lab_symfseries

September 17, 2019

1 Fourier series

This workbook explores the Fourier series representation of signals. It is shown how symbolic methods can be used to calculate series coefficients, and the accuracy of the time-domain reconstruction for different numbers of coefficients is also considered.

1.1 Background

Any signal $x(t)$ that is periodic with a period T can be written in a Fourier series form

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

with $\omega_0 = 2\pi/T = \pi$ radians per second. The coefficients satisfy

$$c_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t} dt.$$

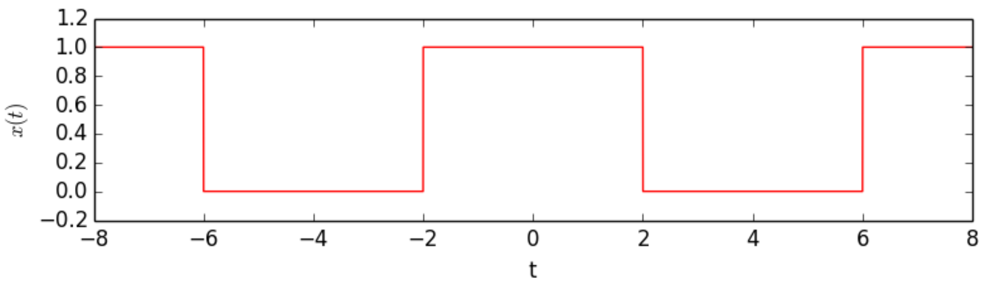
For real signals with $x(t) = x^*(t)$ one can show that $c_{-k} = c_k^*$. Writing in polar form $c_k = |c_k| e^{j\angle c_k}$ the series becomes

$$x(t) = c_0 + \sum_{k=1}^{\infty} (c_k e^{jk\omega_0 t} + c_{-k} e^{-jk\omega_0 t}) = c_0 + \sum_{k=1}^{\infty} |c_k| (e^{jk\omega_0 t} e^{j\angle c_k} + e^{-jk\omega_0 t} e^{-j\angle c_k}) = c_0 + \sum_{k=1}^{\infty} 2|c_k| \cos(\omega_0 t + \angle c_k)$$

The coefficient c_k corresponds to a complex exponential with frequency $k\omega_0$. We call the component of the signal with frequency $k\omega_0$ the kth *harmonic*. The first harmonic is also called the *fundamental*.

1.2 Signal definition and analysis

The following periodic signal is considered throughout this workbook:



The signal is has period $T = 8$, and $\omega_0 = 2\pi/8 = \pi/4$. We can find the corresponding Fourier series coefficients:

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{-2}^{2} e^{-jk\omega_0 t}dt = \frac{1}{jk\omega_0 T}[-e^{-jk\omega_0 t}]_{t=-2}^{2} = \frac{2}{k\omega_0 T}\sin(2k\omega_0).$$

Additionally, the DC coefficient $c_0 = 4/8$.

1.3 Fourier series reconstruction

For a given set of coefficients c_k we want to be able to plot the corresponding $x(t)$. The function defined in the cell below takes a set of Fourier series coefficients for a real signal `ckv` and a fundamental frequency `omega0`, and calculates reconstructed values `xv` at time instants `tv`.

```
In [1]: import numpy as np
        import matplotlib.pyplot as plt
        %matplotlib notebook

        def fsrrec(ckv,omega0,tv):
            """Generate samples from real Fourier series representation
            ckv - 0 to N Fourier series coefficients
            omega0 - fundamental frequency
            tv - input time points
            returns xv - output values
            """

            xv = ckv[0]*np.ones(tv.shape);
            for k in range(1,len(ckv)):
                xv = xv + 2*np.abs(ckv[k])*np.cos(k*omega0*tv + np.angle(ckv[k]));  # update w

            return(np.real(xv));
        # end def
```

```
In [2]: def fsrrec_plots(ckv,omega0,tv):
            """The same as fssrec, but also outputs a plot of the individual harmonic componen
            """
```

```
    xv = ckv[0]*np.ones(tv.shape);
    #tv.shape returns the dimensions of the matrix (or, in this case, array) tv.
    #np.ones creates an array of the input size populated with 1's
    plt.figure(1)
    plt.plot(tv,xv)
    for k in range(1,len(ckv)):
        kh = 2*np.abs(ckv[k])*np.cos(k*omega0*tv + np.angle(ckv[k]));  #create kth har
        plt.plot(tv,kh);

        xv = xv + kh; #add kth harmonic to x
    plt.show()
    return(np.real(xv));
# end def
```

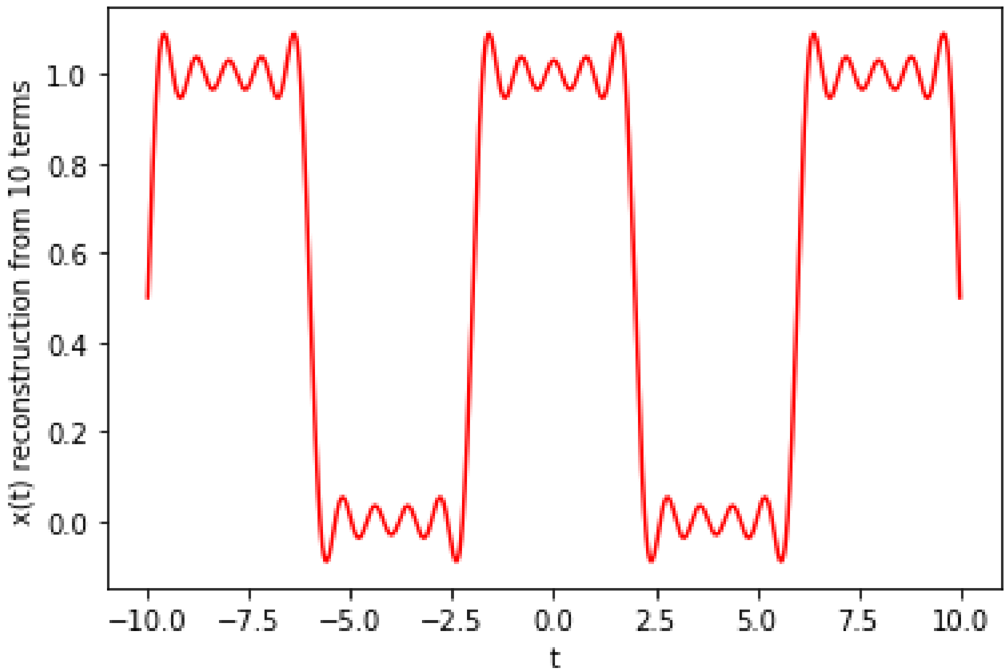
The cell below uses the derived expression for the coefficients of the signal and stores them in the vector `ckv`. The kth element of `ckv` contains the coefficient c_k.

```
In [3]: # Fourier series coefficients for rectangular pulse train
        T = 8;  # period
        N = 10;  # maximum number of terms
        omega0 = 2*np.pi/T;
        ckv = np.zeros(N+1, dtype=np.complex64);
        #np.zeros is like np.ones but with 0's

        for k in range(1,N+1):  ckv[k] = 2/(k*omega0*T)*np.sin(2*k*omega0);
        ckv[0] = 4/8;
```

We can use the `fsrrec` function to find the time-domain representation of the signal $x(t)$ using a finite number of terms in the reconstruction. The code below does this and plots the result.

```
In [4]: # Reconstruct from series representation and plot
        tv = np.linspace(-10,10,10000);
        xv = fsrrec(ckv,omega0,tv);
        #xv = fsrrec_plots(ckv,omega0,tv);
        plt.figure(2)
        plt.plot(tv,xv,'r');
        plt.xlabel('t');  plt.ylabel('x(t) reconstruction from ' + str(N) + ' terms');
```



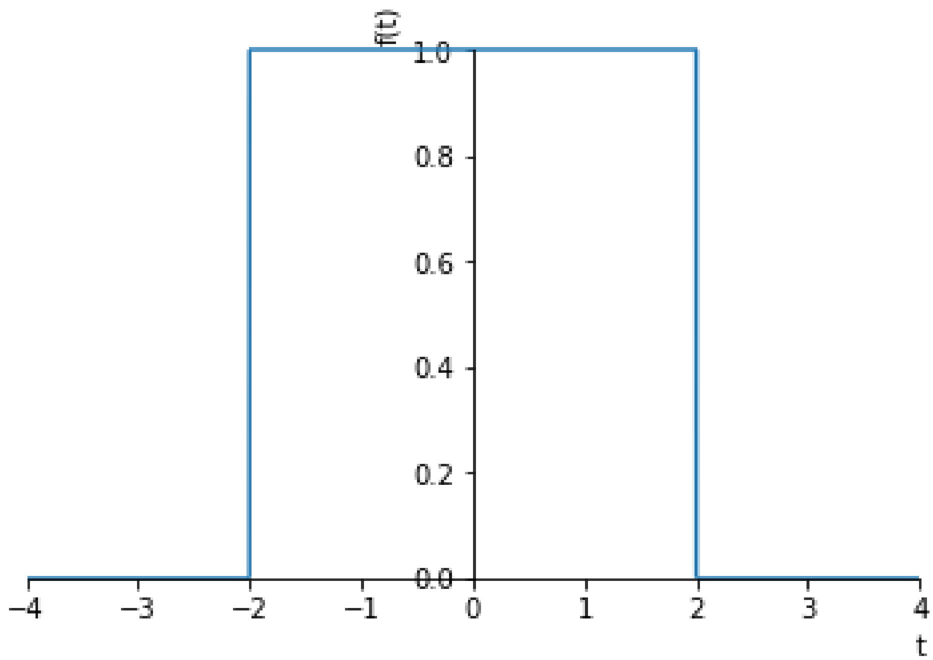
1.4 Finding the coefficients using symbolic math

This section will show how we can calculate the Fourier series coefficients of a signal using symbolic manipulation. The first thing to do is symbolically define the signal. The `Piecewise` function lets you define a signal over different pieces.

```
In [9]: import sympy as sp
        #sp.init_printing();  # pretty printing

        t = sp.symbols('t')
        x = sp.Piecewise( (0, t<=-2), (1, t<2), (0, True));
        # sp.plot(x, (t,-4,4));  # worked in Python 2.8!
        sp.plot(x.subs(t,sp.re(t)), (t,-4,4));  # plot (forcing t real)

        #?sp.Piecewise() #uncomment if you want more information about this function
```



The following cell defines the symbolic integral for computing the coefficients.

```
In [6]: Ts, k, w0 = sp.symbols('Ts k w0');
        w0 = 2*sp.pi/Ts;
        expt = sp.exp(-1j*k*w0*t);
        cke = 1/Ts*sp.integrate(x*expt, (t, -Ts/2, Ts/2));
        #cke = sp.integrate(x*expt, (t, -sp.oo, T/2)) - sp.integrate(x*expt, (t, -sp.oo, -T/2))
        ck = cke.subs(Ts,T).doit();  # set value for period and evaluate
        ck
```

Out[6]:

$$\frac{\begin{cases} -\frac{4.0ie^{0.5i\pi k}}{\pi k} + \frac{4.0ie^{-0.5i\pi k}}{\pi k} & \text{for } k > -\infty \wedge k < \infty \wedge k \neq 0 \\ 4 & \text{otherwise} \end{cases}}{8}$$

We now define a vector `kv` of coefficients of interest, and populate corresponding elements of `ckv` with the coefficient values.

```
In [7]: kv = np.arange(-10,11);  # coefficients to calculate
        #by default, np.arange returns the integers between the given start and end points

        ckvs = np.zeros(kv.shape, dtype=np.complex64);  # corresponding coefficient values
```

```
for i in range(len(kv)):
    ki = kv[i];
    ckvs[i] = ck.subs({k:ki}).evalf();
ckvs
```

```
Out[7]: array([ 0.        +0.j,  0.03536776+0.j, -0.        +0.j, -0.04547284+0.j,
                0.        +0.j,  0.06366198+0.j, -0.        +0.j, -0.10610329+0.j,
                0.        +0.j,  0.31830987+0.j,  0.5       +0.j,  0.31830987+0.j,
                0.        +0.j, -0.10610329+0.j, -0.        +0.j,  0.06366198+0.j,
                0.        +0.j, -0.04547284+0.j, -0.        +0.j,  0.03536776+0.j,
                0.        +0.j], dtype=complex64)
```

Now we can plot the frequency-domain representation of the signal $x(t)$ by displaying the value of c_k for each value k of interest. Since c_k can in general be complex we need two plots: one for magnitude and one for phase.

```
In [8]: fh, ax = plt.subplots(2);
        ax[0].stem(kv, np.abs(ckvs), c='g');  ax[0].set_ylabel(r'$|c_k|$');
        ax[1].stem(kv, np.angle(ckvs), c='g');  ax[1].set_ylabel(r'$\angle c_k$');
        plt.xlabel('$k$');
```

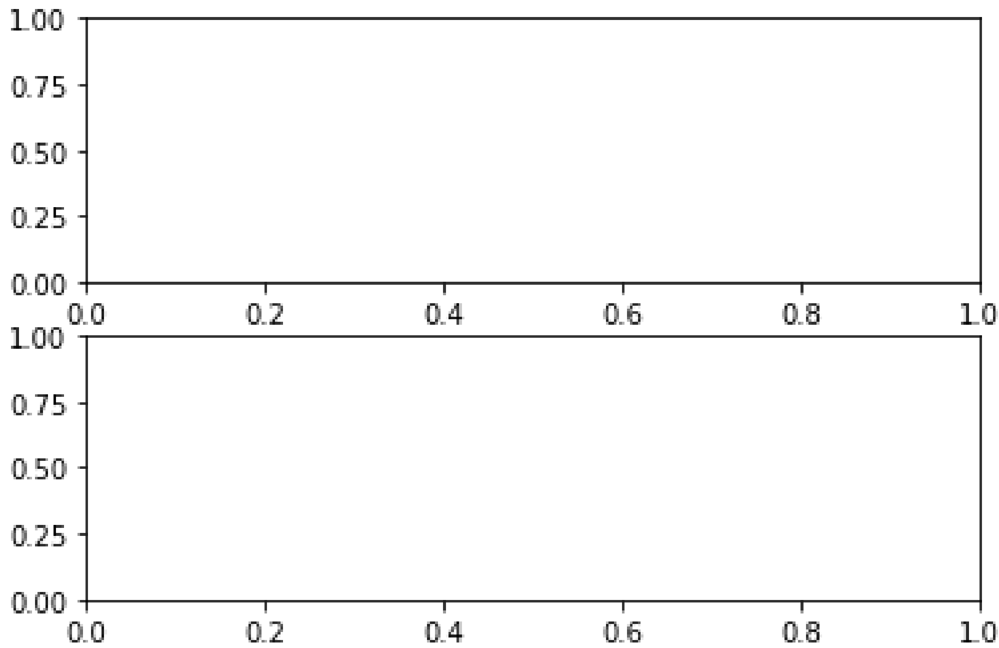
```
        ---------------------------------------------------------------------------

        TypeError                                 Traceback (most recent call last)

        <ipython-input-8-07e1feb2452a> in <module>
          1 fh, ax = plt.subplots(2);
    ----> 2 ax[0].stem(kv, np.abs(ckvs), c='g');  ax[0].set_ylabel(r'$|c_k|$');
          3 ax[1].stem(kv, np.angle(ckvs), c='g');  ax[1].set_ylabel(r'$\angle c_k$');
          4 plt.xlabel('$k$');


        ~/anaconda3/lib/python3.7/site-packages/matplotlib/__init__.py in inner(ax, data, *args
       1587     def inner(ax, *args, data=None, **kwargs):
       1588         if data is None:
    -> 1589             return func(ax, *map(sanitize_sequence, args), **kwargs)
       1590
       1591         bound = new_sig.bind(ax, *args, **kwargs)


        TypeError: stem() got an unexpected keyword argument 'c'
```



We could also have created a lambda function from the symbolic expression. This function takes an array of values for k and calculates c_k directly. Note though that at least on my version of Python `lam_ck(0)` generates a divide-by-zero error.

```
In [ ]: lam_ck = sp.lambdify(k,ck,modules=['numpy']);
        lam_ck(np.array((1,2,3)))
```

With numerical values for the coefficients, obtained via symbolic computation, we can plot the partial sum for the time-domain reconstruction as before. Recall that our `fsrrec` function only takes the coefficients for nonnegative index values.

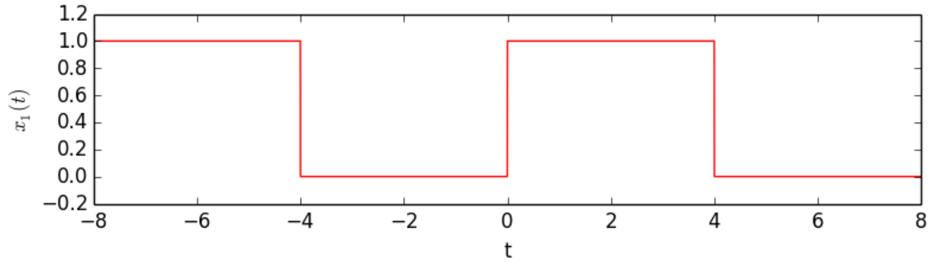
```
In [ ]: kzi = np.where(kv==0)[0][0];  # index for zero element
        ckvsp = ckvs[kzi:];
        tv = np.linspace(-10,10,10000);
        xv = fsrrec(ckvsp,2*np.pi/T,tv);

        fh = plt.figure();
        plt.plot(tv,xv,'r');
        plt.xlabel('t');  plt.ylabel('x(t) reconstruction from ' + str(len(ckvsp)-1) + ' terms
```

2 Tasks

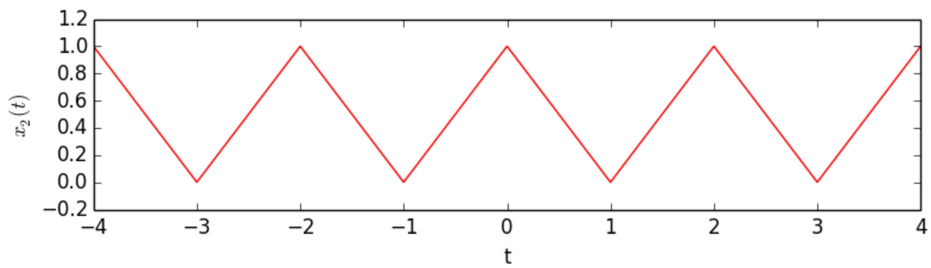
These tasks involve writing code, or modifying existing code, to meet the objectives described.

1. Find and plot the Fourier series frequency-domain representation for the signal $x_1(t)$ below over the range $k = -8, \ldots, 8$:



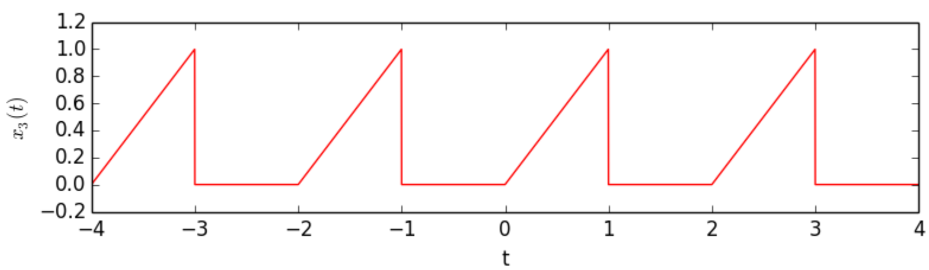
Do this using both symbolic processing and by evaluating the integral for the coefficients by hand. Compare the result with that displayed earlier for $x(t)$. Note that the signals are related in time by $x_1(t) = x(t-2)$. You should observe that shifting a signal only changes the phase in the frequency domain, while the magnitude remains unchanged.

- Use symbolic processing to find and plot the frequency-domain representation of $x_2(t)$ below over the range $k = -8, \ldots, 8$:



Also plot the reconstruction over the range $t = -4$ to $t = 4$ using only components up to and including the 5th harmonic. You should find that as k increases the magnitude of the coefficients in this case falls off much faster than those of $x(t)$. This is because $x_2(t)$ is smoother (it is at least continuous, while $x(t)$ is discontinuous). The reconstruction is therefore also more accurate with a smaller number of terms.

- Find and plot the frequency-domain representation of $x_3(t)$ below over the range $k =$



$-8, \ldots, 8$.

```
In [ ]:
```