2.3 KG field as harmonic oscillator

→ Start with the simplest field; which is the KG field and then quantize it, by re-interpreting variables as operators.

→ The system is then solved by finding the eigenvalues and eigenvectors of these operators.

→ ϕ and π operators and impose specific commutation relations.

~ for a discrete system of one or more particles the comm. relations are:

$$[q_i, p_i] = i\delta_{ij}$$

$$[q_i, q_j] = [p_i, p_j] = 0$$

For a continous system the generalization is quite; since $\pi(x)$ is momentum density

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$$[\phi(x), \pi(y)] = i\delta^{(3)}(x-y)$$

$$[\phi(x), \phi(y)] = [\pi(x), \pi(y)] = 0$$

$$\phi(x,t) = \int \frac{d^3p}{(2\pi)^3} e^{ip\cdot x} \phi(p,t)$$

(with $\phi^*(p) = \phi(-p)$); the K.G eq. becomes:

Ⓐ — $$\left[\frac{\partial^2}{\partial t^2} + \left(|p|^2 + m^2\right)\right]\phi(p,t) = 0$$

$$\omega_p = \sqrt{|p|^2 + m^2}$$ — Ⓑ

Ⓐ is the eq. of SHO with freq Ⓑ.

$$\left(\frac{\partial^2}{\partial t^2} + \omega^2\right)\phi(p,t) = 0$$

$F = -Kx$

$$V = \frac{1}{2} K x^2 \sim \frac{1}{2} \cancel{K} \phi^2$$

$K \to \omega^2$

$$L = T - V$$

$$= \frac{p^2}{2m} + \frac{1}{2} K \phi^{2}$$

$K \to \omega^2$

$$= \frac{p^2}{2} + \frac{1}{2} \omega^2 \phi^2$$

$\Rightarrow$ writing ϕ and p in terms of ladder operator.

$$\phi = \sqrt{\frac{1}{2\omega}} (a + a^{+}) \,;\, p = -i \sqrt{\frac{\omega}{2}} (a - a^{+})$$

$$[\phi, p] = i$$

$$[a, a^{+}] = 1$$

$$H = \frac{P^2}{2} + \frac{1}{2}\omega^2\phi^2$$

$$\Rightarrow \frac{1}{2}\left[-\frac{\omega}{2}\left(aa - a^\dagger a - aa^\dagger + a^\dagger a^\dagger\right) + \frac{\omega^2}{2\omega}\left(aa + a^\dagger a + aa^\dagger + a^\dagger a^\dagger\right)\right]$$

$$\Rightarrow \frac{\omega}{4}\left(-\cancel{aa} + a^\dagger a + aa^\dagger - \cancel{a^\dagger a^\dagger} + \cancel{aa} + a^\dagger a + aa^\dagger + \cancel{a^\dagger a^\dagger}\right)$$

$$\rightarrow \frac{\omega}{4}\left(aa^\dagger + a^\dagger a + aa^\dagger + aa^\dagger\right)$$

$$\Rightarrow \frac{\omega}{2}\left(aa^\dagger + a^\dagger a\right) = \frac{\omega}{2}\left(1 + a^\dagger a + a^\dagger a\right)$$

$$\rightarrow \frac{\omega}{2}\left(2a^\dagger a + 1\right) = \omega\left(a^\dagger a + \frac{1}{2}\right)$$

$\rightarrow$ The state $|0\rangle$ such that $a|0\rangle = 0$ is an eigenstate; the zero-point energy. Furthermore, the commutators

$$[H_{SHO}, a^\dagger] = \omega a^\dagger; \quad [H_{SHO}, a] = -\omega a$$

$$|n\rangle = (a^\dagger)^n |0\rangle$$

$\Rightarrow$ with eigenvalues $\left(n + \frac{1}{2}\right)\omega$

We can find the spectrum of the KG hamiltonian using the same trick; but now each fourier mode of the field is treated as independent oscillator with its own a and $a^\dagger$:

We can write.

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \cdot \frac{1}{\sqrt{2\omega_p}} \left(a_p e^{ip\cdot x} + a_p^\dagger e^{-ip\cdot x} \right)$$

$$\pi(x) = \int \frac{d^3p}{(2\pi)^3} \cdot (-i) \sqrt{\frac{\omega_p}{2}} \left(a_p e^{ip\cdot x} - a_p^\dagger e^{-ip\cdot x} \right)$$

$$[a_p, a_{p'}^\dagger] = (2\pi)^3 \delta^3(p-p')$$

$$[\phi(x), \pi(x')] = \int \frac{d^3p\, d^3p'}{(2\pi)^6} \left(\frac{-i}{2}\right) \sqrt{\frac{\omega_{p'}}{\omega_p}} \left([a_{-p}^\dagger, a_{p'}] - [a_p, a_{-p'}^\dagger] \right) \cdot \exp(i(p\cdot x + p'\cdot x'))$$

$$= i\,\delta^3(x-x')$$

Worked out steps

$$[\phi(x), \pi(x')] = \left[\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left(a_p e^{ipx} + a_p^\dagger e^{-ipx}\right), \frac{d^3p'}{(2\pi)^3}(-i)\left(\sqrt{\frac{\omega_{p'}}{2}}\right)\left(a_{p'} e^{ipx} - a_{p'}^\dagger e^{-ipx}\right)\right]$$

$$\Rightarrow \int \frac{d^3p\, d^3p'}{(2\pi)^6}\left(\frac{-i}{2}\right)\sqrt{\frac{\omega_{p'}}{\omega_p}}\left[\left(a_p + a_{-p}^\dagger\right), \left(a_{p'} - a_{-p'}^\dagger\right)\right] e^{ip\cdot x + ip'\cdot x'}$$

$$\Rightarrow \int \frac{d^3p\, d^3p'}{(2\pi)^6}\left(\frac{-i}{2}\right)\sqrt{\frac{\omega_{p'}}{\omega_p}}\left(\cancel{[a_p, a_{p'}]}^{\,0} - [a_p, a_{-p'}^\dagger] + [a_{-p}^\dagger, a_{p'}] - \cancel{[a_{-p}^\dagger, a_{-p'}^\dagger]}^{\,0}\right) e^{i(p\cdot x + p'\cdot x')}$$

$$\Rightarrow \int \frac{d^3p\, d^3p'}{(2\pi)^6}\left(\frac{-i}{2}\right)\sqrt{\frac{\omega_{p'}}{\omega_p}}\left(e^{i(p\cdot x + p'\cdot x')}\right)\left(-[a_p, a_{-p'}^\dagger] + [a_{-p}^\dagger, a_{p'}]\right)$$

$$\int \frac{d^3p\, d^3p'}{(2\pi)^6}\left(\frac{-i}{2}\right)\sqrt{\frac{\omega_{p'}}{\omega_p}}\left(\quad\right)\left[(2\pi)^3\left[-\delta^3(p+p') - \delta^3(p+p')\right]\right]$$

$$\int \frac{d^3p\, d^3p'}{(2\pi)^6} \left(\frac{\cancel{+}i}{\cancel{2}}\right) \sqrt{\frac{\omega p'}{\omega p}} \left(\quad \right) \left[(2\pi)^3 (\cancel{+}\cancel{2}) \left(\delta(p+p') \right) \right]$$

$$\Rightarrow \int \frac{d^3p\, d^3p'}{(2\pi)^3} (i) \left(\delta(p+p') \right) \sqrt{\frac{\omega p'}{\omega p}} \left(\quad \right)$$

put $p' = -p$

$$-\int \frac{d^3p\, d^3p}{(2\pi)^3} (i) \left(\delta(p-p) \right) \sqrt{\frac{\omega p}{\omega_p}} \left(\exp^{ip(x-x')} \right)$$

$$\int \frac{d^3p}{(2\pi)^3}\, i \cancel{\sqrt{\frac{\omega p}{\omega p}}}^{\,1} \exp^{ip(x-x')}$$

$$i \int \frac{d^3p}{(2\pi)^3} e^{ip(x-x')} \Rightarrow i\,\delta^3(x-x')$$

This confirms that ϕ and π operators satisfy canonical commutation relations!

Writing H in terms of ladder operators

$$H = \int d^3x \, \mathcal{H} = \int d^3x \left(\pi(x) \dot{\phi}(x) - \mathcal{L} \right)$$

$$\Rightarrow \int d^3x \left(\frac{\pi^2}{2} + \frac{(\nabla\phi)^2}{2} - \frac{m^2\phi^2}{2} \right)$$

$$\int d^3x \left(\frac{\dot{\phi}}{2} \; \frac{(+\nabla\phi)^2}{2} - \frac{m^2\phi^2}{2} \right)$$

$$\int \frac{d^3x}{2} \left[\pi^2 + (\nabla\phi)^2 - m^2\phi^2 \right]$$

$$\int \frac{d^3x}{2} \left[\int \frac{d^3p \, d^3p'}{(2\pi)^6} e^{i(p+p')x} \left\{ -\frac{\sqrt{\omega_p \omega_{p'}}}{4} (a_p - a_{-p}^\dagger)(a_{p'} - a_{-p'}^\dagger) \right. \right.$$

$$\left. + \frac{-p \cdot p' + m^2}{4\sqrt{\omega_p \omega_{p'}}} (a_p + a_{-p}^\dagger)(a_{p'} + a_{-p'}^\dagger) \right\}$$

$$\int \frac{d^3p\, d^3p'}{(2\pi)^6} \underbrace{\int \frac{d^3x}{2} \exp(i(p+p')x)}_{\downarrow} \left\{ \qquad \right\}$$

$$\Rightarrow \quad (2\pi)^3 \delta^3(p+p')$$

$$\int \frac{d^3p\, d^3p'}{(2\pi)^3} \delta^3(p+p') \left\{ \qquad \right\}$$

resolve the integral for delta function.

$$\int \frac{d^3p}{(2\pi)^3} \left\{ -\frac{\omega_p}{4} \left[\qquad \right] + \frac{\omega_p}{4} \left(\qquad \right) \right\}$$

$$\int \frac{d^3p}{(2\pi)^3} \left\{ \frac{\omega_p}{4} \left[a_p a_p + a^\dagger_{-p} a_p + a_p a^\dagger_{-p} + a^\dagger_{-p} a^\dagger_{-p} - a_p a_p - a^\dagger_p a^\dagger_{-p} + a^\dagger_{-p} a_p + a_p a^\dagger_{-p} \right] \right.$$

$$\int \frac{d^3p}{(2\pi)^3} \left\{ \frac{\omega_p}{4} \left[\underline{2 (a_p a^\dagger_{-p} + a^\dagger_p a_p)} \right] \right.$$

$$[a_p, a_p^\dagger] = a_p a_p^\dagger - a_p^\dagger a_p$$

$$[H, a_p^\dagger] = \omega_p a_p^\dagger \; ; \; [H, a] = -\omega_p a_p$$

the energy eigenstates are found by applying $a_p^\dagger$ on $|0\rangle$.

Total momentum operators:-

$$P^i = \int T^{0i} d^3x = -\int d^3x \, \pi(x) \partial_i \phi$$

$$P = -\int d^3x \, \pi(x) \nabla \phi(x)$$

$$= \int \frac{d^3p}{(2\pi)^3} \, p \, a_p^\dagger a_p$$

$$\rightarrow -\int d^3x \left[\iint \frac{d^3p \, d^3p'}{(2\pi)^6} \left(\frac{-i}{2} \sqrt{\omega_p} \, (a_p - a_{-p}^\dagger) \right) \left(ip' (a_{p'} + a_{-p'}^\dagger) \right) \cdot \sqrt{\frac{1}{\omega_{p'}}} \cdot \exp(i(p+p') \cdot x) \right]$$

$$\Leftrightarrow \iint -\int \underline{d^3x}\, \exp(i(p+p')x)\, \frac{d^3p\, d^3p'}{(2\pi)^6} \sqrt{\frac{\omega_p}{\omega_{p'}}}$$

$$\underbrace{(pp')\left(a_p a_{p'} + a_p a^\dagger_{-p'} - a^\dagger_{-p} a_p - a^\dagger_{-p} a^\dagger_{-p'}\right)}$$

$$\iint \cancel{(2\pi)^3}\, \delta(p+p')\, \frac{d^3p\, d^3p'}{\cancel{(2\pi)^6}\,^3} \cdot \sqrt{\frac{\omega_p}{\omega_{p'}}} \left(\quad \right)$$

$$\int p \frac{d^3p}{(2\pi)^3} \left(\quad \right)$$

Since we can switch p with $-p$,

we get

$$\int \overset{\text{odd}}{\frac{p}{2}} \left(\frac{d^3p}{(2\pi)^3}\left(\overset{\text{even}}{\cancel{a_p a_{-p}}}^{\,0} + \overset{\text{even}}{\cancel{a^\dagger_{-p} a^\dagger_p}}^{\,0} + \underbrace{a_{-p} a^\dagger_{-p} + a^\dagger_p a_{-p}}_{a^\dagger a_p + [a_p, a_p^\dagger]}\right)\right)$$

ignore

$$\int \frac{p}{2} d^3p \left(a_p^\dagger a_p\right)$$

☆

$$\int \frac{d^3p}{(2\pi)^3} \frac{\omega_p}{2}\left[a_p^\dagger a_p + [a_p, a_p^\dagger] + a_p^\dagger a_p\right]$$

$$\int \frac{d^3p}{(2\pi)^3} \omega_p\left(a_p^\dagger a_p + \underbrace{\frac{1}{2}[a_p, a_p^\dagger]}_{(2\pi)^3\delta(0)}\right)$$

$\hookrightarrow$ infinity

* Since its the sum of zero point energies of all modes with e, $\frac{\omega_p}{2}$; so this was expected; we ignore this we can only measure energy differences.

$$\boxed{P = \int \frac{d^3p}{(2\pi)^3}\, p\, a_p^\dagger a_p}$$

So the $a^\dagger$ creates "particles" with momentum $\vec{p}$; in the momentum eigenstate

→ From now on ω_p is E_p ✓✓

$E_p = \sqrt{|p|^2 + m^2}$

Since $a_p^\dagger a_q^\dagger |0\rangle$; commute we can write $a_q^\dagger a_p^\dagger |0\rangle$; also

A single mode can contain inf particles; like SHO can be excited arbitrarily high levels.

So KG particles follow Bose-Einstein stats

We want to normalize vacuum states so that $\langle 0 | 0 \rangle = 1$; the one particle states $|p\rangle \propto a_p^\dagger |0\rangle$ will also appear quite often.

→ The simplest normalization; $\langle p | q \rangle = (2\pi)^3 \delta^3(p-q)$

→ But this isn't Lorentz invariant under 3-d boost; under such a boost we have $p_3' = \gamma(p_3 - \beta E)$; $E' = \gamma(E + \beta p_3)$.

Using the delta func. identity.

$$\delta(f(x) - f(x_0)) = \frac{1}{|f'(x_0)|} \delta(x - x_0)$$

$$\rightarrow \delta^3(p-q) = \delta^3(p'-q') \cdot \frac{dp_3'}{dp_3}$$

$$\rightarrow \delta^3(p'-q')\,\gamma\left(1+\beta\frac{dE}{dP_3}\right)$$

$$\delta^3(p'-q')\,\frac{\gamma}{E}\left(E+\beta P_3\right)$$

$$\boxed{\delta^3(p'-q')\,\frac{E'}{E}}$$

~ volume is not invariant under lorrentz transformation; but quantity $E_p\delta^3(p-q)$ is lorrentz invariant - we therefore define

$$|p\rangle = \sqrt{2E_p}\; a_p^\dagger|0\rangle$$

$$\langle p|q\rangle = 2E_p(2\pi)^3\delta^3(p-q)$$

→ On the hilbert space of quantum states, a lorrentz transformation Λ will be implemented as some unitary operator $U(\Lambda)$. Our normalization condition the implies that

$$U(\Lambda)|p\rangle = |\Lambda p\rangle$$

$$U(\Lambda)\, a_p^{\dagger}\, U^{-1}(\Lambda) = \sqrt{\frac{E_{\Lambda p}}{E_p}}\, a^{\dagger}_{\Lambda p}$$

→ With this normalization we must divide $2E_p$ in other places.

For 1-KG particle

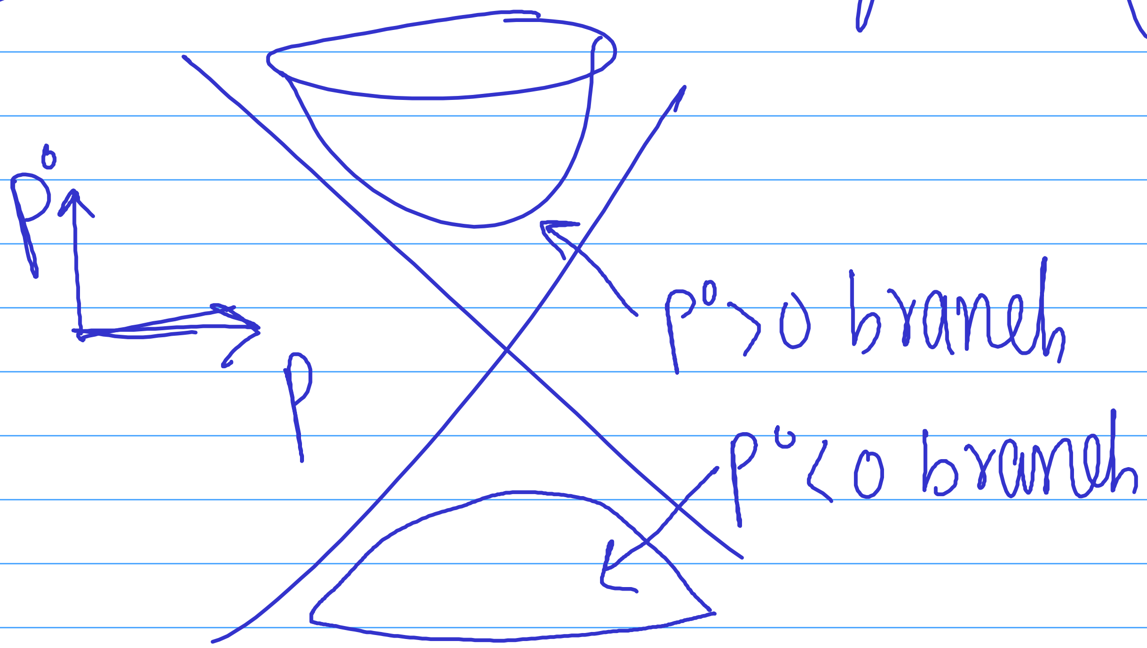
$$(1)_{1\text{-particle}} = \int \frac{d^3p}{(2\pi)^3} |p\rangle \frac{1}{2E_p} \langle p|$$

$$\rightarrow \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} = \int \frac{d^4p}{(2\pi)^4} (2\pi)\delta(p^2-m^2)\Big|_{p^0>0}$$

should be lorentz invariant

→ if $f(p)$ is lorrentz-invariant; so is $\int d^3p\,(f(p)/(2E_p))$.

Can be thought of



as being over the $p^0>0$ branch of the hyperboloid $p^2=m^2$ in 4-momentum. space

Lets see interpretation of $\phi(x)|0\rangle$

$$\int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{1}{2E_p}}\left(a_p e^{ipx} + a_p^{\dagger} e^{-ipx}\right)|0\rangle$$

$$\int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{1}{2E_p}}\, e^{-ipx}\, a_p^{\dagger}|0\rangle$$

$$\therefore |p\rangle = \sqrt{2E_p}\, a_p^{\dagger}|0\rangle$$

$$a_p^{\dagger}|0\rangle = \frac{1}{\sqrt{2E_p}}|p\rangle$$

$$\rightarrow \int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{1}{2E_p}}\cdot\frac{1}{\sqrt{2E_p}}|p\rangle\cdot e^{-ipx}$$

$$\boxed{\int \frac{d^3p}{(2\pi)^3}\left(\frac{1}{2E_p}\right)|p\rangle e^{-ipx}} = \phi(x)|0\rangle$$

So application of $\phi(x)$ creates a particle at position x.

$$\langle 0|\phi(x)|p\rangle = \langle 0| \text{———} \cdot \sqrt{2E_p}\, a_p^\dagger |0\rangle$$

$$= \exp(ip\cdot x)$$

K.G field in space-time

→ In Heisenberg picture operators evolve.

for $\phi(x)$; it evolves:

$$\phi(x) = \phi(x,t) = e^{iHt}\phi(x)e^{-iHt}$$

and similiarly for $\pi(x) = \pi(x,t)$;

for any operator; Heisenberge eq. of motion.

$$i\frac{\partial}{\partial t}O = [O, H]$$

this allows us to compute time dependence; i.e ϕ and π

$$i\frac{\partial}{\partial t}\phi(x,t) = \left[\phi(x,t), \int d^3x \mathcal{H}\right]$$

$$= \left[\phi(x,t), \int d^3x' \left\{\frac{1}{2}\pi^2(x',t) + \frac{1}{2}\left(\nabla\phi(x',t)\right)^2 + \frac{1}{2}m^2\phi^2(x',t)\right\}\right]$$

$$\Rightarrow \int d^3x' \left(i\delta^3(x-x')\pi(x',t)\right)$$

$$\longrightarrow i\pi(x,t)$$

$$\longrightarrow i\frac{\partial}{\partial t}(\pi(x,t)) = \left[\pi(x,t), \int d^3x' \left\{\frac{1}{2}\pi^2(x,t) + \frac{1}{2}\phi(x',t)\left(-\nabla^2 - m^2\right)\phi(x',t)\right\}\right]$$

$$\Rightarrow \int d^3x' \left(-i\delta^3(x-x')\left(-\nabla^2+m^2\right)(\phi(x',t))\right)$$

$$-i\left(-\nabla^2+m^2\right)\phi(x,t)$$

Using those two results gives :-

$$\frac{\partial^2}{\partial t^2}\phi = \left(\nabla^2 - m^2\right)\phi$$

Which is just the Klein-Gordon equation.

$$i\frac{\partial}{\partial t}\phi(x,t) = \left[\phi(x,t), \int d^3x' \left\{ \frac{1}{2}\pi^2(x',t) + \frac{1}{2}(\nabla\phi(x',t))^2 + \frac{1}{2}m^2\phi^2(x',t) \right\}\right]$$

$$\left[\left(e^{iHt}\phi(x)e^{-iHt}\right), \int d^3x' \left\{ \frac{1}{2}\pi^2(x',t) + \frac{1}{2}\left(\underbrace{\nabla\phi(x',t)}_{\nabla\phi = \lim(\phi(x_i,t))}\right)^2 + \underbrace{\frac{1}{2}m^2\phi^2(x',t)}_{\text{will commute}}\right]$$

$\nabla\phi = \lim(\phi(x_i,t))$ ↳ will commute.

↳ will commute.

$$\left[\phi(x,t), \int \frac{d^3x'}{2}\pi^2\right] \sim [A, B^2] \Rightarrow B[A,B]$$

$$\sim \int \frac{d^3x'}{2}\pi(x')\underbrace{[\phi(x,t), \pi(x')]}_{i\delta^3(x-x')} \sim \int \frac{d^3x'}{2}\pi(x')i\delta^3(x-x')$$

for $x = x'$

$\longrightarrow \pi(x)\cdot i$

for

$$i\frac{\partial}{\partial t}\pi(x,t) = \left[\pi(x,t), \int d^3x' \left(\underbrace{\frac{1}{2}\pi^2(x')}_{\text{commute}} + \frac{1}{2}\phi(x')(-\nabla^2+m^2)\phi(x')\right)\right]$$

$$\rightarrow \int \frac{d^3x}{2}\cdot\underbrace{[\pi(x,t), \phi(x)]}_{i\delta^3(x-x')}(-\nabla^2+m^2)\phi(x') \Rightarrow i(-\nabla^2+m^2)\phi(x) = i\frac{\partial}{\partial t}\pi(x,t)$$

$$\frac{\partial^2}{\partial t^2}\phi \Rightarrow \frac{\partial}{\partial t}\pi \Rightarrow -(-\nabla^2+m^2)\phi$$

$$\Rightarrow (\nabla^2 - m^2)\phi$$

We know

$$[H, a_p] = -E_p a_p$$

$H a_p - a_p H = -E_p a_p \rightarrow$ Since E_p is a number

$$H a_p = a_p (H - E_p)$$

for

$$H^n a_p = a_p (H - E_p)^n$$

Useful because; the time evolution e^{iHt} $\mathcal{O}$ e^{-iHt}
↳ Taylor expansion is used

$$e^{iHt} a_p = \left(\sum_{n=0}^{\infty} \frac{(it)^n}{n!} H^n \right) a_p$$

$$= \sum_{n}^{\infty} \frac{(it)^n}{n!} (H^n a_p) \Rightarrow \sum_{n}^{\infty} \frac{(it)^n}{n!} (a_p (H - E_p)^n$$

$$\longrightarrow a_p \sum_{n}^{\infty} \frac{(it)^n}{n!} (H - E_p)^n \sim a_p \exp(i(H - E_p)t)$$

$$e^{iHt} a_p e^{-iHt} \sim a_p (\exp(i(H - E_p)t) \cdot \exp(-iHt)$$

$$\boxed{a_p \exp(iE_p t) = e^{iHt} a_p e^{-iHt}}$$ ~ likewise $$\boxed{a_p^\dagger \exp(iE_p t) = e^{iHt} a_p^\dagger e^{-iHt}}$$

$$\phi(x, t) = e^{iHt} \left[\int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_p e^{i\vec{p}\cdot\vec{x}} + a_p^\dagger e^{-i\vec{p}\cdot\vec{x}} \right) e^{-iHt} \right.$$

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_p e^{-iE_p t} \cdot e^{+i\vec{p}\cdot\vec{x}} + a_p^\dagger e^{-iE_p t} \cdot e^{i\vec{p}\cdot\vec{x}} \right)$$

4-vectors.

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_p e^{\underline{ip\cdot x}} + a_p^\dagger e^{\underline{ip\cdot x}} \right) \Big|_{p_0 = E_p} = \phi(x, t)$$

Causality :-

The prob. amplitude for a particle going from y to x is $\langle 0|\phi(x)\phi(y)|0\rangle$. We will call this quantity $D(x-y)$. Each operator ϕ is a sum of a and $a^\dagger$ operators, but only the term $\langle 0|a_p a_q^\dagger|0\rangle = (2\pi)^3\delta^3(p-q)$ survives in this expression. We are left with

$$\therefore \quad D(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E}e^{-ip\cdot(x-y)}$$

We can get this easily.

$$\phi(x)\phi(y) \Rightarrow \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}} e^{-ip\cdot x}(a_p + a_{-p}^\dagger) \cdot \int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_q}} e^{iq\cdot y}(a_q + a_{-q}^\dagger)$$

$$\Rightarrow \underline{\iint \frac{d^3p\,d^3q}{(2\pi)^6} \cdot e^{-ip\cdot x} e^{iq\cdot y}(a_p + a_{-p}^\dagger)(a_q + a_{-q}^\dagger) \cdot \frac{1}{2\sqrt{E_pE_q}}}$$

$$\downarrow$$

$$\langle 0| \qquad |0\rangle$$

$$\langle 0|\phi(x)\phi(y)|0\rangle = \langle 0|\iint \frac{d^3p\,d^3q}{(2\pi)^6} \frac{e^{ipx}e^{iqy}}{2\sqrt{E_pE_q}}(a_pa_q + a_pa_q^\dagger + a_p^\dagger a_q + a_p^\dagger a_q^\dagger)|0\rangle$$

$$\underbrace{\langle 0|a_pa_q|0\rangle}_{0} + \underbrace{\langle 0|a_pa_q^\dagger|0\rangle}_{\text{non zero.}} + \underbrace{\langle 0|a_p^\dagger a_q|0\rangle}_{0} \qquad \underbrace{\langle 0|a_p^\dagger a_q^\dagger|0\rangle}_{0} \to 0$$

$$\to \iint \frac{d^3p\,d^3q}{(2\pi)^6}\frac{e^{-ipx}e^{iqy}}{\sqrt{4E_pE_q}} \underbrace{\langle p|q\rangle}_{i\delta^3(p-q)(2\pi)^3} \; \Big\} \sim \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}} e^{iP(y-x)} \; ?$$

$$\boxed{\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{-iP(x-y)}}$$

$\therefore$ for timelike case where $x^0 - y^0 = t$ and $\vec{x} - \vec{y} = 0$

$$\sim D(x-y) = \frac{4\pi}{(2\pi)^3}\int \frac{p^2dp}{2E_p}e^{-ip^0t} = \frac{4\pi}{(2\pi)^3}\int \frac{p^2dp}{2\sqrt{p^2+m^2}}\exp(-i\sqrt{p^2+m^2}\,t)$$

$$D(x-y) = \frac{1}{4\pi^2}\int_0^\infty \frac{p^2\,dp\,\exp(-iEt)}{2E}$$

$E^2 = p^2 + m^2$

$2E\,dE = 2p\,dp$

$dp = \frac{E}{p}dE$

$$= \frac{1}{4\pi}\int_m^\infty \frac{\cancel{E}\,p^{\cancel{2}}}{\cancel{p}\,\cancel{E}}\,dE\,\exp(-iEt)$$

$$D(x-y) = \frac{1}{4\pi}\int_m^\infty \sqrt{E^2-m^2}\,\exp(-iEt)\,dE$$

m because for $p=0$ $E=m$

$$\Rightarrow \frac{1}{4\pi}\int_m^\infty \sqrt{E^2-m^2}\,\exp(-iEt)\,dE$$

for $t \to \infty$ } After a lot of time has passed.

$D(x,y) \sim e^{-imt}$

$\therefore$ because $E=m$ gives the most stable point; since e^{-iEt} oscillates between 1 and -1 infinitely fast but it gives the value where this phase change is slowest which is at $E=m$.

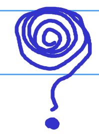
→ It means a quantum state at rest should evolve with e^{-imt};

~ If the integral went to zero, that could mean particles could not exist at rest (atleast in the K.G field).

for causality

↳ Author relied on using the indepence of measurement of $\phi(x)$ and $\phi(y)$; rather than the prob. amplitude of $\phi(x) \to \phi(y)$ in a space like interval.

$$[\phi(x), \phi(y)] \Rightarrow \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\int\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_q}}\underbrace{\left\{\left[a_p e^{-ipx} + a_p^\dagger e^{ipx},\ a_q e^{-iqy} + a_q^\dagger e^{iqy}\right]\right\}}_{C}$$



$$C \to e^{-ipx-iqy}\underbrace{[a_p, a_q]}_{0} + e^{-ipx+iqy}\underbrace{[a_p, a_q^\dagger]}_{i\delta^3(p-q)} + e^{ipx-iqy}\underbrace{[a_p^\dagger, a_q]}_{-i\delta^3(p-q)} + e^{ipx+iqy}\underbrace{[a_p^\dagger, a_q^\dagger]}_{0}$$

Miau?

getting rid of deltas; through integral ($p = q$)

$$\rightarrow \underbrace{\int \frac{d^3p}{2\pi^3} \frac{1}{2E_p}}\left(\underbrace{e^{-ip(x-y)}}_{D(x-y)} - \underbrace{e^{-ip(y-x)}}_{D(y-x)}\right) \quad \text{— I}$$

∴ for space like points $(x-y)^2 < 0$; we can show a continous lorrentz transformation taking $(x-y) \rightarrow (y-x)$; can be done with a boost; so applying that on $D(y-x)$ to get $(D(x-y))$

One can show

$$I = D(x-y) - D(x-y) = 0 = [\phi(x), \phi(y)]$$

Since the measurements are independent ; causality is conserved.

For timelike there is no smooth lorrentz transformation to take $(x-y) \longrightarrow (y-x)$ [you cannot invert their causality] ;

So the $[\phi(x), \phi(y)]$ gives

$\hookrightarrow \sim \exp(-imt) - \exp(imt)$

[so for timelike; measurements can affect one another; which again, makes perfect sense.]

KG propagator :.

lets suppose $x^0 > y^0$; then $\langle 0 | [\phi(x), \phi(y)] | 0 \rangle$ as an integral

↳ commutator

we found it to be

$$\longrightarrow \int \frac{d^3p}{(2\pi)^3} \cdot \frac{1}{2E_p} \cdot \left(\exp(-ip(x-y)) - \exp(-ip(y-x))\right)$$

$$\langle 0| [\phi(x), \phi(y)] |0\rangle = D(x-y) - D(y-x)$$

$$= \int \frac{d^3p}{(2\pi)^3} \cdot \left(\frac{1}{2E_p} \exp(-ip(x-y))\Big|_{E_p=p^0} + \frac{1}{(-2E_p)} \exp(-ip(x-y))\Big|_{-E_p=p_0} \right)$$

$$= \int \frac{d^3p}{(2\pi)^3} \underbrace{\left[\frac{1}{2\pi i} \int_C dp^0 \left(\frac{-1}{p^2-m^2} \right) e^{-ip(x-y)} \right]}_{\to I}$$

As $(p^2-m^2) = (p^{0\,2} - p^2 - m^2) \Rightarrow (p^{0\,2} - E_p)^2 \Rightarrow (p+E_0)(p-E_0)$

$\Rightarrow$

$$I = \left[\frac{1}{2\pi i} \int_C dp^0 \left(\frac{-1}{(p+E_p)(p-E_p)} \right) e^{-ip(x-y)} \right]$$

evaluate integral using Residue theorem.

$$— = -2\pi i \sum \text{Residues} = -2\pi i \left(\frac{-1}{2E_p} \exp(-ip(x-y)) + \frac{1}{2E_p} \exp(i(p(x-y))) \right)$$

putting in the integral

$$\frac{1}{2\pi i} \left(+2\pi i \left(\frac{1}{2E} \left(\exp(-ip(x-y)) - \exp(ip(x-y)) \right) \right) \right)$$

putting in main integral

$$\int \frac{d^3p}{(2\pi)^3} \cdot \frac{1}{2E} \left(\quad \right)$$

So this is why we wrote as contour.

$$D_r(x-y) = \theta(x^0-y^0)\underbrace{\langle 0|\phi(x)\phi(y)|0\rangle}_{\text{comm}}$$

$$(\partial^2+m^2)D_r(x-y) = \partial^2(\theta(x^0-y^0)(\text{comm}))+\partial^\mu\theta(x^0-y^0)\partial_\mu(\text{comm})$$
$$+\underbrace{(\partial^2+m^2)(\text{comm})}_{0}$$

$$= \partial^2\theta(x-y)(\text{Commutator}) + 2\partial^\mu\theta\,\partial_\mu(\text{comm})$$

$$= \dot\delta(x^0-y^0)(\text{comm}) + 2\delta(x^0-y^0)\partial_\mu(\text{comm})$$

$$\dot\delta f(t) = -\dot f\,\delta(t) \quad + \quad 2\delta(x^0-y^0)\partial_\mu(\text{comm})$$

$$-[\pi(x),\phi(y)]\delta(x^0-y^0) + 2\delta(x^0-y^0)[\pi(x),\phi(y)]$$

$$= \delta(x^0-y^0)(-i\delta^3(x-y)) \Rightarrow -i\delta^4(x-y)$$

Also could've derived this as:-

$$D_r(x^0-y^0) = \int d^4p\, \exp(-ip(x-y))\tilde D_R(x-y)$$

and

$$(-p^2+m^2)\tilde D_r = -i \Rightarrow \tilde D_r = \frac{i}{p^2-m^2}$$

So

$$D_R = \int d^4p\, \frac{i}{p^2-m^2}\exp(-ip(x-y)) \quad \Big\}\ \text{easy}$$

$$\rightarrow D_F = \int d^4p\, \frac{i}{p^2-m^2+i\epsilon}\exp(-ip(x-y))$$

the infinitesimal $i\varepsilon$ shifts one pole up and the other plane down in the complex plane

$p^0 = E_p - i\varepsilon$ and $p_0 = E_p + i\varepsilon$

the residue calculations yield

$$D_f = \theta(x^0 - y^0)\langle 0|[\phi(x)\phi(y)]|0\rangle + \theta(y^0 - x^0)\langle 0|[\phi(x), \phi(y)]|0\rangle$$

$\boxed{\text{Stop here to do a project}}$