CSC006P1M: Design and Analysis of Algorithms
Lecture 12 (Polynomial Multiplication)

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Polynomial Multiplication

Let $P = \sum_{i=0}^{m} p_i x^i$, and $Q = \sum_{i=0}^{n} q_i x^i$ be two polynomials of degree m and n respectively.

Problem

Compute the product of two given polynomials of degree m and n respectively.

Polynomial Multiplication

$$
\begin{aligned}
PQ &= (p_m x^m + \cdots + p_0)(q_n x^n + \cdots + q_0) \\
&= p_m x^m (q_n x^n + \cdots + q_0) + \\
&\quad p_{m-1} x^{m-1} (q_n x^n + \cdots + q_0) + \\
&\quad \vdots \\
&\quad + p_0 (q_n x^n + \cdots + q_0)
\end{aligned}
$$

- Number of Multiplications - $(m+1)(n+1)$.
- Number of Additions - Exercise Is it mn?

Can we do better?

For simplicity, assume $m = n$.

Polynomial Multiplication

Let $P = \sum_{i=0}^{n-1} p_i x^i$, and $Q = \sum_{i=0}^{n-1} q_i x^i$ be two polynomials of degree n respectively. Further assume that n is a power of 2.

Now, we divide polynomial into two equal-sized parts.

- $P = P_1 + x^{n/2} P_2$, and
- $Q = Q_1 + x^{n/2} Q_2$

So,

$$\begin{aligned} PQ &= (P_1 + x^{n/2} P_2)(Q_1 + x^{n/2} Q_2) \\ &= P_1 Q_1 + (P_1 Q_2 + P_2 Q_1) x^{n/2} + P_2 Q_2 x^n \end{aligned}$$

Polynomial Multiplication

$$P_1 Q_1 + (P_1 Q_2 + P_2 Q_1)x^{n/2} + P_2 Q_2 x^n$$

Time Complexity:
$T(n) = 4T(n/2) + \Theta(?);\ \ T(1) = 1.$

Time Complexity:
$T(n) = 4T(n/2) + \Theta(n);\ \ T(1) = 1.$
$T(n) = \Theta(n^2).$

No improvement.

Polynomial Multiplication

Let,

- $P_1Q_1 = A$, $P_1Q_2 = B$, $P_2Q_1 = C$ and $P_2Q_2 = D$.

$$P_1Q_1 + (P_1Q_2 + P_2Q_1)x^{n/2} + P_2Q_2x^n = A + (B + C)x^{n/2} + Dx^n.$$

Observations:

- We do not need to compute B and C separately; we need only to know $B + C$.
- $(P_1 + P_2)(Q_1 + Q_2) - A - D = B + C$.

of $(n/2) - 1$-degree Polynomial Multiplications Required

1. $P_1Q_1 = A$.
2. $P_2Q_2 = B$.
3. $(P_1 + P_2)(Q_1 + Q_2) = E$.

$$A + (B + C)x^{n/2} + Dx^n = A + (E - A - D)x^{n/2} + Dx^n$$

Time Complexity:
$T(n) = 3T(n/2) + \Theta(n);\ \ T(1) = 1.$

Time Complexity:
$T(n) = 3T(n/2) + \Theta(n);\ \ T(1) = 1.$
$T(n) = \Theta(n^{\log_2 3}) = \Theta(n^{1.59}).$

Good Job.

Polynomial Multiplication

Let $P = 1 - x + 2x^2 - x^3$, and $Q = 2 + x - x^2 + 2x^3$. Compute PQ.

- $P_1 = 1 - x$, $P_2 = 2 - x$, $Q_1 = 2 + x$, $Q_2 = -1 + 2x$.
- $A = P_1 Q_1 = (1 - x)(2 + x) = 2 - x - x^2$,
- $D = P_2 Q_2 = (2 - x)(-1 + 2x) = -2 + 5x - 2x^2$,
- $E = (P_1 + P_2)(Q_1 + Q_2) = (3 - 2x)(1 + 3x) = 3 + 7x - 6x^2$.
- $B + C = E - A - D = 3 + 3x - 3x^2$.
- $PQ = A + (B + C)x^2 + Dx^4$
- $PQ = (2 - x - x^2) + (3 + 3x - 3x^2)x^2 + (-2 + 5x - 2x^2)x^4$.

  $= 2 - x - x^2 + 3x^2 + 3x^3 - 3x^4 - 2x^4 + 5x^5 - 2x^6$.

  $= 2 - x + 2x^2 + 3x^3 - 5x^4 + 5x^5 - 2x^6$.

Polynomial Multiplication

Can we do better?

Yes. We will discuss in the next class.

Thank You