Chapter : 3 — Dirac field

if phi is some field or a collection of fields and D is some diff. operator, then "$D\phi = 0$; being relativistically invariant means; after boost or rotation $\phi(x,t)$ should satisfy the same eq.

Equation of motion would satisfy Lorentz invariant if it follows a scalar Lagrangian

$$x^\mu \longrightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

for some 4×4 matrix Λ

$$\phi \longrightarrow \phi' = \phi(\Lambda^{-1}x)$$

if field has an extremum at x; then the transformed extremum will be $x \longrightarrow x' = \Lambda x$

The transformation $\partial_\mu \phi$ is

$$\partial_\mu \phi \longrightarrow \partial_\mu(\phi(\Lambda^{-1}x)) \Rightarrow (\Lambda^{-1})^\nu{}_\mu (\partial_\nu \phi)(\Lambda^{-1}x)$$

lets define a dummy variable $y^\nu = \Lambda^\nu_\mu x^\mu$

$\partial_\mu \phi'(x)$

$$\hookrightarrow \partial_\mu(\phi(y^\nu)) \Rightarrow \frac{\partial \phi(y^\nu)}{\partial x^\mu} \Rightarrow \frac{\partial y^\nu}{\partial x^\mu} \cdot \frac{\partial \phi(y^\nu)}{\partial y^\nu}$$

$$\Rightarrow \frac{\partial(\Lambda^{-1}{}^\nu{}_\sigma x^\sigma)}{\partial x^\mu} \cdot \frac{\partial \phi(y^\nu)}{\partial y^\nu} \Rightarrow$$

$$\hookrightarrow (\Lambda^{-1})^\nu{}_\sigma \delta^\sigma_\mu \Rightarrow (\Lambda^{-1})^\nu{}_\mu$$

$$(\Lambda^{-1})^\nu{}_\mu (\partial_\mu \phi)(\Lambda^{-1}x) \longrightarrow A$$

for lagrangian term $(\partial_\mu \phi)(\partial^\nu \phi) \Rightarrow g^{\mu\nu}(\partial_\mu \phi)(\partial_\mu \phi)$

transforming this would give

kinetic term

$$g^{\mu\nu}(\partial_\mu \phi')(\partial_\nu \phi')$$

$$g^{\mu\nu}((\Lambda^{-1})^\lambda{}_\mu(\partial_\lambda \phi(\Lambda^{-1}x))((\Lambda^{-1})^\sigma{}_\nu(\partial_\sigma \phi(\Lambda^{-1}x)))$$

$$= g^{\lambda\sigma}(\partial_\lambda \phi)(\partial_\sigma \phi)(\Lambda^{-1}x)$$

$$\Rightarrow (\partial_\mu \phi)^2 (\Lambda^{-1}x) \quad — (2) \qquad \text{So (2) transformed like a scalar}$$

$$\mathcal{L}(x) \longrightarrow \mathcal{L}(\Lambda^{-1}x)$$

Since lagrange transformed like scalar; it is lorentz invariant.

under 3D rotations, $V^i(x) \longrightarrow R^{ij} V^j (R^{-1}x)$

under lorentz transformation, $V^\mu(x) \rightarrow \Lambda^\mu_\nu V^\nu(\Lambda^{-1}x)$

For a linear tranformations

$$\phi \longrightarrow \phi'(x) = M_{ab}(\Lambda)\phi(\Lambda^{-1}x)$$

we try to find the general form of M_{ab} by finding what satisfies $M(\Lambda)M(\Lambda')$ leading to another transformation $M(\Lambda'')$

$$\phi \rightarrow M(\Lambda)M(\Lambda')\phi = M(\Lambda'')\phi$$

↳ whatever satisfies this, will be allowed form of transformations.

for $\Lambda'' = \Lambda\Lambda'$.

So the correspondance between M and their Λ's must be conserved under multiplication, or we can say that the matrices M must form an n dimensional representation of the lorentz group.

Before moving onto lorentz group; lets look at rotation group; it has a representation for all dimensionalities n; and is related to spin quantum number by $n = 2S+1$; for $S = 1/2$; $n=2$; giving 2×2 matrices with $\det = 1$; which can be expressed as

$$U = \exp(-i\theta^i \sigma^i/2)$$

For any group of transformations that lies inf. close to identity define a vector space with basis called generators. For rotation group; generators are ang. momentum operators. J^i with comm. relations.

$$[J^i, J^j] = \varepsilon_{ijk} J^k$$

By exponentiating J^i we form finite rotation operators.

$$R = \exp(-i\theta^i J^i) \sim \text{gives rotation by angle } |\theta| \text{ around axis } \hat{\theta}.$$

Thus a set of matrices satisfying $[J^i, J^j] = \epsilon_{ijk} J^k$, will form rotations as $\uparrow$; so for spin $\frac{1}{2}$;

$$\boxed{J^i = \frac{\sigma^i}{2}}$$

So if we can find matrices of generators of transformation group we can form matrix rep. of the transformation group. So we need to know commutation relations of gen. of Lorentz group. For rotation we can do this by using operators;

$$J = x \times p \qquad [\text{for } 3d]$$

but generally we can write in form of anti-sym tensor.

$$J^{12} = -i(x^i \nabla^j - x^j \nabla^i) \text{ ; then we can go to 4D using}$$

$$\sim -i(x^\mu \nabla^\nu - x^\nu \nabla^\mu)$$

But to find commutation rules.

$$[J^{\mu\nu}, J^{\sigma\lambda}] = -i\left[(x^\mu \partial^\nu - x^\nu \partial^\mu), (x^\sigma \partial^\lambda - x^\lambda \partial^\sigma)\right]$$ } mf called this trivial

$$-i\left\{(x^\mu \partial^\nu - x^\nu \partial^\mu)(x^\sigma \partial^\lambda - x^\lambda \partial^\sigma) - (x^\sigma \partial^\lambda - x^\lambda \partial^\sigma)(x^\mu \partial^\nu - x^\nu \partial^\mu)\right\}$$

$$\Rightarrow i\left\{x^\mu \partial^\nu x^\sigma \partial^\lambda - x^\mu \partial^\nu x^\lambda \partial^\sigma - x^\nu \partial^\mu x^\sigma \partial^\lambda + x^\nu \partial^\mu x^\lambda \partial^\sigma - x^\sigma \partial^\lambda x^\mu \partial^\nu + x^\sigma \partial^\lambda x^\nu \partial^\mu \right.$$
$$\left. + x^\lambda \partial^\sigma x^\mu \partial^\nu - x^\lambda \partial^\sigma x^\nu \partial^\mu \right\}$$

using

$$[\partial^\mu, x^\nu] = g^{\mu\nu} \quad - \quad \partial^\mu x^\nu = g^{\mu\nu} + x^\nu \partial^\mu$$

$$\rightarrow i^2\left\{x^\mu (x^\sigma \partial^\nu + g^{\nu\sigma})\partial^\lambda - x^\mu (g^{\nu\lambda} + x^\lambda \partial^\nu)\partial^\sigma - x^\nu (g^{\mu\sigma} + x^\sigma \partial^\mu)\partial^\lambda + \right.$$
$$\left. x^\nu (g^{\mu\lambda} + x^\lambda \partial^\mu)\partial^\sigma - x^\sigma (g^{\lambda\mu} + x^\mu \partial^\lambda)\partial^\nu + x^\sigma (g^{\lambda\nu} + x^\nu \partial^\lambda)\partial^\mu - x^\lambda (g^{\sigma\nu} + x^\nu \partial^\sigma)\partial^\mu + x^\lambda (g^{\sigma\mu} + x^\mu \partial^\sigma)\partial^\nu \right\}$$

$$\rightarrow -1\Big\{ x^\mu x^\sigma \underline{\underline{\partial^\nu \partial^\lambda}} + x^\mu g^{\nu\sigma}\partial^\lambda - x^\mu x^\lambda \partial^\nu \partial^\sigma - x^\mu g^{\nu\lambda}\partial^\sigma - x^\nu g^{\mu\sigma}\partial^\lambda - x^\nu x^\sigma \partial^\mu \partial^\lambda + x^\nu g^{\mu\lambda}\partial^\sigma + x^\nu x^\lambda \partial^\mu \partial^\sigma$$
$$- x^\sigma g^{\lambda\mu}\partial^\nu - x^\sigma x^\mu \partial^\lambda \partial^\nu + x^\sigma g^{\lambda\nu}\partial^\mu + x^\sigma x^\nu \partial^\lambda \partial^\mu - x^\lambda g^{\sigma\nu}\partial^\mu - x^\lambda x^\nu \partial^\sigma \partial^\mu + x^\lambda g^{\sigma\mu}\partial^\nu + x^\lambda x^\mu \partial^\sigma \partial^\nu \Big\}$$

double derivatives terms cancel out since $[\partial^i, \partial^j] = 0$ and $[x^\nu, x^\mu] = 0$

$$= -1 \Big\{ x^\mu g^{\nu\sigma}\partial^\lambda - x^\mu g^{\nu\lambda}\partial^\sigma - x^\nu g^{\mu\sigma}\partial^\lambda + x^\nu g^{\mu\lambda}\partial^\sigma + x^\sigma g^{\lambda\nu}\partial^\mu + x^\lambda g^{\sigma\mu}\partial^\nu - x^\lambda g^{\sigma\nu}\partial^\mu - x^\sigma g^{\lambda\mu}\partial^\nu \Big\} \Rightarrow$$

$$i\Big\{ g^{\nu\sigma} i(x^\mu \partial^\lambda - x^\lambda \partial^\mu) + g^{\mu\lambda} i(x^\nu \partial^\sigma - x^\sigma \partial^\nu) + i g^{\lambda\nu}(x^\sigma \partial^\mu - x^\mu \partial^\sigma) + i g^{\sigma\mu}(x^\lambda \partial^\nu - x^\nu \partial^\lambda) \Big\}$$

$$= i\Big\{ g^{\nu\sigma} J^{\mu\lambda} + g^{\mu\lambda} J^{\nu\sigma} + g^{\lambda\nu} J^{\sigma\mu} + g^{\sigma\mu} J^{\lambda\nu} \Big\} \quad (\text{verified}) \text{ — } C$$

We pull a specific representation out of a hat

$$(J^{\mu\nu})_{\alpha\beta} = i(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha)$$

lets see if they satisfy C

Since we are working in minkowski space; lets define.

$$(J^{\mu\nu})^\alpha{}_\beta = g^{\alpha\sigma}(J^{\mu\nu})_{\sigma\beta} \Rightarrow i g^{\alpha\sigma}(\delta^\mu_\sigma \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\sigma)$$

$$i(g^{\alpha\mu}\delta^\nu_\beta - g^{\alpha\nu}\delta^\mu_\beta) = (J^{\mu\nu})^\alpha{}_\beta$$

$$\left[(J^{\mu\nu})^\alpha{}_\beta, (J^{\nu\mu})^\alpha{}_\beta\right] = \underbrace{(J^{\mu\nu})^\alpha{}_\gamma (J^{\nu\mu})^\gamma{}_\beta}_{\mathrm{I}} - \underbrace{(J^{\nu\mu})^\alpha{}_\gamma (J^{\mu\nu})^\gamma{}_\beta}_{\mathrm{II}}$$

$\rightarrow$

$$\mathrm{I} \rightarrow i^2(g^{\alpha\mu}\delta^\nu_\gamma - g^{\nu\alpha}\delta^\mu_\beta)(g^{\gamma\nu}\delta^\mu_\beta - g^{\gamma\mu}\delta^\nu_\beta)$$

$$i^2(g^{\alpha\mu}\delta^\nu_\gamma g^{\gamma\nu}\delta^\mu_\beta - g^{\alpha\mu}\delta^\nu_\gamma g^{\gamma\mu}\delta^\nu_\beta - g^{\nu\alpha}\delta^\mu_\beta g^{\gamma\nu}\delta^\mu_\beta + g^{\alpha\mu}\delta^\mu_\gamma g^{\gamma\mu}\delta^\nu_\beta)$$

$$\rightarrow i(g^{\alpha\mu} g^{\gamma\nu}\delta^\mu_\beta - g^{\alpha\mu} g^{\nu\mu}\delta^\nu_\beta - g^{\nu\alpha} \text{———})$$

Check commutator yourself !!

but it does satisfy; I will try to upload proofs from my handwritten notes (if I'm not able to do so, you can further evaluate products of J.

We are mostly interested in lorrentz group for spin-½ representation; Suppose we have 4 n×n matrices γ^{μ} that satisfy the anti-commutation relations

$$\{\gamma^{\mu},\gamma^{\nu}\} = \gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu}\cdot I_{n\times n}$$

Then we could immediately write down an n-dimensional representation of the lorrentz algebra.

$$S^{\mu\nu} = \frac{i}{4}[\gamma^{\mu},\gamma^{\nu}]$$

We can check $S^{\mu\nu}$ satisfies commutator like J; lets try to check

$$[S^{\mu\nu},S^{\sigma\lambda}] = \frac{i^2}{16}\left[\left[(\gamma^{\mu}\gamma^{\nu}-\gamma^{\nu}\gamma^{\mu})(\gamma^{\sigma}\gamma^{\lambda}-\gamma^{\lambda}\gamma^{\sigma})\right] - \text{opposite}\right]$$

$$= \frac{i^2}{16}\left((\gamma^{\mu}\gamma^{\nu}+\gamma^{\mu}\gamma^{\nu}-2g^{\mu\nu})(\gamma^{\sigma}\gamma^{\lambda}+\gamma^{\sigma}\gamma^{\lambda}-2g^{\sigma\lambda}) - \text{opposite}\right)$$

$$= \frac{i^2}{\cancel{16}_{4}}\left(\cancel{4}(\gamma^{\mu}\gamma^{\nu}-g^{\mu\nu})(\gamma^{\sigma\lambda}-g^{\sigma\lambda})\right) \Rightarrow \frac{i^2}{4}(\gamma^{\mu}\gamma^{\nu}\gamma^{\sigma}\gamma^{\lambda} - \gamma^{\mu}\gamma^{\nu}g^{\sigma\lambda} - g^{\mu\nu}\gamma^{\sigma}\gamma^{\lambda} + g^{\mu\nu}g^{\sigma\lambda}) - \text{opposite}$$

now we arrange gammas so they cancel out and after some (alot more algebra) you will get.

$$\boxed{i(g^{\nu\sigma}S^{\mu\lambda} - g^{\mu\sigma}S^{\nu\lambda} - g^{\nu\lambda}S^{\mu\sigma} + g^{\mu\lambda}S^{\nu\sigma})}$$

This representation goes through in any dimensionality, with the lorrentz or Euclidean metric. It should work in three-dimensional Euclidean space and in fact, we can simply

$$\gamma^j = i\gamma^j \quad \text{(Pauli Sigma matrices)}$$

so that

$$\{\gamma^i, \gamma^j\} = -2\delta^{ij}$$

the factor of i in the first line and the minus sign in the second line are purely conventional.
the matrices representing the lorentz algebra are then:

$$S^{ij} = \frac{1}{2}\epsilon^{ijk}\sigma^k$$

which we recognize as the two-dimensional representation of the rotation group.

Lets find these γ^μ for four-dimensional Minskowski space.
(Whole argument of dimensionality, properties from Dirac eq.)

→ One representation is 2x2 block form.

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad ; \quad \gamma^i = \begin{pmatrix} 0 & +\sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

We call this weyl representation (or Chiral representation).

Without 2x2 block form; lets say 2x2; it won't work because we can't get the fourth matrix.
(In these notes, I will follow that rather than any other forms)
The boost and rotation gen. in this rep. are.

$$S^{0i} = [\gamma^0, \gamma^i] = \begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix}$$

and

$$S^{ij} = [\gamma^i, \gamma^j] = \frac{1}{2}\epsilon_{ijk}\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}$$

A 4D field that transforms under these transformations ~~are~~ is called Dirac Spinor.

We know the transformation laws for ψ now; lets write a field eq; starting with KG

$$(\partial^2+m^2)\psi=0$$

this can work but we can get another eq; with much more info. First we verify another property of γ matrices:

first compute

$$[\gamma^\mu, S^{\sigma\lambda}]=?\quad \gamma^\mu\left(\frac{i}{4}[\gamma^\sigma,\gamma^\lambda]\right)\Rightarrow\frac{i\gamma^\mu}{4}(\gamma^\sigma\gamma^\lambda-\gamma^\lambda\gamma^\sigma)$$

$$=\frac{i}{4}(\gamma^\mu\gamma^\sigma\gamma^\lambda-\gamma^\mu\gamma^\lambda\gamma^\sigma)\Rightarrow\frac{i}{4}\left[\gamma^\mu(2g^{\sigma\lambda}-\gamma^\lambda\gamma^\sigma)-\gamma^\mu\gamma^\lambda\gamma^\sigma\right]$$

$$\Rightarrow\frac{i}{2}\left[\gamma^\mu g^{\sigma\lambda}-\gamma^\mu\gamma^\lambda\gamma^\sigma-\gamma^\mu\gamma^\lambda\gamma^\sigma\right]\Rightarrow\frac{i}{2}\left[\frac{\gamma^\mu g^{\sigma\lambda}}{2}-2\gamma^\mu\gamma^\lambda\gamma^\sigma\right]$$

also compute

$$\frac{i}{4}(\gamma^\sigma\gamma^\lambda-\gamma^\lambda\gamma^\sigma)\gamma^\mu\Rightarrow\frac{i}{4}(\gamma^\sigma\gamma^\lambda\gamma^\mu-\gamma^\lambda\gamma^\sigma\gamma^\mu)\Rightarrow\frac{i}{4}\left((2g^{\sigma\lambda}_\mu-\gamma^\lambda\gamma^\sigma\gamma^\mu-\gamma^\lambda\gamma^\sigma\gamma^\mu)\right)$$

$$=\frac{i}{4}(g^{\sigma\lambda}\gamma^\mu-2\gamma^\lambda\gamma^\sigma\gamma^\mu)$$

now write commutator

$$\frac{i}{4}(\cancel{g^{\sigma\lambda}\gamma^\mu}-\cancel{g^{\sigma\lambda}\gamma^\mu}-2\gamma^\mu\gamma^\lambda\gamma^\sigma+2\gamma^\lambda\gamma^\sigma\gamma^\mu)$$

$$\Rightarrow\frac{i}{2}(\gamma^\lambda\gamma^\sigma\gamma^\mu-\gamma^\mu\gamma^\lambda\gamma^\sigma)\Rightarrow\frac{i}{2}(\gamma^\lambda\gamma^\sigma\gamma^\mu-(2g^{\mu\lambda}+\gamma^\lambda\gamma^\mu)\gamma^\sigma)$$

$$\frac{i}{2}(\gamma^\lambda\gamma^\sigma\gamma^\mu-2g^{\mu\lambda}\gamma^\sigma-\gamma^\lambda\gamma^\mu\gamma^\sigma)\Rightarrow\frac{i}{2}(\gamma^\lambda\gamma^\sigma\gamma^\mu-2g^{\mu\lambda}\gamma^\sigma-\gamma^\lambda(2g^{\mu\sigma}+\gamma^\sigma\gamma^\mu))$$

$$\Rightarrow\frac{i}{2}(\cancel{\gamma^\lambda\gamma^\sigma\gamma^\mu}-2g^{\mu\lambda}\gamma^\sigma\quad-\cancel{\gamma^\lambda\gamma^\sigma\gamma^\mu}-\gamma^\lambda(2g^{\mu\sigma}))$$

$$\Rightarrow\frac{i}{2}(-2g^{\mu\lambda}\gamma^\sigma+2g^{\mu\sigma}\gamma^\lambda)$$

{ might have messed up signs

$$[\gamma^\mu, S^{\sigma\lambda}] = i\left(g^{\mu\sigma}\gamma^\lambda - g^{\mu\lambda}\gamma^\sigma\right) \quad \text{— } \textcircled{I}$$

$$\textcircled{I} \longrightarrow \left(\mathcal{J}^{\rho\sigma}\right)^\mu{}_\nu \gamma^\nu = i\left(\delta^\rho_\nu g^{\sigma\mu} - \delta^\sigma_\nu g^{\rho\mu}\right)\gamma^\nu$$

$$= i\left(g^{\sigma\mu}\gamma^\rho - g^{\rho\mu}\gamma^\sigma\right)$$

then the infinitessimal lorrentz transformation of γ matrices is this

or equivalently,

$$(1 + \underbrace{\tfrac{i}{2}\omega_{\rho\sigma}}_{\varepsilon} S^{\rho\sigma})\gamma^\mu(1 - \tfrac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma}) = (1 - \tfrac{i}{2}\omega_{\rho\sigma}\mathcal{J}^{\rho\sigma})^\mu{}_\nu\gamma^\nu.$$

↳ expand and ignore ε^2 terms

$$(1+\varepsilon S)\gamma^\mu(1-\varepsilon S) \Rightarrow (\gamma^\mu + \varepsilon S\gamma^\mu)(1-\varepsilon S)$$

(ignore)

$$\Rightarrow \gamma^\mu - \gamma^\mu\varepsilon S + \varepsilon S\gamma^\mu - \cancel{\varepsilon^2 S^2\gamma^\mu}$$

$$\Rightarrow \gamma^\mu - \gamma^\mu\varepsilon S + \varepsilon S\gamma^\mu$$

$$\gamma^\mu + \varepsilon(S\gamma^\mu - \gamma^\mu S) \Rightarrow \gamma^\mu + \varepsilon[S, \gamma^\mu]$$

we computed this

$$\Rightarrow \gamma^\mu + \varepsilon(-\mathcal{J}^{\rho\sigma})^\mu{}_\nu\gamma^\nu$$

$$\delta^\mu{}_\nu\gamma^\nu - \frac{i}{2}\omega_{\rho\sigma}(\mathcal{J}^{\rho\sigma})^\mu{}_\nu\gamma^\nu \Rightarrow \left(\delta^\mu_\nu - \frac{i}{2}\omega_{\rho\sigma}(\mathcal{J}^{\rho\sigma})^\mu_\nu\right)\gamma^\nu$$

$$\Rightarrow \left(1 - \frac{i}{2}\omega_{\rho\sigma}\mathcal{J}^{\rho\sigma}\right)^\mu_\nu\gamma^\nu$$

We exponentiate the infinitessimal to get finite trans.

$$\Lambda_{1/2} = \exp\left(-\frac{i}{2}\,\omega_{\mu\nu}S^{\mu\nu}\right)$$

$$\Lambda_{1/2}^{-1}\,\gamma^{\mu}\,\Lambda_{1/2} = \Lambda^{\mu}{}_{\nu}\,\gamma^{\nu}$$

$\Big\{$ you can verify this by using hadamard lemma.

This means γ matrices are invariant under lorrentz transformation; $\Lambda_{1/2}$ is the spinor representation of lorrentz transformation, So we now write the Dirac eq.

$$(i\gamma^{\mu}\partial_{\mu} - m)\,\psi(x) = 0$$

lets check the lorrentz invariance; check by transforming

$$(i\gamma^{\mu}\partial_{\mu} - m)\,\psi(x) \longrightarrow \left\{ \begin{array}{l} \text{remember } \psi(x) \to \psi'(x') = \Lambda_{1/2}\psi(\Lambda^{-1}x) \\ \partial_{\mu} \to \partial'_{\mu} = (\Lambda^{-1})^{\mu}{}_{\nu}\,\partial_{\nu} \end{array}\right.$$

$$\Big((i\gamma^{\mu}(\Lambda^{-1})^{\mu}{}_{\nu}\,\partial_{\nu} - m)\,\Lambda_{1/2}(\psi(\Lambda^{-1}x))\Big)$$

$\times\ \Lambda\Lambda^{-1}$ on leftside.

$$\Lambda\Lambda^{-1}\big(i\gamma^{\mu}(\Lambda^{-1})^{\mu}{}_{\nu}\,\partial_{\nu}\Lambda - m\Lambda\big)\,\psi(\Lambda^{-1}x)$$

component

$$\Lambda\big(i\Lambda^{-1}\gamma^{\mu}\Lambda(\Lambda^{-1})^{\mu}{}_{\nu}\,\partial_{\nu} - \Lambda^{-1}m\Lambda\big)\,\psi(\Lambda^{-1}x)$$

$$\Lambda\big(i\underbrace{(\Lambda)^{\mu}{}_{\sigma}\,\gamma^{\sigma}\,(\Lambda^{-1})^{\mu}{}_{\nu}}_{\delta^{\sigma}_{\nu}}\,\partial_{\nu} - m\big)\,\psi(\Lambda^{-1}x)$$

$$= \Lambda\big(i\gamma^{\nu}\partial_{\nu} - m\big)\,\psi(\Lambda^{-1}x)$$

$\hookrightarrow = 0$ since $(i\gamma^\nu \partial_\nu - m)\psi(x) = 0$
then $\Lambda(\psi) = 0$

Also we can recover KG eq. back

$$0 = (-i\gamma^\nu \partial_\nu - m)(i\gamma^\mu \partial_\mu - m)$$

$$= -i^2 \gamma^\nu \partial_\nu \gamma^\mu \partial_\mu + \cancel{im\gamma^\nu \partial_\nu} - \cancel{im\gamma^\mu \partial_\mu} + m^2$$

$$= \gamma^\nu \partial_\nu \gamma^\mu \partial_\mu + m^2$$

Since $\partial_\mu \partial_\nu$ are symmetric; only symmetric part of $\gamma^\nu \gamma^\mu$ will survive

$$\boxed{\gamma^\nu \gamma^\mu + \gamma^\mu \gamma^\nu = 2g^{\mu\nu}}$$

$\therefore \gamma^\mu \gamma^\nu = \underline{\frac{1}{2}\{\gamma^\mu, \gamma^\nu\}} + \frac{1}{2}[\gamma^\mu, \gamma^\nu]$

$\Rightarrow$

$$\Big(\frac{1}{2}\underbrace{\{\gamma^\mu, \gamma^\nu\}}_{2g^{\mu\nu}} \partial_\mu \partial_\nu + m^2\Big)$$

$\Rightarrow$

$$\frac{1}{\cancel{2}} \cancel{2} g^{\mu\nu} \partial_\mu \partial_\nu + m^2$$

$$\boxed{\Rightarrow (\partial^2 + m^2)\psi(x) = 0} \longrightarrow \text{KG eq}$$

now to form a lorrentz scalar from Dirac spinors; we define

$$\bar{\psi} = \psi^\dagger \gamma^0$$

as $\psi^\dagger \psi$ is not lorrentz scalar as $\psi^\dagger \underbrace{\Lambda^\dagger \Lambda}_{\text{not } I \text{ as } \Lambda \text{ is non unitary}} \psi$

So

$\bar{\psi}\psi$ is a lorrentz scalar; check yourself

$$\bar{\psi}\psi \longrightarrow \psi^\dagger \gamma^0 \Lambda^\dagger \Lambda \psi$$

also check $\bar{\psi}\gamma^\mu \psi$

The Dirac Lagrangian is then.

$$\mathcal{L}_{Dirac} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$$

Weyl spinors:

$$S^{0i} = \frac{i}{4}[\gamma^0, \gamma^i] = -\frac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix},$$

and

$$S^{ij} = \frac{i}{4}[\gamma^i, \gamma^j] = \frac{1}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \equiv \frac{1}{2}\epsilon^{ijk}\Sigma^k.$$

From these two eq. ; it is apparent that Dirac rep of Lorrentz group is reducible ; since its in Block form; we can form 2D rep by considering each block separately , writing

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

the corresponding transformation laws are

$$\psi_L \to \left(1 - i\theta \cdot \frac{\sigma}{2} - \beta \cdot \frac{\sigma}{2}\right)\psi_L$$

$$\psi_R \to \left(1 - i\theta \cdot \frac{\sigma}{2} + \beta \cdot \frac{\sigma}{2}\right)\psi_R$$

Plan ; after brief discussion; move to solutions, some props.

Then we'll move onto quantization, propagators and symmetries; then to perturbation theory.

In terms of ψ_L and ψ_R ; the dirac equation.

$$(i\gamma^\mu \partial_\mu - m)\psi = \begin{pmatrix} -m & i(\partial_0 + \sigma \cdot \nabla) \\ i(\partial_0 - \sigma \cdot \nabla) & -m \end{pmatrix}\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0$$

$$\left.\begin{array}{l} i(\partial_0 + \sigma \cdot \nabla)\psi_R - m\psi_L = 0 \\ i(\partial_0 - \sigma \cdot \nabla)\psi_L - \psi_R = 0 \end{array}\right\}$$ coupled in mass but if $m = 0$; they become decoupled.

will be used for neutrinos later.

→ Free particle solutions of Dirac eq.

Since it forms K.G.; we can write solutions as

$$\psi(x) = U(p)\, e^{-ipx} \quad ; \qquad \text{where } p^2 = m^2$$

plugging into Dirac eq.

$$(i\gamma^\mu \partial_\mu - m)\, U(p)\, e^{-ipx} = 0$$

lets analyze in the rest frame
$p = (m, 0)$

$$(\gamma^0 (m) - m)\, U(p) = 0$$

then we can find the general form of $U(p)$ by boosting this in the rest frame by Λ.

$$(m\gamma^0 - m)\, U(p) = 0$$

writing in matrix form

$$m\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - m\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} U(p) = 0$$

$$m\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} U(p) = 0$$

[solve this, easy $\sim \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \phi \\ \chi \end{pmatrix} = 0 \sim \phi = \chi = \xi$, $\chi = \phi = \xi$, $U(p_0) \sim \begin{pmatrix} \xi \\ \xi \end{pmatrix}$

$$U(p^0) = \sqrt{m}\begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

$\sim \xi$ for any two component spinor.

also normalize

$$\xi^\dagger \xi = 1$$

$$\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Now we can boost to find $U(p)$ from $U(p^0)$

$$U(p) = \Lambda\, U(p_0) \Rightarrow \exp\left(-\frac{1}{2}\eta\, \hat{p}\cdot S\right) \sqrt{m}\begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

lets consider a boost in z-direction.

In infintessimall form; it will transform 4-momentum like.

$$\begin{pmatrix} E \\ p_3 \end{pmatrix} = \left(1 + \eta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \begin{pmatrix} m \\ 0 \end{pmatrix}$$

for finite η. The transformation is

$$\left(\exp\left(\eta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \right) \begin{pmatrix} m \\ 0 \end{pmatrix} \longrightarrow \left[\sinh\left(\eta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) + \cosh\left(\eta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) \right] \begin{pmatrix} m \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} m \cosh \eta \\ m \sinh \eta \end{pmatrix}$$

Remember that

rotations

$$S^{ij} = \varepsilon_{ijk} \begin{pmatrix} \frac{\sigma^k}{2} & 0 \\ 0 & \frac{\sigma^k}{2} \end{pmatrix}$$

$$S^{0i} = \frac{i}{2} \begin{pmatrix} \sigma_i & \\ & -\sigma_i \end{pmatrix}$$

$$S^{03} = \frac{i}{2} \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix} \checkmark$$

Applying the same boost to $U(p^0)$

$$U(p) = \exp\left(-\frac{1}{2} \eta \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix} \right) \begin{pmatrix} \xi \\ \xi \end{pmatrix} \sqrt{m}$$

$$= \left[\cosh\left(\frac{1}{2}\eta\right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \sinh\left(\frac{1}{2}\eta\right) \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix} \right] \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

$$= \left[\frac{e^{\eta/2} + e^{-\eta/2}}{2} \begin{pmatrix} & \\ & \end{pmatrix} - \left(\frac{e^{\eta/2} - e^{-\eta/2}}{2} \right) \begin{pmatrix} & \\ & \end{pmatrix} \right] \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

$$= \begin{pmatrix} \frac{e^{\eta/2} + e^{-\eta/2} - e^{\eta/2}\sigma_3 + e^{-\eta/2}\sigma_3}{2} & 0 \\ 0 & \frac{e^{\eta/2} + e^{-\eta/2} + e^{\eta/2}\sigma_3 - e^{-\eta/2}\sigma_3}{2} \end{pmatrix}$$

$$= \begin{pmatrix} e^{\eta/2}\left(\frac{1-\sigma_3}{2}\right) + e^{-\eta/2}\left(\frac{1+\sigma_3}{2}\right) & 0 \\ 0 & e^{\eta/2}\left(\frac{1+\sigma_3}{2}\right) - e^{\eta/2}\left(\frac{1+\sigma_3}{2}\right) \end{pmatrix}$$

use $\sqrt{m}\, e^{\pm\eta/2} = \sqrt{E \pm p_3}$

$$\Rightarrow \begin{pmatrix} \sqrt{E+p}\,(\quad) + \sqrt{E-p}\,(\quad) & 0 \\ 0 & \sqrt{E+p}\,(\quad) - \sqrt{E-p}\,(\quad) \end{pmatrix} \begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

$$p \cdot \sigma = E - p^3 \sigma^3$$
$$p \cdot \bar{\sigma} = E + p^3 \sigma^3$$

$$= \begin{pmatrix} \sqrt{p\cdot\sigma}\ \xi \\ \sqrt{p\cdot\bar{\sigma}}\ \xi \end{pmatrix} = U(p)$$

$$\rightarrow U^{\dagger}(p)\, U(p) = 2E\, \xi^{\dagger} \xi$$

$$\bar{U}(p)\, U(p) = 2m\, \xi^{\dagger} \xi$$

$$U^{\dagger r}(p)\, U^{s}(p) = 2E\, \delta^{rs}$$

$$\bar{U}^{r}(p)\, U^{s}(p) = 2m\, \delta^{rs}$$

Summing on spins give

$$\sum_{s=1,2} U^{s}(p)\, \bar{U}(p) = \sum \begin{pmatrix} \sqrt{p\cdot\sigma}\ \xi^{s} \\ \sqrt{p\cdot\bar{\sigma}}\ \xi^{s} \end{pmatrix} \begin{pmatrix} \sqrt{p\cdot\bar{\sigma}}\ \xi^{s\dagger}, & \sqrt{p\cdot\sigma}\ \xi^{s} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{p\cdot\sigma}\sqrt{p\cdot\bar{\sigma}} & p\cdot\sigma \\ p\cdot\bar{\sigma} & \end{pmatrix} \Rightarrow \begin{pmatrix} m & p\cdot\sigma \\ p\cdot\bar{\sigma} & m \end{pmatrix}$$

$$= \gamma \cdot p + m$$

for $\sum v(p)\,\bar{v}(p) = \gamma\cdot p - m.$

$\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ ~ commutes with γ^μ, $\gamma^{5\,2} = I$, $\gamma^{5\dagger} = \gamma^5$

Study about the Dirac field bilinears; I'll give a quick overview; so Dirac field bilinears are

$$\bar{\psi} M \psi$$

↳ 4×4 matrix.

* important since these are measurable (dirac field is not 'measurable')

*

$\bar{\psi}$	1	$\underline{1}$	Scalar	ψ
$\bar{\psi}$	4	γ^μ	vector	ψ
$\bar{\psi}$	6	$\sigma^{\mu\nu} \to \frac{i}{2}(\gamma^\mu,\gamma^\nu)$	tensor	ψ
$\bar{\psi}$	4	$\gamma^\mu\gamma^5$	pseudo vector	
$\bar{\psi}$	1	γ^5	" scalar	ψ

} 16 total

I will prove $\bar{\psi}\gamma^\mu\psi$ are vectors.

$$\bar{\psi}\gamma^\mu\psi \longrightarrow \bar{\psi}'\gamma^\mu\psi' \Rightarrow \bar{\psi}S(\Lambda^{-1})\,\gamma^\mu\,S(\Lambda)\psi$$

$$= \bar{\psi}\left(S(\Lambda^{-1})\,\gamma^\mu\,S(\Lambda)\right)\psi$$

$$= \bar{\psi}\,\Lambda^\mu{}_\nu\,\gamma_\nu\,\psi$$

$$= \underbrace{\Lambda^\mu{}_\nu}_{\text{number}}\,\underbrace{(\bar{\psi}\gamma_\nu\psi)}_{\text{Bilinear}}$$

{ transformed like a ~~covariant~~ contravariant vector.

Quantization of Dirac field:-

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$$

$$H = \int d^3x\, \mathcal{H} = \int d^3x\, (\pi(x)\dot{\psi} - \mathcal{L})$$

$$\hookrightarrow \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = \frac{\partial}{\partial \dot{\psi}}\left(\bar{\psi}(i\gamma^0 \partial_0 \psi - i\gamma^i \partial_i \psi - m)\right)$$

$$= \frac{\partial}{\partial \dot{\psi}}\left(\psi \underbrace{\gamma^0 i \gamma^0}_{(\gamma^0)^2 = 1} \partial_0 \psi - i\gamma^i \partial_i \psi - m\psi\right)$$

$$= \frac{\partial}{\partial \dot{\psi}}\left(\psi \dot{\psi} - i\gamma^i \partial_i \psi - m\psi\right)$$

$$\to 0 \qquad 0$$

$$= \psi^\dagger = \pi(x)$$

$$H = \int d^3x\, \pi(x)\dot{\psi} - \mathcal{L} \Rightarrow \int d^3x\, \psi^\dagger \dot{\psi} - \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$$

$$= \int d^3x\, \psi^\dagger \dot{\psi} - \psi^\dagger(\gamma^{0^2} i \partial_0 \psi - \psi^\dagger \gamma^0 (i\gamma^i \nabla_i + m)\psi$$

$$\int d^3x\, \cancel{\psi^\dagger \dot{\psi}} - \cancel{\psi^\dagger \dot{\psi}} - \psi^\dagger \gamma^0 (i\gamma^i \nabla^i + m)\psi \to \int d^3x\, \psi^\dagger [i\gamma^0 \cdot \gamma \cdot \nabla - m\gamma^0]\psi$$

$$\alpha^i = \gamma^0 \gamma^i$$

$$\beta = \gamma^0.$$

$$H = \int d^3x\, \psi^\dagger [i\alpha \cdot \nabla - \beta m]\psi$$

Author, before quantizing the Dirac field; shows that how trying to quantize like KG will not work out; to show this I will go on and try anyways.

The commutatory

$$[\psi_a(x), \psi^\dagger_b(y)] = \delta^3(x-y)\,\delta_{ab} \quad \{\text{At eq. time}\}$$

We need to find representation in terms of annihilation and creation operators. For that we already have eigenfunctions with eigenvalues $\pm E_p$. Expanding $\psi(x)$ in basis.

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \cdot \frac{1}{\sqrt{2E_p}} e^{-ipx} \sum_{s=1,2} \left(a_p^s\, u^s(p) + b_{-p}^s\, v^s(p) \right)$$

$$\left[a_p^r , a_q^{s\dagger} \right] = \left[b_p^r , b_q^{s\dagger} \right] = (2\pi)^3 \delta^3(p-q)\, \delta^{rs}$$

we are working in Schrodinger picture right now; then

$$[\psi(x), \psi^\dagger(y)] =$$

$$\int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{1}{\sqrt{2E_q}} \frac{1}{\sqrt{2E_p}} \sum_{r,s=1,2} \left[\left(a_p^s\, u^s(p) + b_{-p}^s\, v(p) \right), \left(a_q \text{——} \right] \gamma^0 \right.$$

$$\int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{1}{\sqrt{2E_p}} \cdot \frac{1}{\sqrt{2E_q}} e^{-ipx} \cdot e^{+iqy} \sum_{r,s=1,2} \underbrace{\left[a_p^s u^s(p) + b_{-p}^s v^s(p),\; a_q^{r\dagger} \bar{u}^r(q) + b_{-q}^\dagger \bar{v}^r(q) \right]}_{\to (2)} \gamma^0$$

(2) $$\sum_{r,s=1,2} \left(\left[a_p^s , a_q^{r\dagger} + b_{-q}^\dagger \right] + \left[b_{-p}^s , a_q^{r\dagger} + b_{-q}^\dagger \right] \right)$$

$$= \sum_{r,s=1,2} \left(\left[a_p^s , a_q^{r\dagger} \right] + \underline{\left[a_p^s , \cancel{b_{-q}^{\dagger}} \right]}^{\,0} + \underline{\left[b_{-p}^s , \cancel{a_q^{r\dagger}} \right]}^{\,0} + \left[b_{-p}^s , b_{-q}^\dagger \right] \right)$$

$$\Rightarrow \sum_{r,s=1,2} \left(\underbrace{\left[a_p^s , a_q^{r\dagger} \right]}_{\to (2\pi)^3 \delta^3(p-q)} + \underbrace{\left[b_{-p}^s , b_{-q}^{\dagger r} \right]}_{\to (2\pi)^3 \delta^3(p-q)} \right)$$

Now for $\underbrace{\sum u\bar{u}}_{\to (\gamma^0 E_p - \gamma \cdot p + m)}$ and $\underbrace{\sum v\bar{v}}_{\to (\gamma^0 E_p + \gamma \cdot p - m)}$

put back in the commutator.

$$\int \frac{d^3p\, d^3q}{(2\pi)^{6\,3}} \times \exp(+i(px - qy)) \frac{1}{\sqrt{2E_p}} \frac{1}{\sqrt{2E_q}} \cancel{(2\pi)^3} (\gamma^0 E_p - \gamma \cdot p + m)(2\pi^3 \delta^3(p-q)) + \cancel{(2\pi)^3} \delta^3(p-q)\, \gamma^0 (\gamma^0 E_p + \gamma \cdot p - m)$$

resolve delta.

$$\int \frac{d^3p\, \gamma^0}{(2\pi)^3} \exp(-ip(x-y)) \left[(\gamma^0 E_p - \gamma \cdot p + m) + (\gamma^0 E_p + \gamma \cdot p - m) \right] \frac{1}{2E_p}$$

$$\int \frac{d^3p\, \gamma^0}{(2\pi)^3} \exp(-ip(x-y)) \left[\cancel{2}\gamma^0 \cancel{E_p} \cancel{- \gamma \cdot p + \gamma \cdot p}^{\,0} \cancel{- m + m}^{\,0} \right] \frac{1}{\cancel{2E_p}}$$

$$\int \frac{d^3p}{(2\pi)^3} \gamma^0 \exp(-ip(x-y))\, \gamma^0 \Rightarrow \int \frac{d^3p}{(2\pi)^3} \exp(ip(x-y)) \longrightarrow \delta^3(x-y) \times I_{4\times 4}$$

Use this on Hamiltonian

$H = \int d^3x \, \psi^\dagger \left[i\alpha \cdot \nabla - \beta m \right] \psi$ ~ write this in terms of operators.

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \cdot \frac{1}{\sqrt{2E_p}} e^{-ipx} \sum_{s=1,2} \left(a_p^s U^s(p) + b_{-p}^s V^s(p) \right)$$

$$\psi^\dagger(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} e^{ipx} \sum_{s=1,2} \left(a_p^{\dagger s} U^{s\dagger}(p) + b_{-p}^{s\dagger} V^{s\dagger}(-p) \right)$$

$$H = \int d^3x \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} e^{ipx} \sum_{s=1,2} \left(a_p^{\dagger s} U^{s\dagger}(p) + b_{-p}^{s\dagger} V^{s\dagger}(-p) \right)$$

$$\underbrace{(i\alpha \cdot \nabla - \beta m)}_{h_D} \int \frac{d^3p'}{(2\pi)^3} \cdot \frac{1}{\sqrt{2E_p}} e^{+ip'x'} \sum_{s=1,2} \left(a_p^s U^s(p) + b_{-p}^s V^s(p) \right)$$

$\therefore h_D\left(U_s(p) e^{ipx}\right) = E_p \left(U_s(p) \exp(ipx)\right)$

$h_D\left(V_s(p) e^{-ipx}\right) = -E_p \left(V_s(p)\right) e^{-ipx}$

$$= \int d^3x \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} e^{ipx} \sum_{s=1,2} \left(a_p^{\dagger s} U^{s\dagger}(p) + b_{-p}^{s\dagger} V^{s\dagger}(-p) \right)$$

$$\int \frac{d^3p'}{(2\pi)^3} \cdot \frac{1}{\sqrt{2E_{p'}}} e^{+ip'x'} \sum_{s=1,2} \left(2E_{p'} a_{p'}^{s'} U(p') - 2E_p b_{-p}^{s'} V^{s'}(-p) \right) \times E_p$$

$\Rightarrow$ When multiplied we will get $U^{\dagger s}(p) V^s(p)$ terms to be zero; likewise $V^\dagger(p) U(p) \longrightarrow 0$, and $\sum U^\dagger U = 2E\delta^{ss'}$

$$\int d^3x \int \frac{d^3p \, d^3p'}{(2\pi)^6} \frac{1}{\sqrt{2E_p}} \frac{1}{\sqrt{2E_{p'}}} e^{-ix(p-p')} \left(2E_{p'} a_p^\dagger a_{p'} - 2E_{p'} b_{p'}^\dagger b_{p'} \right)$$

Resolve $\int d^3x$ to get $(2\pi)^3 \delta^3(p-p')$

$$= \int \frac{d^3p \, d^3p'}{(2\pi)^3} \frac{\delta^3(p-p')}{\sqrt{2E_p}\sqrt{2E_{p'}}} \left(2E_{p'} \left(a_p^\dagger a_{p'} - b_p^\dagger b_{p'} \right) E_{p'} \right)$$

Resolve delta

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{\cancel{2E_p}} \cdot \cancel{2E_p}\left(a^\dagger a - b^\dagger b\right) E_p \Rightarrow \int \frac{d^3p}{(2\pi)^3} \cdot \left(E_p a^\dagger a - E_p b^\dagger b\right)$$

$$\int \frac{d^3p}{(2\pi)^3} \cdot \left(E_p a^\dagger a - E_p b^\dagger b\right)$$

Now we can see the issue, every time an antiparticle is created, the system loses energy; So we can keep creating anti-particle, and we will end up with no ground state as; a lower energy can still be achieved by applying $b^\dagger$.

This is the problem I talked about.

I assume you are as stubborn as me; and still won't accept the problem. Let me 'prove' explicitly.

Lets say a vacuum.

$$a_p^s |0\rangle = 0 \; ; \quad b_p^s |0\rangle = 0$$

$$H|0\rangle = \int \frac{d^3p}{(2\pi)^3} \left(E_p a^\dagger a + E_p b^\dagger b\right) |0\rangle = 0$$

lets create an anti particle.

$$b_k^{r\dagger} |0\rangle = |b\rangle$$

$$H|b\rangle = \underline{-E_k |b\rangle}$$

You can create as much anti-particles as you want the energy will keep going down.

Lets go a step further to check causality

$$[\psi(x), \bar{\psi}(y)]$$

But we do it in Heisenberg picture

So we see time evolution by

$$e^{+iHt} a^s_p e^{-iHt} = a^s_p e^{-iEt}$$

can be proven by using

$[H, a_p] = -E_p a_p$; opening up taylor expansions. Let me redo this anyways.

$e^{iHt} a^s_p e^{-iHt}$ ~ use $[H, a_p] = Ha_p - a_pH = -Ea_p \Rightarrow a_p(H-E)$

$$Ha_p = a_p(H-E)$$

$$H^n a_p = a_p(H-E)^n \quad \text{—— I}$$

$$= \sum_{n=0}^{\infty} \frac{(it)^n}{n!} H^n a^s_p e^{-iHt}$$

$$= \sum_{n=0}^{\infty} \frac{(it)^n}{n!} a^s_p (H-E)^n e^{-iHt} \Rightarrow a^s_p \sum_{n=0}^{\infty} \frac{(it)^n (H-E)^n}{n!} e^{-iHt} \Rightarrow a^s_p \exp(i(H-E)t) \cdot \exp(-iHt)$$

$$= \boxed{a^s_p \exp(-iEt)}$$

Same goes for b.

So in Heisenberg picture

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left(a^s_p U^s(p) e^{-ipx} + b^s_p V^s(p) e^{ipx} \right)$$

$$\bar{\psi}(x) = \int \frac{d^3p}{2\pi^3} \frac{1}{\sqrt{2E_p}} \sum_s \left(a^{+s}_p \bar{U}^s(p) e^{+ipx} + b^{+s}_p \bar{V}^s(p) e^{-ipx} \right)$$

Now find commutator

$$[\psi(x), \bar{\psi}(y)] = \psi\bar{\psi} - \bar{\psi}\psi$$

$$\int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{1}{\sqrt{2E_p}} \frac{1}{\sqrt{2E_q}} \sum_{r,s=1,2} \left[[a^r, a^{s+}] U^r_{(p)} \bar{U}^s_{(q)} e^{-ipx+iqy} + \underbrace{[b^{+r}, b^s]}_{-[b^s, b^{+r}]} V^r \bar{V}^s e^{ipx-iqy} \right]$$

all other commutators zero.

$$= \int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{1}{\sqrt{4E_pE_q}} \sum_{r,s=1,2} \left[(2\pi)^3 \delta^3(p-q)\, U^r_{(p)} \bar{U}^s_{(q)} e^{-ipx+iqy} \div (2\pi)^3 \delta^3(p-q)\, V^r \bar{V}^s e^{ipx-iqy} \right]$$

resolve deltas

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left[(\gamma\cdot p + m)\, e^{-ip(x-y)} - (\gamma\cdot p - m)\, e^{ip(x-y)} \right]$$

$$\div (i\partial\!\!\!/_x + m) \underbrace{\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left[e^{-ip(x-y)} - e^{ip(x-y)} \right]}_{[\phi(x), \phi(y)]}$$

↳ since this vanishes outside the light cone; so will $[\psi(x), \bar{\psi}(y)]$

But There is another issue; like if we sandwich it in between $\langle 0 |\ 0\rangle$

$$\langle 0 | [\psi^s_a(x), \bar{\psi}^r_b(y)] | 0 \rangle \Rightarrow \langle 0 | \psi_a \bar{\psi}_b | 0 \rangle - \langle 0 | \bar{\psi}_b \psi_a | 0 \rangle$$

$$= \langle 0 | \int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{1}{\sqrt{4E_pE_q}} \sum \left(a^s U^s(p) e^{-ipx} + b^s V^s(p) e^{ipx} \right) \left(a^{\dagger} \bar{U}^r(q) e^{iqy} + b^{\dagger r} \bar{V}(q) e^{-iqy} \right) | 0 \rangle$$

cross terms vanish due to orthogonality.

$$\langle 0 | \int \frac{d^3p\, d^3q}{(\sqrt{4E_pE_q})(2\pi)^6} \sum_{r,s=1,2} \left(a^s a^{\dagger r}\, U\bar{U} e^{-ipx+iqy} + b^r b^{\dagger s}\, V\bar{V} e^{ipx-iqy} \right) | 0 \rangle$$

$$= \int \frac{d^3p\, d^3q}{(\sqrt{4E_pE_q})(2\pi)^6} \sum_{r,s=1,2} \left(\underbrace{\langle 0 | a^s a^{\dagger}_r | 0 \rangle}_{(2\pi)^3 \delta(p-q)} U\bar{U} e^{-ipx+iqy} + \underbrace{\langle 0 | b^{\dagger} b | 0 \rangle}_{(2\pi)^3\delta(p-q)} V\bar{V} e^{ipx-iqy} \right) = \int \frac{d^3p}{2E_p (2\pi)^3} \overbrace{(i\gamma\cdot p + m)\, e^{-ip(x-y)}}^{1st} + \underbrace{(i\gamma\cdot p - m)\, e^{ip(x-y)}}_{2nd}$$

for 2nd term.

$$\langle 0 | \bar{\psi}^r_b \psi^s_a | 0 \rangle$$

⇒ likewise cross terms cancel out; and for same terms

$$\langle 0 | b^{\dagger} b | 0 \rangle = 0$$

$$\langle 0 | a^{\dagger} a | 0 \rangle = 0$$

so 2nd terms contributes nothing; the issue is.

The $\langle 0 | \psi(x) \bar{\psi}(y) | 0 \rangle$ gives both propagators:- ; which show the particle and anti particle going from y to x.

Which doesn't make sense; in KG case particle going from x to y was cancelled by anti-particle from y to x.

So, our commutator relations must be wrong and(or) any assumption we made along the way must've been incorrect.

We make changes like:-

* We drop the idea of a vacuum that gives zero for all a's and b's.

→ look at the term

$\langle 0|\psi^s(x)\bar{\psi}^r(y)|0\rangle$ ~ should represent positive energy particle going from y to x.

So we need

$\psi^{\dagger r}(y)|0\rangle$ to be made of positive energy particle only and thus only $a^\dagger$ must contribute and $b^\dagger$ should take state to zero. Likewise for $\langle 0|\psi(x)$; this should have negative energy components only

Lets write $\langle 0|\psi_{(x)}\bar{\psi}_{(y)}|0\rangle$

$$\longrightarrow \langle 0|\int \frac{d^3p\,d^3q}{(2\pi)^6}\frac{1}{\sqrt{4E_pE_q}}\sum_r\left[a^r u^r(p)e^{-ipx}\right]\times\sum_s\left(a^{\dagger s}\bar{u}^s(p)e^{iqy}\right)|0\rangle$$

we say $|0\rangle = e^{iP\cdot x}|0\rangle$ if $|0\rangle$ is invariant under translation.

$|0\rangle = e^{iP\cdot x}|0\rangle$

$$\langle 0|\int \frac{d^3p\,d^3q}{(2\pi)^6}\frac{1}{\sqrt{4E_pE_q}}\sum_r\left[a^r u^r(p)e^{-ipx}\right]\times\sum_s\left(a^{\dagger s}\bar{u}^s(p)e^{iqy}\right)e^{iPx}|0\rangle$$

Also

$$\langle 0|a^r a^{s\dagger}|0\rangle \longrightarrow \langle 0|e^{-iPx}a^r\left(e^{iP\cdot x}e^{-iP\cdot x}\right)a^{s\dagger}e^{iP\cdot x}|0\rangle$$

We have done this with hamiltonian; lets do with momentum

$$e^{iP\cdot x}a^s_p e^{iP\cdot x} = a^s_p e^{-iPx}$$

$$\overbrace{a^{iP\cdot x}\,a^{s\dagger}_p e^{-iP\cdot x}}^{\text{operator}} = a^{s\dagger}_p \underbrace{e^{iPx}}_{\text{eigenvalue}}$$

$\langle 0|e^{-iPx} = \langle 0|$ and $e^{iPx}|0\rangle = |0\rangle$

$$\langle 0|\, e^{-iqx} a^r_p a^{st}_q\, e^{ipx}\, |0\rangle = \underline{e^{i(p-q)x}\, \langle 0|\, a^r_q a^{st}_p\, |0\rangle} \Big\} \text{— (k)}$$

$$\hookrightarrow = \langle 0|\, a^r_q\, a^{st}_p\, |0\rangle$$

(k) only satisfied if and only if $\dfrac{e^{-i(q-p)x} = 1}{\text{so } q = p}$

So we can write

$$\langle 0|\, a^r_q a^{st}_p\, |0\rangle = (2\pi)^3 \delta^3(p-q)\, A(p)\, \delta^{rs}$$

plug back into the integral.

$$\int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{1}{\sqrt{4E_pE_q}} \sum_{r,s=1,2} U^r \bar{U}^s\, \delta^{rs}\, A(p)\, (2\pi)^3 \delta^3(p-q)\, \delta^{rs}\, e^{-ipx} \cdot e^{iqy}$$

$$= \boxed{\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} (i\gamma\cdot p + m)\, A(p)\, e^{-ip(x-y)}}$$

$A(p)$ is undetermined and we got it because $\langle 0|\, a a^{+} |0\rangle$ can be any function of momentum. But it must be a lorrentz scaler.

$$A^2(\vec{p}) = A^2(p) \text{ and } p^2 = m^2; \text{ so } A \text{ is a constant.}$$

$$\rightarrow \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} (i\gamma\cdot p + m)\, e^{-ip(x-y)} \cdot A$$

likewise for; particle going x to y.

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} (i\gamma\cdot p - m)\, e^{-ip(y-x)} \quad B$$

$$-(i\gamma\cdot p + m) \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip(y-x)}\, B$$

$$\langle 0|\, \psi(x) \bar{\psi}(y)\, |0\rangle - \langle 0|\, \bar{\psi}(y) \psi(x)\, |0\rangle$$

this must be zero; so for $B = 1 = A$ out side the light cone.

So for these to cancel out we must have this
(in spacelike)

$$\boxed{\langle 0 | \psi_{(x)} \bar{\psi}_{(y)} | 0 \rangle = -\langle 0 | \bar{\psi}_{(y)} \psi_{(x)} | 0 \rangle}$$

which is nothing but anti-commutation.

$\{ \psi_{(x)}, \bar{\psi}(y) \} = 0$ at space like seperation

This also works since all the observables still follow

$$[O_1, O_2] = 0 \quad ; \quad (x-y)^2 < 0$$

The eq. time anti-commutation Relations are

$$\{ \psi_a(\vec{x}), \psi_b^\dagger(\vec{y}) \} = \delta^3(x-y)\,\delta_{ab}$$

$$\{ \psi_a(x), \psi_b(y) \} = \{ \bar{\psi}_a(x), \bar{\psi}_b(y) \} = 0.$$

We can expand $\psi(x)$ and $\bar{\psi}(x)$ in terms of operators but these will just follow anti-commutation relations.

$$\{ a_p^r, a_q^{s\dagger} \} = \delta^3(p-q)\,\delta^{rs}\,(2\pi)^3 = \{ b_p^r, b_q^{s\dagger} \}$$

with all other anti-commutators be zero.

We can write; even though I have done this before, but yeah :)

$$H = \int d^3x\, \mathcal{H}$$

$$= \int d^3x \left[\pi(x)\dot{\psi} - \mathcal{L} \right] \Rightarrow \pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} \Rightarrow \frac{\partial}{\partial \dot{\psi}} \{ \psi^\dagger \gamma^0 (\overset{1}{\gamma^0} \partial_0 \psi_{(x)}) + \bar{\psi}(\gamma^i \partial_i + m)\psi$$

$$= \frac{\partial}{\partial \dot{\psi}} \{ \psi^\dagger \dot{\psi} + 0 \} \Rightarrow \psi^\dagger$$

$$= \int d^3x \Big[\underbrace{\psi^\dagger \dot{\psi} - \bar{\psi}\{\gamma^0 \partial_0 \psi\}}_{\text{cancel out}} + \bar{\psi}\{ i\gamma^i \partial_i + m \}\psi$$

$$\int d^3x \left[\cancel{\psi^\dagger \dot\psi} - \cancel{\psi^\dagger \dot\psi} + \bar\psi \{ i\gamma^i \partial_i + m \} \psi \right.$$

$$\int d^3x \; \bar\psi \{ h_D \} \psi$$

↳ Energy operator for $U^{(s)}(p)$

$$\int d^3x \int \frac{d^3p \, d^3q}{(2\pi)^6} \frac{1}{\sqrt{4E_pE_q}} \sum_r \left[a_p^\dagger \bar U e^{+ipx} + b_p \bar V e^{-ipx} \right] h_D \sum_s \left[\sum_s \left[a_q U e^{-iqx} + b_q^\dagger V e^{iqx} \right] \right]$$

$$\int d^3x \int \frac{d^3p \, d^3q}{(2\pi)^6} \frac{1}{\sqrt{4E_pE_q}} \sum_{r,s=1,2} \left[a_p^\dagger \bar U e^{ipx} + b_p^\dagger \bar V e^{-ipx} \right] \left[a_q U e^{-iqx} \cdot E_q - E_q b_q V e^{iqx} \right]$$

cross terms cancel out

$$\int d^3x \int \frac{d^3p \, d^3q}{(2\pi)^6} \frac{1}{\sqrt{4E_pE_q}} \sum_{r,s=1,2} \left[a_p^\dagger a_q \bar U U e^{+ipx} \cdot e^{-iqx} - b_p b_q \bar V V e^{-ipx} \cdot e^{-iqx} \right] E_q$$

Resolving $\int d^3x$ would give $(2\pi)^3 \delta^3(p-q)$

$$\int \frac{d^3p \, d^3q}{(2\pi)^{\cancel{6}\,3}} \frac{1}{\sqrt{4E_pE_q}} \cancel{(2\pi)^3} \delta^3(p-q) \sum_{r,s=1,2} \left[a_p^\dagger a_q \bar U U e^{+ipx} \cdot e^{-iqx} - b_p^\dagger b_q \bar V V e^{-ipx} \cdot e^{-iqx} \right] E_q$$

Resolve delta

$$\int \frac{d^3p \, d^3q}{(2\pi)^3} \frac{1}{\sqrt{4E_pE_q}} \left[a^\dagger a_q \, (2E_p \delta_{rs}) \, e^{i(p-q)x} - (2E_q \delta_{rs}) \, e^{-i(p-q)x} \, b_p^\dagger b_q \right] E_q \, \delta^3(p-q)$$

$$\longrightarrow \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E} \cdot \cancel{2} E_p^2 \left[a^\dagger a - b^\dagger b \right] \Rightarrow \int \frac{d^3p}{(2\pi)^3} \left[E_p a^\dagger a - b^\dagger b \, E_p \right]$$

We still seem to be getting negative enrgy but

$$\{ b_p^r, b_q^{s\dagger} \} = (2\pi)^3 \delta^3(p-q) \delta_{rs} \Rightarrow \text{is symmetric; lets define}$$

$$\tilde b_p^s = b_p^{s\dagger} \quad \text{and} \quad \tilde b_p^{s\dagger} = b_p^s$$

They follow exact same anti-commutation relations, but we will now get

$$-E_p b_p^{\dagger s} b_p^s = +E_p \tilde b \tilde b^{\dagger s}$$

$$\tilde b_p^r \tilde b_p^{\dagger s} + \tilde b_p^{\dagger s} \tilde b_p^r = (2\pi)^3 \delta^3(0)$$

$$\underbrace{E \, \tilde b_p^r \tilde b_p^\dagger}_{-E \,=\, E} = \underbrace{(2\pi)^3 \delta(0)}_{\text{ignore}} - \tilde b_p^{\dagger s} \tilde b_p^s$$

$$\int \frac{d^3p}{(2\pi)^3} \left[E_p\, a^\dagger a + \tilde{b}^\dagger \tilde{b}\, E_p \right]$$

Going forward, I'm putting a note from Schroeder & Peskin that will help you understand this better.

If we choose $|0\rangle$ to be the state that is annihilated by $a^s_{\mathbf{p}}$ and $\tilde{b}^s_{\mathbf{p}}$, then all excitations of $|0\rangle$ have positive energy.

What happened? To better understand this trick, let us abandon the field theory for a moment and consider a theory with a single pair of b and $b^\dagger$ operators obeying $\{b, b^\dagger\} = 1$ and $\{b, b\} = \{b^\dagger, b^\dagger\} = 0$. Choose a state $|0\rangle$ such that $b\,|0\rangle = 0$. Then $b^\dagger\,|0\rangle$ is a new state; call it $|1\rangle$. This state satisfies $b\,|1\rangle = |0\rangle$ and $b^\dagger\,|1\rangle = 0$. So b and $b^\dagger$ act on a Hilbert space of only two states, $|0\rangle$ and $|1\rangle$. We might say that $|0\rangle$ represents an "empty" state, and that $b^\dagger$ "fills" the state. But we could equally well call $|1\rangle$ the empty state and say that $b = \tilde{b}^\dagger$ fills it. The two descriptions are completely equivalent, until we specify some observable that allows us to distinguish the states physically. In our case the correct choice is to take the state of lower energy to be the empty one. And it is less confusing to put the dagger on the operator that creates positive energy. That is exactly what we have done.

Note, by the way, that since $(\tilde{b}^\dagger)^2 = 0$, the state cannot be filled twice. More generally, the anticommutation relations imply that any multiparticle state is antisymmetric under the interchange of two particles: $a^\dagger_{\mathbf{p}} a^\dagger_{\mathbf{q}}\,|0\rangle = -a^\dagger_{\mathbf{q}} a^\dagger_{\mathbf{p}}\,|0\rangle$. Thus we conclude that if the ladder operators obey *anticommutation* relations, the corresponding particles obey *Fermi-Dirac* statistics.

If you are lazy like me, just think ~~of~~ it as; b taking the state a step down so it creates a $-E$ particle from "ground" state [this is an analogy]; $\tilde{b} = b^\dagger$; is kind of, creating a "negative energy" particle; and $\tilde{b}^\dagger = b$ takes a step up in the energy. It is useful since now even the neg. any particle will have positive energy (kind of).

Quantized Dirac field.

Moving forward; from now on I will write $\tilde{b}^s_p$ (which is $b^{\dagger s}_p$); which lowers the energy state; as b^s_p and likewise $\tilde{b}^{s\dagger}_p$ as $b^{s\dagger}_p$.

Giving us $\psi(x)$ as

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \cdot \frac{1}{\sqrt{2E_p}} \cdot \sum_{r=1,2} \left(a^r_p U^r(p) e^{-ipx} + b^{r\dagger}_p V^r(p) e^{ipx} \right)$$

$$\bar{\psi}(x) = \int \frac{d^3p}{(2\pi)^3} \cdot \frac{1}{\sqrt{2E_p}} \cdot \sum_{r=1,2} \left(a^{r\dagger}_p \bar{U}^r(p) e^{+ipx} + b^r_p \bar{V}^r(p) e^{-ipx} \right)$$

$$\{a^r_p, a^{s\dagger}_q\} = (2\pi)^3 \delta^3(p-q)\, \delta^{rs} = \{b^r_{p}, b^{s\dagger}_q\}$$

Equal time commutation relations; then are

$$\{\psi^a(x), \psi^{b\dagger}(y)\} = (2\pi)^3 \delta^3(\vec{x}-\vec{y})\, \delta^{ab}$$

$$\{\psi_a(x), \psi_b(y)\} = \{\psi^\dagger_a, \psi^\dagger_b\} = 0$$

Vacuum is defined to be a state with.

$$a^r_p|0\rangle = 0 = b^r_p|0\rangle$$

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_{s=1,2} \left(E\, a^{\dagger s}_p a^s_p + E\, b^{\dagger s}_p b^s_p \right)$$

[Remember that we dropped an infinity here]

The momentum operator is

$$P = \int d^3x\, \bar{\Psi}(-i\nabla)\Psi \Rightarrow \int \frac{d^3p}{(2\pi)^3} \sum_s p(a^\dagger_p a_p \quad b^\dagger_p b_p)$$

thus $b^\dagger$ is creating a particle with E_p and momentum P.

The one particle states then, are

$$|P,s\rangle = \sqrt{2E_p}\, a^{s\dagger}_p |0\rangle$$

are defined so that their inner product

$$\langle p, r | q, s\rangle = 2E_p \langle 0| a^r_p a^{s\dagger}_q |0\rangle = 2E_p (2\pi)^3 \delta^3(p-q)\delta^{rs}$$

This is lorentz invariant, and implies that operator that implements lorentz transformation is unitary (for Hilbert space), even though for boost $\Lambda_{\frac{1}{2}}$ [$U(\Lambda)$] is non unitary

Yeah, I know, you won't believe it until we prove it.

$$U\Psi(x)U^{-1} = U\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}} \sum_s \left(a^s_p u^s(p)e^{-ipx} + b^{\dagger s}_p v^s(p) e^{ipx}\right)U^{-1}$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}} \sum_s U\left(a^s_p u^s(p)e^{-ipx} + b^{\dagger s}_p v^s(p) e^{ipx}\right)U^{-1}$$

$$\therefore\ U a^s_p U^{-1} = \sqrt{\frac{E_{\Lambda p}}{E_p}}\, a^s_{\Lambda p}$$

I don't feel comfy; writing it without detail; so I'm going to write and prove necessary properties before we move on.

* $p^\mu \cdot x_\mu$ is lorrentz invariant

* Remember $\int \frac{d^3p}{(2\pi)^3 \cdot 2E}$ is lorrentz invariant $\xrightarrow{\text{Proof}}$

$\left\{ \int dp^0 \, \delta(p^2 - m^2) = \int dp^0 (p^{0 2} - \vec{p}^2 - m^2) = \frac{1}{2E_p} \right.$. so $\frac{d^3p}{2E_p}$ doesn't change under boost.

So.

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s U \left(a_p^s u^s(p) e^{-ipx} + b_p^{+s} v^s(p) e^{ipx} \right) U^{-1} \times \sqrt{\frac{2E_p}{2E_p}}$$

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \sum_s U \left| \sqrt{2E_p}\, a_p^s U^s(p) e^{-ipx} + \sqrt{2E_p} \right. \Big) U^{-1}$$

Lets $\tilde{p} = \Lambda p$ then $p = \Lambda^{-1} \tilde{p}$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \sum \left(\sqrt{2E_p}\, U^s(p) e^{-ipx}\, U a_p^s U^{-1} + \cdots\cdots \right)$$

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \sum \left(\sqrt{2\cancel{E_p}}\, U^s(p) e^{-ipx} \sqrt{\frac{E_{\Lambda p}}{\cancel{E_p}}}\, a_{\Lambda p}^s + \cdots\cdots \right)$$

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \sum \left(\sqrt{2}\; U^s(p) e^{-ipx} \sqrt{E_{\Lambda p}}\, a_{\Lambda p}^s + \cdots\cdots \right)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \sum \left(U^s(p) e^{-ipx} \sqrt{2E_{\Lambda p}}\, a_{\Lambda p}^s + \cdots\cdots \right)$$

we know $p \to \Lambda^{-1} \tilde{p}$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \sum \left(U^s(\Lambda^{-1} \tilde{p}) e^{-i(\Lambda^{-1}\tilde{p}) \cdot x} \sqrt{2E_{\tilde{p}}}\, a_{\tilde{p}}^s + \cdots\cdots \right)$$

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \sum \left(\Lambda_{1/2}^{-1} U^s(\tilde{p})\, e^{-i(\Lambda^{-1}\tilde{p}) \cdot x} \sqrt{2E_{\tilde{p}}}\, a_{\tilde{p}}^s + \cdots\cdots \right)$$

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \sum \left(\Lambda_{1/2}^{-1} U^s(\tilde{p})\, e^{-i\tilde{p} \cdot (\Lambda x)} \sqrt{2E_{\tilde{p}}}\, a_{\tilde{p}}^s + \cdots\cdots \right)$$

$\longrightarrow p' \cdot x' = (\Lambda p) \cdot (\Lambda x) = p \cdot x$

$(\Lambda \Lambda^{-1} \tilde{p})(\Lambda x)$

$\tilde{p}^\mu (\Lambda x)_\mu = p^\mu \cdot x_\mu$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \sum \left(\Lambda_{1/2}^{-1} U^s(\tilde{p})\, e^{-i\tilde{p} \cdot (\Lambda x)} \sqrt{2E_{\tilde{p}}}\, a_{\tilde{p}}^s + \cdots\cdots \right)$$

$$U \Psi(x) U^{-1} = \Lambda_{1/2}^{-1} \Psi(\Lambda x)$$

Lets show that these particles have spin $1/2$; and That this spin is related to angular momentum

We use noether theorem and use rotational invariance. Under rotation (or any lorrentz transformation) our field transforms like.

$$\psi(x) \longrightarrow \psi'(x) = \Lambda_{1/2}\,\psi(\Lambda^{-1}x)$$

$$\delta\psi = \psi'(x) - \psi(x) = \Lambda_{1/2}\,\psi(\Lambda^{-1}x) - \psi(x)$$

Lets consider a rotation around z-axis by angle theta,

$$\Lambda_{1/2} \approx \Big(1 - \underbrace{\tfrac{i}{2}\theta}_{\omega^{12}}\,\underbrace{\Sigma^3}_{S^{12}}\Big)$$

Applying this will also transform the coordinates; like now I want $\psi(\Lambda^{-1}x)$

so: lets say $x' = \Lambda^{-1}x$; $x' = x\cos\theta + y\sin\theta$

$y' = y\cos\theta - x\sin\theta$

$z' = z$; $t = t'$

for θ infinitessimal. $x' = x + \theta y$

$y = y - x\theta$

$$\psi(\Lambda^{-1}x) = \underbrace{\psi(t, x+\theta y, y - x\theta, z)}_{\text{Taylor expand this}} - \psi(t,x,y,z)$$

$$\approx \psi(t,x,y,z) + (\theta y)\partial_x\psi + (-\theta x)\partial_y\psi)$$

$$\approx \psi(t,x,y,z) - \theta(x\partial_y - y\partial_x)\psi$$

put in

$$\delta\psi = \left(1 - \tfrac{i}{2}\theta\Sigma^3\right)\left(\psi(x) - \theta(x\partial_y - y\partial_x)\psi(x)\right) - \psi(x)$$

$$\Big(\psi(x) - \theta(x\partial_y - y\partial_x)\psi(x)\Big) - \tfrac{i}{2}\theta\Sigma^3\psi(x) \underbrace{-\theta^2(x\partial_y - y\partial_x)\Sigma^3\psi(x)}_{\to 0 \text{ since } O(\theta^2)\sim 0} - \psi(x)$$

$$\cancel{\psi(x)} - (\theta(x\partial_y - y\partial_x)\psi(x)) - \tfrac{i}{2}\theta\Sigma^3\psi(x) - \cancel{\psi(x)}$$

$$= -\theta\underbrace{\left((x\partial_y - y\partial_x) + \tfrac{i}{2}\Sigma^3\right)}_{\Delta}\psi(x) = \theta\Delta\psi(x)$$

Putting it back in noether current's j^0 formula.

$$j^0 = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} \delta\psi \Rightarrow$$

$$\frac{\partial}{\partial \dot\psi}\left(\bar\psi(\not\partial - m)\psi\right)\delta\psi \Rightarrow \frac{\partial}{\partial\dot\psi}\left(i\psi^+(\dot\psi) + \bar\psi(-i\gamma_i\nabla_i - m)\psi\right)\delta\psi$$

$$\frac{\partial}{\partial\dot\psi}\left((i\psi^+\dot\psi) + 0\right)\delta\psi \Rightarrow i\psi^+ \cdot (\Delta\psi) \Rightarrow i\psi^+\gamma^0\gamma^0 \cdot (\Delta\psi)$$

$$= i\bar\psi\gamma^0 \cdot (\Delta\psi) \Rightarrow (i\bar\psi\gamma^0)\left(-(x\partial_y - y\partial_x + \tfrac{i}{2}\Sigma^3\right)\psi$$

(I dropped theta; since we are interested in generator).

likewise we will get expressions for x and y rotations; remember $P_i = -i\partial_i$; in 4 notation.

$$x(-i\partial_y) - y(-i\partial_x) \Rightarrow (x \times (-\nabla_i))_z$$ { when we add rotations for x and y }

$$(i\psi^+)\left((\vec{x}\times(-i\vec\nabla)) + \tfrac{i}{2}(\vec\Sigma)\right)\psi$$

integrate this density over volume

$$J = \int d^3x\, \psi^+\left(\underbrace{(\vec{x}\times(-i\vec\nabla))}_{\text{orbital}} + \underbrace{\tfrac{i}{2}(\vec\Sigma)}_{\text{spin}}\right)\psi$$

Imagine for fermion at rest

(used Schrodinger picture)

$$J_{z,\,p=0} = \int d^3x\, \psi^+\left(\underbrace{(x\times(-i\nabla))_z}_{\to 0 \text{ since at rest}} + \tfrac{i}{2}\Sigma^3\right)\psi$$

$$\int d^3x\, \psi^+ \left(\tfrac{i}{2}\Sigma^3\right)\psi \Rightarrow \int d^3x \int \frac{d^3p\, d^3p'}{(2\pi)^6}\frac{1}{\sqrt{4E_pE_{p'}}}\sum_{r,s}\left[a_p^{r+}u^r(p) + b_p^r v^r(p)\right]\left[\left(\frac{\Sigma^3}{2}\right)(a^s_{p'}u^s(p') + b^+_{p'}v^s(p'))\right]$$

$$e^{+ip\cdot x}\, e^{-ip\cdot x}$$

We know J_z must anhilate vacuum.

$$J_z|0\rangle = |0\rangle$$

so lets apply this one one particle zero momentum state.

$a_0^{s\dagger}|0\rangle$

$$\underline{J_z a_0^{s\dagger}|0\rangle} = [J_z, a_0^{s\dagger}]|0\rangle$$

$$\left(\int d^3x \int \frac{d^3p\, d^3p'}{(2\pi)^6}\frac{1}{\sqrt{4E_pE_{p'}}}\sum_{r,s}\left[a_p^{r\dagger}u^{r\dagger}(p) + b_p^r v^{r\dagger}(p)\right]\left[\left(\frac{\Sigma^3}{2}\right)\left(a_{p'}^s u^s(p') + b_{p'}^{\dagger s} v^s(p')\right)\right] e^{+ip\cdot x} e^{-ip'\cdot x}\right) \cdot a_0^{s\dagger}|0\rangle$$

$$\left[\int d^3x \int \frac{d^3p\, d^3p'}{(2\pi)^6}\frac{1}{\sqrt{4E_pE_{p'}}}\sum_{r,s}\left(u^{r\dagger}_{(p)}u^s_{(p')}a_p^{r\dagger}a_{p'}^s + b_p^r b_{p'}^{\dagger s} v^{r\dagger}_{(p)}v^s_{(p')}\right)\cdot e^{+ip\cdot x}\cdot e^{-ip'\cdot x}, a_0^{s\dagger}\right]|0\rangle$$

$$\int d^3x \int \frac{d^3p\, d^3p'}{(2\pi)^6}\frac{1}{\sqrt{4E_pE_{p'}}}\sum_{r,s}\left(u^{r\dagger}_{(p)}u^s_{(p')}a_p^{r\dagger}a_{p'}^s + b_p^r b_{p'}^{\dagger s} v^{r\dagger}_{(p)}v^s_{(p')}\right) e^{i(p-p')x}$$

$$\int \frac{d^3p\, d^3p'}{(2\pi)^3}\frac{1}{\sqrt{4E_pE_{p'}}}\delta(p-p')\delta^{rs}\sum_{r,s}\left(u^{r\dagger}_{(p)}u^s_{(p')}a_p^{r\dagger}a_{p'}^s + b_p^r b_{p'}^{\dagger s} v^{r\dagger}_{(p)}v^s_{(p')}\right)$$

There is also $\frac{\Sigma^3}{2}$ term that i forgot

$$\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\sum_r\left(u^{r\dagger}_{(p)}u^r_{(p)}a_p^{r\dagger}a_p^r + b_p^r b_p^{\dagger r} v^{r\dagger}_{(p)}v^r(p)\right)$$

put back in commutator

$$\left[\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\frac{\Sigma^3}{2}\sum_r\left(u^{r\dagger}_{(p)}u^r_{(p)}a_p^{r\dagger}a_p^r + b_p^r b_p^{\dagger r} v^{r\dagger}_{(p)}v^r(p)\right), a_0^{\dagger}\right]|0\rangle$$

$$\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\sum_r\left(u^{r\dagger}_{(p)}\frac{\Sigma^3}{2}u^r_{(p)}\left[a_p^{r\dagger}a_{-p}^r, a_0^{\dagger}\right] + \underbrace{\left[b_{-p}^r b_p^{r\dagger}, a_0^{\dagger}\right]}_{\hookrightarrow \text{commute.}} v^{r\dagger}\frac{\Sigma^3}{2}v^r\right)|0\rangle$$

$$\Rightarrow \int \frac{d^3p}{(2\pi)^3}\cdot\frac{1}{2E_p}\sum_r\left(u^{r\dagger}_{(p)}\frac{\Sigma}{2}u^r_{(-p)}\underline{\left[a_p^{r\dagger}a_{-p}^r, a_0^{\dagger}\right]}\right)|0\rangle$$

$\hookrightarrow [AB, C] = A\{B, C\} - \{A, C\}B$

$a_p^{r\dagger}\{a_{-p}^r, a_0^{\dagger}\} - 0$

$a_p^{r\dagger}(2\pi)^3\delta(p)$

$$\rightarrow \int \frac{d^3p}{\cancel{(2\pi)^3}} \frac{1}{2E_p} \sum_r \left(U^{r\dagger}_{(p)} \frac{\Sigma^3}{2} U^r(p) \cancel{(2\pi)^3} a^{r\dagger}_p \delta(p) \right) |0\rangle$$

$E^2 = p^2 + m^2$
when $p=0$
$E_p = m_p$

evaluate delta $p=0$

$$\Rightarrow \frac{1}{2E_p} \sum_r \left(U^{r\dagger}_{(0)} \frac{\Sigma^3}{2} U^r(0) \right) a^{r\dagger}_0 |0\rangle$$

$$= \frac{1}{2m} \sum_r \left(U^{r\dagger}(0) \frac{\Sigma^3}{2} U^r(0) \right) a^{r\dagger}_0 |0\rangle$$

$$\frac{1}{2m} \sum_r \left(\begin{pmatrix} \xi^{r\dagger} & 0 \end{pmatrix} \sqrt{2m} \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix} \sqrt{2m} \begin{pmatrix} \xi^r \\ 0 \end{pmatrix} \right) a_0^{\dagger r} |0\rangle$$

$$\Rightarrow \sum_r \left(\begin{pmatrix} \xi^{r\dagger} & 0 \end{pmatrix} \frac{1}{2} \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix} \begin{pmatrix} \xi^r \\ 0 \end{pmatrix} \right) a_0^{\dagger r} |0\rangle$$

$$\Rightarrow \sum_r \left(\xi^{r\dagger} \frac{\sigma_3}{2} \xi^r \right) a_0^{\dagger r} |0\rangle$$

$\rightarrow$ for $\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ we get $\frac{1}{2}$ eigenvalue and
$\xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ we get $-\frac{1}{2}$ eigenvalue.

Assuming that you want me to suffer, let me show you explicitly.

$$\sigma_3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \frac{1}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\sigma_3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

So for $\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; we get and for $\xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ we get

$$J_z a_0^{1\dagger} |0\rangle = \frac{1}{2} a_0^{1\dagger} |0\rangle \; ; \; J_z a_0^{2\dagger} = -\frac{1}{2} a_0^{2\dagger} |0\rangle$$

Showing the particle actualy carries $\frac{1}{2}$ spin.

Putting my procastrination aside, lets do it for the charge, too.

We do a phase transformation globally (U(1))

$$\delta\psi = \psi'_{(x)} - \psi_{(x)} = e^{-i\alpha}\psi'(x) - \psi(x)$$

for infinitesimal α, we get $(1 - i\alpha)$

$$= \cancel{\psi(x)} + i\alpha\psi(x) - \cancel{\psi(x)} = -\alpha i\psi(x) \Leftrightarrow \alpha\Delta\psi(x)$$

we are only interested in generator

Using noether theorem.

$$j^0 = \frac{\partial \mathcal{L}}{\partial(\partial_0\psi)}\delta\psi \longrightarrow \frac{\partial}{\partial\dot{\psi}}\left(\psi^\dagger\gamma^0(\gamma^0\partial_0\psi) - \bar{\psi}(\gamma^i\nabla_i - m)\psi\right)\delta\psi$$

$$\Rightarrow -i\psi^\dagger_{(x)}i\psi(x) \Rightarrow \quad +\psi^\dagger_{(x)}\psi(x)$$

$$j^0 = \psi^\dagger_{(x)}\psi_{(x)}$$

$$Q = \int d^3x\ \psi^\dagger\psi \quad \Big\}\ \text{in schrodinger picture.}$$

$$Q = \int d^3x \int \frac{d^3p\, d^3q}{(2\pi)^6}\frac{1}{\sqrt{4E_pE_q}} e^{ipx}e^{-iqx}\sum_{r,s}\left(u^{r\dagger}u^s a^{r\dagger}_p a^s_q + b^r_p b^{\dagger s}_q v^{r\dagger}v^s\right)$$

We have done this so many times; I didn't bother writing cross terms.

$$= \int d^3x \int \frac{d^3p\, d^3q}{(2\pi)^6}\frac{e^{ix(p-q)}}{\sqrt{4E_pE_q}}\sum_{r,s}\left(u^{r\dagger}u^s a^{r\dagger}_p a^s_q + b^r_p b^{\dagger s}_q v^{r\dagger}v^s\right)$$

resolve $\int d^3x$ to give a delta functn $(2\pi)^3\delta^3(p-q)\delta^{rs}$

$$= \int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\sum_r\left(U^{r\dagger}(p)U^r(-p)\, a^{r\dagger}_p a_{-p} + b^r_p b^{\dagger}_{-p}\, V^{\dagger r}(p)V^r(-p)\right)$$

$$= \int\frac{d^3p}{(2\pi)^3}\frac{1}{\cancel{2E_p}}\sum\left(\cancel{2E_p}\ a^{r\dagger}_p a_{-p} + b^r_p b^{\dagger}_{-p}\ \cancel{2E_p}\right)$$

$$= \int\frac{d^3p}{(2\pi)^3}\sum_r\left(a^{r\dagger}_p a^r_{-p} + b^r_p b^{r\dagger}_{-p}\right) = \int\frac{d^3p}{(2\pi)^3}\sum_r\left(a^\dagger_p a_{-p} - b^\dagger_{-p} b_p\right)$$

reminder

$$\{b_p, b^\dagger_p\} = 2\pi^3\delta^3(0) \Rightarrow b_pb^\dagger_p + b^\dagger_pb_p = (2\pi)^3\delta^3(0) \Rightarrow b_pb^\dagger_p = \underbrace{(2\pi)^3\delta^3(0)}_{\text{ignore this infinity.}} - b^\dagger_pb_p$$

$$Q = \int \frac{d^3p}{(2\pi)^3} \sum_r \left(a^{r\dagger}_{p} a^{r}_{-p} - b^{r\dagger}_{-p} b^{r}_{p} \right)$$

Where Q is the charge. $\left\{ \begin{array}{l} \Psi \text{ creates positron} \\ \bar{\Psi} \text{ creates electron.} \end{array} \right.$

The Dirac propagator:-

$$\langle 0 | \psi(x) \bar{\psi}(y) | 0 \rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \sum_s u^s_a(p) \bar{u}^s_a(p)\, e^{-ip(x-y)}$$

$$= \int \frac{d^3p}{(2\pi)^3} \cdot \frac{1}{2E_p} (i\gamma\partial_x + m)\, e^{-ip(x-y)}$$

Using anti-commutation for $\langle 0 | \{\psi(x)\, \bar{\psi}(y)\} | 0 \rangle$; we can find Dirac Propagator.
like we did for KG.
(using fourier transform)

$$S_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p}+m)}{p^2-m^2+i\epsilon}$$

Hopefully you can study about time reversal, parity and charge conjugation.

I will move to Chapter 4 - Perturbation theory.

$$S_R(x-y) = \int \frac{d^4p}{(2\pi)^4} \tilde{S}_R(x-y) \exp(-ip(x-y))$$

and

$$(\not{p}-m)\tilde{S}_R(x-y) = i \Rightarrow \tilde{S}_R(x-y) = \frac{i}{(\not{p}-m)(\not{p}+m)} (\not{p}+m) \Rightarrow \frac{i[\gamma^\mu \partial_\mu + m]}{}$$

The denom.

$$(\gamma^\mu p_\mu - m)(\gamma^\nu p_\nu + m) = \gamma^\mu p_\mu \gamma^\nu p_\nu - m^2 + \cancel{\gamma^\mu p_\mu} - \cancel{\gamma^\nu p_\nu}$$

$$= \gamma^\mu p_\mu \gamma^\nu p_\mu - m^2$$

$$= \gamma^\mu \gamma^\nu p_\mu p_\nu - m^2$$

we can write $\gamma^\mu \gamma^\nu$ in sym and anti sym parts but only sym part surv.

$$= \frac{1}{2}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) p_\mu p_\nu - m^2 \Rightarrow \frac{1}{2}(2g^{\mu\nu} p_\mu p_\nu) - m^2 \Rightarrow p^2 - m^2$$; we get $\tilde{S}_R(x-y) = \frac{i(\not{p}-m)}{p^2-m^2}$

$$\boxed{S_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p}-m)}{p^2-m^2} e^{-ip(x-y)}}$$