CSC006P1M: Design and Analysis of Algorithms (Gray Codes)

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Gray Codes

Gray Code

- We are given a set of n objects and we want to name them. Each name is represented by a unique string of bits.
- We would like to arrange the names in a circular list such that each name can be obtained from the previous names by changing exactly one bit.
- Such a scheme is called **Gray code**.

Gray Code of Length 2

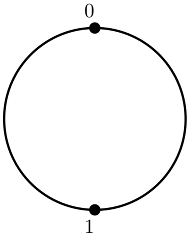


Figure: Gray Code of Length 2

Gray Code of Length 2

1. $0 \rightarrow 0$
2. $1 \rightarrow 1$

Gray Code of Length 4

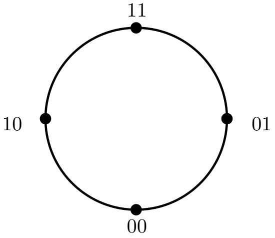


Figure: Gray Code of Length 4

Gray Code of Length 4

1. $0 \rightarrow 00$
2. $1 \rightarrow 01$
3. $2 \rightarrow 11$
4. $3 \rightarrow 10$

Gray Code of Length 3

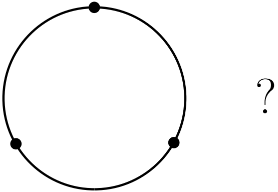
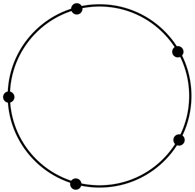


Figure: Gray Code of Length 3

Gray Code of Length 5



$?$

Figure: Gray Code of Length 5

Gray Code of Length 6

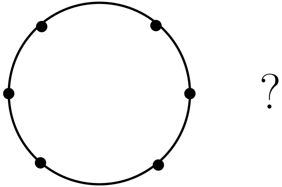


Figure: Gray Code of Length 6

Gray Code of Length $2n$

Theorem

There exists a Gray code of length $2n$ for any $n \geq 1$.

Proof:

- For $n = 1$, the Gray code is: 0, 1.
- Assume that there exists a Gray code for $n = k - 1$, i.e. of length $2k - 2$. Let that Gray code be: $s_1, s_2, \cdots, s_{2k-2}$.
- Construct a code for $n = k$, i.e. of length $2k$ as follows: $0s_1$, $1s_1$, $1s_2$, $0s_2$, $0s_3$, $0s_4$, $\cdots$, $0s_{2k-2}$.
- Now, show that the above code satisfies the criteria for being a Gray code of length $2k$.

Gray Code of Length $2n$

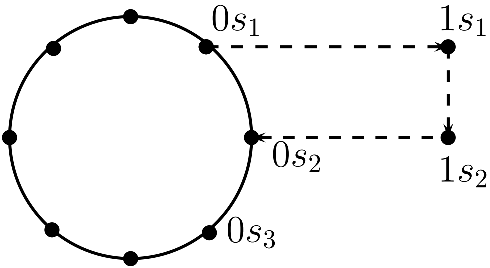


Figure: How to Construct Gray Code of Length $2n$

Gray Code of Length 8 *Revisited*

A Gray code of length 8:
$0000, 1000, 1100, 0100, 0101, 0001, 0011, 0010.$
The code above requires 4 bits for length 8.
But, we need only 3 bits to map 8 different objects uniquely.

0	000	4	100
1	001	5	101
2	010	6	110
3	011	7	111

Exercise

Can we rearrange the strings above in such a manner so that it forms a Gray code of length 8? If no, why? Else, what's the code?

Gray Code of Length 8 *Revisited*

A Gray code of length 8: $101, 001, 011, 010, 000, 100, 110, 111$

How can it be constructed?

Answer

Using the Gray code of length 4.

In-fact, the same idea can be generalised to any Gray code of length 2^l for $l \geq 2$.

Gray Code of Length 2^l having l bits

Proof:

- For $l = 1$, we have a Gray code: 0, 1.
- Inductive Hypothesis: There exists a Gray code of length 2^k having $l = k$ bits for $k \geq 1$.
- Now, we prove that there exists a Gray code of length 2^{k+1}. Let the code of length 2^k be $s_0, s_1, s_2, \cdots, s_{2^k}$. Construct a new code as follows:
  - Construct two sequences - $0s_1, 0s_2, \cdots, 0s_{2^k}$ and $1s_1, 1s_2, \cdots, 1s_{2^k}$
  - Add two sequences in the following manner:

$$1s_2, 0s_2, 0s_3, \cdots, 0s_{2^k}, 0s_1, 1s_1, 1s_{2^k}, 1s_{2^k-1}, \cdots, 1s_3$$

- Observe that the code above is a Gray code of length $2^{k+1} = 2^k + 2^k$ having $k + 1$ bits.

Gray Code of Length 2^l having l bits

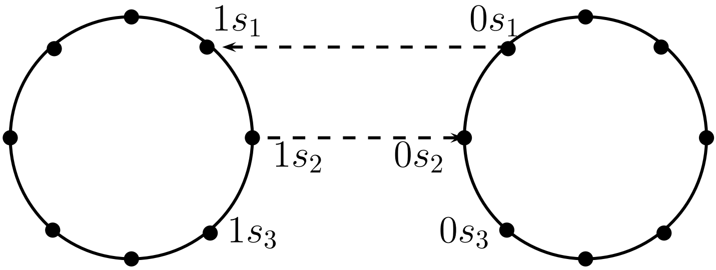


Figure: How to Construct Gray Code of Length 2^l

Gray Code of Even Length $2n$

- Observe that the same idea can be applied not only to the length of 2^l, but also to the length of $2n$ when there exists a code of length n.
- For example, a code of length 12 can be constructed in the same manner from code of length 6.
- But, to construct a code of length 6 requires the existence of code of length 3 which is not possible.

Can't we construct the code of length 6 using the method discussed earlier?

We can.

Gray Code of Odd Length

A Gray code of odd length does not exist.

Why?

The problem with the odd length Gray code was to close the cycle. But, an open cycle exists.

Then?

No problem ... because to construct a code length of $2n$ from the method discussed earlier does not require that the smaller code should be closed. Therefore, a Gray code of length $2n$ can be constructed from an **open** Gray code (abuse of term!!!) of length n also.

Open and Closed Gray Code of Length n

Theorem

1. *There exists an open Gray code of length n having $\lceil \log_2 n \rceil$ bits for odd $n \geq 3$.*
2. *There exists a Gray code of length n having $\lceil \log_2 n \rceil$ bits for even $n \geq 2$.*

Proof:

- For $n = 2$, a Gray code is 0, 1. For $n = 3$, an open Gray code is 00, 01 and 11.
- Inductive Hypothesis (Strong):
  1. For all odd $3 \leq n \leq k$, there exists an open Gray code of length n having $\lceil \log_2 n \rceil$ bits.
  2. For all even $n \leq k$, there exists a Gray code of length n having $\lceil \log_2 n \rceil$ bits.
- Now, we prove for $n = k + 1$.

Open and Closed Gray Code of Length n

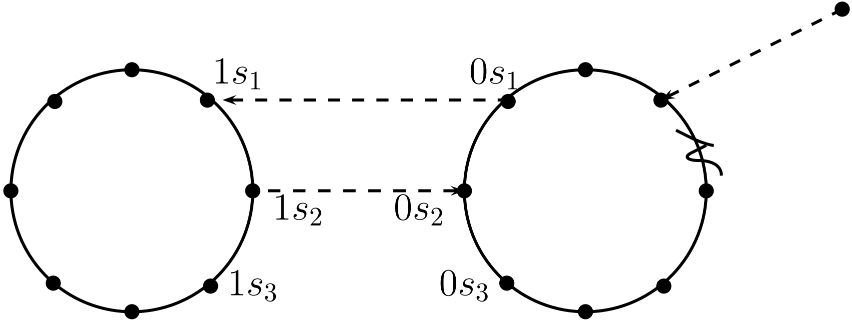
Proof continued:

- Now, we prove for $n = k + 1$. There are two cases:
  1. If $k + 1$ is even: Then take a Gray code (*or an open Gray code depending upon whether* $(k + 1)/2$ *is even or odd*) of length $(k + 1)/2$ having $\lceil \log_2(k + 1)/2 \rceil$ bits. It exists because of our inductive hypothesis. Construct a code of length $k + 1$ having $\lceil \log_2(k + 1)/2 \rceil + 1 = \lceil \log_2(k + 1) \rceil$ bits using the method discussed earlier (*while constructing codes of length* 2^l). Note that the code will be closed.
  2. If $k + 1$ is odd, say $k + 1 = 2r + 1$. There are two cases:
     1. When $2r$ is not a power of 2.
     2. When $2r$ is a power of 2.

Proof continued:

1. When $2r$ is not a power of 2: Then take a Gray code (*or open Gray code*) of length r having $\lceil \log_2 r \rceil$ bits. It exists because of our inductive hypothesis. Construct a Gray code of length $2r$. Now, we need to add one more string to make this Gray code open of length $2r + 1$ having $\lceil log_2(2r + 1) \rceil$ bits. Since, $2r$ is not a power of 2, there exists an unused string connected to the some string used in the Gray code (Why?). Attach this string suitably to make this code open. Note that the length will be $2r + 1$ and bit size will be $\lceil \log_2(2r + 1) \rceil$
2. When $2r$ is a power of 2, i.e. $r = 2^l$: Do the same thing as above. But, this time there is no unused string. Add one more bit to the code. Now, we have unused string left. Do the same as above. Note that, since $2r$ is a power of 2, $\lceil \log_2(2r) \rceil = \log_2(2r)$. Thus, $\lceil \log_2(2r + 1) \rceil = \log_2(2r) + 1$.

An Open Gray of Odd Length



A Gray Code of Length $6 = 2 * 3$

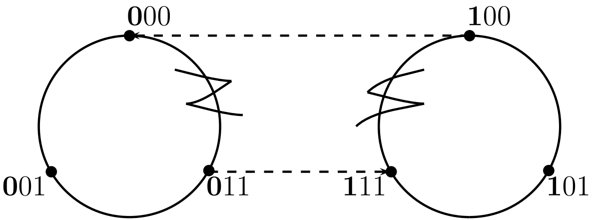


Figure: How to Construct Gray Code of Length $6 = 2 * 3$

A Gray code of length 6: $100, 000, 001, 011, 111, 101$

An Open Gray Code of Length $7 = 2 * 3 + 1$

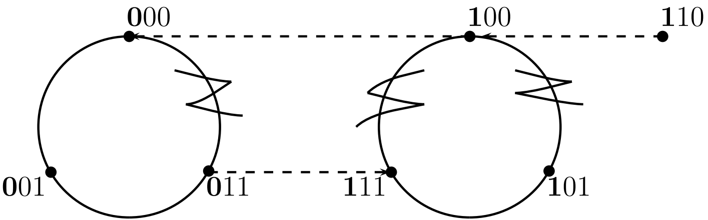


Figure: How to Construct an Open Gray Code of Length $6 = 2 * 3 + 1$

A Gray code of length 6: $100, 000, 001, 011, 111, 101$
Unused strings are: $010, 110$.
Attach 110 with 100.
An open Gray code of length 7: $110, 100, 000, 001, 011, 111, 101$

An Open Gray Code of Length $9 = 2 * 4 + 1$

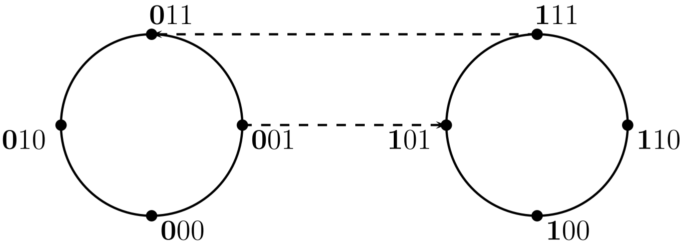


Figure: How to Construct Gray Code of Length $8 = 2 * 4$

A Gray code of length 8: $111, 011, 010, 000, 001, 101, 100, 110$
No unused strings left.
Add one more bit to the code.

An Open Gray Code of Length $9 = 2 * 4 + 1$

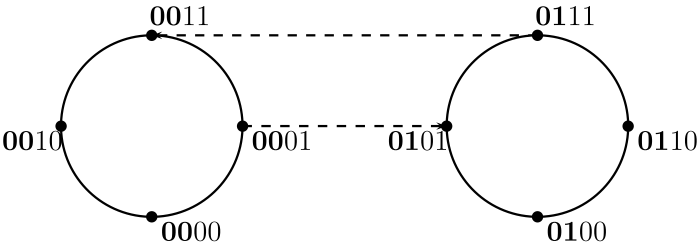


Figure: How to Construct Gray Code of Length $8 = 2 * 4$

After adding one more bit, a Gray code of length 8:
$0111, 0011, 0010, 0000, 0001, 0101, 0100, 0110$
Unused strings are: $1111, 1011, 1010, 1000, 1001, 1101, 1100, 1110$
Attach 1111 with 0111.

An Open Gray Code of Length $9 = 2 * 4 + 1$

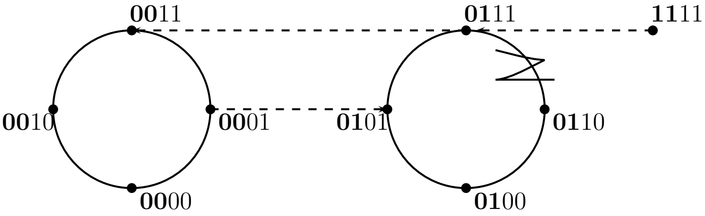


Figure: How to Construct an Open Gray Code of Length $8 = 2 * 4 + 1$

An open Gray code of length 9:
$1111, 0111, 0011, 0010, 0000, 0001, 0101, 0100, 0110$

Thank You