Worksheet 1

Problem 1. ([Al] Ex. 3.2-3.3)

We discussed the discriminant functions $g_i(x), i \in [K]$ where K is the number of classes. When $K = 2$ we can also define a single discriminant

$$g(x) = g_1(x) - g_2(x)$$

and we choose C_1 if $g(x) > 0$ and C_2 if $g(x) < 0$.

1. In a two-class problem, the *likelihood ratio* is

$$\frac{p(x|C_1)}{p(x|C_2)} .$$

   Write a discriminant function in terms of the likelihood ratio.

2. In a two-class problem, the *log odds* is defined as

$$\log \frac{P(C_1|x)}{P(C_2|x)} .$$

   Write a discriminant function in terms of the log odds.

Problem 2. ([Al] Ex. 3.4)

In a two-class, two-action problem, if the loss function is $\lambda_{11} = \lambda_{22} = 0$, $\lambda_{12} = 10$ and $\lambda_{21} = 5$, write the optimal decision rule. How does the rule change if we add a third action of reject with $\lambda = 1$? [Note: we don't have $0/1$ loss for this problem.]

Problem 3. (Poisson MLE)

Let X be a random variable. $X \sim$ Poisson(λ) with the density

$$p(x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

1. Find $\mathbb{E}X$ and $\mathrm{Var}(X)$ if $X \sim$ Poisson(λ).
2. Consider the sample $\mathcal{X} = \{x_n\}_{n=1}^N$ where $x_n \sim^{i.i.d.}$ Poisson(λ). For the parameter λ above, write the likelihood $l(\lambda|\mathcal{X})$ and the log-likelihood $\mathcal{L}(\lambda|\mathcal{X})$.
3. Find the maximum likelihood estimator $\hat{\lambda}_{\text{MLE}}$.
4. Is $\hat{\lambda}_{\text{MLE}}$ biased?

Problem 4. (Uniform MLE)

Let X be a random variable. $X \sim$ Unif(θ) with the density

$$p(x) = \begin{cases} \dfrac{1}{\theta}, & \text{if } 0 \le x \le \theta \\ \\ 0, & \text{otherwise .} \end{cases}$$

1. Find $\mathbb{E}X$ and $\mathrm{Var}(X)$ if $X \sim$ Unif(θ).
2. Consider the sample $\mathcal{X} = \{x_n\}_{n=1}^N$ where $x_n \sim^{i.i.d.}$ Unif(θ). For the parameter θ above, write the likelihood $l(\theta|\mathcal{X})$ and the log-likelihood $\mathcal{L}(\theta|\mathcal{X})$.
3. Find the maximum likelihood estimator $\hat{\theta}_{\text{MLE}}$.
4. Is $\hat{\theta}_{\text{MLE}}$ biased?

Problem 5. (See [AI] Ch.16.2.2)

Find $\hat{q}_{\text{MAP}}$ for the Bernoulli likelihood

$$p(\mathcal{X}|q) = \prod_{n=1}^{N} q^{x_n}(1-q)^{1-x_n}$$

with the beta prior

$$p(q) = beta(q|\alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} q^{\alpha-1}(1-q)^{\beta-1} \; .$$

Problem 6. (Exponential family)

A probability distribution in the *exponential family* is given by

$$p(\boldsymbol{x}|\boldsymbol{\eta}) = h(\boldsymbol{x}) \exp\left(\boldsymbol{\eta}^{\mathsf{T}} T(\boldsymbol{x}) - A(\boldsymbol{\eta})\right) \; ,$$

where $\boldsymbol{\eta}$ is the parameter vector.

1. Prove that $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{I})$ with identity covariance (where $\boldsymbol{\mu}$ is the parameter) is in the exponential family.
2. Prove that
$$\nabla_{\boldsymbol{\eta}} A = \mathbb{E}_{\mathbf{x} \sim p(\boldsymbol{x}|\boldsymbol{\eta})}[T(\mathbf{x})] \; .$$
Hint: Use the fact that $\int p(\boldsymbol{x}|\boldsymbol{\eta}) \, d\boldsymbol{x} = 1$ to get an expression of A first.
3. Verify Part 2 using the example in Part 1.