Bayes Filtering

AN INTRODUCTION BY EXAMPLE

Introduction

The theme of this course is navigation: we want to figure out where we are relative to other things.

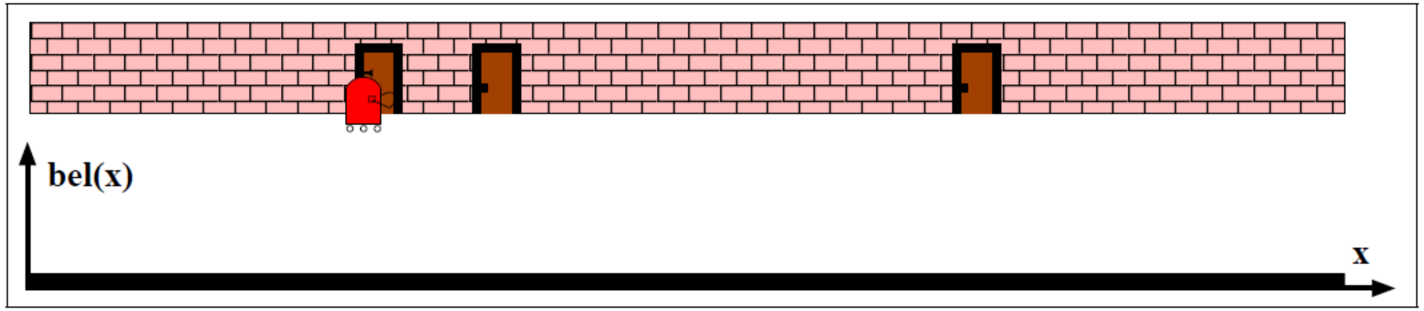
We refer to all possible places you might be as the *state space* and a specific sample of it is referred to as the *state*.

- Typically represented as x

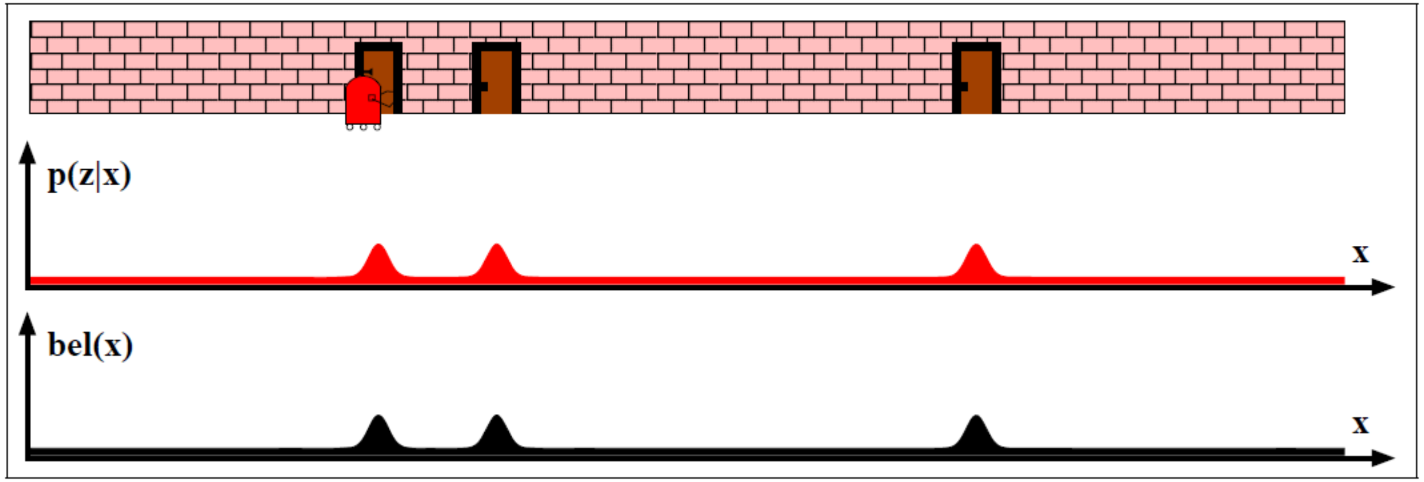
The basic idea behind a probabilistic framework for navigation and the Bayes Filter is to represent your localization as a continuous probability distribution known as the *belief*.

This straightforwardly translates to navigation solutions: you are currently located at the state with the highest belief.

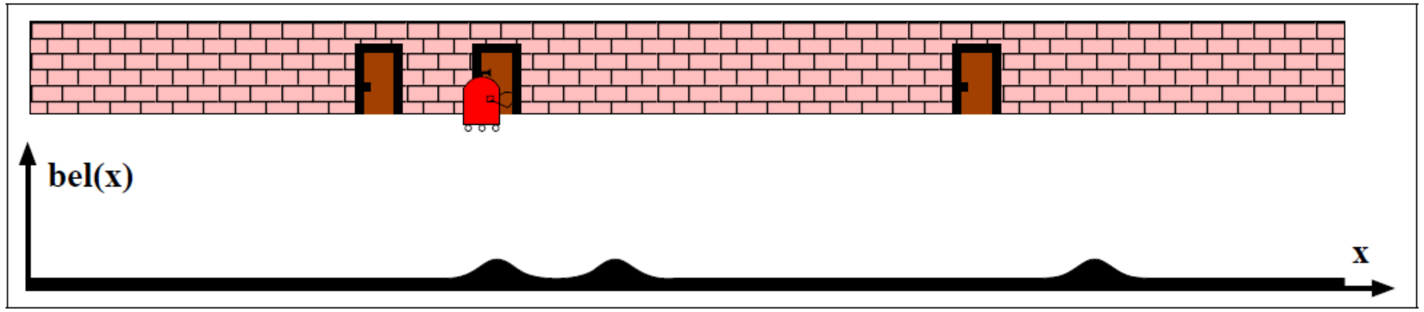
An example in one dimension



An example in one dimension



An example in one dimension



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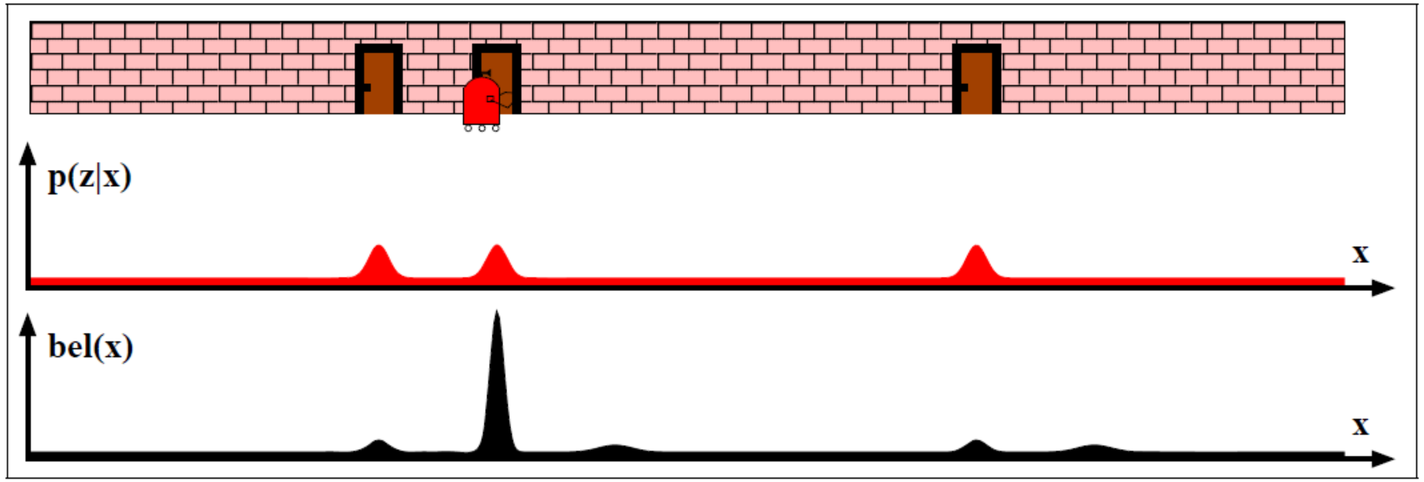

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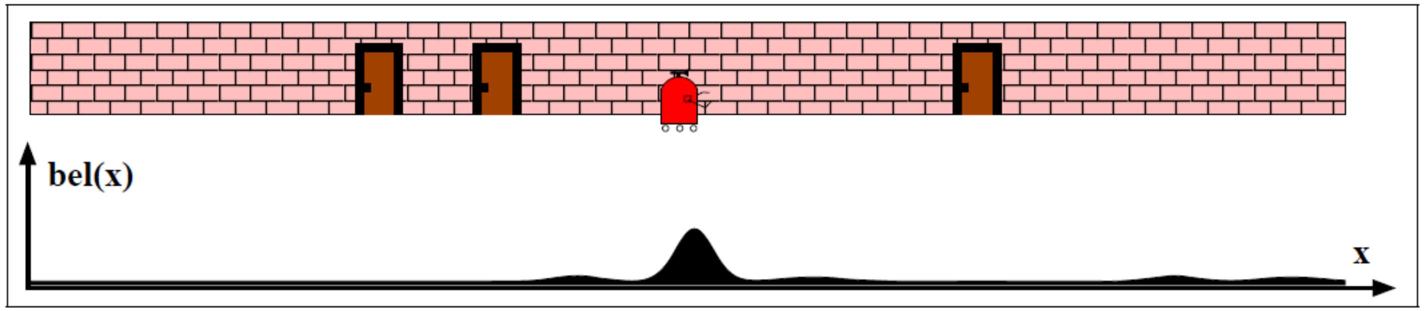
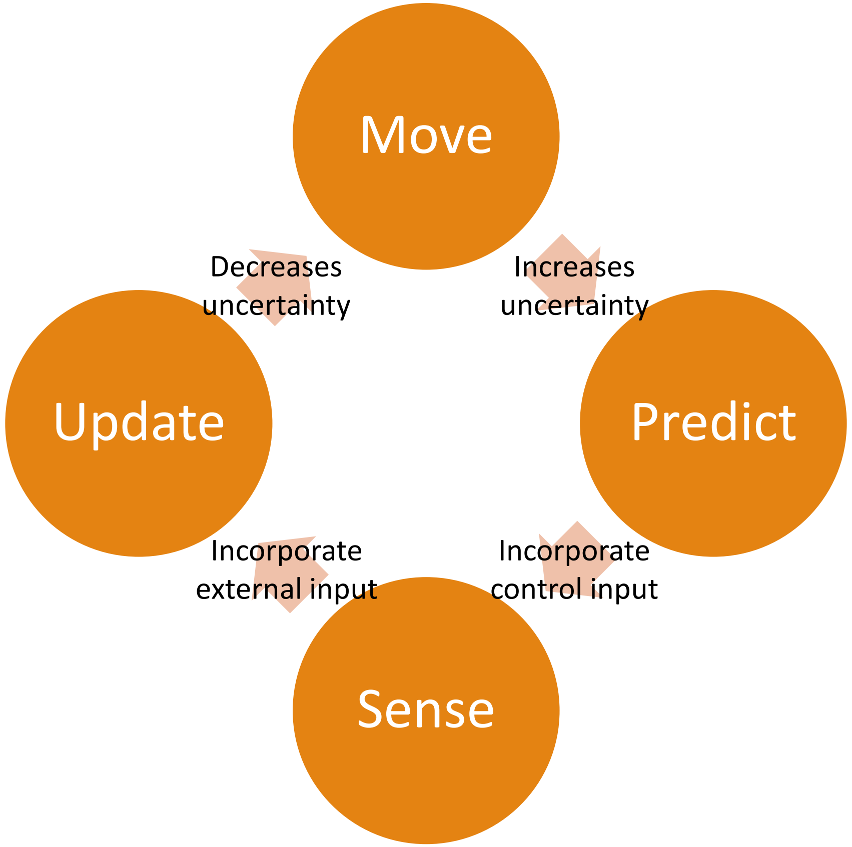


Problem overview



Bayes Filter

PROBABILITY REVIEW

Terminology

Random Variable

- Definition: Variable whose value is subject to change due to randomness or chance
- Properties:
  - Can be continuous (e.g., position in 3D) or discrete (e.g., roll of a die)
  - Observed values of random variables are called realizations
- Example: Pose of a robot, $p(X{=}x)$, or value of a rolled die, $p(D{=}d)$

Probability Density Function

- Definition: Function describing the likelihood that a random variable X will take on a particular value x.
- Properties:
  - Total probability is equal to one
    - Continuous: $\int p(X = x)dx = 1$
    - Discrete: $\sum_x p(X = x) = 1$
  - Non-negativity: $p(X = x) \geq 1$
- Example: Pose of the robot, $p(x) = p(X = x)$

Terminology

Expected Value

- Definition: Probability-weighted average value, the center of mass of the probability distribution
  - $E[X] = \int p(X = x)x\, dx$

Joint Probability Distribution

- Definition: The probability density function of a set of two or more random variables (multivariate distribution)
- $p(X = x, Z = z)$

Covariance

- Definition: A measure of two random variables change together
  - $\sigma(X, Y) = E[(X - E[X])(Y - E[Y])]$
- The variance is a special case where the two random variables are identical: $\sigma^2(X) = \sigma(X, X)$
- Can be thought of as the certainty of the distribution
- Frequently represented as a matrix where $X = [\, x_1, x_2, \dots, x_n]^T$

Terminology

Independence

- Definition: Two random variables are independent if the outcome of one has no effect on the outcome of the other
- Example: If X, Z are the outcomes of two dice rolls, $p(X, Z) = p(X)p(Z)$
- Properties:
  - Independent random variables are uncorrelated $\sigma(X, Z) = 0$
  - Uncorrelated random variables are not necessarily independent

Conditional Probability

- Definition: Probability of an event occurring conditioned on another event occurring.
- $p(z|x) = \frac{p(x,z)}{p(x)} \rightarrow p(x, z) = p(z|x)p(x)$

Terminology

Conditional Independence

- Two random variables are conditionally independent if the outcome of one has no effect on the outcome of the other when conditioned on the outcome of a third random variable
- $p(z_1, z_2|x) = p(z_1|x)\, p(z_2|x)$

Marginal Distribution

- The probability distribution of the subset of a collection of random variables
  - $p(z) = \int p(x, z)dx$
- Also known as the Law of Total Probability
  - $p(z) = \sum_i^n p(z|X = x_i)p(X = x_i)$

Bayes Filtering

DERIVATION

Bayes’ Theorem: A review

Describes how the belief about a random variable should change to account for the collected evidence (measurements).

$$p(x|z) = \frac{p(z|x)p(x)}{p(z)}$$

For a time-indexed system with a state, measurements, and controls:

$$p(x_t|z_{1:t}, u_{1:t}) = \frac{p(z_t|x_t)p(x_t|z_{1:t-1}, u_{1:t-1})}{p(z_t|z_{1:t-1}, u_{1:t})}$$

Markov assumption and simplification

A Bayes' Filter is recursive and calculates the probability accounting for *all* prior measurements and controls.

Mathematically impossible to implement for any significant duration of time.

Simplify the system by using assuming it is a first-order Markov chain:

- Assume each state is conditionally independent
- Only the previous state and current controls effects the current state

Simplification permits recursion via sequential processing making the filter tractable.

Overview

Basic two-step filter consisting of a prediction step and a measurement step.

- Calculates the probability of a state at time t given all previous measurements and inputs.
- $p(x_t | z_{1:t}, u_{1:t})$

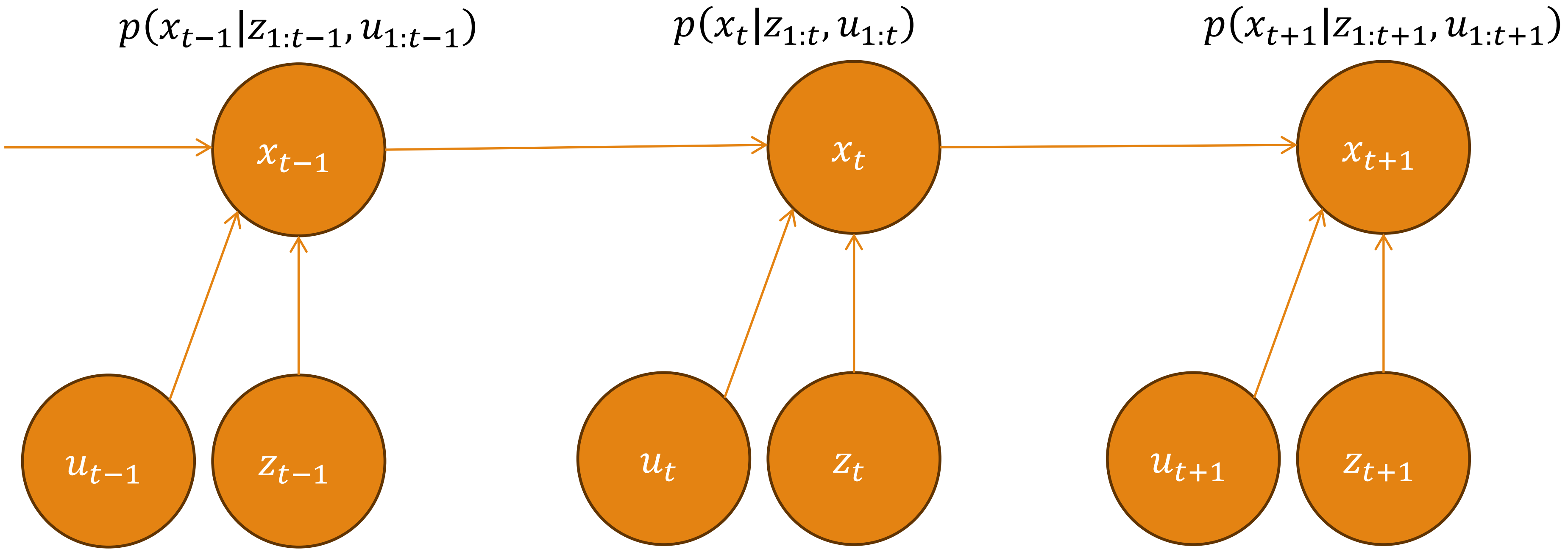
Prediction step

- Takes the prior belief about the state and control input
- Calculates the predicted belief about the state from the process or motion model

Measurement (update) step

- Takes the predicted belief and a measurement
- Calculates the posterior belief about the state according to the measurement model

Overview



$p(x_{t-1}|z_{1:t-1}, u_{1:t-1})$
$p(x_t|z_{1:t}, u_{1:t})$
$p(x_{t+1}|z_{1:t+1}, u_{1:t+1})$
x_{t-1}
x_t
x_{t+1}
u_{t-1}
z_{t-1}
u_t
z_t
u_{t+1}
z_{t+1}

Prediction Step

You can find the current pose via a marginal distribution:

- $p(x_t|z_{1:t-1}, u_{1:t}) = \int p(x_t, x_{t-1}|z_{1:t-1}, u_{1:t})dx_{t-1}$

Expand the integral terms using a conditional distribution:

- $p(x_t|z_{1:t-1}, u_{1:t}) = \int p(x_t|x_{t-1}, z_{1:t-1}, u_{1:t})p(x_{t-1}|z_{1:t-1}, u_{1:t})dx_{t-1}$

First order Markov assumption allows for us to state that the current state is conditionally independent of the *past* measurements and controls given the current state.

- Process model: $p(x_t|x_{t-1}, z_{1:t-1}, u_{1:t}) = p(x_t|x_{t-1}, u_t)$

The prediction step becomes: $p(x_t|z_{1:t-1}, u_{1:t}) = \int p(x_t|z_{1:t-1}, u_{1:t})\, p(x_{t-1})dx_{t-1}$

Update Step

In the update step, we calculate the posterior distribution by multiplying the likelihood and the predicted distribution, and then normalizing.

- $p(x_t|z_{1:t}, u_{1:t}) = \frac{p(z_T|x_t, z_{1:t-1}, u_{1:t})p(x_t|z_{1:t-1}, u_{1:t})}{p(z_t|z_{1:t-1}, u_{1:t})}$

The denominator can be re-written as a marginal distribution of the numerator

- $p(z_t|z_{1t-1}, u_{1:t}) = \int p(z_t|x_t)p(x_t|z_{1:t-1}, u_{1:t})dx_t$

Markov assumption allows for us to treat the measurement as conditionally independent of the past measurement.

- $p(x_t|z_{1:t-1}, u_{1:t}) = p(z_t|x_t)$

Update step becomes:

$$p(x_t|z_{1:t}, u_{1:t}) = \frac{p(z_t|x_t)\, p(x_t|z_{1:t-1}, u_{1:t})}{\int p(z_t|x_t)\, p(x_t|z_{1:t-1}, u_{1:t})dx_t}$$

Pseudocode

A Bayes filter is only two functions: predict and update

```
for (u, z) in (controls, measurements):
    bel(x) = predict(x, u) * bel(x)
    bel(x) = update(x, z) * bel(x)
```

Consider a one-dimensional constant velocity system where the state is position, controls are velocity, and the measurement is position. The sensor is modeled as a normal distribution with accuracy σ.

```
predict(state, u=(control, dt)):
    return p(state + control * dt)

update(state, measurement):
    return (1 / (σ √2π)) * exp(−0.5 ((state − measurement)/σ)^2)
```