BCS405A - Module 4

Principles of counting

4.1 Principles of Inclusion - Exclusion

Introduction:

- If S is a finite set, $|S|$ is a cardinality of S and A_1, A_2 are subsets of S then addition principle or principle of inclusion and exclusion for two sets is given by
  $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$ and $|\overline{A_1} \cap \overline{A_2}| = |\overline{A_1 \cup A_2}| = |S| - |A_1 \cup A_2|$

- If S is a finite set and $A_1, A_2, \ldots, A_n$ are subsets of S then addition principle or principle of inclusion and exclusion for n sets is given by
  $$\begin{aligned} N &= |A_1 \cup A_2 \cup \ldots \cup A_n| \\ &= \sum |A_i| - \sum |A_i \cap A_j| + \sum |A_i \cap A_j \cap A_k| - \cdots + (-1)^n |A_1 \cap A_2 \cap \ldots \cap A_n| \\ &= S_1 - S_2 + S_3 - \ \ldots \ \ (-1)^n S_n \end{aligned}$$

- $$\begin{aligned} \overline{N} = |\overline{A_1} \cap \overline{A_2} \cap \ldots \cap \overline{A_n}| &= |\overline{A_1 \cup A_2 \cup \ldots \cup A_n}| \\ &= |S| - |A_1 \cup A_2 \cup \ldots \cup A_n| \\ &= |S| - N \end{aligned}$$

- The number of elements in S that satisfy exactly m of the n conditions $(0 \le m \le n)$
  $$E_m = S_m - \binom{m+1}{1} S_{m+1} + \binom{m+2}{2} S_{m+2} - \ldots + (-1)^{n-m} \binom{n}{n-m} S_n$$

- The number of elements in S that satisfy at least m of the n conditions $(0 \le m \le n)$
  $$L_m = S_m - \binom{m}{m-1} S_{m+1} + \binom{m+1}{m-1} S_{m+2} - \ldots + (-1)^{n-m} \binom{n-1}{m-1} S_n$$

1. **Among the students in a Hostel, 12 students study Mathematics (A), 20 study Physics(B), 20 study Chemistry(C) and 8 study Biology(D). There are 5 students for A and B, 7 students for A and C, 4 students for A and D, 16 students for B and C, 4 students for B and D and 3 students for C and D. There are 3 students for A, B and C, 2 for A, B and D, 2 for B, C and D, 3 for A,C and D. Finally there are 2 who study all of these subjects. The number of students who study none of the subjects is 71. Find the total number of students in the Hostel.**

Let $|S| =$ Total number of students in the Hostel$= x$.

Let $S_1 = |\mathrm{A}| + |\mathrm{B}| + |\mathrm{C}| + |\mathrm{D}| = 12 + 20 + 20 + 8 = 60$

Let $S_2 = |A \cap B| + |A \cap C| + |A \cap D| + |B \cap C| + |B \cap D| + |C \cap D|$

$= 5 + 7 + 4 + 16 + 4 + 3 = 39$

Let $S_3 = |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D|$

$= 3 + 2 + 3 + 2 = 10$

Let $S_4 = |A \cap B \cap C \cap D| = 2$

Let $\overline{N} =$ The number of students who study none of the subjects $= 71$

By the principle of inclusion-exclusion,

$N = S_1 - S_2 + S_3 - S_4 = 60 - 39 + 10 - 2 = 29$

Total number of students in the Hostel $= |S| = N + \overline{N} = 71 + 29 = 100$

2. **Out of 30 students in a Hostel, 15 study History, 8 study Economics and 6 study Geography. It is known that 3 students study all these subjects. Show that 7 or more students study none of these subjects.**

Let $S_1 = |\mathrm{A}| + |\mathrm{B}| + |\mathrm{C}| = 15 + 8 + 6 = 29$

Let $S_2 = |A \cap B| + |A \cap C| + |B \cap C|$

$\geq |A \cap B \cap C| + |A \cap B \cap C| + |A \cap B \cap C| = 3|A \cap B \cap C|$

Let $S_3 = |A \cap B \cap C| = 3$

By the principle of inclusion-exclusion, $N = S_1 - S_2 + S_3$

$|S| = N + \overline{N}$

No. of students study none of these subjects $= \overline{N}$

$= |S| - N$

$= 30 - S_1 + S_2 - S_3$

$\geq 30 - 29 + 3(3) - 3 = 7$

3. **Determine the number of positive integers n such that $\boldsymbol{1 \leq n \leq 100}$ and n is not divisible by 2, 3 and 5.**

Let A, B, C be the number of positive integers divisible by 2, 3 and 5 respectively.

$|S| = 100,\ S_1 = |A| + |B| + |C| = \left\lfloor \frac{100}{2} \right\rfloor + \left\lfloor \frac{100}{3} \right\rfloor + \left\lfloor \frac{100}{5} \right\rfloor = 50 + 33 + 20 = 103$

$S_2 = |A \cap B| + |B \cap C| + |A \cap C| = \left\lfloor \frac{100}{6} \right\rfloor + \left\lfloor \frac{100}{10} \right\rfloor + \left\lfloor \frac{100}{15} \right\rfloor = 16 + 10 + 6 = 32$

$S_3 = |A \cap B \cap C| = \left\lfloor \frac{100}{30} \right\rfloor = 3.$

Required number of positive integers $= \overline{N} = |S| - N = |S| - (S_1 - S_2 + S_3) = 100 - 103 + 32 - 3 = 26$

4. How many integers between 1 and 300 (inclusive) are
(i) Divisible by at least one of 5, 6, 8? (ii) Divisible by none of 5, 6, 8?

Let A, B, C be the number of positive integers divisible by 5, 6 and 8 respectively.

$|S| = 300,\ S_1 = |A| + |B| + |C| = \left\lfloor \frac{300}{5} \right\rfloor + \left\lfloor \frac{300}{6} \right\rfloor + \left\lfloor \frac{300}{8} \right\rfloor = 60 + 50 + 37 = 147$

$S_2 = |A \cap B| + |B \cap C| + |A \cap C| = \left\lfloor \frac{300}{30} \right\rfloor + \left\lfloor \frac{300}{24} \right\rfloor + \left\lfloor \frac{300}{40} \right\rfloor = 10 + 12 + 7 = 29$

Where $30 = LCM(5,6)$, $24 = LCM(6,8)$, $40 = LCM(5,8)$

$S_3 = |A \cap B \cap C| = \frac{300}{120} = 2.$

(i) $N = S_1 - S_2 + S_3 = 147 - 29 + 2 = 120$

(ii) Number of integers divisible by none of 5, 6, 8 $= \bar{N} = |S| - N = 300 - 120 = 180$

5. In how many ways can the 26 letters of the English alphabet be permuted so that none of the patterns CAR, DOG, PUN or BYTE occurs?

Let S be the set of all permutations of the 26 letters. Then $|S| = 26!$

Let A, B, C and D be the set of all permutations of 26 letters in which CAR, DOG, PUN and BYTE appears respectively.

If CAR is considered as a single block, $|A| = (26 - 3 + 1)! = 24!$

Similarly, $|B| = 24!, |C| = 24!\ and\ |D| = 23!$

If CAR and DOG are considered as two blocks, $|A \cap B| = (26 - 6 + 2)! = 22!$

Similarly, $|A \cap C| = |B \cap C| = 22!$

And $|A \cap D| = |B \cap D| = |C \cap D| = (26 - 7 + 2)! = 21!$

$|A \cap B \cap C| = (26 - 9 + 3)! = 20!$

$|A \cap B \cap D| = |A \cap C \cap D| = |B \cap C \cap D| = (26 - 10 + 3)! = 19!$

$|A \cap B \cap C \cap D| = (26 - 13 + 4)! = 17!$

$S_1 = |A| + |B| + |C| + |D| = 24! + 24! + 24! + 23!$

$S_2 = |A \cap B| + |A \cap C| + |B \cap C| + |A \cap D| + |B \cap D| + |C \cap D| = 3(22! + 21!)$

$S_3 = |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| = 20! + 3(19!)$

$S_4 = |A \cap B \cap C \cap D| = 17!$

Required number of permutations $= \bar{N} = |S| - N = |S| - (S_1 - S_2 + S_3 - S_4)$

$$= 26! - \{3(24!) + 23!\} + \{3(22! + 21!)\} - \{20! + 3(19!)\} + \{17!\}$$

6. Find the number of permutations of English letters which contain (i) Exactly 2 (ii) At least 2 (iii) Exactly 3 and (iv) at least 3 of the patterns CAR, DOG, PUN and BYTE.

Let S be the set of all permutations of the 26 letters. Then $|S| = 26$.

Let A, B, C and D be the set of all permutations of 26 letters in which CAR, DOG, PUN and BYTE appears.

If CAR is considered as a single block, $|A| = (26 - 3 + 1)! = 24!$

Similarly, $|B| = 24!, |C| = 24!\ and\ |D| = 23!$

If CAR and DOG are considered as two blocks, $|A \cap B| = (26 - 6 + 2)! = 22!$

Similarly, $|A \cap C| = |B \cap C| = 22!$

And $|A \cap D| = |B \cap D| = |C \cap D| = (26 - 7 + 2)! = 21!$

$|A \cap B \cap C| = (26 - 9 + 3)! = 20!$

$|A \cap B \cap D| = |A \cap C \cap D| = |B \cap C \cap D| = (26 - 10 + 3)! = 19!$

$|A \cap B \cap C \cap D| = (26 - 13 + 4)! = 17!$

$S_1 = |A| + |B| + |C| + |D| = 24! + 24! + 24! + 23!$

$S_2 = |A \cap B| + |A \cap C| + |B \cap C| + |A \cap D| + |B \cap D| + |C \cap D| = 3(22! + 21!)$

$S_3 = |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| = 20! + 3(19!)$

$S_4 = |A \cap B \cap C \cap D| = 17!$

(i) The number of permutations with exactly two of the patterns

$$E_2 = S_2 - \binom{2+1}{1} S_{2+1} + \binom{2+2}{2} S_{2+2}$$
$$= S_2 - 3S_3 + 6S_4 = 3(22! + 21!) - 3\{20! + 3(19!)\} + 6(17!)$$

(ii) The number of permutations with exactly two of the patterns

$$E_3 = S_3 - \binom{3+1}{1} S_{3+1}$$
$$= S_3 - 4S_4 = \{20! + 3(19!)\} - 4(17!)$$

(iii) The numb er of permutations with exactly two of the patterns

$$L_3 = S_3 - \binom{3}{3-1} S_{3+1}$$
$$= S_3 - 3S_4 = \{20! + 3(19!)\} - 4(17!)$$

7. On how many ways can one arrange the letters in CORRESPONDENTS so that
(i) There is no pair of consecutive identical letters ?
(ii) There are exactly 2 pairs of consecutive identical letters ?
(iii) There are at least 3 pairs of consecutive identical letters ?

CORRESPONDENTS has 14 letters in which 5 letters O, R, E, S, N repeated twice.
Let S be the set of all permutations of these 14 letters. Then $|S| = \frac{14!}{(2!)^5}$
Let A_1, A_2, A_3, A_4, A_5 be the set of all permutations in which O, R, E, S, N appear in pair respectively.

$|A_i| = \frac{13!}{(2!)^4}\ , for\ i = 1, 2, 3, 4$

$|A_i \cap A_j| = \frac{12!}{(2!)^3}\,,\ |A_i \cap A_j \cap A_k| = \frac{11!}{(2!)^2}$

$|A_i \cap A_j \cap A_k \cap A_l| = \frac{10!}{2!}\,,\ |A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5| = 9!$

$S_1 = 5C_1 \times \frac{13!}{(2!)^4} = 5 \times \frac{13!}{(2!)^4}\,,\ S_2 = 5C_2 \times \frac{12!}{(2!)^3} = 10 \times \frac{12!}{(2!)^3}$

$S_3 = 5C_3 \times \frac{11!}{(2!)^2} = 10 \times \frac{11!}{(2!)^2}\,,\ S_4 = 5C_4 \times \frac{10!}{2!} = 5 \times \frac{10!}{2!}\,,$

$S_5 = 5C_5 \times 9! = 9!$

(i) Number of permutations where there is no pair of consecutive identical letters

$$E_0 = S_0 - \binom{0+1}{1} S_{0+1} + \binom{0+2}{2} S_{0+2} - \binom{0+3}{3} S_{0+3} + \binom{0+4}{4} S_{0+4} - \binom{0+5}{5} S_{0+5}$$
$$= S_0 - S_1 + S_2 - S_3 + S_4 - S_5$$
$$= \frac{14!}{(2!)^5} - 5 \times \frac{13!}{(2!)^4} + 10 \times \frac{12!}{(2!)^3} - 10 \times \frac{11!}{(2!)^2} + 5 \times \frac{10!}{2!} - 9!$$

(ii) Number of permutations where there are exactly two pairs of consecutive identical letters

$$E_2 = S_2 - \binom{2+1}{1} S_{2+1} + \binom{2+2}{2} S_{2+2} - \binom{2+3}{3} S_{2+3}$$
$$= S_2 - 3S_3 + 6S_4 - 10S_5$$
$$= 10 \times \frac{12!}{(2!)^3} - 3 \times 10 \times \frac{11!}{(2!)^2} + 6 \times 5 \times \frac{10!}{2!} - 10 \times 9!$$

(ii) Number of permutations where there are atleast three pairs of consecutive identical letters

$$L_3 = S_3 - \binom{3}{2} S_4 + \binom{4}{2} S_5$$
$$= S_3 - 4S_4 + 10S_5$$
$$= 10 \times \frac{11!}{(2!)^2} - 3 \times 5 \times \frac{10!}{2!} + 6 \times 9!$$

8. Determine the number of positive integers n such that $1 \le n \le 300$ which are (i) Divisible by exactly two of 5, 6, 8 (ii) Divisible by at least two of 5, 6, 8.

Let $S = \{1, 2, 3, \dots, 300\}$. Therefore, $|S| = 300$.
Let A, B, C be the set of elements of S that are divisible by 5, 6 and 8 respectively.

$S_1 = |A| + |B| + |C| = \left\lfloor \frac{300}{5} \right\rfloor + \left\lfloor \frac{300}{6} \right\rfloor + \left\lfloor \frac{300}{8} \right\rfloor = 60 + 50 + 37 = 147$

$S_2 = |A \cap B| + |B \cap C| + |A \cap C| = \left\lfloor \frac{300}{30} \right\rfloor + \left\lfloor \frac{300}{24} \right\rfloor + \left\lfloor \frac{300}{40} \right\rfloor = 10 + 12 + 7 = 29$

$S_3 = |A \cap B \cap C| = \frac{300}{120} = 2.$

(i) The number of positive integers n such that $1 \le n \le 300$ which are divisible by exactly two of 5, 6, 8 is $E_2 = S_2 - \binom{3}{1} S_3 = 29 - 6 = 23.$

(ii) The number of positive integers n such that $1 \le n \le 300$ which are divisible by At least two of 5, 6, 8 is $L_2 = S_2 - \binom{2}{1} S_3 = 29 - 4 = 25.$

4.2 Derangements

Introduction:

- A permutation of n distinct objects in which none of the objects is in its natural place is called a derangement.
- Number of possible derangements of n distinct objects is given by

$$d_n = n!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\cdots+\frac{(-1)^n}{n!}\right)$$

- For $n \geq 7$, $d_n = n!\,e^{-1} = 0.3679 \times n!$

1. Find the number of derangements of 1, 2, 3, 4. Also evaluate d_5, d_6, d_7, d_8 (Jan 15)

$d_1 = 0$. If there is only one element, it is in its original place in every arrangement.

$d_2 = 1$. If there are 2 elements, there is one derangement , by interchanging their places.

$d_3 = 2$. For three objects 1, 2, 3 , possible derangements are 231, 312.

$d_4 = 4!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}\right) = 9,\qquad d_5 = 5!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}\right) = 44$

Similarly, $d_6 = 265$, $d_7 = 1854$, $d_8 = 14833$.

2. Thirty students take a quiz. Then for the purpose of grading, the teacher asks the students to exchange papers so that no one is grading his own paper. In how many ways can this be done?

Number of possible derangements of 30 distinct papers is given by

$$d_{30} = 30!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\cdots+\frac{(-1)^n}{30!}\right) \approx 30! \times e^{-1} \approx 30! \times 0.3679$$

3. In how many ways can each of 10 people select a left glove and a right glove out of a total of 10 pairs of gloves so that no person selects a matching pair of gloves.

Number of possible derangements of 10 right gloves is given by

$$d_{10} = 10!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\cdots+\frac{(-1)^n}{10!}\right) \approx 10! \times e^{-1} \approx 10! \times 0.3679$$

4. **While at the race track, a person bets on each of the nine horses in a race to come in accordance to how they are favored. In how many ways can they reach the finish line so that he loses all his bets?**

Number of possible derangements of 9 horses is given by

$$d_9 = 9!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{10!}\right) \approx 9! \times e^{-1} \approx 9! \times 0.3679$$

5. **There are eight letters to eight different people to be placed in eight different addressed envelopes. Find the number of ways of doing this so that at least one letter gets to the right person.**

Number of possible derangements of 8 letters is given by

$$d_8 = 8!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{8!}\right) \approx 8! \times e^{-1} \approx 8! \times 0.3679 \approx 14833.$$

Number of ways of doing this so that at least one letter gets to the right person is

$8! - d_8 = 40320 - 14833 = 25487.$

6. **At a restaurant, 10 men handover their umbrellas to the receptionist. In how many ways can the umbrellas be returned so that (i) no one receives his own umbrella? (ii) at least one of the men receives his own umbrella? (iii) at least two of the men receive their own umbrellas?**

(i) No. of possible derangements of 10 umbrellas is given by

$$d_{10} = 10!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{8!}\right) \approx 10! \times e^{-1} \approx 10! \times 0.3679$$

(ii) Total no. of arrangements is 10! .

No. of ways, no one receives his own umbrella is d_{10} .

No. of ways, at least one of the men receives his own umbrella is $10! - d_{10}$.

(iii) Total no. of arrangements is 10! .

No. of ways, no one receives his own umbrella is d_{10} .

No. of ways, only one man receives his own umbrella is $10C_1 \times d_9$.

Therefore,

No. of ways, at least two men receive their own umbrellas is $10! - d_{10} - 10d_9$.

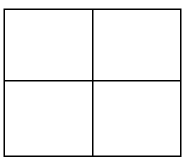
4.3 Rook polynomials

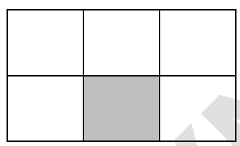
Rook polynomial for the given board

Introduction:

- Let n be the number of squares on the board.
  (which resembles chess board or part of a chess board).
- Two or more squares are said to capture each other if they are in the same row or in the same column of the board.
- For $2 \leq k \leq n$, let r_k be the number of positions of non- capturing rooks (pawns). Then the rook polynomial for the board C is given by $r(c, x) = 1 + r_1 x + r_2 x^2 + \cdots + r_n x^n$.

1. Find the rook polynomials for the following boards:





(i) $r_1 =$ No. of non-capturing positions of one rook $= 4$

$r_2 =$ No. of non capturing positions of two rooks $= 2$

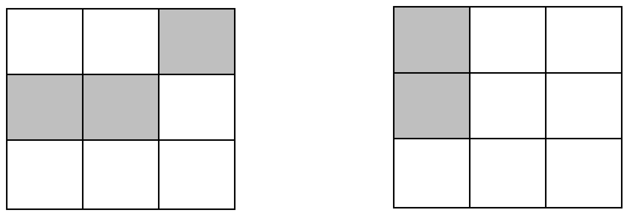
Rook polynomial of the first board is $r(c, x) = 1 + r_1 x + r_2 x^2 = 1 + 4x + 2x^2$

(ii) $r_1 =$ No. of non-capturing positions of one rook $= 5$

$r_2 =$ No. of non capturing positions of two rooks$= 4$

Rook polynomial of the second board is $r(c, x) = 1 + r_1 x + r_2 x^2 = 1 + 5x + 4x^2$

2. Find the rook polynomials for the following boards:



(i) r_1 = No. of non-capturing positions of one rook = 6

$r_2 =$ No. of non capturing positions of two rooks= 8

$r_3 =$ No. of non capturing positions of two rooks= 2

Rook polynomial of the first board is $r(c, x) = 1 + r_1 x + r_2 x^2 + r_3 x^3$

$$= 1 + 6x + 8x^2 + 2x^3$$

(ii) r_1 = No. of non-capturing positions of one rook = 7

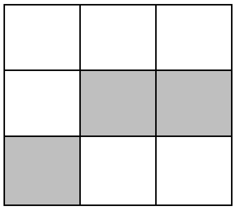
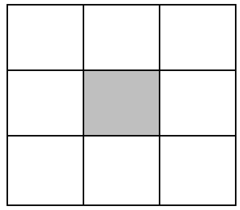
$r_2 =$ No. of non capturing positions of two rooks = 10

$r_3 =$ No. of non capturing positions of two rooks= 2

Rook polynomial of the second board is $r(c, x) = 1 + r_1 x + r_2 x^2 + r_3 x^3$

$$= 1 + 7x + 10x^2 + 2x^3$$

3. Find the rook polynomials for the following boards:

(i) r_1 = No. of non-capturing positions of one rook = 6

$r_2 =$ No. of non capturing positions of two rooks= 8

$r_3 =$ No. of non capturing positions of two rooks= 2

Rook polynomial of the first board is $r(c,x) = 1 + r_1x + r_2x^2 + r_3x^3$

$$= 1 + 6x + 8x^2 + 2x^3$$

(ii) r_1 = No. of non-capturing positions of one rook = 8

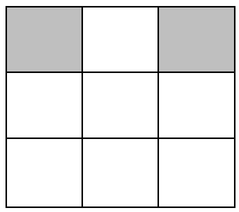
$r_2 =$ No. of non capturing positions of two rooks = 14

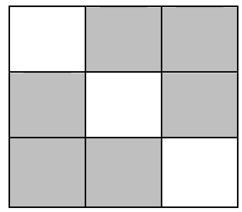
$r_3 =$ No. of non capturing positions of two rooks = 4

Rook polynomial of the second board is $r(c,x) = 1 + r_1x + r_2x^2 + r_3x^3$

$$= 1 + 8x + 14x^2 + 4x^3$$

4. Find the rook polynomials for the following boards:





(i) $r_1 =$ No. of non-capturing positions of one rook = 7

$r_2 =$ No. of non capturing positions of two rooks= 10

$r_3 =$ No. of non capturing positions of two rooks= 2

Rook polynomial of the first board is $r(c,x) = 1 + r_1x + r_2x^2 + r_3x^3$

$= 1 + 7x + 10x^2 + 2x^3$

(ii) $r_1 =$ No. of non-capturing positions of one rook = 3

$r_2 =$ No. of non capturing positions of two rooks = 3

$r_3 =$ No. of non capturing positions of two rooks = 1

Rook polynomial of the second board is $r(c,x) = 1 + r_1x + r_2x^2 + r_3x^3$

$= 1 + 3x + 3x^2 + x^3$

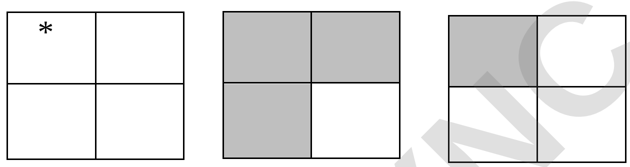
Expansion formula

Introduction:

- In the given board C, mark a particular square by $*$
- Let D be the board obtained by deleting the row and the column containing the square $*$
- Let E be the board obtained by deleting only the square $*$
- Then by expansion formula, the rook polynomial $is\ r(C,x) = x\, r(D,x) + r(E,x)$

5. Find the rook polynomial for 2×2 board using the expansion formula.

C D E



Let the given 2×2 board be C and mark a particular square by $*$

Let D be the board obtained by deleting the row and the column containing the square $*$

The rook polynomial of D is $r(D,x) = 1 + x$.

Let E be the board obtained by deleting only the square $*$

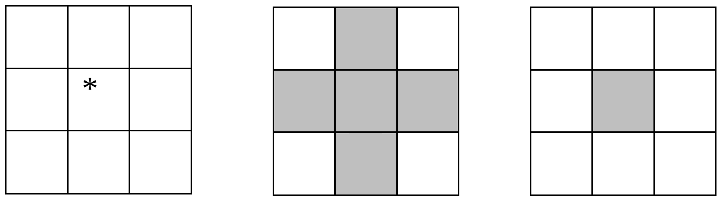
The rook polynomial of E is $r(E,x) = 1 + 3x + x^2$.

Then by expansion formula, the rook polynomial of C is

$$
\begin{aligned}
r(C,x) &= x\, r(D,x) + r(E,x) \\
&= x\, r(D,x) + r(E,x) \\
&= x(1 + x) + (1 + 3x + x^2) \\
&= 1 + 4x + 2x^2
\end{aligned}
$$

6. Find the rook polynomial for 3 × 3 board using the expansion formula.

C D E



Let the given 3×3 board be C and mark the middle square by $*$

Let D be the board obtained by deleting the row and the column containing the square $*$

The rook polynomial of D is $r(D, x) = 1 + 4x + 2x^2$.

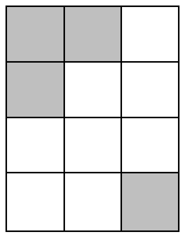
Let E be the board obtained by deleting only the square $*$

The rook polynomial of C is $r(E, x) = 1 + 8x + 14x^2 + 4x^3$.

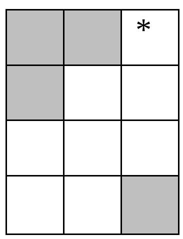
Then by expansion formula, the rook polynomial of C is

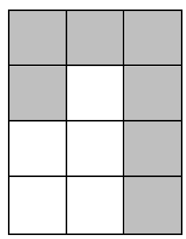
$$\begin{aligned} r(C, x) &= x\, r(D, x) + r(E, x) \\ &= x\, r(D, x) + r(E, x) \\ &= x(1 + 4x + 2x^2) + (1 + 8x + 14x^2 + 4x^3) \\ &= 1 + 9x + 18x^2 + 6x^3 \end{aligned}$$

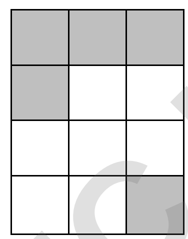
7. Find the rook polynomial for the given board using the expansion formula:



C D E







Let the given board be C and mark a particular square by $*$

Let D be the board obtained by deleting the row and the column containing the square $*$

The rook polynomial of D is $r(D, x) = 1 + 5x + 4x^2$

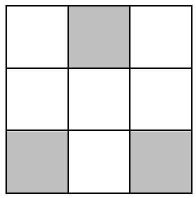
Let E be the board obtained by deleting only the square $*$

The rook polynomial of E is $r(E, x) = 1 + 7x + 11x^2 + 3x^3$

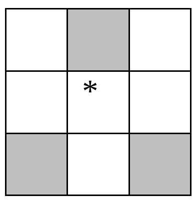
Then by expansion formula, the rook polynomial of C is

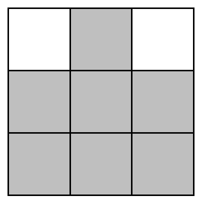
$$
\begin{aligned}
r(C, x) &= x\, r(D, x) + r(E, x) \\
&= x\, r(D, x) + r(E, x) \\
&= x(1 + 5x + 4x^2) + (1 + 7x + 11x^2 + 3x^3) \\
&= 1 + 8x + 16x^2 + 7x^3
\end{aligned}
$$

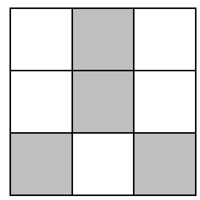
8. Find the rook polynomial for the given board using the expansion formula:



. C D E







Let the given board be C and mark a particular square by $*$

Let D be the board obtained by deleting the row and the column containing the square $*$

The rook polynomial of D is $r(D, x) = 1 + 2x$

Let E be the board obtained by deleting only the square $*$

The rook polynomial of E is $r(E, x) = 1 + 5x + 6x^2 + 2x^3$

Then by expansion formula, the rook polynomial of C is

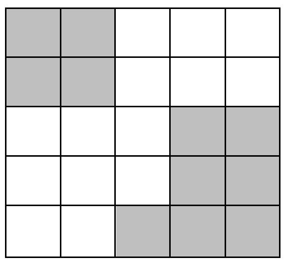
$$r(C, x) = x\, r(D, x) + r(E, x)$$
$$= x\, r(D, x) + r(E, x)$$
$$= x(1 + 2x) + (1 + 5x + 6x^2 + 2x^3)$$
$$= 1 + 6x + 8x^2 + 2x^3$$

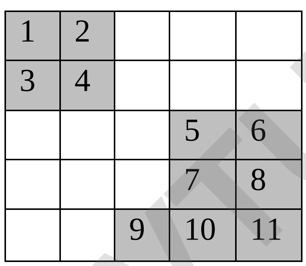
Product formula

Introduction:

- Suppose a board C is made up of two parts C_1 and C_2.
- If C_1 and C_2 have no squares in the same row or column of C, they are called disjoint sub boards of C.
- Suppose a board C is made up of two disjoint sub boards C_1 and C_2 then by product formula, the rook polynomial $r(C, x)$ is given by $r(C, x) = r(C_1, x) \times r(C_2, x)$.

9. Find the rook polynomials for the shaded parts of the given board:





Let C_1 be the board with squares from 1 to 4.

The rook polynomial of C_1 is $r(C_1, x) = 1 + 4x + 2x^2$.

Let C_2 be the board with squares from 5 to 11.

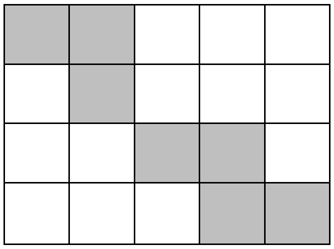
The rook polynomial of C_2 is $r(C_2, x) = 1 + 7x + 10x^2 + 2x^3$

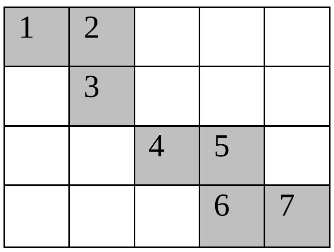
Clearly, C_1 and C_2 are disjoint sub boards of the given board C.

By product formula, Rook polynomial for the shaded parts of the board is

$$r(C, x) = r(C_1, x) \times r(C_2, x).$$
$$= (1 + 4x + 2x^2) \times (1 + 7x + 10x^2 + 2x^3)$$
$$= 1 + 11x + 40x^2 + 56x^3 + 28x^4 + 4x^5$$

10. Find the rook polynomials for the shaded parts of the given board:





Let C_1 be the board with squares from 1 to 3.

The rook polynomial of C_1 is $r(C_1, x) = 1 + 3x + x^2$.

Let C_2 be the board with squares from 4 to 7.

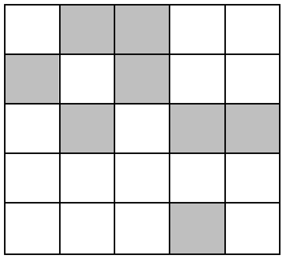
The rook polynomial of C_2 is $r(C_2, x) = 1 + 4x + 3x^2$

Clearly, C_1 and C_2 are disjoint sub boards of the given board C.

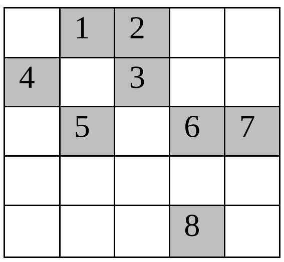
By product formula, Rook polynomial for the shaded parts of the board is

$$\begin{aligned} r(C, x) &= r(C_1, x) \times r(C_2, x). \\ &= (1 + 3x + x^2) \times (1 + 4x + 3x^2) \\ &= 1 + 7x + 16x^2 + 13x^3 + 3x^4 \end{aligned}$$

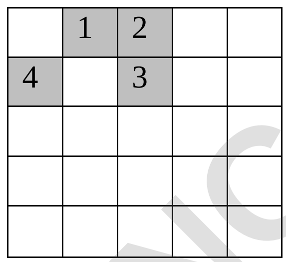
11. Find the rook polynomials for the shaded parts of the given board:



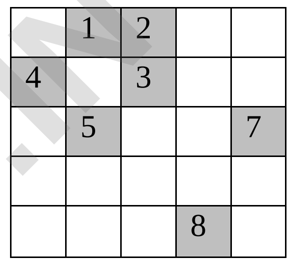
C



D



E



Since board C doesn't have disjoint sub boards, put * in square 6.

Let D be the board obtained by deleting the row and the column containing the square $*$

The rook polynomial of D is $r(D, x) = 1 + 4x + 3x^2$.

Let E be the board obtained by deleting only the square $*$

$E_1 -$ Board containing squares 1, 2, 3, 4, 5, 7. $r(E_1, x) = 1 + 6x + 10x^2 + 4x^3$

$E_2 -$ Board containing square 8. $r(E_2, x) = 1 + x$.

By product formula, The rook polynomial of E is

$$r(E, x) = r(E_1,\ x) \times r(E_2,\ x)$$

$$= (1 + 6x + 10x^2 + 4x^3\ ) \times (1 + x)$$

$$= 1 + 7x + 16x^2 + 14x^3 + 4x^4$$

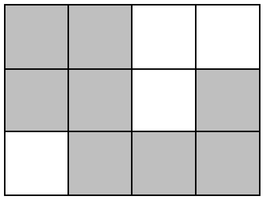
By expansion formula, $r(C, x) = x\, r(D, x) + r(E, x)$

$$= x(1 + 4x + 3x^2) + (1 + 7x + 16x^2 + 14x^3 + 4x^4)$$

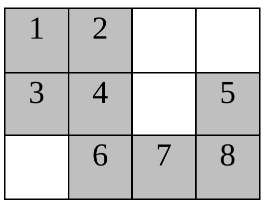
Therefore, Rook polynomial for the shaded parts of the board is

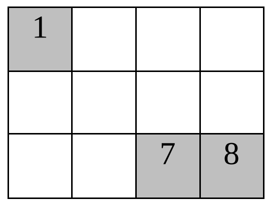
$r(C\ , x)\ = 1 + 8x + 20x^2 + 17x^3 + 4x^4$

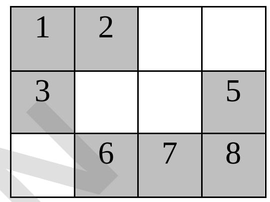
12. Find the rook polynomials for the shaded parts of the given board:



C D E







Since board C doesn't have disjoint sub boards, put * in square 4.

Let D be the board obtained by deleting the row and the column containing the square $*$

The rook polynomial of D is $r(D,x) = 1 + 3x + 2x^2$.

Let E be the board obtained by deleting only the square $*$

The rook polynomial of E is $r(E,x) = 1 + 7x + 13x^2 + 5x^3$

By expansion formula, Rook polynomial for the shaded parts of the board is

$$r(C,x) = x\, r(D,x) + r(E,x)$$

$$= x(1 + 3x + 2x^2) + (1 + 7x + 13x^2 + 5x^3)$$

$$= 1 + 8x + 16x^2 + 7x^3$$

4.4 Arrangements with forbidden positions

Introduction:

Suppose m objects are to be arranged in n places (n ≥ m). Suppose there are constraints under which some objects cannot occupy certain places. Such places are called forbidden positions.

The number of ways of arrangements with forbidden positions is given by

$\overline{N} = S_0 - S_1 + S_2 - S_3 + \cdots + (-1)^n S_n$, where $S_0 = n!\,, S_k = (n-k)!\ \times r_k$.

[r_k is the co efficient of x^k in the rook polynomial].

1. **An apple, a banana, a mango and an orange are to be distributed to four boys $\boldsymbol{B_1}$, $\boldsymbol{B_2}$, $\boldsymbol{B_3}$, $\boldsymbol{B_4}$. The boys $\boldsymbol{B_1}$, $\boldsymbol{B_2}$ do not wish to have apples, the boy $\boldsymbol{B_3}$ does not want banana or mango, and $\boldsymbol{B_4}$ refuses orange. In how many ways the distribution can be made so that no boy is displeased?**

	B_1	B_2	B_3	B_4
Apple	1	2		
Banana			3	
Mango			4	
Orange				5

This board C contains three disjoint sub-boards $C_1, C_2\ and\ C_3$.

C_1 is a sub-board containing squares 1 and 2. $r(C_1, x) = 1 + 2x$

C_2 is a sub-board containing squares 3 and 4. $r(C_2, x) = 1 + 2x$

C_3 is a sub-board containing square 5. $r(C_3, x) = 1 + x$

By product rule, $r(C, x) = r(C_1, x) \times r(C_2, x) \times r(C_3, x) = (1 + 2x)(1 + 2x)(1 + x)$

Therefore, Rook polynomial for the shaded parts of the board is

$r(C\,, x) = 1 + 5x + 8x^2 + 4x^3$

$S_0 = n! = 4! = 24$

$S_1 = (n-1)!\ \times r_1 = 3! \times 5 = 30$

$S_2 = (n-2)!\ \times r_2 = 2! \times 8 = 16$

$S_3 = (n-3)!\ \times r_3 = 1! \times 4 = 4$

$\overline{N} = S_0 - S_1 + S_2 - S_3 = 24 - 30 + 16 - 4 = 6.$

Therefore, No. of ways of distributing the fruits is 6.

2. Five teachers T_1, T_2, T_3, T_4, T_5 are to be made class teachers for five classes C_1, C_2, C_3, C_4, C_5 one teacher for each class. T_1 and T_2 do not wish to become the class teachers for C_1 or C_2. T_3 and T_4 for C_4 or C_5. In how many ways can the teachers be assigned the work without displeasing any teacher?

	C_1	C_2	C_3	C_4	C_5
T_1	1	2			
T_2	3	4			
T_3				5	6
T_4				7	8
T_5			9	10	11

Let $C_1 -$ Board with squares from 1 to 4, $C_2 -$ Board with squares from 5 to 11.

Clearly, C_1 and C_2 are disjoint sub boards of the given board C.

$r(C_1, x) = 1 + 4x + 2x^2$ and $r(C_2, x) = 1 + 7x + 10x^2 + 2x^3$

By product formula, $r(C, x) = r(C_1, x) \times r(C_2, x)$.

$r(C\,, x) = (1 + 4x + 2x^2) \times (1 + 7x + 10x^2 + 2x^3)$

$\qquad\quad = 1 + 11x + 40x^2 + 56x^3 + 28x^4 + 4x^5$

Therefore, Rook polynomial for the shaded parts of the board is

$\boldsymbol{r(C\,, x) = 1 + 11x + 40x^2 + 56x^3 + 28x^4 + 4x^5}$

$S_0 = n! = 5! = 120$

$S_1 = (n-1)! \times r_1 = 24 \times 11 = 264$

$S_2 = (n-2)! \times r_2 = 6 \times 40 = 240$

$S_3 = (n-3)! \times r_3 = 2 \times 56 = 112$

$S_4 = (n-4)! \times r_2 = 1 \times 28 = 28$

$S_5 = (n-5)! \times r_3 = 1 \times 4 = 4.$

$\overline{N} = S_0 - S_1 + S_2 - S_3 + S_4 - S_5 = 120 - 264 + 240 - 112 + 28 - 4 = 8.$

Therefore, the no. of ways in which the work can be assigned = 8.

3. **Four persons P_1 , P_2 , P_3 , P_4 who arrive late for a dinner party. Find that only one chair at each of five tables T_1 , T_2 , T_3 , T_4 and T_5 is vacant. P_1 will not sit at T_1 *or* T_2 , P_2 will not sit at T_2 , P_3 will not sit at T_3 *or* T_4 and P_4 will not sit at T_4 *or* T_5. Find the number of ways they can occupy the vacant chairs.**

	T_1	T_2	T_3	T_4	T_5
P_1	1	2			
P_2		3			
P_3			4	5	
P_4				6	7

Let $C_1 -$ Board with squares from 1 to 3, $C_2 -$ Board with squares from 4 to 7.

Clearly, C_1 and C_2 are disjoint sub boards of the given board C.

$r(C_1, x) = 1 + 3x + x^2$ and $r(C_2, x) = 1 + 4x + 3x^2$

By product formula, $r(C, x) = r(C_1, x) \times r(C_2, x)$.

$r(C\ , x) = (1 + 3x + x^2\ ) \times (1 + 4x + 3x^2) = 1 + 7x + 16x^2 + 13x^3 + 3x^4$

Therefore, Rook polynomial for the shaded parts of the board is

$r(C\ , x)\ = 1 + 7x + 16x^2 + 13x^3 + 3x^4$.

$S_0 = n! = 5! = 120$

$S_1 = (n-1)!\ \times r_1 = 24 \times 7 = 168$

$S_2 = (n-2)!\ \times r_2 = 6 \times 16 = 96$

$S_3 = (n-3)!\ \times r_3 = 2 \times 13 = 26$

$S_4 = (n-4)!\ \times r_2 = 1 \times 3 = 3$

$\overline{N} = S_0 - S_1 + S_2 - S_3 + S_4 = 120 - 168 + 96 - 26 + 3 = 25$.

Therefore, the no. of ways in which the four persons can be the chairs$= 25$.

4. **A girl student has sarees of 5 different colors; blue, green, red, white and yellow. On Mondays she does not wear green; On Tuesdays blue or red; On Wednesdays blue or green; On Thursdays red or yellow; On Fridays red. In how many ways can she dress without repeating a color during a week? (Mon – Fri).**

	Mon	Tue	Wed	Thu	Fri
B		1	2		
G	4		3		
R		5		6	7
W					
Y				8	

C

	1	2		
4		3		

D

	1	2		
4		3		
	5			7
			8	

E

Since board C doesn't have disjoint sub boards, put * in square 6.

Let D be the board obtained by deleting the row and the column containing the square $*$

The rook polynomial of D is $r(D,x) = 1 + 4x + 3x^2$.

Let E be the board obtained by deleting only the square $*$

$E_1 -$ Board containing squares 1, 2, 3, 4, 5, 7. $r(E_1, x) = 1 + 6x + 10x^2 + 4x^3$

$E_2 -$ Board containing square 8. $r(E_2, x) = 1 + x$.

By product formula, The rook polynomial of E is

$$r(E,x) = r(E_1,\ x) \times r(E_2,\ x)$$
$$= (1 + 6x + 10x^2 + 4x^3\ ) \times (1 + x)$$
$$= 1 + 7x + 16x^2 + 14x^3 + 4x^4$$

By expansion formula, $r(C,x) = x\, r(D,x) + r(E,x)$

$$= x(1 + 4x + 3x^2) + (1 + 7x + 16x^2 + 14x^3 + 4x^4)$$

Therefore, Rook polynomial for the shaded parts of the board is

$r(C\ ,x)\ = 1 + 8x + 20x^2 + 17x^3 + 4x^4$

$S_0 = n! = 5! = 120$, $S_1 = (n-1)!\ \times 8 = 24 \times 8 = 192$

$S_2 = (n-2)!\ \times r_2 = 6 \times 20 = 120$, $S_3 = (n-3)!\ \times r_3 = 2 \times 17 = 34$

$S_4 = (n-4)!\ \times r_2 = 1 \times 4 = 4$

$\bar{N} = S_0 - S_1 + S_2 - S_3 + S_4 = 120 - 192 + 120 - 34 + 4 = 18.$

Therefore, the no. of ways of dressing= 18.

4.5 First order recurrence relations

Introduction:

- Any relation connecting the general term of the sequence with its preceding term is called recurrence relation.

  Example: $a_n = 3a_{n-1}, a_0 = 4, n \geq 1$ represents the sequence 4, 12, 36, 108, …
- General solution of the first order homogeneous recurrence relation

  $a_{n+1} = c\, a_n, n \geq 0$ is $a_n = c^n a_0$.
- General solution of the first order non-homogeneous recurrence relation

  $a_{n+1} = c\, a_n + f(n+1), n \geq 0$ is $a_n = c^n a_0 + \sum_{k=1}^{n} c^{n-k} f(k)$.

1. Solve the recurrence relation $4a_n - 5a_{n-1} = 0, n \geq 0, a_0 = 1$

Rewrite the recurrence relation as $a_{n+1} = \frac{5}{4} a_n, n \geq 0$

The general solution is $a_n = \left(\frac{5}{4}\right)^n a_0,\ n \geq 1$. Put $a_0 = 1$.

This implies that $a_n = \left(\frac{5}{4}\right)^n$.

The general solution is $a_n = \left(\frac{5}{4}\right)^n a_0,\ n \geq 1$.

2. Solve the recurrence relation $3a_{n+1} - 4a_n = 0, n \geq 0, a_1 = 5$

Rewrite the recurrence relation as $a_{n+1} = \frac{4}{3} a_n, n \geq 0$

The general solution is $a_n = \left(\frac{4}{3}\right)^n a_0,\ n \geq 1$.

Put $n = 1,\ we\ get\ a_1 = \left(\frac{4}{3}\right) a_0$

Put $a_1 = 5, we\ get\ 5 = \left(\frac{4}{3}\right) a_0$

This implies that $a_0 = \frac{15}{4}$.

Therefore, the general solution is $a_n = \left(\frac{4}{3}\right)^n \left(\frac{15}{4}\right),\ n \geq 1$.

3. Solve the recurrence relation $a_n = 7a_{n-1}, n \geq 1, a_2 = 98$

Rewrite the recurrence relation as $a_{n+1} = 7a_n \,, n \geq 0$

The general solution is $a_n = 7^n a_0,\ n \geq 1.$

Put $n = 2,\ we\ get\ a_2 = 7^2 a_0$

Put $a_2 = 98,\ we\ get\ 98 = (49)a_0$

This implies that $a_0 = \frac{98}{49} = 2$.

Therefore, the general solution is $a_n = (7)^n(2) \,,\ n \geq 1$.

4. Solve the recurrence relation $a_n = na_{n-1},\ n \geq 1,\ a_0 = 1$

$a_1 = 1.a_0 = 1!\,a_0$

$a_2 = 2.a_1 = 2.1.a_0 = 2!\,a_0$

$a_3 = 3.a_2 = 3.2.1.a_0 = 3!\,a_0$

$and\ so\ on.$

Therefore, the general solution is $a_n = n!\,a_0,\ n \geq 1.$

5. Solve the recurrence relation If $a_{n+1} = ka_n, n \geq 0, a_3 = \frac{153}{49}$, $a_5 = \frac{1377}{2401}$, find k.

The general solution is $a_n = k^n a_0 \,,\ n \geq 1.$

Put $n = 3,\ we\ get\ a_3 = k^3 a_0$

Put $n = 5,\ we\ get\ a_5 = k^5 a_0$

Therefore, $\frac{a_5}{a_3} = k^2$

This implies that $k^2 = \frac{a_5}{a_3} = \frac{1377}{2401} \times \frac{49}{153} = \frac{9}{49}$

Therefore, $k = \pm\frac{3}{7}$.

6. **A bank pays 6% annual interest on savings, compounding the interest, if a person deposits 1000 on first day of March, how much will this deposit be worth a year later?**

By data, Monthly interest= $\frac{6\%}{12} = 0.005$ and $a_0 = 1000$

The recurrence relation is $a_n = a_{n-1} + 0.005\ a_{n-1}\ , n \geq 0$

This implies that $a_n = 1.005\ a_{n-1}, n \geq 0.$

The general solution is $a_n = 1.005^n a_0\ ,\ n \geq 1.$

One year later,

Put $n = 12,\ we\ get\ a_{12} = 1.005^{12} a_0 = 1.005^{12} \times 1000$

Therefore, one year later, deposit is worth Rs. 1061

7. **The number of virus affected files in a system is 1000 and their increase 250% every two hours, use the recurrence relation to determine the number of virus affected files in a system after one day?**

By data, $a_0 = 1000$

The recurrence relation is $a_n = a_{n-1} + 250\%\ of\ a_{n-1}\ , n \geq 0$

This implies that $a_n = 3.5\ a_{n-1},\ n \geq 0$

Rewrite the recurrence relation as $a_{n+1} = 3.5 a_n\ , n \geq 0$

The general solution is $a_n = 3.5^n a_0,\ n \geq 1.$

After one day , Put $n = 12,\ we\ get\ a_{12} = 3.5^{12} a_0 = 3.5^{12} \times 1000$

Therefore, after one day, the number of virus affected files in a system is $3.5^{12} \times 1000.$

8. **Find the recurrence relation and initial condition for the sequence $2, 10, 50, 250, \ldots$**

By data, $a_0 = 2$

$a_1 = 5(2) = 5a_0$

$a_2 = 5(10) = 5a_1$

$a_3 = 5(50) = 5a_2\quad And\ so\ on.$

The recurrence relation is $a_n = 5a_{n-1}\ , n \geq 1$

Rewrite the recurrence relation as $a_{n+1} = 5a_n\ , n \geq 0$

The general solution is $a_n = 5^n a_0,\ n \geq 1$. Put $a_0 = 2$.

Therefore, the general solution is $a_n = 5^n(2),\ n \geq 1.$

9. Solve the recurrence relation $a_n - 3a_{n-1} = 5 \times 7^n, n \geq 1$ given that $a_0 = 2$.

Rewrite the recurrence relation as $a_{n+1} = 3a_n + (5 \times 7^{n+1})$, $n \geq 0$

The general solution is $a_n = 3^n a_0 + \sum_{k=0}^{n} 3^{n-k} f(k)$

$$= 3^n a_0 + \sum_{k=0}^{n} 3^{n-k}(5 \times 7^k)$$

$$= 3^n \left\{ a_0 + 5 \sum_{k=0}^{n} \left(\frac{7}{3}\right)^k \right\}$$

$$= 3^n \left\{ 2 + 5 \times \left(\frac{7}{3}\right) \sum_{k=1}^{n} \left(\frac{7}{3}\right)^{k-1} \right\}$$

$$= 3^n \left\{ 2 + \frac{35}{3} \times \frac{\left(\frac{7}{3}\right)^n - 1}{\frac{7}{3} - 1} \right\}$$

$$= 2 \times 3^n + 35 \left(\frac{7^n - 3^n}{7 - 3} \right)$$

Therefore, the general solution is $a_n = \frac{5}{4}(7)^{n+1} - \frac{1}{4}(3)^{n+3},\ n \geq 1.$

10. Solve the recurrence relation $a_n - 3a_{n-1} = 5 \times 3^n, n \geq 1$ given that $a_0 = 2$.

Rewrite the recurrence relation as $a_{n+1} = 3a_n + (5 \times 3^{n+1})$, $n \geq 0$

The general solution is

$$a_n = 3^n a_0 + \sum_{k=0}^{n} 3^{n-k} f(k)$$

$$= 3^n a_0 + \sum_{k=0}^{n} 3^{n-k}(5 \times 3^k)$$

$$= 3^n \left\{ 2 + 5 \sum_{k=0}^{n} 1 \right\}$$

$$= 3^n \{2 + 5n\}$$

Therefore, the general solution is $a_n = 3^n\{2 + 5n\},\ n \geq 1.$

11. Find the recurrence relation and initial condition for the sequence $\mathbf{0, 2, 6, 12, 20, \ldots}$

$a_1 - a_0 = 2 - 0 = 2 = 2(1)$

$a_2 - a_1 = 6 - 2 = 4 = 2(2)$

$a_3 - a_2 = 12 - 6 = 6 = 2(3)$

$and\ so\ on.$

Therefore, the recurrence relation is $a_n - a_{n-1} = 2n,\ n \geq 0$

Rewrite the general solution as $a_{n+1} = a_n + 2(n+1), n \geq 1$

The general solution is

$$a_n = 1^n a_0 + \sum_{k=0}^{n} 1^{n-k} f(k)$$

$$= 0 + \sum_{k=0}^{n} (2k)$$

$$= n(n+1)$$

Therefore, the general solution is $a_n = n(n+1), n \geq 0.$

4.6 Second order homogeneous recurrence relations

Introduction:

- Characteristic equation of the second order homogeneous recurrence relation $c_n a_n + c_{n-1}a_{n-1} + c_{n-2}a_{n-2} = 0,\ n \geq 2$ is
  $c_n k^2 + c_{n-1}k + c_{n-2} = 0$.

- The general solution is

  (i) $a_n = Ak_1^n + Bk_2^n$, if k_1 and k_2 are real nos. and distinct.

  (ii) $a_n = (A + Bn)k^n$, if k_1 and k_2 are real nos. and equal.

  (iii) $a_n = r^n(A\cos n\theta + B\sin n\theta)$, if k_1 and k_2 are complex nos.

1. Solve the recurrence relation: $\boldsymbol{a_n - 6a_{n-1} + 9a_{n-2} = 0,\ n \geq 2, a_0 = 5, a_1 = 12}$.

The characteristic equation is $k^2 - 6k + 9 = 0$.

The general solution is $a_n = (A + Bn)3^n$

Put $n = 0, a_0 = A \Rightarrow A = 5$.

Put $n = 1, a_1 = (5 + B)3 \Rightarrow B = -1$.

Therefore, $a_n = (5 - n)3^n$.

2. Solve the recurrence relation: $\boldsymbol{2a_n = 7a_{n-1} - 3a_{n-2}, n \geq 2, a_0 = 2, a_1 = 5}$.

The characteristic equation is $2k^2 - 7k + 3 = 0$.

The general solution is $a_n = A\left(\frac{1}{2}\right)^n + B(3)^n$

Put $n = 0, a_0 = A + B \Rightarrow A + B = 2$.

Put $n = 1, a_1 = \frac{A}{2} + 3B \Rightarrow A + 6B = 10$.

By solving we get $A = \frac{2}{5}$, $B = \frac{8}{5}$.

Therefore, $a_n = \frac{1}{5}(8(3)^n + 2^{1-n})$.

3. **Solve the recurrence relation $a_n + a_{n-1} - 6a_{n-2} = 0,\ n \geq 2, a_0 = -1, a_1 = 8$.**

The characteristic equation is $k^2 + k - 6 = 0$.

The general solution is $a_n = A(-3)^n + B(2)^n$

Put $n = 0, a_0 = A + B \Rightarrow A + B = -1$.

Put $n = 1, a_1 = -3A + 2B \Rightarrow -3A + 2B = 8$.

By solving we get $A = -2\ ,\ B = 1$.

Therefore, $a_n = (-2)(-3)^n + 2^n$.

4. **Solve the recurrence relation $a_n = 3a_{n-1} - 2a_{n-2},\ n \geq 2, a_1 = 5, a_2 = 3$.**

The characteristic equation is $k^2 - 3k + 2 = 0$.

The general solution is $a_n = A(2)^n + B(1)^n$

Put $n = 1, a_1 = 2A + B \Rightarrow 2A + B = 5$.

Put $n = 2, a_2 = 4A + B \Rightarrow 4A + B = 3$.

By solving we get $A = -1\ ,\ B = 7$.

Therefore, $a_n = -(2)^n + 7$.

5. **Solve the recurrence relation $a_{n+2}^2 - 5a_{n+1}^2 + 4a_n^2 = 0,\ n \geq 0, a_0 = 4, a_1 = 13$.**

Rewrite the equation as $b_n - 5b_{n-1} + 4b_{n-2} = 0\ , n \geq 2$

The characteristic equation is $k^2 - 5k + 4 = 0$.

The general solution is $b_n = A(4)^n + B(1)^n$

Put $n = 0, b_0 = A + B \Rightarrow A + B = 16$.

Put $n = 1, b_1 = 4A + B \Rightarrow 4A + B = 169$.

By solving we get $A = 51\ ,\ B = -35$.

Therefore, $b_n = (51)4^n + (-35)1^n$. $a_n = \pm\sqrt{(51)4^n - 35}$

6. Solve the recurrence relation $F_{n+2} = F_{n+1} + F_n$, $n \geq 0, F_0 = 0, F_1 = 1$.

Rewrite the equation as $F_n - F_{n-1} - F_{n-2} = 0$, $n \geq 2$, $F_0 = 0, F_1 = 1$.

The characteristic equation is $k^2 - k - 1 = 0$.

The general solution is $F_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$

Put $n = 0, F_0 = A + B \Rightarrow A + B = 0 \Rightarrow B = -A$.

Put $n = 1, F_1 = A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right) \Rightarrow A\left(\frac{1+\sqrt{5}}{2}\right) - A\left(\frac{1-\sqrt{5}}{2}\right) = 1$

By solving we get $A = \frac{1}{\sqrt{5}}$, $B = -\frac{1}{\sqrt{5}}$.

Therefore, $F_n = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}$

7. Solve the recurrence relation $a_{n+2} = (a_{n+1})(a_n), n \geq 0$, $a_0 = 1$, $a_1 = 2$

Rewrite the equation as

$\log a_{n+2} = \log a_{n+1} + \log a_n$, $n \geq 0$.

$b_{n+2} = b_{n+1} + b_n$, $n \geq 0$

$b_n - b_{n-1} - b_{n-2} = 0, n \geq 0$

The characteristic equation is $k^2 - k - 1 = 0$.

The general solution is $b_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$

Put $n = 0, b_0 = A + B \Rightarrow A + B = \log a_0 \Rightarrow B = -A$.

$[\because \log a_0 = \log 1 = 0]$

Put $n = 1, b_1 = A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right) \Rightarrow A\left(\frac{1+\sqrt{5}}{2}\right) - A\left(\frac{1-\sqrt{5}}{2}\right) = 1$

$[\because b_1 = \log a_0 = \log_2 2 = 1]$

By solving we get $A = \frac{1}{\sqrt{5}}$, $B = -\frac{1}{\sqrt{5}}$.

Therefore, $b_n = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}$

8. Find the recurrence relation for the number of binary sequences of length n ($n \geq 1$) that has no consecutive zeroes.

$If\ n = 1$, there are 2 binary sequences $0, 1$. Therefore, $a_1 = 2$.

$If\ n = 2$, there are 3 binary sequences 01, 10,11. Therefore, $a_2 = 3$.

Consider a binary sequence of length n.

If n^{th} entry is 1, for the remaining $n - 1$ entries, there are a_{n-1} binary sequences.

If n^{th} entry is 0, the preceding entry must be 1. For the remaining $n - 2$ entries, there are a_{n-2} binary sequences.

Therefore, $a_n = a_{n-1} + a_{n-2}$ is the required recurrence relation.

Rewrite the equation as $a_n - a_{n-1} - a_{n-2} = 0\ , n \geq 2\ , a_1 = 2, a_2 = 3$.

The characteristic equation is $k^2 - k - 1 = 0$.

The general solution is $a_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$

Put $n = 1, a_1 = A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right) \Rightarrow (A + B) + \sqrt{5}(A - B) = 4$

Put $n = 2, a_2 = A\left(\frac{1+\sqrt{5}}{2}\right)^2 + B\left(\frac{1-\sqrt{5}}{2}\right)^2 \Rightarrow 3(A + B) + \sqrt{5}(A - B) = 6$

By solving we get $A = \frac{\sqrt{5}+3}{2\sqrt{5}}\ ,\ B = \frac{\sqrt{5}-3}{2\sqrt{5}}$.

Therefore, $a_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n, where\ A = \frac{\sqrt{5}+3}{2\sqrt{5}}\ , B = \frac{\sqrt{5}-3}{2\sqrt{5}}$.