De Mathematics Competitions

2nd Annual

DMC 10

Friday, May 7, 2021

INSTRUCTIONS

1. DO NOT OPEN THIS BOOKLET UNTIL YOU DECIDE TO BEGIN.
2. This is a twenty-five question multiple choice test. Each question is followed by answers marked A, B, C, D, and E. Only one of these is correct.
3. Mark your answer to each problem on the DMC 10 Answer Form with a keyboard. Check the keys for accuracy and erase errors and stray marks completely.
4. You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
5. No aids are permitted other than writing utensils, blank scratch paper, rulers, compasses, and erasers. No calculators, smartwatches, or computing devices are allowed. No problems on the test will require the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the test, your non-existent proctor will not ask you to record certain information on the answer form.
8. When you give the signal, begin working on the problems. You will have 75 minutes to complete the test. You can discuss only with people that have already taken the test in the private discussion forum until the end of the contest window.
9. When you finish the exam, don't sign your name in the space not provided on the Answer Form.

The Committee on the De Mathematics Competitions reserves the right to re-examine students before deciding whether to grant official status to their scores. The Committee also reserves the right to disqualify all scores from a school if it is determined that the required security procedures were not followed.

Students who score well on this DMC 10 may or may not be invited to the 2022 DIME (De Invitational Mathematics Examination). More details about the DIME and other information are on the back page of this test booklet.

2021 DMC 10

DO NOT OPEN UNTIL FRIDAY, May 7, 2021

Questions and complaints about problems and solutions for this exam should be sent by private message to:

DeToasty3.

The 2022 DIME may or may not be held. It would be a 15-question, 3-hour, integer-answer exam if it was to be held. You may or may not be invited to participate because this contest may or may not exist. *A complete listing of our previous publications may be found at our web site:*

`http://detoasty3.gq/DMC`

****Try Administering This Exam On An Earlier Date. Oh Wait, You Can't.****

1. All the information needed to administer this exam is not contained in the non-existent DMC 10 Teacher's Manual.
2. YOU must not verify on the non-existent DMC 10 COMPETITION CERTIFICATION FORM that you followed all rules associated with the administration of the exam.
3. Send **DeToasty3**, **nikenissan**, **pog**, and **vsamc** a private message submitting your answers to the DMC 10. AoPS is the only way to submit your answers.
4. The publication, reproduction or communication of the problems or solutions of this exam during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, e-mail, World Wide Web or media of any type during this period is a violation of the competition rules.

The **2021 De Mathematics Competitions**
was made possible by the contributions of the following people:

AypraTrip, AT2005, Awesome_guy, DeToasty3, firebolt360, GammaZero, HrishiP, i3435, jayseemath, JustinLee2017, nikenissan, pog, skyscraper, & vsamc

Credit goes to Online Test Seasonal Series (OTSS) for the booklet template.

1. The sum of the first five positive integers and the sum of the first six positive integers are multiplied. What is the resulting product?

(A) 315 **(B)** 335 **(C)** 355 **(D)** 375 **(E)** 395

2. What is the value of

$$\frac{1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + 4 \cdot 4!}{(2 \times 0 \times 2 \times 1) + (2^3 + 0^3 + 2^3 + 1^3)}?$$

(A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

3. Let n be a positive integer less than 2021. It is given that if a regular hexagon is rotated n degrees clockwise about its center, the resulting hexagon coincides with the original hexagon. How many possible values of n are there?

(A) 8 **(B)** 16 **(C)** 17 **(D)** 32 **(E)** 33

4. Today is Toasty's birthday. It is given that the square of his age is 32 less than the number of months he is old. What is the sum of Toasty's possible ages?

(A) 6 **(B)** 8 **(C)** 10 **(D)** 12 **(E)** 14

5. If the product of three distinct positive real numbers forming a geometric progression is equal to 2197, what is the median of the three numbers?

(A) 11 **(B)** 12 **(C)** 13 **(D)** 14 **(E)** 15

6. Four children and four adults are standing in a line, but every child insists on being in between a child and an adult. How many ways can the eight people be arranged to meet these demands? (Assume the children are identical and the adults are identical.)

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

7. A sequence is defined such that the first term is equal to 1, and every subsequent term is equal to 2021 more than 5 times the preceding term. For example, the second term is $5 \cdot 1 + 2021 = 2026$, and the third term is $5 \cdot 2026 + 2021 = 12151$. What is the value of the 2021st term divided by the 2020th term, rounded to the nearest integer?

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

8. A positive integer n exists such that n^3 has four times as many divisors as n. What is the sum of the three smallest values of n with this property?

(A) 26 **(B)** 27 **(C)** 28 **(D)** 29 **(E)** 30

9. In the coordinate plane, let $\mathcal{P}$ be the figure formed by the set of points with coordinates satisfying $0.5x + y = 1$, and let $\mathcal{Q}$ be the figure formed by the set of points with coordinates satisfying $0.25x^2 + y^2 = 1$. How many points lie on both $\mathcal{P}$ and $\mathcal{Q}$?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** infinitely many

10. How many real numbers x satisfy the equation

$$9^x + 3^{3x} = 3^{x+1} + 3?$$

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

11. How many of the following statements are always true?

- The product of the lengths of the diagonals of a rectangle is equal to the area of the rectangle.
- The figure formed by the midpoints of the sides of a rectangle has half the area of the rectangle.
- The figure formed by the intersections of the internal angle bisectors of a rectangle with unequal side lengths is a rectangle with unequal side lengths.
- Consider any point inside a rectangle. If the point is reflected over all its sides, the figure formed by the reflection points has twice the area of the rectangle.

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

12. Bob is at the origin of the coordinate plane with a laser which he can only fire along the positive x-axis. There is also a mirror represented by a line passing through the point $(3, 0)$. The mirror serves to reflect the path of Bob's laser across the line through $(3, 0)$ perpendicular to the mirror. If Bob's laser passes through the point $(6, 3)$, what is the degree measure of the acute angle formed by the mirror and the x-axis?

(A) 15 **(B)** 22.5 **(C)** 30 **(D)** 37.5 **(E)** 45

13. Let ω be the inscribed circle of a rhombus $ABCD$ with side length 4 and $\angle DAB = 60°$. There exist two distinct lines which are parallel to line BD and tangent to ω. Given that the lines intersect sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ at points P, Q, R, and S, respectively, what is the area of quadrilateral $PQRS$?

(A) $2\sqrt{3}$ **(B)** 4 **(C)** $3\sqrt{3}$ **(D)** 6 **(E)** $4\sqrt{3}$

14. Alice goes cherry picking in a forest. For each tree Alice sees, she either picks one cherry or three cherries from the tree and puts them in her basket. Additionally, after every five trees Alice picks from, she finds an extra cherry on the ground and puts it in her basket. At the end, Alice has 45 cherries in her basket. If the smallest possible number of trees Alice could have picked from is n, what is the sum of the digits of n?

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

15. Each of 6 distinct positive integers is placed at each of 6 equally spaced points on the circumference of a circle. If the numbers on every two adjacent points are relatively prime, and the product of the numbers on every two diametrically opposite points is divisible by 3, what is the least possible sum of the 6 integers?

(A) 12 **(B)** 25 **(C)** 26 **(D)** 29 **(E)** 32

16. If a and b are the distinct roots of the polynomial $x^2 + 2021x + 2019$, then

$$\frac{1}{a^2 + 2019a + 2019} + \frac{1}{b^2 + 2019b + 2019} = \frac{m}{n},$$

where m and n are relatively prime positive integers. What is $m + n$?

(A) 2020 **(B)** 2021 **(C)** 4040 **(D)** 6059 **(E)** 6061

17. Let $\triangle ABC$ have $AB = 20$, $AC = 21$, and a right angle at A. Let I be the center of the inscribed circle of $\triangle ABC$. Let point D be the reflection of point B over the line parallel to AB passing through I, and let point E be the reflection of point C over the line parallel to AC passing though I. What is the value of DE^2?

(A) 145 **(B)** 149 **(C)** 153 **(D)** 157 **(E)** 161

18. Let $b > 6$ be an integer. There exist base-b and base-$(b+1)$ numbers such that

$$8\gcd(600_b, 660_b) = 45\gcd(100_{b+1}, 110_{b+1}).$$

What is the sum of the digits of b?

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

19. Let ω_1 and ω_2 be circles with centers O_1 and O_2 and radii 6 and 1, respectively, and let ω_1 and ω_2 intersect at distinct points X and Y. Given that there is a point P on line XY such that $PO_1 = 18$, what is the length PO_2?

(A) 16 **(B)** 17 **(C)** 18 **(D)** 19 **(E)** 20

20. Hanami rolls two standard six-sided dice. If the sum of the numbers she rolled is at least 7, she rolls both dice again (and does not roll again thereafter). Otherwise, she does not roll again. What is the probability that she rolls a 5 on at least one of the dice, on at least one of the rolls?

(A) $\frac{1}{3}$ **(B)** $\frac{13}{36}$ **(C)** $\frac{29}{72}$ **(D)** $\frac{11}{27}$ **(E)** $\frac{4}{9}$

21. In equilateral $\triangle ABC$, let points D and E be on lines AB and AC, respectively, both on the opposite side of line BC as A. If $CE = DE$, and the circumcircle of $\triangle CDE$ is tangent to line AB at D, what is the degree measure of $\angle CDE$?

(A) 70 **(B)** 72 **(C)** 75 **(D)** 80 **(E)** 84

22. If the sum of the digits of the base-three representation of

$$\frac{(3^0+1)^3+1}{(3^0)^2+3^0+1} + \frac{(3^1+1)^3+1}{(3^1)^2+3^1+1} + \cdots + \frac{(3^{15}+1)^3+1}{(3^{15})^2+3^{15}+1}$$

is equal to S, what is the value of S when expressed in base-ten?

(A) 12 **(B)** 13 **(C)** 14 **(D)** 15 **(E)** 16

23. In pentagon $ABCDE$, where all interior angles have a positive degree measure less than 180°, let M be the midpoint of side $\overline{DE}$. It is given that line BM splits $ABCDE$ into two isosceles trapezoids $ABME$ and $CDMB$ such that each one contains exactly three sides of equal length. If $AE = 3$ and $DE = 26$, what is the area of $ABCDE$?

(A) 216 **(B)** 234 **(C)** 288 **(D)** 312 **(E)** 330

24. Let $P(x)$ be a polynomial with degree 3 and real coefficients such that the coefficient of the x^3 term is 1, and $P(x)$ has roots a, b, and c that satisfy

$$\frac{-(a+b)(b+c)(c+a)}{2022} = abc = 2021.$$

What is the minimum possible value of $|P(1)|$?

(A) 2019 **(B)** 2020 **(C)** 2021 **(D)** 2022 **(E)** 2023

25. Ryan has an infinite supply of slips and a spinner with letters O, S, and T, where each letter is equally likely to be spun. Each minute, Ryan spins the spinner randomly, writes on a blank slip the letter he spun, and puts it in a pile. Ryan continues until he has written all 3 letters at least once, at which point he stops. What is the probability that after he stops, he can form the words $OTSS$ and $TOST$ using 4 distinct slips from the pile? (Ryan may reuse slips he used for one word in forming the other.)

(A) $\frac{7}{54}$ **(B)** $\frac{13}{72}$ **(C)** $\frac{2}{9}$ **(D)** $\frac{8}{27}$ **(E)** $\frac{1}{3}$