Module 1: Mathematical Logic

- Propositions: A **proposition** is a declarative sentence (a sentence that declares a fact) that is either true or false, but not both.
- Are the following sentences propositions?
  - Bengaluru is the capital of Karnataka ....Yes
  - Read this carefully. ....No
  - 1+2=3 .....Yes
  - x+1=2 ...... No
  - What time is it? ..... No
- Propositional Logic – the area of logic that deals with propositions
- Propositional Variables – variables that represent propositions: *p*, *q*, *r*, *s*
  - E.g. Proposition *p* – "Today is Friday."
- Truth values – T, F
- Let *p* be a proposition. The negation of *p*, denoted by ¬*p*, is the statement"It is not the case that *p*."
- The proposition ¬*p* is read "not *p*." The truth value of the negation of *p*, ¬*p* is the opposite of the truth value of *p*.

The Truth Table for the Negation of a Proposition.	
p	¬*p*
T F	F T

- Let *p* and *q* be propositions. The *conjunction* of *p* and *q*, denoted by *p* ∧ *q*, is the proposition "*p* and *q*". The conjunction *p* ∧ *q* is true when both *p* and *q* are true and is false otherwise

- Let *p* and *q* be propositions. The *disjunction* of *p* and *q*, denoted by *p* v *q*, is the proposition "*p* or *q*". The conjunction *p* v *q* is false when both *p* and *q* are false and is true otherwise

- Let *p* and *q* be propositions. The *exclusive or* of *p* and *q*, denoted by *p*   *q*, is the proposition that is true when exactly one of *p* and *q* is true and is false otherwise.

- Let *p* and *q* be propositions. The *conditional statement p* → *q*, is the proposition "if *p*, then *q*." The conditional statement is false when *p* is true and *q* is false, and true otherwise. In the conditional statement *p* → *q*, *p* is called the *hypothesis* (or *antecedent* or *premise*) and *q* is called the *conclusion* (or *consequence*).

p	**q**	**p ∧ q** p AND q Conjunction	**p ∨ q** p OR a Disjuntion	**p⊕q** p Ex-OR q	**p→q** Conditional	**p↔q** Binconditional	**¬p** Negation	**¬q**
T	**T**	**T**	**T**	**F**	**T**	**T**	**F**	**F**
T	**F**	**F**	**T**	**T**	**F**	**F**	**F**	**T**
F	**T**	**F**	**T**	**T**	**T**	**F**	**T**	**F**
F	**F**	**F**	**F**	**F**	**T**	**T**	**T**	**T**

Examples

1) Let p and q be the primitive statements for which the conditional p→q is false . determine the truth values of the following compound propositions

p→q is false

p is true   q is false

(i) p∧ q   is false

(ii) ¬p ∧ q   is false

2)Find the Truth values of of p,q and r in the following cases

(i) p → (q ∨ r) is false when p is T and   (q ∨ r)   is F

(q ∨ r) wil be F when both are False

so P is true , q and r are false

(ii) $p \wedge (q \to r)$ is true when p is true and $(q \to r)$ is true

$(q \to r)$ is true whenever,

(i)q is true ,r must be true

(ii) when q is false then r can be true or false

3)Construct the truth table for the following compound propositions

(i) $(p \wedge q) \to (\neg r)$

(ii) $q \wedge ((\neg r) \to p)$

p	q	r	p∧q	¬ r	(p∧q) → (¬ r)
0	0	0	0	1	1
0	0	1	0	0	1
0	1	0	0	1	1
0	1	1	0	0	1
1	0	0	0	1	1
1	0	1	0	0	1
1	1	0	1	1	1
1	1	1	1	0	0

(ii) $q \wedge ((\neg r) \to p)$

p	q	r	¬ r	((¬ r) → p)	q ∧ ((¬ r) → p)
0	0	0	1		0
0	0	1	0		0
0	1	0	1		0
0	1	1	0		1
1	0	0	1		0
1	0	1	0		0
1	1	0	1		1
1	1	1	0		1

4) If a proposition **q has the truth value 1**, determine all the truth value assignments for the primitive propositions p,r and s for which the truth value of the following compound proposition is 1

solution:

Given:

$[q \to \{(\neg p \vee r) \wedge \neg s\}] \wedge \{\neg s \to (\neg r \wedge q)\}$ truth value of this is true and q is also true

$x \equiv [q \rightarrow \{(\neg p \vee r) \wedge \neg s\}]$

$y \equiv \{\neg s \rightarrow (\neg r \wedge q)\}$

$x \wedge y$ is true i.e x is true and y is true

$[q \rightarrow \{(\neg p \vee r) \wedge \neg s)]$ is true ..... since q is true $\{(\neg p \vee r) \wedge \neg s)]$ should be true...

$\{(\neg p \vee r) \wedge \neg s)]$ is true when $(\neg p \vee r)$ is true and $\neg s$ is true i.e **s is false**

$\{\neg s \rightarrow (\neg r \wedge q)]$ since $\neg s$ is true then $\{\neg s \rightarrow (\neg r \wedge q)]$ will be true when $(\neg r \wedge q)$ is true

$(\neg r \wedge q)$ will be true when $\neg r$ is true and q is true .... i.e **r is false**

$(\neg p \vee r)$ is true when $\neg p$ is true because r is false

Therefore **p is false**

i.e p,r,s are **false**

5) Indicate how many rows are needed in the truth table for the compound proposition

$(p \vee \neg q) \leftrightarrow \{(\neg r \wedge s) \rightarrow t)$.

Find the truth value of the propositions if p and r are true and q,s,t are false

Solution:

number of variables is 5 ,therefore $2^5 = 32$ rows needed

$(p \vee \neg q) \leftrightarrow \{(\neg r \wedge s) \rightarrow t)$.

$\underline{(p \vee \neg q)}$ is be true because p is true and $\neg$ q is true

$(\neg r \wedge s)$ is false because $\neg$ r is false and s is also false

$\underline{\{(\neg r \wedge s) \rightarrow t)}$ is true because $(\neg r \wedge s)$ is false and t is false

therefore $(p \vee \neg q) \leftrightarrow \{(\neg r \wedge s) \rightarrow t)$ is true (because $(p \vee \neg q)$ is true and $\{(\neg r \wedge s) \rightarrow t)$ is true

Tautology,Contradiction,Contigency:

Tautology:

A Compound proposition which is True for all possible truth values of its components is called as a tautology

Contradiction:

A Compound proposition which is False for all possible truth values of its components is called as a contradiction

Contigency:

A Compound proposition which is the combination truth values True and False for all possible truth values of its components is called as a contigency. It is neither Tautology nor contradiction

Question:

Prove that ,for any propositions p,q,r the compound proposition

$[(p \to q) \wedge (q \to r)] \to (p \to r)$ is a Tautology

Solution:

p	q	r	$p \to q$	$q \to r$	$[(p \to q) \wedge (q \to r)]$	$p \to r$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$
0	0	0	1	1	1	1	1
0	0	1	1	1	1	1	1
0	1	0	1	0	0	1	1
0	1	1	1	1	1	1	1
1	0	0	0	1	0	0	1
1	0	1	0	1	0	1	1
1	1	0	1	0	0	0	1
1	1	1	1	1	1	1	1

Find the possible truth values of p,q,r,s,t for which the following are contradiction

$[(p \wedge q) \wedge r] \to (s \vee t)$

solution:

since $[(p \wedge q) \wedge r] \to (s \vee t)$ is contradiction ,i.e truth vale of $[(p \wedge q) \wedge r] \to (s \vee t)$ is false

$[(p \wedge q) \wedge r] \to (s \vee t)$ will be false when $[(p \wedge q) \wedge r]$ is true and $(s \vee t)$ is false

now, $(s \vee t)$ wil be false when both **s and t are false** now, **[(p**∧ q) ∧ r] is true when **(p**∧ q) is true and **r is true**

now, **(p**∧ q) is true when both **p and q are true**

Truth values of p,q,r are true

s and t are false

Logical Equivalence:

Two propositions u and v are said to be logically equivalent whenever u and v have the same truth values. In other words when u ↔ v is a Tautology.

logical equivalence is represented by the symbol

$\Longleftrightarrow$

Question: For any two propositions p,q prove that (p → q) ↔¬p ∨ q

p	q	¬p	p → q **u**	¬p ∨ q **v**	(p → q) ↔¬p ∨ q **u↔v**
0	0	1	1	1	1
0	1	1	1	1	1
1	0	0	0	0	1
1	1	0	1	1	1

prove that ,for any three propositions p,q,r

[(p∨q) →r] ↔[(p → r) ∧ (q → r)]

Laws of Logic (Equivalence laws)

- Identity laws, $P \wedge T \equiv P$ $\quad P \vee F \equiv P$
- Inverse law $P \wedge \neg P \equiv F$ $\quad P \vee \neg P \equiv T$
- Domination laws, $P \wedge F \equiv F$, $\quad P \vee T \equiv T$
- Idempotent laws, $P \wedge P \equiv P$, $\quad P \vee P \equiv P$
- Double negation law, $\neg(\neg P) \equiv P$
- Commutative laws, $P \wedge Q \equiv Q \wedge P$, $\quad P \vee Q \equiv Q \vee P$
- Associative laws, $P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$,

  $P \vee (Q \vee R) \equiv (P \vee Q) \vee R$
- Distributive laws, $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$,

  $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$,
- De Morgan's laws, $\neg(P \wedge Q) \equiv (\neg P) \vee (\neg Q)$ $\quad \neg(P \vee Q) \equiv (\neg P) \wedge (\neg Q)$
- Absorption Law $[p \vee (p \wedge q) \equiv p$ $\quad [p \wedge (p \vee q) \equiv p$
- Law with implication (Conditional) $P \to Q \equiv \neg P \vee Q$
- Law for negation of a conditional $\neg (P \to Q) \equiv P \wedge \neg Q$
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Question 1) Let x be a specified number . Write the negation of the following conditional

"If x is an integer, then x is a rational number"

solution: Let p: x is an integer, q: x is a rational numbe

in symbolic form given statement can be written as

$$\mathbf{p \to q}$$

i.e $\quad \mathbf{p \to q} \equiv \neg p \vee q$

i.e. $\quad \neg (P \to Q) \equiv \mathbf{P \wedge \neg Q}$

therefore the negation of the given statement reads as,

"x is an integer and it is not a rational number "

2) Let x be a specified number. Write the down the negation of the following proposition :

"If x is not a real number, then it is not a rational and not an irrational number"

solution:

p: x is a real number q: x is a rational number r: x is irrational number

in symbolic form given statement can be written as

$$\neg p \rightarrow (\neg q \wedge \neg r)$$

negation of this is

$$\neg \{ \neg p \rightarrow (\neg q \wedge \neg r)\} \equiv \neg \{ \neg (\neg p \vee (\neg q \wedge \neg r)\}$$

$$\equiv \neg \{ p \vee (\neg q \wedge \neg r)\}$$

$$\equiv \neg p \wedge \neg (\neg q \wedge \neg r)$$

$$\equiv \neg p \wedge (q \vee r)$$

therefore the negation of the given statement reads as,

"x is a not real number and it is rational number or it is a real number "

3) Simplify the following compound propositions using the laws of logic:

i) $(p \vee q) \wedge [\neg\{(\neg p) \wedge q\}]$ (ii) $(p \vee q)\ [\neg\{(\neg p) \vee q\}]$ (iii) $\neg[\neg\{(p \vee q) \wedge r\} \vee \neg q]$

Solution:

i) $(p \vee q) \wedge [\neg\{(\neg p) \wedge q\}] \equiv (p \vee q) \wedge [\neg\neg p \vee \neg q]$ D'Morgan Law

$\equiv (\mathbf{p} \vee q) \wedge [\mathbf{p} \vee \neg q]$ Law of double negation

$\equiv \mathbf{p} \vee (q \wedge \neg q)$ distribution law

$\equiv \mathbf{p} \vee \mathbf{F}$ Inverse law

$\equiv \mathbf{p}$ Identity Law

(ii) Prove that $[(P \vee Q) \wedge \neg(\neg P \wedge (\neg Q \vee \neg R)\}] \vee (\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R)$ is a Tautology

Solution:

Let z denote the given proposition Then we have $z \equiv x \vee y$, where ,

$x \equiv [(P \vee Q) \wedge \neg(\neg P \wedge (\neg Q \vee \neg R)\}]$ and $y \equiv (\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R)$

By using the laws of logic we find that

$x \equiv [(P \vee Q) \wedge \neg(\neg P \wedge (\neg Q \vee \neg R)\}]$

$\equiv [(P \vee Q) \wedge \neg(\neg P \wedge \neg(Q \wedge R)\}]$ (D'Morgan law)

$\equiv [(P \vee Q) \wedge (P \vee (Q \wedge R)\}]$ (D'Morgan law)

$\equiv P \vee (Q \wedge (Q \wedge R)) \equiv P \vee ((Q \wedge Q) \wedge R)$ (Distributive law) & (Associative law)

$\equiv \mathbf{P \vee (Q \wedge R)}$ (idempotent law)

and

$y \equiv (\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R)$
$\equiv \neg (P \vee Q) \vee \neg (P \vee R)$ (D'Morgan law)
$\equiv \neg \{ (P \vee Q) \wedge (P \vee R) \}$ (D'Morgan law)
$\equiv \neg \{ \mathbf{P \vee (Q \wedge R)} \}$ (Distributive law)
$\equiv \neg x$

Therefore , $z \equiv x \vee y \equiv x \vee (\neg x) \equiv T$

Hence given proposition is a Tautology

Duality:

Suppose u is a compound proposition that contains the connectives $\wedge$ and $\vee$. Suppose we replace each occurrence of $\wedge$ and $\vee$ in u by $\vee$ and $\wedge$ respectively . Also if u contains T and F as components ,suppose we replace each occurrence of T and F by F and T respectively. Then the resulting compound proposition is called the dual of u and is denoted by $\mathbf{u^d}$

For example,

suppose u: $p \wedge (q \vee \neg r) \vee (s \wedge T)$

then the dual of u is u^d : $p \vee (q \wedge \neg r) \wedge (s \vee F)$

Note:

1) $(u^d)^d \equiv u$ (i.e. dual of dual of u is equivalent to u)

2) For any two propositions, u and v ,if $u \equiv v$ then $u^d \equiv v^d$ (Called as principal of duality)

Write down the duals of the following

i) $[(p \vee T) \wedge (q \vee F)] \vee [(r \wedge s) \wedge T]$

Solution: dual is $[(p \wedge F) \vee (q \wedge T)] \wedge [(r \vee s) \vee F]$

ii) $p \rightarrow q$

Solution:

$p \rightarrow q \equiv \neg p \vee q$ (law of conditional)

so the dual is $\neg p \wedge q$

iii) $(p \to q) \to r$

Solution: $(p \to q) \to r \equiv \neg(p \to q) \lor r$ (law of conditional)

$\equiv \neg(\neg p \lor q) \lor r$ (law of conditional)

$\equiv (p \land \neg q) \lor r$ (D Morgan's Law)

dual of this is $(p \lor \neg q) \land r$

iii) Prove that $[(\neg p \lor q) \land (p \land (p \land q)] \equiv p \land q$
Hence deduce that $[(\neg p \land q) \lor (p \lor (p \lor q)] \equiv p \lor q$
Solution: We have

$[(\neg p \lor q) \land (p \land (p \land q)] \equiv [(\neg p \lor q) \land ((p \land p) \land q]$ (Associative law)

$\equiv (\neg p \lor q) \land (p \land q)$ (idempotent law)

$\equiv (p \land q) \land (\neg p \lor q)$ (commutative law)

$\equiv [(p \land q) \land \neg p] \lor [(p \land q) \land q]$(distributive law)

$\equiv [(p \land \neg p) \land q] \lor [(p \land (q \land q)]$ (associative law)

$\equiv (F \land q) \lor (p \land q)$ (inverse law , idempotent law)

$\equiv F \lor (p \land q)$ (domination law)

$\equiv (p \land q)$ (identity law)

Hence proved that $\mathbf{[(\neg p \lor q) \land (p \land (p \land q)] \equiv p \land q}$
Now the dual of the above is,

$\mathbf{[(\neg p \land q) \lor (p \lor (p \lor q)] \equiv p \lor q}$

Hence Proved

Converse ,Inverse and Contrapositive:

Consider a **conditional** $\mathbf{p \to q}$. Then
(i) $\mathbf{q \to p}$ is called the **converse** of $p \to q$
(ii) $\mathbf{\neg p \to \neg q}$ is called the **inverse** of $p \to q$
(iii) $\mathbf{\neg q \to \neg p}$ is called the **contrapositive** of $p \to q$

Truth table for Converse ,Inverse and Contrapositive:

p	q	¬p	¬ q	p → q	q → p	¬p →¬ q	¬ q → ¬p
0	0	1	1	1	1	1	1
0	1	1	0	1	0	0	1
1	0	0	1	0	1	1	0
1	1	0	0	1	1	1	1

conditional (p → q) and contrapositive (¬ q → ¬p) are logically equivalent
i. e $\mathbf{p \to q \equiv \neg q \to \neg p}$
converse (q → p) and inverse (¬p →¬ q) are logically equivalent

i. e $\mathbf{q \to p \equiv \neg p \to \neg q}$

Logical Implication:

When a statement $p \to q$ is such that q is true whenever p is true (i.e . Whenever p is true , q must be true) ,we say that **p implies q**. This is symbolically written as

$p \Rightarrow q$.

Symbol $\Rightarrow$ denotes the word **implies**

$p \Rightarrow q$ (read as **p implies q)**

When a statement $p \to q$ is such that q is not necessarily true whenever p is true (i.e, whenever p is true ,q can be true or false) ,we say that **p doesnt implie q**.

This is symbolically written as $p \nRightarrow q$.

$p \nRightarrow q$ (read as **p doesnt implie q)**

Rules of Inference:

Consider a set of propositions p_1 , p_2 , p_3 ....p_n and a proposition q . Then a compound proposition of the form **(p_1 , p_2 , p_3 ....p_n) → Q** is called an argument .

Here p_1 , p_2 , p_3 ....p_n are called the premises of the arguments and Q is called the conclusion of the argument .

It is a practice to write the above argument in the following form:

$$\begin{array}{l} p_1 \\ p_2 \\ p_3 \\ p_n \\ \hline \therefore Q \end{array}$$

($\therefore$ read it as Therefore)

Above argument is said to be valid if whenever each of the premises p_1 , p_2 , p_3 ....p_n is true, then the conclusion Q is likewise true.

In otherwords, the argument

($p_1 \wedge p_2 \wedge p_3 \wedge \ldots\ldots p_n$) → Q is valid when **($p_1 \wedge p_2 \wedge p_3 \wedge p_n$) ⇒ Q**

<u>Premises</u> are always taken to be true whereas the **<u>conclusion</u> may be true or false**.

The conclusion is true only in the case of valid argument.

There are some rules which can be used to establish the validity of the arguments. These rules are called the Rule of inference.

Following are the rules of inference:

(1) Rule of Conjunctive simplification
(2) Rule of Disjunctive Amplification
(3) Rule of Syllogism
(4) Modus Pones (Rule of Detachement)
(5) Modus Tollens
(6)Rule of Disjunctive Syllogism
(7) Rule of Contradiction

(1) Rule of Conjunctive simplification

This rule states that ,for any two propositions p and q ,if $p \wedge q$ is true ,then p is true

i.e. $(p \wedge q) \Rightarrow p$

(2) Rule of Disjunctive Amplification

This rule states that ,for any two propositions p and q, if p is true then $p \vee q$ is true

i.e. $p \Rightarrow p \vee q$

(3) Rule of Syllogism

This rule states that ,for any three propositions p , q , r , if $p \rightarrow q$ is true and $q \rightarrow r$ is true , then $p \rightarrow r$ is true , p is F q is F and r is also F

i.e., $\{ (p \rightarrow q) \wedge (q \rightarrow r) \} \Rightarrow (p \rightarrow r)$

In Tabular form,

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore (p \rightarrow r) \end{array}$$

(4) Modus Pones (Rule of Detachement)

This rule states that, for any two propositions p and q, if p is true and $p \rightarrow q$ is true then q is true

i.e. $\{ p \wedge (p \rightarrow q) \} \Rightarrow q$

In Tabular form,

$$\begin{array}{l} p \\ p \rightarrow q \\ \hline \therefore q \end{array}$$

(5) Modus Tollens

This rule states that, for any two propositions p and q,if $p \rightarrow q$ is true and q is false then p is false

i.e. $\{ (p \rightarrow q) \wedge \neg q \} \Rightarrow \neg p$

In Tabular form,

$$\begin{array}{l} p \rightarrow q \\ \neg q \\ \hline \therefore \neg p \end{array}$$

(6)Rule of Disjunctive Syllogism

This rule states that, for any two propositions p and q,if $p \vee q$ is true and p is false then q is true

i.e. $\{ (p \vee q) \wedge \neg p \} \Rightarrow q$

In Tabular form,

$$\begin{array}{c} p \vee q \\ \neg p \\ \hline \therefore q \end{array}$$

(7) Rule of Contradiction

This rule states that, for any proposition p, if $\neg p \rightarrow F$ is true then p is true

i.e., $(\neg p \rightarrow F) \Rightarrow p$

In Tabular form,

$$\begin{array}{c} \neg p \rightarrow F \\ \hline \therefore p \end{array}$$

Examples:

Example 1: Test whether the following is a valid argument

If Sachin hits a century, then he gets a free car

Sachin hits a century

Therefore, Sachin gets a free car

Solution:

Let p: Sachin hits a century ,q: sachin gets a free car

Given argument can be written symbolic as

$$\begin{array}{c} p \rightarrow q \\ p \\ \hline \therefore q \end{array}$$

By modus Pones Rule, This is a valid argument

Example 2: Test whether the following is a valid argument

If Sachin hits a century, then he gets a free car

Sachin doesnt get a free car

Therefore, Sachin has not hit a century

Solution:

Let p: Sachin hits a century ,q: sachin gets a free car

Given argument can be written symbolic as

$$\begin{array}{c} p \rightarrow q \\ \neg q \\ \hline \therefore \neg p \end{array}$$

By modus Pones Tollens rule, This is a valid argument

Example 3: Test whether the following is a valid argument
If Sachin hits a century, then he gets a free car
Sachin gets a free car

Therefore, Sachin has hit a century
Solution:
Let p: Sachin hits a century ,q: sachin gets a free car
Given argument can be written symbolic as

$$\begin{array}{l} p \rightarrow q \\ \quad q \\ \hline \therefore\ p \end{array}$$

p	q	p → q	(p → q) ∧ q
1	1	1	1
0	1	1	1

From the above truth table, when q is true and p → q is true then p can be either true or false to make the proposition (p → q) ∧ q true .i.e it is not compulsory that p must be always true. Therefore the given argument is not valid.

Example 4: Test whether the following is a valid argument

If I study ,then I dont fail in the examination
If I dont fail in the examination,then my father gifts me a two-wheeler
----
∴ if I study then my father gifts me a two-wheeler

Solution:
Let p: I Study q: I dont fail in the examination
r: My father gisfts me a two-wheeler
Given argument can be written as,

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$$

By the rule of Syllogism ,this is a valid argument

Example 5: Test whether the following is a valid argument

If Ravi goes out with friends, he will not study
If Ravi doesn't study ,his father becomes angry
His father is not angry

Therefore, Ravi has not gone with friends

Solution:

Let p: Ravi goes out with friends q:Ravi does not study

r: his father becomes angry

Given argument can be written as,

$p \rightarrow q$

$q \rightarrow r$

$\neg r$

$\therefore \neg p$

$(p \rightarrow q) \wedge (q \rightarrow r)$ can be wriiten as $p \rightarrow r$ (Rule of Syllogism)

therefore given argument becomes

$p \rightarrow r$

$\neg r$

$\therefore \neg p$

By Modus Tollens Rule, this is a valid argument

Example 6: Test whether the following is a valid argument

If I study ,then I will not fall in the examination

If I don't watch TV in the evenings, I will study

I failed in the examination

Therefore, I must have watched TV in the evenings

Solution:

Let p: I Study   q: I fail in the examinations

r:I watch TV in the evenings

Therefore, the given argument becomes,

$p \rightarrow \neg q$

$\neg r \rightarrow p$

q

$\therefore r$

By the rule of syllogism

$\neg r \rightarrow p$

$p \rightarrow \neg q$

$\therefore$ $\boldsymbol{\neg r \rightarrow \neg q}$

and since conditional and its contrapositive are logically equivalent (i.e. $p \rightarrow q \equiv \neg q \rightarrow \neg p$)

$\boldsymbol{q \rightarrow r \equiv \neg r \rightarrow \neg q}$

so the given argument becomes

$\boldsymbol{q \rightarrow r}$

q

$\therefore r$

Therefore ,by the rule of Modus pones ,this is valid argument

Example 7: Test whether the following is a valid argument

I will get grade A in this course or I will not graduate

If I don't graduate ,then I wil join the army

I got grade A

∴ I will not join the army

Solution:

Let p: I will get grade A in this course q: I will not graduate r: I wil join the army

Given argument can be written as

$$\begin{array}{l} p \vee q \\ q \to r \\ p \\ \hline \therefore \neg r \end{array}$$

$\neg p \to q \equiv p \vee q$

$q \to r$ can be written as $\neg r \to \neg q$ (using comtarpositive)

$p \vee q$ can be written as $q \vee p$ (commutative law)

and $q \vee p$ can be written as $\neg q \to p$ (law of conditional ... i.e $q \vee p \equiv \neg q \to p$)

(because $\neg p \to q \equiv \neg(\neg p) \vee q \equiv p \vee q$)

therefore given argument becomes

$$\begin{array}{l} \neg r \to \neg q \\ \neg q \to p \\ p \\ \hline \therefore \neg r \end{array}$$

This is logically equivalent to

$$\begin{array}{ll} T & \neg r \to p \\ T & p \\ \hline & \therefore \neg r \end{array}$$

Because $(\neg r \to \neg q) \wedge (\neg q \to p) \Rightarrow \neg r \to p$ rule of syllogism

$\neg r$	p	$\neg r \to p$	$(\neg r \to p) \wedge p$
1	**1**	**1**	1
0	**1**	**1**	1

From the above truth table, when p is true and $\neg r \to p$ is true then $\neg r$ can be either true or false to make the proposition $(\neg r \to p) \wedge p$ as true .i.e it is not compulsory that $\neg r$ must be always true i.e it can be either true or false. Therefore the given argument(IT says $\neg r$ must be true) is not valid.

Example 8: Test whether the following is a valid argument

$p \rightarrow q$
$r \rightarrow s$
$p \vee r$

$\therefore q \vee s$

Solution:

$(p \rightarrow q) \wedge (r \rightarrow s) \wedge (\neg p \vee r) \Leftrightarrow (p \rightarrow q) \wedge (r \rightarrow s) \wedge (\neg p \rightarrow r)$ (conditional law)
$\Leftrightarrow (p \rightarrow q) \wedge (\neg p \rightarrow r) \wedge (r \rightarrow s)$ (Commutative law)
$\Leftrightarrow (p \rightarrow q) \wedge (\neg p \rightarrow s)$ (Rule of Syllogism)
$\Leftrightarrow (\neg q \rightarrow \neg p) \wedge (\neg p \rightarrow s)$ (Contra positive law)
$\Leftrightarrow (\neg q \rightarrow s)$ (Rule of Syllogism)
$\Leftrightarrow q \vee s$ (Conditional)

Therefore the given argument is valid.

Open Statements and Quantifiers

Consider a Statements
(1) p(x): x+3 = 6 **p(3):3+3=6** p(2):2+3 =6
(2) q(x):$x^2 < 10$ q(3):9<10 **q(4):16<10** p(3) V q(4) T V F T
(3)r(x): x divides 4
These statements are not propositions unless the symbol x is specified.
Sentences of these kind are called **open statements** or **open sentences** and the unspecified symbol such as **x** in this example is called as **free variable**.

Consider the sentence 1 above and the set of real numbers R . This sentence becomes a proposition if x is replaced by any element of R. For example ,if x is replaced by 3,the sentence becomes proposition. Here we say R is Universe (or universe of discourse).
Open statements containing a variable x are denoted by p(x),q(x) etc. If U is the universe for the variable x in an open statement p(x) and if $a \in U$, then the proposition got by replacing x by a in p(x) is denoted by p(a).

Compound open propositions can be formed by using the logical connectives.
Thus ¬p(x) is the negation of an open statement p(x).
p(x) ∧ q(x) is conjunction.... p(x) V q(x) is disjunction etc
For a given universe and for a given element of the universe ,the truth values of compound open statements are determined according to the same rules as those valid for compound statements.

Example 1: Suppose the universe consists of all integers. Consider the following open statements:
p(x): x <=3 ,q(x): x+1 is odd, r(x): x> 0
write down the truth values of the following:
1) p(2) T
2) ¬q(2) F
3)p (-1) ∧ q(1) T ∧ F F
4) ¬p(3) V r(0) F V F f
5) p(0) → q(0) T → T T
6) p(1) ↔ ¬ q(2) T ↔F F

Quantifiers:

Consider the following propositions
(1) **All** squares are rectangles
(2) **For every** integer x, x^2 is a non-negative integer
(3) **Some** determinants are equal to zero
(4) **There exists** a real number whose square is equal to itself

In these propositions ,the words "all", "every", "some", "there exists" are associated with the idea of a quantity. Such words are called **quantifiers**.

The proposition (1) can also be written as,

Let S denote the set of all squares and $x \in S$, x is a rectangle

in symbolic form

$\forall x \in S, p(x)$

where $\forall$ denotes the phrase "for all" and $p(x)$ representing the open statement "x is a rectangle".

similarly the proposition (2) can be written as $\forall x \in Z, q(x)$

where $\forall$ denotes the phrase "for every" and $p(x)$ representing the open statement "x^2 is a non-negative integer".

The proposition (3) can also be written as,

Let D denote the set of all determinants and $x \in D$, x is a equal to zero

in symbolic form

$\exists x \in D, p(x)$

where $\exists$ denotes the phrase "some" and $p(x)$ representing the open statement "x is equal to zero".

similarly the proposition (4) can be written as $\exists x \in R, q(x)$

where $\exists$ denotes the phrase "there exist" and let R denotes the set of real numbers and $q(x)$ representing the open statement "x is a real number whose square is equal to itself".

$\forall$ is used to represent the quantifiers "**for all**" ,"**for every**", "**for each**", "**for any**" and these are called as **Universal quantifiers**

$\exists$ is used to represent the quantifiers "**some**" ,"**there exists**", "**for at least one**" and these are called as **Existential quantifiers**

Truth Value of a quantified statement:

Following rules are employed for determining the truth value of a quantified statement

Rule 1: The statement $\forall x \in s$,p(x) is true only when p(x) is true for each $x \in s$

Rule 2: The statement $\exists x \in s$,p(x) is false only when p(x) is false for each $x \in s$

Rule 3: If an open statement p(x) is known to be true for all x in a universe S and if $a \in S$,then p(a) is true. (This is known as Rule of Universal Specification)

Rule 3: If an open statement p(x) is proved to be true for any(arbitrary) x chosen from a set S, then the quantified statement $\forall x \in s$,p(x) is true. (This is known as Rule Universal Generalization)

Rule 5: To construct the negation of a quantified statement , change the quantifier from universal to existential and versa and also replace the open statement by its negation.

i.e. $\neg\{\forall x,p(x)\} \equiv \exists x,\{\neg p(x)\}$ and $\neg\{\exists x, p(x)\} \equiv \forall x,\{\neg p(x)\}$