2.1 Mathematical Induction

Introduction:

- **Well ordering principle:** Every non empty subset of z^+contains a smallest element.
- **Induction principle:** Let $S(n)$ denote an open statement.

  Basis step: Verify that $S(n)$ is true for $n = 1$.
  Induction step: Assuming that $S(k)$ is true, show that $S(k+1)$ is true.

- **Method of mathematical induction**: The method of proving a statement on the basis of the induction principle is called the method of mathematical induction.

1. **Prove by mathematical induction that for all positive integers $n \geq 1$,**

$$\mathbf{1 + 2 + \cdots + n = \frac{n(n+1)}{2}}$$

Consider $S(n): 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$

Basis step: $S(1): 1 = 1\frac{(1+1)}{2}$ is true clearly.
It is verified that $S(n)$ is true for $n = 1$.

Induction step: Assume $S(k): 1 + 2 + \cdots + k = \frac{k(k+1)}{2}$ is true.

$$S(k+1): 1 + 2 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + (k+1)$$

$$= \frac{k(k+1) + 2(k+1)}{2}$$

$$= \frac{k^2 + 3k + 2}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

We proved that $s(k+1)$ is true.
Therefore, by the principle of mathematical induction, $S(n)$ is true for any $n \geq 1$.

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2. **Prove that for each $n \in Z^+, 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$**

Consider $S(n): 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

Basis step: $S(1): 1^2 = \frac{1(1+1)(2+1)}{6}$ is true clearly.

It is verified that $S(n)$ is true for $n = 1$.

Induction step: Assume $S(k): 1^2 + 2^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$ is true.

$$S(k+1): 1^2 + 2^2 + \cdots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$
$$= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$$
$$= \frac{(k+1)(k(2k+1) + 6(k+1))}{6}$$
$$= \frac{(k+1)(2k^2 + 7k + 6)}{6}$$
$$= \frac{(k+1)(k+2)(2k+3)}{6}$$
$$= \frac{(k+1)(k+2)(2(k+1)+1)}{6}$$

It is proved that $s(k+1)$ is true.

Therefore, by the principle of mathematical induction, $S(n)$ is true for any $n \geq 1$.

3. **By mathematical induction P.T. $1.3 + 2.4 + 3.5 + \cdots + n(n+2) = \frac{1}{6}n(n+1)(2n+7)$**

$S(n): 1.3 + 2.4 + 3.5 + \cdots + n(n+2) = \frac{1}{6}n(n+1)(2n+7)$

Basis step: $S(1): 1.3 = \frac{1(1+1)(2+7)}{6}$ is true clearly.

It is verified that $S(n)$ is true for $n = 1$.

Induction step: Assume $S(k): 1.3 + 2.4 + 3.5 + \cdots + k(k+2) = \frac{1}{6}k(k+1)(2k+7)$ is true.

$S(k+1): 1.3 + 2.4 + 3.5 + \cdots + k(k+2) + (k+1)(k+3)$
$$= \frac{1}{6}k(k+1)(2k+7) + (k+1)(k+3)$$
$$= \frac{1}{6}(k+1)(2k^2 + 7k + 6k + 18)$$
$$= \frac{1}{6}(k+1)(2k^2 + 13k + 18)$$
$$= \frac{1}{6}(k+1)(k+2)(2k+9)$$
$$= \frac{1}{6}(k+1)(k+2)(2(k+1)+7)$$

It is proved that $s(k+1)$ is true.

Therefore, by the principle of mathematical induction, $S(n)$ is true for any $n \geq 1$.

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4. Prove by mathematical induction that $\frac{1}{2.5}+\frac{1}{5.8}+\cdots+\frac{1}{(3n-1)(3n+2)}=\frac{n}{6n+4}$

$S(n): \frac{1}{2.5}+\frac{1}{5.8}+\cdots+\frac{1}{(3n-1)(3n+2)}=\frac{n}{6n+4}$

Basis step: $S(1): \frac{1}{2.5}=\frac{1}{6+4}$ is true clearly.

It is verified that $S(n)$ is true for $n=1$.

Induction step: Assume $S(k): \frac{1}{2.5}+\frac{1}{5.8}+\cdots+\frac{1}{(3k-1)(3k+2)}=\frac{k}{6k+4}$ is true. $S(k+1): \frac{1}{2.5}+\frac{1}{5.8}+\cdots+\frac{1}{(3k-1)(3k+2)}+\frac{1}{(3k+2)(3k+5)}$

$$=\frac{k}{6k+4}+\frac{1}{(3k+2)(3k+5)}$$
$$=\frac{1}{(3k+2)}\left(\frac{k}{2}+\frac{1}{(3k+5)}\right)$$
$$=\frac{1}{(3k+2)}\left(\frac{3k^2+5k+2}{6k+10}\right)$$
$$=\frac{1}{(3k+2)}\left(\frac{(3k+2)(k+1)}{6k+10}\right)$$
$$=\frac{k+1}{6(k+1)+4}$$

It is proved that $s(k+1)$ is true.

Therefore, by the principle of mathematical induction, $S(n)$ is true for any $n \geq 1$.

5. Prove by mathematical induction that,
for every positive integer $n \geq 2$, 3 divides n^3-n

$S(n)$: 3 divides $n^3-n, n \geq 2$.

Basis step: $S(2): 3\ divides\ 2^3-2$ is true clearly.

It is verified that $S(n)$ is true for $n=2$.

Induction step: Assume $S(k)$: 3 divides $\mathrm{A_k}=k^3-k$ is true.

We have to show that $S(k+1)$: 3 divides $\mathrm{A_{k+1}}=(k+1)^3-(k+1)$ is true.

$A_{k+1}-A_k=(k+1)^3-(k+1)-k^3+k$

$=k^3+3k^2+3k+1-k-1-k^3+k$

$=3(k^2+k)$

$A_{k+1}=A_k+3(k^2+k)$, divisible by 3.

$\therefore S(k+1)$: 3 divides $\mathrm{A_{k+1}}=(k+1)^3-(k+1)$ is true.

Therefore, by the principle of mathematical induction, $S(n)$ is true for any $n \geq 2$.

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6. **Prove by mathematical induction that, for any positive integer n, the number $\mathbf{11^{n+2} + 12^{2n+1}}$ is divisible by 133.**

$S(n)$: $11^{n+2} + 12^{2n+1}$ is divisible by 133.

Basis step: $S(1)$: $11^{1+2} + 12^{2+1}$ is divisible by 133 is true clearly.

It is verified that $S(n)$ is true for $n = 1$.

Induction step: Assume $S(k)$: $A_k = 11^{k+2} + 12^{2k+1}$ is divisible by 133.

We have to show that $S(k+1)$: $A_{k+1} = 11^{(k+1)+2} + 12^{2(k+1)+1}$ is divisible by 133 is true.

$A_{k+1} - A_k = 11^{(k+1)+2} + 12^{2(k+1)+1} - 11^{k+2} - 12^{2k+1}$

$= 11^{k+2}(11-1) + 12^{2k+1}(12^2 - 1)$

$= 10(11^{k+2} + 12^{2k+1}) + 12^{2k+1}(133)$

$A_{k+1} = A_k + 10A_k + 12^{2k+1}(133)$, divisible by 133.

$\therefore S(k+1)$: $A_{k+1} = 11^{(k+1)+2} + 12^{2(k+1)+1}$ is divisible by 133 is true.

Therefore, by the principle of mathematical induction, $S(n)$ is true for any $n \geq 1$.

7. **Prove by mathematical induction that,for every positive integer $\boldsymbol{n}$, $\boldsymbol{A_n = 5^n + 2.3^{n-1} + 1}$ is a multiple of 8.**

$S(n)$: $A_n = 5^n + 2.3^{n-1} + 1$ is a multiple of 8 , $n > 0$

Basis step: $S(1)$: $A_1 = 5^1 + 2.3^{1-1} + 1$ is a multiple of 8 is true clearly.

It is verified that $S(n)$ is true for $n = 1$.

Induction step: Assume $S(k)$: $A_k = 5^k + 2.3^{k-1} + 1$ is a multiple of 8.

We have to show that $S(k+1)$: $A_{k+1} = 5^{(k+1)} + 2.3^{(k+1)-1} + 1$ is a multiple of 8.

Now, $A_{k+1} - A_k = 5^{(k+1)} + 2.3^{(k+1)-1} + 1 - 5^k - 2.3^{k-1} - 1.$

$= 5^k(5-1) + 2.3^{k-1}(3-1)$

$= 4(5^k + 3^{k-1})$

$=$ 4(Even), which is a multiple of 8.

$A_{k+1} = A_k + 4$(Even) is a multiple of 8.

$\therefore S(k+1)$is true.

Therefore, by the principle of mathematical induction, $S(n)$ is true for any $n \geq 1$.

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8. Prove that $4n < (n^2 - 7)$ for all positive integers $n \geq 6$.

$S(n)$: $4n < (n^2 - 7)$ for all positive integers $n \geq 6$.

Basis step: $S(1): 4(6) < (6^2 - 7)$is a multiple of 8 is true clearly.

It is verified that $S(n)$ is true for $n = 6$.

Induction step: Assume $S(k): 4k < (k^2 - 7)$is a multiple of 8.

We have to show that $S(k+1): 4(k+1) < (k+1)^2 - 7$ is a multiple of 8 is true.

$$4(k+1) = 4k + 4$$
$$< (k^2 - 7) + 4$$
$$< (k^2 - 7) + (2k + 1)$$
$$\therefore 4(k+1) < (k+1)^2 - 7$$

Therefore, it is proved that $S(k+1)$ is true.

Therefore, by the principle of mathematical induction, $S(n)$ is true for any $n \geq 6$.

9. Prove that every positive integer $n \geq 24$ can be written as a sum of 5's and/or 7's.

$S(n)$: n can be written as a sum of 5's and/or 7's.

Basis step: $S(24): 24 = 5 + 5 + 7 + 7$is true clearly.

It is verified that $S(n)$ is true for $n = 24$.

Induction step: Assume $S(k)$: $\mathrm{k} = (7 + 7 + \cdots \text{r times}) + (5 + 5 + \cdots \text{s times})$ is true.

We have to show that $S(k+1)$ is true.

$$k + 1 = (7 + 7 + \cdots (\mathrm{r} - 2)\text{times}) + (5 + 5 + \cdots \text{s times}) + 7 + 7 + 1$$
$$= (7 + 7 + \cdots (\mathrm{r} - 2)\text{times}) + (5 + 5 + \cdots (\mathrm{s} + 3) \text{ times})$$
$$= \text{ sum of 5's and 7's.}$$

Therefore, it is proved that $S(k+1)$ is true.

Therefore, by the principle of mathematical induction, $S(n)$ is true for any $n \geq 24$.

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10. Let $H_1 = 1, H_2 = 1 + \frac{1}{2}, H_3 = 1 + \frac{1}{2} + \frac{1}{3}, \dots, H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. Prove that $\sum_{i=1}^{n} H_i = (n+1)H_n - n$, for all positive integers $n \geq 1$.

$S(n)$: $\sum_{i=1}^{n} H_i = (n+1)H_n - n$, for all positive integers $n \geq 1$.

Basis step: $S(1)$: $H_1 = (1+1)H_1 - 1$ is true clearly.

It is verified that $S(n)$ is true for $n = 1$.

Induction step: Assume $S(k)$: $\sum_{i=1}^{k} H_i = (k+1)H_k - k$

We have to show that $S(k+1)$ is true.

$$\sum_{i=1}^{k+1} H_i = \sum_{i=1}^{k} H_i + H_{k+1}$$

$$= (k+1)H_k - k + H_{k+1}$$

$$= (\mathrm{k}+1)\left(\mathrm{H}_{\mathrm{k}+1} - \frac{1}{k+1}\right) - k + H_{k+1}$$

$$= (\mathrm{k}+2)\mathrm{H}_{\mathrm{k}+1} - (k+1)$$

Therefore, it is proved that $S(k+1)$ is true.

Therefore, by the principle of mathematical induction, $S(n)$ is true for any $n \geq 1$.

Exercise 2.1

1. Prove by mathematical induction that for all positive integers $n \geq 1$,
$$1 + 3 + 5 + \cdots + (2n-1) = n^2$$
2. Prove by mathematical induction that for all positive integers $n \geq 1$,
$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$
3. If n is any positive integer prove that
$$1.2 + 2.3 + 3.4 + \cdots + n(n+1) = \frac{1}{3}n(n+1)(n+2)$$
4. Prove by mathematical induction that for all positive integers $n \geq 1$,
$$\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{1}{2n+1}$$
5. Prove by mathematical induction that for all positive integers n, 5 divides $n^5 - n$.
6. Prove by mathematical induction that for all positive integers n, $6^{n+2} + 7^{2n+1}$ is divisible by 43.
7. Prove by mathematical induction that $n < 2^n$ for all positive integers $n \geq 1$,
8. Prove that every positive integer is greater than or equal to 14 may be

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2.2 Recursive definitions

Introduction:

- In the explicit method, general term of the sequence is explicitly indicated.
- In the recursive method, first few terms of the sequence are explicitly indicated and the general term to get new terms of the sequence is also specified.
- Example:

Sequence	Explicit method	Recursive method
2, 4, 6, 8, …	$a_n = 2n\ , n \in Z^+$	$a_1 = 2\ and\ a_n = a_{n-1} + 2$

- If a sequence is described by a recursive method, we say that the sequence is specified through a recursive definition
- $1 + a + a^2 + a^3 + \cdots + a^{n-1} = \frac{a^n - 1}{a-1}$
- Fibonacci's sequence: 0, 1, 1, 2, 3, 5, 8, 13, …   Luca's sequence: 2, 1, 3, 4, 7, 11, 18, ...

1. Obtain a recursive definition for the sequence $\{a_n\}$ in each of the following:
(i) $a_n = 5n$ (ii) $a_n = 6^n$

(i) $a_1 = 5$

$a_n = 5n$

$a_{n-1} = 5(n-1)$

$a_n - a_{n-1} = 5n - 5(n-1) = 5$

The required recursive definition is

$a_1 = 5, a_n = a_{n-1} + 5, n \geq 2$

(ii) $a_1 = 6$

$a_n = 6^n$

$a_{n-1} = 6^{n-1}$

$\frac{a_n}{a_{n-1}} = \frac{6^n}{6^{n-1}} = 6$

The required recursive definition is

$a_1 = 6, a_n = 6\ a_{n-1}, n \geq 2$

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2. **Obtain a recursive definition for the sequence $\{a_n\}$ in each of the following:**

(i) $a_n = 3n + 7$ (ii) $a_n = 2 - (-1)^n$

(i) $a_1 = 3 + 7$

$a_n = 3n + 7$

$a_{n-1} = 3(n-1) + 7$

$a_n - a_{n-1} = (3n + 7) - (3(n-1) + 7) = 3$

The required recursive definition is

$a_1 = 10, a_n = a_{n-1} + 3, n \geq 2.$

(iii) $a_1 = 2 - (-1)$

$a_n = 2 - (-1)^n$

$a_{n-1} = 2 - (-1)^{n-1} = 2 + (-1)^n$

$a_n - a_{n-1} = 2 - (-1)^n - 2 - (-1)^n$

The required recursive definition is

$a_1 = 3, a_n = a_{n-1} - 2(-1)^n$

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3. Find an explicit definition of the sequence defined recursively by the following:

(i) $\boldsymbol{a_1 = 7, a_n = 2a_{n-1} + 1}$ **for** $\boldsymbol{n \geq 2}$**.**

(ii) $\boldsymbol{a_1 = 0, a_n = a_{n-1} - 2}$ **for** $\boldsymbol{n \geq 2}$**.**

(i)
$$
\begin{aligned}
a_n &= 2a_{n-1} + 1 \\
&= 2(2a_{n-2} + 1) + 1 \\
&= 2^2 a_{n-2} + 2 + 1 \\
&= 2^2(2a_{n-3} + 1) + 2 + 1 \\
&= 2^3 a_{n-3} + 2^2 + 2 + 1
\end{aligned}
$$

In general,

$$
\begin{aligned}
a_n &= 2^{n-1} a_1 + 2^{n-2} + \cdots + 2^2 + 2 + 1 \\
&= 7.2^{n-1} + \frac{2^{n-1} - 1}{2 - 1} \\
&= 8.2^{n-1} - 1
\end{aligned}
$$

This is the required explicit definition.

(ii)
$$
\begin{aligned}
a_n &= a_{n-1} - 2 \\
&= a_{n-1} - 2(1) \\
&= (a_{n-2} - 2) - 2 \\
&= a_{n-2} - 2(2) \\
&= (a_{n-3} - 2) - 2(2) \\
&= a_{n-3} - 3(2)
\end{aligned}
$$

In general,

$$
\begin{aligned}
a_n &= a_{n-(n-1)} - (n-1)2 \\
&= a_1 - 2(n-1) \\
&= -2(n-1)
\end{aligned}
$$

This is the required explicit definition.

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4. Find an explicit definition of the sequence defined recursively by the following:

(i) $\boldsymbol{a_1 = 5, a_n = 7a_{n-1} + 1}$ **for** $\boldsymbol{n \geq 2.}$

(ii) $\boldsymbol{a_1 = 4, a_n = a_{n-1} + 3^n}$**for** $\boldsymbol{n \geq 2.}$

(i) $a_n = 7a_{n-1} + 1$

$= 7(7a_{n-2} + 1) + 1$

$= 7^2 a_{n-2} + 7 + 1$

$= 7^2(7a_{n-3} + 1) + 7 + 1$

$= 7^3 a_{n-3} + 7^2 + 7 + 1$

In general,

$$a_n = 7^{n-1}a_1 + 7^{n-2} + \cdots + 7^2 + 7 + 1$$

$$= 7^{n-1}.5 + \frac{7^{n-1}-1}{7-1}$$

$$= \frac{1}{6}(30.7 + 7^{n-1^{n-1}} - 1)$$

$$= \frac{1}{6}(31.7^{n-1} - 1)$$

This is the required explicit definition.

(ii) $a_n = a_{n-1} + 3^n$

$= (a_{n-2} + 3^{n-1}) + 3^n$

$= (a_{n-3} + 3^{n-2}) + 3^{n-1} + 3^n$

In general,

$$a_n = a_{n-(n-1)} + 3^{n-(n-2)} + \cdots + 3^n$$

$$= a_1 + 3^2 + 3^3 + \cdots + 3^n$$

$$= 4 + 3^2 + 3^3 + \cdots + 3^n$$

$$= 1 + 3 + 3^2 + 3^3 + \cdots + 3^n$$

$$= \frac{3^{n+1} - 1}{3 - 1}$$

$$= \frac{3^{n+1} - 1}{2}$$

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This is the required explicit definition.

5. If $F_0, F_1, F_2, \ldots$ are Fibonacci numbers, Define Fibonacci numbers. Evaluate F_2 to F_9

Recursive definition of Fibonacci's series is $F_n = F_{n-1} + F_{n-2}, F_0 = 0, F_1 = 1$.

$$F_2 = F_1 + F_0 = 1 + 0 = 1$$
$$F_3 = F_2 + F_1 = 1 + 1 = 2$$
$$F_4 = F_3 + F_2 = 2 + 1 = 3$$
$$F_5 = F_4 + F_3 = 3 + 2 = 5$$
$$F_6 = F_5 + F_4 = 5 + 3 = 8$$
$$F_7 = F_6 + F_5 = 8 + 5 = 13$$
$$F_8 = F_7 + F_6 = 13 + 8 = 21$$
$$F_9 = F_8 + F_7 = 21 + 13 = 34$$

6. If $F_0, F_1, F_2, \ldots$ are Fibonacci numbers, P.T. $\sum_{i=0}^{n} F_i^2 = F_n \times F_{n+1}$ for all $n \in N$.

$$S(n): \sum_{i=0}^{n} F_i^2 = F_n \times F_{n+1},$$

for all positive integers $n \geq 1$.

Basis step: $S(1)$: $\sum_{i=0}^{1} F_i^2 = F_1 \times F_{1+1}$

$F_0^2 + F_1^2 = F_1 \times F_2$

$0^2 + 1^2 = 1 \times 1$ is true clearly.

It is verified that $S(n)$ is true for $n = 1$.

Induction step: Assume $S(k)$: $\sum_{i=1}^{k} F_i^2 = F_k \times F_{k+1}$is true.

We have to show that $S(k+1)$ is true.

$$\sum_{i=1}^{k+1} F_i^2 = \sum_{i=1}^{k} F_i^2 + F_{k+1}^2$$

$$= F_k \times F_{k+1} + F_{k+1}^2$$

$$= F_{k+1}(F_k + F_{k+1})$$

$$= F_{k+1} \times F_{k+2}$$

Therefore, it is proved that $S(k+1)$ is true.

Therefore, by the principle of mathematical induction, $S(n)$ is true for any $n \geq 1$.

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7. $\boldsymbol{Prove\ that\ \sum_{i=0}^{n} \frac{F_{i-1}}{2^i} = 1 - \frac{F_{n+2}}{2^n}\ ,\ for\ all\ n \in N}$

$$S(n): \sum_{i=1}^{n} \frac{F_{i-1}}{2^i} = 1 - \frac{F_{n+2}}{2^n}\ , for\ all\ positive\ integers\ n \geq 1$$

Basis step: $S(1): \frac{F_{1-1}}{2^1} = 1 - \frac{F_{1+2}}{2^1}$

$$\frac{F_0}{2^1} = 1 - \frac{F_3}{2}$$

$0 = 1 - \frac{2}{2}$ is true clearly.

It is verified that $S(n)$ is true for $n = 1$.

Induction step: Assume $S(k): \sum_{i=1}^{k} \frac{F_{i-1}}{2^i} = 1 - \frac{F_{k+2}}{2^k}$is true.

We have to show that $S(k+1)$ is true.

$$\sum_{i=0}^{k+1} \frac{F_{i-1}}{2^i} = \sum_{i=0}^{k} \frac{F_{i-1}}{2^i} + \frac{F_k}{2^{k+1}}$$

$$= 1 - \frac{F_{k+2}}{2^k} + \frac{F_k}{2^{k+1}}$$

$$= 1 - \frac{1}{2^{k+1}}(2F_{k+2} - F_k)$$

$$= 1 - \frac{1}{2^{k+1}}(F_{k+2} + F_{k+2} - F_k)$$

$$= 1 - \frac{1}{2^{k+1}}(F_{k+2} + F_{k+1})$$

$$= 1 - \frac{1}{2^{k+1}}(F_{k+3})$$

Therefore, it is proved that $S(k+1)$ is true.

Therefore, by the principle of mathematical induction, $S(n)$ is true for any $n \geq 1$.

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8. Prove that $F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$

$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$, for all positive integers $n \geq 1$.

Basis step: $F_1 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^1 - \left(\frac{1-\sqrt{5}}{2}\right)^1\right]$

$= \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}-1+\sqrt{5}}{2}\right] = 1$ is true clearly.

It is verified that $S(n)$ is true for $n = 1$.

Induction step: Assume $F_k = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right]$ is true.

We have to show that $S(k+1)$ is true.

$$F_{k+1} = F_k + F_{k-1}$$

$$= \frac{1}{\sqrt{5}}(A^k - B^k) + \frac{1}{\sqrt{5}}(A^{k-1} - B^{k-1})$$

$$= \frac{1}{\sqrt{5}}[A^{k-1}(A+1) - B^{k-1}(B+1)]$$

$$= \frac{1}{\sqrt{5}}\left[A^{k-1}\left(\frac{1+\sqrt{5}}{2}+1\right) - B^{k-1}\left(\frac{1-\sqrt{5}}{2}+1\right)\right]$$

$$= \frac{1}{\sqrt{5}}\left[A^{k-1}\left(\frac{3+\sqrt{5}}{2}\right) - B^{k-1}\left(\frac{3-\sqrt{5}}{2}\right)\right]$$

$$= \frac{1}{\sqrt{5}}\left[A^{k-1}\left(\frac{6+2\sqrt{5}}{4}\right) - B^{k-1}\left(\frac{6-2\sqrt{5}}{4}\right)\right]$$

$$= \frac{1}{\sqrt{5}}\left[A^{k-1}\left(\frac{1+\sqrt{5}}{4}\right)^2 - B^{k-1}\left(\frac{1-\sqrt{5}}{4}\right)^2\right]$$

$$= \frac{1}{\sqrt{5}}[A^{k-1}A^2 - B^{k-1}B^2]$$

$$= \frac{1}{\sqrt{5}}[A^{k+1} - B^{k+1}]$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right]$$

Therefore, it is proved that F_{k+1} is true.

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Therefore, by the principle of mathematical induction, F_n is true for any $n \geq 1$.

9. Define Lucas numbers. If $L_i's$ are Lucas numbers, evaluate from L_{11} to L_{15}.

The Lucas numbers are defined recursively by

$$L_0 = 2, L_1 = 1, L_n = L_{n-1} + L_{n-2}, n \geq 2$$

L_0	L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}
2	1	3	4	7	11	18	29	47	76	123

$$L_{11} = L_{10} + L_9 = 123 + 76 = 199$$
$$L_{12} = L_{11} + L_{10} = 199 + 123 = 322$$
$$L_{13} = L_{12} + L_{11} = 322 + 199 = 521$$
$$L_{14} = L_{13} + L_{12} = 521 + 322 = 843$$
$$L_{15} = L_{14} + L_{13} = 843 + 521 = 1364$$

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10. Prove that $\boldsymbol{L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n}$

Consider $L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$

Basis step: $L_0 = \left(\frac{1+\sqrt{5}}{2}\right)^0 + \left(\frac{1-\sqrt{5}}{2}\right)^0 = 1 + 1 = 2$ is true clearly.

Induction step: Assume $L_k = \left(\frac{1+\sqrt{5}}{2}\right)^k + \left(\frac{1-\sqrt{5}}{2}\right)^k$ is true.

We have to prove that $L_{k+1} = A^{k+1} + B^{k+1}$, where $A = \frac{1+\sqrt{5}}{2}$, $B = \frac{1-\sqrt{5}}{2}$

$$\begin{aligned}
L_{k+1} &= L_k + L_{k-1} \\
&= A^k + B^k + A^{k-1} + B^{k-1} \\
&= A^{k-1}(A+1) + B^{k-1}(B+1) \\
&= A^{k-1}\left(\frac{1+\sqrt{5}}{2} + 1\right) + B^{k-1}\left(\frac{1-\sqrt{5}}{2} + 1\right) \\
&= A^{k-1}\left(\frac{3+\sqrt{5}}{2}\right) + B^{k-1}\left(\frac{3-\sqrt{5}}{2}\right) \\
&= A^{k-1}\left(\frac{6+2\sqrt{5}}{4}\right) + B^{k-1}\left(\frac{6-2\sqrt{5}}{4}\right) \\
&= A^{k-1}\left(\frac{1+\sqrt{5}}{4}\right)^2 + B^{k-1}\left(\frac{1-\sqrt{5}}{4}\right)^2 \\
&= A^{k-1}A^2 + B^{k-1}B^2 \\
&= A^{k+1} + B^{k+1} \\
&= \left(\frac{1+\sqrt{5}}{2}\right)^{k+1} + \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}
\end{aligned}$$

Therefore, it is proved that L_{k+1} is true.

Therefore, by the principle of mathematical induction, L_n is true for any $n \geq 1$.

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11. Define Ackermann's numbers. Evaluate the Ackermann's numbers $A_{1,3}$ and $A_{2,3}$. Prove that $A_{1,n} = n + 2$ for all $n \in N$.

(i) The Ackermann's numbers $A_{m,n}$ are defined recursively for $m, n \in N$ as follows:

$A_{0,n} = n + 1 \; for \; n \geq 0,$

$A_{m,0} = A_{m-1,1} \; for \; m > 0,$

$A_{m,n} = A_{m-1,p} \; for \; m, n > 0$

Where $p = A_{m,n-1}, for \; m, n > 0$ Prove that $A_{1,n} = n + 2$ for all $n \in N$.

(ii) $A_{1,0} = A_{0,1} = 1 + 1 = 2$

$A_{1,1} = A(0, A_{1,0}) = A(0,2) = 2 + 1 = 3$

$A_{1,2} = A(0, A_{1,1}) = A(0,3) = 3 + 1 = 4$

$A_{1,3} = A(0, A_{1,2}) = A(0,4) = 4 + 1 = 5$

Therefore, $A(1,3) = 5$

(iii) $A_{2,0} = A_{1,1} = A(0, A_{1,0}) = A(0,2) = 2 + 1 = 3$

$A_{2,1} = A(1, A_{2,0}) = A(1,3) = A(0, A_{1,2}) = A(0,4) = 4 + 1 = 5$

$A_{2,2} = A(1, A_{2,1}) = A(1,5) = A(0, A_{1,4}) = A(0,6) = 6 + 1 = 7$

$A_{2,3} = A(1, A_{2,2}) = A(1,7) = A(0, A_{1,6}) = A(0,8) = 8 + 1 = 9$

Therefore, $A(2,3) = 9$

(iv) Consider $S(n)$: $A_{1,n} = n + 2$.

Basis step: $S(0)$: $A_{1,0} = 0 + 2 = 2$ is true clearly.

Induction step: Assume $S(k)$: $A_{1,k} = k + 2$ is true.

We have to prove that $S(k + 1)$ is true.

$S(k + 1)$: $A_{1,k+1} = A(0, A_{1,k}) = A_{0,k+2}$

$= k + 2 + 1$

$= k + 3 = (k + 1) + 2$

Therefore, it is proved that $S(k + 1)$ is true.

Therefore, by the principle of mathematical induction,

$A_{1,n} = n + 2$ is true for any $n \in N$

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Exercise 2.2

1. Obtain a recursive definition for the sequence $\{a_n\}$ in each of the following:
   (a) $a_n = n(n+2)$ (b) $a_n = 2n+3$ (c) $a_n = (n+1)!$ (d) $a_n = 3^n$

   Answer: (a) $a_1 = 3$ and $a_{n+1} = a_n + 2n + 3$
   (b) $a_1 = 5$ and $a_{n+1} = a_n + 2$ for $n \geq 1$
   (c) $a_1 = 2$ and $a_{n+1} = (n+1)a_n$ for $n \geq 1$
   (d) $a_1 = 3$ and $a_n = 3a_{n-1}$for $n \geq 2$

2. Find an explicit definition of the sequence defined recursively by the following:
   (a) $a_1 = 4, a_n = a_{n-1} + n$ for $n \geq 2$.
   (b) $a_1 = 2, a_n = a_{n-1} + 3$ for $n \geq 2$.
   (c) $a_1 = 8, a_n = a_{n-1} + n^2$for $n \geq 2$.
   (d) $a_1 = 4, a_n = a_{n-1} + (2n+1)$, for $n \geq 2$.

   Answer:
   (a) $a_n = 3 + \frac{1}{2}n(n+1)$
   (b) $a_n = 3(n-1) + 2$
   (c) $a_n = \frac{1}{6}n(n+1)(2n+1) + 7$
   (d) $a_n = (n+1)^2$

3. Evaluate the Fibonacci numbers fromF_{11} to F_{15}.

   Answer: 89, 144, 233, 377, 610.

4. A sequence $\{C_n\}$ is defined recursively by $C_n = 3C_{n-1} - 2C_{n-2}$for all $n \geq 3$, with $C_1 = 5$ and $C_2 = 3$ as the initial conditions. Show that $C_n = -2^n + 7$.

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2.3 The rules of sum and product

The sum rule: Suppose that k tasks $T_1, T_2, T_3, \dots, T_k$ are to be performed such that no two tasks can be performed at the same time. If T_i can be performed in n_i different ways then one of the k tasks can be performed in $n_1 + n_2 + \cdots + n_k$ different ways.

Example: Suppose there are 16 boys and 18 girls in a class and if we select one of these students as the class representative. The no. of ways of selecting a boy = 16, The no. of ways of selecting a girl = 18. By sum rule, the no. of ways of selecting a student= 16+18

The product rule: Suppose that k tasks $T_1, T_2, T_3, \dots, T_k$ are to be performed in a sequence. If T_i can be performed in n_i different ways then the sequence of tasks $T_1, T_2, T_3, \dots, T_k$ can be performed in $n_1 n_2 \dots n_k$ different ways.

Example: Suppose a person has 8 shirts and 5 ties. By product rule, he can choose a shirt and a tie in $8 \times 5 = 40$ different ways.

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1. **There are 20 married couple in a party. Find the number of ways of choosing one woman and one man from the party such that the two are not married to each other.**

   One woman can be selected in 20 ways.
   After neglecting her husband there are 19 men remaining.
   One man can be selected in 19 ways.
   By product rule, required number $= 20 \times 19 = 380$.

2. **A license plate consists of two English letters followed by four digits. If repetitions are allowed, how many of the plates have only vowels and even integers ?**

   Vowels are a, e, i, o, u totally 5 and even integers are 0,2,4,6,8 totally 5.
   Each of the first two positions can be filled in 5 ways.
   Each of the next four positions can be filled in 5 ways.
   By product rule, required number $= 5^2 \times 5^4 = 5^6 = 15{,}625$

3. **Cars of a particular manufacturer come in 4 models, 12 colours, 3 Engine sizes and two transition types. How many distinct cars can be manufactured? Of these how many have the same colour?**

   Number of distinct cars that can be manufactured $= 4 \times 12 \times 3 \times 2 = 288$.
   Number of distinct cars with the same colour that can be manufactured $= 4 \times 3 \times 2 = 24$.

4. **How many 3 digit numbers can be formed by using the 6 digits 2, 3, 4, 5, 6, 8 if the number is to be even and repetitions are not allowed.**

   Since the number is even, unit place can be filled by 2 or 4 or 6 or 8, totally 4 ways.
   Tenth place can be filled in$6 - 1 = 5$ ways.
   Hundredth place can be filled in $6 - 2 = 4$ ways.
   By product rule, the required number $= 4 \times 5 \times 4 = 80$.

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5. A bit is either 0 or 1. A byte is a sequence of 8 bits. Find (i) The no. of bytes. (ii) The no. of bytes that begin with 11 and end with11. (iii) The no. of bytes that begin with 11 and do not end with11. (iv) The number of bytes that begin with 11 or end with11.

(i) Each bit can be filled in 2 ways, either 0 or 1.

Each byte contains 8 bits.

By product rule, the required number $= 2 \times 2 \times \dots 8\ times = 2^8 = 256.$

(ii) In a byte beginning and ending with 11, there are 4 open positions to fill.

By product rule, the required number $= 2 \times 2 \times 2 \times 2 = 2^4 = 16.$

(iii) In a byte beginning with 11, there are 6 positions to fill.

By product rule, this can be done in $2^6 = 64$ ways.

Therefore, the no. of bytes that begin with 11 and do not end with11

=No. of bytes beginning with 11−No. of bytes beginning and ending with 11

$= 64 - 16 = 48.$

(iv) No. of bytes beginning with 11 or ending with 11

=No. of bytes beginning with 11+No. of bytes ending with 11

−No. of bytes beginning and ending with 11

$= 64 + 64 - 16 = 112.$

6. Suppose that a valid computer password consists of 7 characters, the first of which is one of the letters A, B, C, D, E, F, G and the remaining 6 characters are letters chosen from the English alphabet or a digit. How many different passwords are possible?

First character can be chosen in 7 ways.

Each of the remaining 6 characters can be chosen in $26 + 10 = 36$ ways.

By product rule, required number $= 7 \times 36^6$

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7. **Find the total no. of positive integers that can be formed from the digits 1, 2, 3, 4 if no digit is repeated in any one integer.**

By product rule,

Number of integers containing one digit $= 4$

Number of integers containing two digits $= 4 \times 3 = 12$

Number of integers containing three digits $= 4 \times 3 \times 2 = 24$

Number of integers containing four digits $= 4 \times 3 \times 2 \times 1 = 24$

By sum rule,

Total number of integers $= 4 + 12 + 24 + 24 = 64$.

8. **Find the no. of proper divisors of 441000.**

441000= $2^3 3^2 5^3 7^2$

Every divisor is of the form $d = 2^p 3^q 5^r 7^s$

p, q, r and s can be selected in 4,3,4,3 ways respectively.

By product rule, n umber of divisors $= 4 \times 3 \times 4 \times 3 = 144$.

Out of which, 2 of them are improper.

Therefore, the total number of proper divisors $= 142$.

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Exercise 2.3

1. Find the total number of positive integers that can be formed from the digits 1, 2, 3, 4 if no digit is repeated in any one integer.
   Answer: 64

2. A sports committee of 3 in a college is to be formed consisting of one representative each from boy students, girl students and teachers. if there are 3 possible representatives from boy students, 2 from girl students and 4 from teachers, determine how many different committees can be formed.

   Answer: $3 \times 2 \times 4 = 24$

3. A label identifier for a computer programme consists of one letter of the Englishalphabet followed by 2 digits. If repetitions are allowed, how many distinct label identifiers are possible?

   Answer: 2600

4. Find the number of 3 digit even numbers with no repeated digits.

   Answer: $[(1 \times 9) + (4 \times 8)] \times 8 = 328$

5. How many among the first 100,000 positive integers contain exactly one 3, one 4 and one 5in their decimal representations?

   Answer: $5 \times 4 \times 3 \times 7 \times 7 = 2940$

6. There are four bus routes between the places A and B and three bus routes between the places B and C. Find the number of ways a person can make a round trip from A to A via B if he does not use a route more than once.

   Answer: $4 \times 3 \times 2 \times 3 = 72$

7. Find the number of (a) 2 digit even numbers (b) 2 digit odd numbers

   Answer: (a) 9×5 (b) 9×5

8. Find the number of binary sequences of length n that contain an even number of 1's.

   Answer: 2^{n-1}

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2.4 Permutations

Introduction:

- The number of permutations of *n* distinct objects is $n!$ (Taken all at a time)
- The number of circular permutations of *n* distinct objects is $(n-1)!$
- The number of permutations of size *r* of *n* distinct objects is $\frac{n!}{(n-r)!}$.
- The number of permutations of *n* objects of which n_1 are of the first type and n_2 are of the second type is $\frac{n!}{n_1!n_2!}$.

1. **In how many ways can 6 men and 6 women be seated in a row (i) if any person may sit next to any other? (ii) If men and women must occupy alternative seats?**

   (i) If there is no restriction, 12 persons in a row can sit in 12! Ways.

   (ii) 6 men in odd places and 6 women in even places can be seated in $6! \times 6!$ ways. 6 men in even places and 6 women in odd places can be seated in $6! \times 6!$ ways.
   Therefore, total number of arrangements $= 2 \times 6! \times 6!$

2. **In how many ways can three men and three women be seated at a round table if (i) No restriction is imposed? (ii) Two particular women must not sit together? (iii) Each women is to be between two men?**

   (i) If no restriction imposed, 6 persons can be seated in a round table in $(6-1)! = 5! = 120$ ways.

   (ii) Two women can sit together in 2 ways. Consider this as 1 unit.
   One unit and 4 remaining persons can sit in $(5-1)! = 4! = 24$ ways.
   Therefore,
   If two women can sit together, total no. of arrangements is $2 \times 24 = 48$
   If two women can't sit together, total no. of arrangements is $120 - 48 = 72$

   (iii) Three men can be seated in $(3-1)! = 2!$ ways by leaving one seat between them.
   Three women can be seated in the remaining 3 seats in 3! ways.
   Therefore, total number of arrangements is $2! \times 3! = 12$.

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3. A student has three different books on C++ and four different books on Java. In how many ways can he arrange 7 books on a shelf (i) If there are no restrictions? (ii) If the languages should alternate? (iii) If all the C++ books must be next to each other? (iv) If all the C++ books must be next to each other and all the Java books must be next to each other?

(i) If there are no restrictions, three books on C++ and four books on Java, totally 7 books can be arranged in 7! ways.

(ii) Three C++ books in even places and four Java books in odd places can be arranged in $3! \times 4! = 144$ ways.

(iii) Consider three C++ books together as one unit.
Now one unit and four Java books can be arranged in 5! ways.
Therefore, total number of arrangements is $3! \times 5! = 720$.

(iv) Three C++ books . Consider this as one unit. Four Java books. Consider this as one unit. Two units can be arranged in 2 ways.
Therefore, total number of arrangements is 2.

4. Find the number of permutations of the letters of the word MASSASAUGA. In how many of these, all four A's together? How many of them begin with S?

(i) In 10 alphabets, 'S' repeated three times and 'A' repeated 4 times.
Therefore, total no. of permutations $= \frac{10!}{3! \times 4!} = 25{,}200$

(ii) Consider four A's together as one unit. Consider the remaining 6 letters as 6 units.
Now, we have 7 units.Out of 7 units, 'S' repeated three times.
Therefore, total no. of permutations $= \frac{7!}{3!} = 840$

(iii) First alphabet is fixed as S. Now, 9 alphabets remaining.
In 9 alphabets, 'S' repeated twice and 'A' repeated 4 times.
Therefore, total no. of permutations $= \frac{9!}{2! \times 4!} = 7560$

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5. (i) How many arrangements are there for all letters in the word SOCIOLOGICAL? In how many of these arrangements (ii) A and G are adjacent? (iii) All the vowels are adjacent ? (July 2014)

(i) In 12 letters, 'O' repeated thrice and 'C', 'I', 'L' repeated twice each.

Therefore, total number of arrangements$= \frac{12!}{3!\times2!\times2!\times2!} = 99{,}79{,}200$

(ii) A and G together can be arranged in 2 ways. Consider this as one unit.

Consider the remaining 10 letters as 10 units. Now we have 11 units

In 11 units, 'O' repeated thrice and 'C', 'I', 'L' repeated twice each.

Therefore, total number of arrangements$= 2 \times \frac{11!}{3!\times2!\times2!\times2!} = 16{,}63{,}200$

(iii) Two I's and three O's together can be arranged in $\frac{6!}{2!\times3!} = 60$ ways.

Consider this as a single unit. Consider the remaining 6 letters as 6 units.

In 7 units, 'C' and 'L' repeated twice each.

Therefore, total number of arrangements$= 60 \times \frac{7!}{2!\times2!} = 75{,}600$

6. (i) Find the number of permutations of the letters of the word MISSISSIPPI. (ii) How many of these begin with an I? (iii) How many of these begin and end with an S?

(i) In 11 letters, 'S' and 'I' repeated 4 times each and 'P' repeated twice.

Therefore, total no. of permutations $= \frac{11!}{4!\times4!\times2!} = 34{,}650$

(ii) First letter is fixed as I. Now,10 letters remaining. In 10 letters, 'S' repeated four times, 'I' repeated thrice and 'P' repeated twice.

Therefore, total no. of permutations $= \frac{10!}{4!\times3!\times2!} = 12.600$

(iii) Starting and ending letters are fixed as 'S'. Now, 9 letters remaining.

In 9 letters, 'S' and 'P' repeated twice each and 'I' repeated four times.

Therefore, total no. of permutations $= \frac{9!}{2!\times2!\times4!} = 3780$

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7. How many positive integers n can we form using the digits 3, 4, 4, 5, 5, 6, 7 if we want n to exceed 5,000,000? (July 2013)

We have 9 digits, out of which there are two 4's and two 5's.

Let $n = x_1x_2x_3x_4x_5x_6x_7$. x_1 must be 5 or 6 or 7.

Suppose $x_1 = 5$, remaining 6 digits can be arranged in $\frac{6!}{2!} = 360$ ways.

Suppose $x_1 = 6$, remaining 6 digits can be arranged in $\frac{6!}{2!\times 2!} = 180$ ways.

Suppose $x_1 = 7$, remaining 6 digits can be arranged in $\frac{6!}{2!\times 2!} = 180$ ways.

Therefore total no. of arrangements$= 360 + 180 + 180 = 720$.

8. How many different three digit numbers can be formed with 3 four's, 4 two's and 2 three's?

Let $n = x_1x_2x_3$

Suppose $x_1 = 3, x_2 \neq 3$, remaining 2 digits can be arranged in $2 \times 3 = 6$ ways.

Suppose $x_1 = 3, x_2 = 3$, remaining digit can be arranged in 2 ways.

Suppose $x_1 = 4$, remaining 2 digits can be arranged in $3 \times 3 = 9$ ways.

Suppose $x_1 = 2$, remaining 2 digits can be arranged in $3 \times 3 = 9$ ways.

Therefore, total no. of arrangements $= 6 + 2 + 9 + 9 = 26$.

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Exercise 2.4

9. How many 8 digit numbers have one or more repeated digits?

   Answer: $10^8 - \binom{10}{8}$

10. How many different strings (sequences) of length four can be formed using the letters of the word FLOWER?

    Answer: $\binom{6}{4}$

11. How many nine letter words can be formed by using the letters of the word DIFFICULT?

    Answer: $\frac{9!}{2!\times 2!}$

12. Find the number of permutations of all letters of the word BASEBALL if the words are begin and end with a vowel? Answer: 540

13. How many four digit numbers can be formed with the 10 digits 0,1,2,3,4,5,6,7,8,9
    (a) If repetitions are allowed?
    (b) Repetitions are not allowed?
    (c) The last digit must be zero and repetitions are not allowed?

    Answer: (a) 9000 (b) 4536 (c) 504

14. In how many ways can 7 books be arranged on a shelf if
    (a) Any arrangement is allowed
    (b) Three particular books must always be together?
    (c) Two particular books must occupy the ends?

    Answer: (a) 5040 (b) 720 (c) 240

15. In how many ways can three men and three women be seated at a round table if
    (a) No restriction is imposed?
    (b) Two particular women must not sit together?
    (c) Each women is to be between two men?

    Answer: (a) 120 (b) 72 (c) 12

16. Four different mathematics books and five different computer science books and two different control theory books are to be arranged in a shelf. How many different arrangements are possible if
    (a) The books in each particular subject must all be together?
    (b) Only the mathematics books must be together?

    Answer: (a) $4! \times 5! \times 2! \times 3! = 34{,}560$ (b) $8! \times 4!$

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2.5 Combinations

Introduction:

- Selecting r objects from a set of $n \geq r$ objects without regard to order is $\binom{n}{r}$.
- $\binom{n}{r} = \binom{n}{n-r}$, $\binom{n}{0} = \binom{n}{n} = 1$, $\binom{n}{1} = \binom{n}{n-1} = n$

1. **In how many different ways can a committee of 5 teachers and 4 students be selected from 9 teachers and 15 students?**

   No. of ways of selecting 5 teachers from 9 teachers = $\binom{9}{5}$

   No. of ways of selecting 4 students from 15 students = $\binom{15}{4}$.

   By product rule, Total no. of different ways = $\binom{9}{5} \times \binom{15}{4} = 1{,}71{,}990$

2. **A bag contains 5 red marbles and 6 white marbles. Find the number of ways that 4 marbles can be drawn from the bag if the 4 marbles are of the same color.**

   No. of ways of selecting 4 red marbles from 5 red marbles = $\binom{5}{4}$

   No. of ways of selecting 4 white marbles from 6 white marbles= $\binom{6}{4}$

   By sum rule, Total no. of different ways = $\binom{5}{4} + \binom{6}{4} = 20$

3. **How many arrangements of the letters can be made in the word MISSISSIPPI? How many have no consecutive S's?**

   By ignoring four S's, there are 7 letters remaining.

   Among 7 letters, 'P' repeated twice and 'I' repeated four times.

   No. of arrangements of 7 letters= $\frac{7!}{2! \times 4!} = 105$

   There are 8 possible locations for four S's.

   No. of ways of selecting locations for four S's= $\binom{8}{4}$

   $$\left.\begin{matrix}\text{Therefore, total no. of arrangements} \\ \text{having no adjacent S's}\end{matrix}\right\} = 105 \times 70 = 7{,}350$$

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4. Find the number of arrangements of all the letters in TALLAHASSEE. How many of these arrangements have no adjacent A's? (June 2012)

By ignoring three A's, there are 8 letters remaining.

Among 8 letters, 'L','S' and 'E' repeated twice each.

No. of arrangements of 8 letters= $\frac{8!}{2!\times2!\times2!} = 5040$

There are 9 possible locations for three A's.

No. of ways of selecting locations for three A's = $\binom{9}{3}$

$$\left.\begin{array}{r}\text{Therefore, total no. of arrangements}\\ \text{having no adjacent A's}\end{array}\right\} = 5040 \times 84 = 4{,}23{,}360$$

5. From seven consonants and five vowels, how many words consists of four different consonants and three different vowels can be formed? (June 2012)

No. of ways of selecting 4 consonants from 7 consonants = $\binom{7}{4}$

No. of ways of selecting 3 vowels from 5 vowels = $\binom{5}{3}$

No. of arrangements of 4 consonants and 3 vowels = $7!$

Total no. of possible words = $\binom{7}{4} \times \binom{5}{3} \times 7! = 17{,}64{,}000$

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6. **A woman has 11 close relatives and she wishes to invite 5 of them to dinner. In how many ways can she invite them in the following situations:**
(i) Two particular persons will not attend separately.
(ii) Two particular persons will not attend together.

(i) If two particular persons are invited,

No. of ways of selecting 3 more relatives from the remaining 9 $= \binom{9}{3}$

If two particular persons are not invited,

No. of ways of selecting 5 relatives from the remaining 9 relatives $= \binom{9}{5}$

Total no. of ways of selection $= \binom{9}{3} + \binom{9}{5}$

(ii) If two particular persons P_1 and P_2 are not invited,

No. of ways of selecting 5 relatives from the remaining 9 relatives $= \binom{9}{5}$

If P_1 is invited and P_2 is not invited,

No. of ways of selecting 4 more relatives from the remaining 9 $= \binom{9}{4}$

If P_1 is not invited and P_2 is invited,

No. of ways of selecting 4 more relatives from the remaining 9 $= \binom{9}{4}$

By sum rule, Total no. of ways of selection $= \binom{9}{5} + \binom{9}{4} + \binom{9}{4} = 378.$

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7. **Find the number of committees of 5 that can be selected from 7 men and 5 women if the committee is to consist of at least one man and at least one woman.**

No. of ways of selecting 5 persons from 7 men and 5 women $= \binom{12}{5} = 792$

No. of ways of selecting 5 men from 7 men $= \binom{7}{5} = 21$

No. of ways of selecting 5 women from 5 women $= \binom{5}{5} = 1$

No. of ways of selecting a committee consisting at least one man and at least one woman

$$= 792 - 21 - 1 = 770.$$

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8. A certain question paper contains three parts A, B, C with four questions in part A, five questions in part B and six questions in part C. It is required to answer 7 questions selecting at least two questions from each part. In how many ways can a student select his seven questions for answering?

There are 4 questions in part A, 5 questions in part B & 6 questions in part C.

If 2 questions in part A, 2 questions in part B and 3 questions in part C are selected, no. of ways of selecting 7 questions = $\binom{4}{2} \times \binom{5}{2} \times \binom{6}{3} = 1200.$

If 2 questions in part A, 3 questions in part B and 2 questions in part C are selected, no. of ways of selecting 7 questions = $\binom{4}{2} \times \binom{5}{3} \times \binom{6}{2} = 900.$

If 3 questions in part A, 2 questions in part B and 2 questions in part C are selected, no. of ways of selecting 7 questions = $\binom{4}{3} \times \binom{5}{2} \times \binom{6}{2} = 600.$

By sum rule, total no. of ways of selecting 7 questions $= 1200 + 900 + 600$

$= 2700$

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Exercise 2.5

1. There are n married couples attending a party. Each person shakes hands with every person other than his or her spouse. Find the total number of handshakes.

   Answer: $\binom{2n}{2} - n$

2. How many diagonals are there in a regular polygon with n sides? Answer: $\binom{n}{2} - n$
3. Find the number of ways of seating r persons out of n persons around a circular table and the others around another circular table. Answer: $\binom{n}{r} \times (r-1)! \times (n-r-1)!$

4. How many bytes contain (i) exactly two 1's? (ii) Exactly four 1's? (iii) Exactly six 1's? (iv) at least six 1's? Answer: (i) 28 (ii) 70 (iii) 28 (iv) 37
5. A box contains 15 IC chips of which 7 are defective and 8 are non-defective. In how many ways 5 chips can be chosen so that
   (a) all are non-defective?
   (b) All are defective?
   (c) 2 are non-defective
   (d) 3 are non-defective? Answer: (a) 350 (b) 21 (c) 980 (d) 13,431

6. There are 21 consonants and 5 vowels in the English alphabet. Consider only 8 letter words with 3 different vowels and 5 different consonants.
   (a) How many of such words can be formed?
   (b) How many begin with a and end with b?
   (c) How many contain the letters a, b and c?

   Answer: (a) $\binom{5}{3} \times \binom{21}{5} \times 8!$ (b) $\binom{4}{2} \times \binom{20}{4} \times 6!$ (c) $\binom{4}{2} \times \binom{19}{3} \times 8!$

7. A question paper contains 10 questions of which 7 are to be answered. In how many ways a student can select the 7 questions
   (a) If he can choose any 7?
   (b) If he should select 3 questions from the first 5 and 4 questions from the last 5?
   (c) If he should select at least three from the first five?
   Answer: (a) 120 (b) 50 (c) 110

8. Out of 5 mathematicians and 7 Engineers, a committee consisting of two mathematicians and three Engineers to be formed. In how many ways can this be done if
   (a) Any mathematician and any engineer can be included?
   (b) One particular Engineer must be included on the committee?
   (c) Two particular mathematicians cannot be on the committee?
   Answer: (a) 350 (b) 150 (c) 105

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2.6 Combinations with repetitions

Introduction:

$\binom{n+r-1}{r}$ represents any one of the following:

- The number of combinations of n distinct objects, taken r at a time, with repetitions allowed.
- The number of ways in which r identical objects can be distributed among n distinct containers.
- The number of non-negative integer solutions of the equation $x_1 + x_2 + \cdots + x_n = r$.

1. **In how many ways can we distribute 10 identical marbles among 6 distinct containers?**

   By data, $n = 6, r = 10$.

   Therefore, required number = $\binom{n+r-1}{r} = \binom{15}{10} = 3003$

2. **In how many ways can 10 identical pencils be distributed among 5 children so that (i) Each child gets at least 1 pencil? (ii)The youngest child gets at least two pencils?**

   (i) Distribute one pencil to each child. Now, 5 pencils remaining. $n = 5, r = 5$.

   Therefore, required number $= \binom{n+r-1}{r} = \binom{9}{5} = 126$

   (ii) Give 2 pencils to the youngest child. Now,8 pencils remaining. $n = 5, r = 8$.

   Therefore, required number $= \binom{n+r-1}{r} = \binom{12}{8} = 495$

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3. **In how many ways can one distribute 8 identical balls into 4 distinct containers so that (i) No container is left empty? (ii) The 4th container gets an odd no. of balls? (July 2014)**

(i) Distribute one ball to each container. Now, 4 balls remaining. $n = 4, r = 4$.

Therefore, required number $= \binom{n+r-1}{r} = \binom{7}{4} = 35$

(ii) Put 1 ball into 4th container. Now, 7 balls and 3 containers remaining. $n = 3, r = 7$.

No. of ways of distributing 7 balls into 3 containers is $\binom{n+r-1}{r} = \binom{9}{7} = 36$.

Put 3 balls into 4th container. Now, 5 balls and 3 containers remaining. $n = 3, r = 5$.

No. of ways of distributing 5 balls into 3 containers is $\binom{n+r-1}{r} = \binom{7}{5} = 21$

Put 5 balls into 4thcontainer. Now, 3 balls and 3 containers remaining. $n = 3, r = 3$.

No. of ways of distributing 3 balls into 3 containers is $\binom{n+r-1}{r} = \binom{5}{3} = 10$

Put 7 balls into 4thcontainer. Now, 1 ball and 3 containers remaining. $n = 3, r = 1$.

No. of ways of distributing 1 ball into 3 containers is $\binom{n+r-1}{r} = \binom{3}{1} = 3$

By sum rule, the required number$= 36 + 21 + 10 + 3 = 70$.

4. **In how many ways can we distribute 7 apples and 6 oranges among 4 children so that each child gets at least 1 apple? (Jan 2015)**

Distribute 1 apple to each child. Now, 3 apples and 6 oranges remaining.

No. of ways of distributing 3 apples to 4 children $= \binom{n+r-1}{r} = \binom{6}{3} = 20$

No. of ways of distributing 6 oranges to 4 children $= \binom{n+r-1}{r} = \binom{9}{6} = 84$

By product rule, the required number$= 20 \times 84 = 1680$.

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5. **A message is made up of 12 different symbols and is to be transmitted through a communication channel. In addition to the 12 symbols, the transmitter will also send a total of 45 blank spaces between the symbols, with at least 3 spaces between each pair of consecutive symbols. In how many ways can the transmitter send such a message?** (July 2013)

12 symbols can be arranged in 12! ways. There are 11 positions between the symbols.

Distribute 3 spaces to each one of 11 positions. There are 12 spaces remaining. (45-33)

No. of ways of distributing 12 spaces to 11 positions $= \binom{n+r-1}{r} = \binom{22}{12} = 646646$

By product rule, the required number $= 646646 \times 12!$

6. **Find the number of ways of giving 10 identical gift boxes to 6 persons A, B, C, D, E, F in such a way that the total number of boxes given to A and B together does not exceed 4.**

Suppose r boxes out of 10 identical boxes given to A and B,$0 \leq r \leq 4$.

No. of ways of distributing r boxes to 2 persons $\binom{n+r-1}{r} = \binom{r+1}{r} = r+1$

Distribute remaining $10-r$ identical boxes to 4 persons C,D,E and F.

No. of ways of distributing $10-r$ boxes to 4 persons

$$= \binom{n+r-1}{r} = \binom{4+10-r-1}{10-r} = \binom{13-r}{10-r} = \binom{13-r}{3}$$

By sum rule, required number= $\sum_{i=1}^{4}(r+1) \times \binom{13-r}{3}$

7. **Find the number of non negative integer solutions of the inequality $\boldsymbol{x_1+x_2+x_3+\cdots+x_6<10}$.**

$x_1+x_2+x_3+\cdots+x_6<10.$

$\Rightarrow x_1+x_2+x_3+\cdots+x_6 \leq 9$

$\Rightarrow x_1+x_2+x_3+\cdots+x_6 = 9-x_7$

$\Rightarrow x_1+x_2+x_3+\cdots+x_6+x_7 = 9$. Here, $n=7, r=9$.

Therefore, the required number= $\binom{n+r-1}{r} = \binom{7+9-1}{9} = \binom{15}{9} = 5005$

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8. Find the number of integer solutions of $x_1 + x_2 + x_3 + x_4 + x_5 = 30$, where $x_1 \geq 2, x_2 \geq 3, x_3 \geq 4, x_4 \geq 2,\ x_5 \geq 0$.

Let $y_1 = x_1 - 2, y_2 = x_2 - 3\ , y_3 = x_3 - 4, y_4 = x_4 - 2, y_5 = x_5$.

Then $x_1 + x_2 + x_3 + x_4 + x_5 = 30$

$\Rightarrow y_1 + 2 + y_2 + 3 + y_3 + 4 + y_4 + 2 + y_5 = 30$

$\Rightarrow y_1 + y_2 + y_3 + y_4 + y_5 = 30 - 11 = 19$. Where $y_1,\ y_2,\ y_3,\ y_4,\ y_5 \geq 0$.

Here, $n = 5, r = 19$.

Therefore, the required number= $\binom{n+r-1}{r} = \binom{5+19-1}{19} = \binom{23}{19} = 8855$.

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Exercise 2.6

9. A cake shop sells 20 kinds of cakes. If there are at least a dozen cakes of each kind. In how many ways a dozen cakes can be chosen?

   Answer: $\binom{31}{12}$

10. A total amount of ₹1500 is to be distributed to 3 poor students A, B, C of a class. In how many ways the distribution can be made in multiples of ₹100
    (i) If every one of these must get at least ₹300?
    (ii) If A must get at least ₹500, B and C must get at least ₹400 each?

    Answer: (i) $\binom{8}{6}$ (ii) $\binom{4}{2}$

11. How many different outcomes are possible by tossing 10 similar coins?

    Answer: $\binom{2-1+10}{10}$

12. Find the number of ways of placing 20 identical balls into 5 boxes with at least one ball put into each box.

    Answer: $\binom{19}{15}$

13. Find the number of ways of distributing 7 identical pens and 7 identical pencils to 5 children so that each gets at least one pen and at least one pencil.

    Answer: $\binom{6}{4} \times \binom{6}{4}$

14. How many ways are there to place 12 marbles of the same size in five different jars
    (a) If the marbles are all of the same colour?
    (b) If the marbles are all of different colours?

    Answer: (a) $\binom{16}{12}$ (b) 5^{12}

15. Find the number of positive solutions of the equation $x_1 + x_2 + x_3 = 17$.

    Answer: $\binom{3+14-1}{14}$

16. Find the number of distinct terms in the expansion of $(x_1 + x_2 + x_3 + x_4 + x_5)^{16}$.

    Answer: $\binom{5+16-1}{16}$

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2.7 Binomial and Multinomial Theorems

Binomial Theorem:

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r} \text{ , } Where\ r + n - r = n$$

Multinomial Theorem:

$$(x_1 + x_2 + \cdots + x_k)^n = \sum_{n_i} \binom{n}{n_1, n_2, \ldots, n_k} x_1^{n_1}\, x_2^{n_1}\, \ldots x_k^{n_k}, Where\ n_1 + n_2 + \cdots + n_k = n$$

Note:

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!} \quad and \quad \binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1!\ n_2!\ \ldots n_k!}$$

Problems:

1. **Evaluate the following:** $\binom{\mathbf{12}}{\mathbf{5,3,2,2}}$, $\binom{\mathbf{7}}{\mathbf{2,3,2}}$, $\binom{\mathbf{8}}{\mathbf{4,2,2,0}}$, $\binom{\mathbf{10}}{\mathbf{5,3,2,2}}$

$$\binom{12}{5,3,2,2} = \frac{12!}{5!\times 3!\times 2!\times 2!} 166320$$

$$\binom{7}{2,3,2} = \frac{7!}{2!\times 3!\times 2!} = 210$$

$$\binom{8}{4,2,2,0} = \frac{8!}{4!\times 2!\times 2!\times 0!} = 420$$

$\binom{10}{5,3,2,2}$ is meaningless. Because, $5+3+2+2 > 10$.

2. **Determine the coefficient of $\boldsymbol{x^9y^3}$ in the expansion of $\boldsymbol{(2x-3y)^{12}}$ (June 2012)**

By binomial theorem, general term in the expansion of $(2x-3y)^{12}$is

$$\binom{12}{r}(2x)^r(-3y)^{12-r} = \binom{12}{r} 2^r x^r (-3)^{12-r} y^{12-r}$$

Put $r = 9$. The coefficient of $x^9y^3 = \binom{12}{9} 2^9(-3)^3$

3. **Determine the coefficient of $\boldsymbol{x^0}$ in the expansion of $\boldsymbol{\left(3x^2 - \frac{2}{x}\right)^{15}}$ (Jan 2015)**

By binomial theorem, general term in the expansion of $\left(3x^2 - \frac{2}{x}\right)^{15}$ is

$$\binom{15}{r}(3x^2)^r\left(-\frac{2}{x}\right)^{15-r} = \binom{15}{r} 3^r x^{2r-15+r}(-2)^{15-r}$$

Put $r = 5$. The coefficient of $x^0 = \binom{15}{5} 3^5(-2)^{15-5} = \binom{15}{5} 3^5 2^{10}$

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4. **Determine the coefficient of x^{12} in the expansion of $x^3(1-2x)^{10}$.**

By binomial theorem, general term in the expansion of $x^3(1-2x)^{10}$ is

$x^3\binom{10}{r}(1)^r(-2x)^{10-r} = \binom{10}{r}(-2)^{10-r}x^{13-r}$

Put $r = 1$. The coefficient of $x^{12} = \binom{10}{1}(-2)^9 = -5120.$

5. **Determine the coefficient of xyz^2 in the expansion of $(2x-y-z)^4$ (Dec 2012)**

By Multinomial theorem, general term in the expansion of $(2x-y-z)^4$ is

$\binom{4}{n_1,\ n_2,\ n_3}(2x)^{n_1}(-y)^{n_2}(-z)^{n_3}$

Put $n_1 = 1,\ n_2 = 1,\ n_3 = 2$

The coefficient of $xyz^2 = \binom{4}{1,\ 1,\ 2}(2)^1(-1)^1(-1)^2 = \frac{4!}{2!} \times 2 \times -1 = -24$

6. **Determine the coefficient of $x^2y^2z^3$ in the expansion of $(3x-2y-4z)^7$ (Jun 2014)**

By Multinomial theorem, general term in the expansion of $(3x-2y-4z)^7$ is

$\binom{7}{n_1,\ n_2,\ n_3}(3x)^{n_1}(-2y)^{n_2}(-4z)^{n_3}$

Put $n_1 = 2,\ n_2 = 2,\ n_3 = 3$

The coefficient of $xyz^2 = \binom{7}{2,\ 2,\ 3}(3)^2(-2)^2(-4)^3$

$= \frac{7!}{2!\times 2!\times 3!} \times 9 \times 4 \times (-64)$

$= -4,83,840$

7. **Determine the coefficient of $x^{11}y^4$ in the expansion of $(2x^3-3xy^2+z^2)^6$**

By Multinomial theorem, general term in the expansion of $(2x^3-3xy^2+z^2)^6$ is

$\binom{6}{n_1,\ n_2,\ n_3}(2x^3)^{n_1}(-3xy^2)^{n_2}(z^2)^{n_3}$

Put $n_1 = 3,\ n_2 = 2,\ n_3 = 0$

The coefficient of $x^{11}y^4 = \binom{6}{3,\ 2,\ 0}(2)^3(-3)^2 = \frac{6!}{3!\times 2!} \times 8 \times 9 = 4320$

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8. **Determine the coefficient of $a^2b^3c^2d^5$ in the expansion of $(a+2b-3c+2d+5)^{16}$. (Dec 2012)**

By Multinomial theorem, general term in the expansion of

$(a+2b-3c+2d+5)^{16}$ is

$$\binom{16}{n_1, n_2, n_3, n_4, n_5}(a)^{n_1}(2b)^{n_2}(-3c)^{n_3}(2d)^{n_4}(5)^{n_5}$$

Put $n_1 = 2,\ n_2 = 3,\ n_3 = 2,\ n_4 = 5, n_5 = 4$

The coefficient of $a^2b^3c^2d^5 = \binom{16}{2,\ 3,\ 2,\ 5,\ 4}(1)^2(2)^3(-3)^2(2)^5(5)^4$

$$= 3 \times 2^5 \times 5^3 \times \frac{16!}{4! \times 4!}$$

Exercise 2.7

1. Find the coefficient of x^9y^3 in the expansion of $(2x-3y)^{12}$.

   Answer: 6048

2. Find the sum of all coefficients of the expansion of (i) $(x+y)^{10}$ (ii) $(x+y+z+w)^5$ (iii) $(x+2y-3z+2u+3v)^{12}$

   Answer: (i) 2^{10} (ii) 4^5 (iii) 5^{12}

3. Find the coefficient of

   (i) xyz^{-2} in the expansion of $(x-2y+3z^{-1})^4$
   (ii) $x^3y^3z^2$ in the expansion of $(2x-3y+5z)^8$

   Answer: (i) -210 (ii) -3024000

4. Find the coefficient of
   (i) $w^3x^2yz^2$ in the expansion of $(2w-x+3y-2z)^8$
   (ii) $x_1^2x_3x_4^3x_5^4$ in the expansion of $(x_1+x_2+x_3+x_4+x_5)^{10}$

   Answer: (i) 161280 (ii) 12600

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