Analysis of Wind-up Music box

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1 Introduction

This document presents an analysis of an overdamped rotational system where the angular displacement $\theta(t)$ is modeled as a function of time. The physical parameters k (stiffness), I (moment of inertia), and c_g (damping coefficient) were experimentally optimized to fit the observed data. After obtaining the equation for the input rotation, we have obtained the displacement, velocity and acceleration diagrams for the output links, which are the drum and the governor. The kinematic analysis for the appended link is added as well.

2 Torsion Spring Damping Model

2.1 Mathematical model

The governing equation for the angular displacement in an overdamped system is given by:

$$\theta(t) = \theta_0 \left(Ae^{r_1 t} + Be^{r_2 t} \right), \tag{1}$$

where r_1 and r_2 are the roots of the characteristic equation:

$$r^2 + \frac{2400c_g}{I} r + \frac{k}{I} = 0. \tag{2}$$

The constants A and B are defined as:

$$A = \frac{-r_2}{r_1 - r_2}, \quad B = \frac{r_1}{r_1 - r_2}. \tag{3}$$

2.2 Optimized Overdamped Model

The variation of angular displacement with time, using the optimized parameters, is shown in Figure 1.

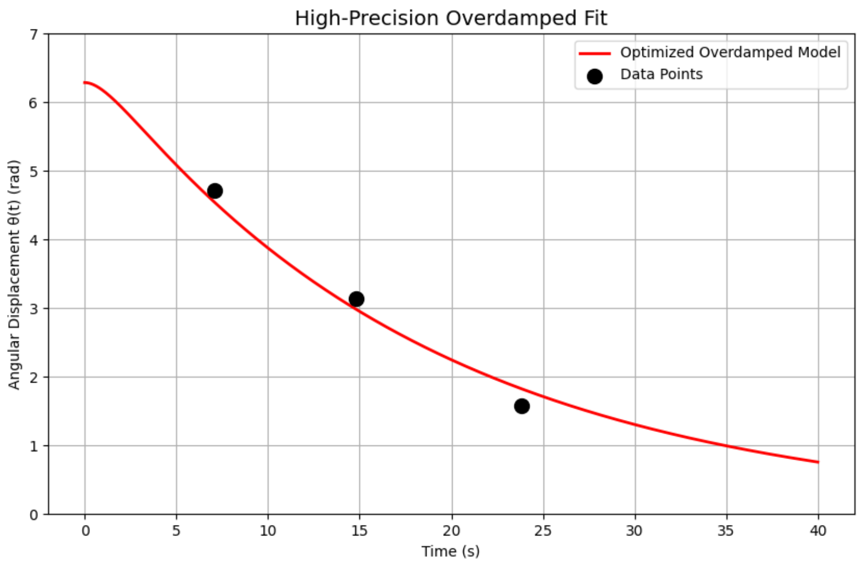

Figure 1: Optimized overdamped model fit to experimental data.

2.3 Optimization of Parameters

Using experimental data points and nonlinear least squares optimization, the best-fit parameters were obtained:

- Optimized stiffness: $k = 0.01$ N·m/rad
- Optimized moment of inertia: $I = 0.2$ kg·m^2
- Optimized damping coefficient: $c_g = 8.1 \times 10^{-5}$ N·m·s/rad
- Computed values:
  - $r_1 = -0.05449565471481052$
  - $r_2 = -0.9175043452851895$
  - $A \approx 1.0632$
  - $B \approx -0.0632$

  Thus, the final expression for $\theta(t)$ is:

$$\theta(t) = \theta_0 \left(1.0632e^{-0.0545t} - 0.0632e^{-0.9175t}\right). \tag{4}$$

2.4 General Equation with Varied Initial Conditions

To further analyze the overdamped system, the initial angular displacement was varied, and the system behavior was studied for $\theta_0 = \{\pi, \frac{3\pi}{2}, 2\pi, \frac{5\pi}{2}, 3\pi\}$. The time at which the angular displacement dropped below 0.5 rad was determined.

Due to friction in the gears, the angular displacement never reaches zero but instead stops abruptly at some displacement greater than zero. This effect is captured in the experimental results and is an important factor in the system's real-world behavior.

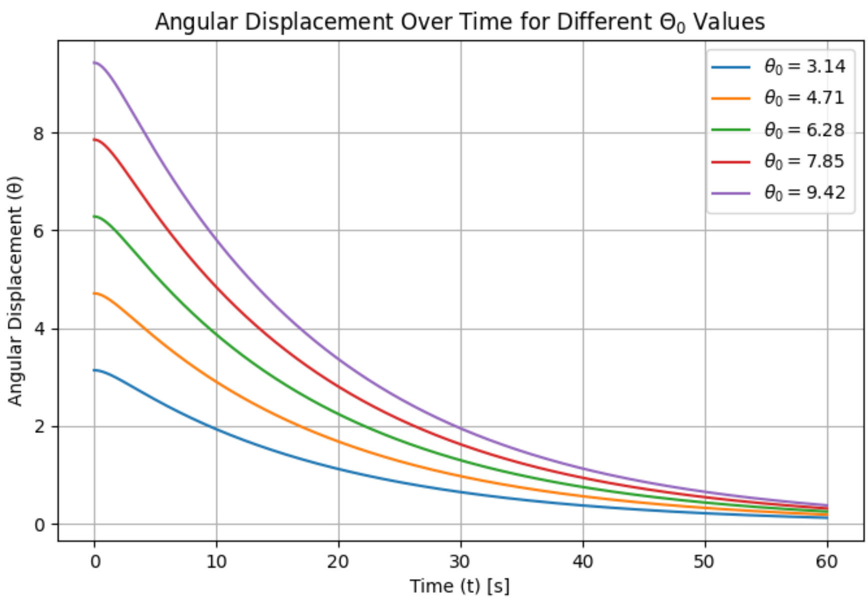

Figure 2: Angular displacement variation over time with θ_0.

3 Gear Ratio Analysis

3.1 Number of teeth

$$\begin{array}{cccc} T_A = 33 & T_B = 12 & T_C = 46 & T_D = 36 \\ T_E = 8 & T_F = 38 & T_G = 6 & T_H = 24 \end{array}$$

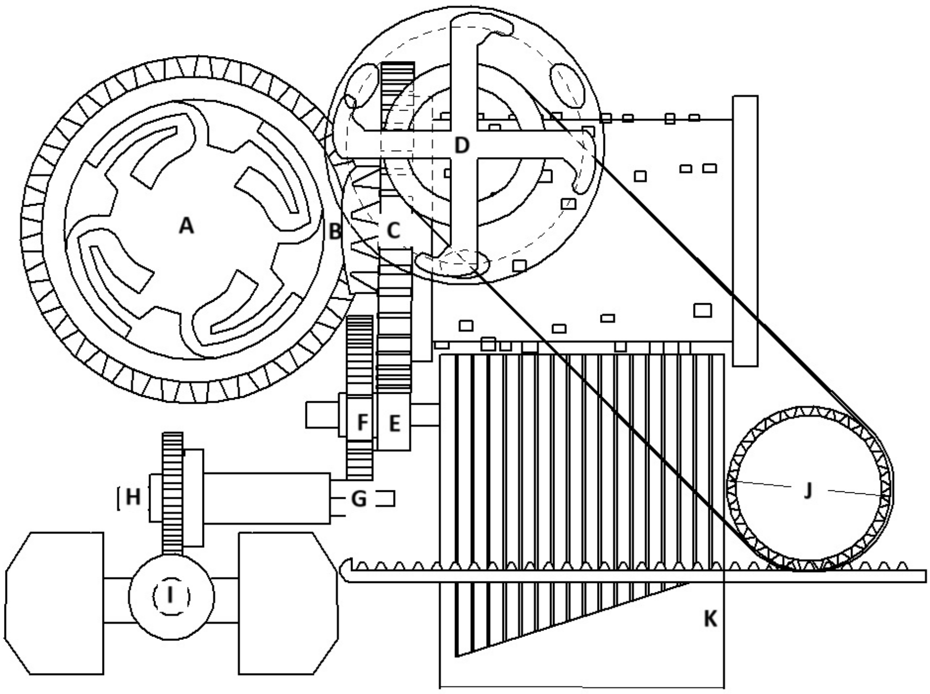

Figure 3: Music Box.

3.2 Input vs Output

To further understand the mechanical transmission in the system, we analyzed the overall angular displacement ratio between Gear A and Gear C (Drum).

First, the gear ratio between A and B:

$$\frac{\theta_B}{\theta_A} = \frac{T_A}{T_B} = \frac{33}{12} = 2.75 \tag{5}$$

Since Gear B and Gear C have the same angular displacement:

$$\theta_B = \theta_C \tag{6}$$

Thus, the overall gear ratio between Gear A and Gear C (Drum) is:

$$\frac{\theta_C}{\theta_A} = 2.75 \tag{7}$$

Next, we try to obtain a relation between gear H and gear A:

$$\frac{\theta_G}{\theta_A} = \frac{T_F \times T_C \times T_A}{T_G \times T_E \times T_B} = 100.1458 \tag{8}$$

where Gear G rotates 100.1458 times for every full rotation of Gear A.

Since Gear H is a worm gear driving Worm I, we use the worm gear ratio:

$$\frac{\theta_I}{\theta_H} = 24 \tag{9}$$

which means Worm I (Governor) rotates 24 times for every full rotation of Gear H, and every full rotation of Gear G is one rotation of Gear H.

Thus, the overall angular displacement ratio is:

$$\frac{\theta_I}{\theta_A} = \frac{\theta_I \times \theta_H}{\theta_H \times \theta_A} = 100.1458 \times 24 = 2403.5 \tag{10}$$

This implies that the governor (Worm I) rotates approximately **2400 times** for every full rotation of Gear A.

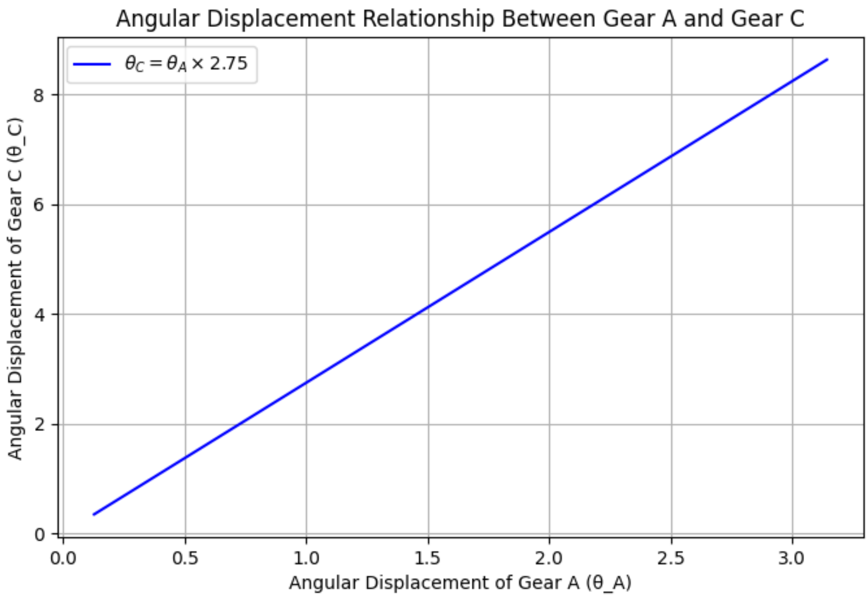

Figure 4: Angular displacement relation between Drum and Key.

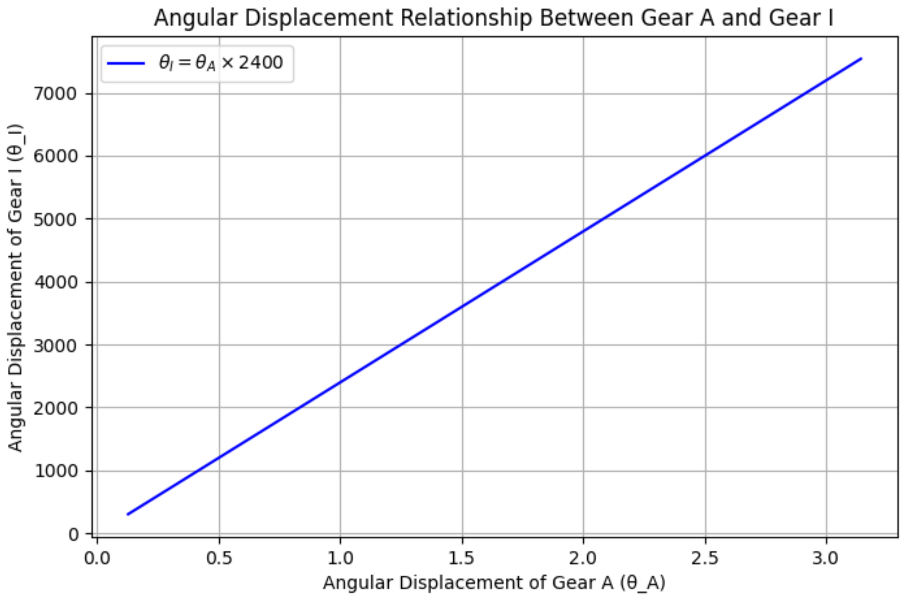

Figure 5: Angular displacement relation between Governor and Key.

The ratio of angular velocities and angular acceleration is between the key and governor is same as the gear ratio.

$$\frac{\theta_I}{\theta_A} = \frac{\omega_I}{\omega_A} = \frac{\alpha_I}{\alpha_A} \tag{11}$$

Similarly for the key and the drum.

$$\frac{\theta_C}{\theta_A} = \frac{\omega_C}{\omega_A} = \frac{\alpha_C}{\alpha_A} \tag{12}$$

Therefore, the plots of the angular velocities and angular acceleration of the input and output gears would be the same as those in the graph obtained for the angular displacement.

4 Inverse Analysis

We try to obtain a relation between Rack K and A. Let the specifications of Gear J be variable. We know that Gear J's angular displacement is the same as Gear D's angular displacement, as they are connected by a belt drive. We obtain the gear ratio of A and D.

$$\frac{\theta_D}{\theta_A} = \frac{\theta_D \times \theta_C}{\theta_C \times \theta_A} = \frac{T_C}{T_D} \times 2.75 = 1.278 \times 2.750 = 3.514 \tag{13}$$

This means that one full rotation of Gear A causes 3.514 rotations of Gear D and Gear J. Gear J drives Rack K. Rack K is a stopping mechanism; hence, we need to decide on the angular displacement of A after which the music stops playing. Let that angular displacement be $\frac{3\pi}{2}$ i.e., 1.5 rotations. Next, we must decide the offset of Rack K from the governor. Let the offset be 7 cm. We know that the displacement of the rack is given as

$$x = r\theta \tag{14}$$

where r is the effective pitch radius of the pinion and θ is the angular displacement of the pinion in radians . Thus,

$$r = \frac{2 \times 0.07}{3\pi \times 3.514} = 4.22\,\text{mm}$$

Let us pick any standard module, let's say 0.5mm, this means that number of teeth is $N = 2r/M \approx 17$ Teeth

5 Results

5.1 Key Velocity and Acceleration Diagrams (INPUT)

$$\theta(t) = \theta_0 \left(1.0632e^{-0.0545t} - 0.0632e^{-0.9175t}\right). \tag{15}$$

Differentiating with respect to t:

$$\frac{d\theta}{dt} = \theta_0 \left(1.0632 \cdot (-0.0545)e^{-0.0545t} - 0.0632 \cdot (-0.9175)e^{-0.9175t}\right)$$

$$\frac{d\theta}{dt} = \theta_0 \left(-0.0580e^{-0.0545t} + 0.0580e^{-0.9175t}\right)$$

$$\frac{d\theta}{dt} = \theta_0 \cdot 0.0580 \left(e^{-0.0545t} - e^{-0.9175t}\right)$$

Differentiating again:

$$\frac{d^2\theta}{dt^2} = \theta_0 \cdot 0.0580 \left(-0.0545e^{-0.0545t} + 0.9175e^{-0.9175t}\right) \tag{16}$$

$$\frac{d^2\theta}{dt^2} = \theta_0 \cdot 0.0580 \left(0.9175e^{-0.9175t} - 0.0545e^{-0.0545t}\right) \tag{17}$$

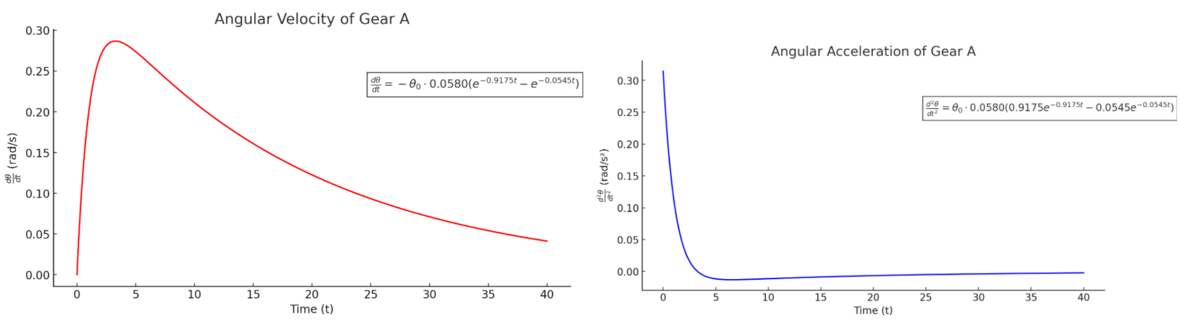

Figure 6: Comparison of Angular Velocity and Angular Acceleration of Gear A

5.2 Governor Velocity and Acceleration Diagrams (OUTPUT)

We plot angular velocity and acceleration of worm which is just 2403 times that of Gear A.

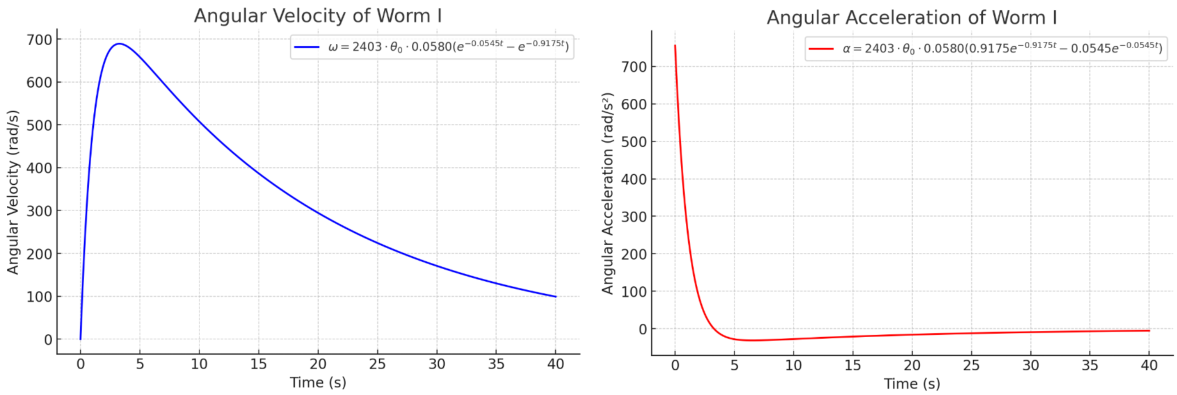

Figure 7: Comparison of Angular Velocity and Angular Acceleration of Worm I

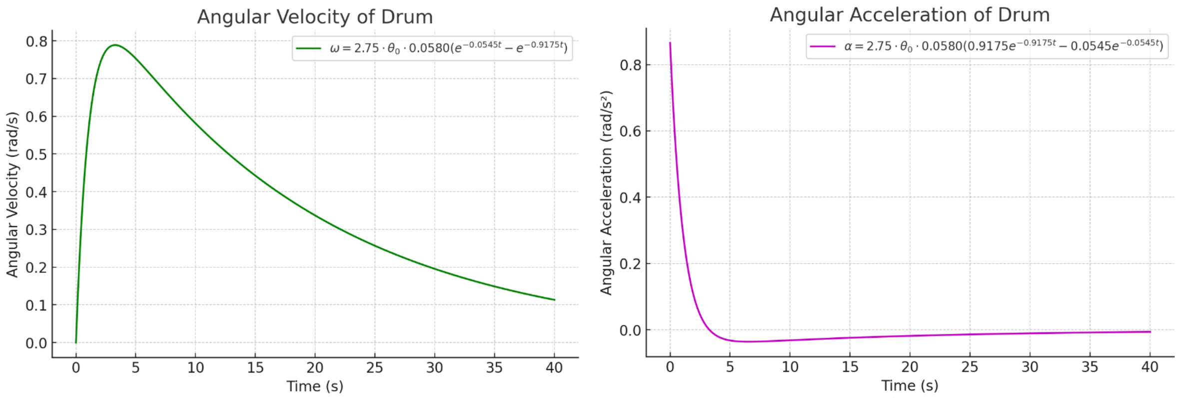

Figure 8: Comparison of Angular Velocity and Angular Acceleration of Drum

5.3 Drum Velocity and Acceleration Diagrams (OUTPUT)

We plot angular velocity and acceleration of the drum which is just 2.75 times that of Gear A.

6 Conclusion

This study provides a comprehensive kinematic analysis of the wind-up music box mechanism, following up on the kinematic diagram and mobility analysis conducted earlier. The analysis included the derivation and optimization of the angular displacement function for the winding key, as well as the corresponding velocity and acceleration diagrams for the key, drum, and governor.

The computed displacement, velocity, and acceleration diagrams show that the system exhibits an overdamped response, with a rapid initial decay in angular velocity followed by a slower approach toward rest. The gear ratio analysis revealed that the drum rotates 2.75 times for every full rotation of the key, while the governor experiences an extreme amplification, rotating approximately 2400 times per key rotation. This explains the rapid spinning motion of the governor, which plays a crucial role in regulating the unwinding speed and ensuring a smooth playback of the music.

A practical observation is that friction in the gears causes the angular displacement to stop before reaching zero, rather than continuously decreasing to rest. This abrupt halt is commonly observed in wind-up toys when they run out of stored energy. Additionally, since the toy's motion is primarily driven by stored elastic potential energy, the damping effect plays a critical role in determining how long the toy operates before stopping.

To enhance the functionality of the toy, an additional mechanism could be introduced—such as a secondary cam-driven linkage that controls small moving figurines or animated elements synchronized with the music. This could provide a more interactive and visually engaging experience while maintaining the core mechanical principles of the wind-up system.