2} \ .$$

Problem 2. Box-Muller [Bi] Ex.11.4

Suppose z_1 and z_2 are uniformly distributed over the unit circle (disk). Show that

$$\mathrm{y}_1 = \mathrm{z}_1 \left(\frac{-2\ln r^2}{r^2}\right)^{1/2} , \quad \mathrm{y}_2 = \mathrm{z}_2 \left(\frac{-2\ln r^2}{r^2}\right)^{1/2}$$

where $r = \mathrm{z}_1^2 + \mathrm{z}_2^2$, has the joint density

$$p_{(\mathrm{y}_1,\mathrm{y}_2)}(y_1, y_2) = \left[\frac{1}{\sqrt{2\pi}}\exp(-y_1^2/2)\right]\left[\frac{1}{\sqrt{2\pi}}\exp(-y_2^2/2)\right]$$

Problem 3. Gibbs sampling

Consider the Gibbs sampler for a vector of parameters $\boldsymbol{x} = (x_1, \cdots, x_M)^{\mathsf{T}}$. Suppose at the s-th step $\boldsymbol{x}^{(s)}$ is sampled from the target distribution $p(\mathbf{x})$ and then $\boldsymbol{x}^{(s+1)}$ is generated using the Gibbs sampler. Show that the marginal probability $P(\boldsymbol{x}^{(s+1)} \in \mathbb{A})$ equals the target distribution $\int_{\mathbb{A}} p(\mathbf{x})d\mathbf{x}$.

Problem 4. Entropy

Recall that the entropy of a discrete random variable X is defined to be

$$H(X) = -\sum_{x\in\mathbb{X}} p(x)\log_2 p(x) \ ,$$

where $\mathbb{X}$ is the set of all possible values of X.

1. A fair coin is flipped until the first head occurs. Let X denote the number of flips required. Find the entropy $H(X)$ in bits.
2. What is the relationship of $H(X)$ and $H(Y)$ if $Y = 2^X$?

Problem 5. Differential entropy

Calculate the (differential) entropy of the following.

1. The exponential density $p(x) = \lambda e^{-\lambda x}$, $x \geq 0$.
2. The sum of x_1 and x_2 where x_1 is independent from x_2 and $p_{\mathsf{x}_i}(x) = \mathcal{N}(x|\mu_i, \sigma_i^2)$ for $i = 1, 2$.

Problem 6. Change of variable

Recall that $H(\mathbf{x}) = -\int p_{\mathbf{x}}(\boldsymbol{x}) \ln p_{\mathbf{x}}(\boldsymbol{x}) d\boldsymbol{x}$. Prove:

$$H(\boldsymbol{A}\mathbf{x}) = \ln |\det(\boldsymbol{A})| + H(\mathbf{x}) \ .$$