2021 DIME Solutions

De Mathematics Competitions

February 14, 2021, to March 7, 2021

Aathreyakadambi, ApraTrip, AT2005, Awesome_guy, DeToasty3, firebolt360, GammaZero, i3435, jayseemath, karate7800, nikenissan, pog, richy, skyscraper, & vsamc

Credit goes to Online Test Seasonal Series (OTSS) for the booklet template, as well as bissue, Bole, djmathman, lrjr24, and P_Groudon for test-solving.

Answer Key:

1. (783)	2. (140)	3. (145)	4. (932)	5. (197)
6. (451)	7. (360)	8. (315)	9. (012)	10. (103)
11. (042)	12. (031)	13. (021)	14. (048)	15. (179)

Solutions:

1. Find the remainder when the number of positive divisors of the value

$$(3^{2020}+3^{2021})(3^{2021}+3^{2022})(3^{2022}+3^{2023})(3^{2023}+3^{2024})$$

is divided by 1000.

Proposed by pog

Answer (783): *Solution by pog*

Let x be equal to 3^{2020}. Thus,

$$(3^{2020}+3^{2021})(3^{2021}+3^{2022})(3^{2022}+3^{2023})(3^{2023}+3^{2024})$$

is equal to $(x+3x)(3x+9x)(9x+27x)(27x+81x)$, or

$$(3+1)x\cdot(3+9)x\cdot(9+27)x\cdot(27+81)x = 4\cdot 12\cdot 36\cdot 108\cdot 3^{8080}.$$

Finding the prime factorization of our expression gives

$$4\cdot 12\cdot 36\cdot 108\cdot 3^{8080} = 2^2\cdot(2^2\cdot 3)\cdot(2^2\cdot 3^2)\cdot(2^2\cdot 3^3)\cdot 3^{8080} = 2^8\cdot 3^{8086}.$$

For each divisor, we can choose a power of 2 from 0 to 8 and a power of 3 from 0 to 8086, so there are $(8+1)(8086+1)$ divisors of $2^8\cdot 3^{8086}$. Thus our answer is equal to $9\cdot 8087 \equiv 9\cdot 87 \equiv \boxed{783} \pmod{1000}$. ∎

2. If x is a real number satisfying the equation

$$9\log_3 x - 10\log_9 x = 18\log_{27} 45,$$

then the value of x is equal to $m\sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. Find $m+n$.

Proposed by pog

Answer (140): *Solution by pog*

Note that $\log_9 x = \frac{1}{2}\log_3 x$ and $\log_{27} x = \frac{1}{3}\log_3 x$, so

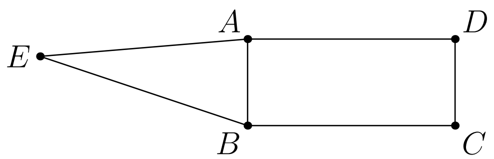
$$9\log_3 x - 10\log_9 x = 18\log_{27} 45 \implies 9\log_3 x - 5\log_3 x = 6\log_3 45.$$

Thus $4\log_3 x = 6\log_3 45 \implies \log_3 x = \frac{3}{2}\log_3 45$, so $x = 45^{\frac{3}{2}} = 45\sqrt{45}$. Noting that $45 = 3^2 \cdot 5$, x is equal to

$$45 \cdot \sqrt{(3^2)} \cdot \sqrt{5} = 45 \cdot 3 \cdot \sqrt{5} = 135\sqrt{5},$$

so our answer is $135 + 5 = \boxed{140}$. ■

3. In the diagram below, rectangle $ABCD$ has $AB = 5$ and $AD = 12$. Also, E is a point in the same plane outside $ABCD$ such that the perpendicular distances from E to the lines AB and AD are 12 and 1, respectively, and $\triangle ABE$ is acute. There exists a line passing through E which splits $ABCD$ into two figures of equal area. Suppose that this line intersects $\overline{AB}$ at a point F and $\overline{CD}$ at a point G. Find FG^2.



Proposed by ApraTrip

Answer (145): *Solution by DeToasty3*

In the coordinate plane, let $A = (0,5)$, $B = (0,0)$, $C = (12,0)$, and $D = (12,5)$. Then, $E = (-12,4)$. The key insight is that the line must pass through the center of the rectangle, $(6, 2.5)$, in order for the line to split $ABCD$ into two figures of equal area. With this in mind, we find that the slope of the line is

$$\frac{2.5-4}{6-(-12)} = \frac{1.5}{18} = \frac{1}{12},$$

so the equation of the line is

$$y - 4 = \frac{1}{12}(x + 12) \implies y = \frac{x}{12} + 5.$$

To find the points F and G, we plug in $x = 0$ and $x = 12$, respectively, giving the points $F = (0, 5)$ and $G = (12, 6)$. Finally, by the Pythagorean Theorem, we get

$$FG^2 = (0 - 12)^2 + (5 - 6)^2 = 144 + 1 = \boxed{145},$$

as desired. ■

4. There are 7 balls in a jar, numbered from 1 to 7, inclusive. First, Richard takes a balls from the jar at once, where a is an integer between 1 and 6, inclusive. Next, Janelle takes b of the remaining balls from the jar at once, where b is an integer between 1 and the number of balls left, inclusive. Finally, Tai takes all of the remaining balls from the jar at once, if any are left. Find the remainder when the number of possible ways for this to occur is divided by 1000, if it matters who gets which ball.

   Proposed by firebolt360 & DeToasty3

   Answer (932): *Solution by firebolt360*

   We will use PIE. First, we calculate the total number of ways to distribute the balls, which is $3^7 = 2187$. Now, we calculate the number of ways for Richard to get 0 balls. Since the balls must go to either Janelle or Tai, we get $2^7 = 128$ total ways. Similarly, the number of ways for Janelle to get no balls is $2^7 = 128$. However, we have overcounted the case where both Janelle and Richard don't get any balls. There is only 1 way for this to happen. So, our answer is $2187-128-128+1 = 1932 \implies \boxed{932}$, as desired. ■

5. Let $\mathcal{S}$ be the set of all positive integers which are both a multiple of 3 and have at least one digit that is a 1. For example, 123 is in $\mathcal{S}$ and 450 is not. The probability that a randomly chosen 3-digit positive integer is in $\mathcal{S}$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

   Proposed by GammaZero

   Answer (197): *Solution by GammaZero*

   We first count the number of 3-digit numbers, which is $999 - 100 + 1 = 900$. Then we break this up into cases.

   Case 1: The hundreds digit is $0 \pmod 3$. If the hundreds digit is $0 \pmod 3$, then we see that the only numbers that work in the tens and ones digit are $12, 15, 18, 21, 51, 81,$

and there are only 3 possible numbers that can be in the hundreds digit: $3, 6, 9$. Thus, the number of possible values for this case is $3 \cdot 6 = 18$.

Case 2: The hundreds digit is $1 \pmod 3$. If the hundreds digit is $1 \pmod 3$, then we see that the only numbers that work in the tens and ones digit are $11, 14, 17, 41, 71$. We see that the only numbers that work are $4, 7$. We don't think about when the hundreds digit is 1 right now, since that case is a bit different. Thus, we get $2 \cdot 5 = 10$.

Case 3: The hundreds digit is $2 \pmod 3$. If the hundreds digit is $2 \pmod 3$, then we see that the only numbers that work in the tens and ones digit are are $1, 10, 13, 16, 19, 31, 61, 91$, and there are only 3 possible values for the hundreds digit: $2, 5, 8$. Thus, the total number of ways is $3 \cdot 8 = 24$.

Case 4: The hundreds digit is 1. If the hundreds digit is 1, then every single multiple of 3 between 100 and 200 will work, and thus, there are 33 numbers that work here.

Thus, we get

$$\frac{18 + 10 + 24 + 33}{900} = \frac{85}{900} = \frac{17}{180},$$

so our answer is $17 + 180 = \boxed{197}$. ■

6. Let ABC be a right triangle with right angle at A and side lengths $AC = 8$ and $BC = 16$. The lines tangent to the circumcircle of $\triangle ABC$ at points A and B intersect at D. Let E be the point on side $\overline{AB}$ such that $\overline{AD} \parallel \overline{CE}$. Then DE^2 can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Proposed by Awesome_guy

Answer (451): *Solution by lrjr24*

Let O be the circumcenter of $\triangle ABC$. Since $OA = OB = OC = 8$, we have that $\triangle ACO$ is an equilateral triangle. This means $\angle OAB = 180° - \angle COA = 120°$. Since $\triangle OAB$ is an isosceles triangle, we have that $\angle OAB = \angle OBA = 30°$, which means $\angle DBA = 90° - \angle OBA = 60°$, and similarly, $\angle DAB = 60°$. This means that $\triangle DAB$ is an equilateral triangle with side length $8\sqrt{3}$. Now since $ACED$ is a trapezoid,

$$\begin{aligned} &\angle CAD + \angle ACE = 180° \\ &\implies \angle OAD + \angle CAO + \angle ACE = 180° \\ &\implies \angle ACE = 30°, \end{aligned}$$

so $\overline{CE}$ is the angle bisector of $\angle ACB$. This means

$$AE = \frac{1}{3} \cdot 8\sqrt{3} = \frac{8\sqrt{3}}{3}.$$

Then by Law of Cosines on $\triangle DAE$, we get

$$DE^2 = 192 + \frac{64}{3} - 2 \cdot 8\sqrt{3} \cdot \frac{8\sqrt{3}}{3} \cdot \cos(60°) = \frac{448}{3},$$

so the answer is $\boxed{451}$. ■

7. In a game, Jimmy and Jacob each randomly choose to either roll a fair six-sided die or to automatically roll a 1 on their die. If the product of the two numbers face up on their dice is even, Jimmy wins the game. Otherwise, Jacob wins. The probability Jimmy wins 3 games before Jacob wins 3 games can be written as $\frac{p}{2^q}$, where p and q are positive integers, and p is odd. Find the remainder when $p+q$ is divided by 1000.

Proposed by firebolt360

Answer (360): *Solution by firebolt360*

Firstly, we calculate the probability that Jimmy wins a single game. Obviously only the parity of the number rolled matters and not what actually comes up.

- If only one rolls (probability $\frac{1}{2}$), then the probability it is even is $\frac{1}{2}$, so if only one rolls, the probability is $\frac{1}{4}$.
- If both roll (probability $\frac{1}{4}$), the probability of it being even is $1 - \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}$, so this gives $\frac{3}{16}$.
- If neither roll, then the probability is 0, because the product is 1.

Thus, Jimmy wins the game with $\frac{1}{4} + \frac{3}{16} = \frac{7}{16}$ chance and Jacob wins with $\frac{9}{16}$ chance.

Now, we use a simple casework to find the answer:

Case 1: Jimmy wins in 3 games. This clearly has probability $\frac{7^3}{16^3}$.

Case 2: Jimmy wins in 4 games. Then Jacob wins 1 game. Since Jimmy wins the last game, there are 3 places for Jacob to win. The probability here is then $3 \cdot \frac{7^3 \cdot 9}{16^3 \cdot 16}$.

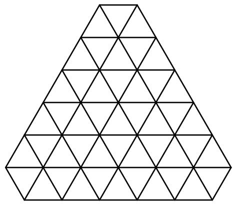
Case 3: Jimmy wins in 5 games. Then Jacob wins 2 games. Analogously, there are $\binom{4}{2}$ ways to place Jacobs wins. The probability here is then $6 \cdot \frac{7^3 \cdot 9^2}{16^3 \cdot 16^2}$.

Now, we sum the probabilities.

$$\begin{aligned}
&\frac{7^3}{16^3} + 3 \cdot \frac{7^3 \cdot 9}{16^3 \cdot 16} + 6 \cdot \frac{7^3 \cdot 9^2}{16^3 \cdot 16^2} \\
&= \frac{7^3(16^2 + 3 \cdot 16 \cdot 9 + 6 \cdot 9^2}{16^5} \\
&= \frac{7^3(256 + 432 + 486)}{16^5} \\
&= \frac{7^3(1174)}{16^5} \\
&= \frac{343 \cdot 587}{2^{19}}.
\end{aligned}$$

Hence, $p + q = 343 \cdot 587 + 19 \equiv 341 + 19 \implies \boxed{360} \pmod{1000}$. ■

8. In the diagram below, a group of equilateral triangles are joined together by their sides. A parallelogram in the diagram is defined as a parallelogram whose vertices are all at the intersection of two grid lines and whose sides all travel along the grid lines. Find the number of distinct parallelograms in the diagram below.



Proposed by Awesome_guy

Answer (315): *Solution by DeToasty3 & richy*

For each pair of vertices not on the same grid line, we see that this determines the longer diagonal of a unique parallelogram. If they are on the same grid line, then we do not form a parallelogram. Thus, it suffices to count the number of pairs of vertices that are not on the same grid line. In total, there are 33 vertices, so the total number of pairs of vertices is $\binom{33}{2} = 528$. Now, we complementary count. Without loss of generality, let the two vertices on the same grid line be parallel to the bottom edge (we will multiply by 3 at the end). Going grid line by grid line, we see that the number of vertices on each edge are $2, 3, 4, \dots, 7, 6$. We have that

$$\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \cdots + \binom{7}{2} + \binom{6}{2} = \binom{8}{3} + \binom{6}{2} = 56 + 15 = 71.$$

Our answer is $528 - 3(71) = 528 - 213 = \boxed{315}$. ■

9. Real numbers a, b, c, and d satisfy the system of equations

$$\begin{aligned} -a - 27b - 8d &= 1, \\ 8a + 64b + c + 27d &= 0, \\ 27a + 125b + 8c + 64d &= 1, \\ 64a + 216b + 27c + 125d &= 8. \end{aligned}$$

Find $12a + 108b + 48d$.

Proposed by firebolt360

Answer (012): *Solution by firebolt360*

Seeing so many cubes, we are naturally motivated to come up with some cubic expression involving $a, b, c,$ and d.

Recall the property that 2 polynomials of degree n are the same if and only if they meet in $n+1$ distinct points. Since the cubes on the LHS are constantly increasing (write the first equation as $a + 27b + 0c + 8d = -1$), we define a function f to be:

$$f(x) = a(x+1)^3 + b(x+3)^3 + cx^3 + d(x+2)^3$$

and define a function g to be

$$g(x) = (x-1)^3.$$

Then note that

$$\begin{aligned} f(0) &= g(0) \\ f(1) &= g(1) \\ f(2) &= g(2) \\ f(3) &= g(3) \end{aligned}$$

Since both are polynomials of degree 3, they are the same. Now we wish to find $12a + 108b + 48d$.

Setting $f(x) = g(x)$ and expanding gives

$$\begin{aligned} (a+b+c+d)x^3 + (3a+9b+6d)x^2 + &(3a+27b+12c)x + a + 27b + 8d \\ &= x^3 - 3x^2 + 3x - 1 \\ \implies 3a + 27b + 12d = 3. \end{aligned}$$

Finally, multiplying by 4 gives $12a + 108b + 48d = \boxed{012}$. ■

10. There exist complex numbers $z_1, z_2, \ldots, z_{10}$ which satisfy

$$|z_k i^k + z_{k+1} i^{k+1}| = |z_{k+1} i^k + z_k i^{k+1}|$$

for all integers $1 \le k \le 9$, where $i = \sqrt{-1}$. If $|z_1| = 9$, $|z_2| = 29$, and for all integers $3 \le n \le 10$, $|z_n| = |z_{n-1} + z_{n-2}|$, find the minimum value of $|z_1| + |z_2| + \cdots + |z_{10}|$.

Proposed by DeToasty3

Answer (103): *Solution by DeToasty3*

If we want to minimize the sum of the magnitudes of the complex numbers, thinking in terms of cis form, intuition tells us that we want z_k and z_{k+1} to be $180°$ apart (i.e. point in opposite directions from the origin in the complex plane). Checking the equation

$$|z_k i^k + z_{k+1} i^{k+1}| = |z_{k+1} i^k + z_k i^{k+1}|,$$

we see that this works. Thus, in order to minimize $|z_1| + |z_2| + \cdots + |z_{10}|$, we must make z_k and z_{k+1} alternate angles by $180°$.

This means that the magnitude of z_{k+2} is the absolute difference between the magnitudes of z_k and z_{k+1}. Since $|z_1| = 9$ and $|z_2| = 29$, we may continue to list them out: $|z_3| = |9-29| = 20$, $|z_4| = |29-20| = 9$, and so on. We find that the remaining magnitudes from z_5 to z_{10} are 11, 2, 9, 7, 2, 5, in that order.

Thus, $|z_1| + |z_2| + \cdots + |z_{10}| = 9+29+20+9+11+2+9+7+2+5 = \boxed{103}$. ■

11. Call a positive integer k *pretty* if for every positive integer a, there exists an integer n such that n^2+n+k is divisible by 2^a but not 2^{a+1}. Find the remainder when the 2021st pretty number is divided by 1000.

    Proposed by i3435

    Answer (042): *Solution by lrjr24*

    Note that the condition says that for every positive integer a, there exists n such that $n^2+n+k \equiv 2^a \pmod{2^{a+1}}$. Note that $a=1$ implies that k must be even. Let $k=2b$. We will try to prove that for $a \geq 2$, the condition gives us no info on a. We will use induction.

    Base Case ($a=2$): $k \equiv 0 \pmod 8$ has a solution for $n=4$. $k \equiv 2 \pmod 8$ has a solution for $n=6$. $k \equiv 4 \pmod 8$ has a solution for $n=8$. $k \equiv 6 \pmod 8$ has a solution for $n=2$, so we get no info.

    Inductive step (a to $a+1$): Fix k. Let $n^2+n+k \equiv 2^a \pmod{2^{a+1}}$. Then we have

    $$n^2+n+k \equiv 2^a, 3\cdot 2^a \pmod{2^{a+2}}.$$

    We also know that

    $$(2^{a+1}+n)^2 + (2^{a+1}+n) + k \equiv 2^a \pmod{2^{a+1}},$$

    as well as

    $$\begin{aligned}&(2^{a+1}+n)^2 + (2^{a+1}+n) + k \\ &\equiv 2^{2a+2} + 2^{a+2}n + n^2 + 2^{a+1} + n + k \\ &\equiv 2^{a+1} + n^2 + n + k \pmod{2^{a+2}}.\end{aligned}$$

    This means that we can achieve both cases of what n^2+n+k can be $\pmod{2^{a+2}}$, so we have finished our induction.

    Thus all even k work and the answer is $4042 \equiv \boxed{042} \pmod{1000}$. ■

12. Let $\omega_1, \omega_2, \omega_3, \ldots, \omega_{2020!}$ be the distinct roots of $x^{2020!}-1$. Suppose that n is the largest integer such that 2^n divides the value

    $$\sum_{k=1}^{2020!} \frac{2^{2019!}-1}{\omega_k^{2020}+2}.$$

Then n can be written as $a! + b$, where a and b are positive integers, and a is as large as possible. Find the remainder when $a + b$ is divided by 1000.

Proposed by vsamc

Answer (031): *Solution by vsamc*

WLOG, let $r_k = e^{2\pi ik/2020!}$. Then, all k equivalent to each other modulo 2019! give the same value of r_k^{2020} because

$$r_{2019!\cdot m+k} = e^{2\pi ik/2020!+m\cdot 2\pi i\cdot 2019!/2020!} = e^{2\pi ik/2020!}e^{m\cdot 2\pi i\frac{1}{2020}},$$

so

$$r_{2019!\cdot m+k}^{2020} = e^{2\pi ik/2019!}e^{m\cdot 2\pi i} = e^{2\pi ik/2019!}.$$

Since there are $\frac{2020!}{2019!} = 2020$ k congruent to each other modulo 2019!, $r_p^{2020} = r_q^{2020}$, for fixed p, has 2020 solutions.

Now, note that $r_k^{2020} = e^{2\pi ik/2019!}$, so r_k^{2020} for $1 \le k \le 2020!$ are all roots of $x^{2019!} - 1$. Since $r_p^{2020} = r_q^{2020}$ for fixed p has 2020 solutions, we can write for each $1 \le j \le 2019!$ that $s_j = r_k^{2020}$ for 2020 values of r, where $s_1, s_2, \ldots, s_{2019!}$ are the solutions to $Q(x) = x^{2019!} - 1$.

Thus, our sum divided by $2^{2019!} - 1$ becomes

$$\frac{2020}{s_1 + 2} + \frac{2020}{s_2 + 2} + \frac{2020}{s_3 + 2} + \cdots + \frac{2020}{s_{2019!} + 2}.$$

Finally, note the roots of $x^{2019!}\left(\left(\frac{1}{x} - 2\right)^{2019!} - 1\right)$ are $\frac{1}{s_i+2}$ for $1 \le i \le 2019!$. Note that the $x^{2019!}$ coefficient of this polynomial is

$$x^{2019!}((-2)^{2019!} - 1) = (2^{2019!} - 1)x^{2019!}$$

and the $x^{2019!-1}$ coefficient of this polynomial is

$$x^{2019!}\binom{2019!}{1}(-2)^{2019!-1}\frac{1}{x} = 2019!(-2)^{2019!-1}x^{2019!-1},$$

thus by Vieta's, the sum of the roots of this polynomial (which is $\sum \frac{1}{s_k+2}$) is

$$-\frac{2019!(-2)^{2019!-1}}{2^{2019!} - 1} = -\frac{2019!(-2)^{2019!-1}}{(-2)^{2019!} - 1} = \frac{2019!2^{2019!-1}}{2^{2019!} - 1},$$

so

$$2020\sum \frac{1}{s_k + 2} = \frac{2020!2^{2019!}}{2(2^{2019!} - 1)}.$$

Thus, the highest power of 2 in the expression when we multiply back that $2^{2019!} - 1$ is the highest power of 2 of $2020! \cdot 2^{2019!-1}$, which is $2019! - 1$ plus the highest power of 2 in 2020!, which can be computed by Legendre to be

$$\sum_{n=1}^{\infty} \left\lfloor \frac{2020}{2^n} \right\rfloor = 1010 + 505 + 252 + 126 + 63 + 31 + 15 + 7 + 3 + 1 = 2013.$$

Thus, $n = 2019! + 2012$, so $a + b = 4031 \equiv \boxed{031} \pmod{1000}$. ■

13. Let $\triangle ABC$ have side lengths $AB = 7$, $BC = 8$, and $CA = 9$. Let D be the projection from A to $\overline{BC}$ and D' be the reflection of D over the perpendicular bisector of $\overline{BC}$. Let P and Q be distinct points on the line through D' parallel to $\overline{AC}$ such that $\angle APB = \angle AQB = 90^\circ$. The value of $AP + AQ$ can be written as $\frac{a+b\sqrt{c}}{d}$, where a, b, c, and d are positive integers such that b and d are relatively prime, and c is not divisible by the square of any prime. Find $a + b + c + d$.

Proposed by i3435

Answer (021): *Solution by ApraTrip*

Suppose that P' is the point such that quadrilateral $ADBP'$ is a parallelogram. Notice that then we have $AP' = BD = CD'$ and $AP' \parallel CD'$, so quadrilateral $ACD'P'$ is a parallelogram (implying $D'P \parallel AC$.) Furthermore, notice that

$$\angle AP'B = \angle ADB = 90^\circ,$$

so either $P' \equiv P$ or $P' \equiv Q$. Without loss of generality, assume that $P' \equiv P$.

Now, notice that by Heron's Formula, $[ABC] = 12\sqrt{5}$, so $AD = 3\sqrt{5}$. Thus, by the Pythagorean Theorem, $BD = AP = 2$. Now, by angle chasing,

$$\angle APQ = \angle APD' = \angle ACB,$$

$$\angle AQP = \angle ABP = 90 - \angle ABC.$$

By the law of sines,

$$\frac{AP}{\sin(\angle AQP)} \cdot \sin(\angle APQ) = AQ,$$

so since $\sin(90 - \angle ABC) = \frac{2}{7}$ and $\sin \angle ACB = \frac{\sqrt{5}}{3}$, $AQ = \frac{7\sqrt{5}}{3}$.

Thus, $AP + AQ = \frac{6+7\sqrt{5}}{3}$, so our answer is $\boxed{021}$. ■

14. For a positive integer n not divisible by 211, let $f(n)$ denote the smallest positive integer k such that $n^k - 1$ is divisible by 211. Find the remainder when

$$\sum_{n=1}^{210} nf(n)$$

is divided by 211.

Proposed by ApraTrip

Answer (048): *Solution by ApraTrip*

Let all sums in this solution be taken $\pmod{211}$.

Clearly, $f(n)$ is the order of n with respect to 211, so thus, $f(n) \mid 210$. Thus, the desired sum can be written as

$$\sum_{k|210} k(\text{sum of numbers with order } k).$$

Dealing with the k term isn't too tricky, so we will try simplify the term in the parenthesis. We will do this by using polynomials.

Case 1: $k = 1$. Clearly, in this case, the sum of numbers with order 1 is 1.

Case 2: k has one prime factor. Note that in this case, all numbers with order k must be factors of the polynomial $x^k - 1 \equiv 0 \pmod{211}$ or in order words $x^k \equiv 1 \pmod{211}$. Clearly, the order of all solutions to $x^k \equiv 1 \pmod{211}$ must divide k, so since k only has one prime factor, the orders must be either 1 or k. Thus, since the sum of numbers with order k and 1 is 0 by Vieta's Formulas and the sum of numbers with order 1 is 1, the sum of numbers with order k is -1.

Case 3: k has two prime factors. Let $k = ab$, where a and b are primes (we can do this as according to our case, k has two prime factors). Note that using the same polynomial idea, the sum of numbers with order k is equal to

$$\begin{aligned}&0 - (\text{sum of numbers with order } 1)\\ &\quad -(\text{sum of numbers with order } a) - (\text{sum of numbers with order } b).\end{aligned}$$

Thus, by our previous case, the sum of numbers with order k is equal to $0-1+2 = 1$.

Note that this logic can be generalized (this is actually due to the binomial theorem) - if a number $k \mid 210$ has n prime factors, then the sum of numbers with order k is equal to $(-1)^n$. (Also, note that we don't ever have to worry about repeating prime factors, as $210 = 2 \cdot 3 \cdot 5 \cdot 7$.)

Thus, our desired sum is $\sum_{k|210} k \cdot (-1)^{\text{the number of prime factors of } k}$. Note that by factoring, this sum is $(1-2)(1-3)(1-5)(1-7)$, so the answer is $\boxed{048}$. ■

15. Let right $\triangle ABC$ have $AC = 3$, $BC = 4$, and right angle at C. Let D be the projection from C to $\overline{AB}$. Let ω be a circle with center D and radius $\overline{CD}$, and let E be a variable point on the circumference of ω. Let F be the reflection of E over point D, and let O be the center of the circumcircle of $\triangle ABE$. Let H be the intersection of the altitudes of $\triangle EFO$. As E varies, the path of H traces a region $\mathcal{R}$. The area of $\mathcal{R}$ can be written as $\frac{m\pi}{n}$, where m and n are relatively prime positive integers. Find $\sqrt{m} + \sqrt{n}$.

    Proposed by AppraTrip

    Answer (179): *Solution by i3435*

    First, note that $ABEF$ is cyclic because $DF \cdot DE = DC^2 = DA \cdot DB$. Thus, we have that $\triangle OEF$ is isosceles. This means that $DO \cdot DH = DF^2 = DE^2$, because of Power of a Point on the circle with diameter $\overline{OH}$. Thus, $DO \cdot DH$ is a constant. Let X and Y be the intersection of ω with the perpendicular bisector of $\overline{AB}$ and let M be the midpoint of $\overline{AB}$. Then H stays on (DXY), since $DX^2 = DY^2 = DF^2$ (inversion/similar triangles should work here too).

    We want to find the circumradius of $\triangle DXY$. Knowing that $DM = \frac{7}{10}$ and $DX = \frac{12}{5}$, we get $XY = \frac{\sqrt{527}}{5}$. The area of $\triangle DXY$ is thus $\frac{7\sqrt{527}}{100}$. The circumradius is thus

    $$\frac{\frac{144}{25} \cdot \frac{\sqrt{527}}{5}}{\frac{7\sqrt{527}}{25}} = \frac{144}{35},$$

    so our answer is $\boxed{179}$. ■