Mathematics for Computer Science (III Semester)

BCS301 - Module 1

Probability distributions

Syllabus:

Module-1: Probability Distributions
Probability Distributions: Review of basic probability theory. Random variables (discrete and continuous), probability mass and density functions. Mathematical expectation, mean and variance. Binomial, Poisson and normal distributions- problems (derivations for mean and standard deviation for Binomial and Poisson distributions only)-Illustrative examples. Exponential distribution. **(12 Hours)**   **(RBT Levels: L1, L2 and L3)**

Review of basic probability theory:

Set theory	Probability
Universal set	Sample space
Set	Event
Elements	Outcome
Disjoint sets	Mutually exclusive events

- ❖ Sample space S is the set of all possible outcomes.
- ❖ The probability P is a real valued function whose domain is S and range is the interval [0,1] satisfying the following axioms:

  (i) For any event E, $P\ (E)\ \geq\ 0$

  (ii) $P\ (S)\ =\ 1$

  (iii) If E and F are mutually exclusive events, then $P(E\ \cup\ F)\ =\ P(E)\ +\ P(F)$.
- ❖ If E and F are equally likely to occur then $P(E) = P(F)$.
- ❖ If E and F are any two events then $P(E \cup F) = P(E) + P(F) - P(E \cap F)$.
- ❖ If E and F are mutually exclusive events then $P(E \cap F) = 0$.

1.1 Discrete probability distribution

Introduction:

- A ***random experiment*** is a process that leads to a single outcome that can't be predicted with certainty. Example: Tossing coin, playing cricket.

- In a random experiment, if a real variable is associated with every outcome then it is called a ***random variable*** (or) stochastic variable.

  Example: Suppose a coin is tossed twice. Now, 1 is associated with head and 0 is associated with tail. $S = \{HH, HT, TH, TT\}$

Outcomes	HH	HT	TH	TT
Random variable (X)	2	1	1	0

- If a random variable takes any discrete value (nonnegative integer) then it is called ***discrete random variable***.

  Example: Number of students in a class, Number of leaves in a tree.

- For each value of x_i of a discrete random variable X, we assign a real number $P(x_i)$ such that $\sum_{i=1}^{n} P(x_i) = 1$ then

X	x_1	x_2	x_3	...	x_n
$P(X)$	$P(x_1)$	$P(x_2)$	$P(x_3)$	...	$P(x_n)$

  is called a ***discrete probability distribution*** of X.

- The function $P(x)$ is called probability mass function.

- The discrete function $F(x)$ defined by $F(x) = P(X \leq x) = \sum_{i=1}^{x} P(x_i)$ is called cumulative distribution function or distribution function.

- Expectation $E(x) = \sum_{i=1}^{n} x_i P(x_i)$ and $E(x^2) = \sum_{i=1}^{n} {x_i}^2 P(x_i)$

- Mean = $E(x)$, Variance = $E(x^2) - [E(x)]^2$, Standard deviation = $\sqrt{Variance}$

1. **Find the mean and standard deviation for the following probability distribution:**

x_i	-5	-4	1	2
$P(x_i)$	**1/4**	**1/8**	**1/2**	**1/8**

$E(x_i) = \Sigma x_i p(x_i) = (-5)\frac{1}{4} + (-4)\frac{1}{8} + (1)\frac{1}{2} + (2)\frac{1}{8} = -1$

$E(x_i^2) = \Sigma x_i^2 p(x_i) = (-5)^2\frac{1}{4} + (-4)^2\frac{1}{8} + (1)^2\frac{1}{2} + (2)^2\frac{1}{8} = 9.25$

Mean $= E(x_i) = -1$

Variance $= E(x_i^2) - [E(x_i)]^2 = 9.25 - 1 = 8.25$

Standard deviation $= \sqrt{8.25} = 2.8723$

2. **Find the mean and standard deviation for the following probability distribution:**

x_i	1	3	4	5
$P(x_i)$	**0.4**	**0.1**	**0.2**	**0.3**

$E(x_i) = \Sigma x_i p(x_i) = 1(0.4) + 3(0.1) + 4(0.2) + 5(0.3) = 3$

$E(x_i^2) = \Sigma x_i^2 p(x_i) = 1^2(0.4) + 3^2(0.1) + 4^2(0.2) + 5^2(0.3) = 12$

Mean $= E(x_i) = -1$

Variance $= E(x_i^2) - [E(x_i)]^2 = 12 - 9 = 3$

Standard deviation $= \sqrt{3} = 1.7320$

3. The probability density function P(x) of a variate x is given by the following table:

x	**0**	**1**	**2**	**3**	**4**	**5**	**6**
p(x)	**k**	**3k**	**5k**	**7k**	**9k**	**11k**	**13k**

(i) For what value of k, does this represent a valid probability distribution.

(ii) Find $p(x < 4), p(x \geq 5)$ and $p(3 < x \leq 6)$

(iii) Determine the minimum value of k so that $P(X \leq 2) \geq 0.3$

Case (i)

Since it is a probability distribution,

$\Sigma p(x_i) = 1 \Longrightarrow 49k = 1 \Longrightarrow k = 1/49$

Case (ii)

$p(x < 4) = p(0) + p(1) + p(2) + p(3) = k + 3k + 5k + 7k = 16k = 16/49$

$p(x \geq 5) = p(5) + p(6) = 11k + 13k = 24k = 24/49$

$p(3 < x \leq 6) = p(4) + p(5) + p(6) = 9k + 11k + 13k = 33k = 33/49$

Case (iii)

$p(x \leq 2) \geq 0.3 \Longrightarrow p(0) + p(1) + p(2) \geq 0.3$

$$\Longrightarrow k + 3k + 5k \geq 0.3 \Longrightarrow 9k \geq 0.3 \Longrightarrow k \geq 1/30$$

Therefore, minimum value of k is $1/30$.

4. The probability distribution of a finite random variable is given by

x_i	-2	-1	0	1	2	3
p(x_i)	0.1	k	0.2	2k	0.3	k

(i) Determine the value of k and find the mean and standard deviation.

(ii) Find $p(x<1), p(-1<x\le 2)$ and $p(x>-1)$

Case (i)

Since it is a probability distribution,

$\Sigma p(x_i) = 1 \Longrightarrow 4k + 0.6 = 1 \Longrightarrow k = 0.1$

$E(x_i) = \Sigma x_i p(x_i) = (-2)0.1 + (-1)k + (0)0.2 + (1)2k + (2)0.3 + (3)k$

$= 4k + 0.4 = 0.8$

$E(x_i^2) = \Sigma x_i^2 p(x_i) = (-2)^2 0.1 + (-1)^2 k + (0)^2 0.2 + (1)^2 2k + (2)^2 0.3 + (3)^2 k$

$= 12k + 1.6 = 2.8$

Mean $= E(x_i) = 0.8$

Variance $= E(x_i^2) - [E(x_i)]^2 = 2.8 - 0.64 = 2.16$

Standard deviation $= \sqrt{2.16} = 1.4697$

Case (ii)

$p(x<1) = p(-2) + p(-1) + p(0) = 0.1 + k + 0.2 = 0.4$

$p(-1<x\le 2) = p(0) + p(1) + p(2) = 0.2 + 2k + 0.3 = 2k + 0.5 = 0.7$

$p(x>-1) = p(0) + p(1) + p(2) + p(3) = 0.2 + 2k + 0.3 + k = 3k + 0.5 = 0.8$

5. **A random variable X has the following probability function:**

x_i	0	1	2	3	4	5	6	7
P(x_i)	**0**	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2 + k$

(i) Find the value of k. (ii) Evaluate $P(X < 6), P(x \geq 6)$ (iii) $P(0 < x < 5)$.

Case (i)

Since it is a probability distribution,

$$\Sigma p(x_i) = 1 \Longrightarrow 0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$$

$$\Longrightarrow 10k^2 + 9k - 1 = 0$$

$$\Longrightarrow k = 0.1 \text{ (The other value is more than 1)}$$

Case (ii)

$p(x < 6) = p(0) + p(1) + p(2) + p(3) + p(4) + p(5)$

$= 0 + k + 2k + 2k + 3k + k^2 = k^2 + 8k = 0.81 \ (\because k = 0.1)$

$p(x \geq 6) = p(6) + p(7) = 2k^2 + 7k^2 + k = 0.19$

Case (iii)

$p(0 < x < 5) = p(0) + p(1) + p(2) + p(3) + p(4) = 0 + k + 2k + 2k + 3k = 8k$

$= 0.8$

6. **A fair coin is tossed 3 times. Let X denote the number of heads showing up. Find the distribution of X. Also find its mean, variance and SD.**

By data, $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

$x =$ Number of heads= {0, 1, 2, 3}

The probability distribution is

x	0	1	2	3
$p(x)$	$1/8$	$3/8$	$3/8$	$1/8$

$$E(x) = \Sigma x p(x) = (0)\frac{1}{8} + (1)\frac{3}{8} + (2)\frac{3}{8} + (3)\frac{1}{8} = 3/2$$

$$E(x^2) = \Sigma x^2 p(x) = (0)^2\frac{1}{8} + (1)^2\frac{3}{8} + (2)^2\frac{3}{8} + (3)^2\frac{1}{8} = 3$$

Mean $= E(x) = \frac{3}{2}$

Variance $= E(x^2) - [E(x)]^2 = 3 - \frac{9}{4} = 3/4$

Standard deviation $= \sqrt{Var} = \sqrt{3/4}$

7. **A random variable x has the density function $P(X) = \begin{cases} kx^2, & x = 0, 1, 2, 3 \\ 0, & elsewhere \end{cases}$ evaluate k. Also find $P(X \le 1), P(1 \le x \le 2),\ P(X > 1)\ and\ P(X > 2)$**

By data,

x	0	1	2	3
$p(x)$	0	k	$4k$	$9k$

Since $p(x)$ is a probability density function,

$\Sigma p(x) = 1 \Rightarrow k + 4k + 9k = 1 \Rightarrow k = 1/14$

$p(x \le 1) = p(0) + p(1) = 0 + k = 1/14$

$p(1 \le x \le 2) = p(1) + p(2) = k + 4k = 5k = 5/14$

$p(x > 1) = p(2) + p(3) = 4k + 9k = 13k = 13/14$

$p(x > 2) = p(3) = 9k = 9/14$

8. **A box contains 12 items of which 4 are defectives. A sample of 3 items is selected from the box. Let X denote the number of defective items in the sample. Find the probability distribution of X. Determine the mean and Standard deviation.**

$x =$ Number of defectives= {0, 1, 2, 3}

$P(0) = \frac{8C_3}{12C_3} = \frac{14}{55}, \qquad P(1) = \frac{4C_1 \times 8C_2}{12C_3} = \frac{28}{55}$

$P(2) = \frac{4C_2 \times 8C_1}{12C_3} = \frac{12}{55}, P(3) = \frac{4C_3}{12C_3} = \frac{1}{55}$

Probability distribution is

x	0	1	2	3
$p(x)$	$14/55$	$28/55$	$12/55$	$1/55$

$E(x) = 0p(0) + 1p(1) + 2p(2) + 3p(3)$

$= 0\left(\frac{14}{55}\right) + 1\left(\frac{28}{55}\right) + 2\left(\frac{12}{55}\right) + 3\left(\frac{1}{55}\right) = 1$

$E(x^2) = 0^2p(0) + 1^2p(1) + 2^2p(2) + 3^2p(3)$

$= 0\left(\frac{14}{55}\right) + 1\left(\frac{28}{55}\right) + 4\left(\frac{12}{55}\right) + 9\left(\frac{1}{55}\right) = \frac{17}{11}$

$Mean = E(x) = 1$

$Variance = E(x^2) - [E(x)]^2 = \frac{17}{11} - 1 = \frac{6}{11}.$

Home work:

9. A die is tossed thrice. A success is getting 1 or 6 on a toss. Find mean and variance of the number of successes.
10. Four coins are tossed. Find the expectation of the number of heads.

1.2 Binomial distribution

Introduction:

- p is the probability of success and q is the probability of failure.

- $q = 1 - p\ (or)\ p + q = 1.$

- $nC_0 p^0 q^n + nC_1 p^1 q^{n-1} + nC_2 p^2 q^{n-2} + \cdots + nC_n p^n q^0 = \sum nC_x p^x q^{n-x} = (p+q)^n = 1$

- The probability density function is said to follow binomial distribution if $P(x)$ satisfies the condition $\boldsymbol{P(x) = nC_x p^x q^{n-x}}$**.** Where p is the probability of success and $q = 1 - p$ is the probability of failure.

- The binomial distribution with 4 Bernoullian trials is given by

x_i	0	1	2	3	4
$P(x_i)$	$4C_0 p^0 q^{4-0}$	$4C_1 p^1 q^{4-1}$	$4C_2 p^2 q^{4-2}$	$4C_3 p^3 q^{4-3}$	$4C_4 p^4 q^{4-4}$

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x is at least 2	x is at most 2
$x \geq 2$	$x \leq 2$

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Mean (μ)	Variance (σ^2)	Standard deviation (σ)
np	npq	$\sqrt{npq}$

1. Find mean and standard deviation of Binomial distribution.

Mean$= \sum xP(x)$

$$= \sum x(nC_x p^x q^{n-x})$$

$$= \sum x\left(\frac{n!}{x!(n-x)!}p^x q^{n-x}\right)$$

$$= np\sum x\left(\frac{(n-1)!}{(x-1)!(n-x)!}p^{x-1} q^{n-x}\right)$$

$$= np\sum (n-1)C_{x-1}p^{x-1}q^{n-x}$$

$$= np\,(p+q)^{n-1}$$

$$= np$$

Variance $= E(x^2) - [E(x)]^2$

$$= E(x^2 - x + x) - [E(x)]^2$$

$$= E[x(x-1) + x] - [E(x)]^2$$

$$= E\big(x(x-1)\big) + E(x) - [E(x)]^2$$

$$= \sum x(x-1)(nC_x p^x q^{n-x}) + np - (np)^2$$

$$= \sum x(x-1)\left(\frac{n!}{x!(n-x)!}p^x q^{n-x}\right) + np - n^2p^2$$

$$= n(n-1)p^2\sum\left(\frac{(n-2)!}{(x-2)!(n-x)!}p^{x-2}q^{n-x}\right) + np - n^2p^2$$

$$= n(n-1)p^2\sum\left(\frac{m!}{r!(m-r)!}p^r q^{m-r}\right) + np - n^2p^2$$

$$= (n^2-n)p^2\sum(mC_r p^r q^{m-r}) + np - n^2p^2$$

$$= (n^2-n)p^2(p+q)^m + np - n^2p^2$$

$$= (n^2-n)p^2 + np - n^2p^2$$

$$= n^2p^2 - np^2 + np - n^2p^2$$

$$= -np^2 + np$$

$$= np(1-p)$$

$$= npq$$

2. **Determine the Binomial distribution for which mean is twice the variance and the sum of mean and variance is 3. Also find $P(x \leq 3)$.**

To find: n, p, q

By data, Mean = 2 Variance

$$np = 2npq \, .$$

By solving, $q = \frac{1}{2}$, $p = 1 - \frac{1}{2} = \frac{1}{2}$

By data, $Mean + Variance = 3$

$np + npq = 3$

$n\left(\frac{1}{2}\right) + n\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = 3$

By solving, $n = 4$.

Therefore, $n = 4, p = \frac{1}{2}, q = \frac{1}{2}$

To find: $p(x)$

It follows Binomial distribution.

Probability function is given by

$$P(x) = nC_x p^x q^{n-x}$$

$$= 4C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{4-x}$$

$$= 4C_x \left(\frac{1}{2}\right)^4$$

$$= \frac{1}{16} \times 4C_x$$

To find: $p(x \leq 3)$

$p(x \leq 3) = 1 - p(x > 3)$

$$= 1 - p(4)$$

$$= 1 - \frac{1}{16} 4C_4$$

$$= \frac{15}{16}$$

3. **Mean and standard deviation of a binomial distribution are 2 and $2/\sqrt{3}$ respectively. Find the corresponding probability density function.**

To find: $\boldsymbol{n, p, q}$

By data, Mean $= np = 2$

Variance $= npq = \frac{4}{3}$.

By dividing, $q = \frac{4}{6} = \frac{2}{3}$, $p = 1 - q = \frac{1}{3}$, $n = \frac{2}{p} = 6$

Therefore, $n = 6, p = \frac{1}{3}$, $q = \frac{2}{3}$

To find: $\boldsymbol{p(x)}$

It follows Binomial distribution.

Probability function is given by

$$p(x) = nC_x p^x q^{n-x} = 6C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{6-x}$$

4. **In 256 sets of 12 tosses of a coin, in how many cases one can expect 8 heads and 4 tails?**

To find: $\boldsymbol{n, p, q}$ *and* $\boldsymbol{N}$

While tossing a coin,

$p = P(Head) = \frac{1}{2}$, $q = P(tail) = \frac{1}{2}$

By data, $n = 12, N = 256$.

To find: $\boldsymbol{p(x)}$

It follows Binomial distribution.

Probability density function is given by

$$P(x) = nC_x p^x q^{n-x} = 12C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{12-x} = \frac{1}{2^{12}} \times 12C_x$$

To find: Number of cases

No. of cases one can expect 8 heads $= NP(8)$

$$= 256\left(\frac{1}{2^{12}}\right)(12C_8)$$

$$= \frac{1}{16} \times 495$$

$$= 31$$

5. **A die is tossed thrice. A success is '$\boldsymbol{getting}$ 1 $\boldsymbol{or}$ 6' on a toss. Find the mean and variance of the number of successes.**

To find: $\boldsymbol{n, p, q}$

$n = 3$

$p = P(getting\ 1\ or\ 6) = \frac{2}{6} = \frac{1}{3}$

$q = 1 - p = 1 - \frac{1}{3} = \frac{2}{3}$

To find: Mean, Variance

It follows Binomial distribution.

Mean $= np = 3\left(\frac{1}{3}\right) = 1$

Variance $= npq = 3\left(\frac{1}{3}\right)\left(\frac{2}{3}\right) = \frac{2}{3}$

6. **A die is thrown 5 times. If getting an odd number is a success, find the probability of getting at least 4 successes.**

To find: $\boldsymbol{n, p, q}$

$n = 5$

$p = P(getting\ odd\ number) = \frac{3}{6} = \frac{1}{2}$

$q = 1 - \frac{1}{2} = \frac{1}{2}$

To find: $\boldsymbol{P(x)}$

It follows Binomial distribution.

Probability function is given by

$$P(x) = nC_x p^x q^{n-x} = 5C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{5-x} = \frac{1}{2^5} \times 5C_x$$

To find: P (at least 4 successes)

$$\begin{aligned}\text{P(at least 4 successes)} &= P(x \geq 4)\\ &= P(4) + P(5)\\ &= \frac{1}{2^5} \times 5C_4 + \frac{1}{2^5} \times 5C_5\\ &= \frac{1}{32}(5+1) = \frac{6}{32} = \frac{3}{16}\end{aligned}$$

7. If the probability that a new-born child is a male is 0.6, find the probability that in a family of 5 children there are exactly 3 boys.

To find: $\boldsymbol{n}, \boldsymbol{p}, \boldsymbol{q}$

By data, $n = 5$

$p = P(male\ child) = 0.6$

$q = 1 - 0.6 = 0.4$

To find: $\boldsymbol{P(x)}$

It follows Binomial distribution.

Probability function is given by

$P(x) = nC_x p^x q^{n-x} = 5C_x (0.6)^x (0.4)^{5-x}$

To find: P (exactly 3 boys)

P(exactly 3 boys) $= P(3)$

$= 5C_3 (0.6)^3 (0.4)^{5-3}$

$= 0.3456$

8. **The probability that a pen manufactured by a company will be defective is 0.1. If 12 such pens are selected, find the probability that (i) exactly 2 will be defective (ii) at least 2 will be defective (iii) none will be defective.**

To find: $\boldsymbol{n, p, q}$

By data, $n = 12$

$p = P(defective\ pen) = 0.1$

$q = 1 - 0.1 = 0.9$

To find: $\boldsymbol{p(x)}$

It follows Binomial distribution.

Probability density function is given by

$P(x) = nC_x p^x q^{n-x} = 12C_x (0.1)^x (0.9)^{12-x}$

To find: Required probabilities

(i) P(exactly 2 defectives) $= P(2)$

$= 12C_2 (0.1)^2 (0.9)^{12-2}$

$= 0.2301$

(ii) P(at least 2 defectives) $= P(x \geq 2)$

$= 1 - P(x < 2)$

$= 1 - [P(0) + P(1)]$

$= 1 - [12C_0 (0.1)^0 (0.9)^{12-0} + 12C_1 (0.1)^1 (0.9)^{12-1}]$

$= 1 - 0.9^{11}(0.9 + 1.2)$

$= 0.3410$

(iii) P(none will be defective) $= P(0)$

$= 12C_0 (0.1)^0 (0.9)^{12}$

$= 0.2825$

9. The number of telephone lines busy at an instant of time is a binomial variate with probability 0.2. If at an instant 10 lines are chosen at random, what is the probability that (i) no line is busy? (ii) 5 lines are busy? (iii) at least one line is busy? (iv) at most 2 lines are busy? (v) all lines are busy?

By data, $n = 10,\ p = 0.2,\ q = 0.8$

It follows Binomial distribution.

Probability function is given by

$P(x) = nC_x p^x q^{n-x} = 10C_x (0.2)^x (0.8)^{10-x}$

(i) P (No line is busy)

$= P(0)$

$= 10C_0 (0.2)^0 (0.8)^{10-0}$

$= 0.1074$

(ii) P(5 lines are busy)

$= P(5)$

$= 10C_5 (0.2)^5 (0.8)^{10-5}$

$= 0.0264$

(iii) P (At least one line is busy) $= P(X \geq 1)$

$= 1 - P(X < 1)$

$= 1 - P(0)$

$= 1 - 0.1074 = 0.8926$

(iv) P (At most two lines are busy)

$= P(x \leq 2)$

$= P(0) + P(1) + P(2)$

$= 10C_0 (0.2)^0 (0.8)^{10} + 10C_1 (0.2)^1 (0.8)^9 + 10C_2 (0.2)^2 (0.8)^8$

$= (0.8)^8 [0.64 + 1.6 + 1.80]$

$= 0.6778$

(v) P (All lines are busy)

$= P(10)$

$= 10C_{10} (0.2)^{10} (0.8)^0$

$= (0.2)^{10}$

10. **In a large number of parts manufactured by a machine, the mean number of defectives in a sample of 20 is 2. Out of 1000 such samples, how many would be expected to contain at least 3 defective parts?**

Hint: $P(x) = 20C_x(0.1)^x(0.9)^{20-x}, f(x \geq 3) = 1000 \times P(x \geq 3) = 566$

Solution:

To find: $\boldsymbol{n}, \boldsymbol{p}, \boldsymbol{q}$

By data, $n = 20,\ \bar{x} = np = 2$

$p = \frac{2}{n} = \frac{2}{20} = 0.1$

$q = 1 - p = 1 - 0.1 = 0.9$

To find: $\boldsymbol{P(x)}$

It follows Binomial distribution.

Probability function is given by

$p(x) = nC_x p^x q^{n-x} = 20C_x(0.1)^x(0.9)^{20-x}$

To find: No. of samples

No. of samples containing at least 3 defective parts

$= N \times P(x)$

$= 1000 \times P(x \geq 3)$

$= 1000 \times [1 - P(x < 3)]$

$= 1000 \times [1 - [P(0) + P(1) + P(2)]]$

$= 1000 \times [1 - (20C_0(0.1)^0(0.9)^{20-0}$

$+20C_1(0.1)^1(0.9)^{20-1} + 20C_2(0.1)^2(0.9)^{20-2})]$

$= 1000 \times [1 - 0.9^{18}(0.81 + 1.8 + 1.9)]$

$= 1000 \times [1 - 0.1501(4.51)]$

$= 1000 \times 0.3231 \cong 323$

11. Fit a binomial distribution to the following data:

x_i	0	1	2	3	4	5
$f(x_i)$	2	14	20	34	22	8

Solution:

Find: n, N, p, q

$n = 5 \; and \; N = \Sigma f(x_i) = 100$

$\bar{x} = \dfrac{\Sigma x_i f(x_i)}{N}$

$= \frac{0(2)+1(14)+2(20)+3(34)+4(22)+5(8)}{100}$

$= 2.84$

For a Binomial distribution, $\bar{x} = np$

Put $n = 5, \bar{x} = 2.84$

we get $p = 0.5680$ and $q = 1 - p = 0.4320$

Therefore, $n = 5, N = 100, p = 0.5680, q = 0.4320$

Find: $f(x)$

Probability density function is given by

$P(x) = nC_x p^x q^{n-x}$

$= 5C_x (0.568)^x (0.432)^{5-x}$

Therefore,

$f(x) = N \times P(x)$

$= 100 \times 5C_x (0.568)^x (0.432)^{5-x}$

Fitting Binomial distribution

Theoretical frequencies are

$f(0) = 100 \times 5C_0 (0.568)^0 (0.432)^{5-0} \cong 2$

$f(1) = 100 \times 5C_1 (0.568)^1 (0.432)^{5-1} \cong 10$

$f(2) = 100 \times 5C_2 (0.568)^2 (0.432)^{5-2} \cong 26$

$f(3) = 100 \times 5C_3 (0.568)^3 (0.432)^{5-3} \cong 34$

$f(4) = 100 \times 5C_4 (0.568)^4 (0.432)^{5-4} \cong 22$

$f(5) = 100 \times 5C_5 (0.568)^5 (0.432)^{5-5} \cong 6$

12. Fit a binomial distribution to the following data:

x_i	0	1	2	3
$f(x_i)$	28	62	10	4

To find: n, p, q

$n = 3 \; and \; N = \Sigma f(x_i) = 104$

$$\bar{x} = \frac{\Sigma x_i f(x_i)}{N}$$

$$= \frac{0(28)+1(62)+2(10)+3(4)}{104}$$

$$= 0.9038$$

For a Binomial distribution, $\bar{x} = np$

Put $n = 3, \bar{x} = 0.9038$

we get $p = 0.3$ and $q = 1 - p = 0.7$

To find: $f(x)$

Probability density function is given by

$$P(x) = nC_x p^x q^{n-x}$$

$$= 3C_x (0.3)^x (0.7)^{3-x}$$

Therefore,

$$f(x) = N \times P(x)$$

$$= 104 \times 3C_x (0.568)^x (0.7)^{3-x}$$

Fitting Binomial distribution

Theoretical frequencies are

$$f(0) = 104 \times 3C_0 (0.3)^0 (0.7)^{3-0} \cong 36$$

$$f(1) = 104 \times 3C_1 (0.3)^1 (0.7)^{3-1} \cong 46$$

$$f(2) = 104 \times 3C_2 (0.3)^2 (0.7)^{3-2} \cong 19$$

$$f(3) = 104 \times 3C_3 (0.3)^3 (0.7)^{3-3} \cong 3$$

1.3 Poisson distribution

Introduction:

- A probability distribution which satisfies the probability density function $P(x) = e^{-m}\frac{m^x}{x!}$ is called ***Poisson distribution***.
- Poisson distribution is a limiting form of binomial distribution.
  That means, Binomial distribution becomes Poisson distribution
  If $n \to \infty, p \to 0, np = m$.
- Mean = Variance = m.
- $\sum_{x=0}^{\infty}\frac{m^x}{x!} = \sum_{x=1}^{\infty}\frac{m^{x-1}}{(x-1)!} = \sum_{x=2}^{\infty}\frac{m^{x-2}}{(x-2)!} = e^m$

Problems:

1. Find the mean and Standard deviation of Poisson distribution.

$$
\begin{aligned}
Mean &= E(x)\\
&= \sum xp(x)\\
&= \sum x\frac{m^x e^{-m}}{x!}\\
&= me^{-m}\sum\frac{m^{x-1}}{(x-1)!}\\
&= me^{-m}(1+\frac{m^1}{1!}+\frac{m^2}{2!}+\cdots)e^m\\
&= me^{-m}e^m\\
&= m.
\end{aligned}
$$

$$
\begin{aligned}
Variance &= E(x^2)-[E(x)]^2\\
&= E[x(x-1)+x]-[E(x)]^2\\
&= E[x(x-1)]+E(x)-[E(x)]^2\\
&= \sum[x(x-1)]p(x)+m-m^2\\
&= \sum[x(x-1)]\frac{m^x e^{-m}}{x!}+m-m^2\\
&= m^2e^{-m}\sum\frac{m^{x-2}}{(x-2)!}+m-m^2\\
&= m^2e^{-m}\,(1+\frac{m^1}{1!}+\frac{m^2}{2!}+\cdots)+m-m^2\\
&= m^2e^{-m}\,e^m+m-m^2\\
&= m^2+m-m^2\\
&= m.
\end{aligned}
$$

2. The probabilities of a Poisson variate taking the values 3 and 4 are equal. Calculate the probabilities of the variate taking the values 0 and 1. Also find the mean and variance of the Poisson distribution.

To find $\boldsymbol{p(x)}$: It follows Poisson distribution. Probability density function is $\boldsymbol{p(x) = e^{-m}\frac{m^x}{x!}}$ By data, $p(3) = p(4)$ $\frac{m^3e^{-m}}{3!} = \frac{m^4e^{-m}}{4!}$ $\Rightarrow m = 4.$ Therefore, $\boldsymbol{p(x) = e^{-4}\frac{4^x}{x!}}$	(i) $p(0) = \frac{4^0e^{-4}}{0!} = 0.0183$ (ii) $p(1) = \frac{4^1e^{-4}}{1!} = 0.0733$ (iii) Mean=Variance= $m = 4.$

3. **X is a Poisson variate and it is found that the probability that $x = 2$ is two third of the probability that $x = 1$. Find the probability that $x = 0$ and the probability that $x = 1$. What is the probability that x exceeds 3?**

To find: $m, p(x)$ It follows Poisson distribution. Probability density function is $p(x) = e^{-m}\frac{m^x}{x!}$ By data, $p(2) = \frac{2}{3}p(1)$ $\frac{m^2e^{-m}}{2!} = \frac{2}{3} \times \frac{m^1e^{-m}}{1!}$ $m = \frac{4}{3}$ Therefore, $p(x) = e^{-\frac{4}{3}}\frac{\left(\frac{4}{3}\right)^x}{x!}$	**To find: $p(0), p(1), p(x > 3)$** (i) $p(0) = e^{-\frac{4}{3}}\frac{\left(\frac{4}{3}\right)^0}{0!} = 0.2636$ (ii) $p(1) = e^{-\frac{4}{3}}\frac{\left(\frac{4}{3}\right)^1}{1!} = 0.3515$ (iii) $p(x > 3)$ $= 1 - p(x \le 3)$ $= 1 - [p(0) + p(1) + p(2) + p(3)]$ $= 1 - e^{-\frac{4}{3}}\left[\frac{\left(\frac{4}{3}\right)^0}{0!} + \frac{\left(\frac{4}{3}\right)^1}{1!} + \frac{\left(\frac{4}{3}\right)^2}{2!} + \frac{\left(\frac{4}{3}\right)^3}{3!}\right]$ $= 0.1506$

4. **2% of the fuses manufactured by a firm are found to be defective. Find the probability that the box containing 200 fuses contains (i) no defective fuse (ii) 3 or more defective fuses.**

To Find: $m, p(x)$ By data, $n = 200, p = 0.02$ It follows Poisson distribution. Probability density function is $p(x) = e^{-m}\frac{m^x}{x!}$ $m = mean = np = 4$ $P(x) = e^{-4}\frac{(4)^x}{x!}$	(i) $p(no\ defective\ fuse)$ $= e^{-4}\frac{(4)^0}{0!} = 0.0183$ (ii) $p(3\ or\ more\ defective\ fuses)$ $= p(x \ge 3)$ $= 1 - p(x < 3)$ $= 1 - [p(0) + p(1) + p(2)]$ $= 1 - e^{-4}\left[\frac{(4)^0}{0!} + \frac{(4)^1}{1!} + \frac{(4)^2}{2!}\right]$ $= 0.7619$

5. **A certain screw manufacturing machine produces on an average of 2 defective screws out of 100 and packs them in boxes of 500. Find the probability that a box contains (i) 3 defective (ii) at least one defective (iii) 15 defective screws.**

To find: $\boldsymbol{m}, \boldsymbol{p}(\boldsymbol{x})$ By data, $n = 500, p = 2/100$ It follows Poisson distribution. Probability density function is $p(x) = e^{-m}\dfrac{m^x}{x!}$ $m = mean = np = 10$ $p(x) = e^{-10}\frac{(10)^x}{x!}$ (i) p(box contains 3 defective screws) $= e^{-10}\frac{(10)^3}{3!}$ $= 0.00757$	(ii) p(box contains at least 1 defective) $= p(x \geq 1)$ $= 1 - p(x < 1)$ $= 1 - p(0)$ $= 1 - e^{-10}$ $= 1$ (iii) p(box contains 15 defective screws) $= p(15)$ $= e^{-10}\frac{(10)^{15}}{15!}$ $= 0.0347$

6. **If the probability of a bad reaction from a certain injection is 0.001. Determine the chance that out of 2000 individuals more than 2 will get a bad reaction.**

To find: $\boldsymbol{m}, \boldsymbol{p}(\boldsymbol{x})$ By data, $n = 2000, p = 0.001$ It follows Poisson distribution. Probability density function is $p(x) = e^{-m}\dfrac{m^x}{x!}$ $m = mean = np = 2$ Therefore, $p(x) = e^{-2}\frac{(2)^x}{x!}$	p(more than 2 will get a bad reaction) $= p(x > 2)$ $= 1 - p(x \leq 2)$ $= 1 - [p(0) + p(1) + p(2)]$ $= 1 - e^{-2}\left[\frac{(2)^0}{0!} + \frac{(2)^1}{1!} + \frac{(2)^2}{2!}\right]$ $= 1 - 0.6767$ $= 0.3233$

7. **In a certain factory turning out razor blades there is a small probability of 1/500 for any blade to be defective. The blades are supplied in packets of 10. Use Poisson distribution to calculate the approximate number of packets containing (i) no defective, (ii) one defective, and (iii) two defective blades in a consignment of 10000 packets.**

To find: $\boldsymbol{m, p(x)}$ By data, $n = 10, p = 0.002$ and $N = 10000.$ $m = mean = np = 0.02$ Since p is too small, It follows Poisson distribution. Probability density function is $p(x) = e^{-m}\frac{m^x}{x!}$ Put $m = mean = np = 0.02$ Therefore, $p(x) = e^{-0.02}\frac{(0.02)^x}{x!}$	(i) Number of packets containing no defectives $= N \times p(0)$ $= 10000 \times e^{-0.02}\frac{(0.02)^0}{0!}$ $= 9802$ (ii) Number of packets containing one defective$= N \times p(1)$ $= 10000 \times e^{-0.02}\frac{(0.02)^1}{1!}$ $= 196$ (iii) Number of packets containing two defectives $= N \times p(2)$ $= 10000 \times e^{-0.02}\frac{(0.02)^2}{2!}$ $= 2$

8. **Alpha particles are emitted by a radioactive source at an average rate of 5 in a 20 minutes' interval. Using Poisson distribution, find the probability that there will be (i) exactly two emissions, and (ii) at least two emissions, in a randomly chosen 20 minutes' interval.**

To find: $\boldsymbol{m, p(x)}$ It follows Poisson distribution. Probability density function is $p(x) = e^{-m}\frac{m^x}{x!}$ By data, m =average rate= 5 Therefore, $p(x) = e^{-5}\frac{5^x}{x!}$	(i) p(exactly two emissions) $= p(2) = e^{-5}\frac{5^2}{2!} = 0.0842$ (ii) p(at least two emissions) $= p(x \geq 2) = 1 - p(x < 2)$ $= 1 - \{p(0) + p(1)\}$ $= 1 - \{e^{-5} + 5e^{-5}\} = 0.9596.$

9. A car hire firm has two cars which it hire out day by day. The number of demand for a car on each day is distributed as a Poisson distribution with mean 1.5, Calculate the proportion of days (i) on which there is no demand (ii) On which demand is refused.

To find: $\boldsymbol{p(x)}$ It follows Poisson distribution. Probability function is $p(x) = e^{-m}\frac{m^x}{x!}$ By data, $m = 1.5$ Therefore, $p(x) = e^{-1.5}\frac{(1.5)^x}{x!}$ (i) P(No demands)= $p(0)$ $= e^{-1.5}\frac{(1.5)^0}{0!} = 0.2231$	(ii) p(demand is refused) $= p(x > 2)$ $= 1 - p(x \leq 2)$ $= 1 - [p(0) + p(1) + p(2)]$ $= 1 - e^{-1.5}\left[\frac{(1.5)^0}{0!} + \frac{(1.5)^1}{1!} + \frac{(1.5)^2}{2!}\right]$ $= 0.1912$

10. The probability that a news reader commits no mistake in reading the news is $\boldsymbol{1/e^3}$. Find the probability that on a particular news broadcast he commits (i) Only 2 mistakes (ii) More than 3 mistakes (iii) At most 3 mistakes.

To find: $\boldsymbol{p(x)}$ By data, $p(0) = e^{-3}$ ------(1) It follows Poisson distribution. Probability function is $p(x) = e^{-m}\frac{m^x}{x!}$ $p(0) = e^{-m}\frac{m^0}{0!} = e^{-m}$-----(2) Equating (1) and (2), $m = 3$ Therefore, $p(x) = e^{-3}\frac{3^x}{x!}$	(i) p(Only 2 mistakes) $= p(2) = e^{-3}\frac{3^2}{2!} = 0.2240$ (ii) p(More than 3 mistakes) $= p(x > 3)$ $= 1 - p(x \leq 3)$ $= 1 - [p(0) + p(1) + p(2) + p(3)]$ $= 1 - e^{-3}\left[\frac{3^0}{0!} + \frac{3^1}{1!} + \frac{3^2}{2!} + \frac{3^3}{3!}\right]$ $= 0.3527$ (iii) p(at most 3 mistakes) $= p(x \leq 3)$ $= p(0) + p(1) + p(2) + p(3)$ $= e^{-3}\left[\frac{3^0}{0!} + \frac{3^1}{1!} + \frac{3^2}{2!} + \frac{3^3}{3!}\right]$ = 0.6472

11. Fit a Poisson distribution for the data

$\boldsymbol{x}$:	**0**	**1**	**2**	**3**	**4**
$\boldsymbol{f}$:	**46**	**38**	**22**	**9**	**1**

To find: $\boldsymbol{N}, \boldsymbol{m}, \boldsymbol{f}(\boldsymbol{x})$

Here, $N = \sum f_i = 46 + 38 + 22 + 9 + 1 = 116$

$\sum x_i f_i = (0 \times 46) + (1 \times 38) + (2 \times 22) + (3 \times 9) + (4 \times 1) = 113.$

m=mean $= \overline{X} = \frac{\sum x_i f_i}{N} = \frac{113}{116} = 0.9741$

Probability function of a Poisson distribution is

$$p(x) = \frac{e^{-m} m^x}{x!} = \frac{e^{-0.9741}\, 0.9741^{\,x}}{x!}$$

$$f(x) = N \times p(x) = 116 \times e^{-0.9741}\ \frac{0.9741^{\,x}}{x!}$$

Fitting Poisson distribution

Theoretical frequencies are

$$f(0) = 116 \times p(0) = 200 \times e^{-0.9741} = 200 \times 0.6065 \cong 44$$

$$f(1) = 116 \times p(1) = 200 \times e^{-0.9741} \times 0.9741 \cong 43$$

$$f(2) = 116 \times p(2) = 200 \times e^{-0.9741} \times \frac{1}{2}(0.9741)^2 \cong 21$$

$$f(3) = 116 \times p(3) = 200 \times e^{-0.9741} \times \frac{1}{3!}(0.9741)^3 \cong 7$$

$$f(4) = 116 \times p(4) = 200 \times e^{-0.9741} \times \frac{1}{4!}(0.9741)^4 \cong 1$$

12. Fit a Poisson distribution to the set of observations:

$\boldsymbol{x}$:	**0**	**1**	**2**	**3**	**4**
$\boldsymbol{f}$:	**122**	**60**	**15**	**2**	**1**

To find: $\boldsymbol{N, m, f(x)}$

Here, $N = \sum f_i = 122 + 60 + 15 + 2 + 1 = 200,$

$\sum x_i f_i = (0 \times 122) + (1 \times 60) + (2 \times 15) + (3 \times 2) + (4 \times 1) = 100.$

m = mean = $\overline{X} = \frac{\sum x_i f_i}{N} = \frac{100}{200} = 0.5$

Probability function of a Poisson distribution is

$$p(x) = \frac{e^{-m} m^x}{x!} = \frac{e^{-0.5} 0.5^x}{x!}$$

$$f(x) = N \times p(x) = 200 \times e^{-0.5} \frac{(0.5)^x}{x!}$$

Fitting Poisson distribution

Theoretical frequencies are

$$f(0) = 200 \times p(0) = 200 \times e^{-0.5} = 200 \times 0.6065 \cong 121$$

$$f(1) = 200 \times p(1) = 200 \times e^{-0.5} \times 0.5 \cong 61$$

$$f(2) = 200 \times p(2) = 200 \times e^{-0.5} \times \frac{1}{2}(0.5)^2 \cong 15$$

$$f(3) = 200 \times p(3) = 200 \times e^{-0.5} \times \frac{1}{3!}(0.5)^3 \cong 3$$

$$f(4) = 200 \times p(4) = 200 \times e^{-0.5} \times \frac{1}{4!}(0.5)^4 \cong 0$$

1.4 Continuous probability distributions

Introduction:

- If a random variable takes any real value in the specified interval then it is called ***continuous random variable***. Ex: height of students in a class, length of leaves in a tree.
- P(x) is the probability density function of the continuous random variable x if (i) $p(x) \geq 0$ (ii) $\int_{-\infty}^{\infty} p(x)\, dx = 1$.
- $E(x) = \int_{-\infty}^{\infty} xp(x)\, dx$, $E(x^2) = \int_{-\infty}^{\infty} x^2 p(x)\, dx$
- If p(x) is the probability density function of the continuous random variable x then cumulative distribution function $F(t) = p(x \leq t) = \int_{-\infty}^{t} p(x)dx$.Then $F'(t) = p(t)$
- $p(a \leq x \leq b) = p(a \leq x < b) = p(a < x \leq b) = p(a < x < b) = F(b) - F(a)$

Problems:

1. **Find the constant k so that** $\boldsymbol{f(x) = \begin{cases} \left(\frac{x}{6}\right) + k, 0 \leq x \leq 3 \\ 0, \quad elsewhere \end{cases}}$ **is a probability density function. Then find** $\boldsymbol{p(1 \leq x \leq 2)}$ **.**

To find: $\boldsymbol{k}$	
$1 = \int_{-\infty}^{\infty} p(x)\, dx$ $= \int_0^3 p(x)dx$ $= \int_0^3 \left(\frac{x}{6} + k\right) dx$ $= \int_0^3 \left(\frac{x}{6}\right) dx + k\int_0^3 dx$ $= \frac{9}{12} + 3k$ $1 - \frac{9}{12} = 3k$ Therefore, $k = \frac{3}{12} = \frac{1}{4}$	$p(1 \leq x \leq 2)$ $= \int_1^2 p(x)dx$ $= \int_1^2 \left(\frac{x}{6} + k\right) dx$ $= \int_1^2 \left(\frac{x}{6} + \frac{1}{12}\right) dx$ $= \frac{1}{3}$

2. A random variable x has the density function $p(x) = \begin{cases} kx^2, 0 \le x \le 3 \\ 0, \quad elsewhere \end{cases}$

Evaluate k and find $p(x \le 1), p(1 \le x \le 2), p(x \le 2) and\ p(x > 1)$

To find: k $1 = \int_{-\infty}^{\infty} p(x)\, dx$ $= \int_0^3 p(x)dx$ $= \int_0^3 kx^2dx$ $= k\int_0^3 x^2dx$ $= 9k$ $k = 1/9.$	(i) $P(x \le 1)$ $= \int_{-\infty}^{1} p(x)dx = k\int_0^1 x^2dx = \frac{1}{9}\left(\frac{1}{3}\right) = \frac{1}{27}$ (ii) $P(1 \le x \le 2)$ $= \int_1^2 p(x)dx = k\int_1^2 x^2dx = \frac{1}{9}\left(\frac{7}{3}\right) = \frac{7}{27}$ (iii) $P(x \le 2)$ $= \int_{-\infty}^{2} p(x)dx = k\int_0^2 x^2dx = \frac{1}{9}\left(\frac{8}{3}\right) = \frac{8}{27}$ (iv) $P(x > 1)$ $= \int_1^3 p(x)dx = k\int_1^3 x^2dx = \frac{1}{9}\left(\frac{26}{3}\right) = \frac{26}{27}$

3. The probability density function of a continuous random variable x is

$$\boldsymbol{p(x) = \begin{cases} kx, & for\ 0 \le x < 2 \\ 2k, & for\ 2 \le x < 4 \\ 6k - kx, & for\ 4 \le x \le 6 \\ 0, & elsewhere \end{cases}}$$ **find k and then determine the mean of x.**

To find: $\boldsymbol{k}$

By data, $1 = \int_{-\infty}^{\infty} p(x)dx$

$$= \int_{-\infty}^{\infty} p(x)dx$$

$$= \int_{0}^{6} p(x)dx$$

$$= \int_{0}^{2} kxdx + \int_{2}^{4} 2kdx + \int_{4}^{6} (6k - kx)dx$$

$$= k\int_{0}^{2} xdx + 2k\int_{2}^{4} dx + 6k\int_{4}^{6} dx - k\int_{4}^{6} xdx$$

$$= 2k + 4k + 12k - 10k$$

Therefore, $1 = 8k$ or $k = 1/8$

To find: Mean

Mean $= \int_{-\infty}^{\infty} x\, p(x)dx$

$= \int_{0}^{6} x\, p(x)dx$

$= \int_{0}^{2} kx^2dx + \int_{2}^{4} 2kxdx + \int_{4}^{6} (6k - kx)xdx$

$$= k\int_{0}^{2} x^2dx + 2k\int_{2}^{4} xdx + 6k\int_{4}^{6} xdx - \mathrm{k}\int_{4}^{6} x^2dx$$

$= \frac{1}{8}\left(\frac{8}{3}\right) + \frac{2}{8}(6) + \frac{6}{8}(10) - \frac{1}{8}\left(\frac{152}{3}\right) = 3.$

4. The frequency distribution of a measurable characteristic varying between 0 and 2 is as $f(x) = \begin{cases} x^3, & 0 \le x \le 1 \\ (2-x)^2, & 1 \le x \le 2 \end{cases}$ Calculate mean and standard deviation.

$E(x) = \int_{-\infty}^{\infty} xf(x)dx$ $= \int_0^2 xf(x)dx$ $= \int_0^1 x^4dx + \int_1^2 (4x + x^3 - 4x^2)dx$ $= 0.2 + 0.4167$ $= 0.6167$	$E(x^2) = \int_{-\infty}^{\infty} x^2f(x)dx$ $= \int_0^2 x^2f(x)dx$ $= \int_0^1 x^5dx + \int_1^2 (4x^2 + x^4 - 4x^3)dx$ $= \frac{1}{6} + 0.5333$ $= 0.7$

$Mean = E(x) = 0.6167$

Standard deviation $= \sqrt{E(x^2) - [E(x)]^2} = \sqrt{0.7 - 0.6167^2} = \sqrt{0.3197} = 0.5654$

5. The probability density function of a continuous random variable x is $p(x) = \begin{cases} kx(1-x)e^x, & 0 \le x \le 1 \\ 0, & elsewhere \end{cases}$. Find the value of k and evaluate mean and standard deviation of the distribution.

$1 = \int_{-\infty}^{\infty} p(x)dx$ $= \int_0^1 p(x)dx$ $= \int_0^1 kx(1-x)e^xdx$ $= k\int_0^1 (x - x^2)e^xdx$ $= k(0.2817)$ $k = 3.5496$	$E(x) = \int_{-\infty}^{\infty} xp(x)dx$ $= k\int_0^1 x^2(1-x)e^xdx$ $= k\int_0^1 (x^2 - x^3)e^xdx$ $= 3.5496 \times 0.1549$ $= 0.5496$	$E(x^2) = \int_{-\infty}^{\infty} x^2p(x)dx$ $= k\int_0^1 x^2(x - x^2)e^xdx$ $= k\int_0^1 (x^3 - x^4)e^xdx$ $= 3.5496 \times 0.0989$ $= 0.3511$

Mean$= E(x) = 0.5496$

Standard deviation $= \sqrt{E(x^2) - [E(x)]^2} = \sqrt{0.3511 - (0.5496)^2}$

$= \sqrt{0.04904} = 0.2214$

6. For the probability distribution given by the cumulative distribution function

$$\boldsymbol{F(t) = \begin{cases} 0, & t > 0 \\ t^2, & 0 \le t \le 1 \\ 1, & t > 1 \end{cases}}$$ **Find the probability density function.**

Also evaluate (i) $\boldsymbol{p(0.5 < x < 0.75)}$ (ii) $\boldsymbol{p(x \le 0.5)}$ (iii) $\boldsymbol{p(x > 0.75)}$

Find: Probability density function

$$F(t) = \begin{cases} 0, & t > 0 \\ t^2 & 0 \le t \le 1 \\ 1, & t > 1 \end{cases}$$

$$p(t) = F'(t) = \begin{cases} 0, & t > 0 \\ 2t, & 0 \le t \le 1 \\ 0, & t > 1 \end{cases}$$

$$p(x) = \begin{cases} 0, & x > 0 \\ 2x, & 0 \le x \le 1 \\ 0, & x > 1 \end{cases}$$

This is the required probability density function.

(i) $p(0.5 < x < 0.75) = F(0.75) - F(0.5) = (0.75)^2 - (0.5)^2 = 0.3125$

(ii) $p(x \le 0.5) = F(0.5) = (0.5)^2 = 0.25$

(iv) $p(x > 0.75) = 1 - p(x \le 0.75) = 1 - F(0.75) = 1 - (0.75)^2 = 0.4375.$

7. **A function is defined as** $f(x) = \begin{cases} \frac{1}{k}, & x_1 < x < x_2 \\ 0, & elsewhere \end{cases}$ **Find the cumulative distribution of the variate x when k satisfies the requirements for $f(x)$ to be a density function.**

$1 = \int_{-\infty}^{\infty} p(x)dx$ $= \int_{x_1}^{x_2} p(x)dx$ $= \int_{x_1}^{x_2} \frac{1}{k} dx$ $= \frac{1}{k}\int_{x_1}^{x_2} dx$ $= \frac{1}{k}(x_2 - x_1)$ $k = x_2 - x_1$	$F(x) = \int_{-\infty}^{x} p(x)dx$ $= \int_{x_1}^{x} p(x)dx$ $= \int_{x_1}^{x} \frac{1}{k} dx$ $= \frac{1}{k}\int_{x_1}^{x} dx$ $= \frac{1}{k}(x - x_1)$ $= \frac{x-x_1}{x_2-x_1}$

8. $f(x) = \begin{cases} \frac{1}{2}(x+1), & -1 < x < 1 \\ 0, & elsewhere \end{cases}$ **represents the density of a r.v X, Find $E(x), Var(x)$.**

$\mathrm{E(x)} = \int_{-\infty}^{\infty} xf(x)dx$ $= \frac{1}{2}\int_{-1}^{1}(x^2 + x)dx$ $= \frac{1}{2}(0.6667)$ $= 0.3334$	$E(x^2) = \int_{-\infty}^{\infty} x^2 p(x)dx$ $= \frac{1}{2}\int_{-1}^{1} x^2(x+1)dx$ $= \frac{1}{2}\int_{-1}^{1}(x^3 + x^2)\,dx$ $= 0.3334$

Variance $= E(x^2) - [E(x)]^2 = 0.3334 - (0.3334)^2 = 0.2222$

Home work:

9. If a random variable x has the probability density function

$f(x) = \begin{cases} 2e^{-2x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$ find $p(1 < x < 3) and\ p(x > 0.5)$. Also find the variance.

10. The frequency function of a continuous r.v is given by $f(x) = y_0 x(2 - x), 0 \leq x \leq 2$.

Find mean and S.D

Solution:

$1 = \int_{-\infty}^{\infty} y_0 (2x - x^2)\, dx = y_0 \int_0^1 (2x - x^2)\, dx = y_0 \left(\frac{2}{3}\right)$

$\therefore \boxed{y_0 = \frac{3}{2}}$

$E(x) = y_0 \int_0^1 x(2x - x^2)\, dx = y_0 \int_0^1 (2x^2 - x^3)\, dx$

$= \frac{3}{2}\left(\frac{2}{3} - \frac{1}{4}\right) = \frac{5}{8}$

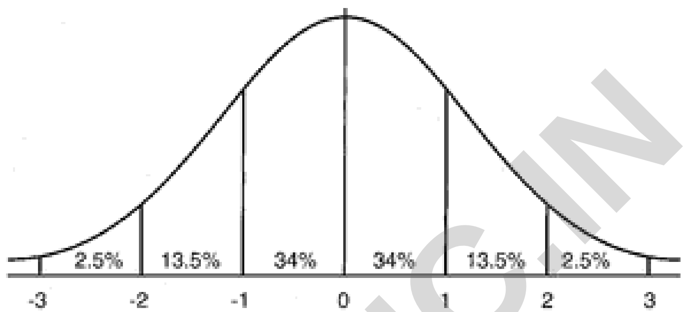
$E(x^2) = y_0 \int_0^1 x^2(2x - x^2)\, dx = \frac{3}{2}\left(\frac{1}{2} - \frac{1}{5}\right) = \frac{9}{20}$

Mean $= \frac{5}{8}$. $SD = \sqrt{\frac{9}{20} - \frac{25}{64}}$

1.5 Normal distribution

Introduction:

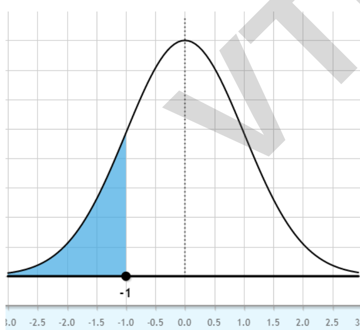
- The continuous probability distribution having the probability density function $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$ is called the normal distribution.
- $f(x) \geq 0, \int_{-\infty}^{\infty} f(x)dx = 1$, Mean $= \mu$, variance $= \sigma^2$
- A normal distribution with $\mu = 0$ *and* $\sigma = 1$ is called standard normal distribution . $z = \frac{x-\mu}{\sigma}$ is called the standard normal variate.
- Standard normal curve is symmetric about the line $z = 0$.
- The total area is 1 and the area on either side of is 0.5

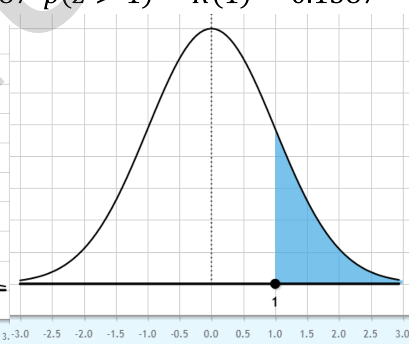


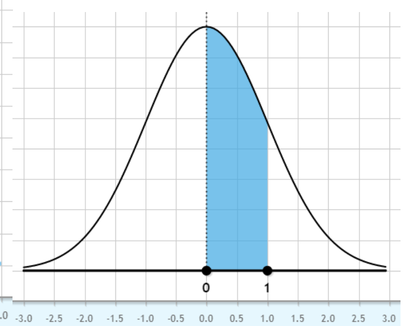
- How to find the area under the standard normal curve using calculator?
  In ms, go to SD, press $\boxed{shift}\boxed{distrn}$ In es, go to stat, $\boxed{shift}\boxed{stat}\boxed{distrn}$
  $P(a) = A(-\infty, a), Q(a) = A(0, a), R(a) = A(a, \infty)$

  Pictorial representation:

$p(z < -1) = P(-1) = 0.1587$ $p(z > 1) = R(1) = 0.1587$ $p(0 < z < 1) = Q(1) = 0.3413$







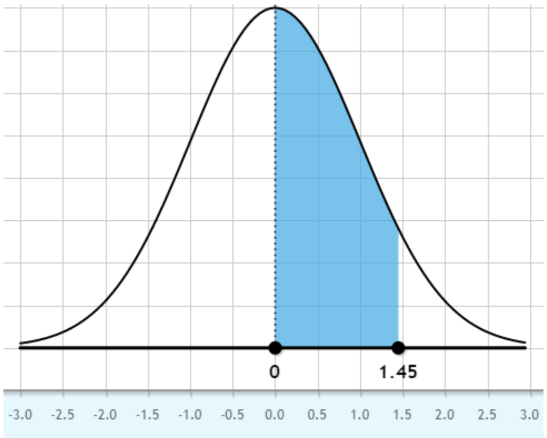
1. **For the standard normal distribution of a random variable z evaluate the following:**

   (i) $\boldsymbol{P(0 \le z \le 1.45)}$ **(ii)** $\boldsymbol{P(-2.60 \le z \le 0)}$ **(iii)**$\boldsymbol{P(-3.40 \le z \le 2.1)}$

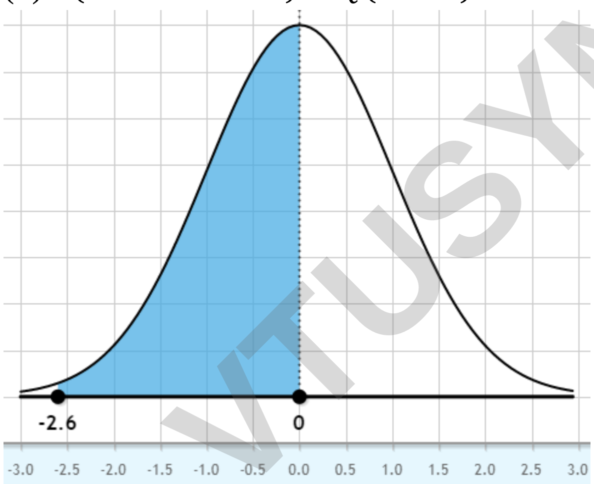
   (iv) $\boldsymbol{P(1.25 \le z \le 2.1)}$ **(v)** $\boldsymbol{P(-2.55 \le z \le -0.8)}$ **(vi)** $\boldsymbol{P(z \ge 1.7)}$

   (vii) $\boldsymbol{P(z \le -3.35)}$ **(viii)** $\boldsymbol{P(|z| \le 1.85)}$

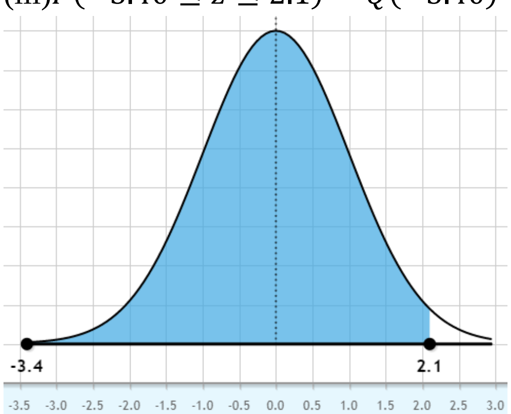
(i) $P(0 \le z \le 1.45) = Q(1.45) = 0.4265$



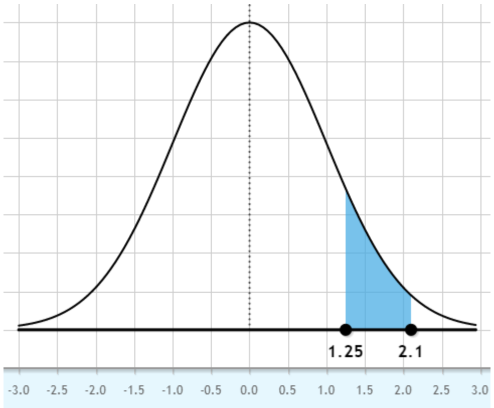
(ii) $P(-2.60 \le z \le 0) = Q(-2.60) = 0.4953$



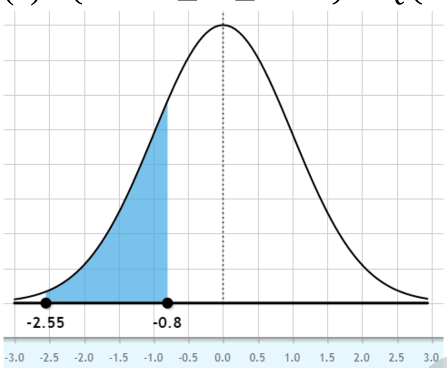
(iii)$P(-3.40 \le z \le 2.1) = Q(-3.40) + Q(2.1) = 0.9818$



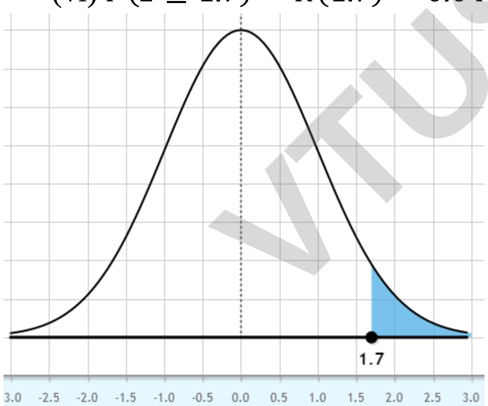
(iv) $P(1.25 \le z \le 2.1) = Q(2.1) - Q(1.25) = 0.0878$



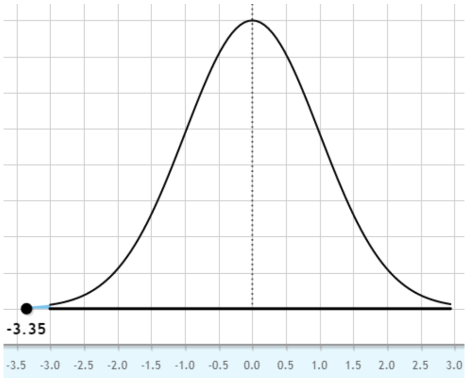
(v) $P(-2.55 \le z \le -0.8) = Q(-2.55) - Q(-0.8) = 0.2065$



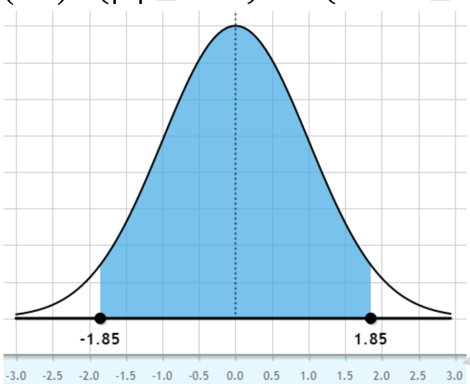
(vi) $P(z \ge 1.7) = R(1.7) = 0.0446$



(vii) $P(z \le -3.35) = P(-3.35) = 0.0004$



(viii) $P(|z| \leq 1.85) = P(-1.85 \leq z \leq 1.85) = Q(-1.85) - Q(1.85) = 0.9357$



2. For the normal distribution with mean 2 and standard deviation 4 evaluate the following probabilities: (i) $P(-6 < x < 3)$ (ii) $P(1 < x < 5)$ (iii) $P(x \geq 5)$ (iv) $P(|x| < 4)$ (v) $P(|x| > 3)$ (vi) $P(|x-2| > 1)$

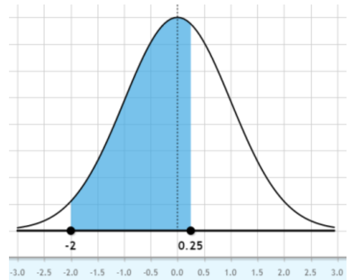
By data, $z = \frac{x-\mu}{\sigma} = \frac{x-2}{4}$

(i) $p(-6 < x < 3) = p\left(\frac{-6-2}{4} < \frac{x-2}{4} < \frac{3-2}{4}\right)$

$= p(-2 < z < 0.25)$

$= Q(-2) + Q(0.25)$

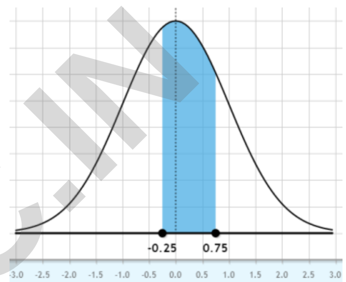
$= 0.5760$



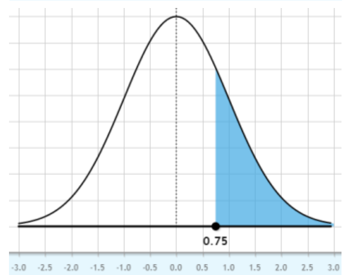
(ii) $P(1 < x < 5) = p\left(\frac{1-2}{4} < \frac{x-2}{4} < \frac{5-2}{4}\right)$

$= p(-0.25 < z < 0.75)$

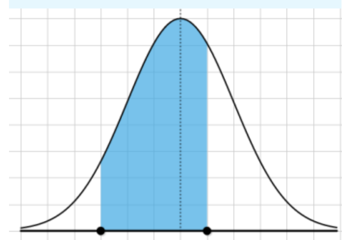
$= Q(-0.25) + Q(0.75)$
$= 0.3721$



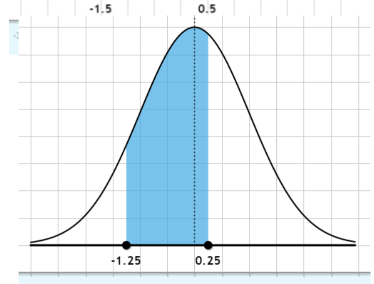
(iii) $P(x \geq 5) = p\left(\frac{x-2}{4} \geq \frac{5-2}{4}\right)$
$= p(z \geq 0.75)$
$= R(0.75)$
$= 0.2266$



(iv) $P(|x| < 4) = p\left(\frac{-4-2}{4} < \frac{x-2}{4} < \frac{4-2}{4}\right)$
$= p(-1.5 < z < 0.5)$
$= Q(-1.5) + Q(0.5)$
$= 0.6247$



(v) $P(|x| > 3) = 1 - p(|x| \leq 3)$
$= 1 - p(-3 \leq x \leq 3)$
$= 1 - p\left(\frac{-3-2}{4} < \frac{x-2}{4} < \frac{3-2}{4}\right)$
$= 1 - p(-1.25 < z < 0.25)$
$= 1 - [Q(-1.25) + Q(0.25)]$
$= 1 - 0.4931$
$= 0.5069$



(vi) $P(|x-2|>1) = 1 - p(|x-2| \le 1)$

$= 1 - p(-1 \le x - 2 \le 1)$

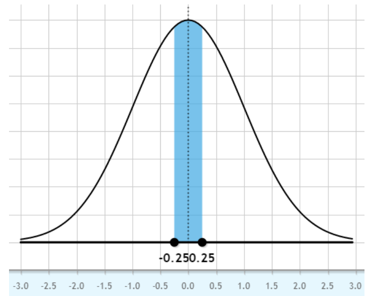
$= 1 - p(-\frac{1}{4} \le \frac{x-2}{4} \le \frac{1}{4})$

$= 1 - p(-0.25 < z < 0.25)$

$= 1 - [Q(-0.25) + Q(0.25)]$

$= 1 - 0.1974$

$= 0.8026$



3. If X is a normal variate with mean 30 and S.D 5, find the probabilities that (i) $26 \le x \le 40$ (ii) $|X-30| > 5$.

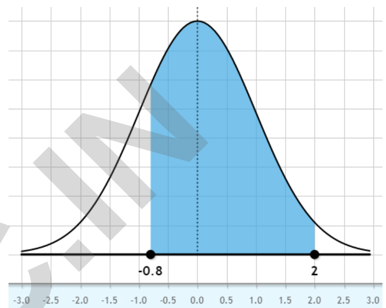
By data, $\mu = 30, \sigma = 5$. Therefore, $z = \frac{x-\mu}{\sigma} = \frac{x-30}{5}$

(i) $P(26 \le x \le 40) = p\left(\frac{26-30}{5} \le \frac{x-30}{5} \le \frac{40-30}{5}\right)$

$= p(-0.8 \le z \le 2)$

$= Q(-0.8) + Q(2)$

$= 0.7654$



(ii) $p(|x-30| > 5) = 1 - p(|x-30| \le 5)$

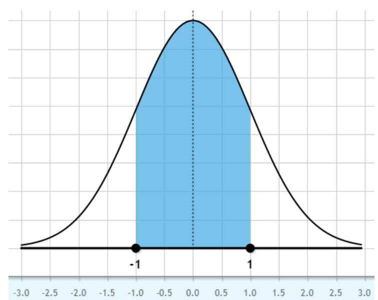
$= 1 - p(-5 \le x - 30 \le 5)$

$= 1 - p\left(-1 \le \frac{x-30}{5} \le 1\right)$

$= 1 - p(-1 \le z \le 1)$

$= 1 - 0.6826$

$= 0.3174$



4. **The mean and standard deviation of the marks obtained by 1000 students in an examination are respectively 34.4 and 16.5. Assuming the normality of the distribution find the approximate number of students expected to obtain the marks between 30 and 60.**

By data, $\mu = 34.4, \sigma = 16.55.$

Therefore, $z = \frac{x-\mu}{\sigma} = \frac{x-34.4}{16.5}$

Approximate number of students

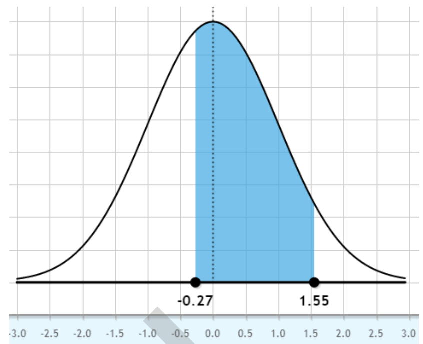
$$= N \times P(30 \le x \le 60)$$

$$= 1000 \times p\left(\frac{30-34.4}{16.5} \le \frac{x-34.4}{16.5} \le \frac{60-34.4.}{16.5}\right)$$

$$= 1000 \times p(-0.2667 \le z \le 1.5515)$$

$$= 1000 \times 0.5458$$

$$\cong 546$$



5. **The mean weight of 500 students at a certain school is 50 kgs and the standard deviation is 6 kgs. Assuming that the weights are normally distributed, find the expected number of students weighing between (i)between 40 and 50 kgs (ii) more than 60 kgs.**

By data, $\mu = 50, \sigma = 6$. Therefore, $z = \frac{x-\mu}{\sigma} = \frac{x-50}{6}$

(i) Expected number of students weighing between 40 and 50 kgs.

$$= N \times P(40 \le x \le 50)$$

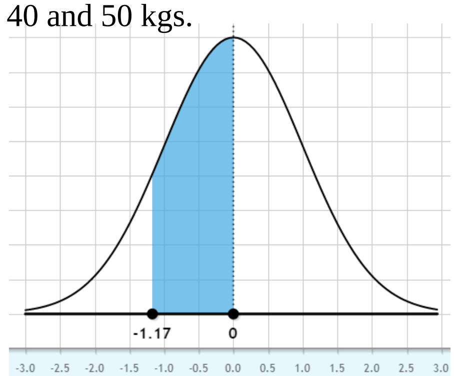
$$= 500 \times p\left(\frac{40-50}{6} \le \frac{x-50}{6} \le \frac{50-50}{6}\right)$$

$$= 500 \times p(-1.6667 \le z \le 0)$$

$$= 500 \times Q(-1.6667)$$

$$= 500 \times 0.4522$$

$$\cong 226$$



(ii) Expected number of students weighing more than 60 kgs.

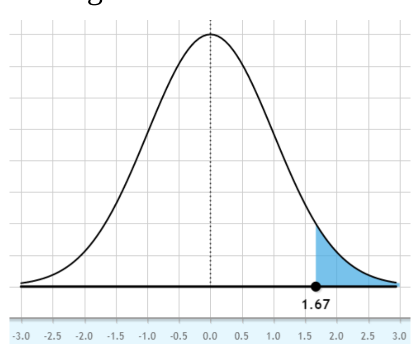
$$= N \times P(x > 60)$$

$$= 500 \times p\left(\frac{x-50}{6} > \frac{60-50}{6}\right)$$

$$= 500 \times p(z > 1.6667)$$

$$= 500 \times R(1.1667)$$

$$= 500 \times 0.0478 \cong 24$$



6. The life of a certain electric lamps is normally distributed with mean of 2040 hours and standard deviation 60 hours. In a consignment of 2000 lamps, find how many would be expected to burn for (i) more than 2150 hours (ii) less than 1950 hours and (iii) between 1920 hours and 2160 hours.

By data, $\mu = 2040, \sigma = 60$. Therefore, $z = \frac{x-\mu}{\sigma} = \frac{x-2040}{60}$

(i) No. of lamps expected to burn more than 2150 hours

$= N \times P(x > 2150)$

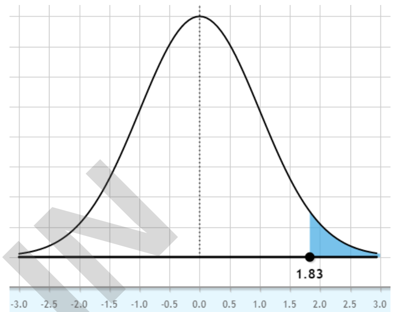
$= 2000 \times p\left(\frac{x-2040}{60} > \frac{2150-2040}{60}\right)$

$= 2000 \times p(z > 1.8333)$

$= 2000 \times R(1.8333)$

$= 2000 \times 0.0334$

$\cong 67$



(ii) No. of lamps expected to burn less than 1950 hours

$= N \times P(x < 1950)$

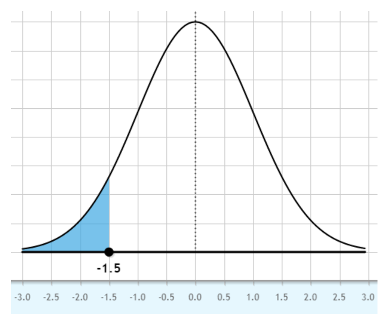
$= 2000 \times p\left(\frac{x-2040}{60} < \frac{1950-2040}{60}\right)$

$= 2000 \times p(z < -1.5)$

$= 2000 \times P(-1.5)$

$= 2000 \times 0.0668$

$\cong 134$



(iii) No. of lamps expected to burn between 1920 hours and 2160 hours

$= N \times P(1920 < x < 2160)$

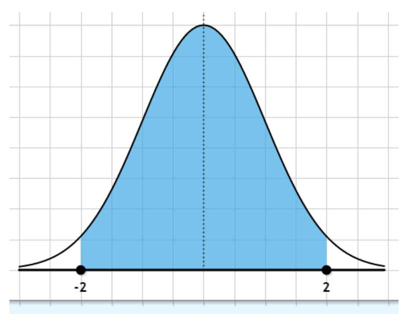
$= 2000 \times p\left(\frac{1920-2040}{60} < \frac{x-2040}{60} < \frac{2160-2040}{60}\right)$

$= 2000 \times p(-2 < z < 2)$

$= 2000 \times [Q(-2) + Q(2)]$

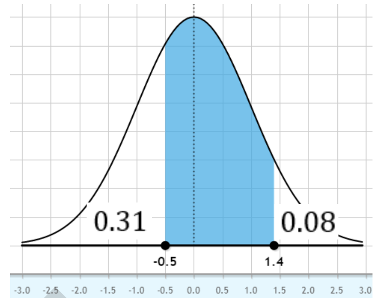
$= 2000 \times 0.9545$

$\cong 1909$



7. In a normal distribution, 31% of the items are under 45 and 8% are over 64. Find the mean and standard deviation.

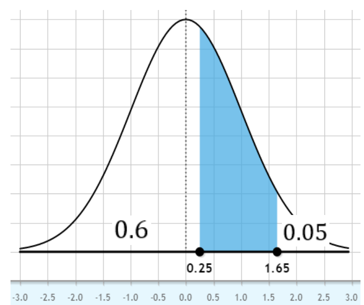
$p(x < 45) = 0.31$	$p(x > 64) = 0.08$
$p\left(\frac{x-\mu}{\sigma} < \frac{45-\mu}{\sigma}\right) = 0.31$	$p\left(\frac{x-\mu}{\sigma} > \frac{64-\mu}{\sigma}\right) = 0.08$
$p\left(z < \frac{45-\mu}{\sigma}\right) = 0.31$	$p\left(z > \frac{64-\mu}{\sigma}\right) = 0.08$
$\frac{45-\mu}{\sigma} = -0.5$	$\frac{64-\mu}{\sigma} = 1.4$
$\mu - 0.5\sigma = 45$ ----- (1)	$\mu + 1.4\sigma = 64$ ---- (2)



By solving (1) and (2) we get $\mu = 50$, $\sigma = 10$
Therefore, mean is 50 and standard deviation is 10.

8. A certain number of articles manufactured in one batch were classified in to three categories according to a particular characteristic, being less than 50, between 50 and 60 and greater than 60. If this characteristic is known to be normally distributed, determine the mean and standard deviation of this batch if 60%, 35% and 5% were found in these categories.

$p(x < 50) = 60\%$	$p(x > 60) = 5\%$
$p\left(\frac{x-\mu}{\sigma} < \frac{50-\mu}{\sigma}\right) = 0.6$	$p\left(\frac{x-\mu}{\sigma} > \frac{60-\mu}{\sigma}\right) = 0.05$
$p\left(z < \frac{50-\mu}{\sigma}\right) = 0.6$	$p\left(z > \frac{60-\mu}{\sigma}\right) = 0.05$
$\frac{50-\mu}{\sigma} = 0.2545$	$\frac{60-\mu}{\sigma} = 1.6450$
$\mu + 0.2545\sigma = 50$ ----- (1)	$\mu + 1.6450\sigma = 60$ ---- (2)



By solving (1) and (2) we get $\mu = 48.17$, $\sigma = 7.19$
Therefore, mean is 48.17 and standard deviation is 7.19

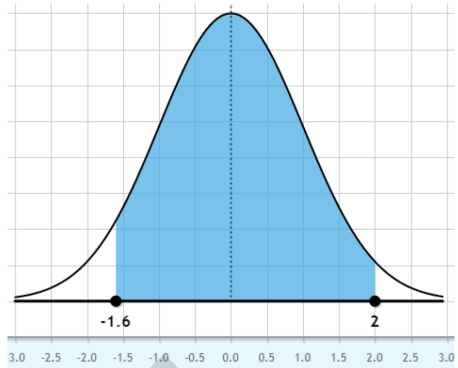
9. **The IQ of students in a certain college is assumed to be normally distributed with mean 70 and variance 25. If two students are selected at random, find the probability that (i) both of them (ii) at least one of them have IQ between 72 and 80.**

$p = P(120 \leq x \leq 155)$

$= p\left(\frac{72-80}{5} \leq \frac{x-70}{5} \leq \frac{80-70}{5}\right)$

$= p(-1.6 \leq z \leq 2)$

$= Q(-1.6) - Q(2)$
$= 0.9225$



(i) Probability that both of them have IQ between 72 and 80

$= p \times p$
$= 0.9225 \times 0.9225$
$= 0.8510$

(ii) the probability that at least one of them have IQ between 72 and 80.

$= 1 - (q \times q)$
$= 1 - (0.0775 \times 0.0775)$
$= 0.9940$

Home work:

10. In a normal distribution 7% are under 35 and 89% are under 60. Find the mean and standard deviation.
Ans: 48.65, 9.22

11. A sample of 100 dry battery cells produced by a certain company was tested for their lengths of life, and the test yielded the following data. Mean life = 12 hours. Standard deviation = 3 hours. Using normal distribution, find how many cells are expected to have their life lengths (i) greater than 15 hours, (ii) between 10 and 14 hours, and (iii) less than 6 hours.

Ans: 16, 50, 2

12. In an examination taken by 500 candidates, the average and S.D. of marks obtained are 40% and 10% respectively. Assuming normal distribution, find (i) how many have scored above 60% (ii) how many will pass if 50% is fixed as the minimum marks for passing (iii) how many will pass if 40% is fixed as the minimum marks for passing and (iv) what should be the minimum percentage of marks for passing so that 350 candidates pass.
Ans: 11, 79, 250, 35

1.6 Exponential distribution

Introduction:

- The continuous probability distribution having the probability density function $f(x) = \begin{cases} \alpha e^{-\alpha x}, 0 \le x < \infty \\ 0, \quad elsewhere \end{cases}$ is called the exponential distribution.
- $f(x) \ge 0,\ \int_{-\infty}^{\infty} f(x)dx = 1$
- Mean $= \frac{1}{\alpha}$, Standard deviation$= \frac{1}{\alpha}$
- $\int_{-\infty}^{k} f(x)dx = \int_{0}^{k} f(x)dx$ ($\because$ It is defined only $0 \le x < \infty$)
- $\int_{k}^{\infty} f(x)dx = 1 - \int_{0}^{k} f(x)dx$ ($\because$ It is defined only $0 \le x < \infty$)

1. **If x is an exponential variate with mean 4, evaluate the following:**
   (i) $P(0 < x < 1)$ (ii) $P(x > 2)$ (iii) $P(-\infty < x < 10)$.

   By data, mean $= \frac{1}{\alpha} = 4$

   Therefore, $\alpha = \frac{1}{4} = 0.25$

   Probability density function is

   $p(x) = \alpha e^{-\alpha x}, x > 0$

   $\quad = 0.25\, e^{-0.25x}, x > 0$

   (i) $P(0 < x < 1) = 0.25 \int_{0}^{1} e^{-0.25x} dx = 0.2212$

   (ii) $P(x > 2) = 1 - P(x < 2) = 1 - 0.25 \int_{0}^{2} e^{-0.25x} dx = 0.6065$

   (iii) $P(-\infty < x < 10) = 0.25 \int_{0}^{10} e^{-0.25x} dx = 0.9179$

2. **In a certain town the duration of a shower is exponentially distributed with mean 5 minutes. What is the probability that a shower will last for (i) less than 10 minutes (ii) 10 minutes or more?**

By data, mean $= \frac{1}{\alpha} = 5$

Therefore, $\alpha = \frac{1}{5} = 0.2$

Probability density function is

$$p(x) = \alpha e^{-\alpha x}, x > 0$$

$$= 0.2e^{-0.2x}, x > 0$$

(i) P (shower lasts for less than 10 minutes)

$$= P(x < 10) = 0.2 \int_0^{10} e^{-0.2x} dx = 0.8647$$

(ii) P (shower lasts for 10 minutes or more)

$$= P(x > 10) = 1 - P(x < 10) = 1 - 0.8647 = 0.1353$$

3. **The length of telephone conversation in a booth has been an exponential distribution and found on an average to be 3 minutes. Find the probability that a random call made from this booth (i) Ends less than 3 minutes (ii) between 3 and 5 minutes.**

By data, mean $= \frac{1}{\alpha} = 3$

Therefore, $\alpha = \frac{1}{3}$

Probability density function is

$$p(x) = \alpha e^{-\alpha x}, x > 0$$

$$= \frac{1}{3} e^{-\frac{1}{3}x}, x > 0$$

(i) $P(\text{Call ends in less than 3mins.}) = \frac{1}{3} \int_0^3 e^{-\frac{1}{3}x} dx = 0.6321$

(ii) $P(\text{Call made between 3 and 5 mins.}) = \frac{1}{3} \int_3^5 e^{-\frac{1}{3}x} dx = 0.1789$

4. **The sales per day for a shop is exponentially distributed with the average sale amounting to ₹100 and net profit is 8%. Find the probability that the net profit exceeds ₹30 on two consecutive days.**

By data, mean $= \frac{1}{\alpha} = 100$

Therefore, $\alpha = \frac{1}{100} = 0.01$

Probability density function is

$p(x) = \alpha e^{-\alpha x}, x > 0$

$\quad\quad = 0.01\, e^{-0.01x}, x > 0$

$8\%\ of\ an\ amount = Profit$

$0.08\ of\ an\ amount = 30$

$Amount = \frac{30}{0.08} = 375$

$P(\text{net profit exceeds ₹30}) = P(\text{Amount exceeds ₹375})$

$= P(x > 375)$

$= 1 - P(x < 375)$

$= 1 - 0.01\int_0^{375} e^{-0.01x}dx = 0.0235$

P(net profit exceeds ₹30 on two consecutive days) $= 0.0235 \times 0.0235 = 0.0005$

5. **The average daily turnout of a departmental store is ₹10,000 and the net profit is 8%. If the turnout has an exponential distribution, find the probability that the net profit will exceed ₹3000 each on two consecutive days chosen at random.**

By data, mean $= \frac{1}{\alpha} = 10000$ Therefore, $\alpha = \frac{1}{10000} = 0.0001$ Probability density function is $p(x) = \alpha e^{-\alpha x}, x > 0$ $= 0.0001\, e^{-0.0001x}, x > 0$ $8\%\ of\ an\ amount = Profit$ $0.08\ of\ an\ amount = 3000$ $Amount = \frac{3000}{0.08} = 37500$	$p(\text{net profit exceeds ₹3000})$ $= P(\text{Amount exceeds ₹37500})$ $= P(x > 37500)$ $= 1 - P(x < 37500)$ $= 1 - 0.0001\int_0^{37500} e^{-0.0001x}dx$ $= 0.0235$ p(net profit exceeds ₹3000 on two consecutive days) $= 0.0235 \times 0.0235$ $= 0.0005$

6. At a certain city bus stop, three buses arrive per hour on an average. Assuming that the time between successive arrivals is exponentially distributed, find the probability that the time between the arrival of successive buses is (i) less than 10 minutes (ii) at least 30 minutes.

Average number of busses per hour = 3

Average time between arrivals = $\frac{60}{3} = 20$ minutes.

By data, mean $= \frac{1}{\alpha} = 20$

Therefore, $\alpha = \frac{1}{20} = 0.05$

Probability density function is

$$p(x) = \alpha e^{-\alpha x}, x > 0$$

$$= 0.05\, e^{-0.05x}, x > 0$$

(i) P (Time between the arrival of successive buses is less than 10 minutes)

$$= P(x < 10) = \int_0^{10} 0.05 e^{-0.05x} dx = 0.3935$$

(ii) P (Time between the arrival of successive buses is at least 30 minutes)

$$= P(x \geq 30) = \int_{30}^{\infty} 0.05 e^{-0.05x} dx$$

$$= 1 - \int_0^{30} 0.05 e^{-0.05x} dx = 1 - 0.7769 = 0.2231$$

7. **After the appointment of a new sales manager, the sales in a two-wheeler show-room is exponentially distributed with mean equal to 4. If two days are selected at random, what is the probability that (i) on both days the sales is over 5 units, (ii) the sales is over 5 units on at least one of the two days?**

By data, mean $= \frac{1}{\alpha} = 4$

Therefore, $\alpha = \frac{1}{4} = 0.25$

Probability density function is

$p(x) = \alpha e^{-\alpha x}, x > 0$

$\quad = 0.25\, e^{-0.25x}, x > 0$

(i) P (The sales is over 5 units on a randomly chosen day)

$$= P(x > 5) = \int_5^\infty 0.25e^{-0.25x}dx = 1 - \int_0^5 0.25e^{-0.25x}dx = 0.2865$$

If two days are selected,

(ii) P (The sales is over 5 units on both days)

$$= P(x > 5) \times P(x > 5) = 0.0821$$

(iii) P (The sales is over 5 units on at least one day)

$= 1 - P(x < 5) \times P(x < 5) = 1 - 0.7135^2 = 0.4909$

8. **The daily turnover in a medical shop is exponentially distributed with ₹6000 as the average with a net profit of 8%. Find the probability that the net profit exceeds ₹500 on a randomly chosen day.**

By data, mean $= \frac{1}{\alpha} = 6000$ Therefore, $\alpha = \frac{1}{6000} = 0.0002$ Probability density function is $p(x) = \alpha e^{-\alpha x}, x > 0$ $\quad = 0.0002\, e^{-0.0002x}, x > 0$ $8\%\ of\ an\ amount = Profit$ $0.08\ of\ an\ amount = 500$ $Amount = \frac{500}{0.08} = 6250$	$P(\text{net profit exceeds ₹500})$ $= P(\text{Amount exceeds ₹6250})$ $= P(x > 6250)$ $= 1 - P(x < 6250)$ $= 1 - 0.0002 \int_0^{6250} e^{-0.0002x}dx$ $= 0.2865$

Home work:

9. If x is an exponential variate with mean 5 evaluate
(i) $P(0 < x < 1)$, (ii)$P(-\infty < x < 10)$ and (iii)$P(x \leq 0 \; or \; x \geq \; 1)$
Ans: 0.1813, 0.8647, 0.8187
10. The life a T.V. tube manufactured by a company is known to have a mean of 200 months. Assuming that the life has an exponential distribution, find the probability that the life of a tube manufactured by the company is (i) less than 200 months, (ii) between 100 and 300 months.

[Ans. $1 - e^{-1}; e^{-0.5} - e^{-1.5}$]

11. If the life time of a certain type of electric bulbs is distributed as an exponential variate with mean of 100 hours, what is the probability that a bulb will last for more than 1500 hours? If two bulbs are selected at random, find the probability that (i) both the bulbs, (ii) at least one bulb will last for more than 1500 hours. [Ans. $e^{-3/2}, e^{-3}, \left(2e^{-3/2} - e^{-3}\right)$]
12. In a certain town, the duration of a shower is exponentially distributed with mean equal to 5 minutes. What is the probability that (i) a shower will last for at least 2 minutes more, given that it has already lasted for 5 minutes, (ii) a shower will last for not more than 6 minutes more if it has already lasted for 3 minutes [Ans. (i) $e^{-2/5}$ (ii) $1 - e^{-6/5}$]