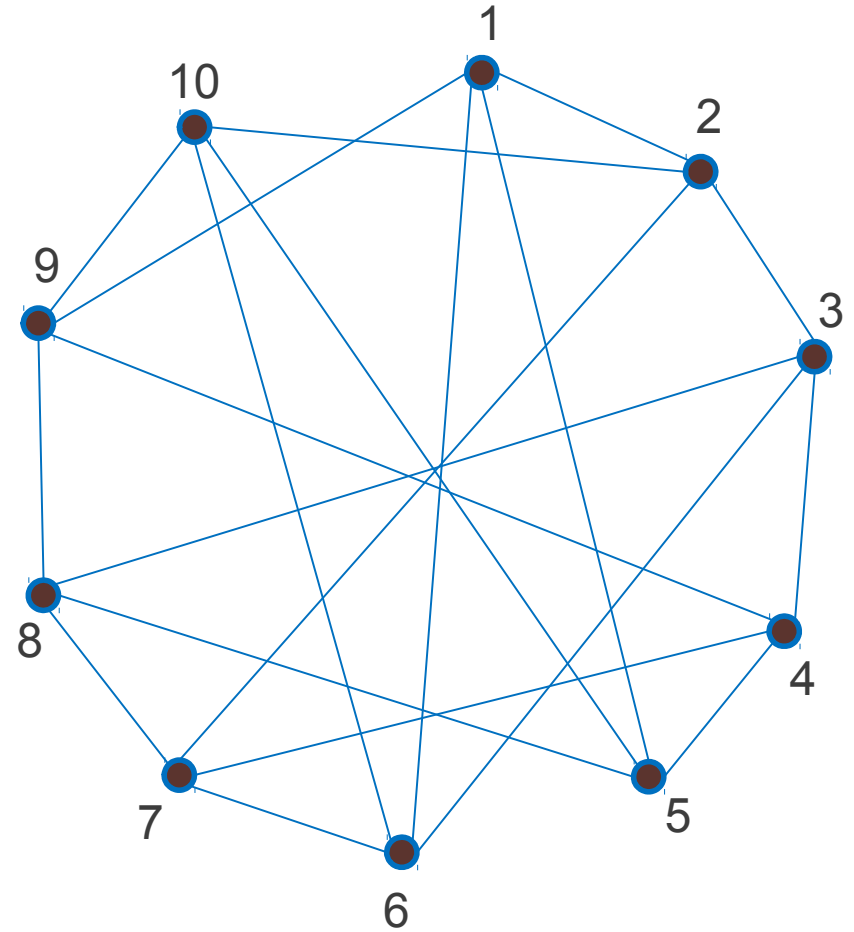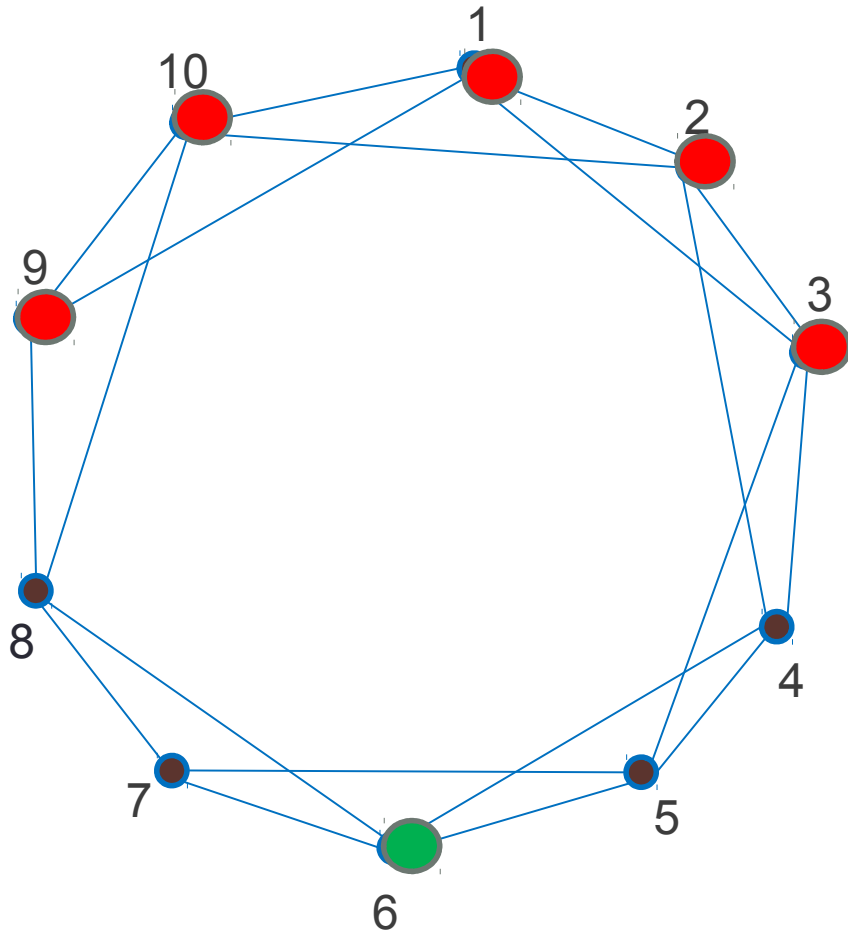# Expanders & Ramanujan Graphs

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Highly Connected Graphs

Connectedness – Why is it important?

- **Efficient Communication Networks** – 'well connected' topologies achieve low latencies with as few links (cost?) as possible

- **Design of Rapidly Mixing Stochastic Processes** – a random walk on a well connected graph is likely to converge to its stationary distribution quickly

- **Pseudorandom Generators** – random walk on a well connected graph can be a very good source of pseudorandom bits – 'randomness extraction'

- **Error Correcting Codes** – a 'richly connected' graph between messages and their codes (bipartite) is likely to lead to enough candidates among the codes that are mutually separated by a minimum distance (such as Hamming distance).

What is 'Connectedness'?

- **Low 'Diameter'** – largest of the minimum distances between pairs of nodes

$$\max_{u,v} \left(\min_{p_{u,v}} l(p_{uv}) \right)$$

- **High 'Expansion'** – any subset of nodes of the graph has 'enough' edges going out to those not in the subset

$$\min_{S \subseteq V, |S| \leq \frac{n}{2}} \frac{|\{E(u,v) \mid u \in S, v \in \bar{S}\}|}{|S|}$$

Isoperimetric Number of the graph

Spectral Graph Theory – Unifying Theme

- Deep relationships between the structural / combinatorial properties of a graph and the algebraic properties of its adjacency matrix. For a d-regular, n-vertex graph G:
  - Adjacency graph A is symmetric, each row/column adding up to d
  - $d = \lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_{n-1} \geq -d$ is the eigen-spectrum of A

  - G is Connected iff $\lambda_0 > \lambda_1$; multiplicity of λ_0 is the number of connected components of G

  - Maximum size of a clique in G is at most $\lambda_1 + 1$

  - G is Bipartite iff $\lambda_{n-1} = -d$; $\chi(G) \geq 1 - \frac{|\lambda_0|}{|\lambda_{n-1}|}$ (Chromatic Number)

    (Hoffman, 1968)

  - Maximum size of a cut in G is at most $\left(\frac{|E|}{2} - (n/4)\lambda_{n-1} \right)$

    (Delorme & Poljak, 1993)

Spectral Gap and Connectivity

- $\lambda = (\lambda_0 - \lambda_1)$ is the **spectral gap** (λ_0 is in fact d)

- Diameter $\delta \leq \dfrac{\log(n-1)}{\log\left(d/\lambda_1\right)} + 1$ (non-bipartite)

$$\delta \leq \dfrac{\log(n-2)/2}{\log\left(d/\lambda_1\right)} + 2 \text{ (bipartite)}$$

   Small $\delta \Rightarrow$ small λ_1

   (Chung, 1989)

- Expansion Ratio $h(G)$

$$\frac{\lambda}{2} \leq h(G) \leq \sqrt{2d\lambda}$$

   Large h(G) $\Rightarrow$ large $\lambda \Rightarrow small\ \lambda_1$

   (Noga Alon & Milman, 1985)

Ramanujan Graphs

- $\lambda_1 \geq 2\sqrt{d-1}.\left(1 - O\left({}^1/_{log^2 n}\right)\right)$

(Noga Alon & Boppanna, 1991)

- For every integer d and ϵ, there exists a constant $c(\epsilon, d)$ such that every (n, d)-graph G has at least $c.n$ eigenvalues greater than $2\sqrt{d-1}$ - ϵ.

(J-.P.Serre, 1991)

- d-regular Graphs with $\lambda_1 \leq 2\sqrt{d-1}$ are **Ramanujan Graphs**

(Lubotzky, Phillips and Sarnak, 1988)

- Ramanujan Graphs have the largest spectral gap possible and therefore are the most 'well connected' (among d-regular graphs)

Examples of Ramanujan Graphs

- K_d, $K_{d,d}$ - are both Ramanujan

- Petersen Graph is Ramanujan

- Random d-regular graphs are 'almost' Ramanujan (Friedman 1991)
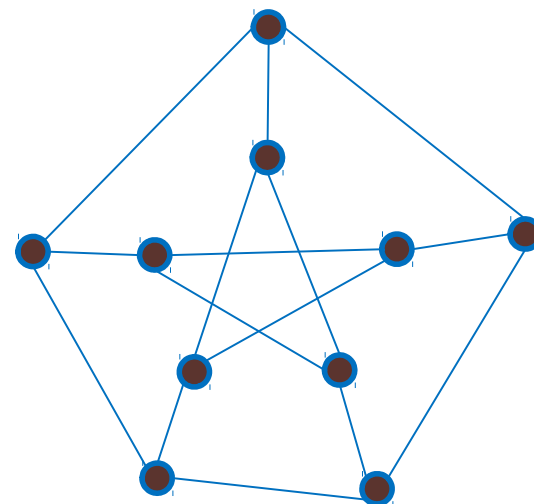
$$\lambda_1 \leq 2\sqrt{d-1} + 2\log d + O(1)$$

Expander Graphs

- Expander Graph Family: family of graphs $G_i, i \in N$ such that:
  - G_i is a d-regular graph of size n_i; $\{n_i\}$ is a monotonically growing series that doesn't grow too fast (say $n_{i+1} \ll n_i^2$)
  - $\forall i, h(G_i) \geq \epsilon > 0$

- Example of an expander family (Super-Concentrators):
  - (n,m,d)-Superconcentrator is a bipartite graph with $|L| = n$, $|R| = m$ and every L-vertex has d neighbors.
  - Known (Pinsker 1973): a random superconcentrator satisfies the following with probability at least 0.9:

$$\text{for every } S \subseteq L, \Gamma(S) \geq \begin{cases} \frac{5d}{8}|S|, & |S| \leq \frac{n}{10d} \\ |S|, & \frac{n}{10d} \leq |S| \leq \frac{n}{2} \end{cases}$$

$\Gamma(S)$ is the set of neighbors of S.

Properties of Expander Graphs

- Expander Families have $\delta = O(\log n)$

- Expander Families are close to random: **Expander Mixing Lemma**: $\forall S, T \subseteq V$,

$$\left| |E(S,T)| - \frac{d|S||T|}{n} \right| \le \lambda_1 \sqrt{|S||T|}$$

Margulis Construction (1973)

●

$$G(Z_m \ X \ Z_m) = (x, y) \rightarrow \begin{Bmatrix} (x \pm y, y) \\ (x \pm (y + 1), y) \\ (x, y \pm x) \\ (x, y \pm (x + 1)) \end{Bmatrix}, \text{ mod m}$$

Graphs with m^2 vertices and $\lambda_1 \leq 8$

Expander Families that are Ramanujan

- What we really need are families of expander graphs that are Ramanujan (Ramanujan Families)

- Constructions exist now – but are highly non-trivial. Proofs of 'Ramanujan-ness' have used a wide range of deep mathematics – Representation Theory of Lie Groups, Number Theory, Algebraic Geometry, …

Examples of Ramanujan Families

- Lubotzky, Phillips and Sarnak (1988): $V_p = Z_p$ for some prime p, $d = 3$, x is connected to x+1, x-1 and x^{-1} modulo p.

- The proof crucially depended on the **Ramanujan-Petersson Conjecture** (now a theorem): that the Ramanujan Tau function,

$$\sum_{n \geq 1} \tau(n)q^n = q \prod_n (1 - q^n)^{24}, \text{ where } q = e^{2\pi i z}$$

satisfies: $|\tau(p)| \leq 2p^{\frac{11}{2}}$, for all primes p.

Hence the name **Ramanujan Graphs**.

Ramanujan Families of Arbitrary size?

- Construction of Ramanujan families for any **n** other than primes and prime powers remains an important open problem.

References

- Sanjeev Arora, and Boaz Barak. Computational Complexity – A Modern Approach. Cambridge University Press. 2009.

- S. Hoory, N. Linial, and A. Wigderson. Expander Graphs and their Applications. Bull. AMS, Vol. 43 (4), (2006).

- Lazlo Lovasz. Eigenvalues of Graphs. Unpublished Manuscript. 2007.

- M. Ram Murty. Ramanujan Graphs. J. Ramanujan Math. Soc. 18 (2003)

Thank You

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