Portfolio Assignment
Honors Linear Algebra (317H)

Purpose:
Building proof writing skills takes practice and feedback. One of the goals of the portfolio assignment is to improve your proof writing skills and understanding of proof techniques by working on writing and refining several proofs. This portfolio gives you the chance to synthesize some ideas from across the semester and demonstrate your ability to write proofs. It also asks you to reflect on your progress in proof writing.

Assignment:

- To earn full credit, your final portfolio should include responses to all of the prompts in the three sections outlined below.
- In order to receive feedback and improve your proofs, you will submit drafts of your proofs (from Section 2 of the Portfolio) to the instructor. These can be submitted individually as the proofs are completed, any time throughout the semester. These drafts must be typed in LaTex. Note that part of the grade on the portfolio assignment is for submitting drafts of the proofs in Section 2. To receive full credit for your draft submissions, by **October 29, 2024** drafts of at least 2 proofs from Section 2 must have been submitted for feedback via gradescope. By **Nov 12, 2024** drafts of all 4 proofs from Section 2 must have been submitted for feedback. You are welcome to also submit the other portions of the portfolio (Section 1 and/or Section 3) for feedback by the November 12th deadline, but this is not required.
- The complete final draft of your portfolio should be submitted during finals week, by **Thursday, December 12th at 1pm**.
- The portfolio should be professional and needs to be typed in LaTex.
  - We will have a LaTex and Proof Portfolio workshop day in recitation on **Monday, October 7th**. Bring your laptops to recitation on that day!
  - There are LaTex resources available on D2L in the Proof Portfolio section.
  - Guidelines for typesetting mathematical proofs can be found at the end of this document.

1 Proof Techniques

Provide responses to each of the prompts below.

1.1 Direct Proof

(a) In a few sentences, describe the components of a good direct proof.

(b) Imagine yourself as a teacher and discuss what aspects of this topic may be confusing to students, discuss common mistakes, and provide hints on how one could avoid common mistakes. Explain how to recognize when direct proof is a good proof strategy for a given statement.

(c) ChatGPT is an AI chatbot that responds when asked to provide a mathematical proof. Click here to view an AI-generated direct proof of the following claim: $\{4n+3 : n \in \mathbb{Z}\} = \{2n+5 : n \in \mathbb{Z}\}$.

Evaluate this proof written by ChatGPT. You can comment on its mathematical correctness, its proof-writing style, and its clarity of argument. In your evaluation, please make sure to discuss whether you believe this claim is true and whether the AI proof is mathematically valid.

1.2 Statements and Negations

(a) Explain why the negation of an "and" statement becomes an "or" statement.

(b) Explain why the negation of a "there exists" statement becomes a "for all" statement.

(c) Explain why the negation of an implication in the form $P \implies Q$ is not the implication $P \implies \neg Q$. In your explanation, provide a simple example where both $P \implies Q$ and $P \implies \neg Q$ are true (which demonstrates they are not negations of each other).

1.3 Proof by Contradiction

(a) Provide a brief explanation of the main idea behind a proof by contradiction and its main components.

(b) Imagine yourself as a teacher and discuss what aspects of this topic may be confusing to students, discuss common mistakes, and provide hints on how one could avoid common mistakes. Explain how to recognize when proof by contradiction is a good proof strategy for a given statement.

(c) For each of the following, write the assumption you would begin with if you tried to prove the statement by contradiction. You do not have to complete the entire proof.

- **Statement 1**: If x and y are odd integers then xy is odd.
- **Statement 2**: For every real number $x \in [0, \pi/2]$, we have $\sin x + \cos x \geq 1$.
- **Statement 3**: There are infinitely many prime numbers.

- **Statement 4**: There are no rational solutions to the equation $x^3 + x + 1 = 0$.

1.4 Proof by Contrapositive

(a) Provide a brief explanation of the main idea behind a proof by contrapositive and its main components.

(b) Imagine yourself as a teacher and discuss what aspects of this topic may be confusing to students, discuss common mistakes, and provide hints on how one could avoid common mistakes. Explain how to recognize when proof by contrapositive is a good proof strategy for a given statement.

(c) For each of the following, decide whether you would use a direct proof, contradiction, or contrapositive to prove the statement. You do not have to prove any of the statements, but your response should include an explanation of why you chose the proof technique that you did.

- **Statement 1**: $\sqrt{2}$ is irrational.
- **Statement 2**: For $n \in \mathbb{Z}$, if n^2 is even then n is even.
- **Statement 3**: There is no largest rational number.
- **Statement 4**: Let $a, b \in \mathbb{Z}$. If $x^2(y+3)$ is even then x is even or y is odd.

1.5 Proof by Induction

(a) Provide a brief explanation of why proof by induction makes sense and what the main components of a proof by induction are.

(b) Imagine yourself as a teacher and discuss what aspects of this topic may be confusing to students, discuss common mistakes, and provide hints on how one could avoid common mistakes. Explain how to recognize when proof by induction is a good proof strategy for a given statement.

(c) Use the following template to finish the proof by induction. Proposition: For all non-negative integers n, $3 \mid (n^3 - 4n + 9)$.
Proof: We proceed by induction.

Base Case: First, we show the base case holds when $n = ...$

Induction assumption: Assume the statement holds for $n = k$.

Inductive step: We must show the statement holds when $n = k + 1$ and verify that 3 divides $((k+1)^3 - 4(k+1) + 9)$ ...

Thus we see by induction that for all non-negative integers n, $3 \mid (n^3 - 4n + 9)$.

2 Proofs

In this section you will write proofs for each of the following four propositions. You must submit drafts of these proofs for feedback by the deadlines outlined at the beginning of the document. Your final versions must incorporate the feedback provided.

Proposition 1. Suppose that $v_1, \ldots, v_m$ is a linearly independent list of vectors in a vector space V. Then the list $v_1, \ldots, v_m, w$ is linearly independent if and only if

$$w \notin \text{ span } (v_1, \ldots, v_m).$$

Proof. We know $v_1, \ldots, v_m$ is a linearly independent list of vectors in a vector space V. We need to show:

1. If $v_1, \ldots, v_m, w$ is linearly independent then $w \notin \text{ span } (v_1, \ldots, v_m)$

   Proof by Contradiction: Let us assume for the sake of contradiction that $v_1, \ldots, v_m, w$ is a linearly independent list and $w \in \text{ span } (v_1, \ldots, v_m)$.

   Now, $w = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m$, as $w \in \text{ span } (v_1, \ldots, v_m)$ where w is not $\vec{0}$ as it is part of a linearly independent list.
   Here,

   $$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m - (1)w = 0 \tag{1}$$

   From 1, we see that $v_1, \ldots, v_m, w$ are Linearly Dependent. This is a contradiction with our original assumption. Thus we have shown that:

   if $v_1, \ldots, v_m, w$ is linearly independent then $w \notin \text{ span } (v_1, \ldots, v_m)$.

2. If $w \notin \text{ span } (v_1, \ldots, v_m)$, then $v_1, \ldots, v_m, w$ is linearly independent.

   Proof by Contradiction: Let us assume for the sake of contradiction that $w \notin \text{ span } (v_1, \ldots, v_m)$ and $v_1, \ldots, v_m, w$ is linearly dependent.

   So,

   $$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m + \alpha_{m+1} w = 0 \tag{2}$$

   Here, at least one of $\alpha_1 \ldots \alpha_m, \alpha_{m+1} \neq 0$, as this is a Linearly Dependent list.

   (a) Case 1: $\alpha_{m+1} \neq 0$

   From, 2, we get:
   $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m + \; = -\alpha_{m+1} w$

Now, because $\alpha_{m+1} \neq 0$, we have:

$$\frac{\alpha_1}{-\alpha_{m+1}}v_1 + \frac{\alpha_2}{-\alpha_{m+1}}v_2 + \cdots + \frac{\alpha_m}{-\alpha_{m+1}}v_m = w \tag{3}$$

Here, from 3, we see that $w \in$ span $(v_1, \ldots, v_m)$ which is a contradiction with our original statement.

(b) Case 2: $\alpha_{m+1} = 0$

Here, we know, at least one of $\alpha_1 \ldots \alpha_m \neq 0$.

Now,

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m + 0w = 0 \tag{4}$$

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m = 0 \tag{5}$$

Thus, from 5, we see that $v_1 \ldots v_m$ is a linearly dependent list. This is a contradiction with our original statement.

Here, we see that both cases lead to a Contradiction. Thus, we can say that If $w \notin$ span $(v_1, \ldots, v_m)$, then $v_1, \ldots, v_m, w$ is linearly independent.

Therefore, after proving the statement both ways, we can conclude that $v1, \ldots, v_m, w$ is Linearly Independent if and only if $w \notin \text{Span}(v_1 \ldots v_m)$.

□

Proposition 2. Let V and W be vector spaces, and $W \neq \{0\}$. Suppose that $v_1, \ldots, v_m$ is a linearly dependent list of vectors in V. Then there exist vectors $w_1, \ldots w_m \in W$ such that no $T \in \mathcal{L}(V, W)$ satisfies $Tv_k = w_k$ for each $k = 1, \ldots m$.

Proof. Your proof goes here. □

Proposition 3. Suppose V is a finite dimensional vector space. Then every linear map on a subspace of V can be extended to a linear map on V. In other words, if U is a subspace of V and $T \in \mathcal{L}(U, W)$, then there exists $S \in \mathcal{L}(V, W)$ such that $Su = Tu$ for all $u \in U$.

Proof. Your proof goes here. □

Proposition 4. The list of polynomials

$$B = 1, (x-3), (x-3)^2, \ldots (x-3)^n.$$

is a basis for $\mathcal{P}_n$, the vectors space of real polynomials with degree less than or equal to n.

Proof. Your proof goes here. $\square$

3 Final Reflection

Write an essay that describes your experience in this course and the progress you made during this semester. The narrative should be 1 to 2 pages, single-spaced, and should address the following questions:

(a) What did you know about the subject of Linear Algebra at the start of the semester? How did you feel about proof-writing during the first few weeks of class? How would you characterize your proof writing confidence and ability at the beginning of the semester?

(b) How have your approaches to writing proofs, generating examples, and understanding advanced mathematics changed during the semester? How do your current feelings about Linear Algebra and proof-writing compare to those from the beginning of the semester? Has this course affected how you view yourself as a mathematician?

(c) What did you find particularly interesting this semester? What was something you learned that was surprising to you? What questions do you still have?

Style Guidelines

Your portfolio should be typed in LaTeX, which you might not have worked with before. This section offers some help getting started on your document **and also gives you some sample code that you could copy, paste, and edit into your own draft to help you typeset**.

In terms of your paragraph writing, the portfolio assignment is not so different from essay writing for subjects other than math. Write to communicate your ideas clearly and effectively to the target audience (AKA me) as well as your own future self. I hope the portfolio will be a useful reference for you in later math courses. All the usual standards for paper writing apply: use proper punctuation and spelling, avoid exceedingly long paragraphs, etc.

The purpose of this document is to give you advice on what the more "mathy" parts of the portfolio should look like, since this is what we expect you to have the least experience with. How should you typeset math equations, what should you write to signal the start and end of a proof, that kind of thing.

Guideline #1 - Use italics when writing equations or math symbols
When you use variables like x or y to represent math quantities, they should be italicized, whether they appear inside a paragraph (like this) or in a sequence of math equations set out from the text, like below.

$$x + 3 = 7 \implies x + 3 - 3 = 7 - 3 \implies x = 4$$

You should NOT write things like x=4 or x=2y without using italics, as I have just done in this sentence.

Guideline #2 - Use superscripts and subscripts where appropriate
Math uses a lot of superscripts (like the 2 in x^2) and subscripts (such as the n in a_n). On a calculator, you would have to type $x \wedge 2$, but in your portfolio this should be written as x^2.

Guideline #3 Do not start a sentence with a mathematical symbol
For example, you should not write, "V is a finite dimensional vector space, so it has a basis $v_1, \dots v_n$." Instead you could write, "Since V is a finite dimensional vector space, it has a basis $v_1, \dots v_n$."

Guideline #4 - Be clear about where proofs start and end
When you prove something in writing in math, if you just jump into it, then your reader might be confused about what you're proving. And if you finish the proof, and then transition immediately into a new paragraph or a new proof, it might not be clear to the reader that the proof is done. Here's what a simple example looks like.

Proposition 5. If x is even, then $x + 1$ is odd.

Proof. Since x is even, we can write it as $x = 2n$ where n is an integer. Then $x + 1 = 2n + 1$, which is odd by the definition of odd integers since n is an integer. □

Notice the following important three details: the word "proposition" signals that I am about to state something which I will later prove. The word "proof" signals that the statement to be proved is done, and now I am going to prove it. The box at the end signals that the proof is done. These three details should appear in some form every time you write a proof in your portfolio.

These three key elements don't have to take exactly this form. The word "proposition" can be replaced by various words: theorem, claim, lemma. The word "proof" doesn't have any alternatives, though. The box at the end can alternately be replaced by the letters QED, which is an abbreviation for the Latin phrase *quod erat demonstrandum.*

Guideline #5 - Do not include the directions for the assignment

If you use this template document for your portfolio, be sure to delete the instructions from the document before turning it in.

Examples of how to write in LaTeX

These examples are included here so that you can see what LaTeX syntax looks like.

Proposition 6. Given sets A and B, defined by $A = \{3x : x \in \mathbb{R}\}$ and $B = \{10x : x \in \mathbb{R}\}$, $A = B$.

Proof. First, we'll prove that $\{3x : x \in \mathbb{R}\} \subseteq \{10x : x \in \mathbb{R}\}$. Take a generic element $a \in \{3x : x \in \mathbb{R}\}$. Thus, there exists $x \in \mathbb{R}$ such that $a = 3x$. We need to show $a \in \{10x : x \in \mathbb{R}\}$. Let us represent a as follows:

$$a = 3x = 10\frac{3}{10}x.$$

Now, note that $x \in \mathbb{R}$ and $\frac{3}{10} \in \mathbb{R}$, so y, defined by $y = \frac{3}{10}x$ is in $\mathbb{R}$. Thus, we can conclude that $a = 10y \in \{10x : x \in \mathbb{R}\}$, so $\{3x : x \in \mathbb{R}\} \subseteq \{10x : x \in \mathbb{R}\}$.

Second, we'll prove that $\{10x : x \in \mathbb{R}\} \subseteq \{3x : x \in \mathbb{R}\}$. Take a generic element $b \in \{10x : x \in \mathbb{R}\}$. Thus, there exists $w \in \mathbb{R}$ such that $b = 10w$. We need to show $b \in \{3x : x \in \mathbb{R}\}$. Let us represent b as follows:

$$b = 10w = 3\frac{10}{3}w.$$

Now, note that $w \in \mathbb{R}$ and $\frac{10}{3} \in \mathbb{R}$, so z, defined by $z = \frac{10}{3}w$ is in $\mathbb{R}$. Thus, we can conclude that $b = 3z \in \{3x : x \in \mathbb{R}\}$, so $\{10x : x \in \mathbb{R}\} \subseteq \{3x : x \in \mathbb{R}\}$. We proved $\{3x : x \in \mathbb{R}\} \subseteq \{10x : x \in \mathbb{R}\}$ and $\{10x : x \in \mathbb{R}\} \subseteq \{3x : x \in \mathbb{R}\}$, thus, $\{10x : x \in \mathbb{R}\} = \{3x : x \in \mathbb{R}\}$. $\square$

Proposition 7. The statements $\neg(P \vee \neg Q)$ and $(\neg P) \wedge Q$ are logically equivalent.

Proof. We will construct a truth table to prove the statements are logically equivalent.

P	Q	$\neg P$	$\neg Q$	$P \vee \neg Q$	$\neg(P \vee \neg Q)$	$(\neg P) \wedge Q)$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	F	T	T
F	F	T	T	T	F	F

Yes they are logically equivalent, since we compare the last two columns of the table, we see that $\neg(P \vee \neg Q)$ and $(\neg P) \wedge Q$ coincide for all possible values of P and Q. $\square$

Proposition 8. If n is an even integer, then $5n + 7$ is odd.

Proof. Assume that $n \in \mathbb{Z}$ is an even integer. Then, by the definition of of an even integer, $\exists k \in \mathbb{Z}$ such that $n = 2k$. Thus,

$$\begin{aligned} 5n + 7 = 5 \cdot 2k + 7 &= 2(5k + 3) + 1 \\ 5n + 7 &= 2s + 1, \end{aligned}$$

where $s = 5k + 3$ is an integer (as the product and sum of integers). Thus $5n + 7$ can be written as $2s + 1$ for some integer s and therefore we conclude that $5n + 7$ is odd. $\square$

Checklist

Before turning in your portfolio, please make sure that you:

- ■ Put your name on the title page of your portfolio.
- ■ Deleted the directions and prompts from the document. Your portfolio should only include your work/responses (and the statements of the results that you are proving).
- ■ Responded to all the questions in Sections 1, 2, and 3 of the portfolio.
- ■ Clearly denoted where your proofs start and end.
- ■ Wrote all equations and math symbols in the portfolio in math mode, and long equations appear on a new line that's centered.
    - There's nothing like $a \wedge 3$ in your portfolio - all super/subscripts are in math mode.
    - There's nothing like 3/4 in your portfolio - all fractions are written in math mode.
    - There's nothing like f(x)=4 written in your portfolio - all functions are in math mode.
- ■ You proof read your portfolio.