BCS405A - Module 3

Relations and functions

Relations and Functions: Cartesian Products and Relations, Functions – Plain and One-to-One, Onto Functions. The Pigeon-hole Principle, Function Composition and Inverse Functions.

Properties of Relations, Computer Recognition – Zero-One Matrices and Directed Graphs, Partial Orders – Hasse Diagrams, Equivalence Relations and Partitions.

Textbook 1: Ch 5:5.1 to 5.3, 5.5, 5.6, Ch 7:7.1 to 7.4

3.1 Cartesian product of sets

Cartesian product:

Let A and B be two sets. Then the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$ is called the Cartesian product of A and B.

$A \times B = \{(a,b) | a \in A \text{ } and \text{ } b \in B\}$, $B \times A = \{(b,a) | \text{ } b \in B \text{ } and \text{ } a \in A\}$
$A \times A = \{(a,b) | \text{ } a \in A \text{ } and \text{ } b \in A\}$, $A \times B \neq B \times A$ and $A^2 = A \times A$

Cardinality:

If $n(A) = |A|, n(B) = |B|$ then $|A \times B| = |A||B|, |B \times A| = |B||A|, |A \times A| = |A|^2$

Equality:

$(a,b) = (c,d) \Rightarrow a = c \text{ } and \text{ } b = d$

1. **Find x and y in each of the following cases: (i) $(2x, x + y) = (6, 1)$ (ii) $(x + 2, 4) = (5, 2x + y)$ (iii) $(y - 2, 2x + 1) = (x - 1, y + 2)$ (iv) $(x, y) = (x^2, y^2)$**

   (i) $x = 3, y = -2$

   (ii) $x = 3, y = -2$

   (iii) Solve $x - y = -1, 2x - y = 1$ we get $x = 0, y = -1$.

   (iv) Solve $x(x - 1) = 0$, $y(y - 1) = 0$ we get $x = 0 \text{ } or \text{ } 1, y = 0 \text{ } or \text{ } 1$.

2. Given $\boldsymbol{A = \{a, b\}}$ and $\boldsymbol{B = \{1, 2, 3\}}$ find $\boldsymbol{A \times B, B \times A, A \times A, B \times B}$.

$A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$

$B \times A = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$

$A \times A = \{(a, a), (a, b), (b, a), (b, b)\}$

$B \times B = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$

3. Given $\boldsymbol{A = \{1, 2\}, B = \{a, b, c\}, C = \{3, 4\}}$ find $\boldsymbol{A \times B \times C}$, $\boldsymbol{B \times C \times A}$.

$$A \times B \times C = \begin{Bmatrix} (1, a, 3), (1, a, 4), (1, b, 3), (1, b, 4), (1, c, 3), (1, c, 4), \\ (2, a, 3), (2, a, 4), (2, b, 3), (2, b, 4), (2, c, 3), (2, c, 4) \end{Bmatrix}$$

$$B \times C \times A = \begin{Bmatrix} (a, 3, 1), (a, 3, 2), (a, 4, 1), (a, 4, 2), \\ (b, 3, 1), (b, 3, 2), (b, 4, 1), (b, 4, 2), \\ (c, 3, 1), (c, 3, 2), (c, 4, 1), (c, 4, 2) \end{Bmatrix}$$

4. Given $\boldsymbol{A = \{1, 3, 5\}, B = \{2, 3\}, C = \{4, 6\}}$

Find $\boldsymbol{A \cap (B \times C)}$, $\boldsymbol{(A \times B) \cup (B \times C)}$, $\boldsymbol{(A \times B) \cap (B \times A)}$, $\boldsymbol{(A \times B) \cap (B \times C)}$.

$A \cap (B \times C)$

$= \{1,3,5\} \cap \{(2,4), (2,6), (3,4), (3,6)\}$

$= \phi$

$(A \times B) \cup (B \times C)$

$= \{(1,2), (1,3), (3,2), (3,3), (5,2), (5,3)\} \cup \{(2,4), (2,6), (3,4), (3,6)\}$

$= \{(1,2), (1,3), (3,2), (3,3), (5,2), (5,3), (2,4), (2,6), (3,4), (3,6)\}$

$(A \times B) \cap (B \times A)$

$= \{(1,2), (1,3), (3,2), (3,3), (5,2), (5,3)\} \cap \{(2,1), (2,3), (2,5), (3,1), (3,3), (3,5)\}$

$= \{(3,3)\}$

$(A \times B) \cap (B \times C)$

$= \{(1,2), (1,3), (3,2), (3,3), (5,2), (5,3)\} \cap \{(2,4), (2,6), (3,4), (3,6)\}$

$= \phi$

5. **Given $\boldsymbol{A = \{\alpha, \beta, \gamma\}, B = \{\theta, \eta\}, C = \{\lambda, \mu, \gamma\}}$.**

Find $\boldsymbol{A \times C}$, $\boldsymbol{(A \cup B) \times C}$, $\boldsymbol{A \cup (B \times C)}$, $\boldsymbol{A \times (B \cup C)}$

$$A \times C = \{(\alpha,\lambda),(\alpha,\mu),(\alpha,\gamma),(\beta,\lambda),(\beta,\mu),(\beta,\gamma),(\gamma,\lambda),(\gamma,\mu),(\gamma,\gamma)\}$$

$$(A \cup B) \times C = \{\alpha,\beta,\gamma,\theta,\eta\} \times \{\lambda,\mu,\gamma\}$$

$$= \begin{Bmatrix} (\alpha,\lambda),(\alpha,\mu),(\alpha,\gamma),(\beta,\lambda),(\beta,\mu),(\beta,\gamma),(\gamma,\lambda), \\ (\gamma,\mu),(\gamma,\gamma),(\theta,\lambda),(\theta,\mu),(\theta,\gamma),(\eta,\lambda),(\eta,\mu),(\eta,\gamma) \end{Bmatrix}$$

$$A \cup (B \times C) = \{\alpha,\beta,\gamma\} \cup \{(\theta,\lambda),(\theta,\mu),(\theta,\gamma),(\eta,\lambda),(\eta,\mu),(\eta,\gamma)\}$$

$$= \{\alpha,\beta,\gamma,(\theta,\lambda),(\theta,\mu),(\theta,\gamma),(\eta,\lambda),(\eta,\mu),(\eta,\gamma)\}$$

$$A \times (B \cup C) = \{\alpha,\beta,\gamma\} \times \{\theta,\eta,\lambda,\mu,\gamma\}$$

$$= \begin{Bmatrix} (\alpha,\theta),(\alpha,\eta),(\alpha,\lambda),(\alpha,\mu),(\alpha,\gamma),(\beta,\theta),(\beta,\eta), \\ (\beta,\lambda),(\beta,\mu),(\beta,\gamma),(\gamma,\theta),(\gamma,\eta),(\gamma,\lambda),(\gamma,\mu),(\gamma,\gamma) \end{Bmatrix}$$

6. **For any non-empty sets A, B and C Prove the following:**

$\boldsymbol{A \times (B \cap C) = (A \times B) \cap (A \times C)}$ **,** $\boldsymbol{A \times (B - C) = (A \times B) - (A \times C)}$

$$\begin{aligned} (x,y) \in A \times (B \cap C) &\Leftrightarrow x \in A \text{ and } y \in B \cap C \\ &\Leftrightarrow x \in A \text{ and } (y \in B \text{ and } y \in C) \\ &\Leftrightarrow (x \in A \text{ and } y \in B) \text{ and } (x \in A \text{ and } y \in C) \\ &\Leftrightarrow x \in A \times B \text{ and } x \in A \times C \\ &\Leftrightarrow x \in (A \times B) \cap (A \times C) \end{aligned}$$

Therefore, $A \times (B \cap C) = (A \times B) \cap (A \times C)$

$$\begin{aligned} (x,y) \in A \times (B - C) &\Leftrightarrow x \in A \text{ and } y \in B - C \\ &\Leftrightarrow x \in A \text{ and } (y \in B \text{ and } y \notin C) \\ &\Leftrightarrow (x \in A \text{ and } y \in B) \text{ and } (x \in A \text{ and } y \notin C) \\ &\Leftrightarrow x \in A \times B \text{ and } x \notin A \times C \\ &\Leftrightarrow x \in (A \times B) - (A \times C) \end{aligned}$$

Therefore, $A \times (B - C) = (A \times B) - (A \times C)$

7. Find $\boldsymbol{A \cap B}$, $\boldsymbol{B \cap C}$, $\overline{\overline{\boldsymbol{A}} \cup \overline{\boldsymbol{C}}}$, $\overline{\boldsymbol{B}} \cup \overline{\boldsymbol{C}}$ if $\boldsymbol{A, B, C \subseteq Z \times Z}$ with

$\boldsymbol{A = \{(x,y) | y = 5x - 1\}, B = \{(x,y) | y = 6x\}, C = \{(x,y) | 3x - y = -7\}}$

(i) $(x,y) \in A \cap B \Rightarrow y = 5x - 1 \; and \; y = 6x$

$\Rightarrow 5x - 1 = 6x$

$\Rightarrow x = -1 \; and \; y = -6$

Therefore, $A \cap B = \{(-1, -6)\}$

(ii) $(x,y) \in B \cap C \Rightarrow \; y = 6x \; and \; 3x - y = -7$

$\Rightarrow 6x = 3x - 7$

$\Rightarrow x = -7/3 \; and \; y = -14$ or 0

Therefore, $B \cap C = \left\{\left(-\frac{7}{3}, -14\right), \left(-\frac{7}{3}, 0\right)\right\}$

(iii) $(x,y) \in \overline{\overline{A} \cup \overline{C}} \Rightarrow \; (x,y) \in A \cap C$

$\Rightarrow y = 5x - 1 \; and \; 3x - y = -7$

$\Rightarrow 5x - 1 = 3x + 7$

$\Rightarrow x = 4 \; and \; y = 19$

Therefore, $\overline{\overline{A} \cup \overline{C}} = \{(4, 19)\}$

(iv) $(x,y) \in \overline{B} \cup \overline{C} \Rightarrow \; (x,y) \in \overline{B \cap C}$

$\Rightarrow (x,y) \notin B \cap C$

$\Rightarrow Z \times Z - (x,y) \in B \cap C$

$\Rightarrow Z \times Z - (y = 6x \; and \; 3x - y = -7)$

$\Rightarrow Z \times Z - (6x = 3x + 7)$

$\Rightarrow Z \times Z - \left(x = \frac{7}{3} \; and \; y = 14\right)$

Therefore, $\overline{B} \cup \overline{C} = Z \times Z - \left\{\left(\frac{7}{3}, 14\right)\right\}$

3.2 Relations

Introduction:

- Let A and B be two sets. Then a subset of $A \times B$ is called a relation from A to B.
- A subset of $A \times A$ is called a binary relation on A.
- $(a, b) \in R \iff aRb$
- Number of relations from A to B = no. of subsets of $A \times B = 2^{mn}$, $(|A| = m, |B| = n)$

1. **Let A and B be sets with $|B| = 3$. If there are 4096 relations from A to B find $|A|$.**

   By data, $|A \times B| = 2^{mn} = 4096, n = 3$

   $4096 = 2^{12} = 2^{3m}, \ m = |A| = 4$

2. **Let $A = \{1, 2, 3\}, B = \{2, 4, 5\}$. Determine (i) $|A \times B|$(ii) Number of relations from A to B (iii) Number of binary relations on A. (iv) Number of relations from A to B that contain exactly 5 ordered pairs.**

   (i) $|A \times B| = mn = 3 \times 3 = 9$

   (ii) Number of relations from A to B $= 2^{\mathrm{mn}} = 2^{3\times 3} = 512$

   (iii) Number of binary relations on A $= 2^{mm} = 2^{3\times 3} = 2^9 = 512$

   (iv) Number of relations from A to B that contain exactly 5 ordered pairs$= 9C_5 = 126$

3. **Binary relation R on N is defined recursively by $(0, 0) \in R$, If $(s, t) \in R$ then $(s + 1, t + 7) \in R$ find R a set of ordered pairs.**

   $(0, 0) \in R \Rightarrow (0 + 1, \ 0 + 7) \in R$

   $(1, 7) \in R \Rightarrow (1 + 1, \ 7 + 7) \in R$

   $(2, 14) \in R \Rightarrow (2 + 1, \ 14 + 7) \in R$

   $(3,21) \in R \Rightarrow (3 + 1, \ 21 + 7) \in R$

   Therefore, $R = \{(0, 0), (1, 7), (2, 14), (3, 21), (4, 28), \ldots\}$

4. **Let $R \subseteq N \times N$ where $(m, n) \in R$ if $n = 5m + 2$. Give a recursive definition for R.**

   R is defined by $(0, 2) \in R$, if $(m, n) \in R$ then $(m + 1, n + 5) \in R$.

3.3 Zero-one matrices and digraphs

Matrix of a relation:

Let $A = \{a_1, a_2, \dots, a_m\}$ and $B = \{b_1, b_2, \dots, b_n\}$then $A \times B = \{(a_i, b_j) | 1 \le i \le m, 1 \le j \le n\}$.

Let R be a relation from A to B which is a subset of $A \times B$.

Then $M(R) = \begin{pmatrix} m_{11} & \cdots & m_{1n} \\ \vdots & \ddots & \vdots \\ m_{m1} & \cdots & m_{mn} \end{pmatrix}$ where $m_{ij} = \begin{cases} 1, & if\ (a_i, b_j) \in R \\ 0, & if\ (a_i, b_j) \notin R \end{cases}$

is called the matrix of the relation R (or) Zero-one matrix for R.

Example:

Let $A = \{0, 1, 2\}, B = \{p, q\}$.Let R be a relation from A to B defined by R= $\{(0, p), (1, q), (2, p)\}$.

Then the matrix of the relation R is $M(R) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Digraph of a relation:

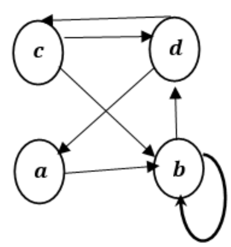
Let R be a binary relation on a finite set A. Draw a small circle for each element of A. These circles are called vertices or nodes. Draw an arrow from a vertex x to vertex y iff$(x, y) \in R$. These arrows are called edges. The pictorial representation of R is called a digraph or directed graph.

Example:

Let $A = \{a, b, c, d\}$

$R = \{a, b), (b, b), (b, d), (c, b), (c, d), (d, a), (d, c)\}$

then the digraph R is



In-degree and out degree:

No. of edges terminating at a vertex is called in-degree.

No. of edges leaving a vertex is called out-degree.

Vertices	a	b	c	d
In - Degree	1	3	1	2
Out - Degree	1	2	2	2

Composite relations:

If A, B and C are sets with $R_1 \subseteq A \times B$ and $R_2 \subseteq B \times C$, then the composition relation $R_1 o R_2$ is a relation from A to C defined by $R_1 o R_2 = \{(x, z) | x \in A, z \in C\}$.

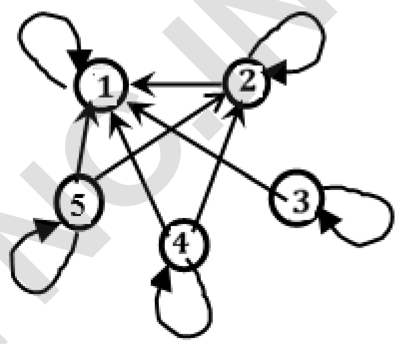
Problems:

1. **Let $A = \{1, 2, 3, 4, 6\}$ and R be a relation on A defined by aRb iff a is a multiple of b. (i) Write down R as a set of ordered pairs (ii) Write matrix of R (iii) Draw the digraph of R.**

(i) $\{(1,1), (2,1), (3,1), (4,1), (6,1), (2,2), (4,2), (6,2), (3,3), (6,3), (4,4), (6,6)\}$

(ii) $M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$

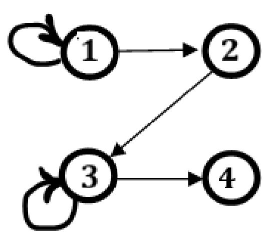
(iii)



2. **Let $R = \{(1,1), (1,2), (2,3), (3,3), (3,4)\}$ be a relation on the set $A = \{1, 2, 3, 4\}$. (i) Write matrix of R (ii) Draw the digraph of R. 5**

(i) $\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$

ii)

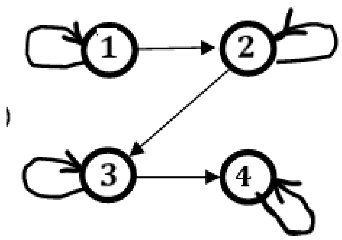


3. **Let $A = \{1, 2, 3, 4\}$. Let R be a relation on A defined by xRyiff $'x\ divides\ y'$. Write down (i) Write down R as a set of ordered pairs (ii) Write matrix of R (iii) Draw the digraph of R (iv) In-degrees and out-degrees of all vertices of the digraph.**

(i) $R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$

(ii) $\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(iii)



(iv)

Vertices	1	2	3	4
In - Degree	1	2	2	2
Out - Degree	2	2	2	1

4. **Let $A = \{a, b, c, d\}$ and R be a relation on A that has the matrix**

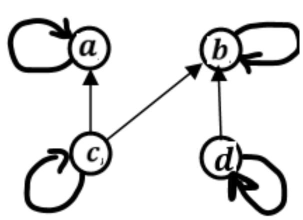
$$\boldsymbol{M_R} = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} \end{pmatrix}$$

(i) Write down R as a set of ordered pairs (ii) Draw the digraph of R

(iii) List in-degrees and out-degrees of all vertices of the digraph.

(i) $R = \{(a,a), ((b,b), (c,a), (c,b), (c,c), (d,b), (d,d)\}$

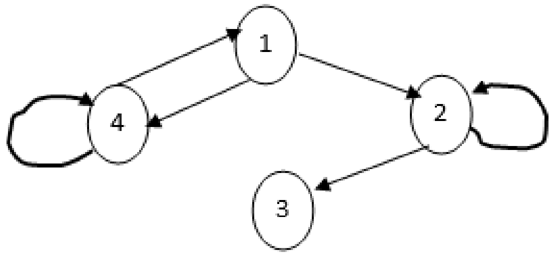
(ii)



(iii)

Vertices	a	b	c	d
In - Degree	2	3	1	1
Out - Degree	1	1	2	3

5. **Find the relation represented by the digraph given below. Also write down its matrix.**



Relation represented by the digraph is $R = \{(1,1),(1,2),(2,2),(2,3),(4,1),(4,4)\}$

Corresponding matrix is $M_R = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$

6. **For $A = \{1,2,3,4\}$ and $R = \{(1,2),(1,3),(2,4),(4,4)\}$. Find $R^2, R^3, R^\infty, M_R, (M_R)^T$.**

$R^2 = \{(1,4),(2,4),(4,4)\}$

$R^3 = \{(1,4),(2,4),(4,4)\}$

$R^\infty = \{(1,4),(1,3),(2,4),(4,4)\}$

$M_R = \begin{pmatrix} 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $(M_R)^T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$

7. **Let $A = \{1,2,3,4\}$ and R, S are relations on A defined by $R = \{(1,2),(1,3),(2,4),(4,4)\}$, $S = \{(1,1),(1,2),(1,3),(1,4),(2,3),(2,4)\}$. Find RoS, SoR, R^2, S^2. Write down their matrices.**

$RoS = \{(1,3),(1,4)\}$

$SoR = \{(1,2),(1,3),(1,4),(2,4)\}$

$R^2 = \{(1,4),(2,4),(4,4)\}$

$S^2 = \{(1,1),(1,2),(1,3),(1,4)\}$

8. Let $A = \{1, 2, 3, 4\}, B = \{w, x, y, z\}$ and $C = \{p, q, r, s\}$. Consider

$R_1 = \{(1, x), (2, w), (3, z)\}$ a relation from A to B,

$R_2 = \{(w, p), (z, q), (y, s), (x, p)\}$ a relation from B to C.

(i) Write relation matrices $M(R_1), M(R_2), M(R_1 o R_2)$

(ii) Verify $M(R_1) \times M(R_2) = M(R_1 o R_2)$.

$R_1 = \{(1, x), (2, w), (3, z)\}, R_2 = \{(w, p), (z, q), (y, s), (x, p)\}$

$R_1 o R_2 = \{(1, p), (2, p), (3, q)\}$

$$M(R_1) = \begin{array}{c c} & \begin{array}{cccc} w & x & y & z \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}, \quad M(R_2) = \begin{array}{c c} & \begin{array}{cccc} p & q & r & s \end{array} \\ \begin{array}{c} w \\ x \\ y \\ z \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \end{array},$$

$$M(R_1 o R_2) = \begin{array}{c c} & \begin{array}{cccc} p & q & r & s \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}, \quad M(R_1) \times M(R_2) = \begin{array}{c c} & \begin{array}{cccc} p & q & r & s \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Therefore, $M(R_1) \times M(R_2) = M(R_1 o R_2)$.

3.4 Properties of relations

Introduction: A relation R on a set A is said to be

- Reflexive if $(a, a) \in R, \forall a \in A$
- Symmetric if $(a, b) \in R \Rightarrow (b, a) \in$ R
- Transitive if $(a, b) \in R, (b, c) \in R \Rightarrow (a, c) \in R$
- Ir-reflexive if $(a, a) \notin R$, for any $a \in A$
- Anti-symmetric if $(a, b) \in R, (b, a) \in R \Rightarrow a = b$
- Asymmetric if it is neither symmetric nor anti-symmetric.

Problems:

1. **Let $A = \{1, 2, 3\}$. Determine the nature of the following relations on A.**

$\boldsymbol{R_1} = \{(\mathbf{1}, \mathbf{2}), (\mathbf{2}, \mathbf{1}), (\mathbf{1}, \mathbf{3}), (\mathbf{3}, \mathbf{1})\}$, $\boldsymbol{R_2} = \{(\mathbf{1}, \mathbf{1}), (\mathbf{2}, \mathbf{2}), (\mathbf{3}, \mathbf{3}), (\mathbf{2}, \mathbf{3})\}$,

$\boldsymbol{R_3} = \{(\mathbf{1}, \mathbf{1}), (\mathbf{2}, \mathbf{2}), (\mathbf{3}, \mathbf{3})\}$, $\boldsymbol{R_4} = \{(\mathbf{1}, \mathbf{1}), (\mathbf{2}, \mathbf{2}), (\mathbf{3}, \mathbf{3}), (\mathbf{2}, \mathbf{3}), (\mathbf{3}, \mathbf{2})\}$,

$\boldsymbol{R_5} = \{(\mathbf{1}, \mathbf{1}), (\mathbf{2}, \mathbf{3}), (\mathbf{3}, \mathbf{3})\}$, $\boldsymbol{R_6} = \{(\mathbf{2}, \mathbf{3}), (\mathbf{3}, \mathbf{4}), (\mathbf{2}, \mathbf{4})\}$, $\boldsymbol{R_7} = \{(\mathbf{1}, \mathbf{3}), (\mathbf{3}, \mathbf{2})\}$

	Reflexive	Symmetric	Transitive	Ir-reflexive	Anti symmetric	Asymmetric
R_1	X	✓	X	✓	X	X
R_2	✓	X	✓	X	X	✓
R_3	✓	✓	✓	X	X	X
R_4	✓	✓	✓	X	X	X
R_5	X	X	✓	X	X	✓
R_6	X	X	✓	✓	✓	X
R_7	X	X	X	✓	✓	X

2. **Let $\boldsymbol{A = \{1, 2, 3, 4\}, R = \{(1, 3), (1, 1), (3, 1), (1, 2), (3, 3), (4, 4)\}}$ be a relation on A. Determine whether R is reflexive, symmetric, transitive, irreflexive or antisymmetric.**

   R is irreflexive because $(1, 1) \in R$ but $(2, 2) \notin R$.

   R is asymmetric because $(1, 3)$, $(3, 1), (1, 2) \in R$ but $(2, 1) \notin R$.

   R is not transitive because $(3, 1), (1, 2) \in R$ but $(3, 2) \notin R$.

3. **Let A be a finite set which consists of n elements. Determine the relations on A which are (i) reflexive (ii) symmetric (iii) antisymmetric.**

   Total number of reflexive relations on A is 2^{n^2-n}

   Total number of symmetric relations on A is $2^{\frac{(n^2+n)}{2}}$

   Total number of anti symmetric relations on A is $2^n \times 3^{\frac{n^2-n}{2}}$

4. **Show that the relation represented by the matrix $\boldsymbol{M_R = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}}$ is transitive.**

   $R = \{(a, a), (a, b), (a, c), (b, c), (c, c)\}$ is a transitive relation.

5. **If $\boldsymbol{A = \{1, 2, 3, . 4\}}$ and $\boldsymbol{R = \{(1, 2), (2, 2), (3, 4), (4, 1)\}}$ verify the relation R on A is whether symmetric or asymmetric or antisymmetric.**

   $R = \{(1, 2), (2, 2), (3, 4), (4, 1)\}$ is an asymmetric relation.

   Because $(2, 2), (1, 2) \in R$ but $(2, 1) \notin R$

3.5 Equivalence relations and partitions

❖ **Equivalence relations:**

A relation R on a set A is said to be an equivalence relation on A if it is reflexive, symmetric and transitive.

❖ **Equivalence classes:**

Let R be an equivalence relation on A and $a \in A$. Then $[a] = \{x \in A | (x, a) \in R\}$ is called the equivalence class of a.

❖ **Partition of a set:**

Let A be a nonempty set and $A_1, A_2, \dots, A_n$ be nonempty subsets of A such that

(i) $A_1 \cup A_2 \cup \dots \cup A_n = A$

(ii) $A_1, A_2, \dots, A_n$ are disjoint then $P = \{A_1, A_2, \dots, A_n\}$ is called a partition of A.

Problems:

1. **Let $\boldsymbol{A = \{1, 2, 3, 4\}}$ and $\boldsymbol{R = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,3), (3,3), (4,4)\}}$ be a relation on A. Verify that R is an equivalence relation.**

   $(1,1), (2,2), (3,3), (4,4) \in R$

   $(a, a) \in R, \forall a \in A,$

   R is **reflexive.**

   $(1,2), (2,1) \in R \Rightarrow (1,1) \in R$

   $(3,4), (4,3) \in R \Rightarrow (3,3) \in R$

   $(a, b) \in R, (b, c) \in R \Rightarrow (a, c) \in R, \forall\, a, b, c \in A$

   R is **transitive.**

   $(1,2) \in R \Rightarrow (2,1) \in R$

   $(3,4) \in R \Rightarrow (3,4) \in R$

   $(a, b) \in R \;\Rightarrow (b, a) \in R, \forall\, a, b \in A$

   R is **symmetric**.

   Therefore, R is an **equivalence relation.**

2. **Let $\boldsymbol{A = \{1, 2, 3, 4\}}$ and $\boldsymbol{R = \{(1,1), (1,2), (2,1), (2,2), (3,1), (3,3), (1,3), (4,1), (4,4)\}}$ be a relation on A.**

   Is R an equivalence relation?

   $(1,1), (2,2), (3,3), (4,4) \in R$

   $(a, a) \in R, \forall a \in A,$

   R is reflexive.

   $(4,1), (1,2) \in R, but\ (4,2) \notin R$

   R is not transitive.

   Therefore, R is not an equivalence relation.

3. **A relation R on a set $\boldsymbol{A = \{a, b, c\}}$ is represented $\boldsymbol{M_R = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$.**

   Determine whether R is an equivalence relation.

   $R = \{(a, a), (b, b), (c, c), (a, c)\}$

   $(a, c) \in R, but\ (c, a) \notin R$

   R is not symmetric.

   Therefore, R is not an equivalence relation.

4. **For a fixed integer $n > 1$, prove that congruent modulo n is an equivalence relation.**

$\boldsymbol{a \equiv b(mod\ n) \Rightarrow a - b\ is\ a\ multiple\ of\ n.}$

$For\ any\ a \in Z,$

$$a - a = 0$$

$$\Rightarrow a - a = 0n$$

$$\Rightarrow a \equiv a\ (mod\ n)$$

$$\Rightarrow (a, a) \in R$$

R is reflexive.

$For\ any\ a, b \in Z,$

$(a, b) \in R$

$$\Rightarrow a \equiv b\ (mod\ n)$$

$$\Rightarrow a - b\ is\ a\ multiple\ of\ n$$

$$\Rightarrow b - a\ is\ a\ multiple\ of\ n$$

$$\Rightarrow b \equiv a\ (mod\ n)$$

$$\Rightarrow (b, a) \in R$$

R is symmetric.

$For\ any\ a, b, c \in Z,$

$(a, b), (b, c) \in R$

$$\Rightarrow a \equiv b\ (mod\ n), b \equiv c\ (mod\ n),$$

$$\Rightarrow\ a \equiv c\ (mod\ n)$$

$$\Rightarrow a - c\ is\ a\ multiple\ of\ n$$

$$\Rightarrow a \equiv c\ (mod\ n)$$

$$\Rightarrow (a, c) \in R$$

R is transitive.

Therefore, R is an equivalence relation.

5. **If $A = A_1 \cup A_2 \cup A_3$, where $A_1 = \{1, 2\}, A_2 = \{2, 3, 4\}, A_3 = \{5\}$. Define relation R on A by xRy if x and y are in the same subset A_i for $1 \leq i \leq 3$. Is R an equivalence relation?**

$A = \{1, 2, 3, 4, 5\}$

$R = \{(1,1), (2,2), (3,3), (4,4), (5,5), (1,2), (2,1), (2,3), (3,2), (2,4), (4,2), (3,4), (4,3)\}$

$(1,1), (2,2), (3,3), (4,4) \in R$

$(a, a) \in R, \forall a \in A,$

R is **reflexive.**

$(1,2), (2,1) \in R \Rightarrow (1,1) \in R$

$(2,3), (3,2) \in R \Rightarrow (2,2) \in R$

$(2,3), (3,4) \in R \Rightarrow (2,4) \in R$

$(a, b) \in R, (b, c) \in R \Rightarrow (a, c) \in R, \forall\, a, b, c \in A$

R is **transitive.**

$(1,2) \in R \Rightarrow (2,1) \in R$

$(3,4) \in R \Rightarrow (4,3) \in R$

$(2,3) \in R \Rightarrow (3,2) \in R$

$(a, b) \in R \Rightarrow (b, a) \in R, \forall\, a, b \in A$

R is **symmetric**.

Therefore, R is an **equivalence relation.**

6. **Let $A = \{1, 2, 3, 4, 5\} \times \{1, 2, 3, 4, 5\}$ and on A by $(x_1, y_1)R(x_2, y_2)$ if $x_1 + y_1 = x_2 + y_2$ Verify that R is an equivalence relation.**

   (ii) Determine the equivalence classes [(1, 3)], [(2, 4)], [(1, 1)].

For any $(a, b) \in A \times A$,

$a + b = a + b$

$\Rightarrow (a, b)R(a, b), \forall a \in A$

R is reflective.

$(a, b)R(c, d)$

$\Rightarrow a + b = c + d$

$\Rightarrow c + d = a + b$

$\Rightarrow (c, d)R(a, b)$

R is symmetric.

$(a, b)R(c, d)$ and $(c, d)R(e, f)$

$\Rightarrow a + b = c + d$ and $c + d = e + f$

$\Rightarrow a + b = e + f$

$\Rightarrow (a, b)R(e, f)$

R is transitive.

Therefore, R is an **equivalence relation.**

7. **Let $A = \{1, 2, 3, 4, 5\} \times \{1, 2, 3, 4, 5\}$ and on A by $(x_1, y_1)R(x_2, y_2)$ if $x_1 + y_1 = x_2 + y_2$, Determine the equivalence classes [(1, 3)], [(2, 4)], [(1, 1)].**

[(1, 3)]$= \{(x, y) \mid (x, y)R\ (1, 3)\}$

$= \{(x, y) \mid x + y = 1 + 3\}$

$= \{(2, 2), (3, 1), (1, 3)\}$

$[(2, 4)] = \{(x, y) \mid (x, y)R\ (2, 4)\}$

$= \{(x, y) \mid x + y = 2 + 4\}$

$= \{(1, 5), (5, 1), (3, 3), (2, 4), (4, 2)\}$

$[(1, 1)] = \{(x, y) \mid (x, y)R\ (1, 1)\}$

$= \{(x, y) \mid x + y = 1 + 1\} = \{(1, 1)\}$

8. **For $\boldsymbol{A = \{a, b, c, d, x, y, z\}}$ define an equivalence relation on A. Hence find equivalence classes. Also find a partition of A.**

Equivalence relation:

$A = \{a, b, c, d, x, y, z\}$

$R = \{(a,a),(a,b),(a,c),(b,a),(b,b),(b,c),(d,d),(c,a),(c,b),(c,c),(x,x),(y,y),(z,z)\}$

Since R is reflexive, symmetric and transitive, R is an equivalence relation.

Equivalence classes:

$[a] = \{x \mid x\ R\ a\} = \{a, b, c\}$ $[b] = \{x \mid x\ R\ b\} = \{a, b, c\}$ $[c] = \{x \mid x\ R\ c\} = \{a, b, c\}$	$[d] = \{x \mid x\ R\ d\} = \{d\}$ $[x] = \{x \mid x\ R\ x\} = \{x\}$ $[y] = \{x \mid x\ R\ y\} = \{y\}$ $[z] = \{x \mid x\ R\ z\} = \{z\}$

Partition of A: Partition of A $= \{\{a, b, c\}, \{d\}, \{x\}, \{y\}, \{z\}\}$

9. **If R is a relation defined on $\boldsymbol{A = \{1, 2, 3, 4\}}$ by $\boldsymbol{R = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,3),(3,3),(4,4)\}}$. Determine the partition induced by the equivalence relation.**

Equivalence classes are

$[1] = \{x \mid x\ R\ 1\} = \{1, 2\}$ $[2] = \{x \mid x\ R\ 2\} = \{1, 2\}$	$[3] = \{x \mid x\ R\ 3\} = \{3, 4\}$ $[4] = \{x \mid x\ R\ 4\} = \{3, 4\}$

Partition induced by the given equivalence relation is $\{\{1, 2\}, \{3, 4\}\}$

10. **Let $A = \{a, b, c, d, e, f, g\}$ and consider the partition $P = \{\{a, c, d\}, \{b\}, \{e, g\}, \{f\}\}$.**

Determine the corresponding equivalence relation R.

$$R = \begin{Bmatrix} (a,a),(a,c),(a,d),(c,a),(c,c),(c,d),(d,a),(d,c), \\ (d,d),((b,b),(e,e),(e,g),(g,e),(g,g),(f,f) \end{Bmatrix}$$

11. **Let R be an equivalence relation on A and $a, b \in A$. Then prove the following:**

(i) $a \in [a]$ (ii) aRb iff $[a] = [b]$ (iii) If $[a] \cap [b] \neq \phi$ then $[a] = [b]$

Proof:

(i) R is reflexive

$\Rightarrow (a, a) \in R \Rightarrow a \in \{a\}$

(ii) **To prove:** $aRb \Rightarrow [a] = [b]$

Let $(a, b) \in R$

$x \in [a]$ $\Rightarrow (x, a) \in R$ $\Rightarrow (x, a), (a, b) \in R$ $\Rightarrow (x, b) \in R$ $\Rightarrow x \in [b]$ $\therefore [a] \subseteq [b]$.	$x \in [b]$ $\Rightarrow (x, b) \in R$ $\Rightarrow (x, b), (a, b) \in R$ $\Rightarrow (x, b), (b, a) \in R$ $\Rightarrow (x, a) \in R$ $\Rightarrow x \in [a]$ $\therefore [b] \subseteq [a]$.

Therefore, $[a] = [b]$.

To prove: $[a] = [b] \Rightarrow aRb$

$[a] = [b]$

$a \in [a] \Rightarrow a \in [b]$

$\Rightarrow (a, b) \in R$

$\Rightarrow a\ R\ b$

(iii) $[a] \cap [b] \neq \phi$

$\Rightarrow$ Let $x \in [a] \cap [b]$

$\Rightarrow x \in [a]$ and $x \in [b]$

$\Rightarrow (x, a), (x, b) \in R$ for some $x \in R$ $(\because By\ definition)$

$\Rightarrow (a, x), (x, b) \in R$ for some $x \in R$ $(\because R\ is\ symmetric)$

$\Rightarrow (a, b) \in R$ $(\because R\ is\ transitive)$

$\Rightarrow [a] = [b]$ $(\because By\ (ii))$

3.6 Partial orders and Hasse diagram

Partial orders:

A relation R on a set A is said to be a partial order on Aif R reflexive, antisymmetric and transitive. A set with a partial order R is called a partially ordered set or a poset. It is denoted by (A, R). (e-g) $(Z, \leq)$ is a poset.

Note: A digraph of a partial order has no cycle of length greater than one.

Hasse diagram:

The digraph of a partial order with the following conditions is called Hasse diagram.

- ❖ All vertices have a cycle of length one. (Since it is known, need not produce loop)
- ❖ If there is an edge from a to b and from b to c then there is an edge from a to c. (Since it is known, need not produce an edge from a to c)
- ❖ All edges point upward (Since it is known, need not put arrows in the edges)

1. **Let $A = \{1, 2, 3, 4\}$ and $R = \{(1,1), (1,2), (2,2), (2,4), (1,3), (3,3), (3,4), (1,4), (4,4)\}$. Verify that R is a partial order on A. Write down the Hasse diagram for R.**

$(1,1), (2,2), (3,3), (4,4) \in R$.

Therefore, $(x,x) \in R, \forall x \in A$.

$(1,3), (3,4) \in R \Longrightarrow (1,4) \in R$

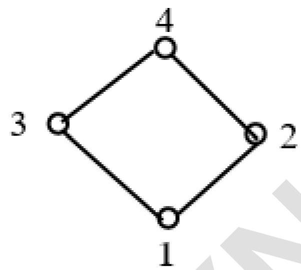
$(x,y), (y,z) \in R \;\Longrightarrow (x,z) \in R\,,\; for\ any\ x\ y, z \in A$

R does not contain (x,y) and (y,x) with $x \neq y$.

Therefore, R is reflexive, anti - symmetric and transitive.

Therefore, R is a partial order on A.

Hasse diagram:



2. **If R is a relation on a set $A = \{1, 2, 3, 4\}$ defined on xRyiff $x|y$. Prove that (A, R) is a poset. Draw its Hasse diagram.**

$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\}$

$(1,1), (2,2), (3,3), (4,4) \in R$.

Therefore, $(x,x) \in R, \forall x \in A$.

$(1,2), (2,4) \in R \Longrightarrow (1,4) \in R$

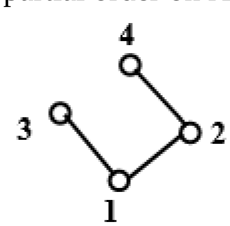
$(x,y), (y,z) \in R \;\Longrightarrow (x,z) \in R\,,\; for\ any\ x\ y, z \in A$

R does not contain (x,y) and (y,x) with $x \neq y$.

Therefore, R is reflexive, anti - symmetric and transitive.

Therefore, R is a partial order on A.

Hasse diagram:



3. Let $A = \{1, 2, 3, 4, 6, 12\}$. On A define relation R by aRb iff $a|b$. Prove that R is a partial order on A. Draw the Hasse diagram for this relation.

$$R = \left\{ \begin{matrix} (1,1),(1,2),(1,3),(1,4), \ (1,6),(1,12),(2,2),(2,4),(2,6), \\ (2,12),(3,3),(3,6),(3,12),(4,4),(4,12),(6,6),(6,12),(12,12) \end{matrix} \right\}$$

$(1,1),(2,2),(3,3),(4,4)\ ,(6,6),(12,12) \in R.$

Therefore, $(x,x) \in R, \forall x \in A.$

$x|y, y|z \in R \Longrightarrow x|z$

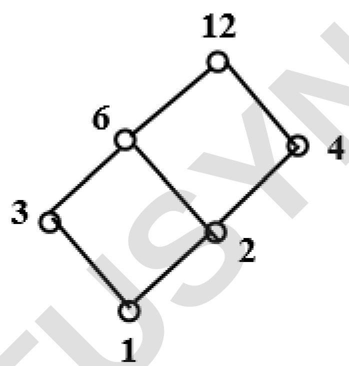
$(x,y),(y,z) \in R \ \Longrightarrow (x,z) \in R\ ,\ \ for\ any\ x,y,z \in A$

R does not contain (x,y) and (y,x) with $x \neq y$.

Therefore, R is reflexive, anti - symmetric and transitive.

Therefore, R is a partial order on A.

Haase diagram:



4. Let $S = \{1, 2, 3\}$ and $P(S)$ the power set of S. On $P(S)$ define the relation R by xRy iff $X \subseteq Y$. Show that the relation is a partial order on $P(S)$. Draw its Hasse diagram.

$$P(S) = R = \{\{\ \}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$

$$= \{S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8\}$$

(i) $S_i \subseteq S_j$, $\forall\, 1 \le i, j \le 8$

$(S_i, S_i) \in R, \forall\, 1 \le i \le 8$

(ii) $S_i \subseteq S_j$, $S_j \subseteq S_k \implies S_i \subseteq S_k$, $\forall\, 1 \le i, j, k \le 8$

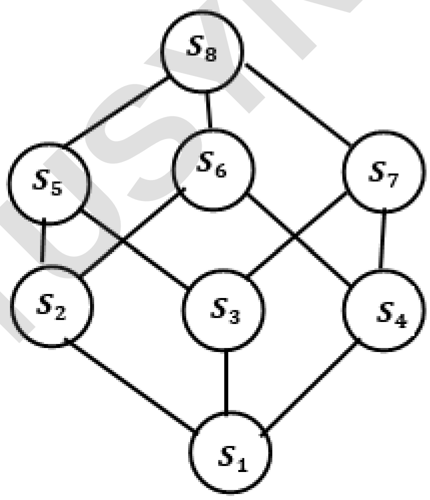
$(S_i, S_j), (S_j, S_k) \in R \implies (S_i, S_k) \in R,\ for\ any\ 1 \le i, j, k \le 8$

(iii) R does not contain $S_i \subseteq S_j$ and $S_j \subseteq S_i$ with $S_i \ne S_j$

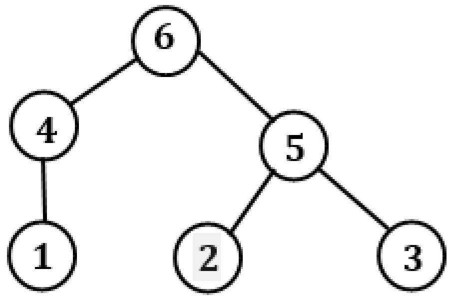
R does not contain (S_i, S_j) and (S_j, S_i) with $S_i \ne S_j$

Therefore, R is reflexive, anti-symmetric and transitive.

Therefore, R is partial order relation.



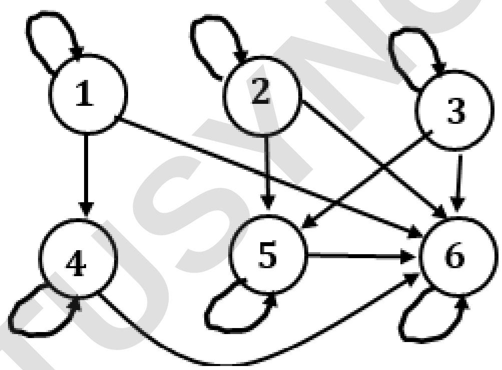
5. The Hasse diagram of the partial order R on a set $A = \{1, 2, 3, 4, 5, 6\}$ is a s given below:



Write down R as a subset of $A \times A$. Construct its digraph.

$R =$
$\{(1,1),(1,4),(1,6),(2,2),(2,5),(2,6),(3,3),(3,5),(3,6)(4,4),(4,6),(5,5),(5,6),(6,6)\}$

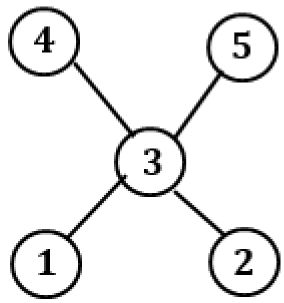
Digraph:



6. **Draw the Hasse diagram of the relation R on $A = \{1, 2, 3, 4, 5\}$ whose matrix is as given below:** $M_R = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

$R = \{(1,1), (1,3), (1,4), (1,5), (2,2), (2,3), (2,4), (2,5), (3,3), (3,4), (3,5), (4,4), (5,5)\}$

Hasse diagram**:**

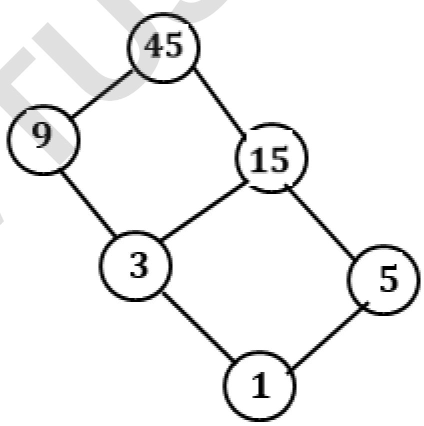


7. **Write down the Hasse diagram for the positive divisors of 45.**

$A = Set\ of\ all\ divisors\ of\ 45 = \{1, 3, 5, 9, 15, 45\}$

$R = \{(1,1), (1,3), (1,5), (1,15), (1,45), (3,3), (3,15), (3,45), (9,9), (9,45), (15,15), (15,45), (45,45)\}$

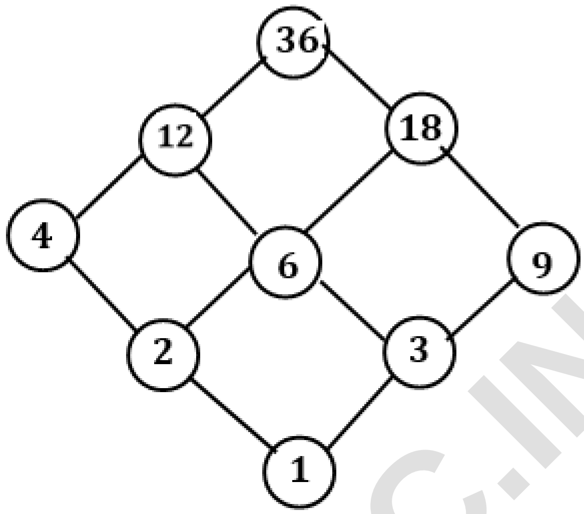
Hasse diagram:



8. Draw the Hasse diagram to represent the positive divisors of 36.

$A = Set\ of\ all\ divisors\ of\ 45 = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$

Hasse diagram:



3.7 Functions

Definition:

Let A and B be any two non empty sets. Then a function f from A to B is a relation such that for each $a \in A$, there is a unique $b \in B$.

Note:

- If $(a, b) \in f$ then $b = f(a)$ is the image a and a is a pre-image of b.
- If $f: A \to B$ is a function then A-Domain of f, B-Codomain of f, $f(A)$-Range of f.

Example:

- $\{(1, a), (2, a), (2, c), (3, b)\}$is not a function.
- $\{(x, z), (y, x), (y, y)\}$is not a function.
- $\{(p, q), (q, q), (r, q)\}$is a function.

Problems:

1. **Let $\boldsymbol{A = \{1, 2, 3, 4\}}$. Determine whether or not the following relations on A are functions: (i) $\boldsymbol{f = \{(2, 3), (1, 4), (2, 1), (3, 2), (4, 4)\}}$ (ii) $\boldsymbol{g = \{(3, 1), (4, 2), (1, 1)\}}$ (iii) $\boldsymbol{h = \{(2, 1), (3, 4), (1, 4), (4, 4)\}}$.**

   (i) No. 2 has two images.
   (ii) No. 2 doesn't have image.
   (iii) Yes. Each element has a unique image.

2. **Let a function $\boldsymbol{f: R \to R}$ be defined by $\boldsymbol{f(x) = x^2 + 1}$. Determine the images of the following subsets of R: $\boldsymbol{A = \{2, 3\}, B = \{-2, 0, 3\}, C = (0, 1), D = [-6, 3]}$.**

   $f(A) = \{2^2 + 1, 3^2 + 1\} = \{5, 10\}$,

   $f(B) = \{(-2)^2 + 1, 0^2 + 1, 3^2 + 1\} = \{5, 1, 10\}$

   $f(C) = \{x^2 + 1 \,|0 < x < 1\}$

   $f(D) = \{x^2 + 1| - 6 \leq x \leq 3\}$

3. **Let $A = \{1, 2, 3, 4, 5, 6\}$and $B = \{6, 7, 8, 9, 10\}$. If a function $f: A \to B$ is defined by {(1, 7), (2, 7), (3, 8), (4, 6), (5, 9), (6, 9)} determine $f^{-1}(6)$and $f^{-1}(9)$. If $B_1 = \{7, 8\}$and $B_2 = \{8, 9, 10\}$, find $f^{-1}(B_1)$ and $f^{-1}(B_2)$.**

$f^{-1}(6) = Pre\ image\ of\ 4 = \{4\}$

$f^{-1}(9) =\ Set\ of\ pre\ images\ of\ 9 = \{5, 6\}$

$f^{-1}(B_1) = Set\ of\ pre\ images\ of\ 7\ and\ 9 = \{1, 2, 3\}$

$f^{-1}(B_2) = Set\ of\ pre\ images\ of\ 8, 9\ and\ 10 =\ \{3, 5, 6\}$

4. **Let $A = \{1, 2, 3, 4, 5\}$and $B = \{3, 4, 5\}$. A function $f: A \to B$ is given by $f = \{(1, 3), (2, 3), (3, 4), (4, 5), (5, 4)\}$.Find $f^{-1}(3), f^{-1}(4), f^{-1}(B_1), f^{-1}(B_2)$. Where $B_1 = \{3, 4\}, B_2 = \{4, 5\}$.**

$f^{-1}(3) = Pre\ image\ of\ 3 = \{1, 2\}$

$f^{-1}(4) =\ Set\ of\ pre\ images\ of\ 4 = \{3, 5\}$

$f^{-1}(B_1) = Set\ of\ pre\ images\ of\ 3\ and\ 4 = \{1, 2, 3, 5\}$

$f^{-1}(B_2) = Set\ of\ pre\ images\ of\ 4\ and\ 5 =\ \{3, 4, 5\}$

5. **Let $f: R \to R$ be defined by $f(x) = \begin{cases} 3x - 5, & for\ x > 0 \\ -3x + 1, & for\ x \le 0 \end{cases}$.**

Determine $f(0), f(-1), f\left(\frac{5}{3}\right), f\left(-\frac{5}{3}\right), f(2)$

$$f(0) = -3(0) + 1 = 1$$

$$f(-1) = -3(-1) + 1 = 4$$

$$f\left(\frac{5}{3}\right) = 3\left(\frac{5}{3}\right) - 5 = 0$$

$$f\left(-\frac{5}{3}\right) = -3\left(-\frac{5}{3}\right) + 1 = 6$$

$$f(2) = 3(2) - 5 = 1$$

6. **Let $f: R \to R$ be defined by $f(x) = \begin{cases} 3x-5, & for\ x>0 \\ -3x+1, & for\ x \le 0 \end{cases}$.**
Find $f^{-1}(0), f^{-1}(1), f^{-1}(-1), f^{-1}(3), f^{-1}(-3), f^{-1}(-6)$

$f(x) = 3x-5, x>0$ $\Rightarrow 0 = 3x-5, x>0$ $\Rightarrow x = 5/3$	$f(x) = -3x+1, x \le 0$ $\Rightarrow 0 = -3x+1, x \le 0$ $\Rightarrow x = 1/3, impossible.$

Therefore, $f^{-1}(0) = \frac{5}{3}$

$f(x) = 3x-5, x>0$ $\Rightarrow 1 = 3x-5, x>0 \Rightarrow$ $\Rightarrow x = 2$	$f(x) = -3x+1, x \le 0$ $\Rightarrow 1 = -3x+1, x \le 0$ $\Rightarrow x = 0$

Therefore, $f^{-1}(1) = \{2, 0\}$

$f(x) = 3x-5, x>0$ $\Rightarrow -1 = 3x-5, x>0$ $\Rightarrow\Rightarrow x = 4/3$	$f(x) = -3x+1, x \le 0$ $\Rightarrow -1 = -3x+1, x \le 0$ $\Rightarrow x = 2/3$

Therefore, $f^{-1}(-1) = \{4/3\}$

$f(x) = 3x-5, x>0$ $\Rightarrow 3 = 3x-5, x>0$ $\Rightarrow x = \frac{8}{3}$	$f(x) = -3x+1, x \le 0$ $\Rightarrow 3 = -3x+1, x \le 0$ $\Rightarrow x = -\frac{2}{3}, impossible.$

Therefore, $f^{-1}(3) = \left\{\frac{8}{3}, -\frac{2}{3}\right\}$

$f(x) = 3x-5, x>0$ $\Rightarrow -3 = 3x-5, x>0$ $\Rightarrow x = \frac{2}{3}$	$f(x) = -3x+1, x \le 0$ $\Rightarrow -3 = -3x+1, x \le 0$ $\Rightarrow x = \frac{4}{3}, impossible.$

Therefore, $f^{-1}(-3) = \left\{\frac{2}{3}\right\}$

$f(x) = 3x-5, x>0$ $\Rightarrow -6 = 3x-5, x>0$ $\Rightarrow x = -\frac{1}{3}, impossible.$	$f(x) = -3x+1, x \le 0$ $\Rightarrow -6 = -3x+1, x \le 0$ $\Rightarrow x = \frac{7}{3}, impossible.$

Therefore, $f^{-1}(-6) = \phi$

7. **If $f: Z \to R$ is defined by $f(x) = x^2 + 5$ find $f^{-1}(\{6\}), f^{-1}([6,7]), f^{-1}([6,10])$, $f^{-1}([-4,5]), f^{-1}([5,\infty])$.**

$x = f^{-1}(6)$ $\Rightarrow f(x) = 6$ $\Rightarrow x^2 + 5 = 6$ $\Rightarrow x^2 = 1$ $\Rightarrow x = \pm 1$ $\Rightarrow x = \{-1, 1\}$	$x = f^{-1}([6,7])$ $\Rightarrow f(x) = [6,7]$ $\Rightarrow 6 \leq f(x) \leq 7$ $\Rightarrow 6 \leq x^2 + 5 \leq 7$ $\Rightarrow 1 \leq x^2 \leq 2$ $\Rightarrow x = \{-1, 1\}$	$x = f^{-1}([6,10])$ $\Rightarrow f(x) = [6,10]$ $\Rightarrow 6 \leq f(x) \leq 10$ $\Rightarrow 6 \leq x^2 + 5 \leq 10$ $\Rightarrow 1 \leq x^2 \leq 4$ $\Rightarrow x = \{-2, -1, 1, 2\}$

$x = f^{-1}([-4,5])$ $\Rightarrow f(x) = [-4,5]$ $\Rightarrow -4 \leq f(x) \leq 5$ $\Rightarrow -4 \leq x^2 + 5 \leq 5$ $\Rightarrow -9 \leq x^2 \leq 0$ $\Rightarrow x = \{0\}$	$x = f^{-1}\big([-5,\infty)\big)$ $\Rightarrow f(x) = [-5,\infty)$ $\Rightarrow -5 \leq f(x) < \infty$ $\Rightarrow -5 \leq x^2 + 5 < \infty$ $\Rightarrow -25 \leq x^2 < \infty$ $\Rightarrow x = Z$

8. **Let $f: R \to R$ be defined by $f(x) = \begin{cases} 3x - 5, & for\ x > 0 \\ -3x + 1, & for\ x \le 0 \end{cases}$.**
   Find $f^{-1}([-5, 5])$ and $f^{-1}([-6, 5])$

$-5 \le f(x) \le 5$ $\Longrightarrow -5 \le 3x - 5 \le 5, x > 0$ $\Longrightarrow 0 \le 3x \le 10, x > 0$ $\Longrightarrow 0 < x \le 10/3$ $\Longrightarrow x \in \left(0, \frac{10}{3}\right]$	$-5 \le f(x) \le 5$ $\Longrightarrow -5 \le -3x + 1 \le 5, x \le 0$ $\Longrightarrow -6 \le -3x \le 4, x \le 0$ $\Longrightarrow -4 \le 3x \le 6, x \le 0$ $\Longrightarrow -\frac{4}{3} \le x \le 2, x \le 0$ $\Longrightarrow x \in [-4/3, 0]$

Therefore, $f^{-1}([-5, 5]) = \left[-\frac{4}{3}, \frac{10}{3}\right]$

$-6 \le f(x) \le 5$ $\Longrightarrow -6 \le 3x - 5 \le 5, x > 0$ $\Longrightarrow -1 \le 3x \le 10, x > 0$ $\Longrightarrow -\frac{1}{3} \le x \le \frac{10}{3}, x > 0$ $\Longrightarrow x \in \left(0, \frac{10}{3}\right]$	$-5 \le f(x) \le 5$ $\Longrightarrow -6 \le -3x + 1 \le 5, x \le 0$ $\Longrightarrow -7 \le -3x \le 4, x \le 0$ $\Longrightarrow -4 \le 3x \le 7, x \le 0$ $\Longrightarrow -\frac{4}{3} \le x \le \frac{7}{3}, x \le 0$ $\Longrightarrow x \in [-4/3, 0]$

Therefore, $f^{-1}([-5, 5]) = \left[-\frac{4}{3}, \frac{10}{3}\right]$

9. **Let $f: X \to Y$ be a function, A be a arbitrary non empty subset of X and C, D be arbitrary non empty subsets of Y. Then prove that $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$ (ii) $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$ (iii) $f^{-1}(\bar{c}) = \overline{f^{-1}(C)}$.**

$$x \in f^{-1}(C \cup D) \Leftrightarrow f(x) \in C \cup D$$
$$\Leftrightarrow f(x) \in C\ or\ f(x) \in D$$
$$\Leftrightarrow x \in f^{-1}(C)\ or\ x \in f^{-1}(D)$$
$$\Leftrightarrow x \in f^{-1}(C) \cup f^{-1}(D)$$

Therefore, $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$

10. **Let $\boldsymbol{f}: \boldsymbol{X} \rightarrow \boldsymbol{Y}$ be a function, A be a arbitrary non empty subset of X and C, D be arbitrary non empty subsets of Y. Then prove that $\boldsymbol{f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)}$.**

$x \in f^{-1}(C \cap D) \Leftrightarrow f(x) \in C \cap D$

$\Leftrightarrow f(x) \in C \text{ and } f(x) \in D$

$\Leftrightarrow x \in f^{-1}(C) \text{ and } x \in f^{-1}(D)$

$\Leftrightarrow x \in f^{-1}(C) \cap f^{-1}(D)$

Therefore, $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$

11. **Let $\boldsymbol{f}: \boldsymbol{X} \rightarrow \boldsymbol{Y}$ be a function, A be a arbitrary non empty subset of X and C, D be arbitrary non empty subsets of Y. Then prove that $\boldsymbol{f^{-1}(\bar{c}) = \overline{f^{-1}(C)}}$.**

$x \in f^{-1}(\bar{C}) \Leftrightarrow f(x) \in \bar{C}$

$\Leftrightarrow f(x) \notin C$

$\Leftrightarrow x \notin f^{-1}(C)$

$\Leftrightarrow x \in \overline{f^{-1}(C)}$

Therefore, $f^{-1}(\bar{c}) = \overline{f^{-1}(C)}$.

12. **If A and B are finite sets with $|\boldsymbol{A}| = \boldsymbol{m}$ and $|\boldsymbol{B}| = \boldsymbol{n}$. Find how many functions are possible from A to B.**

Let $A = \{a_1, a_2, \dots, a_m\}, B = \{b_1, b_2, \dots, b_n\}, |A| = m, |B| = n$

The function f is of the form, $f = \{(a_1, x), (a_2, x), \ (a_n, x)\}$

x has n chances in (a_1, x), x has n chances in (a_2, x) and so on.

Total number of functions from A to B = $n \times n \times \dots \times n = n^m$

13. **If there are 2187 functions from A to B and $|B| = 3$. What is $|A|$?**

If $|A| = m, |B| = n$ then the total number of functions from A to B $= n^m$

By data, the total number of functions from A to B $= 2187$

$n^m = 2187$

By data, $n = 3$.

$3^m = 2187$

$m \log 3 = \log 2187$

$$m = \frac{\log 2187}{\log 3} = 7$$

14. **If $A = \{0, \pm 1, \pm 2\}$ and $f: A \to R$ is defined by $f(x) = x^2 - x + 1, x \in R$. find the range of f.**

$f(0) = 0^2 - 0 + 1 = 1$

$f(-1) = (-1)^2 - (-1) + 1 = 3$

$f(1) = 1^2 - 1 + 1 = 1$

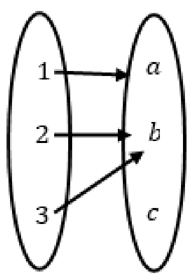
$f(-2) = (-2)^2 - (-2) + 1 = 7$

$f(2) = 2^2 - 2 + 1 = 3$

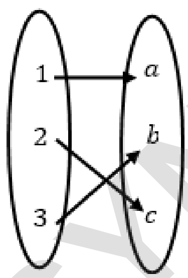
Therefore, $x = \{1, 3, 7\}$

3.8 1-1 and onto function

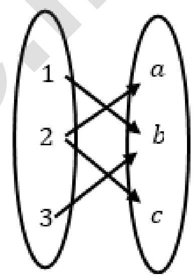
- **1-1 function and onto function or injective function:** Different elements of A have different images in B under f.That is, $f(a_1) = f(a_2) \Rightarrow a_1 = a_2, for\ a_1, a_2 \in A$.
- **Onto function or surjective function:** Every element of B has a pre image in A under f. That is, $f(A) = B$.
- **1-1 correspondence or bijective function:** f is 1-1 and onto. That is, Every element of A has a unique image in B and every element of B has a unique element in A.
- **Identity function:**$f: A \to A$ such that $f(a) = a$, for every $a \in A$. It is denoted by I_A.
- **Constant function:** $f: A \to A$ such that $f(a) = c$, for every $a \in A$.
- **Constant function:** $f: A \to A$ such that $f(a) = c$, for every $a \in A$.



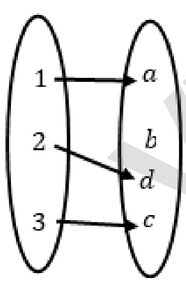
Neither 1-1 nor onto



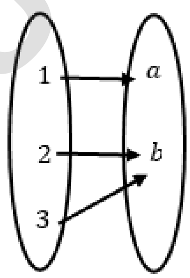
1-1 and onto



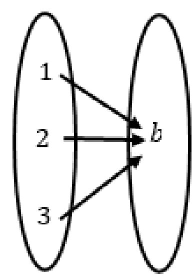
Not a function



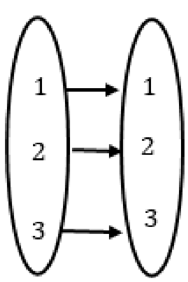
1-1 but not onto



Onto but not 1-1



Constant function



Identity function

Problems:

1. **Find the nature of the following functions defined on $\boldsymbol{A = \{1, 2, 3\}}$,**

(i) $\boldsymbol{f = \{(1, 1), (2, 2), (3, 3)\}}$

(ii) $\boldsymbol{g}$ **={(1, 2), (2, 2), (3, 2)}**

(iii) $\boldsymbol{h}$ **={(1, 2), (2, 2), (3, 1)}**

(iv) $\boldsymbol{p}$ **= {(1, 2), (2, 3), (3, 1)}**

(i) $a = f(a), \forall a \in A$. Therefore, f is a identity function.

(ii) $g(a) = 2, \forall a \in A$. Therefore, g is a constant function.

(iii) $h(1) = h(2) = 2$. 3 has no pre-image. Therefore, h is neither 1-1 nor onto.

(iv) $p(1) = 2, p(2) = 3, p(3) = 1$. All are distinct images. Range and co-domain are same. Therefore, p is 1-1 and onto.

2. **In each of the following cases determine f is 1-1 or onto or both or neither:**

(i) $\boldsymbol{A = B = \{1, 2, 3, 4\}\ and\ f = \{(1, 1), (2, 3), (3, 4), (4, 2)\}}$

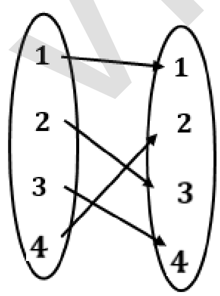
(ii) $\boldsymbol{A = \{a, b, c\}, B = \{1, 2, 3, 4\}, f = \{(a, 1), (b, 1), (c, 3)\}}$

(iii) $\boldsymbol{A = \{1, 2, 3\}, B = \{1, 2, 3, 4, 5\}, f = \{(1, 1), (2, 3), (3, 4)\}}$

(iv) $\boldsymbol{A = \{1, 2, 3\}, B = \{1, 2, 3, 4, 5\}, f = \{(1, 1), (2, 3), (3, 3)\}}$

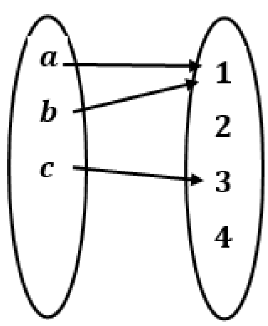
(v) $\boldsymbol{A = \{1, 2, 3, 4\}, B = \{a, b, c, d\} f = \{(1, a), (2, a), (3, d), (4, c)\}}$

(i)

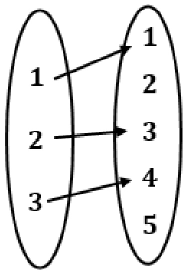


Different elements of A have different elements in B. Also $f(A) = B$. Therefore, it is 1-1 correspondence (or) Bijective function.

(ii)



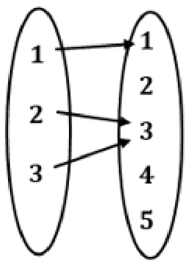
$f(a) = f(b) = 1$ and $f(A) = \{1, 3\} \neq B$. Therefore, this function is neither 1-1 nor onto.

(iii) 

Different elements of A have different elements in B.

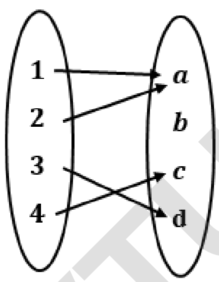
But $f(A) = \{1, 3, 4\} \neq B$.

Therefore, this function is 1-1 but not onto.

(iv) 

$f(2) = f(3) = 3 \text{ and } f(A) = \{1, 3\} \neq B$.

Therefore, this function is 1-1 but not onto.

(v) 

$f(1) = f(2) = a \text{ and } f(A) = \{a, c, d\} \neq B$.

Therefore, this function is 1-1 but not onto.

3. **The functions $f: R \rightarrow R$ and $g: R \rightarrow R$ are defined by $f(x) = 3x + 7$ for all $x \in R$ and $g(x) = x(x^3 - 1)$ for all $x \in R$. Verify f is 1-1 but g is not.**

$f(x_1) = f(x_2) \Longrightarrow 3x_1 + 7 = 3x_2 + 7$

$\Longrightarrow 3x_1 = 3x_2$

$\Longrightarrow x_1 = x_2\,,\ \forall\, x_1, x_2 \in R$

Therefore, f is 1-1.

$g(0) = g(1) = 0, but\ 0 \neq 1.$

Therefore, g is not 1-1.

4. **The function $f: Z \rightarrow Z$ be defined by $f(a) = a + 1$ for $a \in Z$. Find whether 1-1 or onto.**

$f(a) = f(b) \Longrightarrow a + 1 = b + 1$

$\Longrightarrow a = b\,, \forall\, a, b \in Z$

Therefore, f is 1-1.

For any $b \in Z$, we can find a pre image $b - 1$ as $f(b - 1) = b - 1 + 1 = b$.

Therefore, f is onto.

5. **The function $f: Z \times Z \rightarrow Z$ is defined by $f(x, y) = 2x + 3y$. P.T f is onto but not 1-1.**

To prove: f is 1-1

$n = 4n - 3n = 2(2n) + 3(-n), \forall n \in Z.$

Therefore, for any $n \in Z$, we can find a pre-image$(2n, -n)$ such that

$f(2n, -n) = 2(2n) + 3(-n) = n$. Therefore, f is onto.

To prove: f is not onto

$f(0, 2) = 2(0) + 3(2) = 6$

$f(3, 0) = 2(3) + 3(0) = 6$

$f(0, 2) = f(3, 0) = 6. But\ (2, 0) \neq (3, 0)$

Therefore, f is onto.

6. If $f: R \to R$ in each of the following cases determine f is 1-1 or onto or both:

(i) $f(x) = 2x - 3$ (ii) $f(x) = x^2 + x$ (iii) $f(x) = e^x$ (iv) $f(x) = \sin x$

(i) Check whether f is 1-1.

$$f(x_1) = f(x_2) \Longrightarrow 2x_1 - 3 = 2x_2 - 3$$
$$\Longrightarrow 2x_1 = 2x_2$$
$$\Longrightarrow x_1 = x_2, \forall\, x_1, x_2 \in R$$

Therefore, f is 1-1.

Check whether f is onto.

$$y = 2x - 3 \Longrightarrow x = \frac{y+3}{2}$$

For any $y \in R$, there exist $x \in R$ such that $f(x) = f\left(\frac{y+3}{2}\right) = 2\left(\frac{y+3}{2}\right) - 3 = y$.

Therefore, f is onto.

(ii) Check whether f is 1-1.

$$f(-1) = (-1)^2 - 1 = 0$$
$$f(0) = 0^2 + 0 = 0$$
$$f(-1) = f(0), But -1 \neq 0$$

Therefore, f is not 1-1.

Check whether f is onto.

If $f(x) = -2$, then $x^2 + x + 2 = 0$ has complex roots.

Therefore, f is not onto.

7. **If $f: R \to R$ in each of the following cases determine f is 1-1 or onto or both:**
   (i) $f(x) = e^x$ (ii) $f(x) = \sin x$

(i) Check whether f is 1-1 .

$f(x_1) = f(x_2) \Longrightarrow e^{x_1} = e^{x_2}$

$\Longrightarrow x_1 = x_2, \forall\, x_1, x_2 \in R$

Therefore, f is 1-1.

Check whether f is onto.

$y = e^x \Longrightarrow x = \log y$

For any $y \in R$, there exist $x \in R$ such that $f(x) = f(\log y) = e^{\log y} = y$.

Therefore, f is onto.

(ii) Check whether f is 1-1

$f(0) = \sin 0 = 0$

$f(\pi) = \sin \pi = 0$

$f(0) = f(\pi), But\ 0 \neq \pi$

Therefore, f is not 1-1.

Check whether f is onto.

Range = $[-1, 1] \subset R$. Range is a subset of co-domain.

Therefore, f is not onto.

8. Let $\boldsymbol{f}: \boldsymbol{X} \to \boldsymbol{Y}$ be a function and A and B be arbitrary non empty subsets of X. then prove the following: (i) If $\boldsymbol{A} \subseteq \boldsymbol{B}$ then $\boldsymbol{f}(\boldsymbol{A}) \subseteq \boldsymbol{f}(\boldsymbol{B})$ (ii) $\boldsymbol{f}(\boldsymbol{A} \cup \boldsymbol{B}) = \boldsymbol{f}(\boldsymbol{A}) \cup \boldsymbol{f}(\boldsymbol{B})$.

(i) $y \in f(A) \Longrightarrow y = f(x)\ for\ some\ x \in A$

$\Longrightarrow y = f(x)\ for\ some\ x \in B \quad (\because A \subseteq B\ )$

$\Longrightarrow y \in f(B)$

$Therefore, f(A) \subseteq f(B)$

(ii) $y \in f(A \cup B) \Longrightarrow y = f(x)\ for\ some\ x \in A \cup B$

$\Longrightarrow y = f(x)\ for\ \ x \in A\ or\ x \in B$

$\Longrightarrow y \in f(A)\ or\ y \in f(B)$

$\Longrightarrow y \in f(A) \cup f(B)$

Therefore, $f(A \cup B) \subseteq f(A) \cup f(B)$ ------------- (1)

$A \subseteq A \cup B, B \subseteq A \cup B$

$\Longrightarrow f(A) \subseteq f(A \cup B), f(B) \subseteq f(A \cup B)$

$\Longrightarrow f(A) \cup f(B) \subseteq f(A \cup B)$ ----------------------- (2)

9. Let A and B be finite sets and f be a function from A to B. Then prove the following: (i) If f is 1-1, $|A| \leq |B|$ (ii) If f is onto, $|B| \leq |A|$ (iii) If f is bijection, $|A| = |B|$.

(i) Let $|A| = m, |B| = n\ and\ A = \{a_1, a_2, \dots, a_m\}$.

$a_1 \neq a_2 \neq \cdots \neq a_m \Rightarrow f(a_1) \neq f(a_2) \neq \cdots \neq f(a_m)$

$\Rightarrow$ All images in B are distinct.

$\Rightarrow$ B has at least m elements.

$\Rightarrow |B| \geq m$

$\Rightarrow |B| \geq |A|$

$\Rightarrow |A| \leq |B|$

(ii) f is onto $\Rightarrow$ For each $b \in B, t$here exists $a \in A$ such that $f(a) = b$.

$\Rightarrow$ As f is a function, all pre-images in A are distinct.

$\Rightarrow$ A has at least n elements.

$\Rightarrow |A| \geq n$

$\Rightarrow |A| \geq |B|$

$\Rightarrow |B| \leq |A|$

(iii) f is a bijection $\Rightarrow f$ is 1-1 and onto

$\Rightarrow |A| \leq |B|$ and $|B| \leq |A|$

$\Rightarrow |A| = |B|$

10. Let A and B be finite sets with $|A| = m$ and $|B| = n$ How many 1-1 functions are possible from A to B?

Let $A = \{a_1, a_2, \dots, a_m\}$ and $B = \{b_1, b_2, \dots, b_m\}$

f is 1-1 $\Longrightarrow |A| \leq |B| \Longrightarrow m \leq n$.

$f: A \to B$ is of the form $f = \{(a_1, x), (a_2, x), \dots, (a_m, x)\}$

There are n choices for x in (a_1, x).

There are $n - 1$ choices for x in (a_2, x).

....

There are $n - (m - 1)$ choices for x in (a_m, x).

The total number of choices for $x = n(n-1)(n-2) \dots \big(n - (m-1)\big)$

$$= \frac{n!}{(n-m)!}$$

Therefore, number of 1−1 functions from A to B $= \frac{n!}{(n-m)!}$

11. If there are 60 1-1 functions from A to B and $|A| = 3$, what is $|B|$?

By data, $m = |A| = 3$.

Number of 1−1 functions from A to B $= \frac{n!}{(n-m)!}$

$$60 = \frac{n!}{(n-3)!}$$

$$60 = n(n-1)(n-2)$$

Therefore, $n = 5$.

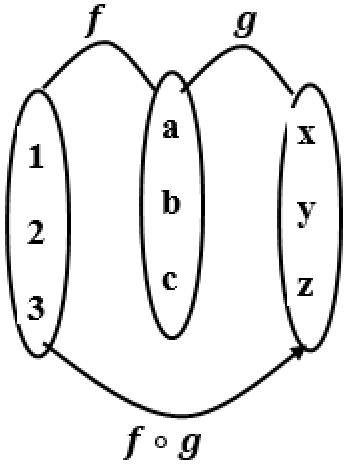
3.9 composition of functions

Introduction:

- Consider three non empty sets A, B and C and functions $f: A \to B$and $g: B \to C$.

  The composition of these two functions is defined as

  $gof: A \to C$ with $(gof)(a) = g[f(a)], \forall a \in A$



Problems:

1. **Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c\}$ and $C = \{w, x, y, z\}$ with $f: A \to B$ and $g: B \to C$ given by $f = \{(1, a), (2, a), (3, b), (4, c)\}$ and $g = \{(a, x), (b, y), (c, z)\}$ find gof**

   $(g \circ f)(1) = g[f(1)] = g(a) = x$

   $(g \circ f)(2) = g[f(2)] = g(a) = x$

   $(g \circ f)(3) = g[f(3)] = g(b) = y$

   $(g \circ f)(4) = g[f(4)] = g(c) = z$

   Therefore, $g \circ f = \{(1, x), (2, x), (3, y), (4, z)\}$

2. **If $A = \{1, 2, 3, 4\}$ and $f = \{(1, 2), (2, 2), (3, 1), (4, 3)\}$ find f^2 and f^3.**

To find: f^2

$f^2(1) = (f \circ f)(1) = f[f(1)] = f(2) = 2$

$f^2(2) = (f \circ f)(2) = f[f(2)] = f(2) = 2$

$f^2(3) = (f \circ f)(3) = f[f(3)] = f(1) = 2$

$f^2(4) = (f \circ f)(4) = f[f(4)] = f(3) = 1$

Therefore, $f^2 = \{(1, 2), (2, 2), (3, 2), (4, 1)\}$

To find: f^3

$f^3(1) = (f \circ f^2)(1) = f[f^2(1)] = f(2) = 2$

$f^3(2) = (f \circ f^2)(2) = f[f^2(2)] = f(2) = 2$

$f^3(3) = (f \circ f^2)(3) = f[f^2(3)] = f(2) = 2$

$f^3(4) = (f \circ f^2)(4) = f[f^2(4)] = f(1) = 2$

Therefore, $f^3 = \{(1, 2), (2, 2), (3, 2), (4,2)\}$

3. **Consider the functions f and g defined by $f(x) = x^3$ and $g(x) = x^2 + 1, \forall x \in R$. Find gof, fog, f^2 and g^2.**

$(g \circ f)(x) = g[f(x)] = g(x^3) = (x^3)^2 + 1 = x^6 + 1$

$(f \circ g)(x) = f[g(x)] = f(x^2 + 1) = (x^2 + 1)^3$

$f^2(x) = (f \circ f)(x) = f[f(x)] = f(x^3) = (x^3)^3 = x^9$

$g^2(x) = (g \circ g)(x) = g[g(x)] = g(x^2 + 1) = (x^2 + 1)^2 + 1$

4. **Consider the functions f and g defined by $f(x) = ax + b$ and $g(x) = 1 - x + x^2$. If $(gof)(x) = 9x^2 - 9x + 3$ determine a and b.**

$9x^2 - 9x + 3 = (g \circ f)(x) = g[f(x)] = g(ax + b)$

$= 1 - (ax + b) + (ax + b)^2$

$= a^2x^2 + (2ab - a)x + (1 - b + b^2)$

Equating coefficients on both the sides,

$a^2 = 9,\ 2ab - a = -9,\ 1 - b + b^2 = 3$

$a = \pm 3,\ a(2b - 1) = -9,\ b(b - 1) = 2$

If $a = 3,\ 2b - 1 = -3, b = -1$

If $a = -3, 2b - 1 = 3, b = 2$

Therefore, $a = 3, b = -1\ (or)\ a = -3, b = 2$.

5. **Let $\boldsymbol{f, g}$ *and* $\boldsymbol{h}$ be three functions from Z to Z defined by $\boldsymbol{f(x) = x - 1, g(x) = 2x}$ and $\boldsymbol{h(x) = \begin{cases} 7, & if\ x\ is\ even \\ 4, & if\ x\ is\ odd \end{cases}}$ Find $\boldsymbol{fog, gof, goh, hog, fo(goh), (fog)oh}$.**

$(f \circ g)(x) = f[g(x)] = f(2x) = 2x - 1$

$(g \circ f)(x) = g[f(x)] = g(x - 1) = 2(x - 1)$

$(g \circ h)(x) = g[h(x)] = 2[h(x)] = \begin{cases} 2(7), & if\ x\ is\ even \\ 2(4), & if\ x\ is\ odd \end{cases} = \begin{cases} 14, & if\ x\ is\ even \\ 8, & if\ x\ is\ odd \end{cases}$

$(h \circ g)(x) = h[g(x)] = h(2x) = 7$

$[f \circ (g \circ h)](x) = f[(g \circ h)(x)]$

$$= \begin{cases} f(14), & if\ x\ is\ even \\ f(8), & if\ x\ is\ odd \end{cases}$$

$$= \begin{cases} 14 - 1, & if\ x\ is\ even \\ 8 - 1, & if\ x\ is\ odd \end{cases}$$

$$= \begin{cases} 13, & if\ x\ is\ even \\ 7, & if\ x\ is\ odd \end{cases}$$

$[(f \circ g) \circ h](x) = (f \circ g)[h(x)]$

$$= \begin{cases} (f \circ g)(7), & if\ x\ is\ even \\ (f \circ g)(4), & if\ x\ is\ odd \end{cases}$$

$$= \begin{cases} f[g(7)], & if\ x\ is\ even \\ f[g(4)], & if\ x\ is\ odd \end{cases}$$

$$= \begin{cases} f[2(7)], & if\ x\ is\ even \\ f[2(4)], & if\ x\ is\ odd \end{cases}$$

$$= \begin{cases} f(14), & if\ x\ is\ even \\ f(8), & if\ x\ is\ odd \end{cases}$$

$$= \begin{cases} 14 - 1, & if\ x\ is\ even \\ 8 - 1, & if\ x\ is\ odd \end{cases}$$

$$= \begin{cases} 13, & if\ x\ is\ even \\ 7, & if\ x\ is\ odd \end{cases}$$

Basic results:

(i) $f \circ (g \circ h) = (f \circ g) \circ h$

(ii) $f \circ g \neq g \circ f$

6. Let $f: A \to B$ and $g: B \to C$ be any two functions. Prove the following:
(i) If f and g are 1-1, so is gof. (ii) If gof is 1-1 then f is 1-1.

(i) Since f is 1-1, $f(x_1) = f(x_2) \Longrightarrow x_1 = x_2$ ---------------- (1)

Since g is 1-1, $g(y_1) = g(y_2) \Longrightarrow y_1 = y_2$ ---------------- (2)

$(g \circ f)(x_1) = (g \circ f)(x_2) \Longrightarrow g[f(x_1)] = g[f(x_2)]$

$\Longrightarrow f(x_1) = f(x_2)$ $[\because by\ (2)]$

$\Longrightarrow x_1 = x_2$ $[\because by\ (1)]$

Therefore, $g \circ f$ is 1-1.

(ii) Since $g \circ f$ is 1-1, $(g \circ f)(x_1) = (g \circ f)(x_2) \Longrightarrow x_1 = x_2$----- (3)

$f(x_1) = f(x_2) \Longrightarrow g[f(x_1)] = g[f(x_2)]$

$\Longrightarrow (g \circ f)(x_1) = (g \circ f)(x_2)$

$\Longrightarrow x_1 = x_2$ $[\because by\ (2)]$

Therefore, f is 1-1.

7. Let $f: A \to B$ and $g: B \to C$ be any two functions. Prove the following:
(i) If f and g are onto, so is gof. (ii) If gof is onto, then g is onto.

(i) Since f is onto, for any $b \in B$, there exists $a \in A$ such that $f(a) = b$.

Since g is onto, for any $c \in C$, there exists $b \in B$ such that $g(b) = c$.

For any $c \in C$, there exists $a \in A$ such that

$(g \circ f)(a) = g[f(a)] = g(b) = c$.

Therefore, $g \circ f$ is onto.

(ii) Since $g \circ f$ is onto, for any $c \in C$, there exists $a \in A$ such that $(g \circ f)(a) = c$.

For any $c \in C$, there exists $b \in B$ such that

$g[b] = g[f(a)] = (g \circ f)(a) = c$.

Therefore, $g \circ f$ is onto.

8. **Let $\boldsymbol{f: A \to B, g: B \to C, h: C \to D}$ be three functions. Then $\boldsymbol{(hog)of = ho(gof)}$**

$[(hog)of](a) = (h \circ g)[f(a)] = (h \circ g)(b) = h[g(b)] = h(c) = d\ , \forall a \in A$

$[ho(gof)](a) = h[(g \circ f)(a)] = h\{g[f(a)]\} = h[g(b)] = h(c) = d\ , \forall a \in A$

Therefore, $(hog)of = ho(gof)$

9. **Let $\boldsymbol{f, g\ and\ h}$ be three functions from Z to Z defined by $\boldsymbol{f(x) = x - 1, g(x) = 3x}$ and $\boldsymbol{h(x) = \begin{cases} 0, & if\ x\ is\ even \\ 1, & if\ x\ is\ odd \end{cases}}$ Verify that $\boldsymbol{fo(goh) = (fog)oh}$.**

$$[fo(goh)](x) = f\{(g \circ h)(x)\}$$
$$= f\{g[h(x)]\}$$
$$= \begin{cases} f[g(0)], & if\ x\ is\ even \\ f[g(1)], & if\ x\ is\ odd \end{cases}$$
$$= \begin{cases} f[3(0)], & if\ x\ is\ even \\ f[3(1)], & if\ x\ is\ odd \end{cases}$$
$$= \begin{cases} f(0), & if\ x\ is\ even \\ f(3), & if\ x\ is\ odd \end{cases}$$
$$= \begin{cases} 0 - 1, & if\ x\ is\ even \\ 3 - 1, & if\ x\ is\ odd \end{cases}$$
$$= \begin{cases} -1, & if\ x\ is\ even \\ 2, & if\ x\ is\ odd \end{cases}$$

$$[(f \circ g) \circ h](x) = (f \circ g)[h(x)]$$
$$= \begin{cases} (f \circ g)(0), & if\ x\ is\ even \\ (f \circ g)(1), & if\ x\ is\ odd \end{cases}$$
$$= \begin{cases} f[g(0)], & if\ x\ is\ even \\ f[g(1)], & if\ x\ is\ odd \end{cases}$$
$$= \begin{cases} f[3(0)], & if\ x\ is\ even \\ f[3(1)], & if\ x\ is\ odd \end{cases}$$
$$= \begin{cases} f(0), & if\ x\ is\ even \\ f(3), & if\ x\ is\ odd \end{cases}$$
$$= \begin{cases} 0 - 1, & if\ x\ is\ even \\ 3 - 1, & if\ x\ is\ odd \end{cases}$$
$$= \begin{cases} -1, & if\ x\ is\ even \\ 2, & if\ x\ is\ odd \end{cases}$$

10. Let f and g be functions from R to R defined by $f(x) = x^2$ and $g(x) = x + 5$. Prove that $gof \neq fog$.

$(g \circ f)(x) = g[f(x)] = g(x^2) = x^2 + 5 , \forall x \in R$

$(f \circ g)(x) = f[g(x)] = f(x + 5) = (x + 5)^2, \forall\, x \in R$

Therefore, $g \circ f \neq f \circ g$

11. Let f and g be functions from R to R defined by $f(x) = x^2$ and $g(x) = x + 5$, $h(x) = \sqrt{x^2 + 2}$. Verify that $fo(goh) = \ (fog)oh$.

$$\begin{aligned}[fo(goh)](x) &= f\{(g \circ h)(x)\} \\ &= f\{g[h(x)]\} \\ &= f\left\{g\left(\sqrt{x^2+2}\right)\right\} \\ &= f\left\{\sqrt{x^2+2}+5\right\} \\ &= \left(\sqrt{x^2+2}+5\right)^2\end{aligned}$$

$$\begin{aligned}[(f \circ g) \circ h](x) &= (f \circ g)[h(x)] \\ &= (f \circ g)[\sqrt{x^2+2}] \\ &= f\left\{g\left(\sqrt{x^2+2}\right)\right\} \\ &= f\left\{\sqrt{x^2+2}+5\right\} \\ &= \left(\sqrt{x^2+2}+5\right)^2\end{aligned}$$

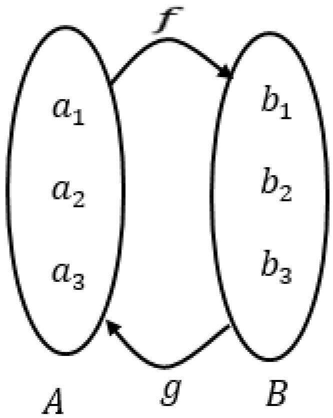
$[fo(goh)](x) = [(fog)oh](x), for\ any\ x \in R$

Therefore, $fo(goh) = \ (fog)oh$.

3.10 Invertible function

A function $f: A \to B$ is said to be inverible if there exists a function $g: B \to A$ such that

$gof = I_A$ and $fog = I_B$. Where I_A- identity function on A, I_B- Identity function on B.



$g = f^{-1}$

1. **Let $A = \{1, 2, 3, 4\}$ and f, g be functions from A to A given by $f = \{(1, 4), (2, 1), (3, 2), (4, 3)\}$ and $g = \{(1, 2), (2, 3), (3, 4), (4, 1)\}$. Prove that g and f inverses of each other.**

   $(g \circ f)(1) = g[f(1)] = g(4) = 1 = I_A(1)$
   $(g \circ f)(2) = g[f(2)] = g(1) = 2 = I_A(2)$
   $(g \circ f)(3) = g[f(3)] = g(2) = 3 = I_A(3)$
   $(g \circ f)(4) = g[f(4)] = g(3) = 4 = I_A(4)$
   $(g \circ f)(x) = I_A(x), \forall x \in A$
   Therefore, $g \circ f = I_A$ and hence g is the inverse of f.

   $(f \circ g)(1) = f[g(1)] = f(2) = 1 = I_A(1)$
   $(f \circ g)(2) = f[g(2)] = f(3) = 2 = I_A(2)$
   $(f \circ g)(3) = f[g(3)] = f(4) = 3 = I_A(3)$
   $(f \circ g)(4) = f[g(4)] = f(1) = 4 = I_A(4)$
   $(f \circ g)(x) = I_A(x), \forall x \in A$
   Therefore, $f \circ g = I_A$ and hence f is the inverse of g.

2. **Consider the function $f: R \rightarrow R$ defined by $f(x) = 2x + 5$. Leta function $g: R \rightarrow R$ be defined by $g(x) = \frac{1}{2}(x - 5)$. Prove that g is an inverse of f.**

$$(g \circ f)(x) = g[f(x)] = g(2x+5) = \frac{1}{2}(2x+5-5) = x = I_R(x), \qquad \forall x \in R.$$

$$(f \circ g)(x) = f[g(x)] = g\left(\frac{x-5}{2}\right) = 2\left(\frac{x-5}{2}\right) + 5 = x = I_R(x), \qquad \forall x \in R.$$

Therefore, $g \circ f = f \circ g = I_R$ and hence g is an inverse of h.

3. **Prove that if a function $f: A \rightarrow B$ is invertible then it has a unique inverse.**

Let g be an inverse of f such that $g \circ f = I_A$ and $f \circ g = I_B$

Let h be another inverse of f such that $h \circ f = I_A$ and $f \circ h = I_B$

Then $h = h \circ I_B = h \circ (f \circ g) = (h \circ f) \circ g = I_A \circ g = g$.

Therefore, f has a unique inverse.

4. **Prove that If $f(a) = b$ then $f^{-1}(b) = a$, provided f is invertible.**

Let f be invertible and g be the inverse of f. Then $g \circ f = I_A$

Let $f(a) = b$.

Then $a = I_A(a) = (g \circ f)(a) = g[f(a)] = g(b) = f^{-1}(b)$

Therefore, $f^{-1}(b) = a$

5. **A function $f: A \to B$ is invertible if and only if it is 1-1 and onto.**

IF PART:

Given: f is invertible.

To prove: f is 1-1 and onto.

f is invertible.

$\Longrightarrow$ There exist a unique function $g: B \to A$ such that $g\,of = I_A, fog = I_B$.

$f(a_1) = f(a_2) \Longrightarrow g[f(a_1)] = g[f(a_2)]$

$\Longrightarrow (gof)(a_1) = (gof)(a_2)$

$\Longrightarrow I_A(a_1) = I_A(a_2)$

$\Longrightarrow a_1 = a_2 \; for \; any \; a_1, a_2 \in A$

$Therefore, f \; is \; 1-1.$

$For \; any \; b$ in B, there exists $a \in A$ such that $b = I_B(b) = (fog)(b) = f[g(b)] = f(a)$.

$Therefore, f \; is \; onto.$

ONLY IF PART:

Given: f is 1-1 and onto.

To prove: f is invertible.

f is 1-1 and onto.

$\Longrightarrow For \; any \; b$ in B, there exists a unique $a \in A$ such that $b = f(a)$.

$Define \; g: B \to A$ by $g(b) = a$.

$Then \; (gof)(a) = g[f(a)] = g(b) = a = I_A(a), for \; any \; a \in A$

$(fog)(b) = f[g(b)] = f(a) = b = I_B(b), for \; any \; b \in B$

Therefore, $gof = I_A$ and $fog = I_B$ and hence f is invertible.

Note:

$f \; is \; invertible \Leftrightarrow f \; is \; 1-1 \; and \; onto.$

6. **Consider the function $f: R \to R$ defined by $f(x) = x^2, \forall x \in R$. Is f invertible?**

$f(a) = f(-a) = a^2. But \; a \neq -a.$

$Therefore, f \; is \; not \; 1-1.$

$Therefore, f \; is \; not \; invertible.$

Note:

If $f: A \to B$ is an invertible function then $fof^{-1} = I_B$ and $f^{-1}of = I_A$.

If $g: B \to C$ is an invertible function then $gog^{-1} = I_C$ and $g^{-1}og = I_B$.

7. **If $\boldsymbol{f}: \boldsymbol{A} \to \boldsymbol{B}$ ao nd $\boldsymbol{g}: \boldsymbol{B} \to \boldsymbol{C}$ are invertible functions then $\boldsymbol{gof}: \boldsymbol{A} \to \boldsymbol{C}$ is an invertible function and $(\boldsymbol{gof})^{-1} = \boldsymbol{f^{-1}og^{-1}}$.**

f and g are invertible functions

$\Longrightarrow f$ and g are 1-1 and onto.

$\Longrightarrow gof$ is 1-1 and onto.

$\Longrightarrow gof$ is invertible.

$f: A \to B$ and $g: B \to C$ are invertible functions.

$\Longrightarrow f^{-1}: B \to A$ and $g^{-1}: C \to B$ are invertible.

$\Longrightarrow f^{-1}og^{-1}: C \to A$ is invertible.

$\Longrightarrow h: C \to A$ is invertible. Where $h = f^{-1}og^{-1}$.

Therefore,

$(gof)\, oh = (gof)\, o(f^{-1}og^{-1}) = go(fof^{-1})og^{-1} = goI_Bog^{-1} = gog^{-1} = I_C$

$ho(gof) = (f^{-1}og^{-1})o(gof) = f^{-1}o(g^{-1}og)of = f^{-1}oI_Bof = f^{-1}of = I_A$

Therefore,

h is the inverse of gof.

$\Longrightarrow (gof)^{-1} = h$

$\Longrightarrow (gof)^{-1} = f^{-1}og^{-1}$

8. **Let $A = B = R$. The set of all real numbers and the functions $f: A \to B$ and $g: B \to A$ be defined by $f(x) = 2x^3 - 1, \forall x \in A$ and $g(y) = \left\{\frac{1}{2}(y+1)\right\}^{\frac{1}{3}}, \forall y \in B$. Show that g and f are inverse to each other.**

$$(gof)(x) = g[f(x)] = g(2x^3 - 1) = \left[\frac{1}{2}(2x^3 - 1 + 1)\right]^{1/3} = x = I_A(x), for\ any\ x \in A$$

$$(fog)(y) = f[g(y)] = f\left[\frac{1}{2}(y+1)\right]^{1/3} = 2\left[\frac{1}{2}(y+1)\right] - 1 = y = I_B(y), for\ any\ y \in B$$

Therefore, $gof = I_A$ and $fog = I_B$ and hence f and g are inverse to each other.

9. **Let $A = B = C = R$. Let $f: A \to B$ and $g: B \to C$ be defined by $f(a) = 2a + 1, \forall a \in A$ and $g(b) = \frac{1}{3}b, \forall b \in B$.**
Compute gof. Show that gof is invertible. Also find $(gof)^{-1}$.

(i) $(gof)(a) = g[f(a)] = g(2a+1) = \frac{1}{3}(2a+1)$.

(ii) $f(a_1) = f(a_2) \Longrightarrow 2a_1 + 1 = 2a_2 + 1 \Longrightarrow 2a_1 = 2a_2 \Longrightarrow a_1 = a_2$

For any $b \in B$, there exists $a \in A$ such that $2a + 1 = b \Longrightarrow a = \frac{b-1}{2}$

Therefore, f is 1-1 and onto, hence f is invertible.

$g(b_1) = g(b_2) \Longrightarrow \frac{1}{3}b_1 = \frac{1}{3}b_2 \Longrightarrow b_1 = b_2$

For any $c \in C$, there exists $b \in B$ such that $\frac{1}{3}b = c \Longrightarrow b = 3c$.

Therefore, g is 1-1 and onto, hence g is invertible.

Since f and g are invertible, gof is also invertible.

(iii) $(gof)^{-1}(c) = (f^{-1}og^{-1})(c) = f^{-1}[g^{-1}(c)] = f^{-1}(3c) = \frac{3c-1}{2}$.

10. For the functions f and g from R to R defined by $f(x) = 2x$ and $g(x) = 3x - 2$. Verify that $(gof)^{-1} = f^{-1}og^{-1}$.

$f(x) = 2x$ $\Longrightarrow y = 2x$ $\Longrightarrow \frac{y}{2} = x$ $\Longrightarrow \frac{y}{2} = f^{-1}(y)$ Therefore, $f^{-1}(x) = \frac{x}{2}$	$g(x) = 3x - 2$ $\Longrightarrow z = 3x - 2$ $\Longrightarrow \frac{z+2}{3} = x$ $\Longrightarrow \frac{z+2}{3} = g^{-1}(z)$ Therefore, $g^{-1}(x) = \frac{x+2}{3}$

$(gof)(x) = g[f(x)] = g(2x) = 3(2x) - 2 = 6x - 2$

$\Longrightarrow p = 6x - 2$ [Assume $(gof)(x) = p$]

$$\Longrightarrow \frac{p+2}{6} = x$$

$$\Longrightarrow \frac{p+2}{6} = (gof)^{-1}(p)$$

Therefore, $(gof)^{-1}(x) = \frac{x+2}{6}$ ---------------------------- (1)

$(f^{-1}og^{-1})(x) = f^{-1}[g^{-1}(x)] = f^{-1}\left(\frac{x+2}{3}\right) = \frac{x+2}{6}$ ------------ (2)

$From$ (1), (2) it is clear that $(gof)^{-1}(x) = (f^{-1}og^{-1})(x), for\ any\ x \in R$.

Therefore, it is verified that $(gof)^{-1} = f^{-1}og^{-1}$.

3.11 The pigeonhole principle

- If m Pigeons occupies n pigeonholes and $m > n$ then atleast one pigeonhole has two or more pigeons in it. In general, there are at least $p + 1$ pigeons in it. Where $p = \left\lfloor \frac{m-1}{n} \right\rfloor$.

Problems:

1. **If five colours are used to paint 26 doors, show that at least 6 doors will have the same colours.**

   $m =$ Number of Pigeons = Number of doors = 26

   $n =$ Number of Pigeonholes = Number of colours = 5

   $p + 1 = \left\lfloor \frac{m-1}{n} \right\rfloor + 1 = \left\lfloor \frac{26-1}{5} \right\rfloor + 1 = 5 + 1 = 6$

   By Pigeonhole principle, there are at least $p + 1 = 6$ doors will have the same colours.

2. **If $n + 1$ numbers from 1 to $2n$ are selected then show that at least two of them will have their sum equal to $2n + 1$.**

   Consider $A_1 = \{1, 2n\}, A_2 = \{2, 2n - 1\}, \dots, A_{n-1} = \{n - 1, n + 2\}$ *and* $A_n = \{n, n + 1\}$

   There are n sets containing exactly two elements with their sum $2n + 1$.

   $m =$ Number of Pigeons = Numbers from 1 to $2n = n + 1$

   $n =$ Number of Pigeonholes = Number of sets = n

   $p + 1 = \left\lfloor \frac{m-1}{n} \right\rfloor + 1 = \left\lfloor \frac{n+1-1}{n} \right\rfloor + 1 = 1 + 1 = 2$

   By Pigeonhole principle,

   There are at least $p + 1 = 2$ of them will have their sum equal to $2n + 1$.

3. **Show that if any seven numbers from 1 to 12 are chosen then at least two of them will add to 13.**

   Consider $A_1 = \{1, 12\}, A_2 = \{2, 11\}, A_3 = \{3, 10\}, A_4 = \{4, 9\}, A_5 = \{5, 8\}, A_6 = \{6, 7\}$

   There are 6 sets containing exactly two elements with their sum 13.

   $m =$ Number of Pigeons = Numbers selected= 7

   $n =$ Number of Pigeonholes = Number of sets = 6

   $p + 1 = \left\lfloor \frac{m-1}{n} \right\rfloor + 1 = \left\lfloor \frac{7-1}{6} \right\rfloor + 1 = 1 + 1 = 2$

   By Pigeonhole principle, there are at least $p + 1 = 2$ of them will add to 13.

4. **Show that if any 14 integers are selected from the set $S = \{1, 2, 3, \ldots, 25\}$ there are at least two integers whose sum is 26.**

Consider

$A_1 = \{1, 25\}, A_2 = \{2, 24\}, A_3 = \{3, 23\}, A_4 = \{4, 22\}, A_5 = \{5, 21\}, A_6 = \{6, 20\},$

$A_7 = \{7. 19\}, A_8 = \{8, 18\}, A_9 = \{9, 17\}, A_{10} = \{10, 16\}, A_{11} = \{11, 15\}, A_{12} = \{12, 14\}$

There are 12 sets containing exactly two elements with their sum 26.

$m =$ Number of Pigeons = Numbers selected= 14

$n =$ Number of Pigeonholes = Number of sets = 12

$p + 1 = \left\lfloor \frac{m-1}{n} \right\rfloor + 1 = \left\lfloor \frac{14-1}{12} \right\rfloor + 1 = 1 + 1 = 2$

By Pigeonhole principle, there are at least $p + 1 = 2$ integers whose sum is 26.

5. **Shirts numbered consecutively from 1 to 20 are worn by 20 students of a class. When any three of these students are chosen to a debating team from the class, the sum of their numbers is used as the code number of the team. Show that if any 8 of the 20 are selected then from these 8, we may from at least two different teams having the same code number.**

Total number of teams with 3 students= $\binom{8}{3} = 56$

Smallest possible code to a team= $1 + 2 + 3 = 6$

highest possible code to a team= $18 + 19 + 20 = 57$

We can generate codes varying from 6 to 57 and there are 52 codes.

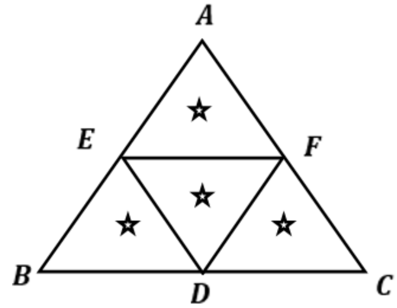
$m =$ Number of Pigeons = Number of teams $= 56$

$n =$ Number of Pigeonholes = Number of codes = 52

$p + 1 = \left\lfloor \frac{m-1}{n} \right\rfloor + 1 = \left\lfloor \frac{56-1}{52} \right\rfloor + 1 = 1 + 1 = 2$

By Pigeonhole principle, there are at least $p + 1 = 2$ different teams having the same code number.

6. **Let ABC be an equilateral triangle with $AB = 1cm$. Show that if we select 5 points in the interior of the triangle , there must be at least two points whose distance is less than ½ cm.**



Consider the triangle ABC with $AB = BC = CA = 1\ cm$.

Consider another triangle DEF formed by joining mid points of BC, AB and AC.

$\Delta\ ABC$ is partitioned into 4 small equivalent triangles each of which have sides ½ cm.

$m =$ Number of Pigeons $=$ Number of points $= 5$

$n =$ Number of Pigeonholes $=$ Number of partitions $= 4$

$p + 1 = \left\lfloor \frac{m-1}{n} \right\rfloor + 1 = \left\lfloor \frac{5-1}{4} \right\rfloor + 1 = 1 + 1 = 2$

By Pigeonhole principle,

There are at least $p + 1 = 2$ points whose distance is less than ½ cm.

7. **How many persons must be chosen in order that at least 5 of them will have birthdays in the same calendar month.**

$m =$ Number of Pigeons $=$ Number of persons $= m$

$n =$ Number of Pigeonholes $=$ Number of calendar months $= 12$

By Pigeonhole principle,

$p + 1 = \left\lfloor \frac{m-1}{n} \right\rfloor + 1 = \left\lfloor \frac{m-1}{12} \right\rfloor + 1 = 5$

$\left\lfloor \frac{m-1}{12} \right\rfloor = 4$

Least possible value of m is 49.

49 persons must be chosen in order that at least 5 of them will have birthdays in the same calendar month.