Problem Set 2: Markov Chains

Warm up

Questions at the end of Chapter 10 of Baier-Katoen's book, in particular, do try questions 10.1, 10.2, 10.3 (for DTMCs), 10.22, 10.25 (for MDPs).

Problem 0

Can you give an example of a Markov chain that is irreducible, has 7 states, and period 5? If no such example is possible, explain why.

Problem 1

A certain calculating machine uses only the digits 0 and 1. It is supposed to transmit one of these digits through several stages. However, at every stage, there is a probability p that the digit that enters this stage will be changed when it leaves and a probability $q = 1 - p$ that it won't. Form a Markov chain to represent the process of transmission by taking as states the digits 0 and 1. What is the matrix of transition probabilities?

Now draw a tree and assign probabilities assuming that the process begins in state 0 and moves through two stages of transmission. What is the probability that the machine, after two stages, produces the digit 0 (i.e., the correct digit)?

Problem 2

Consider the knight's tour on a chess board: A knight selects one of the next positions at random independently of the past.

(i) Why is this process a Markov chain?

(ii) What is the state space?

(iii) Is it irreducible? Is it aperiodic?

(iv) Find the stationary distribution. Give an interpretation of it: what does it mean, physically?

(v) Which are the most likely states in steady-state? Which are the least likely ones?

Problem 3

Discuss the topological properties of the graphs of the following Markov chains:

(a)

$$P = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

(b)

$$P = \begin{bmatrix} 0.5 & 0.5 \\ 1 & 0 \end{bmatrix}$$

(c)

$$P = \begin{bmatrix} 1/3 & 0 & 2/3 \\ 0 & 1 & 0 \\ 0 & 1/5 & 4/5 \end{bmatrix}$$

(d)

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(e)

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

Problem 4: Umberella's problem

I have 4 umbrellas, some at home, some in the office. I keep moving between home and office. I take an umbrella with me only if it rains. If it does not rain I leave the umbrella behind (at home or in the office). It may happen that all umbrellas are in one place, I am at the other, it starts raining and must leave, so I get wet.

(1) If the probability of rain is p, what is the probability that I get wet?

(2) Current estimates show that $p = 0.6$ in Edinburgh. How many umbrellas should I have so that, if I follow the strategy above, the probability I get wet is less than 0.1?

Problem 5

A. Assume that an experiment has m equally probable outcomes. Show that the expected number of independent trials before the first occurrence of k consecutive occurrences of one of these outcomes is

$$\frac{m^k - 1}{m - 1}$$

Hint: Form an absorbing Markov chain with states $1, 2, \ldots, k$ with state i representing the length of the current run. The expected time until a run of k is 1 more than the expected time until absorption for the chain started in state 1.

B. It has been found that, in the decimal expansion of $\pi = 3.14159\ldots$, starting with the 24,658,601st digit, there is a run of nine 7's. What would your result say about the expected number of digits necessary to find such a run if the digits are produced randomly?

Problem 6

Suppose that $\xi_0, \xi_1, \xi_2, \ldots$ are independent random variables with common probability function $f(k) = P(\xi_0 = k)$ where $k \in \mathbb{Z}$. Let $S = \{1, \ldots, N\}$. Let X_0 be another random variable, independent of the sequence (ξ_n), taking values in S, and let $f : S \times \mathbb{Z} \to S$ be a certain function. Define new random variables $X_1, X_2, \ldots$ by

$$X_{n+1} = f(X_n, \xi_n), \quad n = 0, 1, 2, \ldots$$

(i) Show that the X_n form a Markov chain.

(ii) Find its transition probabilities.

Note: Can ignore if you didn't have idea about Random Variables.

Problem 7

A rat runs through the maze shown below. At each step it leaves the room it is in by choosing at random one of the doors out of the room.

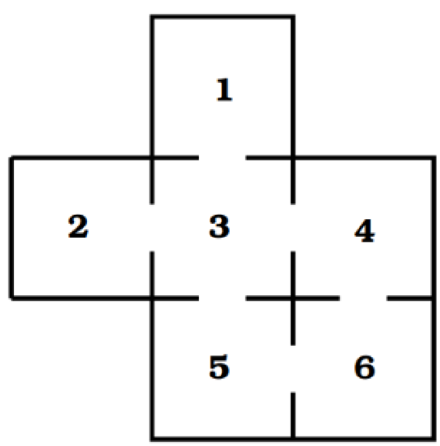

Figure 1: Diagram for problem 7.

(a) Give the transition matrix P for this Markov chain.

(b) Show that it is irreducible but not aperiodic.

(c) Find the stationary distribution.

(d) Now suppose that a piece of mature cheddar is placed on a deadly trap in Room 5. The mouse starts in Room 1. Find the expected number of steps before reaching Room 5 for the first time, starting in Room 1.

(e) Find the expected time to return to room 1.

Problem 8: Gambler's Ruin Problem

A gambler plays a game in which on each play he wins one dollar with probability p and loses one dollar with probability $q = 1 - p$. The Gambler's Ruin Problem is the problem of finding

$\phi(x) :=$ the probability of winning an amount b before losing everything, starting with state $x = \mathbb{P}_x(S_b < S_0)$

(1) Show that this problem may be considered to be an absorbing Markov chain with states $0, 1, 2, \ldots, b$, with 0 and b absorbing states.

(2) Write down the equations satisfied by $\phi(x)$.

(3) If $p = q = \frac{1}{2}$, show that

$$\phi(x) = \frac{x}{b}$$

(4) If $p \neq q$, show that

$$\phi(x) = \frac{(q/p)^x - 1}{(q/p)^b - 1}$$