| 姓名 | 学号 | 班级 | 选题 | 论述 | 结论 | 总分 |
| --- | --- | --- | --- | --- | --- | --- |
| 李震宇 | 2015301020127 | 物基二班 | | | | |

# Demonstration of Chaos in The Damped, Driven, Nonlinear Pendulum

李震宇 2015301020127 物基二班

**Abstract**

The physical systems whose characteristics change in a seemingly erratic manner gives rise to the study of chaotic systems. The characteristics of these systems are due to their hypersensitivity to changes in initial conditions. In order to understand the phenomena of chaos, some sort of simulation and visualization is necessary. Consequently, in this work, the system of a damped, driven and nonlinear physical pendulum is introduced to demonstrate the chaotic system in a graphically way. The visualized results obtained which highlight the hypersensitivity of the pendulum to initial conditions can be used to effectively introduce the physics of chaotic system. Time series curves, phase-space plot, Poincare section and bifurcation diagram are made to illustrate the traits of chaotic behaviors. Besides, the simulation and visualization program is written in Python codes and Euler-Cromer method is applied.

**Key words**

Chaos, initial conditions, physical pendulum, Python, Euler-Cromer Method

## I. INTRODUCTION

Chaos is the field of study in mathematics that studies the behavior of dynamical systems that are highly sensitive to initial conditions—a response popularly referred to as the butterfly effect.[1] Small differences in initial conditions yield widely diverging outcomes for such dynamical systems, rendering long-term prediction impossible in general.[2] This happens even though these systems are deterministic, meaning that their future behavior is fully determined by their initial conditions, with no random elements involved. In other words, the deterministic nature of these systems does not make them predictable. This behavior is known as deterministic chaos, or simply chaos. The theory was summarized by Edward Lorenz as:' When the present determines the future, but the approximate present does not approximately determine the future'.[3]

Chaotic behavior exists in many natural systems, such as weather and climate.[4][5] This behavior can be studied through analysis of a chaotic mathematical model, or through analytical techniques such as recurrence plots and Poincare maps.

However, some physical systems that can exhibit chaotic behavior can be introduced and visualized to demonstrate chaotic systems with computer programming. This is the purpose of this project: to simulate and visualize a damped, driven and nonlinear pendulum exhibiting chaotic behavior using a computer program written in Python codes.

In this study, the pendulum is approximated to a relatively simple one with only the damping, driving force and nonlinearity. The mathematical model of the pendulum used in this study is made of a second-order nonlinear differential equation which has no exact analytical solution yet. Consequently, the equation was solved by breaking it into two first order differential equations and then solving them in the computer programming. Thus, a computer program simulation is utilized to implement the model of the pendulum in order to test it under different initial conditions. For the visualization part, time series curves, phase-space plot, Poincare section and bifurcation diagram are employed to describe the model of the driven nonlinear pendulum used. The results obtained is used to introduce chaos in dynamical systems.

In the study, SI units are assumed for all physical quantity and approximately $g = 9.8m/s^2$.

## II. PHYSICAL MODEL OF THE PENDULUM

For an ideal pendulum, the friction is ignored and the angle the string makes with the vertical is assumed to be small. Such a pendulum undergoes what is known as simple harmonic oscillation. However, elementary-level treatments usually do not consider the behavior of a real pendulum. So in this project, a physical pendulum is introduced as one with dissipation, an external driving force and nonlinearity and it is allowed to swing to large angles. The implication is that changing the initial conditions of these parameters could alter the behavior and pattern of the pendulum.

The damping is introduced with a magnitude proportional to the velocity. The frictional force employed here has the form of $q(d\theta / dt)$. The driving force is assumed to be sinusoidal with time in a form of $F_D \sin(\Omega_D t)$. The small-angle approximation is not employed and thus the term $\sin\theta$ does not expand as $\theta$. Now putting all these parameters together, we have the equation of motion of the damped, driven, nonlinear pendulum as:

$$\begin{gathered} \frac{d^2\theta}{dt^2} = f(\theta, \omega, t) \\ f(\theta, \omega, t) = -\frac{g}{l}\sin\theta - q\omega + F_D \sin(\Omega_D t) \end{gathered} \tag{1}$$

where $l$ is the length of string, $g$ is the acceleration due to gravity. The natural frequency of the un-damped pendulum is

$$\Omega = \sqrt{g/l} \tag{2}$$

Nonlinear equations such as Equation (1) describes many things in nature and are difficult, if not impossible, to solve analytically. This implies that a solution does not always exist and when it does, it is not always unique.

## III. NUMERICAL METHOD

The original equation of motion, Equation (1), can be broken down into two first-order differential equations as follows:

$$\begin{gathered} \frac{d\omega}{dt} = f(\theta, \omega, t) \\ \frac{d\theta}{dt} = \omega \end{gathered} \tag{3}$$

where

$$f(\theta,\omega,t) = -\frac{g}{l}\sin\theta - q\omega + F_D\sin(\Omega_D t) \tag{4}$$

For this kind of problem, Euler-Cromer method, Runge-Kutta method and Verlet method are all suitable. But in this project, Euler-Cromer method is utilized. The procedure can be interpreted as:

For each time step $i$ (beginning with $i=1$ ), calculate $\omega$ and $\theta$ at time step $i+1$ with the following difference equation:

$$\begin{aligned} \omega_{i+1} &= \omega_i + f(\theta_i,\omega_i,t_i)\Delta t \\ \theta_{i+1} &= \theta_i + \omega_{i+1}\Delta t \\ t_{i+1} &= t_i + \Delta t \end{aligned} \tag{5}$$

And repeat the above calculation for the desired number of time steps.

During the operation, the value of $\theta$ can be adjusted in each iteration so as to keep it in the between $-\pi$ and $\pi$. Now the pendulum is free to swing all the way around its pivot point, and values of $\theta$ differed by $2\pi$ correspond to the same position. So this adjustment is done because for plotting purpose it is convenient to keep $\theta$ in the range.

Equation (4) and (5) and the above procedure can then be coded and used to obtain a solution to the second order differential equation of the pendulum model in our system. All of the calculations below are based on this.

## IV. THE EFFECT OF DISSIPATION, DRIVING AND NONLINEARITY RESPECTIVELY

For this part, the separate influences of the dissipation, the external driving force and nonlinearity are investigated to have a brief understanding of the physical pendulum model. When there is no nonlinearity in the system, we rewrite the equation of motion, Equation (1) as:

$$\begin{gathered} \frac{d^2\theta}{dt^2} = f(\theta,\omega,t) \\ f(\theta,\omega,t) = -\frac{g}{l}\theta - q\omega + F_D\sin(\Omega_D t) \end{gathered} \tag{6}$$

### i. Results and Interpretation for Dissipation

First of all, only the dissipation, also known as damping, is considered in the pendulum. This can be done by setting the parameters in Equation (6) as $F_D = 0$ and $l = 1.0$. The initial conditions for each lines are the same $\omega_0 = (d\theta/dt)_{t=0} = 0$ and $\theta_0 = \theta_{t=0} = 0.2$. The parameter $q$ is related to damping. The results are shown in Fig.1.

It can be seen from Fig.1 that as the damping coefficient increases, the oscillation is less obvious. When $q = 1,3$, the pendulum is under damped. When $q = 10,30$, it is over damped. When $q = 2\Omega = 2\sqrt{g/l}$, it is critically damped and there is no oscillation at all.

### ii. Results and Interpretation for Damping and Driving Force

Next, the damped driven pendulum is introduced without the influence of nonlinearity. The parameters in Equation (6) are set as $q = 1.0$ and $l = 1.0$. The parameters $F_D$ and $\Omega_D$ are related to external force and are changeable here. The results are shown in Fig.2.

From Fig.2, it is obvious that the motion of pendulum is approaching a steady state. The driving force will pump energy into (or out of) the system and the imposed frequency will compete with the natural frequency of the pendulum. The frequency of oscillation is related to the frequency of driving force. The steady oscillation frequency of the blue line is half as others, just as the driving force frequency. Besides, as $F_D$ increases, the amplitude will also increase, which is due to the fact that driving force is pumping energy into the system.

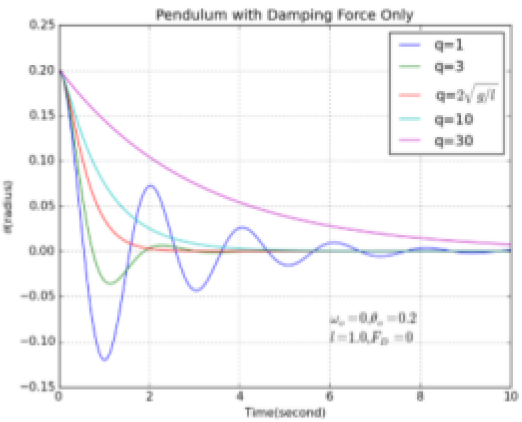



Fig.1.This shows a damped pendulum without driving force. The difference in the damping coefficient $q$ is illustrated in the figure.

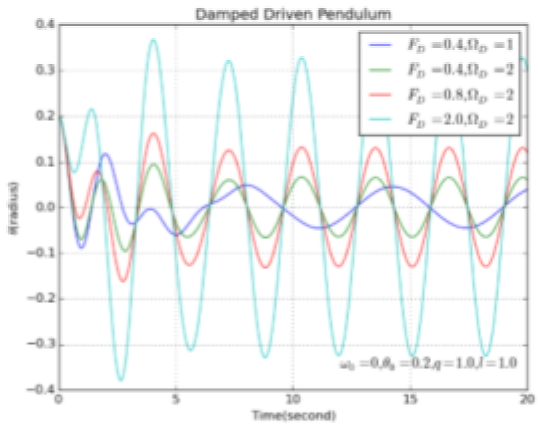



Fig.2.This shows the motions of four damped driven pendulums with different driving force. Each has initial condition as $\omega_0$=0and $\theta_0$=0. 2, while other parameters are illustrated in the figure.

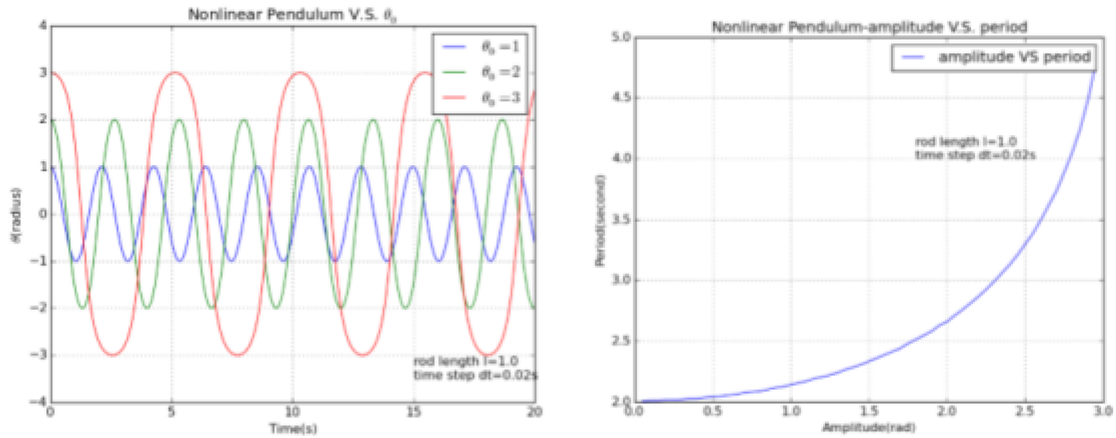



Fig.3 Left: the oscillation of pendulum with different values of $\theta_0$. Right: the period differs with initial amplitude.

### iii. Results and Interpretation for Nonlinearity

Then, the nonlinearity is introduced to investigate the motion of the pendulum. So in Equation (4), the parameters related to damping and driving forces are all zero, that is, q=0 and $F_D$=0. Besides, we can set l=1.0. A program is built to show the motion of the nonlinear pendulum with time and to detect how the period grows with the amplitude of the initial displacement. The results are in Fig.3.

From Fig.3, it can be seen that as the initial amplitude grows, the period and amplitude of the pendulum increases accordingly. Besides, the period increases with the initial amplitude in an exponential way.

## V. THE CHARACTERISTIC OF PHYSICAL PENDULUM AND CHAOS

We add an external driving force, dissipation and nonlinearity to the pendulum at the same time rather than discuss these ingredients separately and what we get is the so-called physical pendulum under Equation (3) and (4). For this part, both the displacement $\theta$ and the angular velocity $\omega$ are investigated. And we are supposed to see: (a) it is possible for a system to be both deterministic and unpredictable and this is what the term chaos means; (b) the behavior in the chaotic regime is not completely random, but can be describe by a strange attraction phase space.

### i. External Force and Chaos

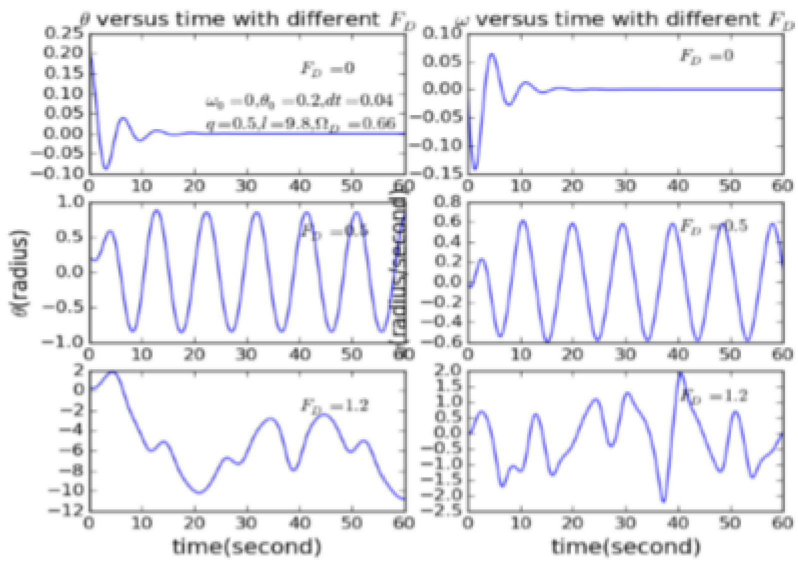



Fig.4. The above set of figure is the behavior of the driven, damped, nonlinear pendulum in 60s. The left figures are the behavior of $\theta$ as a function of time while the right are the behavior of $\omega$ .

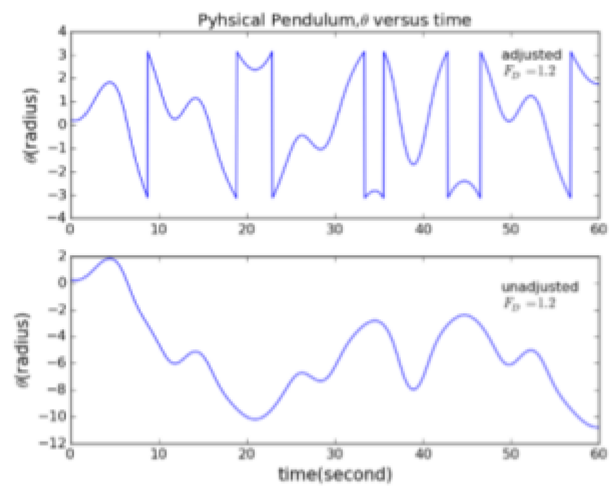



Fig.5. This is the behavior of $\theta$ under $F_D$=1.2, with and without the resetting of the angle.

In this part, we get the results with different amplitudes of the sinusoidal driving force, $F_D$. The results are shown in Fig.4 and the parameters are set as $q = 0.5$ , $l = 9.8$ and $\Omega_D = 0.66$ . The time interval is $dt = 0.04$ and the initial condition is $\omega_0 = 0, \theta_0 = 0.2$. And if we adjust $\theta$ to keep it between $-\pi$ and $\pi$ , Fig.5 is obtained.

From Fig.4 and Fig.5, the following conclusion can be reached. When there is no driving force, the motion is damped and the pendulum comes to rest after several oscillation. Its frequency is close to the natural frequency of the un-damped pendulum $\Omega = \sqrt{g/l}$ . With small driving force, say $F_D = 0.5$ , the initial displacement of the pendulum leads to a component of the motion that decays with time and has frequency of $\Omega$ , too. After the transient is damped away, the pendulum settles into a steady oscillation with a frequency the same as its driving force. At high driving amplitude, say $F_D = 1.2$ , the motion is no longer simple. The pendulum does not settle into any sort of repeating steady-state behavior, at least in the range shown here. And that is chaos.

So we can say that as the driven force increases, the motion of the physical pendulum transforms from simple harmonic oscillation to chaotic ones. And the concept of the behavior of chaotic systems in terms of hypersensitivity to initial conditions is successfully captured by our dynamical system as it can be seen that there is a hypersensitive change in the system's behavior when the driving force is changed

**ii. Phase-Space Plot**

Phase plane is a two-dimensional projection of the phase space. It represents each of the state variable's instantaneous state to each other. The different motions of the dynamical system can be easy distinguished visually from each other. So, a fixed-point solution is a point in the phase plane. A periodic solution is a closed curve, while a quasi-periodic solution is a curve on a torus. In the end, chaotic solutions are distinct and complicated curves in phase plane. Here, the phase-space plot shows the change of angular velocity $\omega$ as a function of $\theta$ .

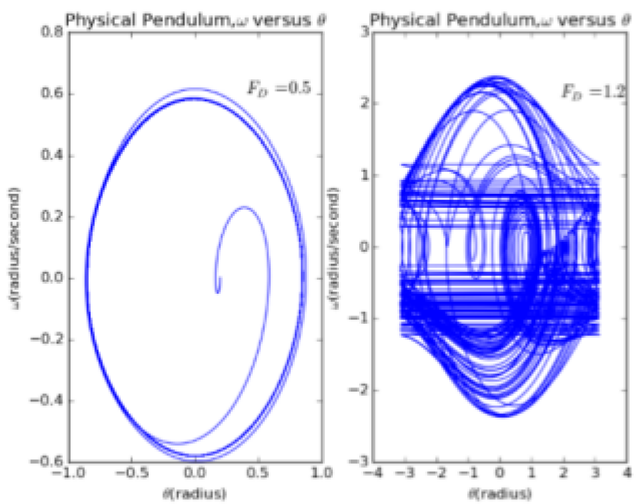



Fig.6. This is the phase-space plots, showing angular velocity $\omega$ as a function of $\theta$ in a physical pendulum in 500 seconds. Left: $F_D$=0.5. Right: $F_D$=1.2. The horizontal jumps are due to the adjustment of $\theta$ .

We keep the parameters fixed as in Fig.4 and Fig.5, assign $F_D = 0.5, 1.2$ respectively, and get the phase-space plot in Fig.6.

From Fig.6, we can see that for the small driving force shown in the left figure above, there is a transient depending on the initial conditions, but the pendulum quickly settles into a regular orbit in phase space corresponding to the oscillatory motion of both $\omega$ and $\theta$. This final orbit is independent of the initial conditions. For the figure on the right with large driving force, it is obvious that the behavior of pendulum is in the chaotic regime. The pattern is not a simple one, but it is not completely random. Meanwhile, it exhibits phase-space trajectories with significant structure.

### iii. Poincare Section

Poincare Section is a method of displaying the character of a particular trajectory without examining its complete time development, in which the trajectory is sampled periodically, and the rate of change of a quantity under study is plotted against the value of that quantity at the beginning of each period. In fact, the different types of motions appear as finite number of points for periodic orbits, curve filling points for quasi-periodic motions and area filling points for chaotic trajectories. For a better understanding, we'll plot the two-dimensional Poincare section of the phase-space plot at times that are in phase with the driving force. So only the points with $\Omega_D t = 2n\pi (n \in \text{integer})$ are displayed.

Set the interest period as long as 1000 seconds and the time interval $dt = 0.01$. Other parameters are the same as Fig.4 and can be seen in the graph. The result is shown in Fig.7.

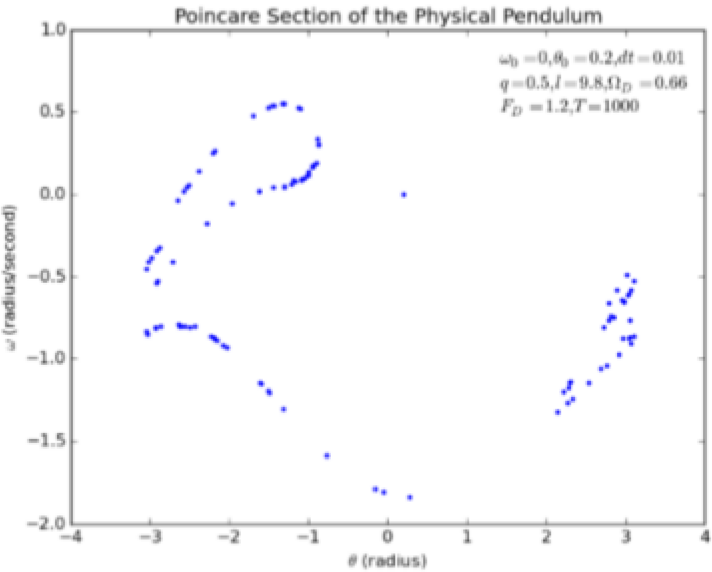



Fig.7. Poincare section of the physical pendulum.

### iv. Routes to Chaos-Bifurcation Diagram

This part deals with the question: how does the transition from simple to chaotic behavior take place with drive amplitude?

First, we draw the change of $\theta$ as a function of time with the driving force a little bit different. Keep the parameters the same as in Fig.4 and the results are shown in Fig.8. As is shown, the left figure has a period as the driving force, the middle figure has a period twice the drive period, while the right figure has a period that is four times the drive period.

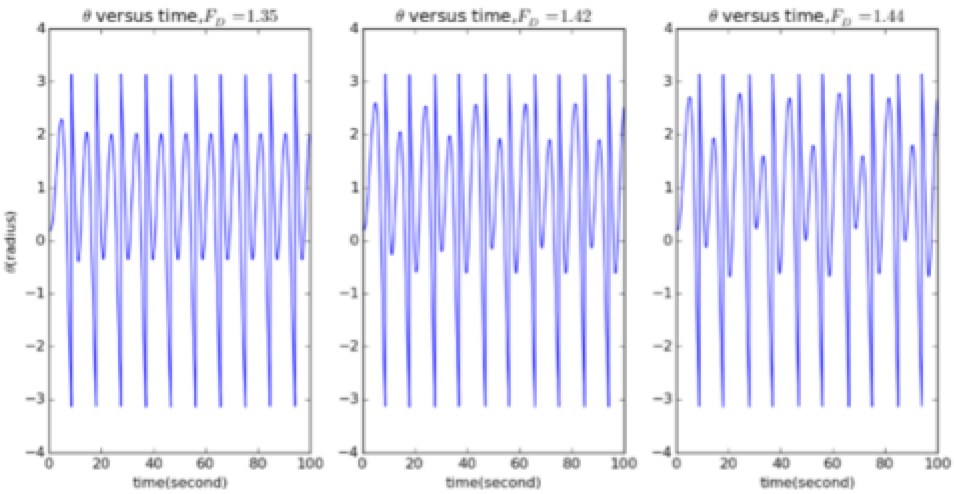
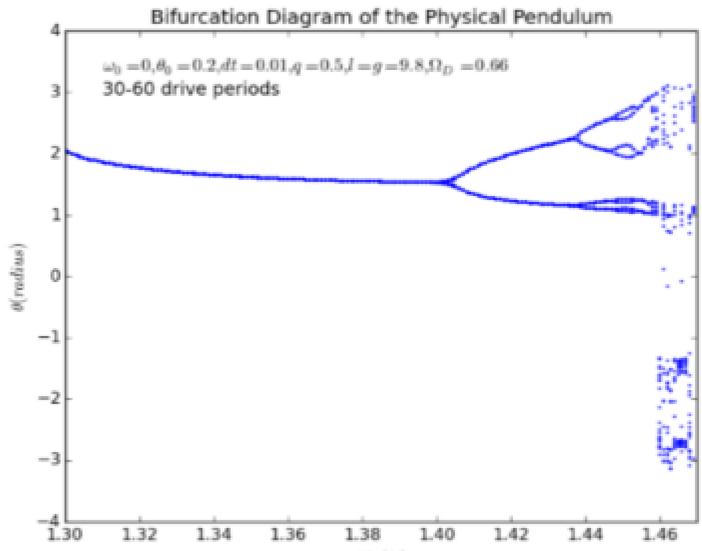



Fig.8.This is the results for $\theta$ as a function of time for the pendulum with different drive amplitude within 100 seconds. From left to right, the driving amplitudes are $F_D$=1.35, $F_D$=1.42 and $F_D$=1.44 respectively.




To know this deeper, bifurcation diagram is drawn. For each value of $F_D$ , the displacement $\theta$ is calculated as a function of time. After waiting for 30 driving periods so that the initial transients have decayed away, we plot $\theta$ at times that are in phase with the driving force as a function of $F_D$. Here we plot the points up to the 60th drive period and the result is in Fig.9.

Motion whose period is $n$ times the drive period $T_D = 2\pi / \Omega_D$ will yield n points on the bifurcation diagram for that value of $F_D$. From Fig.9, we know that the period is $T_D$ up to approximately $F_D = 1.402$, followed by a transition to period-$2T_D$ motion up to $F_D = 1.437$. The motion with period $4T_D$ does not end until $F_D = 1.447$ . The range $1.447 < F_D < 1.454$ corresponds to period- $8T_D$ motion.

## VI. CODING PROGRAM

All the computational programs and figures used in the study can be found on my Github:

https://github.com/AaalgerLee/compuational_physics_N2015301020127/tree/master/fanal

## VII. SUMMARY ANE CONCLUSION

Separately, the damping, driving or nonlinearity has different influences on the motion of the pendulum. As the dissipation coefficient $q$ increases, the oscillation is weaker. The amplitude and frequency of the pendulum is concerned with those of the driving force. Under the influence of nonlinearity, both the amplitude and the frequency are influenced by the initial displacement.

As for a physical pendulum with dissipation, external sinusoidal driving force and nonlinearity at the same time, chaos can be demonstrated when the parameters are chosen appropriately. Time series curves, phase-space plot, Poincare section and bifurcation diagram can be utilized to illustrate the existence of chaotic behaviors. The phase-space plot affirms the concept of the behavior of chaotic systems in terms of hypersensitivity to initial conditions. The Poincare section indicates some regularity behind the erratic and random behavior of chaotic systems. In particular, in the bifurcation diagram, when $F_D$ increases from 1.30 to 1.45, interesting transitions take place.

## VIII. REFERENCES


[1] Boeing (2015). "Chaos Theory and the Logistic Map". Retrieved 2015-07-16.

[2] Kellert, Stephen H. (1993). In the Wake of Chaos: Unpredictable Order in Dynamical Systems. University of Chicago Press. p. 32. ISBN 0-226-42976-8.

[3] Danforth, Christopher M. (April 2013). "Chaos in an Atmosphere Hanging on a Wall". Mathematics of Planet Earth 2013. Retrieved 4 April 2013.

[4] Lorenz, Edward N. (1963). "Deterministic non-periodic flow". Journal of the Atmospheric Sciences20(2):130–141.

[5] Ivancevic, Vladimir G.; Tijana T. Ivancevic (2008). Complex nonlinearity: chaos, phase transitions, topology change, and path integrals. Springer. ISBN 978-3-540-79356-4.

[6] Nicholas J. Giordano & Hisao Nakanishi. Computational Physics: 2nd ed.[J]. 2011.