

Midterm Report:
Markov Decision Processes and their
Applications in Artificial Intelligence
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Abstract

This report details the progress made during the first phase of the Summer of Science 2025 project focusing on Markov Decision Processes (MDPs) and their applications in Artificial Intelligence. It covers foundational concepts from logic and automata theory, stochastic processes like Markov Chains, and dives deep into the theory and implementation of algorithms for solving MDPs, including Multi-Armed Bandits, Dynamic Programming techniques (Policy Iteration and Value Iteration) applied to custom GridWorld environments and classic RL problems like Jack's Car Rental and the Gambler's Problem. The report outlines the theoretical understanding gained, practical implementations developed, key challenges encountered, and learnings acquired.

Keywords: Reinforcement Learning, Markov Decision Processes, Dynamic Programming, Policy Iteration, Value Iteration, Multi-Armed Bandits, Artificial Intelligence.

Contents

1	Introduction	4
1.1	Motivation and Personal Journey	4
1.2	Overview of Reinforcement Learning	4
1.3	Project Objectives and Scope (First Half)	5
1.4	Report Structure	5
2	Theoretical Foundations	7
2.1	Logic and Automata Theory	7
2.1.1	Propositional Logic	7
2.1.2	Predicate Logic	9
2.1.3	Finite Automata	10
2.1.4	Linear Temporal Logic (LTL)	11
2.2	Stochastic Processes: Markov Chains	11
2.3	Markov Decision Processes (MDPs)	13
2.3.1	Bellman Equations	14
2.3.2	Optimality	15
3	Exploration and Exploitation: Multi-Armed Bandits	17
3.1	The k-Armed Bandit Problem	17
3.2	Action-Value Methods Implemented	18
3.2.1	Epsilon-Greedy Action Selection	18
3.2.2	Handling Non-Stationary Problems	18
3.2.3	Optimistic Initial Values	19
3.2.4	Upper Confidence Bound (UCB) Action Selection	20
3.2.5	Gradient Bandit Algorithms	21
3.3	Summary of Bandit Experiments	22
4	Dynamic Programming for Solving MDPs	23
4.1	Overview of Dynamic Programming in RL	23
4.2	Policy Iteration	23
4.2.1	Policy Evaluation (Prediction)	23
4.2.2	Policy Improvement	24
4.2.3	The Policy Iteration Algorithm	24
4.3	Value Iteration	25
4.4	Implementation Case Study 1: GridWorld Environment	25
4.4.1	Environment Description	25
4.4.2	Policy Iteration Implementation and Results	26
4.4.3	Value Iteration Implementation and Results (If completed)	27

4.5	Implementation Case Study 2: Jack's Car Rental	28
4.5.1	Problem Description	28
4.5.2	Policy Iteration Implementation Approach	28
4.5.3	Results and Analysis	29
4.6	Implementation Case Study 3: Gambler's Problem	31
4.6.1	Problem Description	31
4.6.2	Value Iteration Implementation Approach	31
4.6.3	Results and Analysis	31
5	Challenges, Learnings, and Conclusion	33
5.1	Challenges Encountered	33
5.2	Key Learnings and Insights	33
5.3	Conclusion and Transition to Future Work	34
A	Rules for natural deduction	35

List of Figures

3.1	Comparison of ϵ -greedy algorithms with varying ϵ values over 1,000 steps, averaged over 10,000 runs.	18
3.2	Performance comparison on a non-stationary bandit problem: sample-average vs. constant step-size α	19
3.3	Effect of optimistic initial values (e.g., $Q_1(a) = 5, \epsilon = 0$) compared to realistic initial values with ϵ -greedy (e.g., $Q_1(a) = 0, \epsilon = 0.1$).	20
3.4	Performance of UCB action selection (with $c = 2$) compared to ϵ -greedy.	21
3.5	Performance of Gradient Bandit algorithms with different α values and with/without a baseline.	22
4.1	A sample 16×16 map used in the custom GridWorld environment, illustrating different tile types (e.g., NORMAL (brown), WALL (grey rocks), LAVA (red puddles), TREASURE (chest), COIN (yellow)).	26
4.2	Converged state-value function (v_*) for the GridWorld environment obtained using Policy Iteration. Brighter colors indicate higher values. . . .	27
4.3	Optimal policy for the GridWorld environment derived from Policy Iteration. Arrows indicate the optimal action in each state.	27
4.4	Jack's Car Rental: Policy iteration. Positive values indicate moving cars from location 1 to 2; negative values indicate movement from 2 to 1. . . .	29
4.5	Jack's Car Rental: Converged state-value function (v_*).	30
4.6	Solution to extended Jack's car rental problem.	30
4.7	Gambler's problem value function sweeps and optimal policies for $p_h \in \{0.25, 0.4, 0.55\}$	32

Chapter 1

Introduction

1.1 Motivation and Personal Journey

Like everyone, I was introduced to AI through the hype around it being built up for years. Being new to Computer science, I wanted to explore everything, and that was when I got to RL. During the WiDS (Winter in Data Science) after the first semester, I learned what exactly RL was in my “AI snake game” project. Although it was not a theoretically dense program focussing more on learning to code by following tutorials, it sparked my interest. I had tasted the blood, and I needed more. Time passed, and I tried to keep using the same tutorial method. Unthinkingly writing code to do simple stuff. Then came this summer at the end of the first semester, and with that came my first opportunity to take up an SoS project. I had had enough of following someone writing code for me, and It was time for me to learn what I wanted properly. Thus, I began my theoretical journey into RL and MDPs, building my understanding from basic CS logic to MDP algorithms week by week.

1.2 Overview of Reinforcement Learning

The idea that we learn by interacting with our environment is probably the first to occur to us when we think about the nature of learning.

— Sutton and Barto [1]

This opening thought from Sutton and Barto’s seminal work on reinforcement learning resonates deeply with our intuition about how natural learning occurs. Indeed, when an animal learns, it typically does so through interaction and consequence, without an explicit teacher. Nature has endowed us with an intrinsic learning mechanism: actions taken in an environment often elicit feedback, perhaps akin to hormonal responses in biological systems, signaling the ‘goodness’ of the outcome. This natural feedback loop is a biological counterpart to the ‘reward’ signal central to RL.

This concept echoes Alan Turing’s proposition in his paper “Computing Machinery and Intelligence”, where he suggested that building a “child machine” capable of learning like a human might be a more feasible path to artificial intelligence than attempting to construct a fully formed “adult-like” machine. Reinforcement Learning (RL), a prominent branch of Machine Learning (ML), beautifully formalizes this interactive learning paradigm.

In RL, an **agent** (the learner or decision-maker) interacts with a simulated or real **environment**. The agent takes an **action**, and as a consequence, it perceives **observations** (or transitions to new **states**) from the environment. Following this, the agent receives a scalar **reward** signal, which indicates the immediate desirability of the action taken or the state reached. The agent’s objective is to learn a **policy**—a strategy dictating its actions based on its observations—that maximizes the *cumulative reward* it receives over the long run. Typically, the agent begins with a random (or specifically initialized) policy. Through continuous interaction, it evaluates the effectiveness of its current policy and iteratively refines it to achieve its environmental goals more effectively. After sufficient learning, the agent can develop a highly effective policy.

Thus, the core pursuit of RL is to develop computational methods that mimic this natural, trial-and-error learning process, enabling an agent to autonomously improve its behavior and decision-making capabilities within an environment.

1.3 Project Objectives and Scope (First Half)

The first four weeks of this project laid the theoretical groundwork for understanding Reinforcement Learning, progressing through key concepts:

- **Week 1:** Foundational Logic and Automata. This week focused on the syntax and semantics of propositional and predicate logic, which was followed by an emphasis on finite automata as an introduction to state-based systems.
- **Week 2:** Markov Chains. Building upon formal methods, this week delved into the definition, properties, and dynamics of Markov Chains, establishing the foundation for MDPs.
- **Week 3:** MDP Introduction and k-Armed Bandits. This period involved understanding the application of Markov Chains within the context of problem-solving algorithms and marked the beginning of core RL study with the practical exploration of the K-Armed Bandit problem.
- **Week 4:** Algorithm Implementation and Reporting. This culminating week of the first phase focused on translating learned MDP algorithms (such as Policy Iteration and Value Iteration) into code through various projects, alongside compiling this midterm report.

1.4 Report Structure

Chapter 2 entails a summary of my theoretical journey during the project. It is divided into three sections, with section 1 being Logic and Automata Theory. Section 2.2 introduces Markov chains through stochastic processes—finally, Section 2.3 moves on to MDPs and their formal definitions.

Chapter 3 moves on to the first practical aspect of the project. It goes through the theoretical introduction of RL via k K-armed Bandit and its implementation.

Chapter 4 involves the bulk of practical work in the project. It begins with an introduction to dynamic programming and its algorithms(Policy Iteration and Value Iteration). Then come 3 case studies: Gridworld(4.4), Jack’s Car Rental(4.5) and Gambler’s Problem(4.6).

Chapter 5 ends the report with my key challenges and learnings from the project and my future path.

Chapter 2

Theoretical Foundations

2.1 Logic and Automata Theory

Logic is an essential pillar across all Computer Science disciplines, and this foundational role extends directly to Reinforcement Learning (RL). Correctly understanding RL relies on the theoretically rigorous modelling of sequential decision-making problems, for which the Markov Decision Process (MDP) provides a powerful and convenient framework. Therefore, a complete understanding of basic Logic and Automata Theory becomes a hard prerequisite for comprehending the structure and dynamics of MDPs, as these concepts provide methods to describe formally and reason about states, transitions, and system properties.

The primary resources consulted for this section include “Principles of Model Checking” by Baier and Katoen [2], and “Logic in Computer Science” by Huth and Ryan [3].

2.1.1 Propositional Logic

Propositional logic forms the bedrock of logical reasoning by providing a formal language for representing statements that can be either true or false. Key concepts explored include:

- **Syntax and Semantics:** Understanding the well-formed formulas (propositions, connectives like \neg , \wedge , \vee , \rightarrow , \leftrightarrow) and how their truth values are determined by truth assignments to atomic propositions using truth tables.

\neg The **negation** of p is given by $\neg p$

\vee Representation of **either p or q** is given by $p \vee q$.

\wedge Representation of **p and q both being True** is given by $p \wedge q$.

\rightarrow q being the logical consequence of p being true, given by $p \rightarrow q$. In other words **if p then q** . It is equivalent to $\neg p \vee q$.

\leftrightarrow Double implication of **p implies q and q implies p** , is given by $p \leftrightarrow q$.

- **Logical Equivalence:** Identifying when different formulas have the same truth

value under all interpretations (e.g., De Morgan's laws). Some examples

$$p \rightarrow q \vdash \neg p \vee q \quad (2.1)$$

$$\neg(p \vee q) \vdash \neg p \wedge \neg q \quad \text{De Morgan's law} \quad (2.2)$$

$$\neg(p \wedge q) \vdash \neg p \vee \neg q \quad \text{De Morgan's law} \quad (2.3)$$

$$\neg\neg p \vdash p \quad (2.4)$$

- **Satisfiability, Tautologies, Contradictions:** A formula is satisfiable if there's at least one truth assignment making it true; a tautology is true under all assignments; a contradiction is false under all assignments.

Thus ψ is **satisfiable** iff:

$$\Phi \vdash \psi \quad \text{Where, } \Phi = \{\phi_1, \phi_2 \dots \phi_n\}$$

For some specific values of atomic propositions $\phi_i \forall 1 \leq i \leq n$, in other words ψ evaluates to **T** for some valuation.

A **tautology** can be induced without any premise, as \top is a tautology:

$$\vdash \top$$

A **contradiction** is not **True** under any premise. If assumed true can be used to prove anything.

$$\Phi \not\vdash \perp$$

$$\perp \vdash \psi$$

Where $\Phi = \{\phi_1, \phi_2 \dots \phi_n\}$ can be any set of formulas and ψ can be any formula.

- **Logical Consequence (Entailment):** The concept of $\Gamma \models \phi$ (semantic entailment), meaning ϕ is true in all models where all formulas in Γ are true.

It can be easily checked by building a Truth table. Example:

$$p \rightarrow q \models \neg p \vee q$$

p	q	$\neg p$	$p \rightarrow q$	$\neg p \vee q$
F	F	T	T	T
T	F	F	F	F
F	T	T	T	T
T	T	F	T	T

- **Proof Systems (Natural Deduction):** H&R introduces formal proof systems like Natural Deduction, which provide rules for deriving conclusions from premises syntactically ($\Gamma \vdash \phi$). The goal of such systems is to match semantic entailment. The rules can be found in Appendix A.
- **Soundness and Completeness:** Conceptually, a proof system is sound if everything provable is semantically true (\vdash implies \models), and complete if every semantic truth is provable (\models implies \vdash). This ensures the reliability and expressive power of the formal deduction system.

- **Normal Forms:** Conjunctive normal forms(CNF) and other normal forms are easier to build proofs using an automated system. Proper algorithms can be formed by first converting all formulas into normal forms and then prove equivalence automatically.

Relevance to RL/MDPs: While not directly modeling stochasticity, propositional logic provides the basic tools for defining conditions and properties that might characterize states or desired outcomes in simpler, deterministic settings. It also lays the groundwork for more expressive logics.

2.1.2 Predicate Logic

Predicate Logic, or First-Order Logic (FOL), significantly extends the expressive power of propositional logic. It allows us to reason about objects, their properties, and the relations between them, which is crucial for describing more complex systems. Key concepts studied include:

Syntax Beyond Propositions FOL introduces:

- **Terms:** Representing objects, which can be constants, variables, or functions applied to terms (e.g., x , Alice, $\text{father}(x)$).
- **Predicates:** Representing properties of objects or relations between them, which evaluate to true or false (e.g., $\text{IsRed}(x)$, $\text{Loves}(\text{Alice}, \text{Bob})$).
- **Quantifiers:**
 - **Universal Quantifier** (\forall): “For all” - $\forall x P(x)$ means $P(x)$ is true for every object x in the domain.
 - **Existential Quantifier** (\exists): “There exists” - $\exists x P(x)[o]$ means $P(x)$ is true for at least one object x in the domain.
- Formulas are constructed using predicates applied to terms, logical connectives ($\neg, \wedge, \vee, \rightarrow, \leftrightarrow$), and quantifiers.

Semantics and Interpretations Unlike propositional logic where semantics are given by truth assignments to atoms, FOL semantics require an interpretation (or model). An **interpretation** consists of:

- A non-empty **domain of discourse** (the set of objects we are talking about).
- Assignments of meaning to constant symbols (mapping them to objects in the domain), function symbols (mapping them to functions over the domain), and predicate symbols (mapping them to relations over the domain).
- The truth of a quantified formula depends on whether its inner part holds for all or some assignments of domain objects to its variables.

Validity, Satisfiability, and Entailment in FOL These concepts are analogous to propositional logic but are evaluated with respect to all possible interpretations or within a specific interpretation. A formula is valid if true in all interpretations. An argument $\Gamma \models \phi$ holds if ϕ is true in every interpretation where all formulas in Γ are true.

Relevance to RL/MDPs: FOL provides a much richer language for describing states, especially when states have complex internal structures or involve relationships between multiple entities (e.g., in relational RL or AI planning). While core MDPs often use simpler state representations, the principles of FOL are fundamental to more advanced AI reasoning and knowledge representation that can intersect with RL. My work on the Week 1 problem sheet, particularly question 2.7(ii) involving FO-definability of languages, provided practical experience in applying these concepts to describe properties of sequences.

2.1.3 Finite Automata

Finite Automata (FA) serve as fundamental mathematical models for systems that process information sequentially and transition between a finite set of states based on inputs. They are key to understanding regular languages and form a conceptual bridge to more complex state-based models like Markov Chains. My understanding was primarily built from Baier & Katoen [2] and supplemented by Huth & Ryan [3]. Key concepts include:

- **Core Components:**
 - **Deterministic Finite Automata (DFA):** Defined by a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, representing states, input alphabet, a unique transition function, a start state, and a set of accept states.
 - **Nondeterministic Finite Automata (NFA):** Similar to DFAs, but the transition function can lead to multiple states (or no state) for a given state-input pair, and can include ϵ -transitions (transitions without consuming an input symbol).
- **Language Recognition:** An FA accepts a string if, starting from the start state, it reaches an accept state after processing the entire string according to its transition rules. The set of all strings accepted by an FA is the language it recognizes.
- **Equivalence of DFAs and NFAs:** A crucial insight is that NFAs, despite their nondeterminism, recognize the same class of languages as DFAs (the regular languages). The subset construction (or powerset construction) provides an algorithm to convert any NFA into an equivalent DFA.
- **Regular Expressions (REs):** A concise algebraic notation for specifying regular languages, using operations like concatenation, union ($|$ or $+$), and Kleene star ($*$).
- **Kleene's Theorem:** This fundamental theorem establishes the equivalence in expressive power between finite automata and regular expressions: any language definable by an RE can be recognized by an FA, and vice-versa. Conceptual understanding of the constructive proofs (e.g., RE to NFA \rightarrow FA, FA to RE via state elimination or other methods) was a focus.

Relevance to RL/MDPs: The concept of a system defined by a set of states, with transitions between them triggered by events (inputs in FA, actions/probabilities in MDPs), is central to both automata theory and RL. Understanding FAs helps build intuition for state-space diagrams, reachability, and how sequences of events can lead to specific outcomes or “accepting” (goal) states. This paradigm directly informs the way we model environments and agent interactions in MDPs.

2.1.4 Linear Temporal Logic (LTL)

Linear Temporal Logic (LTL) extends propositional logic to allow reasoning about sequences of events or states over time. It is particularly useful for specifying dynamic properties of systems. My initial exploration covered:

- **Temporal Modalities:** LTL introduces operators to describe how truth values of propositions change along a linear sequence of time points (or states):
 - $X\phi$ (NeXt): ϕ holds in the immediate next state.
 - $F\phi$ (Eventually/Finally): ϕ holds at some point in the future (or currently).
 - $G\phi$ (Globally/Always): ϕ holds at all points in the future (including currently).
 - $\phi U \psi$ (ϕ Until ψ): ϕ must hold continuously from the current point until ψ becomes true, and ψ must eventually become true.
- **Interpretation over Traces:** LTL formulas are typically interpreted over infinite sequences of states, known as traces, where each state in the trace assigns truth values to atomic propositions.
- **Expressing Properties:** LTL provides a formal way to express common system properties, such as:
 - Safety properties (e.g., $G\neg(\text{critical_error})$ – “a critical error never occurs”).
 - Liveness properties (e.g., $F(\text{request_granted})$ – “a request will eventually be granted”).

Relevance to RL/MDPs: While not always explicitly used in the design of basic RL algorithms, LTL is fundamental in areas like formal verification of AI systems, model checking of MDPs, and specifying complex, temporally extended goals or constraints in RL (e.g., “always avoid unsafe states while eventually reaching the goal”). Understanding LTL provides a richer language for thinking about the desired long-term behavior of an agent.

2.2 Stochastic Processes: Markov Chains

Markov Chains (MCs) are fundamental stochastic models that describe sequences of possible events where the probability of transitioning to a future state depends only on the current state, not on the sequence of events that preceded it (the Markov Property). They form a critical theoretical underpinning for understanding the dynamics of environments in Reinforcement Learning before introducing agent control. My understanding of MCs was developed through solving problems from the provided practice sheet (Practice Sheet 2) and consulting texts such as “Principles of Model Checking” by Baier & Katoen [2] for formal definitions, and “Reinforcement Learning: An Introduction” by Sutton & Barto [1] for their context within RL. Key concepts explored include:

- **Formal Definition:** A Discrete-Time Markov Chain (DTMC) is typically defined by a tuple $\mathcal{M} = (S, \mathbf{P}, \iota_{\text{init}}, AP, L)$:
 - S : is a finite or countable set of States.

- **P**: is a Transition Probability Matrix (TPM)¹ **P** where each entry $\mathbf{P}_{ss'} = \mathbf{P}(S_{t+1} = s' | S_t = s)$ represents the probability of moving from state s to state s' in one time step. The rows of **P** must sum to 1.
- ι_{init} : Gives the initial distribution of states.
- **AP**: gives the set of Atomic Propositions which are true in specific states. For now the only case concerning us is one where atomic propositions are of the states itself.
- **L**: can be defined as a function $S \rightarrow 2^{AP}$ such that $L(s)$ gives the set of atomic propositions true for that state s . It can also be defined as a matrix where L_{sp} gives 1 if proposition p is true for state s and 0 if not.
- **The Markov Property**: The conditional probability distribution of future states of the process, given the present state and all past states, depends only upon the present state and not on any past states:

$$P(S_{t+1} = s' | S_t = s, S_{t-1} = s_{t-1} \dots S_0 = s_0) = \mathbf{P}(S_{t+1} = s' | S_t = s)$$

- **N-Step Transition Probabilities**: The probability of transitioning from state s to s' in exactly n steps is given by $\mathbf{P}_{ss'}^{(n)}$, can be found from the n -th power of the TPM \mathbf{P}^n . These probabilities satisfy the **Chapman-Kolmogorov equations**.
- **Classification of States**: States in an MC can be classified based on their long-term properties:
 - **Accessibility and Communication**: State s' is *accessible* from s if there is a path from s to s' . States s and s' *communicate* if both of them are accessible from other. Communicating states form communicating classes.
 - **Recurrence and Transience**: A state is *recurrent* if, starting from that state, the probability of eventually returning to it is 1. Otherwise, it is *transient*.
 - **Periodicity**: A state s has a period d if any recurrence of state s must occur in time steps that are multiples of d , where d is the largest such number. An MC is *aperiodic* if it has a period of 1.
 - **Irreducibility**: A Markov Chain is *Irreducible* if all the states are accessible from every other state. In other words it must form a single Communicating class.
 - **Absorbing States**: A state s of MC is called *absorbing* if it is impossible to leave it after arrival i.e. $P_{ss} = 1$.
- **Stationary Distribution(π)**: For some MCs \exists a distribution π such that if initialized according to it, it remains in the same distribution π for all following time steps ($\pi = \pi\mathbf{P}$). A unique such distribution π exists for every periodic and irreducible finite Markov Chain, and independent of initialization the probability of such MC being in state s converges to $\pi(s)$.

¹Many sources define **P** as a Transition Probability *Function* such that $\mathbf{P} : S \times S \rightarrow [0, 1]$ and $\mathbf{P}(s, s')$ represents the probability of transitioning from state s to s' .

Relevance to RL/MDPs: Markov Chains model the inherent dynamics of an environment when no actions are taken or when a fixed policy effectively reduces an MDP to an MC (known as a Markov Reward Process if rewards are included). Understanding MC properties like recurrence, transience, and stationary distributions is crucial for analyzing the long-term behavior of policies and the convergence of RL algorithms. The Gambler's Problem and aspects of Jack's Car Rental, for instance, can be analyzed using MC concepts.

2.3 Markov Decision Processes (MDPs)

Markov Decision Processes (MDPs) extend Markov Chains by incorporating an agent that can make decisions (take actions), influencing state transitions and receiving rewards. MDPs provide the standard mathematical formalism for Reinforcement Learning problems involving sequential decision-making under uncertainty. The primary reference for this section is Baier & Katoen [2], Chapter 10, section 10.6 and Sutton & Barto [1], Chapter 3. An MDP is formally defined as a tuple $\mathcal{M} = (S, A, \mathbf{P}, \iota_{\text{init}}, AP, L)$

S is the finite set of states the system can be in.

A is the set of actions the agent can take. It can also be defined as a function $S \rightarrow 2^{\mathcal{A}}$ where \mathcal{A} is the set of all actions possible and $A(s)$ gives the actions available at state s .

$\mathbf{P} : S \times \mathcal{A} \times S \rightarrow [0, 1]$ is known as the **Transition probability function**. For all $s \in S$ and $a \in A(s)$, $\sum_{s' \in S} \mathbf{P}(s', s, a) \in \{0, 1\}$. It defines the dynamics of the problem.

$\iota_{\text{init}} : S \rightarrow [0, 1]$ is the initial distribution which defines the probability($\iota(s)$) of MDP starting from a state(s).

AP is the set of atomic propositions that can be true at various states, for present practical considerations we will be assuming AP to represent a set of rewards possible on reaching a reward.

$L : S \rightarrow 2^{AP}$ is the labeling function for atomic propositions.

Our agent based MDPs have a goal to maximise the reward it gets not only in present but in all time including present and future. Thus we define a goal G_t as the goal the MDP needs to maximise at time t by choosing some action. G_t can be written as $\mathbb{E} \left[\sum_{i=t+1}^{i \rightarrow \infty} R_i \right]$, however it has some problems:

- Defining $G_t = \mathbb{E} \left[\sum_{i=t+1}^{i \rightarrow \infty} R_i \right]$ makes all rewards in present and future equally important, thus the agent might have to look too far in future to decide and in case of infinitely long episode the agent will take infinite time to decide its action.
- If some a sequence of actions give the agent a steady reward, over infinity the value of G_t will be undefined as it will be a sum of non converging infinite sequence.

Thus another idea to define the goal introduces the concept of **Discounting**. We assume a constant γ such that $G_t = \mathbb{E} \left[\sum_{i=t+1}^{i \rightarrow \infty} \gamma^{i-t-1} R_i \right]$. This directly translates to taking present reward as whole($i = t + 1, \gamma^{i-t-1} = 1$) R_{t+1} and future awards are discounted by factor of γ^n where n is larger for in future rewards making future rewards less important in present. This solves both of our problems in most cases(except the ones where R_t diverges faster than the exponent converges, but it is unlikely to occur). At a certain point we can cut off our calculation and oversight as the exponent converges the sequence to make all terms after some terms to make negligibly small.

For practical purposes we will be defining MDPs as a tuple $\mathcal{M} = (S, A, \mathbf{P}, R, \gamma)$, where R is a function $S \times A \rightarrow \mathcal{R}$ such that \mathcal{R} is set of all rewards possible and $R(s, a)$ gives the reward for taking action a in state s . Therefore $R(s, a) = \mathbb{E}[R_{t+1} | S_t = s, A_t = a]$. γ gives the discount factor that the agent uses to decide the worth of future action.

Here are some more important frameworks for agent based MDPs:

Policy (π) A policy is agent's behavior function in the system. It maps the states to a probability distribution over actions to take. $\pi(a|s) = \text{Pr}\{A_t = a | S_t = s\}$. A policy can be stochastic or non stochastic(where states are mapped to unique actions to take).

Value Functions is a function defined to measure the goodness of certain states and actions. It is of two types:

- **State Value Function** v_π maps states to expected cumulative reward from that state under policy π .

$$v_\pi(s) \doteq \mathbb{E}_\pi[G_t | S_t = s] = \mathbb{E}_\pi \left[\sum_{i=0}^{\infty} R_{t+i+1} \gamma^i | S_t = s \right]$$

- **Action State value function** q_π gives the expected cumulative reward for choosing an action a in state s .

$$q_\pi(a, s) \doteq \mathbb{E}_\pi[G_t | S_t = s, A_t = a] = \mathbb{E}_\pi \left[\sum_{i=0}^{\infty} R_{t+i+1} \gamma^i | S_t = s, A_t = a \right]$$

2.3.1 Bellman Equations

Bellman Equations were given by "Richard E. Bellman". These equations express a recursive relationship between the value of a state (or state-action pair) and the values of its successor states, under a given policy π . They are fundamental for most RL algorithms.

For value function ($v_\pi(s)$), we know

$$v_\pi(s) = \mathbb{E}_\pi[G_t | S_t = s] \quad (2.5)$$

$$= \mathbb{E}_\pi \left[\sum_{i=0}^{\infty} R_{t+i+1} \gamma^i | S_t = s \right] \quad (2.6)$$

$$= \sum_{a \in A} \Pr(A_t = a | S_t = s) \mathbb{E}_\pi[G_t | S_t = s, A_t = a] \quad (2.7)$$

$$= \sum_{a \in A} \pi(a|s) \sum_{s' \in S, r \in R} \Pr(S_{t+1} = s', R_{t+1} = r | S_t = s, A_t = a) G_t \quad (2.8)$$

$$= \sum_{a \in A} \pi(a|s) \sum_{s' \in S, r \in R} \mathbf{P}(s', r | s, a) \mathbb{E}_\pi[G_t | S_t = s, A_t = a, R_{t+1} = r, S_{t+1} = s] \quad (2.9)$$

$$= \sum_{a \in A} \pi(a|s) \sum_{s' \in S, r \in R} \mathbf{P}(s', r | s, a) \mathbb{E}_\pi \left[\sum_{i=0}^{\infty} R_{t+i+1} \gamma^i | S_t = s, A_t = a, R_{t+1} = r, S_{t+1} = s \right] \quad (2.10)$$

$$= \sum_{a \in A} \pi(a|s) \sum_{s' \in S, r \in R} \mathbf{P}(s', r | s, a) \mathbb{E}_\pi \left[(R_{t+1} + \sum_{i=1}^{\infty} R_{t+i+1} \gamma^i) | S_t = s, A_t = a, R_{t+1} = r, S_{t+1} = s \right] \quad (2.11)$$

$$= \sum_{a \in A} \pi(a|s) \sum_{s' \in S, r \in R} \mathbf{P}(s', r | s, a) (r + \mathbb{E}_\pi \left[\gamma \sum_{i=0}^{\infty} R_{t+2+i} \gamma^i | S_t = s, A_t = a, R_{t+1} = r, S_{t+1} = s \right]) \quad (2.12)$$

$$= \sum_{a \in A} \pi(a|s) \sum_{s' \in S, r \in R} \mathbf{P}(s', r | s, a) (r + \gamma v_\pi(s')) \quad (2.13)$$

Therefore the Bellman equation for state value function is

$$v_\pi(s) = \sum_{a \in A} \pi(a|s) \sum_{s' \in S, r \in R} \mathbf{P}(s', r | s, a) (r + \gamma v_\pi(s'))$$

Similar method can be taken to prove the Bellman equation for state action value function:

$$q_\pi(s, a) = \sum_{s' \in S, r \in R} \mathbf{P}(s', r | s, a) (r + \gamma \sum_{a' \in A} \pi(a' | s') q_\pi(a', s'))$$

or

$$q_\pi(s, a) = \sum_{s' \in S, r \in R} \mathbf{P}(s', r | s, a) (r + \gamma v_\pi(s'))$$

2.3.2 Optimality

An optimal policy, denoted π_* , is a policy that achieves higher or equal expected return than any other policy from all states. It is the best the agent can do in order to work toward its goal. It maximizes the G_t an agent can get right from initial state.

Following this idea we can define the **Optimal value functions** (v_* and q_*) as:

$$v_*(s) = \max_{\pi} v_\pi(s) \quad \forall s \in S$$

$$q_*(a, s) = \max_{\pi} q_{\pi}(a, s) \quad \forall a \in A, s \in S$$

With slight variation we can alter Bellman's equations to **Bellman's optimality equations**:

$$v_{\pi}(s) = \max_{a \in A} \sum_{s' \in S, r \in R} \mathbf{P}(s', r | s, a) (r + \gamma v_{\pi}(s'))$$

$$q_{\pi}(s, a) = \sum_{s' \in S, r \in R} \mathbf{P}(s', r | s, a) (r + \gamma \max_{a' \in A} q_{\pi}(a', s'))$$

Relevance to RL: MDPs are the mathematical framework for almost all problems Reinforcement Learning aims to solve. Understanding this formalism, especially the Bellman equations, is absolutely critical for developing and analyzing RL algorithms, including the Dynamic Programming methods (Policy Iteration, Value Iteration) discussed in subsequent chapters, as well as model-free methods like Q-learning. All my practical implementations (GridWorld, Jack's Car Rental, Gambler's Problem) are direct applications of solving MDPs.

Chapter 3

Exploration and Exploitation: Multi-Armed Bandits

The k -Armed Bandit problem serves as a simplified, yet fundamental, introduction to several core concepts in Reinforcement Learning, most notably the critical trade-off between exploration (trying out actions to learn more about the environment) and exploitation (choosing the action currently believed to give best reward). This chapter details the problem formulation and discusses several algorithms implemented to address it, based on Chapter 2 of Sutton and Barto [1]. My implementations and experiments are available in the associated GitHub repository [4].

3.1 The k -Armed Bandit Problem

A k -armed bandit problem models a scenario where an agent is faced with k different options, or “arms” (like slot machine levers, as slot machine is a one armed bandit). At each time step t , the agent chooses an action $A_t \in \{1, \dots, k\}$, and receives a numerical reward R_t drawn from a normal probability distribution (with fix means and unity variance) associated with the chosen arm A_t . The true mean reward for action a , denoted $q_*(a)$, is initially unknown to the agent. The agent’s goal is to maximize the total reward obtained over a series of time steps. Another point of view involves calculating regret (Δ_a) which is the difference between the reward received by agent and the best reward it could have received. The point is to minimize the quantity of total regret $\left(\sum_{t=0}^{\infty} \Delta_{A_t}\right)$ i.e. the sum of all regrets. According to Lai and Robbins theorem total regret grows at least logarithmically. Thus the minimum total regret an agent can aim for is logarithmic.

The agent maintains an estimate $Q_t(a)$ of the true action value $q_*(a)$. A common way to estimate this is the sample-average method:

$$Q_t(a) \doteq \frac{\text{sum of rewards when action } a \text{ taken prior to step } t}{\text{number of times action } a \text{ taken prior to step } t} = \frac{\sum_{i=1}^{t-1} R_i \cdot \mathbb{I}(A_i = a)}{\sum_{i=1}^{t-1} \mathbb{I}(A_i = a)} \quad (3.1)$$

where $\mathbb{I}(\cdot)$ is an indicator function. The challenge lies in balancing the desire to exploit arms with high current estimates versus exploring other arms to potentially discover even better options.

3.2 Action-Value Methods Implemented

Several action-value methods were implemented and tested to understand their behavior in balancing exploration and exploitation.

3.2.1 Epsilon-Greedy Action Selection

The ϵ -greedy method is a simple, basic and most used strategy. With a small probability ϵ , the agent chooses an action uniformly at random from all k actions (exploring the environment). With probability $1 - \epsilon$, the agent chooses the action with the highest current estimated value (exploiting it's knowledge).

The performance of ϵ -greedy algorithms was compared for varying values of ϵ , as shown in Figure 3.1. Both non zero values of ϵ (0.1 and 0.01) performs better than pure greedy $\epsilon = 0$. Also higher ϵ allows sooner realization of true mean values but over longer run converge and lag behind lower ϵ as once low ϵ can exploit it's later found knowledge more that high ϵ .

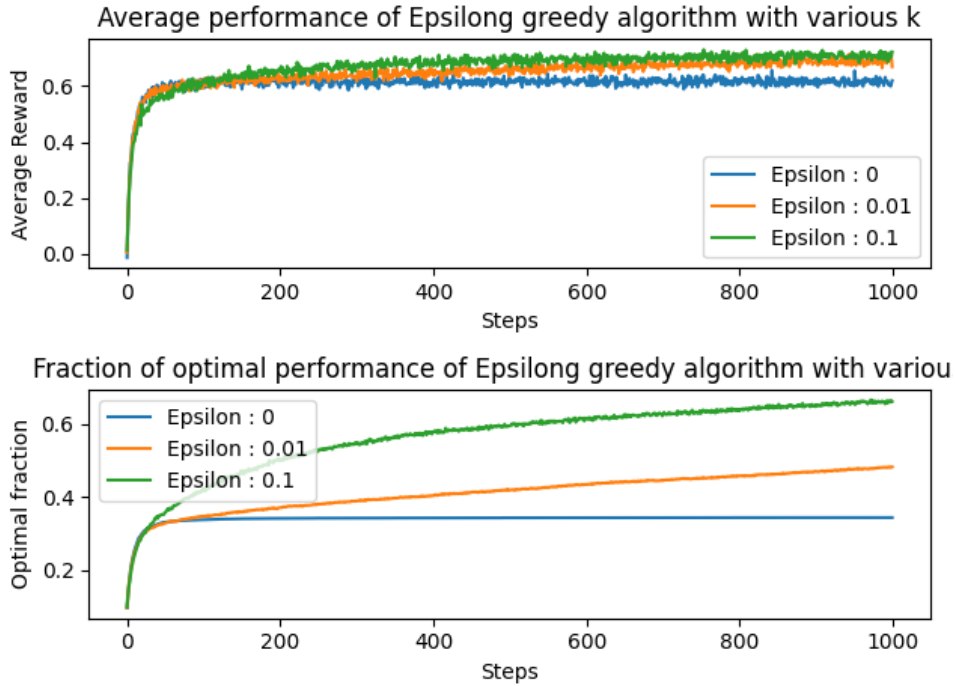


Figure 3.1: Comparison of ϵ -greedy algorithms with varying ϵ values over 1,000 steps, averaged over 10,000 runs.

3.2.2 Handling Non-Stationary Problems

In many real-world scenarios, the reward distributions for actions can change over time (non-stationary problems). The standard sample-average method is not well-suited for this, as it gives equal weight to all past rewards. A more effective approach is to use a constant step-size parameter $\alpha \in (0, 1]$ in the update rule for $Q_{t+1}(A_t)$:

$$Q_{n+1} \doteq Q_n + \alpha[R_n - Q_n] \quad (3.2)$$

This exponentially weighted average gives more weight to recent rewards. Figure 3.2 illustrates the performance difference. It is worth noting that α can also be a function of n . In sample-average case $\alpha = 1/n$.

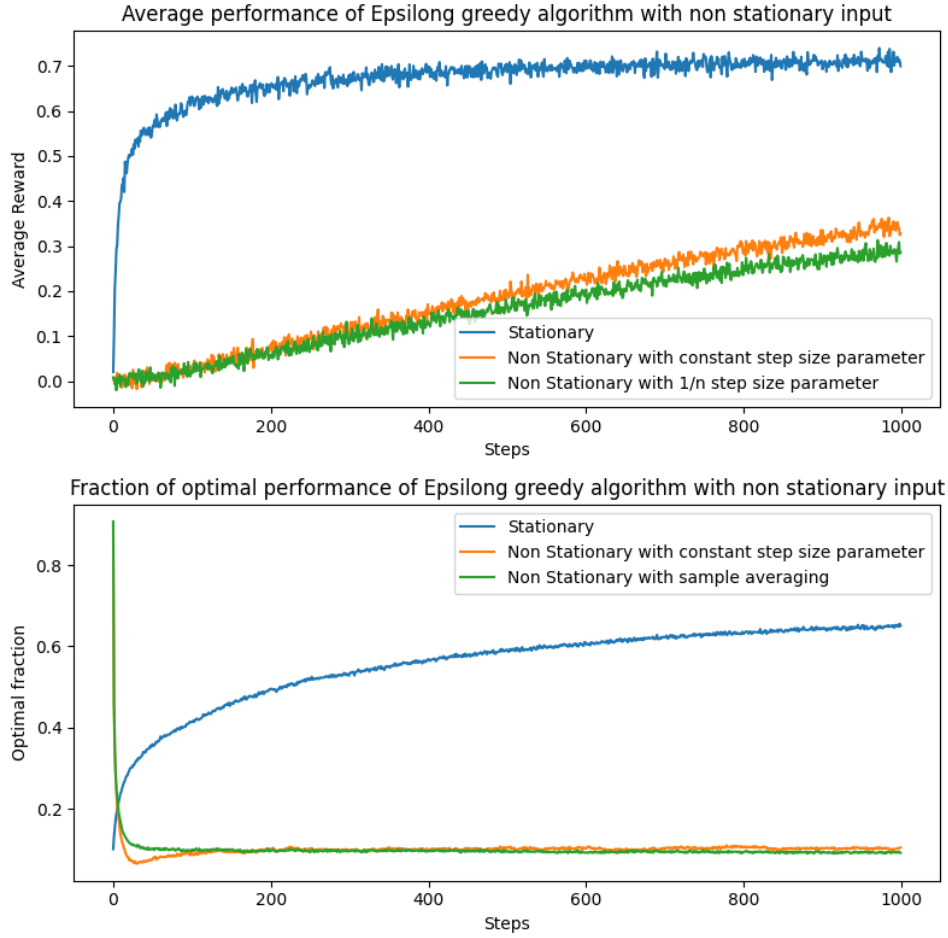


Figure 3.2: Performance comparison on a non-stationary bandit problem: sample-average vs. constant step-size α .

The figure shows that for normal ϵ -greedy stationary case ($\epsilon = 0.1$) performing very well. However for a non-stationary input the performance deteriorates with constant step size α case performing better.

3.2.3 Optimistic Initial Values

Initializing action-value estimates $Q_1(a)$ to a high, optimistic value (e.g., higher than any possible true mean reward) can encourage early exploration. When an arm is chosen, if the reward received is less than the initial optimistic value, the estimate for that arm decreases, making other (still optimistically valued) arms more likely to be chosen. Figure 3.3 shows its effect.

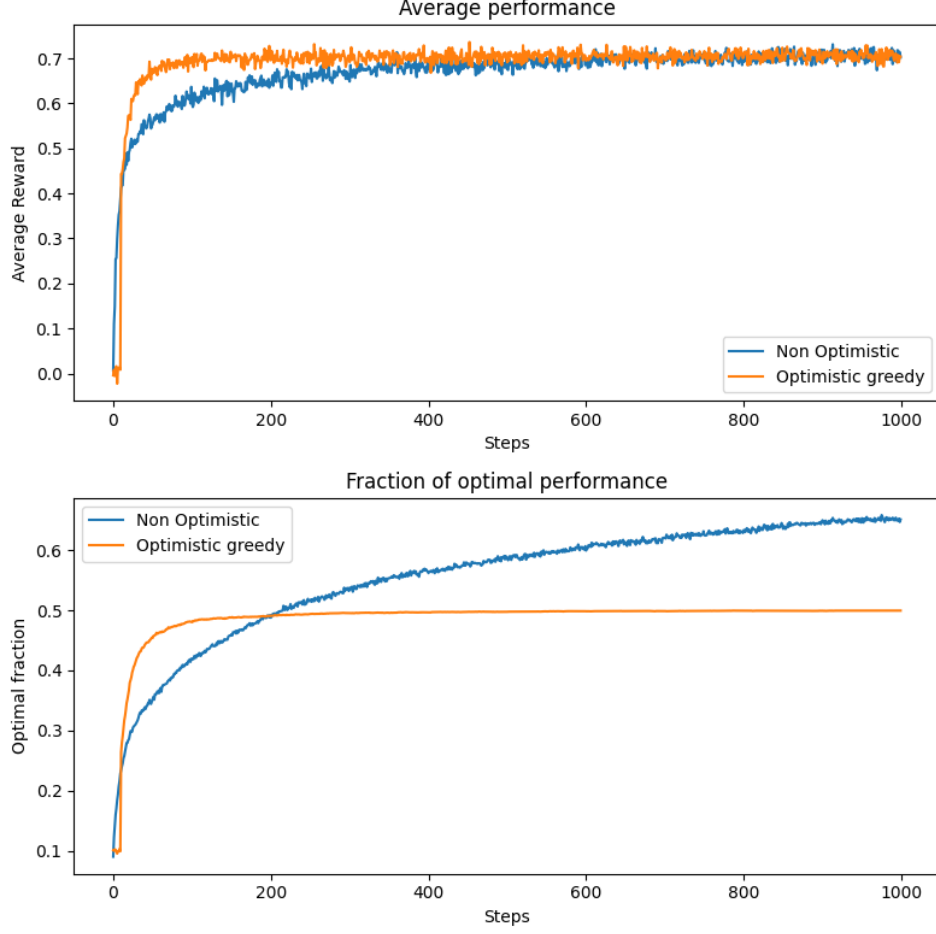


Figure 3.3: Effect of optimistic initial values (e.g., $Q_1(a) = 5, \epsilon = 0$) compared to realistic initial values with ϵ -greedy (e.g., $Q_1(a) = 0, \epsilon = 0.1$).

The graph shows that optimistic greedy policy trains very fast owing to its early high exploration rate. However because of being completely greedy ($\epsilon = 0$) it sticks to a policy after certain turns and then ϵ -greedy policy outperforms it.

3.2.4 Upper Confidence Bound (UCB) Action Selection

UCB addresses exploration more strategically by selecting actions according to:

$$A_t \doteq \operatorname{argmax}_a \left[Q_t(a) + c \sqrt{\frac{\ln t}{N_t(a)}} \right] \quad (3.3)$$

where $N_t(a)$ is the number of times action a has been selected prior to time t , and $c > 0$ controls the degree of exploration. The square-root term is a measure of uncertainty or potential for arm a ; UCB favors arms that are either estimated to be good ($Q_t(a)$ is high) or have not been tried often ($N_t(a)$ is low). According to Lai and Robbins theorem the total regret in case of UCB grows logarithmically making it mathematically best algorithm, the results are shown in Figure 3.4.

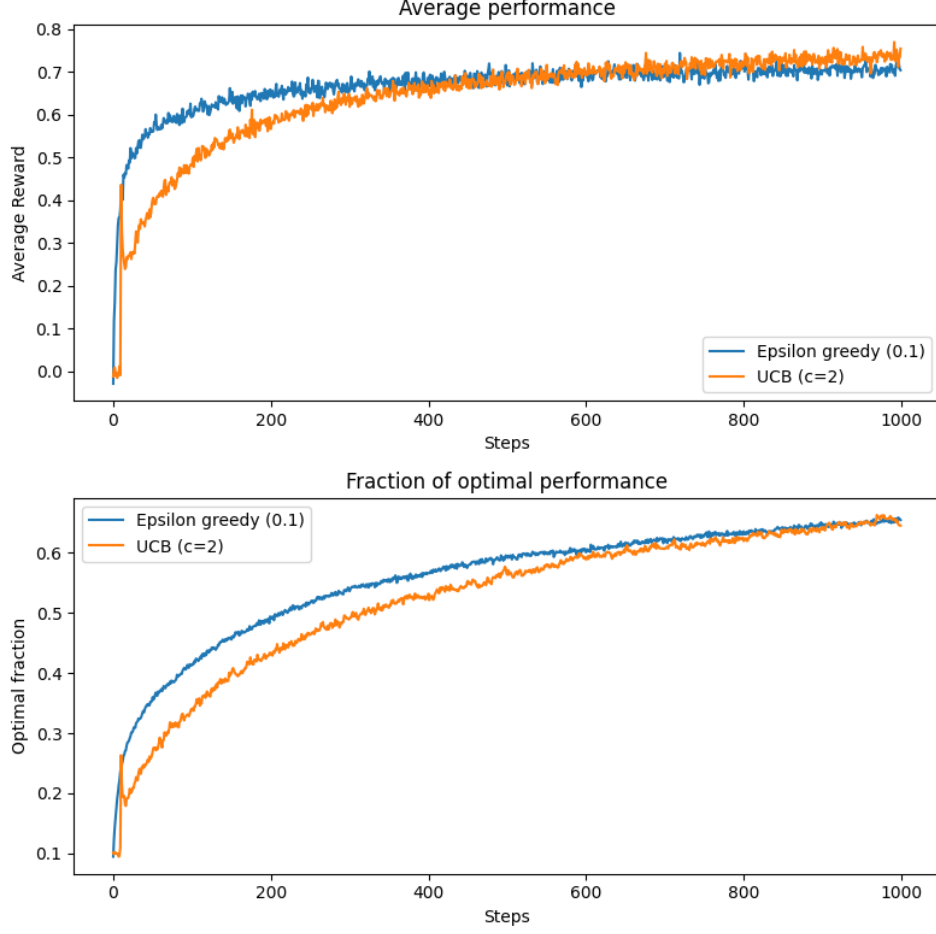


Figure 3.4: Performance of UCB action selection (with $c = 2$) compared to ϵ -greedy.

3.2.5 Gradient Bandit Algorithms

Gradient bandit algorithms learn a numerical preference $H_t(a)$ for each action a . Actions are selected stochastically according to a softmax distribution over these preferences:

$$\pi_t(a) \doteq \Pr\{A_t = a\} = \frac{e^{H_t(a)}}{\sum_{b=1}^k e^{H_t(b)}} \quad (3.4)$$

The preferences are updated using stochastic gradient ascent:

$$H_{t+1}(A_t) \doteq H_t(A_t) + \alpha(R_t - \bar{R}_t)(1 - \pi_t(A_t)) \text{ for the chosen action } A_t$$

and

$$H_{t+1}(a) \doteq H_t(a) - \alpha(R_t - \bar{R}_t)\pi_t(a) \quad \forall a \neq A_t$$

\bar{R}_t is a baseline, often the average of rewards received up to time t . The performance with and without a baseline is compared in Figure 3.5.

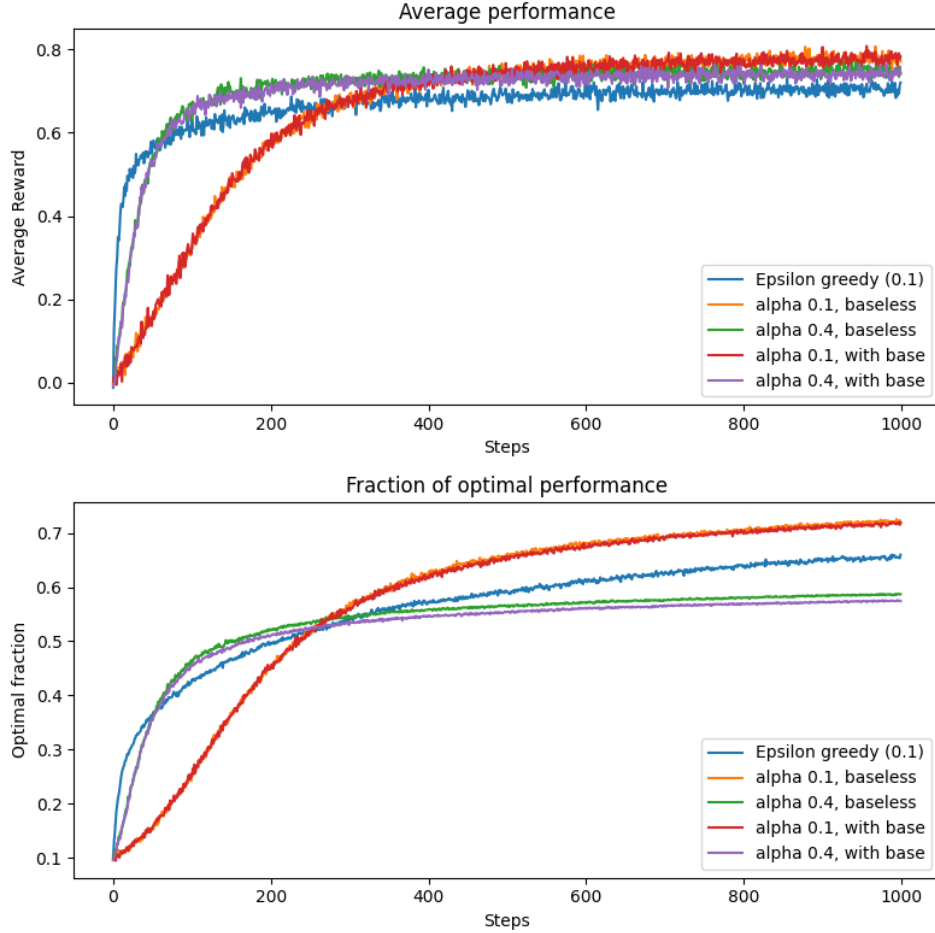


Figure 3.5: Performance of Gradient Bandit algorithms with different α values and with/without a baseline.

The plots show that average performance of gradient algorithm is better than ϵ -greedy but slower to train. higher α performs much better than other policies.

3.3 Summary of Bandit Experiments

The k-armed bandit problem, despite its simplicity, encapsulates the fundamental exploration-exploitation dilemma in RL. Implementing various strategies like ϵ -greedy, UCB, and gradient bandits provided practical insights into their mechanisms and performance characteristics. Optimistic initialization demonstrated a simple way to encourage early exploration. For non-stationary environments, using a constant step-size for value updates proved crucial for adaptability. Gradient bandit algorithms offer a different approach by learning action preferences and giving an optimal stochastic policy, and the use of a baseline was shown to stabilize learning. These experiments highlight that no single method is universally superior; the best choice often depends on the specifics of the problem, such as stationary and the desired balance between exploration and long-term reward, with mathematically derived methods being slightly better than naive approaches.

Chapter 4

Dynamic Programming for Solving MDPs

Dynamic Programming (DP) refers to a collection of algorithms that can compute optimal policies given a perfect model of the environment as a Markov Decision Process (MDP). While DP methods are often computationally expensive and require a full model (which is not always available in RL), they are theoretically important as they provide a foundation for understanding more advanced RL algorithms. This chapter, based on Sutton & Barto Chapter 4 [1], discusses the core DP algorithms and their application to several case studies.

4.1 Overview of Dynamic Programming in RL

DP algorithms in the context of RL leverage the Bellman equations to iteratively find optimal value functions and policies. Key assumptions include:

- The environment dynamics $p(s', r|s, a)$ are fully known.
- The state and action spaces are finite (though extensions exist).

DP methods work by turning Bellman equations into update rules that are applied repeatedly to improve approximations of the desired value functions or policies until convergence to the optimal solution.

4.2 Policy Iteration

Policy Iteration is an algorithm that alternates between two main steps: policy evaluation and policy improvement, until the policy is stable and optimal. It can be quite computationally expensive but are sure to give an optimal policy.

4.2.1 Policy Evaluation (Prediction)

Given a policy π , policy evaluation (or prediction) is the process of computing the state-value function v_π . This is achieved by iteratively applying the Bellman expectation

equation as an update rule:

$$v_{k+1}(s) \doteq E_\pi[R_{t+1} + \gamma v_k(S_{t+1}) | S_t = s] \quad (4.1)$$

$$v_{k+1}(s) = \sum_a \pi(a|s) \sum_{s',r} p(s',r|s,a)[r + \gamma v_k(s')] \quad (4.2)$$

This update is applied to all states $s \in S$ in each iteration k , until v_k converges to v_π however some tolerance θ needs to be specified.

4.2.2 Policy Improvement

Once v_π for a policy π is known, we can try to improve the policy. For each state s , we consider if selecting an action $a \neq \pi(s)$ and then following π would be better. This involves computing $q_\pi(s, a)$ for all $a \in A(s)$:

$$q_\pi(s, a) \doteq E[R_{t+1} + \gamma v_\pi(S_{t+1}) | S_t = s, A_t = a] \quad (4.3)$$

$$q_\pi(s, a) = \sum_{s',r} p(s',r|s,a)[r + \gamma v_\pi(s')] \quad (4.4)$$

A new greedy policy π' is then formed by choosing $a' = \operatorname{argmax}_a q_\pi(s, a)$ for each state s . The Policy Improvement Theorem guarantees that if $\pi' \neq \pi$, then $v_{\pi'}(s) \geq v_\pi(s)$ for all $s \in S$. It can be observed as if we take the greedy action at this position it will improve or at least give same result as following policy π , and then on using π will continue to give same results. Thus the new policy is going to an improvement on the previous policy π .

4.2.3 The Policy Iteration Algorithm

Policy Iteration combines these two steps:

1. Initialization: Start with an arbitrary policy π_0 and value function v_0 .
2. Policy Evaluation: Compute v_{π_k} using iterative application of Equation 4.1 until convergence.
3. Policy Improvement: Construct a new policy π_{k+1} by acting greedily with respect to v_{π_k} .
4. If $\pi_{k+1} = \pi_k$, then the algorithm has converged to the optimal policy π_* and optimal value function v_* . Otherwise, repeat from step 2 with $k \leftarrow k + 1$.¹

¹It is worth noting that the algorithm may not converge if the policies keep switching between two optimal policies. Thus a check may be introduced to prevent this.

4.3 Value Iteration

Value Iteration combines aspects of policy evaluation and policy improvement into a single, simpler update rule. It is much simpler computationally compared to policy iteration. It directly applies the Bellman optimality equation as an update:

$$v_{k+1}(s) \doteq \max_a E[R_{t+1} + \gamma v_k(S_{t+1}) | S_t = s, A_t = a] \quad (4.5)$$

$$v_{k+1}(s) = \max_a \sum_{s', r} p(s', r | s, a) [r + \gamma v_k(s')] \quad (4.6)$$

This update is applied iteratively for all states s until v_k converges to v_* . Once v_* is found, an optimal deterministic policy can be extracted by choosing actions greedily with respect to v_* . Value Iteration often converges faster than Policy Iteration if policy evaluation steps are computationally expensive.

4.4 Implementation Case Study 1: GridWorld Environment

To practically explore these DP algorithms, a custom GridWorld environment was developed using Pygame.

4.4.1 Environment Description

The GridWorld environment consists of a $N \times M$ grid. States are represented by (row, col) tuples. The agent can take one of four deterministic actions: UP, DOWN, LEFT, or RIGHT. Transitions move the agent one cell in the chosen direction unless a WALL tile is encountered, in which case the agent remains in its current position getting a reward of -1. Specific tiles offer rewards or penalties: LAVA tiles incur a large negative reward (-10) and reset the episode, TREASURE tiles provide a large positive reward (+10) and reset, COIN tiles offer a smaller positive reward (+1) and reset, and NORMAL tiles provide a small negative step cost (or zero reward here). An example map is shown in Figure 4.1.

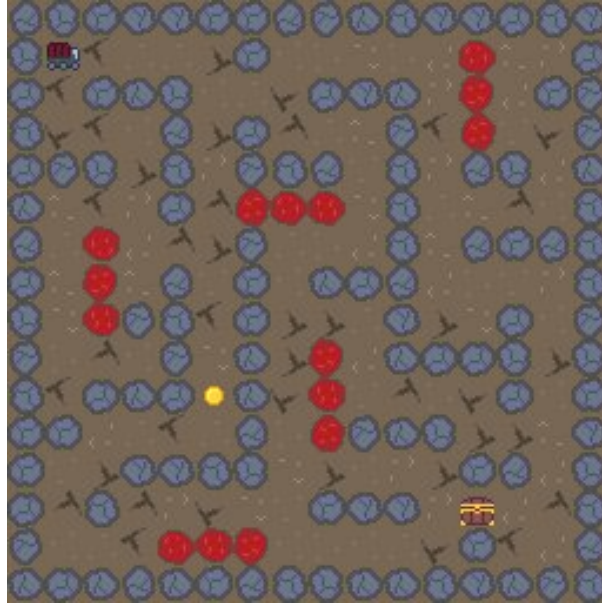


Figure 4.1: A sample 16×16 map used in the custom GridWorld environment, illustrating different tile types (e.g., NORMAL (brown), WALL (grey rocks), LAVA (red puddles), TREASURE (chest), COIN (yellow)).

```

1 class TileTypes(enum.Enum):
2     NORMAL = 0
3     LAVA = -10
4     TREASURE = 10
5     COIN = 1
6     WALL = -1 # Effectively an invalid state to transition into
7 # Rewards are directly from the map: self.reward = self.mapping[*self.
   truck_pos]
8 # Transitions are deterministic based on action, with wall-bumping.

```

Listing 4.1: GridWorld Tile Types and Rewards (Conceptual)

4.4.2 Policy Iteration Implementation and Results

Policy Iteration was implemented for the GridWorld. The ‘evaluate_policy’ method iteratively computed the state values under the current policy, and ‘improve_policy’ updated the policy to be greedy with respect to these values. This process was repeated until the policy stabilized. The converged value function and the derived optimal policy for the map in Figure 4.1 are shown in Figures 4.2 and 4.3 respectively.

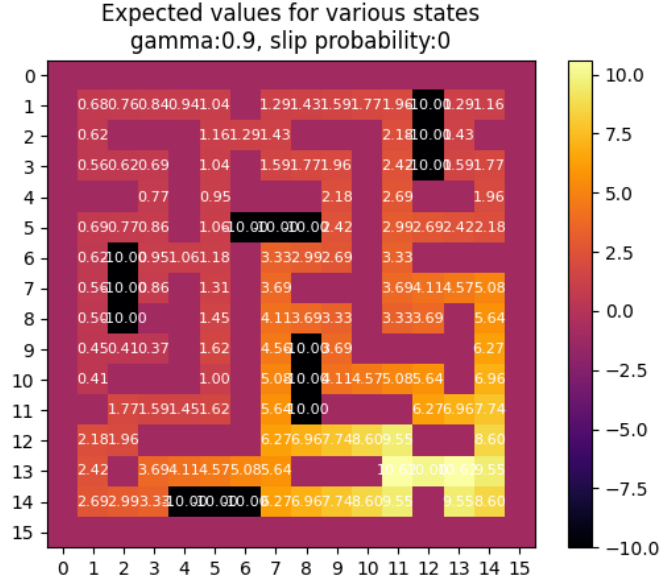


Figure 4.2: Converged state-value function (v_*) for the GridWorld environment obtained using Policy Iteration. Brighter colors indicate higher values.

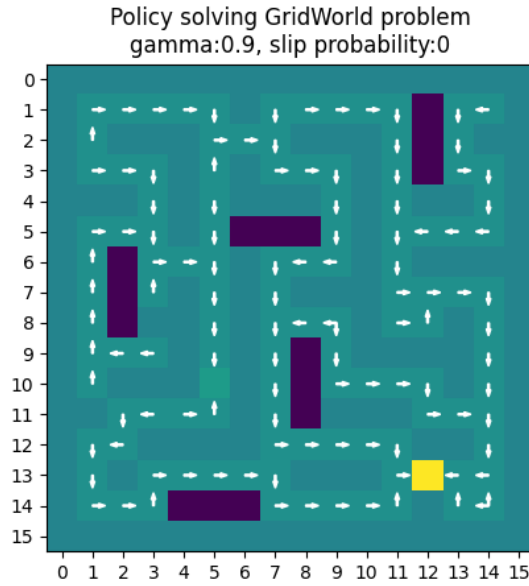


Figure 4.3: Optimal policy for the GridWorld environment derived from Policy Iteration. Arrows indicate the optimal action in each state.

4.4.3 Value Iteration Implementation and Results (If completed)

Value Iteration was also implemented, directly applying the Bellman optimality update (Equation 4.5) until the value function converged. The resulting optimal value function and policy were identical to those found by Policy Iteration, as expected, though convergence characteristics (number of iterations/time) might differ depending on the problem specifics.

4.5 Implementation Case Study 2: Jack’s Car Rental

The Jack’s Car Rental problem (Sutton & Barto, Example 4.2) is a more complex MDP involving stochastic demand and returns for rental cars at two locations. This was a significantly difficult problem to implement and run with least expected complexity.

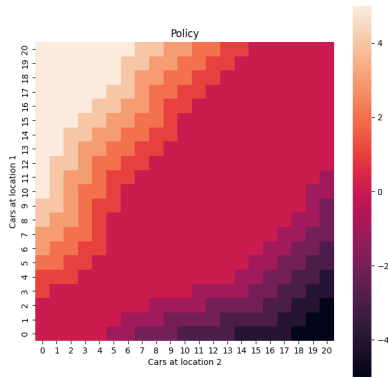
4.5.1 Problem Description

As the name suggests the problems involves Jack who controls two car rental locations for a company. Every car rented gives a 10 credit reward to him. Thus the state is defined by the number of cars at each of two locations at the end of the day, (n_1, n_2) , where $0 \leq n_1, n_2 \leq 20$. Jack has the option of moving a number of cars from one location to other overnight, each move costing 2 credits. Hence, actions are the number of cars moved overnight from location 1 to location 2, ranging from -5 (move 5 from loc2 to loc1) to +5 (move 5 from loc1 to loc2). Rewards include income from rentals (\$10 per car rented) minus the cost of moving cars (\$2 per car moved). Car rental requests and returns at each location follow independent Poisson distributions with specified mean values ($\lambda_{req1} = 3, \lambda_{req2} = 4, \lambda_{ret1} = 3, \lambda_{ret2} = 2$). There’s a maximum capacity of 20 cars at each location; excess returns are lost, and requests are capped by availability. There is also an extension to problem which involves dynamics that Jack can move one car from first to second location, and having more than 10 cars resulting in extra parking fees(Sutton & Barto, Exercise 4.7).

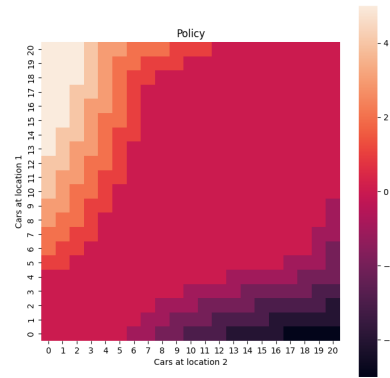
4.5.2 Policy Iteration Implementation Approach

The problem was modeled with an **Agent** class managing the policy and value function, and an **Env** class (specifically, its `simulate_day` method) calculating the expected one-step return $E[R_{t+1} + \gamma v_k(S_{t+1}) | S_t = s, A_t = a]$. The core challenge lies in computing this expectation, which involves summing over all possible numbers of rental requests and returns at both locations. A naive implementation required multiple nested loops, which was my first attempt resulting in 4 nested loops within the `simulate_day` function. This resulted in a total of 8 nested loops and the program ran very slowly. Finding the solution with my only computational resource, my laptop(Intel Ultra 9 processes) was to take too long. It was then that I scraped it to try a new a vectorized approach using NumPy developed within the `simulate_day` method (Listing 4.2). This involves creating a 4-dimensional arrays for Poisson probabilities of requests/returns and broadcasting operations to compute outcomes for all combinations simultaneously, then summing to get the final expectation. The discount factor γ was set to 0.9. This took a significant debugging effort as errors were prone because of difficult imagination of a 4-dimensional array and required some AI help to generate debugging code.

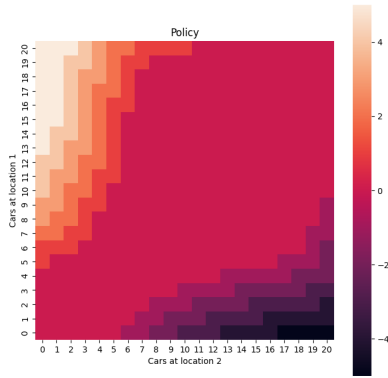
```
1 # ... (setup of prob1, prob2, prob3, prob4 4D arrays for Poisson
   # probabilities using scipy function) ...
2 # ... (setup of ncars and credits 4D arrays based on movement, requests
   # , returns) ...
3 ncars = ncars.astype(np.int32) # Ensure integer indices for value array
4 future_value = values[ncars[0], ncars[1]] # Advanced indexing for V(s')
5 # Element-wise product of joint probabilities and (immediate rewards +
   # discounted future values)
6 result_matrix = prob1 * prob2 * prob3 * prob4 * (credits + gamma *
   future_value)
```



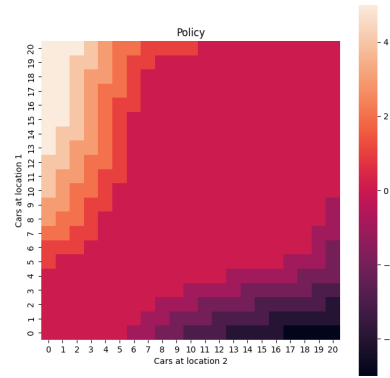
(a) policy after 1st iteration



(b) policy after 2nd iteration



(c) policy after 3rd iteration



(d) converged policy after 4th iteration

Figure 4.4: Jack’s Car Rental:Policy iteration. Positive values indicate moving cars from location 1 to 2; negative values indicate movement from 2 to 1.

```
7 return result_matrix.sum() # Sum over all dimensions of requests/
   returns
```

Listing 4.2: Key Vectorized Expectation Snippet from Jack’s Car Rental `simulate_day`

4.5.3 Results and Analysis

Policy Iteration was run until the policy stabilized. Figures 4.4a through 4.4d show the evolution of the policy (average number of cars moved) across iterations. The corresponding converged value function is shown in Figure 4.5. The implementation took approximately one hour to converge on the available hardware. The solution to extended problem is visible in Figure 4.6

It can be seen that the policy learns to equalize the number of cars in both locations in order for maximum cars to rent but still preferring more cars at location with higher mean renting requests.

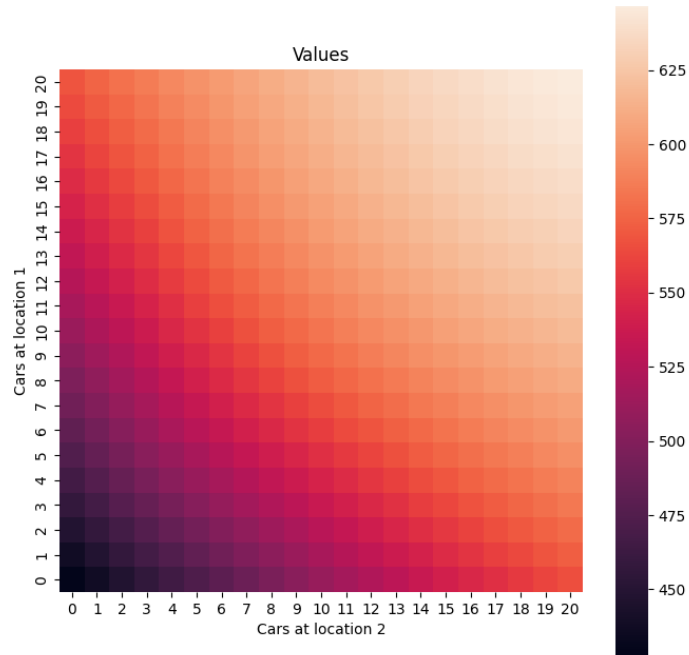
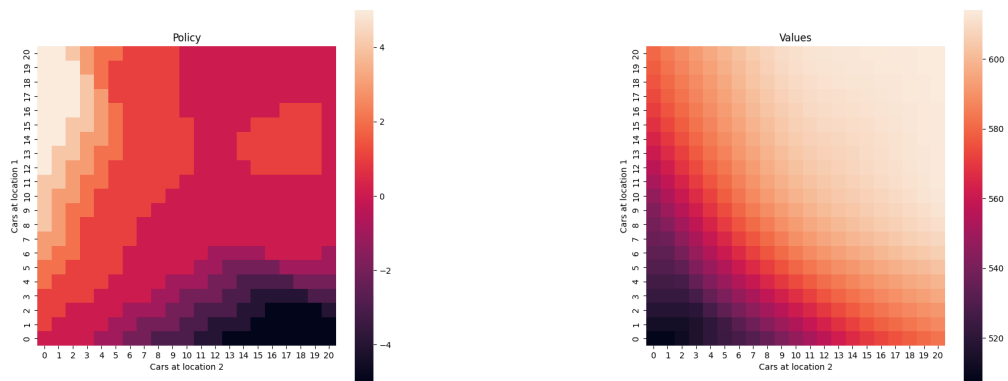


Figure 4.5: Jack's Car Rental: Converged state-value function (v_*).



(a) Extended problem solution policy

(b) Extended problem value function

Figure 4.6: Solution to extended Jack's car rental problem.

4.6 Implementation Case Study 3: Gambler's Problem

The Gambler's Problem (Sutton & Barto, Example 4.3) involves a gambler betting on coin flips to reach a capital goal.

4.6.1 Problem Description

The state $s \in \{1, 2, \dots, 99\}$ is the gambler's capital. The goal is to reach a capital of \$100. Reaching \$0 means ruin. The action a is the stake, $a \in \{0, 1, 2, \dots, \min(s, 100 - s)\}$. If the coin flip is heads (probability p_h), the gambler wins the stake; otherwise (probability $1 - p_h$), he loses it. Reaching \$100 yields a reward of +1, and all other transitions yield 0 reward. The discount factor $\gamma = 1$.

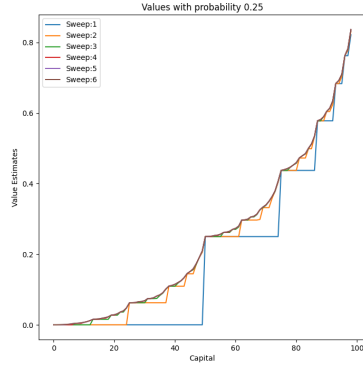
4.6.2 Value Iteration Implementation Approach

Value Iteration was implemented to find the optimal value function and policy. The `iterate_values` method repeatedly sweeps through all states, updating the value of each state s by taking the maximum expected value over all possible valid bets from that state, using the Bellman optimality equation: $v_{k+1}(s) = \max_{a \in A(s)} [p_h(R_{win} + \gamma v_k(s + a)) + (1 - p_h)(R_{loss} + \gamma v_k(s - a))]$ where $v_k(100) = 1$ and $v_k(0) = 0$. The policy is then derived by choosing the bet(s) that achieve this maximum.

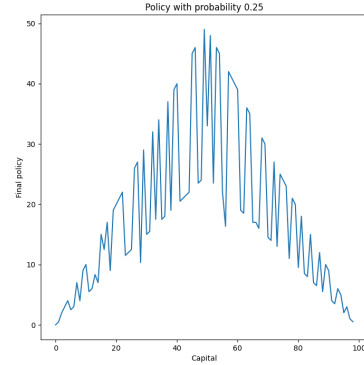
4.6.3 Results and Analysis

The algorithm was run for different probabilities of heads (p_h). Figures 4.7 shows the convergence of the value function and the optimal policies obtained for various values of p_h . For $p_h < 0.5$ (e.g., 0.25, 0.4), the value function is concave and the policy is often to bet boldly, either everything or just enough to reach the goal, especially when capital is low or high. The policy exhibits a characteristic jagged pattern. For $p_h > 0.5$ (e.g., 0.55), the game is favorable, the value function is convex (or near linear), and the optimal policy tends to be more conservative, often betting small amounts (e.g., \$1) to gradually accumulate capital. These results align closely with those presented in Sutton & Barto (Figure 4.3). The optimal policy is non uniform as it is the weighed average of actions with their probabilities as weights. There is no unique optimal policy for this problem hence the non-uniformity.

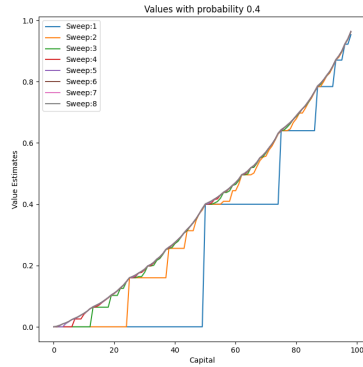
Figure 4.7: Gambler's problem value function sweeps and optimal policies for $p_h \in \{0.25, 0.4, 0.55\}$



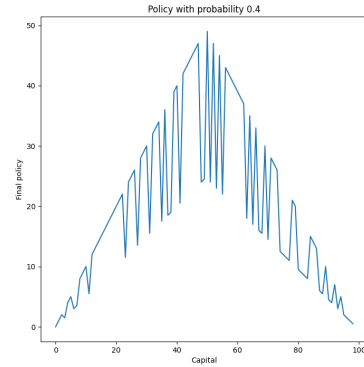
(a) Value function sweeps for $p_h = 0.25$.



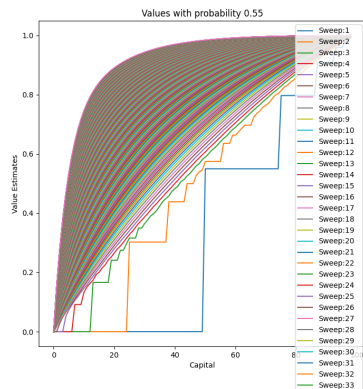
(b) Optimal policy (stake) for $p_h = 0.25$.



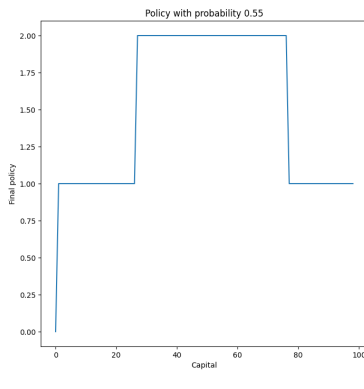
(c) Value function sweeps for $p_h = 0.4$.



(d) Optimal policy (stake) for $p_h = 0.4$.



(e) Value function sweeps for $p_h = 0.55$.



(f) Optimal policy (stake) for $p_h = 0.55$.

Chapter 5

Challenges, Learnings, and Conclusion

5.1 Challenges Encountered

Throughout this initial phase, several challenges were encountered. Initially, grappling with the dense formalism of texts like Baier & Katoen (Which turned out to be nightmarish during initial period) for certain topics required significant effort, prompting a strategy to supplement with more intuitively explained resources like Sutton & Barto and Huth & Ryan. The implementation of Dynamic Programming algorithms, particularly for Jack’s Car Rental, presented considerable debugging challenges. Moving from a conceptual understanding of the nested expectations to an efficient vectorized NumPy implementation was a non-trivial task, involving several iterations of code design to manage the multi-dimensional state and probability arrays. Ensuring correct convergence for both Policy and Value Iteration also required careful attention to termination conditions and update rules. Balancing the demanding theoretical study and implementations for this SoS project alongside commitments to other ongoing projects (Blockchain Arena, unofficial compiler exploration, and DSA Bootcamp) necessitated strict time management and prioritization.

5.2 Key Learnings and Insights

This project has been immensely rewarding in terms of learning. A primary insight is the profound difference between implementing algorithms by following tutorials and truly understanding the underlying theory. Delving into foundational formal logic, automata, and then the formalisms of MCs and MDPs has provided a much deeper appreciation for *why* RL algorithms work. The importance of fine tuning hyperparameters was also evident from the various experiments. Thus the power of Dynamic Programming for solving MDPs with known models became evident through practical implementation; it highlighted the importance of precise problem formulation, including careful definition of states, actions, rewards, and transition dynamics.

The k-Armed Bandit experiments underscored the nuances of the exploration-exploitation trade-off and how same problem can be solved by different and yet same strategies. Implementing Policy and Value Iteration for the GridWorld provided a clear, visual understanding of how value functions and policies evolve. It gave a clear picture of RL algorithm tackling a task scalable at more practical levels. Tackling Jack’s Car Rental was particularly instructive in managing larger state spaces and complex stochastic dynamics, and

it drove home the practical benefits of vectorization (using NumPy) for performance. It taught the importance of optimizing the approach to same method of solving. The Gambler's Problem illustrated how optimal policies can sometimes be counter-intuitive and highly sensitive to problem parameters. A key recurring theme has been the central role of Bellman equations in structuring solutions.

5.3 Conclusion and Transition to Future Work

The first phase of this Summer of Science project has successfully established a strong theoretical and practical foundation in Markov Decision Processes and exact solution methods. Through rigorous study and implementation of various algorithms from Multi-Armed Bandits to Dynamic Programming techniques on problems like Jack's Car Rental and the Gambler's Problem, a solid understanding of core RL principles has been achieved.

This groundwork is crucial for the next phase of the project, which will focus on model-free Reinforcement Learning algorithms (such as Monte Carlo methods, Temporal Difference learning and Q-learning) and function approximation techniques, including an introduction to Deep Reinforcement Learning. As state spaces become too large for exact DP methods, the principles of function approximation, as suggested by my mentor in the context of Approximate Dynamic Programming, will be key. The ultimate practical goal for the second half is to develop an RL agent, likely using Deep Q-Networks (DQN), to play a game like Flappy Bird. The updated Plan of Action, submitted separately, details these future steps.

Appendix A

Rules for natural deduction

Natural deduction is a proof system for propositional logic which both sound and complete. It allows us to prove many formulas using a basic set of premise statements. All the defined rules for natural deduction are given in Table A.1. These rules define the possible steps in a natural deduction proof. Here is an example:

Proof of $p \wedge \neg q \rightarrow r, \neg r, p \vdash q$

1	$p \wedge \neg q \rightarrow r$	premise
2	$\neg r$	premise
3	p	premise
<hr/>		
4	$\neg q$	assumption
5	$p \wedge \neg q$	$\wedge i_{3,4}$
6	r	$\rightarrow e_{1,5}$
7	\perp	$\neg e_{6,2}$
<hr/>		
8	$\neg \neg q$	$\neg i_{4-7}$
9	q	$\neg \neg e_8$

Some useful derived rules:

$$\frac{\phi \rightarrow \psi \quad \neg \psi}{\neg \phi} \text{MT(Modus Tollens)}$$

$$\frac{\boxed{\begin{array}{c} \neg \phi \\ \vdots \\ \perp \end{array}}}{\phi} \text{PBC(Proof by Contradiction)}$$

$$\frac{}{\phi \vee \neg \phi} \text{LEM(law of excluded middle)}$$

Table A.1: Rules for Propositional Deduction

Connective	Introduction	Elimination
\wedge	$\frac{\phi \quad \psi}{\phi \wedge \psi} \wedge i$	$\frac{\phi \wedge \psi}{\phi} \wedge e_1 \quad \frac{\phi \wedge \psi}{\psi} \wedge e_2$
\vee	$\frac{\phi}{\phi \vee \psi} \vee i_1 \quad \frac{\psi}{\phi \vee \psi} \vee i_2$	$\frac{\phi \vee \psi \quad \begin{array}{ c } \phi \\ \vdots \\ \chi \end{array} \quad \begin{array}{ c } \psi \\ \vdots \\ \chi \end{array}}{\chi} \vee e$
\rightarrow	$\frac{\begin{array}{ c } \phi \\ \vdots \\ \psi \end{array}}{\phi \rightarrow \psi} \rightarrow i$	$\frac{\phi \rightarrow \psi \quad \phi}{\psi} \rightarrow e$
\neg	$\frac{\begin{array}{ c } \phi \\ \vdots \\ \perp \end{array}}{\neg \phi} \neg i$	$\frac{\neg \phi \quad \phi}{\perp} \neg e$
\perp	(no introduction rule for \perp)	$\frac{\perp}{\phi} \perp e$
$\neg\neg$	$\frac{\phi}{\neg\neg\phi} \neg\neg i$	$\frac{\neg\neg\phi}{\phi} \neg\neg e$

References

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- [4] A. Singh, *Mdps and their applications in ai - sos 2025 implementations*, https://github.com/Aadeshveer/MDPs_and_their_applications_in_AI_SOS_2025, 2025.