Module - 4 - ODE of higher Order

Linear D.E of second and higher order with constant coefficients

- A Differential Equations in which the dependent variable and its deterivatives appears only in first degree is called a linear D.E.

A.D.E of the form

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots a^{n-1} \frac{dy}{dx} + a_n y = \phi(x)$$

is called a linear D.E of nth order with constant coefficients, where $a_1, a_2 \dots a_n$ are all constants.

- If $\phi(x) = 0$, then the eq. so obtained is called a linear Homogeneous D.E and the solution of such an equation is called the complementary function CF of eq^n (1).

Using the notation of diff. operators.

$$D = \frac{d}{dx}, \quad D^2 = \frac{d^2}{dx^2} \dots D^n = \frac{d^n}{dx^n}$$ eq (1) becomes

If $\phi(x) = 0$, $(D^n + a_1 D^{n-1} + \dots + a_{n-1} D + a_n) y = 0$

- Method of finding complementary functions:

i) Find the Auxilary Eq. (A.E) - $f(m) = 0$
ii) The following 3 cases arises in eq^n

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case i) If the roots of the A.E are real and distinct, say $m_1, m_2, \dots m_n$, then the general solution of eq ② is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_n e^{m_n x}.$$

case ii) If the auxilary has real and equal roots say $m_1 = m_2$, the G.S will be

$$y = (c_1 + c_2 x) e^{m_1 x} + c_3 e^{m_3 x} + \dots c_n e^{m_n x}.$$

case iii) If two of the roots of the A.E $f(m) = 0$ are imaginary, then they will occur in pairs. Let $\alpha \pm i\beta$ be a pair of imaginary root, then the general solution of eq^n ② is

$$y = e^{\alpha x}[c_1 \cos\beta x + c_2 \sin\beta x] + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$$

* If these two pairs of imaginary roots, say $\alpha \pm i\beta$, $\gamma \pm i\delta$ then the general sol will be.

$$y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x) + e^{\gamma x}[c_3 \cos\delta x + c_4 \sin\delta x) + c_5 e^{m_5 x} + \dots + c_n e^{m_n x}.$$

* If the A.E $f(m) = 0$ has two equal pairs of imaginary roots say $\alpha + i\beta$ and $\alpha - i\beta$ occuring twice, then the general solution of eq^n ② is

$$y = e^{\alpha x}[(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x] + c_5 e^{m_5 x} + \dots + c_n e^{m_n x}.$$

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Solve the following differential equations.

1) $\frac{d^3y}{dx^2} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = 0$

Sol^n:

$D^3y - 3D^2y + 3Dy - y = 0$ where $D = \frac{d}{dx}$

$(D^3 - 3D^2 + 3D - 1)y = 0$ is of the form.

$f(D)y = 0$

where $f(D) = D^3 - 3D^2 + 3D - 1$.

A.E. $f(m) = 0$

$m^3 - 3m^2 + 3m - 1 = 0$

By inspection $\boxed{m = 1}$ is a root.

$m = 1$	1	-3	3	-1
	0	1	-2	1
	1	-2	1	0

$\therefore m^2 - 2m + 1 = 0$

$\boxed{m = 1, 1}$

3 roots are 1, 1, 1

$\therefore$ the general solution of the Given D.E is

$CF = y = (c_1 + c_2x + c_3x^2)e^{1x}$

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2) $(4D^4 - 4D^3 - 23D^2 + 12D + 36)y = 0$

Sol^n:

$A.E = 4m^4 - 4m^3 - 23m^2 + 12m + 36 = 0$

By inspection $\boxed{m = 2}$ is a root

$m = 2$	4	-4	-23	12	36
	0	8	8	-30	-36
	4	4	-15	-18	0

$\therefore 4m^3 + 4m^2 - 15m - 18 = 0$ ——— (1)

By inspection $\boxed{m = 2}$ is a root

$m = 2$	4	4	-15	-18
	0	8	24	18
	4	12	9	0

$\therefore 4m^2 + 12m + 9 = 0$ ——— (2)

$4m^2 + 6m + 6m + 9 = 0$

$2m(2m+3) + 3(2m+3) = 0$

$(2m+3)(2m+3) = 0$

$(2m+3) = 0$, $(2m+3) = 0$

$\boxed{m = \frac{-3}{2}}$, $\boxed{m = \frac{-3}{2}}$

4 roots are $2, 2, \frac{-3}{2}, \frac{-3}{2}$

General Sol^n is given by

$$\boxed{y = CF = (c_1 + c_2 x)e^{2x} + (c_3 + c_4 x)e^{-3/2 x}}$$

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3). $(4D^4 - 8D^3 - 7D^2 + 11D + 6)\,y = 0$

Sol^n:

$AE = (4m^4 - 8m^3 - 7m^2 + 11m + 6) = 0$

By inspection $\boxed{m = -1}$ is a root

$m=-1$	4	-8	-7	11	6
	0	-4	12	-5	-6
	4	-12	5	6	0

$4m^3 - 12m^2 + 5m + 6 = 0$

By inspection $\boxed{m = 2}$ is a root

$m=2$	4	-12	5	6
	0	8	-8	-6
	4	-4	-3	0

$4m^2 - 4m - 3 = 0$

$4m^2 - 6m + 2m - 3 = 0$

$2m(2m-3) + 1(2m-3) = 0$

$(2m+1)(2m-3)$

$\boxed{m = \frac{-1}{2}}$ $\boxed{m = \frac{+3}{2}}$

The roots are $-1, 2, \frac{-1}{2}, \frac{3}{2}$.

General Sol^n of the given D.E is

$CF = c_1 e^{-x} + c_2 e^{2x} + c_3 e^{-1/2} + c_4 e^{3/2}$

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4) $(D^4 - m^4)y = 0$

Sol^n:

A.E $= p^4 - m^4 = 0$ [$\because$ not replaced by m since question involves m)

$\Rightarrow (p^2)^2 - (m^2)^2 = 0$

$(p^2 + m^2)(p^2 - m^2) = 0$

$\therefore p^2 + m^2 = 0$

$p^2 = -m^2$

$p = \pm\sqrt{-m^2}$

$\boxed{p = \pm mi}$

$p^2 - m^2 = 0$

$p^2 = m^2$

$p = \pm\sqrt{m^2}$

$\boxed{p = \pm m}$

Roots are $+m, -m, \pm mi, -mi$

General Sol^n of the Given D.E is

$y = CF = C_1 e^{mx} + C_2 e^{-mx} + e^{0x}[C_3 \cos mx + C_4 \sin mx]$

5) $(D^4 + 64)y = 0$

Sol^n A.E :- $m^4 + 64 = 0$

$(m^2)^2 + 8^2 = 0$

W.K.T $a^2 + b^2 = (a+b)^2 - 2ab$

Hence $a = m^2$, $b = 8$

$(m^2 + 8)^2 - 2 \times m^2 \times 8 = 0$

$(m^2 + 8)^2 - 16m^2 = 0$

$(m^2 + 8) - (4m)^2 = 0$

$(m^2 + 8 + 4m)(m^2 + 8 - 4m) = 0$

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$m^2+4m+8=0$

$m = \frac{-4 \pm \sqrt{16-4\times1\times8}}{2\times1}$

$= \frac{-4 \pm \sqrt{-16}}{2}$

$= \frac{-4 \pm 4i}{2}$

$\boxed{m = -2 \pm 2i}$

$m^2-4m+8=0$

$m = \frac{+4 \pm \sqrt{16-4\times1\times8}}{2\times1}$

$= \frac{+4 \pm \sqrt{-16}}{2}$

$= \frac{+4 \pm 4i}{2}$

$\boxed{m = 2 \pm 2i}$

Discriminant formula $d = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

4 ranks are $-2 \pm 2i$, $2 \pm 2i$

a) Sol^n: General is

$$\boxed{y = e^{-2x}(C_1\cos 2x + C_2\sin 2x) + e^{2x}[C_3\cos 2x + C_4\sin 2x]}$$

6). $\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 0$ given that $x(0)=0$, $\frac{dx}{dt}(0)=15$

Sol^n:- Given $\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 0$.

$(D^2+5D+6)x = 0$, where $D = \frac{d}{dt}$

AE: $m^2+5m+6=0$

$m^2-3m-2m+6=0$

$\boxed{m = -2, -3}$

General Sol^n is $= C_1e^{-2t} + C_2e^{-3t}$ —— (1)

differentiate w.r.t. t.

$\frac{dx}{dt} = C_1 \times -2 \cdot e^{-2t} + C_2 \times -3e^{-3t}$

i.e $\frac{dx}{dt} = -2C_1e^{-2t} - 3C_2e^{-3t}$ —— (2)

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Cond^n ① $x = 0$ at $t = 0$

eq ① $0 = c_1 e^0 + c_2 e^0$

$$\boxed{0 = c_1 + c_2} \quad \text{——} \quad ③$$

Cond^n ② $\frac{dx}{dt} = 15$ at $t = 0$

eq^n ② $15 = -2c_1 e^0 - 3c_2 e^0$

$$\boxed{15 = -2c_1 - 3c_2} \quad \text{——} \quad ④$$

Solving ③ and ④,

$$c_1 + c_2 = 0 \longrightarrow \times 2.$$
$$-2c_1 - 3c_2 = 15$$

$$2c_1 + 2c_2 = 0$$
Add (+) $$-2c_1 - 3c_2 = 15$$
$$-c_2 = 15$$

$$\boxed{c_2 = -15} \qquad \boxed{c_1 = 15}$$

substituting c_1 and c_2 in eq^n ①

$$\boxed{x = 15e^{-2t} - 15e^{-3t}}$$

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Inverse Differential operators and the Particular Integral Sol^n of Non-Homogeneous linear D.E with constant coefficients.

- Consider a non-homogeneous linear D.E

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_n y = \phi(x)$$

i.e. $(D^n + a_1 D^{n-1} + \dots + a_n) y = \phi(x)$

$$\Rightarrow f(D) y = \phi(x)$$

→ Form the A.E $f(m) = 0$, find the roots and write the CF.

→ find PI by P.I $= \frac{1}{f(D)} \phi(x)$, where $\frac{1}{f(D)}$ is the inverse Differential operators.

→ Write the general sol^n as $y = CF + PI$

- Particular Sol^n of $f(D)y = \phi(x)$ (or) $P.I = y_p = \frac{\phi(x)}{f(D)}$

Type of PI	Method of getting the P.I
1) $\frac{e^{ax}}{f(D)}$ (Also applicable for any constant $K = K \cdot e^{0x}$, e^{ax+b}, $a^x = e^{\log a(x)}$ sinhax, coshax by using their definitions.	$\frac{e^{ax}}{f(a)}$, $f(a) \neq 0$ (D is replaced by a) if $f(a) = 0$, then $x \cdot \frac{e^{ax}}{f'(a)}$; $f'(a) \neq 0$ if $f'(a) = 0$, then $x^2 \frac{e^{ax}}{f(x)}$ $f'(a)$; $f''(a) \neq 0$
2) $\frac{\sin ax}{f(D^2)}$, $\frac{\cos ax}{f(D^2)}$ Also applicable for $\sin(ax+b)$, $\cos(ax+b)$, $\sin ax$, $\cos bx$, $\sin ax$, $\sin bx$, $\cos ax$ $\cos bx$ by changing into sum	$\frac{\sin ax}{f(-a^2)}$, $\frac{\cos ax}{f(-a^2)}$, where $f(-a^2) \neq 0$ (D^2 is replaced by $-a^2$) If $f(-a^2) = 0$, then $x \cdot \frac{\sin ax}{f'(-a^2)}$. $\frac{x \cdot \cos ax}{f'(-a^2)}$
3) $\frac{\phi(x)}{f(D)}$, where $\phi(x)$ is a polynomial in x.	Divide $\phi(x)$ by $f(D)$ by writing $\phi(x)$ in desc. power of x and $f(D)$ is asc. power of D. Quotient = PI as remainder will be zero.

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Solve the following D.E

1) $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 6e^{3x}$

Sol: $(D^2 - 6D + 9)y = 6e^{3x}$ where $D = \frac{d}{dx}$: $f(D) = D^2 - 6D + 9$

$A.E = f(m) = 0$ $\quad m^2 - 6m + 9 = 0$ $\quad (m-3)^2 = 0$

$m - 3 = 0$; $m - 3 = 0$

$m(3,3)$

$\therefore CF = y(c_1 + c_2 x)e^{3x}$

To find P.I. $= \frac{\phi(x)}{f(D)}$ $\quad P.I = \frac{6e^{3x}}{D^2 - 6D + 9}$

$D \rightarrow a = 3$ $\quad$ deno $= 3$

$P.I = 6 \cdot x \cdot \frac{e^{3x}}{2D - 6}$

$D \rightarrow a = 3$ $\quad$ deno $= 0$

$PI = \frac{{}^{3}6x^2 \cdot e^{3x}}{\cancel{2}} \rightarrow PI = 3x^2 e^{3x}$

Solution is $y = CF + PI = (c_1 + c_2 x)e^{3x} + 3x^2 e^{3x}$

2) $(D^2 - 4D + 13)y = e^{3x}\cosh 2x + 3^x$

$A.E = f(m) = 0$ $\quad$ Here $f(D) = D^2 - 4D + 13$

$m^2 - 4m + 13 = 0$

$m = \frac{4 \pm \sqrt{16 - 4 \times 1 \times 13}}{2 \times 1} = \frac{4 \pm \sqrt{-36}}{2}$

$m = \frac{4 \pm 6i}{2}$ $\quad \boxed{m = 2 \pm 3i}$

$CF = e^{2x}[c_1 \cos 3x + c_2 \sin 3x]$

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$$P.I = \frac{\phi(x)}{f(D)} = \frac{e^{3x}\cosh 2x + 3^x}{D^2-4D+13} = P_1 + P_2 \longrightarrow (1)$$

$$P_1 = \frac{1}{D^2-4D+13} e^{3x}\cosh 2x$$

$$= \frac{1}{D^2-4D+13} \cdot e^{3x}\left\{\frac{e^{2x}+e^{-2x}}{2}\right\}$$

$$= \frac{1}{2} \cdot \frac{1}{D^2-4D+13}\left\{e^{5x}+e^{x}\right\} =$$

$$= \frac{1}{2}\left[\frac{1}{D^2-4D+13} \cdot e^{5x} + \frac{1}{D^2-4D+13} e^{x}\right]$$

$D \to a = 5$ $\qquad$ $D \to a = 1$

$$= \frac{1}{2}\left[\frac{e^{5x}}{5^2-20+13} + \frac{e^{x}}{1-4+13}\right] = \frac{1}{2}\left[\frac{e^{5x}}{18} + \frac{e^{x}}{10}\right]$$

$$P_2 = \frac{1}{D^2-4D+13} \cdot 3^x$$

$$= \frac{1}{D^2-4D+13} \cdot e^{\log 3^x} = \frac{1}{D^2-4D+13} \cdot e^{x\log 3}$$

$D \to a = \log 3$

$$P_2 = \frac{1}{(\log 3)^2 - 4(\log 3) + 13} \cdot e^{x\log 3} = \frac{3^x}{(\log 3)^2 - 4\log 3 + 13}$$

$$\therefore PI = P_1 + P_2 = \frac{1}{2}\left[\frac{e^{5x}}{18} + \frac{e^{x}}{10}\right] + \frac{3^x}{(\log 3)^2 - 4\log 3 + 13}$$

General solution.

$$y = e^{2x}\left[c_1\cos 3x + c_2 \sin 3x\right] + \frac{1}{2}\left[\frac{e^{5x}}{18} + \frac{e^{x}}{10}\right] + \frac{3^x}{(\log 3)^2 - 4\log 3 + 13}$$

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3) $\frac{d^3y}{dx^3} + \frac{6d^2y}{dx^2} + 11\frac{dy}{dx} + 6y = e^x + 1.$

Sol^n $(D^3 + 6D^2 + 11D + 6)y = e^x + 1$

A.E $= m^3 + 6m^2 + 11m + 6 = 0$

$\boxed{m = -1, -2, -3}$

$CF = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-3x}$

$PI = \frac{1}{D^3 + 6D^2 + 11D + 6}\{e^x + 1\}$

$= \frac{e^x}{D^3 + 6D^2 + 11D + 6} + \frac{1 \cdot e^{0x}}{D^3 + 6D^2 + 11D + 6}$

$D \longrightarrow a = 1 \qquad D \longrightarrow a = 0$

$PI = \frac{e^x}{1+6+11+6} + \frac{1 \cdot e^{0x}}{0+0+0+6} = PI = \frac{e^x}{24} + \frac{1}{6}$

General solution $y = CF + PI$

$$\boxed{y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-3x} + \frac{e^x}{24} + \frac{1}{6}}$$

4) $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 4y = \sinh(2x+3)$

Sol^n: $(D^3 - D^2 + 4D - 4)y = \sinh(2x+3)$

A.E $= m^3 - m^2 + 4m - 4 = 0$

$\Rightarrow m = +1 \pm 2i$

$$\boxed{CF = c_1 e^x + e^{0x}[c_2 \cos 2x + c_3 \sin 2x]}$$

$PI = \frac{1}{D^3 - D^2 + 4D - 4} \sinh(2x+3)$

$= \frac{1}{D^3 - D^2 + 4D - 4}\left\{\frac{e^{2x+3} - e^{(2x+3)}}{2}\right\}$

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$$PI = \frac{1}{2}\left[\frac{e^{2x+3}}{D^3-D^2+4D-4} - \frac{e^{-2x+3}}{D^3-D^2+4D-4}\right]$$

$D \to a = 2$ $\qquad$ $D \to a = -2$

$$PI = \frac{1}{2}\left[\frac{e^{2x+3}}{8-4+8-4} - \frac{e^{-(2x+3)}}{-8-4-8-4}\right]$$

$$= \frac{1}{2}\left[\frac{e^{2x+3}}{8} + \frac{e^{-(2x+3)}}{24}\right]$$

$\therefore$ General solⁿ $y = CF + PI$

$$y = c_1 e^{x} + c_2 \cos 2x + c_3 \sin 2x + \frac{1}{2}\left[\frac{e^{2x+3}}{8} + \frac{e^{-(2x+3)}}{24}\right]$$

Problems on Type 2. : $\sin ax / \cos ax$

Solve the following D.E

1) $y'' - 4y' + 4y = 8\cos 2x$

solⁿ. $(D^2 - 4D + 4)y = 8\cos 2x.$

A.E $= m^2 - 4m + 4 = 0 \rightarrow (m-2)^2 = 0$

$\boxed{m = 2, 2}$

$CF = (c_1 + c_2 x)e^{2x}$

$PI = \frac{\phi(x)}{f(D)} = \frac{8\cos 2x}{D^2 - 4D + 4}$

$D^2 \to -a^2 = -2^2 = -4.$

$$PI = 8 \cdot \frac{1}{-4-4D+4} \cos 2x$$

$$= 8 \cdot \frac{1}{-4D} \cos 2x$$

$$= -2 \cdot \frac{1}{D} \cdot \cos 2x$$

$D = \frac{d}{dx}$ = diff. operator

$\frac{1}{D}$ = Inverse diff operator integration

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$= -2\int \cos 2x\, dx$

$= -2 \times \frac{\sin 2x}{2} = -\sin 2x$

Solution $y = CF + PI = (c_1 + c_2 x)e^{2x} - \sin 2x$.

2) $(D^2 - 3D + 2)y = \sin 2x$.

Sol^n: A.E $= m^2 - 3m + 2 = 0$

$\boxed{m = 1, 2}$

$CF = c_1 e^{x} + c_2 e^{2x}$

$PI = \frac{\phi(x)}{f(D)} = \frac{1}{D^2 - 3D + 2}(\sin 2x)$

$\therefore D^2 \to -a^2 = -2^2 = -4$

$PI = \frac{1}{-4 - 3D + 2}(\sin 2x) = \frac{1}{-3D - 2}(\sin 2x) = \frac{1}{-(3D+2)}(\sin 2x)$

$= \frac{-1}{(3D+2)} \times \frac{(3D-2)}{(3D-2)}\{\sin 2x\} = \frac{(-3D-2)}{(3D)^2 - 4}\{\sin 2x\}$

$= \frac{-(3D-2)}{9D^2 - 4}\{\sin 2x\}$ $D^2 \to -a^2 = -2^2 = -4$

$= \frac{-(3D-2)}{(9 \times -4) - 4}\{\sin 2x\} = \frac{1}{4D}(3D - 2)(\sin 2x)$

$PI = \frac{1}{4D}\{3D(\sin 2x) - 2\sin 2x\} = \frac{1}{4D}\{3 \times 2\cos 2x - 2\sin 2x\}$

$= \frac{1}{4D}\{6\cos 2x - 2\sin 2x\} = \frac{1}{4D}\{3\cos 2x - \sin 2x\}$

$= \frac{1}{2D}\{3\cos 2x - \sin 2x\}$

General solution : $y = CF + PI$

$$\boxed{y = c_1 e^{x} + c_2 e^{2x} + \frac{1}{2D}\{3\cos 2x - \sin 2x\}}$$

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3) $(D^2-4D+4)y = e^{2x} + \sin x.$

Sol:- AE $m^2 - 4m + 4 = 0$

$\boxed{m = 2, 2}$

$CF = (c_1 + c_2 x)e^{2x}.$

$PI = \frac{\phi(x)}{f(1)} = \frac{1}{D^2-4D+4}\{e^{2x} + \sin x\}$

$PI = P_1 + P_2$ —— ①

Consider, $P_1 = \frac{e^{2x}}{D^2-4D+4}$

$D \rightarrow a = 2$ denominator $= 0$

$P_1 = x \cdot \frac{1}{2D-4} e^{2x}.$

$D \rightarrow a = 2$, denominator $= 0$

$P_1 = x^2 \cdot \frac{1}{2} \cdot e^{2x}.$

$\boxed{P_1 = \frac{x^2 e^{2x}}{2}}$

Consider $P_2 = \frac{1}{D^2-4D+4}\{\sin x\}$

$D^2 \rightarrow -a^2 = -1^2 = -1$

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$$P_2 = \frac{1}{-1-4D+4} \cdot \sin x$$

$$= \frac{1}{3-4D} \{\sin x\}$$

$$P_2 = \frac{1}{(3-4D)} \cdot \frac{(3+4D)}{(3+4D)} \{\sin x\}$$

$$= \frac{(3+4D)(\sin x)}{3^2-(4D)^2}$$

$$= \frac{1}{9-16D^2} \times (3+4D)(\sin x)$$

$$D^2 \longrightarrow -a^2 = -1^2 = -1$$

$$P_2 = \frac{1}{9-(16\times -1)} (3+4D)(\sin x)$$

$$= \frac{1}{25}\left[3\sin x + 4D(\sin x)\right] \qquad \boxed{D = \frac{d}{dx}}$$

$$PI = \frac{x^2 e^{2x}}{2} + \frac{1}{25}(3\sin x + 4\cos x)$$

General Sol^n is $y = CF + PI$.

$$\boxed{y_1 = (c_1 + c_2 x)e^{2x} + \frac{x^2 e^{2x}}{2} + \frac{1}{25}(3\sin x + 4\cos x)}$$

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4) $(D^3-1)y = 3\cos 2x + e^{-2x}$

Sol^n: $\therefore AE = m^3 - 1$

$$\begin{array}{r|rrrr} m=1 & 1 & 0 & 0 & -1 \\ & 0 & 1 & 1 & 1 \\ \hline & 1 & 1 & 1 & \boxed{0} \end{array}$$

$$m^2 + m + 1 = 0$$

$$m = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$= \frac{-1 \pm \sqrt{-3}}{2}$$

$$\boxed{m = \frac{-1}{2} \pm i\frac{\sqrt{3}}{2}}$$

$$CF = c_1 e^{x} + e^{-\frac{1}{2}x}\left[c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right]$$

$$PI = \frac{1}{f(D)} \cdot \phi(x)$$

$$= \frac{1}{D^3-1} \cdot \{3\cos 2x + e^{-2x}\}$$

$$\Rightarrow P_1 + P_2 \longrightarrow (1)$$

Consider $P_1 = \frac{1}{D^2-1} \cdot 3\cos 2x$

$$D^2 \longrightarrow -a^2 = -2^2 = -4$$

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$$P_1 = \frac{1}{D \cdot D^2 - 1}$$

$$P_1 = 3 \cdot \frac{1}{-4D - 1} \cdot \cos 2x$$

$$= -3 \cdot \frac{1}{4D + 1} \cdot \cos 2x$$

$$= -3 \cdot \frac{1}{(4D+1)} \times \frac{(4D-1)}{(4D-1)} \cos 2x$$

$$= \frac{-3 \cdot (4D - 1)}{16D^2 - 1} \cdot \cos 2x$$

$$D^2 \longrightarrow -a^2 \longrightarrow = -2^2 = -4$$

$$= \frac{-3 \cdot (4D - 1)}{(16 \times -4) - 1} \cdot \cos 2x$$

$$P_1 = \frac{-3}{-65} (4D - 1) \cos 2x$$

$$P_1 = \frac{3}{65} \left[4D(\cos 2x) - \cos 2x \right]$$

$$= \frac{3}{65} \left[4 \times -2 \sin 2x - \cos 2x \right]$$

$$= \frac{3}{65} \left[-8 \sin 2x - \cos 2x \right]$$

$$P_1 = \frac{-3}{65} \left[8 \sin 2x + \cos 2x \right]$$

$$P_2 = \frac{1}{D^3 - 1} \cdot e^{-2x}$$

$$D \longrightarrow a = -2.$$

$$P_2 = \frac{1}{(-2)^3 - 1} \cdot e^{-2x} \Rightarrow \boxed{P_2 = \frac{-e^{-2x}}{9}}$$

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General Solution

$$y = CF + PI$$

$$y = c_1 e^{x} + e^{-1/2 x}\left[c_2 \cos\frac{\sqrt{3}}{2} + c_3 \sin\frac{\sqrt{3}}{2}\right] + \frac{-3(8\sin 2x - \cos 2x)}{65} - \frac{e^{-2x}}{9}$$

5) $(D^2+3D+2)\,y = 4\cos^2 x$

Soln. $(D^2+3D+2)y = 4\left[\frac{1+\cos 2x}{2}\right]$

$$= 2(1+\cos 2x)$$

$$= 2(e^{0x} + \cos 2x)$$

$$AE = m^2 + 3m + 2 = 0$$

$$\boxed{m = -1, -2}$$

$$CF = c_1 e^{-1x} + c_2 e^{-2x}$$

$$PI = \frac{\phi(x)}{f(D)}$$

$$PI = \frac{1}{D^2+3D+2}\{2(e^{0x} + \cos 2x)\}$$

$$= P_1 + P_2 \quad \text{—— (1)}$$

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$$P_1 = \frac{1}{D^2+3D+2} \cdot 2e^{0x}$$

$$D \longrightarrow a = 0$$

$$P_1 = \frac{1}{2} \cdot 2e^{0x}$$

$$\boxed{P_1 = 1}$$

$$P_2 = \frac{1}{D^2+3D+2} \cdot 2\cos 2x$$

$$D^2 \longrightarrow -a^2 = -2^2 = -4$$

$$P_2 = \frac{1}{-4+3D+2} \cdot 2\cos 2x$$

$$= 2 \cdot \frac{1}{3D-2} (\cos 2x)$$

$$= \frac{2}{(3D-2)} \frac{(3D+2)}{(3D+2)}$$

$$= \frac{2(3D+2)(\cos 2x)}{9D^2-4}$$

$$D^2 \longrightarrow -a^2 = -4$$

$$= \frac{2 \cdot (3D+2)(\cos 2x)}{(9x-4)-4}$$

$$P_2 = 2 \cdot \frac{1}{-40} \{3D(\cos 2x) + 2\cos 2x\}$$

$$P_2 = \frac{-1}{20} [(3x - 2\sin 2x) + 2\cos 2x]$$

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$$= \frac{-2}{20}\left[-3\sin 2x + \cos 2x\right]$$

$$\boxed{P_2 = \frac{-1}{10}\left[-3\sin 2x + \cos 2x\right]}$$

$$PI = 1 + \left(\frac{-1}{10}\right)\left(-3\sin 2x + \cos 2x\right).$$

General Solution,

$$y = PI + CF \qquad \boxed{y = CF + PI}$$

$$\boxed{y \Rightarrow c_1 e^{-2x} + c_2 e^{-x} + 1 - \frac{1}{10}(\cos 2x + \sin 2x)}$$

□ Problem on type 3 to find $PI = \frac{\phi(x)}{f(D)}$.

1) $(D^2+4)y = x^2 + e^{-x}$.

Sol^n: A.E. $m^2 + 4 = 0$

$$m^2 = -4$$

$$\boxed{m = \pm 2i}$$

$$CF = e^{0x}\left[c_1 \cos 2x + c_2 \sin 2x\right]$$

$$PI = \frac{\phi(x)}{f(D)} = \frac{x^2 + e^{-x}}{D^2+4}$$

$$= \frac{x^2}{4+D^2} + \frac{e^{-x}}{D^2+4} = P_1 + P_2 \quad \text{— (1)}$$

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$$P_1 = \frac{x^2}{4+D^2}$$

$$4+D^2 \overline{\big)\, x^2 \;\; x^2 + 1/2 \;\; -1/2 \;\; -1/2 \;\; 0} \quad \Big(x^2/4 - \frac{1}{8}$$

$(-)$ $(-)$ $(+)$

$$\frac{x^2}{4} \rightarrow Q$$

$$\frac{4 \times x^2}{4} \rightarrow = x^2$$

$$D^2\left(\frac{x^2}{4}\right) = D\left(\frac{2x}{4}\right)$$

$$= \frac{2 \times 1}{4} = \frac{1}{2}$$

$$\frac{-1}{2}/4 = \frac{-1}{8} \rightarrow Q$$

$$4 \times \frac{-1}{8} = -1/2$$

$$D^2\left(\frac{-1}{8}\right) = 0$$

$$P_1 = \frac{x^2}{4} - \frac{1}{8}$$

$$P_2 = \frac{e^{-x}}{D^2+4}$$

$$D \longrightarrow a = -1$$

$$\frac{e^{-x}}{1+4}$$

$$P_2 = \frac{e^{-x}}{5}$$

The general solution is $y = CF + PI$

$$\boxed{y = C_1 \cos 2x + C_2 \sin 2x + \frac{x^2}{4} - \frac{1}{8} + \frac{e^{-x}}{5}}$$

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2) $y'' + 2y' + y = 2x + x^2$

Sol^n: $(D^2 + 2D + 1)y = 2x + x^2$

AE: $m^2 + 2m + 1 = 0$

$(m+1)^2 = 0$

$\boxed{m = -1, -1}$

$CF = (c_1 + c_2 x)e^{-x}$

$PI = \frac{1}{f(D)} \phi(x)$

$PI = \frac{1}{D^2 + 2D + 1} \{2x + x^2\}$

$PI = \frac{x^2 + 2x}{1 + 2D + D^2}$

$$
\begin{array}{r|l}
1 + 2D + D^2 & x^2 + 2x \quad (x^2 - 2x + 2 \\
 & \underline{x^2 + 4x + 2} \\
 & (-) \quad (-) \quad (-) \\
 & -2x - 2 \\
 & \underline{-2x - 4} \\
 & (+) \quad (+) \\
 & 2 \\
 & \underline{(-) \; 2} \\
 & 0
\end{array}
$$

$\frac{x^2}{1} = x^2 \rightarrow Q$

$1 \times x^2 = x^2$

$2D(x^2) = 2 \times 2x = 4x$

$D^2(x^2) = D(2x) = 2$

$\frac{-2x}{1} = -2x \rightarrow Q$

$1 \times -2x = -2x$

$2D(-2x) = 2 \times -2 = -4$

$PI = x^2 - 2x + 2$

General Sol^n.

$\boxed{\rightarrow (C_1 + C_2 x)e^{-x} + x^2 - 2x + 2}$

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3) $(D^2+4)y = x^2 + \cos 2x + 2^{-x}$

Sol^n AE : $m^2+4=0$

$$\boxed{m = \pm 2i}$$

$$CF = e^{0x}\left[c_1\cos 2x + c_2 \sin 2x\right]$$

$$PI = \frac{1}{f(D)}\phi(x)$$

$$= \frac{1}{D^2+4}\left\{x^2 + \cos 2x + 2^{-x}\right\}$$

$$= P_1 + P_2 + P_3$$

Considering $P_1 = \dfrac{x^2}{D^2+4}$

$$\begin{array}{r|l} 4+D^2 \;\big)\; x^2 & \left(x^2/4 - 1/8\right. \\ \underline{x^2 + 1/2} & \\ (-) \quad (-) & \\ -1/2 & \\ \underline{-1/2} & \\ (+) & \\ 0 & \end{array}$$

$\dfrac{x^2}{4} \longrightarrow Q$

$4 \times x^2/4 \rightarrow x^2$

$D^2\left(\dfrac{x^2}{4}\right) = D\left(\dfrac{2x}{4}\right) = \dfrac{2}{4} = \dfrac{1}{2}$

$\dfrac{-1/2}{4} = \dfrac{-1}{8} \rightarrow Q$

$4 \times \dfrac{1}{8} = -1/2$

$$\boxed{P_1 = \frac{x^2}{4} - \frac{1}{8}}$$

$$P_2 = \frac{1}{D^2+4}\left\{\cos 2x\right\}$$

$$D^2 \longrightarrow -a^2 = -2^2 = -4$$

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$$P_2 = x \cdot \frac{1}{2D}\{\cos 2x\}$$

$$= \frac{x}{2}\int \cos 2x \, dx$$

$$= \frac{x}{2} \times \frac{\sin 2x}{2}$$

$$\boxed{P_2 = \frac{x \sin 2x}{4}}$$

$$P_3 = \frac{1}{D^2+4}\{2^{-x}\}$$

$$= \frac{1}{D^2+4}\{e^{\log 2^{-x}}\}$$

$$= \frac{1}{D^2+4} e^{-x(\log 2)}$$

$$= \frac{1}{D^2+4}\{e^{(-\log 2)x}$$

$$D \longrightarrow a = -\log 2$$

$$P_3 = \frac{1}{(-\log 2)^2+4} \cdot 2^{-x}$$

$$\boxed{PI = \frac{x^2}{4} - \frac{1}{8} + \frac{x \sin 2x}{4} + \frac{1}{(-\log 2)^2+4} \cdot 2^{-x}}$$

General solution $y = CF + PI$

$$\boxed{y = e^{0x}\left[c_1 \cos 2x + c_2 \sin 2x\right] + \frac{x^2}{4} - \frac{1}{8} + \frac{x \sin 2x}{4} + \frac{1}{(-\log 2)^2+4} \cdot 2^{-x}}$$

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4) $(D-2)^2 y = 8(e^{2x} + \sin 2x + x^2)$

Sol^n:- AE: $(m-2)^2 = 0$

$m-2=0$ $\quad$ $m-2=0$

$\boxed{m = 2, 2}$

$CF = (c_1 + c_2 x)e^{2x}$

$PI = \frac{1}{(D-2)^2}\{8(e^{2x} + \sin 2x + x^2)\}$

$= \frac{1}{D^2-4D+4}\{8(e^{2x} + \sin 2x + x^2)\}$

$\Rightarrow P_1 + P_2 + P_3$

Consider $P_1 = \frac{1}{D^2-4D+4} \cdot e^{2x}$

$D \rightarrow a = 2$ $\quad$ deno $= 0$

$P_1 = 8 \cdot x \cdot \frac{1}{2D-4} \cdot e^{2x}$

$D \rightarrow a = 2$ $\quad$ deno $= 0$

$P_1 = 8x^2 \cdot \frac{1}{2} e^{2x}$

$\boxed{P_1 = 4x^2 e^{2x}}$

$P_2 = \frac{1}{D^2-4D+4} \cdot 8\sin 2x$

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$$D^2 \rightarrow -a^2 = -2^2 = -4$$

$$P_2 = \frac{1}{-4-4D+4} \cdot 8\sin 2x$$

$$= 8 \cdot \frac{1}{-4D} \cdot \sin 2x$$

$$= -2\int \sin 2x \, dx$$

$$= -2x - \frac{\cos 2x}{2}$$

$$\boxed{P_2 = \cos 2x}$$

$$P_3 = \frac{1}{D^2-4D+4}\{8x^2\} = \frac{8x^2}{4-4D+D^2}$$

$$\begin{array}{r|l} & 2x^2 \\ \hline 4-4D+D^2 & 8x^2 \\ & \underline{\overset{(-)}{8x^2} \overset{(+)}{-16x} \overset{(-)}{+4}} \\ & \overset{(+)}{16x} \overset{(-)}{-4} \\ & \underline{\overset{(-)}{16x} \overset{(+)}{-16}} \\ & 12 \\ & \underline{12} \\ & 0 \end{array}$$

$$\frac{8x^2}{4} = 2x^2 \rightarrow Q$$

$$4 \times 2x^2 = 8x^2$$

$$-4D(2x^2) = -4 \times 2 \times 2x = -16x$$

$$D^2(2x^2) = D(4x) = 4$$

$$\frac{16x}{4} = 4x \rightarrow Q$$

$$-4D(4x) = -16 \times 1 = -16$$

$$\frac{12}{4} = 3 \rightarrow Q$$

$$\boxed{P_3 = 2x^2 + 4x + 3}$$

General Solution $y = CF + PI$.

$$\boxed{(c_1 + c_2 x)e^{2x} + \frac{x^2}{2} 8e^{2x} + \cos 2x + 2x^2 + 4x + 3}$$

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Linear D.E with Variable coefficients

I) Solution of Legendre's linear D.E

* An equation of the form
$a_0(ax+b)^2y'' + a_1(ax+b)y' + a_2 y = \phi(x)$ is called legendre linear eq^n of 2nd order.

* Take the substitution $t = \log(ax+b)$ or $e^t = ax+b$
* We have $(ax+b)y' = aDy$ where $D = d/dt$
similarly $(ax+b)^2y'' = a^2D(D-1)y$
$(ax+b)^3y''' = a^3D(D-1)(D-2)y$ etc.

* Substitute these eqn ①, which reduces to linear D.E with constant coefficients.

II) Solution of Cauchy's linear D.E :- (Take $a=1$, $b=0$ in eq^n ①)

* An equation of the form $a_0x^2y'' + a_1xy' + a_2y = \phi(x)$ — ②
is called cauchy's linear eq^n of 2nd order.

* Take the substitution $t = \log x$ or $e^t = x$.
we have $xy' = Dy$ where $D = d/dt$
similarly $x^2y'' = D(D-1)y$
$x^3y''' = D(D-1)(D-2)y$ etc.

* Substitute these in equation ②, which reduces to linear D.E with constant coefficients.

NOTE :- Solve the linear D.E with constant coefficients to obtain $y = CF + PI$ in terms of 't'. substitute for 't' and obtain the sol 'y' in terms of 'x'

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Problems on Cauchy's and Legendre's Linear Differential Equations

1) $x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \log x$

Sol^n: Given $x^2y'' - xy' + y = \log x$ —— (1)

Eq (1) is a Cauchy's linear D.E

Substitute $t = \log x$

$e^t = x$

We have $xy' = Dy$ where $D = \frac{d}{dt}$

$x^2y'' = D(D-1)y$

Substituting in eq^n (1).

$D(D-1)y - Dy + 1\cdot y = t$

$(D^2 - D - D + 1)y = t$

$(D^2 - 2D + 1)y = t$ —— (2) where $D = \frac{d}{dt}$

A.E $m^2 - 2m + 1 = 0$

$\boxed{m = 1, 1}$

$CF = (c_1 + c_2 t)e^t$

$PI = \frac{1}{f(D)}\phi t = \frac{t}{1-2D+D^2}$

We ll. form the division.

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$$
\begin{array}{r|l}
 & t+2 \\
\hline
1-2D+D^2 & t \\
 & \underset{(-)}{t} \underset{(+)}{-2} \\
\hline
 & 2 \\
 & \underset{(-)}{2} \\
\hline
 & 0
\end{array}
$$

$$\left\{\begin{array}{l} \frac{t}{1} = t \rightarrow Q. \\ -2D(t) = -2 \times 1 = -2 \\ \frac{2}{1} = 2 \rightarrow Q. \end{array}\right.$$

$$\boxed{PI = t+2.}$$

G.sol^n is $y = CF + PI$.

$$y = (c_1 + c_2 t)e^t + t + 2.$$

$$\boxed{y = (c_1 + c_2 \log x)x + \log x + 2}$$

2) $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + 9y = \sin(3\log x) + 3x^2$

sol^n:- Given $x^2y'' + xy' + 9y = \sin(3\log x) + 3x^2$

eq ① is cauchy's D.E.

Substitute $t = \log x \Rightarrow e^t = x$.

we have $xy' = Dy$ where $D = \frac{d}{dt}$.

$x^2y'' = D(D-1)y$.

Substitute in eq ①

$$D(D-1)y + Dy + 9y = \sin(3t) + 3(e^t)^2.$$

$$(D^2 - D + D + 9)y = \sin 3t + 3e^{2t}$$

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$$(D^2+9)y = \sin 3t + 3e^{2t} \quad \text{—} (2) \qquad \text{where } D = \frac{d}{dt}$$

$$AE = m^2 + 9 = 0$$

$$\boxed{m = +3i, -3i}$$

$$\boxed{CF = C_1 \cos 3t + C_2 \sin 3t}$$

$$PI = \frac{1}{f(D)} \phi(x) = \frac{\sin 3t + 3e^{2t}}{D^2+9}$$

$$\Rightarrow P_1 + P_2$$

$$P_1 = \frac{1}{D^2+9} \sin 3t$$

$$D^2 \longrightarrow -a^2 = -3^2 = -9 \quad , \text{deno} = 0$$

$$P_1 = t \cdot \frac{1}{2D} \cdot \sin 3t$$

$$P_1 = \frac{t}{2} \int \sin 3t \, dt$$

$$= \frac{t}{2} \times \frac{-\cos 3t}{3}$$

$$\boxed{P_1 = \frac{-t \cos 3t}{6}}$$

$$P_2 = \frac{3e^{2t}}{D^2+9}$$

$$D \rightarrow a = 2.$$

$$P_2 = 3e^{2t}$$

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3) $x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = (1+x)^2$

Sol^n $x^2y'' - 3xy' + 4y = (1+x)^2$ —— ①

Eq^n ① is Cauchy's linear D.E

Substitute $t = \log x$.
$e^t = x$

We have $xy' = Dy$
$x^2y'' = D(D-1)y$

where $D = d/dt$

① $\Rightarrow D(D-1)y - 3Dy + 4y = (1+e^t)^2$

$(D^2 - D - 3D + 4)y = 1 + (e^t)^2 + 2\times1\times e^t$

$(D^2 - 4D + 4)y = 1 + e^{2t} + 2e^t$ —— ②

A.E. $m^2 - 4m + 4 = 0$
$\boxed{m = 2, 2}$

$CF = (C_1 + C_2 t)e^{2t}$ $\qquad PI = \frac{1}{f(D)}\phi(t)$

$PI = \frac{1}{D^2 - 4D + 4}\{1 + e^{2t} + 2e^t\}$

$= P_1 + P_2 + P_3$

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$$P_1 = \frac{1}{D^2-4D+4} \cdot 1 \cdot e^{0t} \longrightarrow D \longrightarrow a = 0$$

$$P_1 = \frac{1}{4} \cdot e^{0t}$$

$$\boxed{P_1 = \frac{1}{4}}$$

$$P_2 = \frac{1}{D^2-4D+4} \cdot e^{2t}$$

$D \longrightarrow a = 2.$, deno $= 0$

$$P_2 = t \cdot \frac{1}{2D-4} \cdot e^{2t}$$

$D \longrightarrow a = 2$ deno $= 0$

$$P_2 = t^2 \cdot \frac{1}{2!} e^{2t}$$

$$\boxed{P_2 = \frac{t^2 e^{2t}}{2}}$$

$$P_3 = \frac{1}{D^2-4D+4} \cdot 2e^{t}$$

$D \longrightarrow a = 1.$

$$P_3 = \frac{1}{1-4+4} \cdot 2e^{t}$$

$$\boxed{P_3 = 2e^{t}}$$

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$PI = P_1 + P_2 + P_3$

$$\boxed{PI = \frac{1}{4} + \frac{t^2 e^{2t}}{2} + 2e^t}$$

General Solution $\quad y = CF + PI$

$$\boxed{y = (c_1 + c_2 t)e^t + \frac{1}{4} + \frac{t^2 e^{2t}}{2} + 2e^t}$$

$$\boxed{y = (c_1 + c_2 \log x)x^{\cancel{\log x}} + \frac{1}{4} + \frac{t^{\log x} e^{2\log x}}{2} + 2x}$$

4) $(1+x)^2 y'' + (1+x)y' + y = 2\sin[\log(1+x)]$

Sol^n. Eq^n ① is legendre's differential equation.

Substitute $t = \log(ax+b)$ — general

here $a = 1 \quad b = 1$

$t = \log(x+1)$

$e^t = x+1 \quad \Rightarrow x = e^t - 1$

we have $(x+1)y' = 1 \cdot Dy$

$(x+1)^2 y'' = 1^2 D(D-1)y \quad ; \quad D = \frac{d}{dt}$

subs. in eq^n ①,

$D(D-1)y + Dy + y = 2\sin t$

$(D^2 - D + D + 1)y = 2\sin t$

$(D^2+1)y = 2\sin t$ —— ② $\quad D = d/dt$

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$A.E = m^2 + 1 = 0$

$$\boxed{m = \pm i}$$

$$CF = e^{0t}[c_1 \cos t + c_2 \sin t]$$

$$PI = \frac{1}{f(D)} \cdot \phi(t)$$

$$= \frac{1}{D^2+1} \, 2\sin t$$

$D^2 \longrightarrow -a^2 = -1^2 = -1$, deno $= 0$

$$PI = t \cdot \frac{1}{2D} \cdot 2\sin t$$

$$PI = t \int \sin t \, dt$$

$$\boxed{PI = -t\cos t}$$

General solution is $y = CF + PI$.

$$y = c_1 \cos t + c_2 \sin t - t\cos t$$

$$\boxed{y = c_1 \cos[\log(x+1)] + c_2 \sin[\log(x+1)] - \log(x+1)\cos[\log(x+1)]}$$

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5) $(3x+2)^2 y'' + 3(3x+2)y' - 36y = 8x^2 + 4x + 1$

Sol^n: Eq^n ① is legendre's D.E.

substitute $t = \log(ax+b)$ - general

here $= a = 3$ $b = 2$

Sub $t = \log(3x+2)$

$e^t = 3x+2$

$e^t - 2 = 3x$

$$\boxed{x = \frac{e^t - 2}{3}}$$

we have $(3x+2)y' = 3 \cdot Dy$

$(3x+2)y'' = 3^2 D(D-1)y$ where $D = \frac{d}{dt}$

substitute in eq ①,

$\Rightarrow 9D(D-1)y + 3 \times 3Dy - 36y = 8\left(\frac{e^t-2}{3}\right) + 4\left(\frac{e^t-2}{3}\right) + 1$

$\Rightarrow (9D^2 - \cancel{9D} + \cancel{9D} - 36)y = \frac{8}{9}(e^{2t} + 4 - 2 \times e^t \times 2) + \frac{4}{3}(e^t - 2) + 1$

$(9D^2 - 36)y = \frac{8}{9}e^{2t} + \frac{32}{9} - \frac{32}{9}e^t + \frac{4}{3}e^t - \frac{8}{3} + 1$

$(9D^2 - 36)y = \frac{8}{9}e^{2t} - \frac{20}{9}e^t + \frac{17}{9}$

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$$= 9(D^2-4)y = \frac{8}{9}e^{2t} - \frac{20}{9}e^{t} + \frac{17}{9}$$

$$(D^2-4)y = \frac{8}{81}e^{2t} - \frac{20}{81}e^{t} + \frac{17}{81} \quad \text{——— (2)}$$

$$A.E = m^2 - 4 = 0$$

$$\boxed{m = \pm 2}$$

$$\boxed{CF = c_1 e^{2t} + C_2 e^{-2t}}$$

$$PI = \frac{1}{f(D)}\phi(t)$$

$$PI = \frac{1}{D^2-4}\left[\frac{8}{81}e^{2t} - \frac{20}{81}e^{t} + \frac{17}{81}\right]$$

$$= P_1 + P_2 + P_3$$

$$P_1 = \frac{1}{D^2-4}\left(\frac{8}{81}e^{2t}\right)$$

$D \rightarrow a = 2$ deno $= 0$

$$P_1 = \frac{8}{81}t \cdot \frac{1}{2D} \cdot e^{2t}$$

$D \rightarrow a = 2$

$$P_1 = \frac{8}{81}t \cdot \frac{1}{2\times 2} \cdot e^{2t}$$

$$\boxed{P_1 = \frac{2}{81}te^{2t}}$$

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$$P_2 = \frac{1}{D^2-4}\left\{-\frac{20}{81}e^t\right\}$$

$D \longrightarrow a = 1$

$$P_2 = \frac{1}{-3} \times \frac{-20}{81} e^t$$

$$\boxed{P_2 = \frac{20}{243} \cdot e^t}$$

$$P_3 = \frac{1}{D^2-4}\left\{\frac{17}{81}e^{0t}\right]$$

$D \rightarrow a = 0$

$$P_3 = \frac{1}{-4} \times \frac{17}{81}$$

$$\boxed{P_3 = \frac{-17}{324}}$$

$$\boxed{PI = \frac{2}{81}te^{2t} + \frac{20}{243} \cdot e^t - \frac{17}{324}}$$

General Soln. = $y = CF + PI$.

$$\boxed{y = C_1e^{2t} + C_2e^{-2t} + \frac{2}{81}te^{2t} + \frac{20}{243}e^t - \frac{17}{324}}$$

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Method of Variation of Parameters

Working Rule :

1) Given $f(D)y = \phi(x)$ where $f(D)$ is of 2nd order, write CF in the form $y_c = c_1 y_1 + c_2 y_2$ and assume $y = Ay_1 + By_2$ to be the complete solution of the D.E where $A = A(x)$ & $B = B(x)$.

2) complete y_1', y_2' and Wronskian $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

3) Assume the expression for A' and B' in the form $A' = \frac{-y_2 \phi(x)}{W}$

$B' = \frac{y_1 \phi(x)}{W}$ and similarly to obtain A & B by integration

4) sub. A, B in $y = Ay_1 + By_2$ to get the complete solution.

Solve the Linear D.E by Variation of Parameters.

1) $y'' + a^2 y = \sec(ax)$

Sol^n: $(D^2 + a^2)y = \sec(ax)$

A.E. $m^2 + a^2 = 0$

$m^2 = -a^2$

$m = \pm\sqrt{-a^2}$

$\boxed{m = \pm ia}$

CF $= y_c = c_1 \cos ax + c_2 \sin ax$

$= c_1 y_1 + c_2 y_2$

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Here $y_1 = \cos ax$; $y_2 = \sin ax$.

$y_1' = -a\sin ax$; $y = a\cos ax$.

$$\therefore W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix}$$

$$= a\cos^2 ax - (-a\sin^2 ax)$$
$$= a[\cos^2 ax + \sin^2 ax]$$
$$= a[1]$$

$$\boxed{W = a}$$

Let complete Sol^n be $y = Ay_1 + By_2$ —— (2)

To find A & B

Consider $A' = \frac{-y_2 \phi(x)}{W} = \frac{-\sin ax \times \sec ax}{a}$

$$A = \frac{-1}{a}\int \sin ax \times \sec ax \, dx + k_1$$
$$= \frac{-1}{a}\int \sin ax \frac{1}{\cos ax} dx + k_1$$
$$= \frac{-1}{a}\int \tan ax \, dx + k_1$$
$$= \frac{-1}{a} \frac{\log(\sec ax)}{a} + k_1$$

$$\boxed{A = \frac{-1}{a^2}\log(\sec ax) + k_1}$$

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$$B' = \frac{y_1 \phi(x)}{W} = \frac{\cos ax \times \sec ax}{a}$$

$$= \frac{\cos ax}{a} \times \frac{1}{\cos ax}$$

$$B' = \frac{1}{a}$$

$$B = \int \frac{1}{a}\, dx + k_2$$

$$B = \frac{1}{a}\int 1 \cdot dx + k_2 \qquad \boxed{B = \frac{x}{a} + k_2}$$

Substituting in eq (1),

$$y = \left[\frac{-1}{a^2}\log(\sec ax) + k_1\right](\cos ax) + \left[\frac{x}{a} + k_2\right](\sin ax)$$

$$\boxed{y = k_1 \cos ax + k_2 \sin ax - \frac{1}{a^2}\log(\sec ax) \times \cos ax + \frac{x}{a}\sin ax}$$

2) $y'' + 4y = \tan 2x$

$(D^2 + 4)y = \tan 2x$

A.E. $m^2 + 4 = 0$

$m^2 = -4$

$\boxed{m = \pm 2i}$

$CF = y_c = c_1 \cos 2x + c_2 \sin 2x$

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Here $y_1 = \cos 2x$; $y_2 = \sin 2x$

$y_1' = -2\sin 2x$; $y_2' = 2\cos 2x$

$$\therefore W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$$

$$= 2\cos^2 2x + 2\sin^2 2x$$

$$= 2(\cos^2 2x + \sin^2 2x)$$

$$\boxed{W = 2}$$

Let complete Sol^n be $y = Ay_1 + By_2$ —— (1)

To find (A) and (B)

Consider $A' = \dfrac{-y_2\,\phi(x)}{W} = \dfrac{-\sin 2x \times \tan 2x}{2}$

$$A' = \frac{-\sin 2x}{2} \times \frac{\sin 2x}{\cos 2x}$$

$$= \frac{-\sin^2 2x}{2\cos 2x}$$

$$= \frac{-(1-\cos^2 2x)}{2\cos 2x}$$

$$= \frac{-1}{2}\left[\sec 2x - \cos 2x\right]$$

$$A' = \frac{1}{2}\left[\cos 2x - \sec 2x\right]$$

$$\therefore A = \frac{1}{2}\int (\cos 2x - \sec 2x)\,dx + k_1$$

$$A = \frac{1}{2}\left[\frac{\sin 2x}{2} - \frac{\log(\sec 2x + \tan 2x)}{2}\right] + k_1$$

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$$\boxed{A = \frac{1}{4}\left[\sin 2x - \log(\sec 2x + \tan 2x)\right] + k_1}$$

$$B' = \frac{y_1 \phi(x)}{W} = \frac{\cos 2x \times \tan 2x}{2}$$

$$B' = \frac{1}{2}\sin 2x$$

$$B = \frac{1}{2}\int \sin 2x \, dx + k_2$$

$$B = \frac{1}{2} \times -\frac{\cos 2x}{2} + k_2$$

$$\boxed{B = \frac{-\cos 2x}{4} + k_2}$$

Sub in eqⁿ ①

$$y = \left\{\frac{1}{4}\left[\sin 2x - \log(\sec 2x + \tan 2x)\right] + k_1\right\}(\cos 2x) + \left[-\frac{\cos 2x}{4} + k_2\right](\sin 2x)$$

3) $y'' - 6y' + 9y = \frac{e^{3x}}{x^2}$

Sol: $(D^2 - 6D + 9)y = \frac{e^{3x}}{x^2}$

A.E. $m^2 - 6m + 9 = 0$

$(m-3)^2 = 0$

$\boxed{m = 3, 3}$

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$$CF = (c_1 + c_2 x)e^{3x}$$
$$= c_1 e^{3x} + c_2 x e^{3x}$$
$$= c_1 y_1 + c_2 y_2 \quad \text{(say)}$$

where $y_1 = e^{3x}$, $y_2 = xe^{3x}$

$y_1' = 3e^{3x}$, $y_2' = x \cdot 3e^{3x} + e^{3x} \cdot 1$.

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & x3e^{3x} + e^{3x} \end{vmatrix}$$

$$W = e^{3x}(3xe^{3x} + e^{3x}) - (3e^{3x})(xe^{3x})$$

$$= \cancel{3xe^{6x}} + e^{6x} - \cancel{3xe^{6x}}$$

$$\boxed{W = e^{6x}}.$$

Let the complete sol^n be $y = Ay_1 + By_2$ is the sol^n. $\longrightarrow$ (1)

To find A and B,

$$A' = \frac{-y_2 \phi(x)}{W} = \frac{-xe^{3x}}{e^{6x}} \times \frac{e^{3x}}{x^2}$$

$$= \frac{-x \cdot e^{6x}}{e^{6x} \cdot x^2}$$

$$\Rightarrow \frac{-1}{x} \qquad A = -\int \frac{1}{x} dx + k_1$$

$$\boxed{A = -\log x + k_1}$$

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$$B' = \frac{y_1 \phi(x)}{W} = \frac{e^{3x}}{e^{6x}} \times \frac{e^{3x}}{x^2}$$

$$B' = \frac{1}{x^2} \qquad B = \int \frac{1}{x^2}\, dx + k_2$$

$$B = \int x^{-2}\, dx + k_2$$

$$B = \int \frac{x^{-2+1}}{-2+1} + k_2$$

$$= \frac{x^{-1}}{-1} + k_2$$

$$\boxed{B = \frac{-1}{x} + k_2}$$

$$y = \left[-\log x + k_1\right] e^{3x} + \left[\frac{-1}{x} + k_2\right] x e^{3x}$$

$$y = k_1 e^{3x} + k_2 x e^{3x} - \log x \cdot e^{3x} - e^{3x}$$

4) $y'' + y = \sec x \cdot \tan x$

sol^n. $(D^2+1)y = \sec x \cdot \tan x$

A.E. $m^2 + 1 = 0$

$$\boxed{m = \pm i}$$

$$CF = C_1 \cos x + C_2 \sin x.$$

$$= c_1 y_1 + c_2 y_2 \text{ (say)}$$

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$y_1 = \cos x$; $y_2 = \sin x$

$y_1' = -\sin x$; $y_2' = \cos x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}.$$

$$W = \cos x(\cos x) - (-\sin x)(\sin x)$$
$$= \cos^2 x + \sin^2 x$$
$$= 1$$

$$\boxed{W = 1}$$

Let $y = Ay_1 + By_2$ be the sol^n

To find A and B.

$$A' = \frac{-y_2 \phi(x)}{W} = \frac{-\sin x \sec x \tan x}{1}$$

$$A' = -\sin x \times \frac{1}{\cos x} \times \frac{\sin x}{\cos x}.$$

$$= \frac{-\sin^2 x}{\cos^2 x} = -\tan^2 x.$$

$1 + \tan^2\theta = s$

$\tan^2\theta = \sec$

$$A' = -(\sec^2 x - 1)$$
$$A' = 1 - \sec^2 x.$$

$$A' = 1 - \sec^2 x.$$

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On Integration

$$A = \int (1 - \sec^2 x)\, dx + k_1$$

$$\boxed{A = x - \tan x + k_1}$$

$$B' = \frac{y_1 \phi(x)}{W} = \frac{\cos x \times \sec x \tan x}{1}$$

$$B' = \tan x$$

$$B = \int \tan x \, dx + k_2$$

$$\boxed{B = \log(\sec x) + k_2}$$

$$y = [x - \tan x + k_1]\cos x + [\log(\sec x) + k_2]\sin x$$

$$y = k_1 \cos x + k_2 \sin x + x\cos x - \sin x + \sin x \cdot \log(\sec x)$$

5) $y'' + 2y' + 2y = e^{-x}\sec^3 x$.

Sol. $(D^2 + 2D + 2)y = e^{-x}\sec^3 x$.

A.E. $m^2 + 2m + 2 = 0$ $\boxed{m = -1 \pm i}$

$$CF = e^{-x}(c_1 \cos x + c_2 \sin x)$$

$$= c_1 e^{-x}\cos x + c_2 e^{-x}\sin x$$

$$= c_1 y_1 + c_2 y_2 \quad (\text{say})$$

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where $y_1 = e^{-x}\cos x$; $y_2 = e^{-x}\sin x$

$$y_1' = e^{-x}\sin x + \cos x \cdot -e^{-x}$$
$$= -e^{-x}\sin x - e^{-x}\cos x$$

$$y_2' = e^{-x}\cos x + \sin x \cdot -e^{-x}$$
$$= e^{-x}\cos x - e^{-x}\sin x$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x}\cos x & e^{-x}\sin x \\ -e^{-x}\sin x - e^{-x}\cos x & e^{-x}\cos x - e^{-x}\sin x \end{vmatrix}$$

$$= e^{-x}\cos x\left(e^{-x}\cos x - e^{-x}\sin x\right) - e^{-x}\sin x\left(-e^{-x}\sin x - e^{-x}\cos x\right)$$
$$= e^{-2x}\cos^2 x - \cancel{e^{-2x}\cos x\sin x} + e^{-2x}\sin^2 x + \cancel{e^{-2x}\sin x\cos x}$$

$$= e^{-2x}\left(\cos^2 x + \sin^2 x\right)$$
$$= e^{-2x}$$

$$\boxed{W = e^{-2x}}$$

Let $y = Ay_1 + By_2$

To find A and B.

$$A' = \frac{-y_2\,\phi(x)}{W} = \frac{-\cancel{e^{-x}}\sin x \times \cancel{e^{-x}}\sec 3x}{\cancel{e^{-2x}}}$$

$$A' = -\sin x \cdot \sec 3x$$
$$A' = -\sin x \cdot \frac{1}{\cos x} \cdot \sec^2 x$$

$$A' = -\tan x \sec^2 x$$

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$$A = -\int \tan x \sec^2 x \, dx + k_1$$

Subs $\tan x = t$

$\rightarrow \sec^2 x \, dt = dt$

$$A = -\int t \, dt + k_1$$

$$= \frac{-t^2}{2} + k_1 \quad \boxed{A = \frac{-\tan^2 x}{2} + k_1}$$

$$B = \frac{y_1 \phi(x)}{W} = \frac{e^{-x}\cos x - e^{-x}\sec^3 x}{e^{-2x}}$$

$B' = \cos x \cdot \sec^3 x$

$B' = \sec^2 x$

$$B = \int \sec^2 x \, dx$$

$$\boxed{B = \tan x + k_2}$$

Substituting in eq (1) $\quad y = Ay_1 + By_1$

$$= \left[\frac{-\tan^2 x}{2} + k_1\right] e^{-x}\cos x + \left[\tan x + k_2\right] e^{-x}\sin x$$

$$\Rightarrow \frac{-\tan^2 x \cdot e^{-x}\cos x}{2} + e^{-x}\cos x \, k_1 + \tan x \cdot e^{-x}\sin x + e^{-x}\sin x \, k_2$$

$$\boxed{\Rightarrow k_1 e^{-x}\cos x + k_2 e^{-x}\sin x - \frac{\tan^2 x}{2}(e^{-x}\cos x) + e^{-x}\sin x \tan x}$$

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6) $y'' + y = \frac{1}{1+\sin x}$

Sol^n

$(D^2+1) = \frac{1}{1+\sin x}$

$AE = m^2+1 = 0$

$\boxed{m^2 = \pm i}$

$CF = C_1 \cos x + C_2 \sin x.$

$C_1 y_1 + C_2 y_2$ (say)

$y_1 = \cos x$; $y_2 = \sin x.$

$y_1' = -\sin x$ $y_2' = \cos x.$

$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}.$

$\boxed{W = 1}$

Let the complete Sol^n be $y = Ay_1 + By_2$

$A' = \frac{-y_2 \phi(x)}{W}$

$A' = -\sin x \times \frac{1}{1+\sin x}$

$A' = \frac{-\sin x}{1+\sin x}$

$A' = \frac{-\sin x}{1+\sin x} \times \frac{1-\sin x}{1-\sin x}$

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$$A' = \frac{-\sin x(1-\sin x)}{(1-\sin^2 x)}$$

$$A' = \frac{-\sin x + \sin^2 x}{\cos^2 x}$$

$$A' = \frac{-\sin x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x}$$

$$A' = -\tan x \cdot \sec x + \tan^2 x$$

$$= -\sec x \tan x + (1 - \sec^2 x)$$

$$A = \int \left[-\sec x \tan x - 1 + \sec^2 x\right] dx + k_1$$

$$\boxed{A = -\sec x - x + \tan x + k_1}$$

$$B' = \frac{y_1 \phi(x)}{W}$$

$$= \frac{\cos x}{1} \times \frac{1}{1+\sin x}$$

$$=$$

Substituting $1 + \sin x = t$

$\cos x \, dx = dt$

$$B = \int \frac{dt}{t} + k_2$$

$$B = \log t + k_2$$

$$\boxed{B = \log(1+\sin x) + k_2}$$

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Substituting in eq ①,

$$y = (-\sec x + \tan x - x + k_1)\cos x + (\log(1+\sin x) + k_2)\sin x$$

$$y = -\sec x \cos x + \sin x - x\cos x + k_1\cos x + \log(1+\sin x)(\sin x) + k_2(\sin x)$$

$$\boxed{y = k_1\cos x + k_2\sin x + \sin x - x\cos x + \sin x\log(1+\sin x)}$$

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SELF STUDY

☐ Application to oscillation of a spring and LCR circuit.

① In a LCR circuit, the charge Q (q) on a plate of the condenser, is given by $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E\sin\omega t$.

where $i = \frac{dq}{dt}$. The circuit is tuned to resonance so that $\omega^2 = \frac{1}{LC}$. Initially $q = 0$, $i = 0$, when $t = 0$. Show that

$$q = \frac{E}{R\omega}\left[e^{-\frac{Rt}{2L}}\left(\cos pt + \frac{R}{2Lp}\sin pt\right) - \cos\omega t\right]$$

where $p^2 = \frac{1}{LC} - \frac{R^2}{4L^2}$.

Sol^n: $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E\sin\omega t$

$$\left(LD^2 + RD + \frac{1}{C}\right)q = E\sin\omega t$$

$$A.E = Lm^2 + Rm + \frac{1}{C} = 0$$

$\div L$ throughout

$$m^2 + \frac{R}{L}m + \frac{1}{LC} = 0$$

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$$m = \frac{-\frac{R}{L} \pm \sqrt{\frac{R^2}{L^2} - 4 \times 1 \times \frac{1}{LC}}}{2 \times 1}$$

$$m = \frac{-\frac{R}{L} \pm \sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}}{2}$$

$$m = \frac{-R}{2L} \pm \frac{1}{2}\sqrt{-\left(\frac{4}{LC} - \frac{R^2}{L^2}\right)}$$

$$= \frac{-R}{2L} \pm \sqrt{\frac{-1}{4}\left(\frac{4}{LC} - \frac{R^2}{L^2}\right)}$$

$$= \frac{-R}{2L} \pm \sqrt{-\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)}$$

$$= \frac{-R}{2L} \pm \sqrt{-p^2}$$

$$\boxed{m = \frac{-R}{2L} \pm ip}$$

$$CF = e^{-\frac{Rt}{2L}}\left(c_1 \cos pt + c_2 \sin pt\right)$$

from ①, $PI = \dfrac{1}{LD^2 + RD + \frac{1}{C}} \cdot E\sin\omega t$

$$D^2 \longrightarrow -a^2 = -\omega^2 = \frac{-1}{LC}$$

$$PI = \frac{1}{\left(L \times \frac{-1}{LC}\right) + RD + \frac{1}{C}} \cdot E\sin\omega t$$

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$$PI = \frac{1}{R \cdot D} \cdot E \sin \omega t$$

$$= \frac{E}{R} \int \sin \omega t \cdot dt$$

$$= \frac{E}{R} \times \frac{-\cos \omega t}{\omega}$$

$$\boxed{PI = \frac{-E}{\omega R} \cdot \cos \omega t}$$

General solⁿ is :- $q = CF + PI$

$$q = e^{-\frac{Rt}{2L}} \{ C_1 \cos pt + C_2 \sin pt \} \frac{-E}{\omega R} \cos \omega t \longrightarrow (2)$$

Differentiate eq (2) w.r.t t

$$\frac{dq}{dt} = i = e^{-\frac{Rt}{2L}} \{ -C_1 p \sin pt + C_2 p \cos pt \} + \{ C_1 \cos pt + C_2 \sin pt \} \times \left(\frac{-R}{2L}\right) e^{-\frac{Rt}{2L}} + \frac{E}{\omega R} \cdot \omega \sin \omega t \longrightarrow (3)$$

Given Condⁿ 1 :- $q = 0$ when $t = 0$

$$(2) \Rightarrow 0 = e^0 [(C_1 \times 1) + (C_2 \times 0)] \frac{-E}{\omega R} \times 1$$

$$0 = C_1 - \frac{E}{\omega R} \Rightarrow \boxed{C_1 = \frac{E}{\omega R}}$$

given Condⁿ (2) : $i = 0$ when $t = 0$

$$0 = 1 \{ (C_1 \times 0) + (C_2 p \times 1) \} + \{ (C_1 \times 1) + (C_2 \times 0) \} \left(\frac{-R}{2L} \times 1 \right) + 0$$

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$$0 = c_2 p + c_1 \times \frac{-R}{2L}$$

$$c_2 p = \frac{c_1 R}{2L}$$

$$c_2 = \frac{c_1 R}{2pL} = \frac{R}{2pL} \times \frac{E}{\omega R}$$

$$\boxed{c_2 = \frac{E}{2pL\omega}}$$

Substituting c_1 and c_2 in ②

$$q = e^{-\frac{Rt}{2L}} \left\{ \frac{E}{\omega R} \cos pt + \frac{E}{2pL\omega} \sin pt \right\} - \frac{E}{\omega R} \cos \omega t$$

taking $\frac{E}{\omega R}$ common

$$q = \frac{E}{\omega R} \left[e^{-\frac{Rt}{2L}} \left(\cos pt + \frac{R}{2pL} \sin pt \right) - \cos \omega t \right]$$

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