DISCRETE MATHEMATICAL STRUCTURES

SEMESTER – III **Subject Code: BCS405A**

CREDITS – 04

Course objectives: This course will enable students to

- Prepare for a background in abstraction, notation, and critical thinking for the mathematics most directly related to computer science.
- Understand and apply logic, relations, functions, basic set theory, countability and counting arguments, proof techniques,
- Understand and apply mathematical induction, combinatorics, discrete probability, recursion, sequence and recurrence, elementary number theory
- Understand and apply graph theory and mathematical proof techniques.

Module -1 **10 Hours**

Fundamentals of Logic: Basic Connectives and Truth Tables, Logic Equivalence – The Laws of Logic, Logical Implication – Rules of Inference. The Use of Quantifiers, Quantifiers, Definitions and the Proofs of Theorems, **Textbook 1: Ch 2**

Module -2 **10Hours**

Properties of the Integers: Mathematical Induction, The Well Ordering Principle – Mathematical Induction, Recursive Definitions. **Fundamental Principles of Counting:** The Rules of Sum and Product, Permutations, Combinations – The Binomial Theorem, Combinations with Repetition. **Textbook 1: Ch 4: 4.1, 4.2 Ch 1.**

Module – 3 **10 Hours**

Relations and Functions: Cartesian Products and Relations, Functions – Plain and One-to-One, Onto Functions. The Pigeon-hole Principle, Function Composition and Inverse Functions. **Properties of Relations**, Computer Recognition – Zero-One Matrices and Directed Graphs, Partial Orders – Hasse Diagrams, Equivalence Relations and Partitions.
Textbook 1: Ch 5:5.1 to 5.3, 5.5, 5.6, Ch 7:7.1 to 7.4

Module-4 **10 Hours**

The Principle of Inclusion and Exclusion: The Principle of Inclusion and Exclusion, Generalizations of the Principle, Derangements – Nothing is in its Right Place, Rook Polynomials. **Recurrence Relations:** First Order Linear Recurrence Relation, The Second Order Linear Homogeneous Recurrence Relation with Constant Coefficients.
Textbook 1: Ch 8: 8.1 to 8.4, Ch 10:10.1 to 10.2

Module-5 **10 Hours**

Introduction to Groups Theory: Definitions and Examples of particular groups (The Klein 4-group, Additive group of Integers modulo n, Multiplicative group of Integers modulo p and permutation groups), Properties of groups, Subgroups, cyclic groups Lagrange's Theorem.
Coding theory: Preliminaries, Encoding and Decoding of a message, The Hamming Metric, Generator Matrix, parity-check Matrix, Group codes, Hamming Matrices.
Textbook 1: Ch 11: 11.1 to 11.3, Ch 12: 12.1 to 12.4

BCS405A - Module 1

Fundamentals of logic

Syllabus: Basic Connectives and Truth Tables, Logic Equivalence – The Laws of Logic, Logical Implication – Rules of Inference. The Use of Quantifiers, Quantifiers, Definitions and the Proofs of Theorems.

1.1 Basic connectives and truth tables

Introduction:

- ❖ Declarative statement which is either true or false but not both is called ***proposition*** or ***statement***.
- ❖ The simplest statement which cannot be broken down further is called ***primitive statement***.
- ❖ The truth or falsity of a statement is called ***truth value***.

Truth value	Notation
True	1
False	0

- ❖ Connective words which are used to combine two or more propositions are called ***logical connectives***.

logical connectives	Notations
and	$\wedge$
Or	$\vee$
exclusive or	$\veebar$
If then	$\rightarrow$
If and only if	$\leftrightarrow$

- ❖ A proposition containing one or more connectives is called ***compound proposition***.

Compound proposition	Notation	Meaning
Negation	$\neg p$	Not p
Conjunction	$p \wedge q$	p and q
Disjunction	$p \vee q$	p or q or both
Exclusive disjunction	$p \veebar q$	p or q but not both
Conditional	$p \rightarrow q$	If p then q
Biconditional	$p \leftrightarrow q$	p if and only if q

- ❖ Example: Consider p: Sun is shining, q: Humidity is low

$\neg p$	Sun is not shining.
$p \wedge q$	Sun is shining and humidity is low
$p \vee q$	Sun is shining or humidity is low or both.
$p \veebar q$	Sun is shining or humidity is low but not both.
$p \rightarrow q$	If sun is shining then humidity is low.
$p \leftrightarrow q$	*Sun is shining* if and only if humidity is low

❖ Truth table:

p	q	$p \wedge q$	$p \vee q$	$p \veebar q$	$p \rightarrow q$	$p \leftrightarrow q$
0	0	0	0	0	1	1
0	1	0	1	1	1	0
1	0	0	1	1	0	0
1	1	1	1	0	1	1

❖ Compound proposition which is always true for all possible combinations of truth values is called ***Tautology***.

❖ Compound proposition which is always false for all possible combinations of truth values is called ***contradiction.***

❖ Compound proposition which is neither tautology nor contradiction is called ***contingency***.

1. **Consider the following propositions concerned with a certain triangle** ***p*****: ΔABC is isosceles,** ***q*****: ΔABC is equilateral,** ***r*** **: ΔABC is equiangular. Write down the following propositions in words:**
(i) $p \wedge \neg q$ **(ii)** $p \wedge \neg q$ **(iii)** $p \to q$ **(iv)** $q \to p$ **(v)** $\neg r \to \neg q$ **(vi)** $p \to \neg q$

(i) ΔABC is isosceles and is not equilateral.
(ii) ΔABC is either not isosceles or equilateral.
(iii) If ΔABC is isosceles then it is equilateral.
(iv) If ΔABC is equilateral then it is isosceles.
(v) If ΔABC is not equiangular then it is not equilateral.
(vi) If ΔABC is isosceles then it is not equilateral.

2. **Given that p is true and q is false find the truth values of the following:**
(i) $\neg p \wedge q$ **(ii)** $\neg(p \wedge q) \vee \neg(p \leftrightarrow q)$ **(iii)** $\neg(p \to \neg q)$

(i) $F \wedge F \Leftrightarrow F$
(ii) $\neg(T \wedge F) \vee \neg(T \leftrightarrow F) \Leftrightarrow T \vee T \Leftrightarrow T$
(iii) $\neg(T \vee T) \Leftrightarrow F$

3. **Given that p is true and q is false find the truth values of the following:**
(i) $(p \to q) \vee \neg(p \leftrightarrow \neg q)$ **(ii)** $(p \wedge q) \to (p \vee q)$ **(iii)** $(p \to \neg q) \vee (q \to \neg p)$

(i) $(T \to F) \vee \neg(T \leftrightarrow T) \Leftrightarrow F \vee F \Leftrightarrow F$
(ii) $(T \wedge F) \to (T \vee F) \Leftrightarrow F \to T \Leftrightarrow T$
(iii) $(T \to T) \vee (F \to F) \Leftrightarrow (T \vee T) \Leftrightarrow T$

4. **Determine the truth values of the following:**

(i) $p \wedge q$ **is false and** ***q*** **is true. Find the truth value if** ***p*****.**
(ii) $p \vee q$ **is false and** ***q*** **is false. Find the truth value of** ***p*****.**
(iii) $p \to q$ **is true and** ***q*** **is false. Find** ***p*****.**
(iv) $p \leftrightarrow q$ **is true and** ***p*** **is false. Find** ***q*****.**
(v) $p \to q$ **is false and** ***p*** **is false.**

(a) $p \wedge T$ is false $\Leftrightarrow$ p is false.
(b) $p \vee F$ is false $\Leftrightarrow p$ is false.
(c) $p \to F$ is true $\Leftrightarrow$ p is false.
(d) $p \leftrightarrow F$ is true.
(e) $F \to q$ is false $\Leftrightarrow q$ may be true or false.

5. Find the possible truth values of *p,q and r* in the following cases:
(a) $p \to (q \vee r)$ **is false** **(b)** $p \wedge (q \to r)$ **is true.**

(a)

$p \to (q \vee r)$	p	$q \vee r$	q	r
0	1	0	0	0

Truth values of p, q and r are 1, 0, 0 respectively.

(b)

$p \wedge (q \to r)$	p	$q \to r$	q	r
1	1	1	0 Or 1	1 or 0 1

p	q	r
1	1	1
1	0	1
1	0	0

6. Let *p, q* and *r* be propositions having truth values 0, 0, 1 respectively. Find the truth values of the following compound propositions: (a) $p \to (q \wedge r)$ **(b)** $p \wedge (r \to q)$
(c) $(p \wedge q) \to r$ **(d)** $p \to (q \to \neg r)$

(a)

p	q	r	$q \wedge r$	$p \to (q \wedge r)$
0	0	1	0	1

(b)

p	q	r	$r \to q$	$p \wedge (r \to q)$
0	0	1	0	0

(c)

p	q	r	$p \wedge q$	$(p \wedge q) \to r$
0	0	1	0	1

$\neg$

(d)

p	q	r	$q \to \neg r$	$p \to (q \to \neg r)$
0	0	1	1	1

7. Let *p* and *q* primitive statements for which the conditional $p \to q$ is false. Determine the truth values of the following compound propositions: (a) $p \wedge q$ **(b)** $\neg p \vee q$ **(c)** $q \to p$ **(c)** $\neg q \to \neg p$

Since $p \to q$ is false, *p* is 1 and *q* is 0.

Therefore, (a) $p \wedge q$ is 0 (b) $\neg p \vee q$ is 0 (c) $q \to p$ is 1 (d) $\neg q \to \neg p$ is 0

8. Form the truth tables for the following:

(i) $(p \vee q) \wedge \neg p$ **(ii)** $\neg(p \vee \neg q)$ **(iii)** $p \to (q \to r)$ **(d)** $(p \to q) \to r$ **(e)** $[(p \wedge q) \vee \neg r] \leftrightarrow p$

(i)

p	q	$p \vee q$	$\neg p$	$(p \vee q) \wedge \neg p$
0	0	0	1	0
0	1	1	1	1
1	0	1	0	0
1	1	1	0	0

(ii)

p	q	$\neg q$	$p \vee \neg q$	$\neg(p \vee \neg q)$
0	0	1	1	0
0	1	0	0	1
1	0	1	1	0
1	1	0	1	0

(iii)

p	q	r	$q \to r$	$p \to (q \to r)$
0	0	0	1	1
0	0	1	1	1
0	1	0	0	1
0	1	1	1	1
1	0	0	1	1
1	0	1	1	1
1	1	0	0	0
1	1	1	1	1

9. Form the truth tables for the following:
(i) $(p \to q) \to r$ **(ii)** $[(p \wedge q) \vee \neg r] \leftrightarrow p$

(i)

p	q	r	$p \to q$	$(p \to q) \to r$
0	0	0	1	0
0	0	1	1	1
0	1	0	1	0
0	1	1	1	1
1	0	0	0	1
1	0	1	0	1
1	1	0	1	0
1	1	1	1	1

(ii)

p	q	r	$p \wedge q$	$\neg r$	$(p \wedge q) \vee \neg r$	$[(p \wedge q) \vee \neg r] \leftrightarrow p$
0	0	0	0	1	1	0
0	0	1	0	0	0	1
0	1	0	0	1	1	0
0	1	1	0	0	0	1
1	0	0	0	1	1	1
1	0	1	0	0	0	0
1	1	0	1	1	1	1
1	1	1	1	0	1	1

10. Show that for any propositions p, q and r $[(p \to q) \wedge (q \to r)] \to (p \to r)$ **is a tautology by constructing truth table.**

p	q	r	$p \to q$	$q \to r$	$(p \to q) \wedge (q \to r)$ (1)	$p \to r$ (2)	(1) → (2)
0	0	0	1	1	1	1	1
0	0	1	1	1	1	1	1
0	1	0	1	0	0	1	1
0	1	1	1	1	1	1	1
1	0	0	0	1	0	0	1
1	0	1	0	1	0	1	1
1	1	0	1	0	0	0	1
1	1	1	1	1	1	1	1

The given compound proposition is true for all possible combinations of truth values. Therefore it is a ***Tautology***.

11. Show that for any propositions p, q and r $[p \wedge (p \to q) \wedge r] \to [(p \vee q) \to r]$ is a tautology by constructing truth table.

p	q	r	$p \to q$	$p \wedge (p \to q) \wedge r$ (1)	$p \vee q$	$(p \vee q) \to r$ (2)	$(1) \to (2)$
0	0	0	1	0	0	1	1
0	0	1	1	0	0	1	1
0	1	0	1	0	1	0	1
0	1	1	1	0	1	1	1
1	0	0	0	0	1	0	1
1	0	1	0	0	1	1	1
1	1	0	1	0	1	0	1
1	1	1	1	1	1	1	1

The given compound proposition is true for all possible combinations of truth values. Therefore, it is a ***Tautology***.

12. Show that for any propositions p, q and r $\{(p \vee q) \wedge [(p \to r) \wedge (q \to r)]\} \to r$ is a tautology by constructing truth table.

p	q	r	$p \vee q$ (1)	$p \to r$	$q \to r$	$(p \to r) \wedge (q \to r)$ (2)	$(1) \wedge (2)$	$[(1) \wedge (2)] \to r$
0	0	0	0	1	1	1	0	1
0	0	1	0	1	1	1	0	1
0	1	0	1	1	0	0	0	1
0	1	1	1	1	1	1	1	1
1	0	0	1	0	0	0	0	1
1	0	1	1	1	1	1	1	1
1	1	0	1	0	0	0	0	1
1	1	1	1	1	1	1	1	1

The given compound proposition is true for all possible combinations of truth values. Therefore it is a ***Tautology.***

13. Show that for any propositions p, q and r $[p \to (q \to r)] \to [(p \to q) \to (p \to r)]$ is a tautology by constructing truth table.

p	q	r	$q \to r$	$p \to (q \to r)$ (1)	$p \to q$	$p \to r$	$(p \to q) \to (p \to r)$ (2)	$(1) \to (2)$
0	0	0	1	1	1	1	1	1
0	0	1	1	1	1	1	1	1
0	1	0	0	1	1	1	1	1
0	1	1	1	1	1	1	1	1
1	0	0	1	1	0	0	1	1
1	0	1	1	1	0	1	1	1
1	1	0	0	0	1	0	0	1
1	1	1	1	1	1	1	1	1

The given compound proposition is true for all possible combinations of truth values. Therefore it is a ***Tautology***.

14. Show that for any propositions p, q and r $[(p \to q) \wedge (q \to r)] \to [(p \vee q) \to r]$ is a tautology by constructing truth table.

p	q	r	$p \to q$	$q \to r$	$(p \to q) \wedge (q \to r)$ (1)	$p \vee q$	$(p \vee q) \to r$ (2)	$(1) \to (2)$
0	0	0	1	1	1	0	1	1
0	0	1	1	1	1	0	1	1
0	1	0	1	0	0	1	0	1
0	1	1	1	1	1	1	1	1
1	0	0	0	1	0	1	0	1
1	0	1	0	1	0	1	1	1
1	1	0	1	0	0	1	0	1
1	1	1	1	1	1	1	1	1

The given compound proposition is true for all possible combinations of truth values. Therefore it is a ***Tautology***.

15. Show that for any propositions p, q and r $[\neg(p\vee q)\vee(\neg p\wedge q)]\vee p$ **is a tautology by constructing truth table.**

p	q	$\neg p$	$p\vee q$	$\neg(p\vee q)$	$\neg p\wedge q$	$\neg(p\vee q)\vee(\neg p\wedge q)$ (1)	$(1)\vee p$
0	0	1	0	1	0	1	1
0	1	1	1	0	1	1	1
1	0	0	1	0	0	0	1
1	1	0	1	0	0	0	1

The given compound proposition is true for all possible combinations of truth values. Therefore it is a ***Tautology***.

16. Show that for any propositions p, q and r $[(p\to r)\wedge(q\to r)]\to[(p\vee q)\to r]$ **is a tautology by constructing truth table.**

p	q	r	$p\to r$	$q\to r$	$(p\to q)\wedge(q\to r)$ (1)	$p\vee q$	$(p\vee q)\to r$ (2)	$(1)\to(2)$
0	0	0	1	1	1	0	1	1
0	0	1	1	1	1	0	1	1
0	1	0	1	0	0	1	0	1
0	1	1	1	1	1	1	1	1
1	0	0	0	1	0	1	0	1
1	0	1	0	1	0	1	1	1
1	1	0	1	0	0	1	0	1
1	1	1	1	1	1	1	1	1

The given compound proposition is true for all possible combinations of truth values. Therefore it is a ***Tautology***.

17. Show that for any proposition *p* and *q*, $(p\veebar q)\vee(p\leftrightarrow q)$ **is a tautology.**

p	q	$p\veebar q$	$p\leftrightarrow q$	$(p\veebar q)\vee(p\leftrightarrow q)$
0	0	0	1	1
0	1	1	0	1
1	0	1	0	1
1	1	0	1	1

The given compound proposition is true for all possible combinations of truth values. Therefore it is a ***Tautology***.

18. Show that for any proposition *p* and *q*, $(p \veebar q) \wedge (p \leftrightarrow q)$ is a contradiction.

p	q	$p \veebar q$	$p \leftrightarrow q$	$(p \veebar q) \wedge (p \leftrightarrow q)$
0	0	0	1	0
0	1	1	0	0
1	0	1	0	0
1	1	0	1	0

Compound proposition is false for all possible combinations of truth values.
Therefore it is a ***contradiction.***

19. Show that for any proposition *p* and *q*, $(p \veebar q) \wedge (p \to q)$ is a contingency.

p	q	$p \veebar q$	$p \to q$	$(p \veebar q) \wedge (p \to q)$
0	0	0	1	0
0	1	1	1	1
1	0	1	0	0
1	1	0	1	0

Compound proposition is neither tautology nor contradiction.
Therefore it is a ***contingency***.

1.2 Logical equivalence - The laws of logic

Definition:

Two propositions u and v are said to be logically equivalent if u and v have the same truth values for all possible combinations of truth values. It is denoted by $u \Leftrightarrow v$ or $u \equiv v$.

Laws of logic:

Law of double negation $\neg\neg p \Leftrightarrow p$	Idempotent law $p \vee p \Leftrightarrow p$ $p \wedge p \Leftrightarrow p$
Demorgan's law	Identity law
$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$ $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$	$p \vee F_0 \Leftrightarrow p$ $p \wedge T_0 \Leftrightarrow p$
Associative law	Inverse law
$p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$ $p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$	$p \vee \neg p \Leftrightarrow T_0$ $p \wedge \neg p \Leftrightarrow F_0$
Commutative law	Domination law
$p \vee q \Leftrightarrow q \vee p$ $p \wedge q \Leftrightarrow q \wedge p$	$p \vee T_0 \Leftrightarrow T_0$ $p \wedge F_0 \Leftrightarrow F_0$
Distributive law	Absorbtion law
$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$	$p \vee (p \wedge q) \Leftrightarrow p$ $p \wedge (p \vee q) \Leftrightarrow p$

1. **By constructing truth tables prove that** $[(p \vee q) \to r] \Leftrightarrow [\neg r \to \neg(p \vee q)]$

<u>Proof</u>:

Let $u \equiv [(p \vee q) \to r]$ and $v \equiv [\neg r \to \neg(p \vee q)]$

p	q	r	$p \vee q$	$[(p \vee q) \to r]$ (u)
0	0	0	0	1
0	0	1	0	1
0	1	0	1	0
0	1	1	1	1
1	0	0	1	0
1	0	1	1	1
1	1	0	1	0
1	1	1	1	1

p	q	r	$p \wedge q$	$\neg r$	$\neg(p \vee q)$	$[\neg r \to \neg(p \vee q)]$ (v)
0	0	0	0	1	1	1
0	0	1	0	0	1	1
0	1	0	1	1	0	0
0	1	1	1	0	0	1
1	0	0	1	1	0	0
1	0	1	1	0	0	1
1	1	0	1	1	0	0
1	1	1	1	0	0	1

Since, and v have the same truth values for all possible combinations, $u \Leftrightarrow v$.

Therefore, $[(p \vee q) \to r] \Leftrightarrow [\neg r \to \neg(p \vee q)]]$

2. By constructing truth tables prove that

$$[(p \leftrightarrow q) \wedge (q \leftrightarrow r) \wedge (r \leftrightarrow p)] \Leftrightarrow [(p \to q) \wedge (q \to r) \wedge (r \to p)]$$

Proof:

Let $u \equiv [(p \leftrightarrow q) \wedge (q \leftrightarrow r) \wedge (r \leftrightarrow p)]$ and $v \equiv [(p \to q) \wedge (q \to r) \wedge (r \to p)]$

p	q	r	$[(p \leftrightarrow q)$	$(q \leftrightarrow r)$	$(r \leftrightarrow p)$	$[(p \leftrightarrow q) \wedge (q \leftrightarrow r) \wedge (r \leftrightarrow p)]$ (u)
0	0	0	1	1	1	1
0	0	1	1	0	0	0
0	1	0	0	0	1	0
0	1	1	0	1	0	0
1	0	0	0	1	1	0
1	0	1	0	0	0	0
1	1	0	1	0	0	0
1	1	1	1	1	1	1

p	q	r	$(p \to q)$	$(q \to r)$	$(r \to p)$	$[(p \to q) \wedge (q \to r) \wedge (r \to p)]$ (v)
0	0	0	1	1	1	1
0	0	1	1	1	0	0
0	1	0	1	0	1	0
0	1	1	1	1	0	0
1	0	0	0	1	1	0
1	0	1	0	1	1	0
1	1	0	1	0	1	0
1	1	1	1	1	1	1

Since,u and v have the same truth values for all possible combinations, $u \Leftrightarrow v$.

Therefore, $[(p \leftrightarrow q) \wedge (q \leftrightarrow r) \wedge (r \leftrightarrow p)] \Leftrightarrow [(p \to q) \wedge (q \to r) \wedge (r \to p)]$

3. By constructing truth tables prove that $[(p \vee q) \rightarrow r] \Leftrightarrow [(p \rightarrow r) \wedge (q \rightarrow r)]$

<u>Proof</u>:

Let $u \equiv (p \vee q) \rightarrow r$ and $v \equiv (p \rightarrow r) \wedge (q \rightarrow r)$

p	q	r	$p \vee q$	$(p \vee q) \rightarrow r$ (u)
0	0	0	0	1
0	0	1	0	1
0	1	0	1	0
0	1	1	1	1
1	0	0	1	0
1	0	1	1	1
1	1	0	1	0
1	1	1	1	1

p	q	r	$p \rightarrow r$	$q \rightarrow r$	$(p \rightarrow r) \wedge (q \rightarrow r)$ (v)
0	0	0	1	1	1
0	0	1	1	1	1
0	1	0	1	0	0
0	1	1	1	1	1
1	0	0	0	1	0
1	0	1	1	1	1
1	1	0	0	0	0
1	1	1	1	1	1

Since,u and v have the same truth values for all possible combinations, $u \Leftrightarrow v$.

Therefore, $[(p \vee q) \rightarrow r] \Leftrightarrow [(p \rightarrow r) \wedge (q \rightarrow r)]$

4. By constructing truth tables prove that $p \to (q \to r) \Leftrightarrow (p \wedge q) \to r$

Proof:

Let $u \equiv p \to (q \to r)$ and $v \equiv (p \wedge q) \to r$

p	q	r	$(q \to r)$	$p \to (q \to r)$ (u)
0	0	0	1	1
0	0	1	1	1
0	1	0	0	1
0	1	1	1	1
1	0	0	1	1
1	0	1	1	1
1	1	0	0	0
1	1	1	1	1

p	q	r	$p \wedge q$	$(p \wedge q) \to r$ (v)
0	0	0	0	1
0	0	1	0	1
0	1	0	0	1
0	1	1	0	1
1	0	0	0	1
1	0	1	0	1
1	1	0	1	0
1	1	1	1	1

Since,u and v have the same truth values for all possible combinations, $u \Leftrightarrow v$.

Therefore, $p \to (q \to r) \Leftrightarrow (p \wedge q) \to r$

5. By constructing truth tables prove that $p \to (q \vee r) \Leftrightarrow (p \wedge q) \to r$

Proof:

Let $u \equiv p \to (q \vee r)$ and $v \equiv (p \wedge \neg q) \to r$

p	q	r	$q \vee r$	$p \to (q \vee r)$ (u)
0	0	0	0	1
0	0	1	1	1
0	1	0	1	1
0	1	1	1	1
1	0	0	0	0
1	0	1	1	1
1	1	0	1	1
1	1	1	1	1

p	q	r	$\neg q$	$p \wedge \neg q$	$(p \wedge \neg q) \to r$ (v)
0	0	0	1	0	1
0	0	1	1	0	1
0	1	0	0	0	1
0	1	1	0	0	1
1	0	0	1	1	0
1	0	1	1	1	1
1	1	0	0	0	1
1	1	1	0	0	1

Since,u and v have the same truth values for all possible combinations, $u \Leftrightarrow v$.

Therefore, $p \to (q \vee r) \Leftrightarrow (p \wedge \neg q) \to r$

6. **Prove the following using the laws of logic:**

$$(\neg p \vee \neg q) \wedge (F_0 \vee p) \wedge p \Leftrightarrow (p \wedge \neg q)$$

<u>Proof</u>:

$(\neg p \vee \neg q) \wedge (F_0 \vee p) \wedge p$

$\Leftrightarrow (\neg p \vee \neg q) \wedge p \wedge p$ [By Identity law

$\Leftrightarrow (\neg p \vee \neg q) \wedge p$ [By Idempotent law

$\Leftrightarrow (\neg p \wedge p) \vee (\neg q \wedge p)$ [By distribution law

$\Leftrightarrow F_0 \vee (\neg q \wedge p)$ [By inverse law

$\Leftrightarrow (\neg q \wedge p)$ [By Identity law

$\Leftrightarrow (p \wedge \neg q)$ [By commutative law

7. **Prove the following using the laws of logic:**

$p \rightarrow [q \rightarrow (p \wedge q)]$ **is a tautology.**

<u>Proof</u>:

$p \rightarrow [q \rightarrow (p \wedge q)]$

$\Leftrightarrow p \rightarrow [\neg q \vee (p \wedge q)]$ [By the definition of conditional

$\Leftrightarrow \neg p \vee [\neg q \vee (p \wedge q)]$ [By the definition of conditional

$\Leftrightarrow (\neg p \vee \neg q) \vee (p \wedge q)]$ [By Associative law

$\Leftrightarrow \neg (p \wedge q) \vee (p \wedge q)]$ [By Demorgan's law

$\Leftrightarrow T_0$ [By inverse law

8. **Prove the following using the laws of logic:**

$$p \to (q \to r) \Leftrightarrow (p \wedge q) \to r$$

Proof:

$p \to (q \to r)$

$\Leftrightarrow p \to (\neg q \vee r)$ [By the definition of conditional

$\Leftrightarrow \neg p \vee (\neg q \vee r)$ [By the definition of conditional

$\Leftrightarrow (\neg p \vee \neg q) \vee r$ [By Associative law

$\Leftrightarrow \neg(p \wedge q) \vee r$ [By Demorgan's law

$\Leftrightarrow (p \wedge q) \to r$ [By the definition of conditional

9. **Prove the following using the laws of logic:**

$$[\neg p \wedge (\neg q \wedge r)] \vee [(q \wedge r) \vee (p \wedge r)] \Leftrightarrow r$$

Proof:

$[\neg p \wedge (\neg q \wedge r)] \vee [(q \wedge r) \vee (p \wedge r)]$

$\Leftrightarrow [(\neg p \wedge \neg q) \wedge r)] \vee [(q \wedge r) \vee (p \wedge r)]$ [By Associative law

$\Leftrightarrow [(\neg p \wedge \neg q) \wedge r)] \vee [(q \vee p) \wedge r]$ [By Distributive law

$\Leftrightarrow [(\neg(p \vee q) \wedge r)] \vee [(p \vee q) \wedge r)]$ [By Demorgan's law

$\Leftrightarrow [\neg(p \vee q) \vee (p \vee q)] \wedge r$ [By Distributive law

$\Leftrightarrow T_0 \wedge r$ [By inverse law

$\Leftrightarrow r$ [By Identity law

10. Prove the following using the laws of logic:

$\{(p \vee q) \wedge \neg[\neg p \wedge (\neg q \vee \neg r)]\} \vee \{(\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)\}$ **is** T_0

Proof:

$\{(p \vee q) \wedge \neg[\neg p \wedge (\neg q \vee \neg r)]\} \vee \{(\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)\}$

$\Leftrightarrow \{(p \vee q) \wedge \neg[\neg p \wedge \neg(q \wedge r)]\} \vee \{\neg(p \vee q) \vee \neg(p \vee r)\}$ [By Demorgan's law

$\Leftrightarrow \{(p \vee q) \wedge [p \vee (q \wedge r)]\} \vee \neg\{(p \vee q) \wedge (p \vee r)\}$ [By Demorgan's law

$\Leftrightarrow \{(p \vee q) \wedge (p \vee q) \wedge (p \vee r)\} \vee \neg\{(p \vee q) \wedge (p \vee r)\}$ [By Distributive law

$\Leftrightarrow \{(p \vee q) \wedge (p \vee r)\} \vee \neg\{(p \vee q) \wedge (p \vee r)\}$ [By Idempotent law

$\Leftrightarrow T_o$ [By inverse law

11. Prove the following using the laws of logic: $[(p \vee q) \wedge (p \vee \neg q)] \vee q \Leftrightarrow (p \vee q)$

Proof:

$[(p \vee q) \wedge (p \vee \neg q)] \vee q$

$\Leftrightarrow [(p \vee (q \wedge \neg q)] \vee q$ [By Distributive law

$\Leftrightarrow [p \vee F_0] \vee q$ [By inverse law

$\Leftrightarrow p \vee q$ [By Identity law

12. Prove the following using the laws of logic: $[\neg(p \wedge q)] \rightarrow [\neg p \vee (\neg p \vee q)] \Leftrightarrow \neg p \vee q$

Proof:

$[\neg(p \wedge q)] \rightarrow [\neg p \vee (\neg p \vee q)]$

$\Leftrightarrow (p \wedge q) \vee [\neg p \vee (\neg p \vee q)]$ [By the definition of conditional

$\Leftrightarrow (p \wedge q) \vee [(\neg p \vee \neg p) \vee q]$ [By Associative law

$\Leftrightarrow (p \wedge q) \vee [\neg p \vee q]$ [By Idempotent law

$\Leftrightarrow (p \wedge q) \vee [q \vee \neg p]$ [By commutative law

$\Leftrightarrow [(p \wedge q) \vee q] \vee \neg p$ [By Associative law

$\Leftrightarrow q \vee \neg p$ [By Absorption law law

$\Leftrightarrow \neg p \vee q$ [By commutative law

13. Prove the following using the laws of logic: $[(\neg p \vee \neg q) \to (p \wedge q \wedge r)] \Leftrightarrow p \wedge q$

Proof:

(a) $[(\neg p \vee \neg q) \to (p \wedge q \wedge r)]$

$\Leftrightarrow [\neg(\neg p \vee \neg q) \vee (p \wedge q \wedge r)]$ [By the definition of conditional]

$\Leftrightarrow [(p \wedge q) \vee ((p \wedge q) \wedge r)]$ [By Demorgan's law and associative law]

$\Leftrightarrow (p \wedge q)$ [By Absorbtion law]

14. Prove the following using the laws of logic: $(p \to q) \wedge [\neg q \wedge (r \vee \neg q)] \Leftrightarrow \neg(q \vee p)$

Proof:

$(p \to q) \wedge [\neg q \wedge (r \vee \neg q)]$

$\Leftrightarrow (p \to q) \wedge [\neg q \wedge (\neg q \vee r)]$ [By commutative law]

$\Leftrightarrow (p \to q) \wedge \neg q$ [By Absorbtion law]

$\Leftrightarrow (\neg p \vee q) \wedge \neg q$ [By the definition of conditional]

$\Leftrightarrow \neg(p \wedge \neg q) \wedge \neg q$ [By Demorgan's law]

$\Leftrightarrow \neg[(p \wedge \neg q) \vee q]$ [By Demorgan's law]

$\Leftrightarrow \neg[(p \wedge q) \vee (\neg q \wedge q)]$ [By distributive law]

$\Leftrightarrow \neg[(p \wedge q) \vee F_0]$ [By inverse law]

$\Leftrightarrow \neg(p \wedge q)$ [By identity law]

$\Leftrightarrow \neg(q \wedge p)$ [By commutative law]

15. Prove the following using the laws of logic:

$[(P \vee Q) \wedge \neg(\neg P \wedge (\neg Q \vee \neg R))] \vee (\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R)$ is a tautology.

Proof:

$[(P \vee Q) \wedge \neg(\neg P \wedge (\neg Q \vee \neg R))] \vee (\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R)$

$\Leftrightarrow [(P \vee Q) \wedge (P \vee (Q \wedge R))] \vee \neg(P \vee Q) \vee \neg(P \vee R)$ [By Demorgan's

$\Leftrightarrow [(P \vee Q) \wedge \{(P \vee Q) \wedge (P \vee R)\}] \vee \neg(P \vee Q) \vee \neg(P \vee R)$

[By distributive law]

$\Leftrightarrow [\{(P \vee Q) \wedge (P \vee Q)\} \wedge (P \vee R)\}] \vee \neg(P \vee Q) \vee \neg(P \vee R)$

[By Associative law]

$\Leftrightarrow [(P \vee Q) \wedge (P \vee R)] \vee \neg[(P \vee Q) \wedge (P \vee R)]$

[By idempotent law]

$\Leftrightarrow T_0$

16. Show that the compound propositions $p \wedge (\neg q \vee r)$ and $p \vee (q \wedge \neg r)$ are not logically equivalent.

p	q	r	$\neg q$	$\neg q \vee r$	$p \wedge (\neg q \vee r)$ (u)
0	0	0	1	1	0
0	0	1	1	1	0
0	1	0	0	0	0
0	1	1	0	1	0
1	0	0	1	1	1
1	0	1	1	1	1
1	1	0	0	0	0
1	1	1	0	1	1

p	q	r	$\neg q$	$p \wedge \neg q$	$(p \wedge \neg q) \to r$ (v)
0	0	0	1	0	1
0	0	1	1	0	1
0	1	0	0	0	1
0	1	1	0	0	1
1	0	0	1	1	0
1	0	1	1	1	1
1	1	0	0	0	1
1	1	1	0	0	1

u and v do not have the same truth values in all possible situations.

Therefore, $p \wedge (\neg q \vee r)$ and $p \vee (q \wedge \neg r)$ are not logically equivalent.

1.3 Duality, NAND and NOR, Converse, inverse and contra positive

Introduction:

- Suppose u is a compound proposition , dual of u is a compound proposition obtained by replacing(i) $\wedge\, by \vee$ (ii) $\vee\, by\ \wedge$ (iii) $T_0\ by\ F_0$ (iv) $F_0\ by\ T_0$ [Jan 09]

- (i) $(u^d)^d = u$ (ii) If $u \Leftrightarrow v$ then $u^d \Leftrightarrow v^d$ (Principle of duality)

- NAND is the combination of NOT and AND . NOR is the combination of NOT and OR.

Name	Symbol	Example	Explanation
NAND	$\uparrow$	$p \uparrow q$	$\neg(p \wedge q)$
NOR	$\downarrow$	p $\downarrow q$	$\neg(p \vee q)$

- Truth table for NAND and NOR: [Jan '09]

p	q	$p \vee q$	$p \wedge q$	p $\downarrow q$	$p \uparrow q$
0	0	0	0	1	1
0	1	1	0	0	1
1	0	1	0	0	1
1	1	1	1	0	0

- Converse, inverse and contra positive of the conditional $p \to q$: [July '09]

Converse	inverse	contra positive
$q \to p$	$\neg p \to \neg q$	$\neg q \to \neg p$

- Truth table for converse, inverse and contra positive:

p	q	$p \to q$	$q \to p$	$\neg p \to \neg q$	$\neg q \to \neg p$
0	0	1	1	1	1
0	1	1	0	0	1
1	0	0	1	1	0
1	1	1	1	1	1

- Laws of logic in conditional:

$p \to q \equiv \neg q \to \neg p$	$q \to p \equiv \neg p \to \neg q$

1. **Write the duals of the following propositions:**

   (i) $\boldsymbol{p \to q}$ **(ii)** $\boldsymbol{p \leftrightarrow r}$ **(iii)** $\boldsymbol{p \veebar q}$

(i) $u \equiv p \to q \equiv \neg p \vee q$
$u^d \equiv \neg p \wedge q$

(ii) $u \equiv p \leftrightarrow r \equiv (p \to r) \wedge (r \to p) \equiv (\neg p \vee q) \wedge (\neg r \vee p)$
$u^d \equiv (\neg p \wedge q) \vee (\neg r \wedge p)$

(iii) $u \equiv p \veebar q \equiv (\neg p \wedge q) \vee (p \wedge \neg q)$
$u^d \equiv (\neg p \vee q) \wedge (p \vee \neg q)$

2. **Write the duals of the following propositions:**
   (i) $(\boldsymbol{p} \vee \boldsymbol{T_0}) \wedge (\boldsymbol{q} \vee \boldsymbol{F_0}) \vee [(\boldsymbol{r} \wedge \boldsymbol{s}) \vee \boldsymbol{F_0}]$ **(ii)** $(\boldsymbol{p} \wedge \boldsymbol{q}) \wedge [(\neg \boldsymbol{p} \vee \boldsymbol{q}) \wedge (\neg \boldsymbol{r} \vee \boldsymbol{s})] \vee (\boldsymbol{r} \wedge \boldsymbol{s})$
   (iii) $\neg(\boldsymbol{p} \vee \boldsymbol{q}) \wedge [\boldsymbol{p} \vee \neg(\boldsymbol{q} \wedge \neg \boldsymbol{s})]$ **(iv)** $\boldsymbol{p} \to (\boldsymbol{q} \to \boldsymbol{r})$

(i) $u \equiv (p \vee T_0) \wedge (q \vee F_0) \vee [(r \wedge s) \vee F_0]$
$u^d \equiv (p \wedge F_0) \vee (q \wedge T_0) \wedge [(r \vee s) \wedge T_0]$

(ii) $u \equiv (p \wedge q) \wedge [(\neg p \vee q) \wedge (\neg r \vee s)] \vee (r \wedge s)$
$u^d \equiv (p \vee q) \vee [(\neg p \wedge q) \vee (\neg r \wedge s)] \wedge (r \wedge s)$

(iii) $u \equiv \neg(p \vee q) \wedge [p \vee \neg(q \wedge \neg s)]$
$u^d \equiv \neg(p \wedge q) \vee [p \wedge \neg(q \vee \neg s)]$

(iv) $u \equiv p \to (q \to r) \equiv \neg p \vee (q \to r) \equiv \neg p \vee (\neg q \vee r)$
$u^d \equiv \neg p \wedge (\neg q \wedge r)$

3. **Verify the principle of duality for the logical equivalence:**
   $[\neg(\boldsymbol{p} \wedge \boldsymbol{q}) \to \neg \boldsymbol{p} \vee (\neg \boldsymbol{p} \vee \boldsymbol{q})] \Leftrightarrow \neg \boldsymbol{p} \vee \boldsymbol{q}$

Principle of duality: If $u \Leftrightarrow v$ then $u^d \Leftrightarrow v^d$

If $u \Leftrightarrow v$ then

u^d	$\equiv [\neg(p \wedge q) \to \neg p \vee (\neg p \vee q)]^d$	[By data
	$\equiv [(p \wedge q) \vee \neg p \vee (\neg p \vee q)]^d$	[By the definition of conditional
	$\equiv [(p \wedge q) \vee (\neg p \vee \neg p) \vee q]^d$	[By Associative law
	$\equiv [(p \wedge q) \vee \neg p \vee q]^d$	[By idempotent law
	$\equiv [(p \wedge q) \vee q \vee \neg p]^d$	[By commutative law
	$\equiv [q \vee \neg p]^d$	[By Absorbtion law
	$\equiv [\neg p \vee q]^d$	[By Commutative law
	$\equiv v^d$	[By data

4. For any proposition p, q and r prove the following:

(i) $p \uparrow q \equiv q \uparrow p$ (ii) $p \downarrow q \equiv q \downarrow p$ (iii) $\neg(p \uparrow q) \equiv \neg q \downarrow \neg p$

Proof:

(i) $p \uparrow q$	$\equiv \neg(p \wedge q)$	[By the definition of NAND
	$\equiv \neg(q \wedge p)$	[By commutative law
	$\equiv q \uparrow p$	[By the definition of NAND

(ii) $p \downarrow q$	$\equiv \neg(p \vee q)$	[By the definition of NOR
	$\equiv \neg(q \vee p)$	[By commutative law
	$\equiv q \downarrow p$	[By the definition of NOR

(iii) $\neg(p \uparrow q)$	$\equiv \neg\neg(p \wedge q)$	[By the definition of NAND
	$\equiv \neg(\neg p \vee \neg q)$	[By Demorgan's law
	$\equiv \neg(\neg q \vee \neg p)$	[By commutative law
	$\equiv \neg q \downarrow \neg p$	[By the definition of NOR

5. For any proposition p, q and r prove the following:

(i) $\neg(p \downarrow q) \equiv \neg q \uparrow \neg p$ (ii) $p \uparrow (q \uparrow r) \equiv \neg p \vee (q \wedge r)$
(iii) $\neg(p \uparrow q) \equiv \neg p \downarrow \neg q$ (iv) $p \downarrow (q \downarrow r) \equiv \neg p \wedge (q \vee r)$

Proof:

(i) $\neg(p \downarrow q)$	$\equiv \neg\neg(p \vee q)$	[By the definition of NOR
	$\equiv \neg(\neg p \wedge \neg q)$	[By Demorgan's law
	$\equiv \neg q \uparrow \neg p$	[By the definition of NAND

(ii) $p \uparrow (q \uparrow r)$	$\equiv \neg(p \wedge (q \uparrow r))$	[By the definition of NAND
	$\equiv \neg(p \wedge \neg(q \wedge r))$	[By the definition of NAND
	$\equiv \neg p \vee (q \wedge r)$	[By Demorgan's law

(iii) $\neg(p \uparrow q)$	$\equiv \neg\neg(p \wedge q)$	[By the definition of NAND
	$\equiv \neg(\neg p \vee \neg q)$	[By Demorgan's law
	$\equiv \neg p \downarrow \neg q$	[By the definition of NOR

(iv) $p \downarrow (q \downarrow r)$	$\equiv \neg(p \vee (q \uparrow r))$	[By the definition of NAND
	$\equiv \neg(p \vee \neg(q \vee r))$	[By the definition of NAND
	$\equiv \neg p \wedge (q \vee r)$	[By Demorgan's law

17. Represent $p \vee q, p \wedge q$ and $p \rightarrow q$ using only $\downarrow$.

$$
\begin{aligned}
p \vee q \quad &\Leftrightarrow \neg\neg(p \vee q) \\
&\Leftrightarrow \neg(\neg p \wedge \neg q) \\
&\Leftrightarrow \neg(p \downarrow q) \\
&\Leftrightarrow \neg(p \downarrow q) \wedge \neg(p \downarrow q) \\
&\Leftrightarrow (p \downarrow q) \downarrow (p \downarrow q)
\end{aligned}
$$

$$
\begin{aligned}
p \wedge q \quad &\Leftrightarrow \neg\neg(p \wedge q) \\
&\Leftrightarrow \neg(\neg p \vee \neg q) \\
&\Leftrightarrow \neg(\neg p) \wedge \neg(\neg q) \\
&\Leftrightarrow (\neg p) \downarrow (\neg q) \\
&\Leftrightarrow (\neg p \wedge \neg p) \downarrow (\neg q \wedge \neg q) \\
&\Leftrightarrow (p \downarrow p) \downarrow (q \downarrow q)
\end{aligned}
$$

$$
\begin{aligned}
p \rightarrow q \quad &\Leftrightarrow \neg p \vee q \\
&\Leftrightarrow \neg\neg(\neg p \vee q) \\
&\Leftrightarrow \neg(p \wedge \neg q) \\
&\Leftrightarrow \neg(\neg p \downarrow q) \\
&\Leftrightarrow \neg(\neg p \downarrow q) \wedge \neg(\neg p \downarrow q) \\
&\Leftrightarrow (\neg p \downarrow q) \downarrow (\neg p \downarrow q) \\
&\Leftrightarrow ((p \downarrow p) \downarrow q) \downarrow ((p \downarrow p) \downarrow q)
\end{aligned}
$$

18. Represent $p \vee q, p \wedge q$ and $p \rightarrow q$ using only $\uparrow$.

$$
\begin{aligned}
p \vee q &\Leftrightarrow \neg\neg(p \vee q) \\
&\Leftrightarrow \neg(\neg p \wedge \neg q) \\
&\Leftrightarrow \neg(\neg p) \vee \neg(\neg q) \\
&\Leftrightarrow (\neg p) \uparrow (\neg q) \\
&\Leftrightarrow (\neg p \vee \neg p) \uparrow (\neg q \vee \neg q) \\
&\Leftrightarrow (p \uparrow p) \uparrow (q \uparrow q)
\end{aligned}
$$

$$
\begin{aligned}
p \wedge q &\Leftrightarrow \neg\neg(p \wedge q) \\
&\Leftrightarrow \neg(\neg p \vee \neg q) \\
&\Leftrightarrow \neg(p \uparrow q) \\
&\Leftrightarrow \neg(p \uparrow q) \vee \neg(p \uparrow q) \\
&\Leftrightarrow (p \uparrow q) \uparrow (p \uparrow q)
\end{aligned}
$$

$$
\begin{aligned}
p \rightarrow q &\Leftrightarrow \neg p \vee q \\
&\Leftrightarrow \neg p \vee \neg\neg q \\
&\Leftrightarrow p \uparrow (\neg q) \\
&\Leftrightarrow p \uparrow \neg(q \wedge q) \\
&\Leftrightarrow p \uparrow (q \uparrow q)
\end{aligned}
$$

19. Write the converse, inverse and contra positive of the conditional:

If a quadrilateral is a parallelogram, then its diagonals bisect each other.

Let p: Quadrilateral is a parallelogram and q:Diagonals of the quadrilateral bisect each other. The given conditional is $p \rightarrow q$.

<u>Converse</u>: $(q \rightarrow p)$ If the diagonals of the quadrilateral bisect each other, then it is a parallelogram.

<u>Inverse</u>: $(\neg p \rightarrow \neg q)$ If a quadrilateral is not a parallelogram, then its diagonals do not bisect each other.

<u>Contra positive</u>: $(\neg q \rightarrow \neg p)$ If the diagonals of the quadrilateral do not bisect each other, then it is not a parallelogram.

20. Write the converse, inverse and contra positive of the conditional:

If a real number x^2 is greater than zero, then x is not equal to zero.

Let p: A real number x^2 is greater than zero and q: x is not equal to zero

The given conditional is $p \to q$.

Converse: $(q \to p)$ If a real number$x$ is not equal to zero, then x^2 is greater than zero.

Inverse: $(\neg p \to \neg q)$ If a real number x^2 is not greater than zero, then x is equal to zero.

Contra positive: $(\neg q \to \neg p)$ If a real number$x$ is equal to zero, then x^2 is not greater than zero.

21. Write the converse, inverse and contra positive of the conditional:
If a triangle is not isosceles, then it is not equilateral.

Let p: $(q \to p)$ A triangle is not isosceles and q:Triangle is not equilateral

The given conditional is $p \to q$.

Converse: $(q \to p)$ If a triangle is not equilateral, then it is not isosceles.

Inverse: $(\neg p \to \neg q)$ If a triangle is isosceles, then it is equilateral.

Contra positive: $(\neg q \to \neg p)$ If a triangle is equilateral, then it is isosceles.

22. Write the converse, inverse and contra positive of the conditional:
If Ram can solve the puzzle, then Ram can solve the problem.

Let p : Ram can solve the puzzle. q : Ram can solve the problem.

The given conditional is $p \to q$.

Converse: $(q \to p)$ If Ram can solve the problem, then he can solve the puzzle.

Inverse: $(\neg p \to \neg q)$ If Ram cannot solve the puzzle, then he cannot solve theproblem.

Contra positive: $(\neg q \to \neg p)$ If Ram cannot solve the problem, then he cannot solve the puzzle.

1.4 Rules of inference

Introduction:

There exist rules of logic which can be employed for establishing validity of arguments. These rules are called rules of inference.

Modus ponens	Modus Tollens	Rule of syllogism	Rule of disjunctive syllogism	Disjunctive amplification	Conjunctive simplification
p $p \to q$ ——— $\therefore q$	$p \to q$ $\neg q$ ——— $\therefore \neg p$	$p \to q$ $q \to r$ ——— $\therefore p \to r$	$p \vee q$ $\neg q$ ——— $\therefore p$	p ——— $\therefore p \vee q$	$p \wedge q$ ——— $\therefore p$

$p \vee q \equiv \neg p \to q$

1. **Test the validity for the following:**
   If there is a strike by students then exam will be postponed. Exam was not postponed. Therefore there were no strike by students.

   Let p: There is a strike by students and$q$:exam is postponed
   Given premises are $p \to q$, $\neg q$

S No.	Steps used	Step	Rule
1	---	$p \to q$	Premise 1
2	---	$\neg q$	Premise 2
3	1, 2	$\neg p$	Modus Tollens

Therefore, there were no strike by students. Therefore, the given argument is valid.

2. **Test the validity for the following:**
   If Sachin hits a century then he gets a free car. Sachin gets a free car. Therefore Sachin has hit a century.

   Let p: Sachin hits a century and q: Sachin gets a free car
   Given premises are $p \to q$, q

S No.	Steps used	Step	Rule
1	---	$p \to q$	Premise 1
2	---	q	Premise 2
3	1, 2	p need not be true	

$\therefore p$ need not be true.

Therefore Sachin may not hit a century.Therefore, the given argument is invalid.

3. **Test the validity for the following:**
   If I drive to work, then I will arrive tired. I am not tired. Therefore I do not drive to work.

Let p: I drive to work and q: I will arrive tired
Given premises are $p \to q,\ \neg q$

S No.	Steps used	Step	Rule
1	---	$p \to q$	Premise 1
2	---	$\neg q$	Premise 2
3	1, 2	$\neg p$	Modus Tollens

Therefore I do not drive to work. Therefore, the given argument is valid.

4. **Test the validity for the following:**
   If interest rate falls then stock market will rise. The stock market will not rise. Therefore the interest rates will not fall.

Let p: interest rate falls and q: stock market will rise. Given premises are $p \to q,\ \neg q$.

S No.	Steps used	Step	Rule
1	---	$p \to q$	Premise 1
2	---	$\neg q$	Premise 2
3	1, 2	$\neg p$	Modus Tollens

Therefore the interest rates will not fall. Therefore, the given argument is valid.

5. **Test the validity for the following:**
If Ravi studies, then he will pass in DMS. If Ravi does not play cricket, then he will study. Ravi failed in DMS. Therefore, Ravi played cricket.

Let p: Ravi studies, q:He will pass in DMS and r:Ravi plays cricket
Given premises are $p \to q,\ \neg r \to p,\ \neg q$.

S No.	Steps used	Step	Rule
1	---	$p \to q$	Premise 1
2	---	$\neg q$	Premise 3
3	1, 2	$\neg p$	Modus Tollens
4	---	$\neg r \to p$	Premise 2
5	3, 4	r	Modus Tollens

Therefore Ravi played cricket. Therefore, the argument is valid.

6. **Test the validity for the following:**
If I study then I will not fail in exam. If I do not watch TV in the evening then I will study. I failed in exam. Therefore I must have watched TV in the evening.

Let p: I study, q: I fail in exam. and r: I watch TV in the evening.
Given premises are $p \to \neg q,\ \ \neg r \to p,\ \ q$

S No.	Steps used	Step	Rule
1	---	$p \to \neg q$	Premise 1
2	---	q	Premise 2
3	1, 2	$\neg p$	Modus Tollens
4	---	$\neg r \to p$	Premise 3
5	3, 4	r	Modus Tollens

Therefore I watch TV in the evening. Therefore, the given argument is valid.

7. Test the validity for the following:
If Rochelle gets the supervisor's position and work hard , then she will get a rise in her payment. If she gets a rise, then she will buy a car. She has not purchased the car. Therefore either Rochelle did not get the supervisor's position or she did not work hard.

Let p: Rochelle gets the supervisor's position,q:Rochelle works hard

r:Rochelle gets a rise in her payment,s:Rochelle purchased the car.

Given premises are $(p \wedge q) \rightarrow r,\ r \rightarrow s,\ \neg s$

S No.	Steps used	Step	Rule
1	---	$r \rightarrow s$	Premise 2
2	---	$\neg s$	Premise 3
3	1, 2	$\neg r$	Modus Tollens
4	---	$(p \wedge q) \rightarrow r$	Premise 1
5	3, 4	$\neg(p \wedge q)$	Modus Tollens
6	5	$\neg p \vee \neg q$	Demorgen's law

Therefore either Rochelle did not get the supervisor's position or she did not work hard.

Therefore, the given argument is valid.

8. Test the validity for the following:

Let p, q and r be the primitive statements. p: Ram studies q: Ram plays tennis, r: Ram passes in DMS. Let H_1, H_2, H_3 be the premises. H_1: If Ram studies then he will pass in DMS. H_2: If Ram does not play Tennis then he will study. H_3: Ram did not pass in DMS. Show that q follows from H_1, H_2, H_3.

Let p: Ram studies q: Ram plays tennis, r: Ram passes in DMS

Given premises are $H_1: p \to r,\ H_2: \neg q \to p, H_3: \neg r$

S No.	Steps used	Step	Rule
1	---	$p \to r$	Premise H_1
2	---	$\neg r$	Premise H_3
3	1, 2	$\neg p$	Modus Tollens
4	---	$\neg q \to p$	Premise H_2
5	4,3	q	Modus Tollens

Therefore, q follows from H_1, H_2 and H_3.

9. Test the validity for the following:

I will get grade A in this course or I will not graduate.

If I do not graduate, I will join army.

I got grade A.

Therefore, I will not join the army.

Is this valid argument? Prove using rules of inference.

p: I get grade A in this course.

q: I am not graduate.

r: I will join army.

Given premises are $p \vee q$, $q \to r$, p

S No.	Steps used	Step	Rule
1	---	$p \vee q$	Premise 1
2	---	$\neg p \to q$	Conditional law
3	1	$q \to r$	Premise 2
4	1, 2	$\neg p \to r$	Rule of syllogism
5	---	p	Premise 3

$\therefore \neg r$ need not be true.

Therefore, this argument is not valid.

10. Test the validity of the following argument:

I will become famous or I will not become a musician.

I will become a musician. Therefore, I will become famous.

Let p: I will become famous

q: I will become a musician.

Given premises are $p \vee \neg q$, q

S No.	Steps used	Step	Rule
1	---	$p \vee \neg q$	Premise 1
2	---	q	Premise 2
3	1, 2	p	Disjunctive syllogism

Therefore, I will become a musician. Therefore, this argument is valid.

11. Test the validity of the following: $\boldsymbol{p \to q}$, $\boldsymbol{\neg r \vee s}$, $\boldsymbol{p \vee r}$ $\boldsymbol{\therefore \neg q \to s}$

Given premises are $p \to q$, $\neg r \vee s$, $p \vee r$

S No.	Steps used	Step	Rule
1	---	$p \to q$	Premise 1
2	---	$p \vee r$	Premise 2
3	---	$\neg r \vee s$	Premise 3
4	1	$\neg q \to \neg p$	Contra positive
5	2	$\neg p \to r$	Conditional law
6	3	$r \to s$	Conditional law
7	1,2, 3	$\neg q \to s$	Rule of syllogism

Therefore, this argument is valid.

12. Test the validity of the following: $p \vee q, \quad \neg p \vee r, \quad \neg r \quad \therefore q$

S No.	Steps used	Step	Rule
1	---	$\neg p \vee r$	Premise 2
2	---	$\neg r$	Premise 3
3	1, 2	$\neg p$	Disjunctive syllogism
4	---	$p \vee q$	Premise 1
5	3, 4	q	Disjunctive syllogism

Therefore, this argument is valid.

13. Test the validity of the following: $p \wedge \neg q, \quad p \to (q \to r) \quad \therefore \neg r$

S No.	Steps used	Step	Rule
1	---	$p \wedge \neg q$	Premise 1
2	---	$p \to (q \to r)$	Premise 2
3	1	p	Conjunctive simplification
4	2, 3	$q \to r$	Modus pones
5	1	$\neg q$	Conjunctive simplification
6	4, 5	r or $\neg r$	---

Therefore, this argument is invalid.

14. Test the validity of the following: $p, p \to q, \quad s \vee r, \quad r \to \neg q \quad \therefore s \vee t$

S No.	Steps used	Step	Rule
1	---	p	Premise p_1
2	---	$p \to q$	Premise p_2
3	1, 2	q	Contra positive
4	---	$r \to \neg q$	Premise p_3
5	3, 4	$\neg r$	Conditional law
6	---	$s \vee r$	Disjunctive syllogism
7	5, 6	s	

Therefore, this argument is valid.

15. Test the validity of the following: $p \wedge q, \quad p \to (q \to r) \qquad \therefore r$

S No.	Steps used	Step	Rule
1	---	$p \wedge q$	Premise p_1
2	1	p	Conjunctive simplification
3	---	$p \to (q \to r)$	Premise p_2
4	2, 3	$q \to r$	Modus ponens
5	1	q	Conjunctive simplification
6	4, 5	r	Modus ponens

Therefore, this argument is valid.

16. Test the validity of the following: $\boldsymbol{p \to r}$, $\boldsymbol{q \to r}$ $\quad \therefore (\boldsymbol{p \vee q}) \to \boldsymbol{r}$

S No.	Steps used	Step	Rule
1	---	$p \to r$	Premise p_1
2	1	$\neg p \vee r$	Conditional law
3	---	$q \to r$	Premise p_2
4	3	$\neg q \vee r$	Conditional law
5	2, 4	$(\neg p \wedge \neg q) \vee r$	Distributive law
6	5	$\neg(p \vee q) \vee r$	Demorgen's law
7	6	$(p \vee q) \to r$	Conditional law

Therefore, this argument is valid.

17. Test the validity of the following: $\boldsymbol{p}$, $\boldsymbol{p \to r}$, $\boldsymbol{p} \to (\boldsymbol{q} \vee \neg \boldsymbol{r})$, $\neg \boldsymbol{q} \vee \neg \boldsymbol{s}$ $\quad \therefore \boldsymbol{s}$

S No.	Steps used	Step	Rule
1	---	p	Premise p_1
2	---	$p \to r$	Premise p_2
3	---	$p \to (q \vee \neg r)$	Premise p_3
4	1, 2	r	Modus ponens
5	1, 3	$q \vee \neg r$	Modus ponens
6	4, 5	q	Disjunctive syllogism
7	---	$\neg q \vee \neg s$	Conditional law
8	6, 7	$\neg s$	Disjunctive syllogism

Therefore, this argument is invalid.

18. Test the validity of the following: $\boldsymbol{p \leftrightarrow q}$, $\boldsymbol{q \rightarrow r}$, $\boldsymbol{r \vee \neg s}$, $\boldsymbol{\neg s \rightarrow q}$ $\boldsymbol{\therefore s}$

S No.	Steps used	Step	Rule
1	---	$r \vee \neg s$	Premise p_3
2	1	$\neg r \rightarrow \neg s$	Conditional law
3	---	$\neg s \rightarrow q$	Premise p_4
4	---	$q \rightarrow r$	Premise p_2
5	2, 3, 4	$\neg r \rightarrow r$	Rule of syllogism
6	5	$r \vee r$	Conditional law
7	5	r	Disjunctive simplification
8	1, 7	s need not be true	Contra positive

Therefore, this argument is invalid.

19. Test the validity of the following:
$C \vee D, (C \vee D) \rightarrow \neg H,\ \neg H \rightarrow (A \wedge \neg B)$ and $(A \wedge \neg B) \rightarrow R \vee S.\ \therefore R \vee S$

S No.	Steps used	Step	Rule
1	---	$(C \vee D) \rightarrow \neg H$	Premise p_2
2	---	$\neg H \rightarrow (A \wedge \neg B)$	Premise p_3
3	---	$(A \wedge \neg B) \rightarrow (R \vee S)$	Premise p_4
4	1, 2, 3	$(C \vee D) \rightarrow (R \vee S)$	Rule of syllogism
5	---	$C \vee D$	Premise p_1
6	4, 5	$R \vee S$	Modus ponnes

Therefore, this argument is valid.

1.5 The use of quantifiers

Introduction:

- A declarative sentence is an open statement if it contains one or more variables.
- Open statements are not propositions unless the variables are not specified.
- Some propositions containing words like 'all' , 'for every' , 'for any' , 'for each' 'some' , 'there exists' , 'at least one' . These words are called quantifiers.
- Examples for existential quantifier ($\exists$) are some, there exists, at least one, at least.
- Examples for Universal quantifier ($\forall$) are for, for each, for every, for any.
- $\neg[\forall x, p(x)] = \exists x, \neg p(x)$ and $\neg[\exists x, p(x)] = \forall x, \neg p(x)$

	Universal	Existential
Specification	$\forall x \in S, p(x) \Rightarrow p(a), a \in S$	$\exists x \in S, p(x) \Rightarrow p(a), a \in S$
Generalisation	$p(a), a \in S \Rightarrow \forall x \in S, p(x)$	$p(a), a \in S \Rightarrow \exists x \in S, p(x)$

Logical equivalence:

$\forall x, [p(x) \wedge q(x)] \equiv \forall x\, p(x) \wedge \forall x\, q(x)$	$\forall x, [p(x) \vee q(x)] \equiv \forall x\, p(x) \vee \forall x\, q(x)$
$\exists x, [p(x) \wedge q(x)] \equiv \exists x\, p(x) \wedge \exists x\, q(x)$	$\exists x, [p(x) \vee q(x)] \equiv \exists x\, p(x) \vee \exists x\, q(x)$

1. **For the universe of all integers let $p(x)$: $x > 0$, $q(x)$: x is even, $r(x)$: x is a perfect square, $s(x)$: x is divisible by 3, $t(x)$: x is divisible by 7.Write the following statements in symbolic form:**

   (a) At least one integer is even, (b) There exists a positive integer that is even.

   (c) Some even integers are divisible by 3. (d) Every integer is either even or odd.

   (e) If x is even and a perfect square then x is not divisible by 3.

   (f) If x is odd or is not divisible by 7 then x is divisible by 3.

   (a) $\exists x, q(x)$

   (b) $\forall x, \ [p(x) \wedge q(x)]$

   (c) $\exists x, [q(x) \wedge s(x)]$

   (d) $\forall x, [q(x) \vee \neg q(x)]$

   (e) $\forall x, [q(x) \wedge r(x)] \rightarrow \neg s(x)$

   (f) $\forall x, [q(x) \wedge t(x)] \rightarrow s(x)$

2. **Consider the universe of all polygons with 3 or 4 sides and define the following open statements for this universe. a(x): All interior angles of x are equal, e(x): x is an equilateral triangle, i(x): x is an isosceles triangle, p(x): x has an interior angle that exceeds 180^0, q(x): x is a quadrilateral, r(x): x is a rectangle, s(x): x is a square, t(x): x is a triangle.(a)** $\forall \boldsymbol{x}, [\boldsymbol{q}(\boldsymbol{x}) \vee \boldsymbol{t}(\boldsymbol{x})]$ **(b)** $\forall \boldsymbol{x}, [\boldsymbol{i}(\boldsymbol{x}) \vee \boldsymbol{e}(\boldsymbol{x})]$ **(c)** $\exists \boldsymbol{x}, [\boldsymbol{t}(\boldsymbol{x}) \wedge \boldsymbol{p}(\boldsymbol{x})]$ **(d)** $\exists \boldsymbol{x}, [\boldsymbol{q}(\boldsymbol{x}) \wedge \neg$**r(x)]** **(e)** $\forall \boldsymbol{x}, \{[\boldsymbol{a}(\boldsymbol{x}) \wedge \boldsymbol{t}(\boldsymbol{x})] \leftrightarrow \boldsymbol{e}(\boldsymbol{x})\}$ **(f)** $\forall \boldsymbol{x}, \boldsymbol{t}(\boldsymbol{x}) \rightarrow \neg \boldsymbol{p}(\boldsymbol{x})$

   (a) For any x, x is a quadrilateral or a triangle.

   (b) For any x, x is an isosceles triangle or an equilateral triangle.

   (c) For some x, x is a triangle and a quadrilateral.

   (d) For some x, x is a quadrilateral and not a rectangle.

   (e) For any x, All interior angles of x are equal and is a triangle if and only if x is an equilateral triangle.

   (f) For any x, if x is a triangle then x does not have an interior angle that exceeds 180^0.

3. Consider the following statements with a set of all real numbers as the universe. $p(x): x \geq 0, q(x): x^2 \geq 0, r(x): x^2 - 3x - 4 = 0, s(x): x^2 - 3 > 0$. Determine the truth values of the following: (a) $\exists x, p(x) \wedge q(x)$ (b) $\forall x, p(x) \to q(x)$ (c) $\forall x, q(x) \to s(x)$ (d) $\forall x, r(x) \vee s(x)$(e) $\exists x, p(x) \wedge r(x)$.

(a) For $x = 1$, $p(x): x \geq 0$ *and* $q(x): x^2 \geq 0$ are true.
Therefore, $\exists x, p(x) \wedge q(x)$ is true.

(b) $q(x): x^2 \geq 0$ cannot be false for any real number x.
Therefore,$p(x) \to q(x)$ cannot be false for any real number x.
Therefore, $\forall x, p(x) \to q(x)$ is true.

(c) For $x = 1, q(x): x^2 \geq 0$ is true but $s(x): x^2 - 3 > 0$is false.
Therefore, $\forall x, q(x) \to s(x)$ is false.

(d) For $x = 1, r(x): x^2 - 3x - 4 = 0$ *and* $s(x): x^2 - 3 > 0$ are false.
Therefore, $\forall x, r(x) \vee s(x)$is false.

(e) For $x = 4, p(x): x \geq 0$ is true and$r(x): x^2 - 3x - 4 = 0$ is true.
Therefore, $\exists x, p(x) \wedge r(x)$ is true.

4. Negate and simplify each of the following:
(a)$\exists x, p(x) \vee q(x)$**(b)** $\forall x, p(x) \to q(x)$
(c)$\forall x, p(x) \to \neg q(x)$**(d)** $\exists x, [(p(x) \vee q(x)) \to r(x)]$

(a) $u \equiv \exists x, p(x) \vee q(x)$
$$\neg u \equiv \forall x, \neg p(x) \wedge \neg q(x)$$

(b) $u \equiv \forall x, p(x) \to q(x) \equiv \forall x, \neg p(x) \vee q(x)$
$$\neg u \equiv \exists x, p(x) \wedge \neg q(x)$$

(c) $u \equiv \forall x, p(x) \to \neg q(x) \equiv \forall x, \neg p(x) \vee \neg q(x)$
$$\neg u \equiv \exists x, p(x) \wedge q(x)$$

(d) $u \equiv \exists x, [(p(x) \vee q(x)) \to r(x)] \equiv \exists x, [\neg(p(x) \vee q(x)) \vee r(x)]$
$$\neg u \equiv \forall x, [(p(x) \vee q(x)) \wedge \neg r(x)]$$

5. **Write down the following propositions in symbolic form and find their negation:**

(a) For all integers, if n is not divisible by 2, then n is odd.

(b) If l, m, n are any integers where $l - m$ and $m - n$ are odd then $l - n$ is even .

(c) If x is a real number where x^2 is greater than 16 then x is less than -4 or x is more than 4.

(a) Let $p(n)$: n is divisible by 2 and $q(n)$: n is odd

$u \equiv \text{For all integers, if n is not divisible by 2, then n is odd.}$

$\equiv \forall\, n, \neg p(n) \rightarrow q(n)$

$\equiv \forall n, p(n) \vee q(n)$

$\neg u \equiv \exists n, \neg p(n) \wedge \neg q(n)$

$\equiv$ Some integers are neither divisible by 2 nor odd.

(b) Let $p(x)$: $l - m$ is odd , $q(x)$: $m - n$ is odd and $r(x)$: $l - n$ is odd.

u $\equiv$For any integers ,if $l - m$ and $m - n$ are odd then $l - n$ is even

$\equiv \forall x, [p(x) \wedge q(x)] \rightarrow r(x)$

$\equiv \forall x, \neg[p(x) \wedge q(x)] \vee r(x)$

$\neg u \equiv \exists x, \quad [p(x) \wedge q(x)] \wedge \neg r(x)$

$\equiv$ For some $l, m, n, l - m$ and $m - n$ are odd and $l - n$ is also odd.

(c) Let $p(x)$: $x^2 > 16$, $q(x)$: $x < -4$, $r(x)$: $x > 4$

u $\equiv$If x is a real number where $x^2 > 16$ then x < -4 or x > 4.

$\equiv \forall x \in R, p(x) \rightarrow [q(x) \vee r(x)]$

$\equiv \forall x \in R, \neg p(x) \vee [q(x) \vee r(x)]$

$\neg u \equiv \exists x \in R, p(x) \wedge \neg[q(x) \vee r(x)]$

$\equiv$For some real number x, $x^2 > 16$, $x \geq -4$ *and* $x \leq 4$.

6. Write down the following propositions in symbolic form and find their negation:

(a) All rational numbers are real and some real numbers are not rational.

(b) No real number is greater than its square.

(c) All integers are rational numbers and some rational numbers are not integers

(d) Some straight lines are parallel or all straight lines intersect.

(a) Let $p(x)$: x is real and $q(x)$: x is rational

$u \equiv$ All rational numbers are real and some real numbers are not rational

$\equiv \forall x \in Q, p(x) \wedge \exists x \in R, \neg q(x)$

$\neg u \equiv \exists x \in Q, \neg\, p(x) \wedge \forall\, x \in R, q(x)$

$\equiv$ Some rational numbers are not real or all real numbers are rational.

(b) Let $p(x)$: $x > x^2$

$u \equiv$ No real number is greater than its square.

$\equiv \forall x \in R, \neg p(x)$

$\neg u \equiv \exists x \in R, p(x)$

$\equiv$ Some real numbers are greater than its square.

(c) Let $p(x)$: x is rational number and $q(x)$: x is integer.

$u \equiv$ All integers are rational numbers and some rational numbers are not integers.

$\equiv \forall x \in Z, p(x) \wedge \exists x \in Q, \neg q(x)$

$\neg u \equiv \exists x \in Z, \neg\, p(x) \vee \forall\, x \in Q, q(x)$

$\equiv$ Some integers are not rational numbers or every rational number is an integer.

(d) Let $p(x)$: x is parallel and $q(x)$: x intersect.

$u \equiv$ Some straight lines are parallel or all straight lines intersect.

$\equiv \exists x \in L, p(x) \vee \forall x \in L, q(x)$

$\neg u \equiv \forall x \in L, \neg\, p(x) \wedge \exists\, x \in L, \neg q(x)$

$\equiv$ All straight lines are not parallel and some straight lines do not intersect.

7. Prove the following logical equivalences:

(a) $\exists \boldsymbol{x}, \boldsymbol{p}(\boldsymbol{x}) \to \boldsymbol{q}(\boldsymbol{x}) \equiv \forall \boldsymbol{x}, \boldsymbol{p}(\boldsymbol{x}) \to \exists \boldsymbol{x}, \boldsymbol{q}(\boldsymbol{x})$

(b) $\exists \boldsymbol{x}, \boldsymbol{p}(\boldsymbol{x}) \to \forall \boldsymbol{x}, \boldsymbol{q}(\boldsymbol{x}) \equiv \forall \boldsymbol{x}, [\boldsymbol{p}(\boldsymbol{x}) \to \boldsymbol{q}(\boldsymbol{x})]$

(c) $\neg[\exists \boldsymbol{x}, \neg \boldsymbol{p}(\boldsymbol{x})] \equiv \forall \boldsymbol{x}, \boldsymbol{p}(\boldsymbol{x})$

(d) $\forall \boldsymbol{x}, \{\boldsymbol{p}(\boldsymbol{x}) \wedge [\boldsymbol{q}(\boldsymbol{x}) \wedge \boldsymbol{r}(\boldsymbol{x})]\} \equiv \forall \boldsymbol{x}, [\{\boldsymbol{p}(\boldsymbol{x}) \wedge \boldsymbol{q}(\boldsymbol{x})\} \wedge \boldsymbol{r}(\boldsymbol{x})]$

(a) $\exists x, p(x) \to q(x)$

$\equiv \exists x, \neg p(x) \vee q(x)$

$\equiv \exists x, \neg p(x) \vee \exists x, q(x)$

$\equiv \neg \forall x, p(x) \vee \exists x, q(x)$

$\equiv \forall x, p(x) \to \exists x, q(x)$

(b) $\exists x, p(x) \to \forall x, q(x)$

$\equiv \neg \exists x, p(x) \vee \forall x, q(x)$

$\equiv \forall x, \neg p(x) \vee \forall x, q(x)$

$\equiv \forall x, [\neg p(x) \vee q(x)]$

$\equiv \forall x, [p(x) \to q(x)]$

(c) $\neg[\exists x, \neg p(x)]$

$\equiv \forall x, \neg\neg p(x)$

$\equiv \forall x, p(x)$

(d) $\forall x, \{p(x) \wedge [q(x) \wedge r(x)]\}$

$\equiv \forall x, p(x) \wedge \forall x, [q(x) \wedge r(x)]$

$\equiv \forall x, p(x) \wedge [\forall x, q(x) \wedge \forall x, r(x)]$

$\equiv [\forall x, p(x) \wedge \forall x, q(x)] \wedge \forall x, r(x)$

$\equiv \forall x, [\, p(x) \wedge q(x)] \wedge \forall x, r(x)$

$\equiv \forall x, [\{p(x) \wedge q(x)\} \wedge r(x)]$

8. Find whether the following arguments valid:

If a triangle has two equal sides, then it is isosceles.

If a triangle is isosceles, then it has two equal angles.

<u>A certain triangle ABC does not have two equal angles.</u>

∴ The triangle ABC does not have two equal sides.

Let $p(x): x$ has equal sides, $q(x):$ x is isosceles, $r(x):$ x has two equal angles.

Let c denote the triangle ABC.

By data, $\forall x,[p(x) \to q(x)]$

$\forall x,[q(x) \to r(x)]$

$\neg r(c)$

$\therefore \neg p(c)$

S No.	Steps used	Step	Rule
1	---	$\forall x,[p(x) \to q(x)]$	Premise 1
2	---	$\forall x,[q(x) \to r(x)]$	Premise 2
3	1	$p(c) \to q(c)$	Universal specification
4	2	$q(c) \to r(c)$	Universal specification
5	1, 2	$p(c) \to r(c)$	Rule of syllogism
6	---	$\neg r(c)$	Premise 3
7	5, 6	$\neg p(c)$	Modus Tollens

∴ The triangle ABC does not have two equal sides.

9. Find whether the following arguments valid:

No Engineering student of first or second semester studies logic.

<u>Anil is an Engineering student who studies Logic.</u>

$\therefore$ Anil is not in second semester.

$p(x):x$ is in first semester,

$q(x):$ x is in second semester,

$r(x):$ x studies logic

and c: Anil

By data, $\forall x,[p(x)\vee q(x)]\rightarrow\neg r(x)$

$r(c)$

$\therefore \neg q(c)$

S No.	Steps used	Step	Rule
1	---	$\forall x,[p(x)\vee q(x)]\rightarrow\neg r(x)$	Premise 1
2	1	$[p(c)\vee q(c)]\rightarrow\neg r(c)$	Universal specification
3	---	$r(c)$	Premise 2
4	2, 3	$\neg[p(c)\vee q(c)]$	Modus Tollens
5	4	$\neg p(c)\wedge\neg q(c)$	Demorgen's law
6	5	$\neg q(c)$	Conjunctive simplification

$\therefore$ Anil is not in second semester.

10. Find whether the following arguments valid:

All Mathematics professors have studied calculus.

<u>Ramanujan is a mathematics professor.</u>

$\therefore$ Ramanujan have studied calculus.

$p(x): x$ is a Mathematics professor

$q(x):$ x is studies calculus,

c : Ramanujan.

By data, $\forall x,[p(x) \to q(x)]$

$p(c)$

$\therefore q(c)$

S No.	Steps used	Step	Rule
1	---	$\forall x,[p(x) \to q(x)]$	Premise 1
2	1	$p(c) \to q(c)$	Universal specification
3	---	$p(c)$	Premise 2
4	2, 3	$q(c)$	Modus ponens

$\therefore$ Ramanujan have studied calculus.

11. Find whether the following arguments valid:

All employers pay their employees.

<u>Anil is an employer.</u>

$\therefore$ Anil pays his employees.

$p(x): x$ is an employer

$q(x):$ x pays his employee

c : Anil

By data, $\forall x,[p(x) \to q(x)]$

$p(a)$

$\therefore \neg q(c)$

Step	Steps used	Step	Rule
1	---	$\forall x,[p(x) \to q(x)]$	Premise 1
2	---	$\forall x,[q(x) \to r(x)]$	Premise 2
3	1	$p(c) \to q(c)$	Universal specification
4	2	$q(c) \to r(c)$	Universal specification
5	1, 2	$p(c) \to r(c)$	Rule of syllogism
6	---	$\neg r(c)$	Premise 3
7	5, 6	$\neg p(c)$	Modus Tollens

12. Prove that the following arguments are valid:

$$\forall x,[p(x) \to q(x)]$$
$$\forall x,[q(x) \to r(x)]$$
$$\overline{\qquad\qquad\qquad\qquad}$$
$$\therefore \forall x,[p(x) \to r(x)]$$

Step	Steps used	Step	Rule
1	---	$\forall x,[p(x) \to q(x)]$	Premise 1
2	---	$\forall x,[q(x) \to r(x)]$	Premise 2
3	1	$p(c) \to q(c)$	Universal specification
4	2	$q(c) \to r(c)$	Universal specification
5	1, 2	$p(c) \to r(c)$	Rule of syllogism
6	5	$\forall x,[p(x) \to r(x)]$	Universal generalisation

$$\therefore \forall x,[p(x) \to r(x)]$$

13. Prove that the following arguments are valid:

$$\forall x,\{p(x) \rightarrow [q(x) \wedge r(x)]\}$$
$$\forall x,[p(x) \wedge s(x)]$$
$$\overline{\hspace{5cm}}$$
$$\therefore \forall x,[r(x) \wedge s(x)]$$

Step	Steps used	Step	Rule
1	---	$\forall x,\{p(x) \rightarrow [q(x) \wedge r(x)]\}$	Premise 1
2	---	$\forall x,[p(x) \wedge s(x)]$	Premise 2
3	1	$p(c) \rightarrow [q(c) \wedge r(c)]$	Universal specification
4	2	$p(c) \wedge s(c)$	Universal specification
5	4	$p(c)$	Conjunctive simplification
6	3, 5	$q(c) \wedge r(c)$	Modus pones
7	6	$r(c)$	Conjunctive simplification
8	4	$s(c)$	Conjunctive simplification
9	7, 8	$r(c) \wedge s(c)$	----
10	9	$\therefore \forall x,[r(x) \wedge s(x)]$	Universal generalisation

14. Prove that the following arguments are valid:

$$\forall x,[p(x)\vee q(x)]$$
$$\forall x,[\{\neg p(x)\wedge q(x)\}\rightarrow r(x)]$$
$$\overline{}$$
$$\therefore \forall x,[\neg r(x)\rightarrow p(x)]$$

Step	Steps used	Step	Rule
1	---	$\forall x,[\{\neg p(x)\wedge q(x)\}\rightarrow r(x)]$	Premise 1
2	1	$\{\neg p(c)\wedge q(c)\}\rightarrow r(c)$	Universal specification
3	2	$\neg\{\neg p(c)\wedge q(c)\}\vee r(c)$	Conditional law
4	3	$\{p(c)\vee\neg q(c)\}\vee r(c)$	Demorgen's law
5	4	$\{p(c)\vee\{\neg q(c)\vee r(c)\}$	Associative law
6	---	$\{p(c)\vee\{q(c)\rightarrow r(c)\}$	Conditional law
7	6	$\forall x,[p(x)\vee q(x)]$	Premise 2
8	5, 7	$p(c)\vee q(c)$	Universal specification
9	6, 8	$p(c)\vee\{[q(c)\rightarrow r(c)]\wedge q(c)\}$	Distributive law
10	9	$p(c)\vee r(c)$	Modus ponens
11	10	$r(c)\vee p(c)$	Commutative law
12	11	$\neg r(c)\rightarrow p(c)$	Conditional law

15. Prove that the following arguments are valid:

$$\forall x,[p(x) \vee q(x)]$$
$$\exists x, \neg p(x)$$
$$\forall x,[\neg q(x) \vee r(x)]$$
$$\forall x,[s(x) \rightarrow \neg r(x)]$$

$$\exists x, \neg s(x)$$

Steps	Steps used	Step	Rule
1	---	$\forall x,[p(x) \vee q(x)]$	Premise 1
2	1	$p(c) \vee q(c)$	Universal specification
3	---	$\exists x, \neg p(x)$	Premise 2
4	3	$\neg p(c)$	Universal specification
5	2, 4	$q(c)$	Rule of syllogism
6	---	$\forall x,[\neg q(x) \vee r(x)]$	Premise 3
7	6	$\neg q(c) \vee r(c)$	Universal specification
8	5, 7	$r(c)$	Disjuctive syllogism
9	---	$\forall x,[s(x) \rightarrow \neg r(x)]$	Premise 3
10	9	$s(c) \rightarrow \neg r(c)$	Universal specification
11	8, 10	$\neg s(c)$	Rule of syllogism
12	11	$\forall x, \neg s(x)$	Universal generalisation

1.6 Proofs of theorems

- **Direct proof of conditional $p \rightarrow q$:**

  Assume p is true.
  Prove q is true using the laws of logic.
  Conclusion: $p \rightarrow q$ is true.

- **Indirect proof of conditional $p \rightarrow q$:**

  Assume $\neg q$ is true.
  Prove $\neg p$ is true using the laws of logic.
  Therefore, $\neg q \rightarrow \neg p$ is true.
  Conclusion: $p \rightarrow q$ is true.

  $$\boldsymbol{p \rightarrow q \equiv \neg q \rightarrow \neg p}$$

- **Proof of conditional $p \rightarrow q$ by contradiction:**

  Assume $p \rightarrow q$ is false. That is, assume p is true and q is false.
  By assuming q is false, prove p is false using the laws of logic.
  This contradicts to our assumption that p is true.
  Conclusion: $p \rightarrow q$ is true.

1. **Give a direct proof for the following statements:**
   (a) The sum of two odd integers is an even integer.
   (b) The square of an odd integer is an odd integer.
   (c) If an integer a is such that $a - 2$ is divisible by 3, then $a^2 - 1$ is divisible by 3.

(a) Let $p: x$ is a sum of two odd integers and $q: x$ is an even integer.

Assume p is true.
$\Rightarrow x$ is a sum of two odd integers $\Rightarrow x = (2m+1) + (2n+1), where\ m, n \in I$
$\Rightarrow x = 2m + 1 + 2n + 1$
$\Rightarrow x = 2m + 2n + 2$
$\Rightarrow x = 2(m + n + 1)$
$\Rightarrow x$ is an even integer
$\Rightarrow q$ is true.
$\therefore\ p \rightarrow q$ is true.
$\therefore$ The given statement is true by direct proof.

(b) Let $p: x$ is a square of an odd integer and $q: x$ is an odd integer.
Assume p is true.
$\Rightarrow x$ is a square of an odd integer
$\Rightarrow x = (2n+1)^2, where\ n \in I$
$\Rightarrow x = 4n^2 + 4n + 1$
$\Rightarrow x$ is an odd integer
$\Rightarrow q$ is true.
$\therefore\ p \rightarrow q$ is true.
$\therefore$ The given statement is true by direct proof.

(c) Let $p: a - 2$ is divisible by 3 and $q: a^2 - 1$ is divisible by 3

Assume p is true.
$\Rightarrow a - 2$ is divisible by 3
$\Rightarrow a - 2 = 3n, n \in I$
$\Rightarrow a = 3n + 2$
$\Rightarrow a^2 - 1 = (3n+2)^2 - 1$
$= (9n^2 + 12n + 4) - 1$
$= 9n^2 + 12n + 3$
$= 3(3n^2 + 4n + 1)$
$\Rightarrow a^2 - 1$ is divisible by 3.
$\Rightarrow q$ is true.

$\therefore\ p \rightarrow q$ is true. $\therefore$ The given statement is true by direct proof.

2. **Give a direct proof for the following statements:**

(a) For all positive integers m and n, if m and n are perfect squares, then mn is also a perfect square.

(b) For all integers k and l, if $k$and l are both even, then $k + l$ is even.

(a) Let p: m and n are perfect squares and q: mn is a perfect square.

Assume p is true.

$\Rightarrow m$ and n are perfect squares

$\Rightarrow m = a^2, n = b^2, where\ a, b \in I$

$\Rightarrow mn = a^2 b^2, where\ a, b \in I$

$\Rightarrow m = (ab)^2, where\ ab \in I$

$\Rightarrow mn$ is a perfect square.

$\Rightarrow q$ is true.

$\therefore\ p \rightarrow q$ is true.

$\therefore$ The given statement is true by direct proof.

(b) For all integers k and l, if $k$and l are both even, then $k + l$ is even.

Assume p is true.

$\Rightarrow k$ and l are both even

$\Rightarrow k = 2m, l = 2n, where\ m, n \in I$

$\Rightarrow k = 2m + 2n$

$\Rightarrow k = 2(m + n)$

$\Rightarrow k + l$ is even

$\Rightarrow q$ is true.

$\therefore\ p \rightarrow q$ is true.

$\therefore$ The given statement is true by direct proof.

3. **Give an indirect proof for the following statements:**
   (a) For any real number x, if $x^2 > 0$ then $x \neq 0$.
   (b) If m is an even integer then $m + 7$ is an odd integer.
   (c) If x and y are integers such that xy is odd, then x and $y$are both odd.

(a) Let for any real number x, $p: x^2 > 0$and $q: x \neq 0$.

Assume $\neg q$ is true.

$\Rightarrow x = 0, x \in R$

$\Rightarrow x^2 = 0, x \in R$

$\Rightarrow \neg p$ is true.

$\therefore$ $\neg q \rightarrow \neg p$ is true and hence $p \rightarrow q$ is true.

$\therefore$ The given statement is true by indirect proof.

(b) Let $p: m$ is an even integer and $q: m + 7$ is an odd integer.

Assume $\neg q$ is true.

$\Rightarrow m + 7$ is an even integer.

$\Rightarrow m + 7 = 2k, k \in I$.

$\Rightarrow m = 2k - 7, k \in I$

$= 2k - 8 + 1, k \in I$

$= 2(k - 4) + 1, k \in I$

$\Rightarrow m$ is an odd integer

$\Rightarrow \neg p$ is true.

$\therefore$ $\neg q \rightarrow \neg p$ is true and hence $p \rightarrow q$ is true.

$\therefore$ The given statement is true by indirect proof.

(c) Let if x and y are integers$p: xy$ is odd and $q: x$ and y are both odd.

Assume $\neg q$ is true.

$\Rightarrow x$ is even or y is even

$\Rightarrow x = 2\text{m}$ or$y = 2n,\ m, n \in I$

$\Rightarrow xy = 2my$ or $xy = x2n$.

$\Rightarrow xy = 2(my)$ or $2(xn)$

$\Rightarrow xy$ is even

$\Rightarrow \neg p$ is true.

$\therefore \neg q \rightarrow \neg p$ is true and hence $p \rightarrow q$ is true.

$\therefore$ The given statement is true by indirect proof.

4. Give an indirect proof for the following statements:

(a) If an integer n is such that n^2 is odd then n is odd.

(b) If n is a product of two positive integers a and b, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

(a) Let $p: n^2$ is odd integer and $q: n$ is odd integer.

Assume $\neg q$ is true.

$\Rightarrow n$ is an even integer

$\Rightarrow n = 2\text{k}\, , k \in I$

$\Rightarrow n^2 = (2k)^2 = 4k^2$

$\Rightarrow n^2$ is even integer

$\Rightarrow \neg p$ is true.

$\therefore \neg q \rightarrow \neg p$ is true and hence $p \rightarrow q$ is true.

$\therefore$ The given statement is true by indirect proof.

(b) Let p: n is a product of two positive integers a and b, q: $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

Assume $\neg q$ is true.

$\Rightarrow a > \sqrt{n}$ and $b > \sqrt{n}$.

$\Rightarrow ab > n$

$\Rightarrow n$ is not a product of a and b.

$\Rightarrow \neg p$ is true.

$\therefore \neg q \rightarrow \neg p$ is true and hence $p \rightarrow q$ is true.

$\therefore$ The given statement is true by indirect proof.

5. **Prove the following statements by the method of contradiction:**

(a) For all positive real nos., x and y, if the product xy>25, then $x > 5$ or $y > 5$.

(b) The sum of two prime numbers, each larger than 2, is not a prime number.

(a) Let Let $p: xy > 25$ and $q: x > 5$ or $y > 5$.

Assume $p \rightarrow q$ is false. That is, p is true and q is false.

$\Rightarrow$ $x \leq 5$ and $y \leq 5$.

$\Rightarrow xy \leq 25$

$\Rightarrow p$ is false.

Which is a contradiction to our assumption.

Therefore, the given statement is true by the method of contradiction.

(b) Let $p: a \ and \ b$ are prime numbers, greater than 2

$q: a + b$ is not a prime number.

Assume $p \rightarrow q$ is false. That is, p is true and q is false.

$\Rightarrow a + b$ is a prime number.

$\Rightarrow a + b = odd \ number$

$\Rightarrow a = odd - b = odd - odd = even$

$\Rightarrow a = even, a > 2$

$\Rightarrow a$ is not a prime number

$\Rightarrow p$ is false.

Which is a contradiction to our assumption.

Therefore, the given statement is true by the method of contradiction.

6. Prove that 'The sum of two odd integers is an even integer' by the method of contradiction.

Let $p: a \text{ and } b$ are odd integers.

$q: a + b$ is an even integers.

Assume $p \rightarrow q$ is false. That is, p is true and q is false.

q is false $\Rightarrow a + b$ is an odd integer.

$\Rightarrow a = odd - b = odd - odd = even$

$\Rightarrow a = even$

$\Rightarrow p$ is false.

Which is a contradiction to our assumption.

Therefore, the given statement is true by the method of contradiction.

7. **Give (i) a direct proof, (ii) an indirect proof (iii) proof by contradiction for the following statement: If n is an odd integer, then $n + 9$ is an even integer.**

Let p: n is an odd integer

q: $n + 9$ is an even integer.

Direct proof:

Assume p is true

$\Rightarrow n$ is an odd integer.

$\Rightarrow n = 2k + 1, k \in I$

$\Rightarrow n + 9 = 2k + 1 + 9 = 2k + 10 = 2(k + 5)$

$\Rightarrow n + 9$ is an even integer.

$\Rightarrow q$ is true. $\therefore p \to q$ is true.

Therefore, the given statement is true by a direct proof.

Indirect proof:

Assume $\neg q$ is true

$\Rightarrow n + 9$ is an odd integer.

$\Rightarrow n + 9 = 2k + 1, for\ some\ k \in I$

$\Rightarrow n = 2k + 1 - 9 = 2k - 8 = 2(k - 4), for\ some\ k \in I$

$\Rightarrow n$ is an even integer.

$\Rightarrow \neg p$ is true. $\therefore \neg q \to \neg p$ is true.

Therefore, the given statement is true by an indirect proof.

Method of contradiction:

Assume $p \to q$ is false. That is, p is true and q is false.

$\Rightarrow$ $n + 9$ is an odd integer.

$\Rightarrow n + 9 = 2k + 1, for\ some\ k \in I$

$\Rightarrow n = 2k - 8 = 2(k - 4)$

$\Rightarrow n$ is an even integer.

$\Rightarrow p$ is false.

Which is a contradiction to our assumption.

Therefore, the given statement is true by the method of contradiction.