A Summary of Linear Algebra

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This document is a list of some material in linear algebra that you should be familiar with. Throughout, we will take A to be the 3 x 4 matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & 3 & -1 & 5 \\ 3 & -1 & 4 & -1 \end{bmatrix}$$

I assume you are familiar with matrix and vector addition and multiplication.

- All vectors will be **column** vectors.
- Given a vector v, if we say that $v \neq 0$, we mean that v has at least one nonzero component.
- The **transpose** of a vector or matrix is denoted by a superscript T. For example,

$$A^T = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ 3 & -1 & 4 \\ 4 & 5 & -1 \end{bmatrix}$$

- The **inner product** or **dot product** of two vectors u and v in $\mathbb{R}^n$ can be written $u^T v$; this denotes $\sum_{i=1}^n u_i v_i$. If $u^T v=0$ then u and v are **orthogonal**.
- The **null space** of A is the set of all solutions x to the matrix-vector equation $Ax=0$.
- To solve a system of equations $Ax=b$, use Gaussian elimination. For example, if $b = [4,\ -3,\ 7]^T$, then we solve $Ax=b$ as follows: (We set up the augmented matrix and row reduce (or pivot) to upper triangular form.)

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 4 \\ -2 & 3 & -1 & 5 & -3 \\ 3 & -1 & 4 & -1 & 7 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 4 \\ 0 & 7 & 5 & 13 & 5 \\ 0 & -7 & -5 & -13 & 7 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 4 \\ 0 & 7 & 5 & 13 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

  Thus, the solutions are all vectors x of the form

$$x = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 11 \\ 5 \\ -7 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 13 \\ 0 \\ -7 \end{bmatrix}$$

  for any numbers s and t.
- The **span** of a set of vectors is the set of all linear combinations of the vectors. For example, if $v^1 = [11,\ 5,\ -7,\ 0]^T$ and $v^2 = [2,\ 13,\ 0,\ -7]^T$ then the span of v^1 and v^2 is the set of all vectors of the form sv^1+tv^2 for some scalars s and t.
- The span of a set of vectors in $\mathbb{R}^n$ gives a **subspace** of $\mathbb{R}^n$. Any nontrivial subspace can be written as the span of any one of uncountably many sets of vectors.
- A set of vectors $\{v^1, \ldots, v^m\}$ is **linearly independent** if the only solution to the vector equation $\lambda_1 v^1 + \ldots + \lambda_m v^m = 0$ is $\lambda_i = 0$ for all i. If a set of vectors is not linearly independent, then it is **linearly dependent**. For example, the rows of A are *not* linearly independent, since

$$-\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \\ -1 \\ 5 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

  To determine whether a set of vectors is linearly independent, write the vectors as columns of a matrix C, say, and solve $Cx=0$. If there are any nontrivial solutions then the vectors are linearly dependent; otherwise, they are linearly independent.
- If a linearly independent set of vectors spans a subspace then the vectors form a **basis** for that subspace. For example, v^1 and v^2 form a basis for the span of the rows of A. Given a subspace S, every basis of S contains the same number of vectors; this number is the **dimension** of the subspace. To find a basis for the span of a set of vectors, write the vectors as rows of a matrix and then row reduce the matrix.
- The span of the rows of a matrix is called the **row space** of the matrix. The dimension of the row space is the **rank** of the matrix.
- The span of the columns of a matrix is called the **range** or the **column space** of the matrix. The row space and the column space always have the same dimension.
- If M is an m x n matrix then the null space and the row space of M are subspaces of $\mathbb{R}^n$ and the range of M is a subspace of $\mathbb{R}^m$.
- If u is in the row space of a matrix M and v is in the null space of M then the vectors are orthogonal. The dimension of the null space of a matrix is the **nullity** of the matrix. If M has n columns then rank(M)+nullity(M)=n. Any basis for the row space together with any basis for the null space gives a basis for $\mathbb{R}^n$.
- If M is a square matrix, λ is a scalar, and x is a vector satisfying $Mx = \lambda x$ then x is an **eigenvector** of M with corresponding **eigenvalue** λ. For example, the vector $x = [1,\ 2]^T$ is an eigenvector of the matrix

$$M = \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}$$

with eigenvalue $\lambda = 7$.

- The eigenvalues of a symmetric matrix are always real. A nonsymmetric matrix may have complex eigenvalues.
- Given a symmetric matrix *M*, the following are equivalent:

  1. All the eigenvalues of *M* are positive.
  2. $x^T M x > 0$ for any $x \neq 0$.
  3. M is **positive definite**.

- Given a symmetric matrix *M*, the following are equivalent:

  1. All the eigenvalues of *M* are nonnegative.
  2. $x^T M x \geq 0$ for any *x*.
  3. M is **positive semidefinite**.

- About this document ...

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