Lusin’s Theorem

Lusin’s Theorem has not been formalised yet in Lean4. As such, we are taking on the challenge of doing so.

Proof we are working from:

In particular, we endeavour to prove the formulation of Lusin’s Theorem provided in the MA359 Measure Theory lecture notes, written by Josephine Evans. The notes can be found here: https://avocadoade.github.io/ma359/index.html.

Lusin’s Theorem can be found as Theorem 4.4 in section 4.2 of these notes. The theorem, along with the proof, has been pasted below:

Theorem 4.4 (Lusin's Theorem) *Suppose that f is a measurable function and $A \subseteq \mathbb{R}^d$ is a Borel set and $\lambda(A) < \infty$ then for any $\epsilon > 0$ there is a compact subset K of A with $\lambda(A \setminus K) < \epsilon$ such that the restriction of f to K is continuous.*

Remark. This theorem can be generalised to locally compact Hausdorff spaces, see Cohn’s book.

Proof. Suppose first that f only takes countably many values, $a_1, a_2, a_3, \ldots$ on A the let $A_k = \{x \in A : f(x) = a_k\}$, by measurablility of f we can see that $A_k = f^{-1}(\{a_k\})$ is measurable. We know that $A = \bigcup_n A_n$ so by continuity of measure $\lambda(\bigcup_{k=1}^n A_k) \uparrow \lambda(A)$. Since $\lambda(A) < \infty$ we have that for any $\epsilon > 0$ there exists n such that $\lambda(A \setminus \bigcup_{k=1}^n A_k) < \epsilon/2$. By the regularity of Lebesgue measure we can find compact subsets $K_1, \ldots, K_n$ such that $\lambda(A_n \setminus K_n) \leq \epsilon/2n$. Then let $K = \bigcup_{k=1}^n K_k$. This is a compact subset of A and

$$\lambda(A \setminus K) \leq \lambda(A \setminus \bigcup_{k=1}^{n} A_k) + \lambda(\bigcup_{k=1}^{n} A_k \setminus \bigcup_{k=1}^{n} K_k) < \epsilon/2 + \epsilon/2.$$

Now f restricted to K is continuous since the K_i are disjoint and f is constant on each K_i.

Now we have proved the special case where f takes countably many values we can use this to prove the theorem for general f. Let $f_n = 2^{-n}\lfloor 2^n f \rfloor$ then $2^{-n} \geq f(x) - f_n(x) \geq 0$ so $f_n(x) \to f(x)$, uniformly. Now, f_n can only take countably many values, so by our special case of Lusin’s theorem there exists a $K_n \subseteq K$, compact, such that $\lambda(A \setminus K_n) \leq \epsilon 2^{-n}$, and f_n is continuous on K_n. Now let $K_\infty = \bigcap_n K_n$, then K_∞ is compact and $\lambda(A \setminus K_\infty) = \lambda(\bigcup_n (A \setminus K_n)) \leq \sum_n \epsilon 2^{-n} = \epsilon$. Now we have that f_n converges uniformly to f on K_∞ and f_n is continuous on K_∞ for each n. As the uniform limit of continuous functions is continuous this shows that f is continuous on K. $\square$

Lusin states that for any measurable function from a Borel subset of $\mathbb{R}^d$ to $\mathbb{R}$ is almost continuous - i.e. it is continuous except for an arbitrarily small set.

There is also the more general topological version of which the MA359 version is based off, which can be found in Donald L. Cohn’s “Measure Theory”:

Theorem 7.4.4 (Lusin's Theorem). *Let X be a locally compact Hausdorff space, let $\mathscr{A}$ be a σ-algebra on X that includes $\mathscr{B}(X)$, let μ be a regular measure on $(X, \mathscr{A})$, and let $f: X \to \mathbb{R}$ be $\mathscr{A}$-measurable. If A belongs to $\mathscr{A}$ and satisfies $\mu(A) < +\infty$ and if ε is a positive number, then there is a compact subset K of A such that $\mu(A - K) < \varepsilon$ and such that the restriction of f to K is continuous. Moreover, there is a function g in $\mathscr{K}(X)$ that agrees with f at each point in K; if $A \neq \varnothing$ and f is bounded on A, then the function g can be chosen so that*

$$\sup\{|g(x)| : x \in X\} \leq \sup\{|f(x)| : x \in A\}. \tag{1}$$

Definitions needed for the statement

- **Define** a function f from a measurable space $(\mathbb{R}^d, B(\mathbb{R}^d))$ to another measurable space $(\mathbb{R}, B(\mathbb{R}))$
  - Includes the definition of $\mathbb{R}, \mathbb{R}^d$, a measurable space, sigma algebra, Borel set, functions between measurable spaces.
  - Borel set: https://leanprover-community.github.io/mathlib4_docs/Mathlib/MeasureTheory/Constructions/BorelSpace/Basic.html
  - Measurable space: https://leanprover-community.github.io/mathlib4_docs/Mathlib/MeasureTheory/MeasurableSpace/Basic.html#MeasurableSpace.map
- **Define** a finite subset $A \in B(\mathbb{R}^d))$ and $\lambda(A) < \infty$.
  - Includes the definition of a an element of the Borel set with finite measure.
  - https://leanprover-community.github.io/mathlib4_docs/Mathlib/MeasureTheory/Measure/FiniteMeasure.html#MeasureTheory.FiniteMeasur
- **Define** $\epsilon > 0$ a non-negative real number
- **Define** a countable union
  - https://leanprover-community.github.io/mathlib4_docs/Mathlib/Data/Set/Countable.html#Set.Countable
- **Define** a compact [closed also works] subset $K \subset A$ such that $\lambda(A/K) < \epsilon$
  - Includes the definition of a compact subset and the measure of operations of sets
  - https://leanprover-community.github.io/mathlib4_docs/Mathlib/Topology/Compactness/Compact.html#IsCompact
- **Define** the restriction of the function f to the set K [maybe call it g]
- **Define** epsilon-delta definition of convergence - this will be applied following the application of continuity of measure
- **Define** the floor function
  - https://leanprover-community.github.io/mathlib4_docs/Std/Data/Rat/Basic.html#Rat.floor
- **Define** uniform convergence on our space of functions
  - https://leanprover-community.github.io/mathlib_docs/topology/uniform_space/uniform_convergence.html#tendsto_uniformly_on

Pre-existing results we will import from Mathlib

- That a singleton is a measurable subset
- Continuity of measure
- Finite unions and infinite intersections of compact sets are compact.
- Sum of a geometric series
- Restriction of a continuous function is continuous
- Uniform limit of continuous functions is continuous

Decomposition of the proof:

Part 1 - The case where f takes countably many values

We start by trying to prove an easier case.

- **Define** f as above and $f(A) = \bigcup_k a_k$
- **Define** $A_k = \{x \in A : f(x) = a_k\}$
- Show the set of countable singletons $a_1, a_2, a_3, ...$ is Borel

  Definition 4.1 (Mesasurable functions)**.** If $(E, \mathcal{E})$ and $(F, \mathcal{F})$ are two measurable spaces and f is a function $E \to F$, then we say f is *measurable* if for every $A \in \mathcal{F}$ we have $f^{-1}(A) \in \mathcal{E}$.

  - Use measurability of f to show sets $A_k = f^{-1}(\{a_k\})$ are measurable

$$f^{-1}\left(\bigcup_i A_i\right) = \bigcup_i f^{-1}(A_i)$$

- Use above to show that $A = f^{-1}(\bigcup_k a_k) = \bigcup_k f^{-1}(\{a_k\}) = \bigcup_k A_k$

  Theorem 2.1 (Continuity of measure)**.** *Let $(E, \mathcal{E}, \mu)$ be a measure space. Suppose that $(A_n)_n$ is a sequence of measurable sets with $A_1 \subseteq A_2 \subseteq ...$ and $(B_n)_n$ is a sequence of measurable sets with $B_1 \supseteq B_2 \supseteq ...$, and $\mu(B_1) < \infty$ then we have*

$$\mu\left(\bigcup_n A_n\right) = \lim_n \mu(A_n)$$

- Use continuity of measure to show $\lim_n \lambda(\bigcup_{k=1}^n A_k) \to \lambda(\bigcup_k A_k) = \lambda(A)$
- Use the finiteness of A and the definition of limit to show $\forall \epsilon > 0$ such that $\lambda(A / \bigcup_{k=1}^n A_k) < \epsilon/2$

Definition 4.11. Let E be a topological space and μ be a measure on $(E, \mathcal{B}(E))$ then say μ is *regular* if for every $A \in \mathcal{B}(E)$ we have

- $$\mu(A) = \inf\{\mu(U) \,:\, A \subseteq U, U \text{ is open}\},$$
- $$\mu(A) = \sup\{\mu(K) \,:\, K \subseteq A, K \text{ is compact}\}.$$

- Show that there exist compact [closed also] subsets $K_k \subset A_k$ such that $\lambda(A_k / K_k) < \epsilon/2n$, using the regularity of the Lebesgue measure above.
- **Define** $K = \bigcup_{k=1}^n K_k$ and this is compact because finite unions of compact subsets are compact.

$$\lambda(A \setminus K) \leq \lambda(A \setminus \bigcup_{k=1}^n A_k) + \lambda(\bigcup_{k=1}^n A_k \setminus \bigcup_{k=1}^n K_k) < \epsilon/2 + \epsilon/2.$$

- The above uses the fact any measure is countably sub-additive.
- The restriction of f to K is the restriction of f to a finite collection of disjoint compact sets K_k and f is constant on each set, therefore f is continuous and we have proven Lusin's theorem for the special case where f takes countably many values.

Part 2 - Extending to all measurable f

- Let f be defined as in the theorem statement.
- Show that for $f_n = 2^{-n}\lfloor 2^n f \rfloor$, f_n converges uniformly to f. Not sure if this will have been proven in lean4, if not we use the following to prove it:

$$\text{Let } f_n = 2^{-n}\lfloor 2^n f \rfloor \text{ then } 2^{-n} \geq f(x) - f_n(x) \geq 0 \text{ so } f_n(x) \to f(x) \text{, uniformly.}$$

- Show that f_n takes countably many values
- Show that there exist compact sets $K_k \subset K$ such that $\lambda(A/K_k) < \epsilon/2^n$ and f_n is continuous on K_k using part 1.
- **Define** $K_\infty = \bigcap_n K_n$, which is compact since the intersection of any number of compact sets is compact.

$$\lambda(A \setminus K_\infty) = \lambda(\bigcup_n (A \setminus K_n)) \leq \sum_n \epsilon 2^{-n} = \epsilon.$$

- Show the statement above, using rules about set operations.
- Show that f_n converges uniformly to f on K_∞ [should be just because it is a subset and it converges uniformly on the larger set]
- Show that f is continuous using the fact that the uniform limit of continuous functions is continuous.

Distributing the workload

The following is a rough outline of how we plan to distribute the workload. We will refine this in our individual video submissions in a few week's time:

- Aadam: research the utility of the packages we plan on importing from Mathlib. Introduce the main imports and test their implementations before we progress to proving the Lusin-specific components. Decompose the proof of the theorem into the aforementioned constituents in the final .lean file.
- Ameer: proofs for part 1:
  - Countable union of singletons is a Borel set.
  - Pre-image of each singleton under f is measurable, using imported definitions along with the measurability of f.
  - Utilising the regularity of the Lebesgue measure for the sequence of compact sets K_k
- Louis: proofs for part 2:
  - Uniform convergence of the sequence of floor functions which converges to f.
  - Verifying that f_n takes countably many values.
  - Applying result from part 1.
- Giovanni: supporting Ameer and Louis with the two main parts, in particular:
  - Part 1: applying the continuity of measure to pass the limit inside the $\lambda(\bigcup A_k)$ and concluding the "almost continuous" property of f on the given domain.
  - Part 2: convergence of f_n on the constructed restriction K_∞.

Other avenues of support

We shall utilise the Zulip Lean4 threads to ask questions if we're stuck with specific issues during our project: https://leanprover-community.github.io/archive/. If we ask for support in this way, we will ensure proper referencing in our code to reflect this.