MODULE-2

JOINT PROBABILITY DISTRIBUTION:-

If X and Y are discrete random variables, we define the Joint Probability function of X and Y by

$$P(X=x_i, Y=y_j) = f(x_i, y_j) = J_{ij}$$

Here, $f(x,y)$ satisfies the conditions,

$$f(x,y) \geq 0$$

$$\sum_x \sum_y f(x,y) = 1$$

Joint Probability Table:

X\Y	y_1	y_2	----	y_n	Sum
x_1	J_{11}	J_{12}	----	J_{1n}	$f(x_1)$
x_2	J_{21}	J_{22}	----	J_{2n}	$f(x_2)$
$\vdots$					
x_m	J_{m1}	J_{m2}	----	J_{mn}	$f(x_m)$
Sum	$g(y_1)$	$g(y_2)$	----	$g(y_n)$	

Here, $f(x_1) = J_{11} + J_{12} + \cdots + J_{1n}$

$f(x_2) = J_{21} + J_{22} + \cdots + J_{2n}$

$g(y_1) = J_{11} + J_{21} + \cdots + J_{m1}$

$g(y_2) = J_{12} + J_{22} + \cdots + J_{m2}$

Marginal Probability Distribution of X and Y:-

Marginal Distribution of X

$X = x_i$	x_1	x_2	----
$f(x)$	$f(x_1)$	$f(x_2)$	----

Marginal Distribution of Y

$Y = y_j$	y_1	y_2	----
$g(y)$	$g(y_1)$	$g(y_2)$	----

IMPORTANT RESULTS:-

Expectation

Expectation in X, $E(X) = \sum x_i f(x_i)$

Expectation in Y, $E(Y) = \sum y_j g(y_j)$

Expectation in XY, $E(XY) = \sum\sum x_i y_j J_{ij}$

Variance

Variance in X, $V(X) = E(X^2) - [E(X)]^2$

Variance in Y, $V(Y) = E(Y^2) - [E(Y)]^2$

Standard Deviation

S.D in X, $\sqrt{V(X)}$

S.D in Y, $\sqrt{V(Y)}$

Covariance

$cov(X, Y) = E(XY) - E(X)E(Y)$

Correlation

$$\rho(X, Y) = \frac{cov(X, Y)}{\sigma_x \sigma_y}$$

PROBLEMS:-

1) The Joint Distribution of two random variables x and y is as follows:

x\y	-4	2	7
1	1/8	1/4	1/8
5	1/4	1/8	1/8

Compute

i) $E(x), E(y)$

ii) $E(xy)$

iii) σ_x, σ_y

iv) $cov(x,y)$

v) $\rho(x,y)$

⇒ Marginal Distribution of x and y

Distribution of x

$X = x_i$	$x_1 = 1$	$x_2 = 5$
$f(x)$	$f(x_1) = \frac{1}{2}$	$f(x_2) = \frac{1}{2}$

$$f(x_1) = \frac{1}{8} + \frac{1}{4} + \frac{1}{8} = \frac{1}{2}$$

$$f(x_2) = \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$$

Distribution of y

$Y = y_j$	$y_1 = -4$	$y_2 = 2$	$y_3 = 7$
$g(y)$	$g(y_1) = 3/8$	$g(y_2) = 3/8$	$g(y_3) = 1/4$

$$g(y_1) = \frac{1}{8} + \frac{1}{4} = \frac{3}{8}$$

$$g(y_2) = \frac{1}{4} + \frac{1}{8} = \frac{3}{8}$$

$$g(y_3)\ \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

i) $E(X), E(Y), E(XY)$

$$E(X) = \Sigma x_i f(x_i)$$

$$= \left[(1)\left(\frac{1}{2}\right) + (5)\left(\frac{1}{2}\right)\right]$$

$$\boxed{E(X) = 3}$$

$$E(Y) = \Sigma y_j g(y_j)$$

$$= \left[(-4)\left(\frac{3}{8}\right) + (2)\left(\frac{3}{8}\right) + (7)\left(\frac{1}{4}\right)\right]$$

$$\boxed{E(Y) = 1}$$

ii) $E(XY)$

$$E(XY) = \Sigma x_i y_j J_{ij}$$

$$E(XY) = [(x_1)(y_1)J_{11} + (x_1)(y_2)J_{12} + (x_3)(y_3) + J_{13} + (x_2)(y_1)J_{21} + (x_2)(y_2)J_{22} + (x_2)(y_3)J_{23}]$$

$$\boxed{E(XY) = \frac{3}{2}}$$

(iii) σ_x, σ_y

$$\sigma_x = \sqrt{V(x)}$$

$$V(x) = E(x^2) - [E(x)]^2$$

$$E(x^2) = \Sigma x^2 f(x)$$

$$= \left[1^2\left(\frac{1}{2}\right) + 5^2\left(\frac{1}{2}\right)\right]$$

$$= \left[\frac{1}{2} + 25 \times \frac{1}{2}\right]$$

$$\boxed{E(x^2) = 13}$$

$$\therefore V(x) = E(x^2) - [E(x)]^2$$

$$= 13 - (3)^2$$

$$= 13 - 9$$

$$\boxed{V(x) = 4}$$

$$\sigma_x = \sqrt{V(x)}$$

$$= \sqrt{4}$$

$$\boxed{\sigma_x = 2}$$

$$\sigma_y = \sqrt{V(y)}$$

$$V(y) = E(y^2) - [E(y)]^2$$

$$E(y^2) = \Sigma y^2 g(y)$$

$$= \left[(-4)^2\left(\frac{3}{8}\right) + (1)^2\left(\frac{3}{8}\right) + 7^2\left(\frac{1}{4}\right)\right]$$

$$= \left(6 + \frac{3}{8} + \frac{49}{4}\right)$$

$$\boxed{E(y^2) = \frac{79}{4}}$$

$$\therefore V(y) = E(y^2) - [E(y)]^2$$

$$= \frac{79}{4} - 1^2$$

$$\boxed{V(y) = \frac{75}{4}}$$

$$\sigma_y = \sqrt{V(y)}$$

$$= \sqrt{\frac{75}{4}}$$

$$\boxed{\sigma_y = 4.33}$$

(iv) $cov(x, y)$

$$cov(x,y) = E(xy) - E(x)E(y)$$

$$= \frac{3}{2} - 3 \times 1$$

$$= \frac{3}{2} - 3 \Rightarrow \frac{3-6}{2} = \frac{-3}{2}$$

$$\boxed{cov(x,y) = -3/2}$$

v) $\rho(x,y) = \dfrac{cov(x,y)}{\sigma_x \cdot \sigma_y}$

$= \dfrac{\left(\frac{-3}{2}\right)}{2 \times 4.33}$

$= \dfrac{-3}{2} \times \dfrac{1}{8.66}$

$\boxed{\rho(x,y) = -0.1732}$

2) Q Suppose X and Y are independent random variables with the following respective distribution. Find the Joint distribution of X and Y and also verify that cov(X,Y)=0

x_i	1	2
$f(x_i)$	0.7	0.3

y_j	-2	5	8
$g(y_j)$	0.3	0.5	0.2

<u>NOTE</u>:- • If X and Y are independent random variable, then

$$f(x_i) \cdot g(y_j) = J_{ij}$$

• X and Y are dependent random variable, then

$$f(x_i) \cdot g(y_j) \neq J_{ij}$$

⇒ Here, x and y are independent random variables.
Therefore, it satisfies $f(x_i)g(y_j) = J_{ij}$

Probability Table:
↳ Joint

$x \backslash y$	$y_1 = -2$	$y_2 = 5$	$y_3 = 8$	$f(x_i)$
$x_1 = 1$	$J_{11} = 0.21$	$J_{12} = 0.35$	$J_{13} = 0.14$	$f(x_1) = 0.7$
$x_2 = 2$	$J_{21} = 0.09$	$J_{22} = 0.15$	$J_{23} = 0.06$	$f(x_2) = 0.3$
$g(y_j)$	$g(y_1) = 0.3$	$g(y_2) = 0.5$	$g(y_3) = 0.2$	1

$J_{11} = f(x_1) \cdot g(y_1)$
$= 0.7 \times 0.3 = 0.21$

$J_{12} = f(x_1) \cdot g(y_2)$
$= 0.7 \times 0.5 = 0.35$

$J_{13} = f(x_1) \cdot g(y_3)$
$= 0.7 \times 0.2 = 0.14$

$J_{21} = f(x_2) \cdot g(y_1)$
$= 0.3 \times 0.3 = 0.09$

$J_{22} = f(x_2) \cdot g(y_2)$
$= 0.3 \times 0.5 = 0.15$

$J_{23} = f(x_2) \cdot g(y_3)$
$= 0.3 \times 0.2 = 0.06$

$cov(X,Y) = E(XY) - E(X)E(Y)$

$E(X) = \Sigma x_i f(x_i)$

$= (1)(0.7) + (2)(0.3) = 1.3$

$E(Y) = \Sigma y_j g(y_j)$

$= (-2)(0.3) + (5 \times 0.5) + (8 \times 0.2) = 3.5$

$E(XY) = \Sigma x_i y_j J_{ij}$

$= (1 \times -2 \times 0.21) + (1 \times 5 \times 0.35) + (1 \times 8 \times 0.14) + (2 \times -2 \times 0.09)$

$+ (2 \times 5 \times 0.15) + (2 \times 8 \times 0.06)$

$= -0.42 + 1.75 + 1.12 - 0.36 + 1.5 + 0.96 = 4.55$

$cov(XY) = E(XY) - E(X) \cdot E(Y)$

$= 4.55 - 1.3(3.5)$

$= 0$

Hence, verified

3) The Joint Probability Distribution Table X and Y is as follows:

X\Y	-2	-1	4	5
1	0.1	0.2	0	0.3
2	0.2	0.1	0.1	0

Write the Marginal Probability Distribution of X and Y and also compute

i) $E(X), E(Y), E(XY)$

ii) σ_x, σ_y

iii) $\rho(XY)$

Further, verify that X and Y are dependent random variables. Also, find $P(X+Y>0)$

⇒ Distribution of X

$X = x_i$	$x_1 = 1$	$x_2 = 2$
$f(x)$	$f(x_1) = 0.6$	$f(x_2) = 0.4$

$f(x_1) = 0.1 + 0.2 + 0 + 0.3 = 0.6$

$f(x_2) = 0.2 + 0.1 + 0.1 + 0 = 0.4$

Distribution of Y

$Y = y_j$	$y_1 = -2$	$y_2 = -1$	$y_3 = 4$	$y_4 = 5$
$g(y)$	$g(y_1) = 0.3$	$g(y_2) = 0.3$	$g(y_3) = 0.1$	$g(y_4) = 0.3$

$g(y_1) = 0.1 + 0.2 = 0.3$

$g(y_2) = 0.2 + 0.1 = 0.3$

$g(y_3) = 0 + 0.1 = 0.1$

$g(y_4) = 0.3 + 0 = 0.3$

i) $E(x), E(y), E(xy)$

$E(x) = \Sigma x_i f(x_i)$

$= (1)(0.6) + 2(0.4)$

$\boxed{E(x) = 1.4}$

$E(y) = \Sigma y_j g(y_j)$

$= (-2)(0.3) + (-1)(0.3) + (4)(0.1) + (5)(0.3)$

$\boxed{E(y) = \cancel{0.9}\ 1}$

$E(xy) = \Sigma x_i y_j J_{ij}$

$E(xy) = (x_1)(y_1)J_{11} + (x_1)(y_2)J_{12} + (x_1)(y_3)J_{13} + (x_1)(y_4)J_{14} +$
$(x_2)(y_1)J_{21} + (x_2)(y_2)J_{22} + (x_2)(y_3)J_{23} + (x_2)(y_4)J_{24}$

$\boxed{E(xy) = 09}$

ii) σ_x, σ_y

$\sigma_x = \sqrt{V(x)}$

$V(x) = E(x^2) - [E(x)]^2$

$E(x^2) = \Sigma x^2 f(x)$

$= 1^2(0.6) + (2)^2(0.4)$

$\boxed{E(x^2) = 2.2}$

$V(x) = E(x^2) - [E(x)]^2$

$= 2.2 - (1.4)^2$

$\boxed{V(x) = 0.24}$

$\sigma_x = \sqrt{V(x)}$

$\boxed{\sigma_x = 0.4898}$

$\sigma_y = \sqrt{V(y)}$

$V(y) = E(y^2) - [E(y)]^2$

$E(y^2) = \Sigma y^2 g(y)$

$= (-2)^2(0.3) + (-1)^2(0.3) + (4)^2(0.1) + (5)^2(0.3)$

$\boxed{E(y^2) = 10.6}$

$V(y) = E(y^2) - [E(y)]^2$

$= 10.6 - (1)^2$

$= 10.6 - 1$

$\boxed{V(y) = 9.6}$

$\sigma_y = \sqrt{V(y)}$

$= \sqrt{9.6}$

$\sigma_y = 3.09 = 3.1$

$\boxed{\sigma_y = 3.1}$

iii) $\rho(xy) = \dfrac{cov(x,y)}{\sigma_x \cdot \sigma_y}$

$cov(x,y) = E(xy) - E(x)E(y)$

$= 0.9 - (1.4)(1)$

$\boxed{cov(x,y) = -0.5}$

$\rho(xy) = \dfrac{-0.5}{0.4898 \times 3.1}$

$\boxed{\rho(xy) = -0.3292}$

$\rightarrow f(x_i)\,g(y_j) \neq J_{ij}$

Taking, $f(x_1)g(y_2) = (0.6)(0.3) = 0.18$

$$J_{12} = 0.2$$

$\therefore f(x_1)g(y_2) \neq J_{12} \rightarrow$ so it is dependent random variable.

$\rightarrow P(X+Y>0)$ is possible only when we takes the values

$X = \{x_1, x_2\} = \{1, 2\}$

$Y = \{y_1, y_2, y_3, y_4\} = \{-2, -1, 4, 5\}$

$(x_1, y_3) = J_{13}$

$(x_1, y_4) = J_{14}$

$(x_2, y_2) = J_{22}$

$(x_2, y_3) = J_{23}$

$(x_2, y_4) = J_{24}$

4) The joint probability distribution of 2 discrete random variables X & Y is given by $f(x,y) = K(2x+y)$ where x, y are integers such that $0 \leq x \leq 2$, $0 \leq y \leq 3$.

i) Find the values constant K.

ii) Find the Marginal probability distribution of X & Y.

iii) Show that the random variables of X and Y are dependent.

iv) $E(X)$, $E(Y)$, $E(XY)$, $E(X^2)$, σ_x, σ_y, $P(x \geq 1, y \leq 2)$, $P(x+y>2)$

$\Rightarrow$ $X = \{x_i\} = \{0, 1, 2\}$

$Y = \{y_j\} = \{0, 1, 2, 3\}$

Given: $f(x,y) = K(2x+y)$

Joint Probability Table:

$x \backslash y$	$y_1=0$	$y_2=1$	$y_3=2$	$y_4=3$	$f(x_i)$
$x_1=0$	$J_{11}=0$	$J_{12}=K$	$J_{13}=2K$	$J_{14}=3K$	$f(x_1)=6$
$x_2=1$	$J_{21}=2K$	$J_{22}=3K$	$J_{23}=4K$	$J_{24}=5K$	$f(x_2)=K$
$x_3=2$	$J_{31}=4K$	$J_{32}=5K$	$J_{33}=6K$	$J_{34}=7K$	$f(x_3)=22$
$g(y_j)$	$g(y_1)=6K$	$g(y_2)=9K$	$g(y_3)=12K$	$g(y_4)=15K$	1

$$f(x,y) = K(2x+y)$$
$$= K(2(0)+1)$$
$$= K$$

$$f(x,y) = K(2x+y)$$
$$= K(0+2)$$
$$= 2K$$

$$f(x,y) = K(2x+y)$$
$$= K(0+3)$$
$$= 3K$$

Distribution of X:

$X=x_i$	$x_1=0$	$x_2=1$	$x_3=2$
$f(x)$	$f(x_1)=6K$	$f(x_2)=14K$	$f(x_3)=22K$
	$f(x_1)=\frac{1}{7}$	$f(x_2)=\frac{1}{3}$	$f(x_3)=\frac{11}{21}$

Distribution of y:

$Y=y_j$	$y_1=0$	$y_2=1$	$y_3=2$	$y_4=3$
$g(y_j)$	$g(y_1)=6K$	$g(y_2)=9K$	$g(y_3)=12K$	$g(y_4)=15K$
	$g(y_1)=\frac{1}{7}$	$g(y_2)=\frac{3}{14}$	$g(y_3)=\frac{2}{7}$	$g(y_4)=\frac{5}{14}$

(i) $\Sigma P(x) = 1$

$\Rightarrow 6k + 14k + 22k = 1$

$42k = 1$

$\boxed{k = \frac{1}{42}}$

Rewriting Joint Distribution Table:

x\y	$y_1=0$	$y_2=1$	$y_3=2$	$y_4=3$	$f(x_i)$
$x_1=0$	$J_{11}=0$	$J_{12}=1/42$	$J_{13}=\frac{1}{21}$	$J_{14}=\frac{1}{14}$	$f(x_1)=1/7$
$x_2=1$	$J_{21}=1/21$	$J_{22}=1/14$	$J_{23}=2/21$	$J_{24}=\frac{5}{42}$	$f(x_2)=1/3$
$x_3=2$	$J_{31}=\frac{2}{21}$	$J_{32}=\frac{5}{42}$	$J_{33}=1/7$	$J_{34}=1/6$	$f(x_3)=\frac{11}{21}$
$g(y_j)$	$g(y_1)=1/7$	$g(y_2)=3/14$	$g(y_3)=2/7$	$g(y_4)=\frac{5}{14}$	1

(iii) To show random variable x and y are dependent it must satisfy, $f(x_i)\cdot g(y_j) \neq J_{ij}$

$$f(x_1)\cdot g(y_2) = \left(\frac{1}{7}\right)\left(\frac{3}{14}\right) = \frac{3}{98}$$

$$J_{12} = \frac{1}{42}$$

$$\therefore f(x_1)\cdot g(y_2) \neq J_{12}$$

iv) $E(X) = \sum x f(x_i)$

$= \left[0\left(\frac{1}{7}\right) + 1\left(\frac{1}{3}\right) + 2\left(\frac{11}{21}\right)\right]$

$\boxed{E(X) = \frac{29}{21}}$

$E(y) = \sum y\, g(y_j)$

$= \left[0\left(\frac{1}{7}\right) + 1\left(\frac{3}{14}\right) + 2\left(\frac{2}{7}\right) + 3\left(\frac{5}{14}\right)\right]$

$\boxed{E(y) = \frac{13}{7}}$

$E(xy) = \sum x_i y_j J_{ij}$

$= x_1y_1J_{11} + x_1y_2J_{12} + x_1y_3J_{13} + x_1y_4J_{14} +$
$x_2y_1J_{21} + x_2y_2J_{22} + x_2y_3J_{23} + x_2y_4J_{24} +$
$x_3y_1J_{31} + x_3y_2J_{32} + x_3y_3J_{33} + x_3y_4J_{34}$

$= 0+0+0+0+0+(1)(1)\left(\frac{1}{14}\right) + (1)(2)\left(\frac{2}{21}\right) + (1)(3)\left(\frac{5}{42}\right) +$
$0 + (2)(1)\left(\frac{5}{42}\right) + (2)(2)\left(\frac{1}{7}\right) + (2)(3)\left(\frac{1}{6}\right)$

$\boxed{E(xy) = \frac{17}{7}}$

$E(X^2) = \sum x^2 f(x)$

$= \left[0\left(\frac{1}{7}\right) + 1\left(\frac{1}{3}\right) + 2^2\left(\frac{11}{21}\right)\right]$

$\boxed{E(X^2) = \frac{17}{7}}$

$\sigma_x = \sqrt{V(X)}$

$V(X) = E(X^2) - [E(X)]^2$

$= \frac{17}{7} - \left(\frac{29}{21}\right)^2$

$V(X) = 0.52$

$\sigma_x = \sqrt{0.52}$

$\boxed{\sigma_x = 0.72}$

$\sigma_y = \sqrt{V(Y)}$

$V(Y) = E(Y^2) - [E(Y)]^2$

$= 0\left(\frac{1}{21}\right) + 1^2\left(\frac{3}{14}\right) + 2^2\left(\frac{2}{7}\right) + 3^2\left(\frac{5}{14}\right)$

$\boxed{E(Y^2) = \frac{32}{7}}$

$V(Y) = \frac{32}{7} - \left(\frac{13}{7}\right)^2$

$V(Y) = \frac{55}{49}$

$\boxed{\sigma_y = 1.06}$

(iv) $P(x \geq 1, y \leq 2)$

$x = \{\overset{x_1}{0}, \overset{x_2}{1}, \overset{x_3}{2}\}$

$y = \{\overset{y_1}{0}, \overset{y_2}{1}, \overset{y_3}{2}, \overset{y_4}{3}\}$

$x = (1,2), \; y = (0,1,2)$

$(x,y) = (1,0)\,(1,1)\,(1,2)\,(2,0)\,(2,1)\,(2,2)$

$= (x_2, y_1)\,(x_2, y_2)\,(x_2, y_3)\,(x_3, y_1)\,(x_3, y_2)\,(x_3, y_3)$

$= J_{21} + J_{22} + J_{23} + J_{31} + J_{32} + J_{33}$

$= \frac{1}{21} + \frac{1}{14} + \frac{2}{21} + \frac{3}{21} + \frac{5}{42} + \frac{1}{7} = \boxed{\frac{4}{7}}$

$P(X+Y > 2) = (0,3)\,(1,2)\,(1,3)\,(2,1)\,(2,2)\,(2,3)$

$= (x_1, y_4)\,(x_2, y_3)\,(x_2, y_4)\,(x_3, y_2)\,(x_3, y_3)\,(x_3, y_4)$

$= J_{14} + J_{23} + J_{24} + J_{32} + J_{33} + J_{34}$

$= J_{14} + J_{23} + J_{24} + J_{32} + J_{33} + J_{34}$

$= \frac{1}{14} + \frac{2}{21} + \frac{5}{42} + \frac{5}{42} + \frac{1}{7} + \frac{1}{6}$

$= \boxed{\frac{5}{7}}$

5) A fair coin is tossed thrice with the random variables x and y are defined as follows:

x = 0 (or) 1 according as head (or) tail occurs on the first toss, y = number of heads.

⇒ Sample space = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

H → 0, T → 1

x = 0 → H, 1 → T, y → no. of heads

Sample space	HHH	HHT	HTH	HTT	TTT	TTH	THT	THH
x	0	0	0	0	1	1	1	1
y	3	2	2	1	0	1	1	2

Joint Probability Table:

x|y	$y_1 = 0$	$y_2 = 1$	$y_3 = 2$	$y_4 = 3$	$f(x_i)$
$x_1 = 0$	$J_{11} = 0$	$J_{12} = \frac{1}{8}$	$J_{13} = \frac{1}{4}$	$J_{14} = 1/8$	$f(x_1) = 1/2$
$x_2 = 1$	$J_{21} = \frac{1}{8}$	$J_{22} = \frac{1}{4}$	$J_{23} = \frac{1}{8}$	$J_{24} = 0$	$f(x_2) = 1/2$
$g(y_j)$	$g(y_1) = \frac{1}{8}$	$g(y_2) = \frac{3}{8}$	$g(y_3) = \frac{3}{8}$	$g(y_4) = \frac{1}{8}$	1

$J_{11} = f(x_1)g(y_1)$ [$P(x=0, y=0)$]

$= H$ $\quad 0$

$J_{12} = f(x_1)g(y_2)$ [$P(X=0, y=1)$]

$= HTT$ $\quad 1/8$

$J_{13} = f(x_1)g(y_3)$ [$P(X=0, y=2)$]

$= HHT, HTH$ $\quad 1/4$

$J_{14} = f(x_1)g(y_4)$ [$P(X=0, y=2)$]

$= HHH$ $\quad 1/8$

$J_{21} = f(x_2)g(y_1)$ [$P(X=1, y=0)$]

$= TTT$ (o no. of heads) $= 1/8$

$J_{22} = f(x_2)g(y_2)$ [$P(X=1, y=1)$]

$= THT, TTH$, $2/8 = 1/4$

$J_{23} = f(x_2)g(y_3)$ [$P(X=1, y=2)$]

$= THH$ $= 1/8$

$J_{24} = f(x_2)g(y_4)$ [$P(X=1, y=3)$]

0 (Impossible event)

(i) Determine the marginal Distribution of X & Y.

(ii) Determine the joint distribution of X and Y

(iii) obtain the expectation of X, $E(Y)$, $E(XY)$, σ_X, σ_Y.

(iv) Compute the covariance and co-relation of X & Y.

1) Distribution in X

$X = x_i$	$x_1 = 0$	$x_2 = 1$
$f(x_i)$	$f(x_1) = 1/2$	$f(x_2) = 1/2$

Distribution in Y

$Y = y_j$	$y_1 = 0$	$y_2 = 1$	$y_3 = 2$	$y_4 = 3$
$g(y_j)$	$g(y_1) = 1/8$	$g(y_2) = 3/8$	$g(y_3) = 3/8$	$g(y_4) = 1/8$

$$E(X) = \sum x f(x)$$
$$= \left(0 \times \frac{1}{2}\right) + \left(1 \times \frac{1}{2}\right)$$
$$\boxed{E(X) = \frac{1}{2}}$$

$$E(y) = \sum y g(y)$$
$$= \left(0 \times \frac{1}{8}\right) + \left(1 \times \frac{3}{8}\right) + \left(2 \times \frac{3}{8}\right) + \left(3 \times \frac{1}{8}\right)$$
$$\boxed{E(y) = \frac{3}{2}}$$

$$E(xy) = \sum x_i y_j J_{ij}$$
$$= x_1 y_1 J_{11} + x_1 y_2 J_{12} + x_1 y_3 J_{13} + x_1 y_4 J_{14} + x_2 y_1 J_{21} + x_2 y_2 J_{22} + x_2 y_3 J_{23} + x_2 y_4 J_{23}$$
$$= (0 \times 0 \times 0) + \left(0 \times 1 \times \frac{1}{8}\right) + \left(0 \times 2 \times \frac{1}{4}\right) + \left(0 \times 3 \times \frac{1}{8}\right)$$
$$+ \left(1 \times 0 \times \frac{1}{8}\right) + \left(1 \times 1 \times \frac{1}{4}\right) + \left(1 \times 2 \times \frac{1}{8}\right) + (1 \times 3 \times 0)$$
$$\boxed{E(xy) = \frac{1}{2}}$$

$$E(x^2) = \sum x_i^2 f(x)$$
$$= \left(0^2 \times \frac{1}{2}\right) + \left(1^2 \times \frac{1}{2}\right)$$
$$\boxed{E(x^2) = \frac{1}{2}}$$

$$E(y^2) = \sum y_j^2 g(y)$$
$$= \left(0^2 \times \frac{1}{8}\right) + \left(1^2 \times \frac{3}{8}\right) + \left(2^2 \times \frac{3}{8}\right) + 3^2\left(\frac{1}{8}\right)$$
$$\boxed{E(y^2) = 3}$$

$$V(x) = E(x^2) - [E(x)]^2$$
$$= \frac{1}{2} - \left(\frac{1}{2}\right)^2$$
$$V(x) = \frac{1}{4}$$

$$V(y) = E(y^2) - [E(y)]^2$$
$$= 3 - \left(\frac{3}{2}\right)^2$$
$$V(y) = \frac{3}{4}$$

$$\sigma_x = \sqrt{V(x)}$$
$$= \sqrt{\frac{1}{4}}$$
$$\boxed{\sigma_x = 0.5}$$

$$\sigma_y = \sqrt{V(y)}$$
$$= \sqrt{\frac{3}{4}}$$
$$\boxed{\sigma_y = 0.8660}$$

(iv) $cov = E(xy) - E(x)\,E(y)$

$$= \frac{1}{2} - \frac{1}{2} \times \frac{3}{2}$$

$$\boxed{cov(x,y) = \frac{-1}{4}}$$

correlation $\rho(x,y) = \dfrac{cov(x,y)}{\sigma_x \cdot \sigma_y}$

$$= \frac{-1/4}{0.5 \times 0.8660}$$

$$\boxed{\rho(x,y) = -0.5774}$$

STOCHASTIC PROCESS (RV changing acc to time)

Anything which is changing w.r.t time is called Stochastic Process. [Business, share market]

Stochastic Process is a set of random variables $\{x(t), t \in T\}$ defined on S with parameter t. (Time)

The values assumed by the random variable $x(t)$ are called states.

$x_0 = x(0)$ is called the initial state of the system.

If the state space of a stochastic process is discrete, then it is called discrete state process (or chain).

If the state space is continuous, then the stochastic process is continuous state process.

PROBABILITY VECTOR:

A vector $v = (v_1, v_2, \ldots, v_n)$ is called a Probability vector if each one of its component are non-negative & their sum is equal to unity.

$$u = (1, 0)$$

$$v = \left(\frac{1}{2}, \frac{1}{2}\right)$$

$$w = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)$$

NOTE: If v is not a probability vector and contains non-negative, then λv is the probability vector where

$$\lambda = \frac{1}{\sum_{i=1}^{n} v_i}$$

Ex: If $v = (1, 2, 3)$

$$\lambda = \frac{1}{6} \qquad \lambda v = \left(\frac{1}{6}, \frac{2}{6}, \frac{3}{6}\right)$$

$$\underline{\lambda v = 1}$$

STOCHASTIC MATRIX:

A square matrix $P = P_{ij}$ having every row in the form of a probability vector is called a stochastic matrix.

$$v = \begin{bmatrix} 1/2 & 1/2 \\ 0 & 1 \end{bmatrix}$$

$$w = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

Regular Stochastic Matrix:

A Stochastic Matrix P is said to be regular stochastic matrix if all the entries of some power p^n are positive.

Ex:

$$A = \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}$$

$$A^2 = A \cdot A = \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix}$$

A is a regular stochastic matrix and here

$$\boxed{n=2}$$

Problems:

1) If $A = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$ is a stochastic Matrix and $V = [v_1, v_2]$ is a probability vector. Show that VA is a Probability vector.

→ Given: A is a stochastic Matrix

$$\Rightarrow \quad a_1 + a_2 = 1$$

$$b_1 + b_2 = 1$$

again, from the given data, V is a probability vector implies $v_1+v_2=1$

To show that, VA is Probability vector.

$$VA = \underset{(1\times 2)}{[v_1, v_2]} \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}_{(2\times 2)}$$

$$[v_1a_1+v_2b_1 \quad v_1a_2+b_2v_2]$$

we have to show that $v_1a_1+v_2b_1+v_1a_2+v_2b_2=1$

$$v_1(a_1+a_2)+v_2(b_1+b_2)=1$$

$$v_1+v_2=1$$

$$1=1$$

$\therefore$ Hence, VA is a P. vector.

2) Prove with reference to two 2nd order stochastic matrices that there product is also stochastic matrix.

$\Rightarrow$ Let, $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ be two stochastic matrices.

$$\Rightarrow \left.\begin{matrix} a_{11}+a_{12}=1 \\ a_{21}+a_{22}=1 \end{matrix} \quad \text{and} \quad \begin{matrix} b_{11}+b_{12}=1 \\ b_{21}+b_{22}=1 \end{matrix}\right\} \rightarrow (1)$$

$$AB = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$AB = \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21}, & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21}, & a_{21}b_{12}+a_{22}b_{22} \end{bmatrix}$$

we have to show that,

$$a_{11}b_{11}+a_{12}b_{21}+a_{11}b_{12}+a_{12}b_{22}=1$$

$$a_{21}b_{11}+a_{22}b_{21}+a_{21}b_{12}+a_{22}b_{22}=1$$

$$\Rightarrow a_{11}(b_{11}+b_{12})+a_{12}(b_{21}+b_{22})=1$$

$$a_{21}(b_{11}+b_{12})+a_{22}(b_{21}+b_{22})=1$$

using ①;

$$a_{11}+a_{12}=1$$

$$a_{21}+a_{22}=1$$

$$1=1$$

$\therefore$ AB is stochastic matrix.

NOTE: To find unit fixed probability vector for a given matrix, we assume the p. vector

$$V = [x, y, z]$$

$$\Rightarrow x + y + z = 1 \rightarrow VA = V$$

Here, A is a stochastic matrix of 3x3 will be finding for x, y, z.

3) Find the unique fixed probability vector for the regular stochastic matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix}$$

$\Rightarrow$ To find the unique fixed probability vector, let $V = [x, y, z]$ be the unique fixed probability vector.

$$\Rightarrow x + y + z = 1 \ \& \ VA = V$$

$$[x, y, z] \begin{bmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix} = [x, y, z]$$

$$\left[0 + y/6 + 0, \ x + \frac{y}{2} + \frac{2z}{3}, \ 0 + y/3 + z/3\right] = [x, y, z]$$

$$\left[y/6, \ x + \frac{y}{2} + \frac{2z}{3}, \ \frac{y}{3} + \frac{z}{3}\right] = [x, y, z]$$

Comparing both the sides.

$\frac{y}{6} = x$

$\boxed{y = 6x} \rightarrow (1)$

$x + \frac{y}{2} + \frac{2z}{3} = y$

$\frac{6x + 3y + 4z}{6} = y$

$6x + 3y + 4z = 6y$

$6x - 3y + 4z = 0 \rightarrow (2)$

$\frac{y}{3} + \frac{z}{3} = z$

$y + z = 3z$

$y = 2z \rightarrow (3)$

$z = \frac{y}{2} = \frac{6x}{2} = 3x$

$\boxed{z = 3x} \rightarrow (4)$

From, $\boxed{x + y + z = 1}$

$x + 6x + 3x = 1$

$10x = 1$

$\boxed{x = \frac{1}{10}}$

$\boxed{y = \frac{6}{10}}$ $\boxed{z = \frac{3}{10}}$ $\therefore$ the required probability vector is

$\therefore V = \left(\frac{1}{10}, \frac{6}{10}, \frac{3}{10}\right)$

4) Find the unique fixed probability vector of the regular stochastic matrix $x = 2/3, y = 1/3$

$$A = \begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix}$$

⇒ To find the unit fixed p-vector, let

$v = [x, y]$ be the unit fixed p-vector

$\Rightarrow x+y=1 \ni vA = v$

$$[x, y]\begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix} = [x, y]$$

$$\left[\frac{3x}{4} + \frac{y}{2}, \frac{x}{4} + \frac{y}{2}\right] = [x, y]$$

comparing on both the sides,

$$\frac{3x}{4} + \frac{y}{2} = x, \qquad \frac{x}{4} + \frac{y}{2} = y$$

$$\frac{y}{2} = \frac{x}{4} \qquad 2\frac{x}{4} = \frac{y}{2}$$

$$y = \frac{2x}{4} \qquad \boxed{x = 2y} \rightarrow (3)$$

$$\boxed{y = \frac{x}{2}} \rightarrow (2) \qquad 2y = \frac{x}{2}$$

Sub in ①:

$$x+y=1$$

$$2y+y=1$$

$$3y=1$$

$$\boxed{y=\frac{1}{3}}$$

$$x=2y$$

$$x=2\left(\frac{1}{3}\right)$$

$$\boxed{x=\frac{2}{3}}$$

$$\therefore \quad V=\left(\frac{2}{3}, \frac{1}{3}\right)$$

5) Find the unique fixed probability vector of the regular stochastic matrix

$V = [a,b,c,d]$
$a = \frac{1}{3}, b = \frac{1}{3}, c = \frac{1}{6}$
$d = \frac{1}{6}$

$$P = \begin{bmatrix} 0 & 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/4 & 1/4 \\ 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ $V = [a,b,c,d]$ & $a+b+c+d = 1$; $VA = V$

$$[a\ b\ c\ d] \begin{bmatrix} 0 & 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/4 & 1/4 \\ 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{bmatrix} = [a,b,c,d]$$

$$\left[0 + \frac{b}{2} + \frac{c}{2} + \frac{d}{2},\ \frac{a}{2} + 0 + \frac{c}{2} + \frac{d}{2},\ \frac{a}{4} + \frac{b}{4} + 0 + 0,\ \frac{a}{4} + \frac{b}{4} + 0 + 0\right] = [a,b,c,d]$$

$$\left[\frac{b}{2} + \frac{c}{2} + \frac{d}{2},\ \frac{a}{2} + \frac{c}{2} + \frac{d}{2},\ \frac{a}{4} + \frac{b}{4},\ \frac{a}{4} + \frac{b}{4}\right] = [a,b,c,d]$$

Comparing on both sides,

$b+c+d = 2a$, $a+c+d = 2b$, $a+b = 4c$, $a+b = 4d$

$c = \frac{a+b}{4}$ $\quad$ $d = \frac{a+b}{4}$

$$b+c+d=2a$$

$$b+\frac{a+b}{4}+\frac{a+b}{4}=2a$$

$$\frac{4b+a+b+a+b}{4}=2a$$

$$2a+6b=8a$$

$$6b=6a$$

$$\boxed{a=b} \rightarrow ②$$

$$a+b+c+d=2b$$

$$a+\frac{a+b}{4}+\frac{a+b}{4}=2b$$

$$\frac{4a+a+b+a+b}{4}=2b$$

$$6a+2b=8b$$

$$6a=8b-2b$$

$$6a=6b$$

$$\boxed{a=b}$$

$$c=\frac{a+b}{4}$$

$$c=\frac{2b}{4}$$

$$\boxed{c=\frac{b}{2}} \rightarrow ③$$

$$d=\frac{a+b}{4}$$

$$d=\frac{2b}{4}$$

$$\boxed{d=\frac{b}{2}} \rightarrow ④$$

Sub ②, ③, ④ in equ^n ①

$$a+b+c+d=1$$

$$b+b+\frac{b}{2}+\frac{b}{2}=1$$

$$3b=1$$

$$\boxed{b=\frac{1}{3}},\ \boxed{a=\frac{1}{3}}\ \boxed{c=\frac{b}{2}=\frac{1}{6}=d}$$

$$\therefore v=\left(\frac{1}{3},\frac{1}{3},\frac{1}{6},\frac{1}{6}\right)$$

6) Show that $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$ is a regular stochastic matrix. Also, find the unique fixed probability vector.

$\Rightarrow$ To show that, P is a regular stochastic matrix,

$$P^2 = P \cdot P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} = \begin{bmatrix} 0+0+0 & 0+0+0 & 0+ \\ 0+0+1/2 & 0+0+1/2 & 0+ \\ 0+0+0 & \frac{1}{2}+0+0 & 0+ \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$P^3 = P \cdot P^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 0+\frac{1}{2}+0 & 0+\frac{1}{2}+0 & 0+0+0 \\ 0+0+0 & 0+0+\frac{1}{2} & 0+0+\frac{1}{2} \\ 0+\frac{1}{4}+0 & 0+\frac{1}{4}+0 & \frac{1}{2}+0+0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

$$P^4 = P \cdot P^3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

$$P^4 = \begin{bmatrix} 0+0+\overset{0}{\frac{1}{4}} & 0+1/2+0 & 0+1/2+0 \\ 0+0+1/4 & 0+0+1/4 & 0+0+1/2 \\ \frac{1}{4}+0+0 & \frac{1}{4}+\frac{1}{4}+0 & 0+\frac{1}{4}+0 \end{bmatrix} = \begin{bmatrix} \overset{0}{\frac{1}{4}} & \frac{1}{2} & \frac{1}{2} \\ 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}$$

$$P^5 = P.P^4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}$$

$$P^5 = \begin{bmatrix} 0+\frac{1}{4}+0 & 0+\frac{1}{4}+0 & 0+\frac{1}{2}+0 \\ 0+0+\frac{1}{4} & 0+0+\frac{1}{2} & 0+0+\frac{1}{4} \\ 0+\frac{1}{8}+0 & \frac{1}{4}+\frac{1}{8}+0 & \frac{1}{4}+\frac{1}{4}+0 \end{bmatrix} = \begin{bmatrix} 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 1/8 & 3/8 & \frac{1}{2} \end{bmatrix}$$

Hence, the given matrix is regular stochastic matrix.

To find unique fixed probability vector,

Let $V = [x, y, z]$ be the unique fixed probability vector

$\Rightarrow \boxed{x+y+z=1} \ni VP = V$

$$[x, y, z] \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} = [x, y, z]$$

$$[0+0+z/2,\ x+0+z/2,\ 0+y+0] = [x, y, z]$$

$$[z/2,\ x+z/2,\ y] = [x, y, z]$$

Comparing on both the sides.

$\frac{z}{2} = x \qquad x + \frac{z}{2} = y \qquad y = z$

$\boxed{z = 2x} \rightarrow (2) \qquad \boxed{y = z = 2x} \rightarrow (3)$

Using (2) and (3) in (1)

$x + y + z = 1$

$x + 2x + 2x = 1$

$5x = 1$

$\boxed{x = \frac{1}{5}}, \boxed{y = \frac{2}{5}}, \boxed{z = \frac{2}{5}}$

thus, $v = \left(\frac{1}{5}, \frac{2}{5}, \frac{2}{5}\right)$

7) Find the unique fixed probability vector of

MQP

$$P = \begin{bmatrix} 0 & 3/4 & 1/4 \\ 1/2 & 1/2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$\Rightarrow$ let $v = [x, y, z]$ be the unique fixed probability vector

$\boxed{x + y + z = 1} \rightarrow (1) \ni VP = V$

$$(x, y, z)\begin{bmatrix} 0 & 3/4 & 1/4 \\ 1/2 & 1/2 & 0 \\ 0 & 1 & 0 \end{bmatrix} = [x, y, z]$$

$[0 + y/2 + 0,\ 3x/4 + y/2 + z,\ x/4] = [x, y, z]$

$[y/2,\ 3x/4 + y/2 + z,\ x/4] = [x, y, z]$

$$\frac{y}{2} = x \qquad \frac{3x}{4} + \frac{y}{2} + z = y \qquad \frac{x}{4} = z$$

$$\boxed{y = 2x} \rightarrow (2) \qquad \boxed{z = \frac{x}{4}} \rightarrow (3)$$

using eqn (2) and (3) in (1)

$$x + y + z = 1$$

$$x + 2x + \frac{x}{4} = 1$$

$$\frac{4x + 8x + x}{4} = 1$$

$$\frac{13x}{4} = 1$$

$$13x = 4$$

$$x = \frac{4}{13}, \quad y = \frac{8}{13}, \quad z = \frac{4}{52} = \frac{1}{13}$$

Thus, $V = \left(\frac{4}{13}, \frac{8}{13}, \frac{1}{13}\right)$

MARKOV CHAINS [or PROCESS]

Defn: A stochastic process which is, generation of
|such that|
the probability distribution depend only in the present state is called Markov Process i.e;

A stochastic process in which the occurrence of future state depends on the current state and only on it is known as Markov chain.

STATE:

Defn: It is a condition (or) location of an object in the system at a particular time.

Eg:- i) Behaviour of consumer's in terms of their brand loyalty and switching pattern.

ii) Machines used to manufacture a product [Here, two states – working (or) not-working at any point].

Assumption:

1) Finite number of state
2) States are mutually exclusive
3) States are collectively exhaustive.
4) Probability of moving from one state to other state is constant over time.

Transition Probability: The prob. of moving from one state to another state (or) remaining in the same state during a single time period is called the Transition prob.

Mathematically, we write it as

$$P_{ij} = P(\text{initial state} - s_i - t=0, \text{Next state} - s_j \cdot \text{at } t=1)$$

Transition Probability Matrix (TPM):

The Probabilities which forms a square matrix where we predict the movement of system from one state to the next state.

$$P = [P_{ij}] \quad \begin{array}{ll} & \quad\ \ s_1 \quad\ s_2 \quad\ s_3 \ [\text{next state } n=1] \\ \text{Initial} \ \ s_1 & \\ \text{state} \ \ \ s_2 & \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \\ [n=0] \ \ s_3 & \end{array}$$

$P_{11} = P[s_1 \text{ at time } t=0, s_1 \text{ at time } t=1]$

$P_{12} = \text{Prob}[\text{initial state} - s_1 \text{ at } t=0, \text{Next state} - s_2 \text{ at } t=1]$

$P_{21} = \text{Prob}[\text{initial state} - s_2 \text{ at } t=0, \text{Next state} - s_1 \text{ at } t=1]$

These are called <u>one-step</u> <u>Transition</u> <u>probability</u>.

[we are switching from $t=0$ to $t=1$]

Say, 2-step transition probability.

$P_{11}^{(2)}$ = Prob(initial state S_1 at $t=0$ and next state S_1 at $t=2$]

$P_{21}^{(2)}$ = Prob(initial state S_2 at $t=0$ and next state S_1 at $t=2$]

Next state $S_j \longrightarrow$

$$P^{(2)} = \begin{array}{cc} & \begin{array}{ccc} S_1 & S_2 & S_3 \end{array} \\ \text{Initial state } S_i \quad \begin{array}{c} S_1 \\ S_2 \\ S_3 \end{array} & \begin{bmatrix} P_{11}^{(2)} & P_{12}^{(2)} & P_{13}^{(2)} \\ P_{21}^{(2)} & P_{22}^{(2)} & P_{23}^{(2)} \\ P_{31}^{(2)} & P_{32}^{(2)} & P_{33}^{(2)} \end{bmatrix} \end{array}$$

n-step transition Probability - The prob. that the system changes from the S_i to the next state S_j in exactly n-step.

$P^{(n)} = P_{ij}^{(n)}$

Next state S_j

$$\begin{array}{cc} & \begin{array}{ccc} S_1 & S_2 & S_3 \end{array} \\ \text{Initial state } S_i \quad \begin{array}{c} S_1 \\ S_2 \\ S_3 \end{array} & \begin{bmatrix} P_{11}^{(n)} & P_{12}^{(n)} & P_{13}^{(n)} \\ P_{21}^{(n)} & P_{22}^{(n)} & P_{23}^{(n)} \\ P_{31}^{(n)} & P_{32}^{(n)} & P_{33}^{(n)} \end{bmatrix} \end{array}$$

$P_{31}^{(n)}$ = Prob(initial state S_3 at $t=0$ and next state S_1 at $t=n$]

i.e; from one state to another state with 'n' steps.

Assumption of TPM:

i) Row sum = 1

ii) Each element of TPM is probability

i.e; $0 \leq P_{ij} \leq 1$ $\sum_{i,j=1}^{m} P_{ij} = 1$

iii) Square matrix

row represents initial and column represents next state.

NOTE:

Transition probability matrix.

$P^{(1)}$ - 1-step TPM $= P^{(0)}P$

$P^{(2)}$ - 2 step TPM $= P^{(1)}P$ (or) $P^{(0)}P^2$

$P^{(3)}$ - 3 step TPM $= P^{(2)}P$ (or) $P^{(0)}P^3$

Here, $P^{(0)} = [P_1^{(0)}, P_2^{(0)}, ---- P_m^{(0)}]$ denotes initial Probability distribution.

- Stationary distribution of regular Markov chains (fixed)

A Markov chain is said to be regular if the associated transition probability matrix P is regular.

A stationary distribution of a Markov chain is a prob. distribution that remains unchanged in the Markov chain as time process.

If P is a regular stochastic matrix of the Markov chain, then the sequence of n-step transition matrices $P^2, P^3 --- P^n$ approaches the matrix V whose rows are each the unique fixed probability vector of P.

i.e; $p^{(n)} = p^{(0)} P^n$

As $n \to \infty$ $P_i^{(n)} = V_i$ where $i = 1, 2, 3 -- m$ is called the stationary distribution of the Markov chain and $V = (V_1, V_2, -- V_m)$ is called the stationary (fixed) prob-vector of the markov chain.

NOTE: Markov chain is irreducible if the associated transition prob. matrix is regular.

$p^{(2)} \to$ transition prob.

$P^{(1)} \to$ one step transition p...

· Absorbing state of a Markov chain:

In a Markov chain, the process reaches to a certain state after which it continues to remain in the same state, such a state is called an Absorbing state of a Markov chain.

i.e; $P_{ij} = 1$ for $i = j$ and
0 otherwise

Eg: $$P = \begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array} \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$ The state a_1 is absorbing state of a markov chain.

Eg:- Three boys A, B, C are throwing ball to each other. A always throws ball to B and B always throws the ball to C. C is just likely to throw the ball to B as to A.

State space = {A, B, C}

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \end{array}$$

· Suppose C was the 1st person having the ball 1st then, $p^{(0)} = (0, 0, 1)$

- $P^5 = \begin{bmatrix} 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 1/8 & 3/8 & 1/2 \end{bmatrix}$ → P is regular stochastic matrix

$$p^{(5)} = p^{(0)}P^5 = [0,0,1] \begin{bmatrix} 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 1/8 & 3/8 & 1/2 \end{bmatrix} = [1/8, 3/8, 1/2]$$

$$p^{(5)} = [P_A^{(5)}, P_B^{(5)}, P_C^{(5)}]$$

The Prob. that the ball is with A is 1/8, with B is 3/8, with C = 1/2 after 5 throws.

- Unique fixed Prob. vector of P is (1/5, 2/5, 2/5)

We can conclude that as $n \to \infty$, A will have thrown the ball 20% of the time, B and C have thrown the ball 40% of the time.

NOTE:

$$p^{(1)} = p^{(0)}P = [P_1^{(1)}, P_2^{(1)} \ldots]$$

$$p^{(2)} = p^{(0)}P^2 = [P_1^{(2)}, P_2^{(2)}, P_3^{(2)} \ldots]$$

$$p^{(n)} = p^{(0)}P^n = [P_1^{(n)}, P_2^{(n)}, P_3^{(n)} \ldots]$$

PROBLEM:

1) The TPM of a Markov chain is given by

$$P = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}$$

and, the initial prob. distribution is $p^{(0)}(1/2, 1/2, 0)$. Find $P_{13}^{(2)}, P_{23}^{(2)}, p^{(2)}, p_1^{(2)}$.

↳ choosen from matrix form.

⇒

NOTE: $P^2 = P \cdot P = \begin{bmatrix} P_{11}^{(2)} & P_{12}^{(2)} & P_{13}^{(2)} \\ P_{21}^{(2)} & P_{22}^{(2)} & P_{23}^{(2)} \\ P_{31}^{(2)} & P_{32}^{(2)} & P_{33}^{(2)} \end{bmatrix}$

$$p^{(2)} = p^{(0)} \cdot P^2 = [p_1^{(2)}, p_2^{(2)}, p_3^{(2)}]$$

$$P^3 = P \cdot P^2 = \begin{bmatrix} P_{11}^{(3)} & P_{12}^{(3)} & P_{13}^{(3)} \\ P_{21}^{(3)} & P_{22}^{(3)} & P_{23}^{(3)} \\ P_{31}^{(3)} & P_{32}^{(3)} & P_{33}^{(3)} \end{bmatrix}$$

$$p^{(3)} = p^{(0)} \cdot P^3 = [p_1^{(3)}, p_2^{(3)}, p_3^{(3)}]$$

$$P^2 = P \cdot P = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} \frac{1}{4}+0+\frac{1}{8} & 0+0+\frac{1}{4} & \frac{1}{4}+0+\frac{1}{8} \\ 1/2+0+0 & 0+0+0 & 1/2+0+0 \\ \frac{1}{8}+\frac{1}{2}+\frac{1}{16} & 0+0+1/4 & 1/8+0+\frac{1}{16} \end{bmatrix} =$$

$$P^2 = \begin{bmatrix} \frac{3}{8} & \frac{1}{4} & \frac{3}{8} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{11}{16} & \frac{1}{8} & \frac{3}{16} \end{bmatrix} = \begin{bmatrix} P_{11}^{(2)} & P_{12}^{(2)} & P_{13}^{(2)} \\ P_{21}^{(2)} & P_{22}^{(2)} & P_{23}^{(2)} \\ P_{31}^{(2)} & P_{32}^{(2)} & P_{33}^{(2)} \end{bmatrix}$$

$$\therefore \quad P_{13}^{(2)} = \frac{3}{8}$$

$$P_{23}^{(2)} = \frac{1}{2}$$

To find $p^{(2)}$:

$$p^{(2)} = p^{(0)} \cdot P^2$$

$$p^{(2)} = \begin{bmatrix} 1/2 & 1/2 & 0 \end{bmatrix}_{1\times 3} \begin{bmatrix} \frac{3}{8} & \frac{1}{4} & \frac{3}{8} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{11}{16} & \frac{1}{8} & \frac{3}{16} \end{bmatrix}_{3\times 3} = 1\times 3$$

$$p^{(2)} = \begin{bmatrix} \frac{3}{16}+\frac{1}{4}+0, & \frac{1}{8}+0+0, & \frac{3}{16}+\frac{1}{4}+0 \end{bmatrix}$$

$$p^{(2)} = \begin{bmatrix} \frac{7}{16}, & \frac{1}{8}, & \frac{7}{16} \end{bmatrix}$$

$$p^{(2)} = \begin{bmatrix} p_1^{(2)}, & p_2^{(2)}, & p_3^{(2)} \end{bmatrix}$$

$$\therefore \quad p^{(2)} = \begin{bmatrix} \frac{7}{16} & \frac{1}{8} & \frac{7}{16} \end{bmatrix}$$

$$p_1^{(2)} = \frac{7}{16}$$

5) Three boys A, B, C are throwing balls to each other. A always throws the ball to B and B always throws the ball to C. C is just as likely to throw the ball to B as to A. If C was a 1st person to throw the ball, find the probability that after 3 throws,

i) A has the ball

ii) B has the ball

iii) C has the ball

Prove that TPM is irreducible (regular). find the corresponding stationary probability vector?

⇒ State space = {A, B, C}

$$TPM = P = P_{ij} = \begin{array}{c c} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \end{array}$$

C is the first person to throw the ball. this gives us the initial condition i.e; $p^{(0)} = [0, 0, 1]$

To find the probabilities after 3 throws implies 3 step transition probabilities i.e; $\boxed{p^{(3)} = p^{(0)} \cdot p^3} \rightarrow (1)$

$$P^2 = P \cdot P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

$$p^3 = p \cdot p^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

Substituting p^3 in eqn ①,

$$p^{(3)} = p^{(0)} \cdot p^3$$

$$= [0 \ 0 \ 1] \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix} = [1/4 \ \ 1/4 \ \ 1/2]$$

$$= [p_A^{(3)}, p_B^{(3)}, p_C^{(3)}]$$

Thus, after 3 throws, the probability that the ball is with A is 1/4 and with B is 1/4, with the C is 1/2.

To prove it is irreducible, P has to be regular

[It is regular for P^5, refer the last problem of stochastic] Q.no. ⑥

$$P^5 = \begin{bmatrix} 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 1/8 & 3/8 & 1/2 \end{bmatrix}$$

$\therefore$ P is irreducible.

To find stationary probability vector i.e., to find unique fixed prob. vector

[Refer last problem of stochastic] Q.no. ⑥

$\boxed{x = \frac{1}{5}}$ $\boxed{y = \frac{2}{5}}$ $\boxed{z = \frac{2}{5}}$

Thus, the required stationary prob. vector is

$$\left(\frac{1}{5}, \frac{2}{5}, \frac{2}{5}\right)$$

3) Every year, a man trades his car for a new car, if he has a Maruti, he trades it for an Ambassador if he has an Ambassador, he trades it for Santro however if he had Santro he is just as likely to trade it for a new Santro as to trade it for a Maruti or an Ambassador. In 2000 he bought his 1st car which was a santro. Find the prob. that he has

(i) 2002 Santro
(ii) 2002 Maruti
(iii) 2003 Ambassador
(iv) 2003 Santro

$M \to A$
$A \to S$
$S \to$ new S/M/A

⇒ State space = {Maruti, Ambassador, Santro}

Associated Transition Matrix $= P = P_{ij} =$

$$\begin{array}{c|c} & \begin{matrix} m & A & S \end{matrix} \\ \begin{matrix} m \\ A \\ S \end{matrix} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} \end{array}$$

He bought his 1st car in 2000

2002 ⇒ this implies 2 years after his 1st car.
P^2 matrix

2000 → 2002

$$P^2 = P \cdot P = \begin{array}{c|c} & \begin{matrix} m & A & S \end{matrix} \\ \begin{matrix} m \\ A \\ S \end{matrix} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} \end{array} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \\ \boxed{1/9} & 4/9 & \boxed{4/9} \end{bmatrix} = \begin{bmatrix} P_{11}^{(2)} & P_{12}^{(2)} & P_{13}^{(2)} \\ P_{21}^{(2)} & P_{22}^{(2)} & P_{23}^{(2)} \\ P_{31}^{(2)} & P_{32}^{(2)} & P_{33}^{(2)} \end{bmatrix}$$

3 1

$\frac{1}{3} + \frac{1}{9}$

$\frac{3+1}{9} = \frac{4}{9}$

$$P^3 = P \cdot P^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1/3 & 1/3 & 1/3 \\ 1/9 & 4/9 & 4/9 \end{bmatrix}$$

③ + ①

$\frac{1}{9} + \frac{1}{27} = \frac{4}{27}$

$$= \begin{array}{c|ccc} & m & A & S \\ \hline m & 1/3 & 1/3 & 1/3 \\ A & 1/9 & 4/9 & 4/9 \\ S & 4/27 & 7/27 & 16/27 \end{array} = \begin{bmatrix} P_{11}^{(3)} & P_{12}^{(3)} & P_{13}^{(3)} \\ P_{21}^{(3)} & P_{22}^{(3)} & P_{23}^{(3)} \\ P_{31}^{(3)} & P_{32}^{(3)} & P_{33}^{(3)} \end{bmatrix}$$

In 2000-2002

(i) Santro-Santro $= P_{33}^{(2)} = 4/9$

(ii) Santro-maruti $= P_{31}^{(2)} = 1/9$

In 2000-2003

(iii) Santro-Ambassador

$$= P_{32}^{(3)} = \boxed{\frac{7}{27}}$$

(iv) Santro-Santro $= P_{33}^{(3)} = \boxed{\frac{16}{27}}$

In the long run of probability of having Santro is,

$$\boxed{P(S) = \frac{3}{6} = \frac{1}{2} = 0.5}$$

In the long run, 50 % of the time he will have Santro

To this particular case, we find unique fixed probability

i.e; As $n \to \infty$

$$V = (x, y, z) \Rightarrow VP = V \quad \& \quad x + y + z = 1$$

$$\boxed{x = \frac{1}{6}} \quad \boxed{y = \frac{2}{6}} \quad \boxed{z = \frac{3}{6}}$$

$$V = \left[\frac{1}{6}, \frac{2}{6}, \frac{3}{6} \right]$$

$$V = \left[P(m), P(A), P(S) \right]$$

5) A gamblers luke follows the pattern if he wins a game, the probability of winning the next game is 0.6. However, if he loss a game, the probability of losing the next game is 0.7. There is an even chance of gambler winning the first game. If so,

(i) What is the probability of he winning the second game?

(ii) What is the probability of he winning the third game?

(iii) In the long run, how often he will win?

⇒ State space = {win, lose} = {w, l}

Associated transition matrix = $P = P_{ij} = \begin{array}{c|cc} & W & L \\ \hline W & 0.6 & 0.4 \\ L & 0.3 & 0.7 \end{array}$

WKT, $0.6 + \_ = 1 \Rightarrow 0.4$

$\_ + 0.7 = 1 \Rightarrow 0.3$

$$P = P_{ij} = \begin{array}{c|cc} & W & L \\ \hline W & 6/10 & 4/10 \\ L & 3/10 & 7/10 \end{array} = \frac{1}{10}\begin{bmatrix} 6 & 4 \\ 3 & 7 \end{bmatrix}$$

From the given, Probability of even chance of winning the first game is, $p^{(0)} = [1/2, 1/2]$.

(i) Probability of winning the second game

$$p^{(1)} = p^{(0)} \cdot P = \left[\frac{1}{2}, \frac{1}{2}\right] \frac{1}{10} \begin{bmatrix} 6 & 4 \\ 3 & 7 \end{bmatrix}$$

$$p^{(1)} = \frac{1}{2}[1, 1] \frac{1}{10} \begin{bmatrix} 6 & 4 \\ 3 & 7 \end{bmatrix}$$

$$p^{(1)} = \frac{1}{20}[1, 1] \begin{bmatrix} 6 & 4 \\ 3 & 7 \end{bmatrix}$$

$$p^{(1)} = \frac{1}{20}\left[6+3 \quad 4+7\right]$$

$$p^{(1)} = \left[\frac{9}{20}, \frac{11}{20}\right]$$

$$p^{(1)} = \left[p(W), p(L)\right]$$

$\therefore$ Probability of winning the second game is $\underline{9/20}$.

(ii) Probability of ~~losing~~ winning the third game is

Probability of he winning the third game is,

$$\boxed{p^{(2)} = p^{(0)} \cdot P^2} \rightarrow (1)$$

Let us find P^2

$$P^2 = P \cdot P = \frac{1}{10} \begin{bmatrix} 6 & 4 \\ 3 & 7 \end{bmatrix} \cdot \frac{1}{10} \begin{bmatrix} 6 & 4 \\ 3 & 7 \end{bmatrix}$$

$$P^2 = \frac{1}{100} \begin{bmatrix} 36+12 & 24+28 \\ 18+21 & 12+49 \end{bmatrix}$$

$$P^2 = \frac{1}{100} \begin{bmatrix} 48 & 52 \\ 39 & 61 \end{bmatrix}$$

$\therefore$ equn ① $\Rightarrow$ $p^{(2)} = p^{(0)} \cdot p^2$

$$= [1/2, 1/2] \cdot \frac{1}{100}\begin{bmatrix} 48 & 52 \\ 39 & 61 \end{bmatrix}$$

$$= \frac{1}{2}[1, 1] \cdot \frac{1}{100}\begin{bmatrix} 48 & 52 \\ 39 & 61 \end{bmatrix}$$

$$= \frac{1}{200}[48+39 \quad 52+61]$$

$$p^{(2)} = \frac{1}{200}[87, 113]$$

$$p^{(2)} = \left[\frac{87}{200}, \frac{113}{200}\right]$$

$$p^{(2)} = [p(w), p(L)]$$

$\therefore$ Probability of winning the third game is $\underline{87/200}$.

(iii) Probability of $n \to \infty$, we have to find unique fixed probability vector.

Let, $V = (x, y)$ be the unique fixed probability vector $\Rightarrow$ $\boxed{x+y=1} \to (1)$ $\ni VP = V$

$$VP = V$$

$$[x, y] \frac{1}{10}\begin{bmatrix} 6 & 4 \\ 3 & 7 \end{bmatrix} = [x, y]$$

$$\frac{1}{10}[6x+3y, 4x+7y] = [x, y]$$

Comparing both the sides,

$\frac{1}{10}(6x+3y)=x$ $\qquad\qquad$ $\frac{1}{10}(4x+7y)=y$

$6x+3y=10x$

$3y=4x$

$\boxed{y=\frac{4x}{3}}$ $\rightarrow (2)$

Using (2) in (1),

$\boxed{x=\frac{3}{7}}$, $\boxed{y=\frac{4}{7}}$

$\therefore V=(x,y)$

$\boxed{V=\left(\frac{3}{7},\frac{4}{7}\right)}$ — verify using $x+y=1$

$V=(p(W), p(L))$

$\therefore$ In the long run, he wins $3/7$th of the time.