De Mathematics Competitions

2nd Annual

DMC 10B

Friday, July 2, 2021

INSTRUCTIONS

1. DO NOT OPEN THIS BOOKLET UNTIL YOU DECIDE TO BEGIN.
2. This is a twenty-five question multiple choice test. Each question is followed by answers marked A, B, C, D, and E. Only one of these is correct.
3. Mark your answer to each problem on the DMC 10 Answer Form with a keyboard. Check the keys for accuracy and erase errors and stray marks completely.
4. You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
5. No aids are permitted other than writing utensils, blank scratch paper, rulers, compasses, and erasers. No calculators, smartwatches, or computing devices are allowed. No problems on the test will require the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the test, your non-existent proctor will not ask you to record certain information on the answer form.
8. When you give the signal, begin working on the problems. You will have 75 minutes to complete the test. You can discuss only with people that have already taken the test in the private discussion forum until the end of the contest window.
9. When you finish the exam, don't sign your name in the space not provided on the Answer Form.

The Committee on the De Mathematics Competitions reserves the right to re-examine students before deciding whether to grant official status to their scores. The Committee also reserves the right to disqualify all scores from a school if it is determined that the required security procedures were not followed.

Students who score well on this DMC 10 may or may not be invited to the 2022 DIME (De Invitational Mathematics Examination). More details about the DIME and other information are on the back page of this test booklet.

1. What is the value of

$$4^1 - 3^2 + 2^3 - 1^4?$$

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

2. Let a, b, c, and d be real numbers that satisfy

$$a + 1 = b,\ b + 2 = c,\ c + 3 = d,\ d + 4 = 21.$$

What is a?

(A) 11 **(B)** 12 **(C)** 13 **(D)** 14 **(E)** 15

3. A square has a perimeter which is twice the area of the circle inscribed in the square. What is the circumference of the circle?

(A) 2 **(B)** 4 **(C)** 8 **(D)** 16 **(E)** 32

4. A group of 200 people were invited to see a movie, where each person either had first row seats, second row seats, or neither. It is given that four-fifths of the people invited chose to watch the movie, one-ninth of the viewers were not invited, one-fifth of the viewers had first row seats, and 60 of the viewers had neither first nor second row seats. What is the probability that a randomly selected viewer had second row seats?

(A) $\frac{3}{10}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{7}{15}$ **(D)** $\frac{1}{2}$ **(E)** $\frac{4}{5}$

5. Hideki has three weightless boxes and four pebbles, each of which has a weight of either 3, 4, or 5 ounces. He puts each pebble in one of the boxes such that each box has at least one pebble in it. If the weights of the boxes form an increasing arithmetic sequence, what is the largest possible sum of the weights of the four pebbles, in ounces?

(A) 14 **(B)** 15 **(C)** 16 **(D)** 17 **(E)** 18

6. At Test Academy, there are 4 classes numbered 1, 2, 3, and 4. For each $1 \leq n \leq 3$, class $n + 1$ has twice as many students and an average grade half of that of class n. If the average grade of class 1 is 90, what is the average grade of all 4 classes combined?

(A) 15 **(B)** 18 **(C)** 21 **(D)** 24 **(E)** 27

7. How many ordered pairs of positive integers (a, b) exist such that the product

$$(2a - 2020)(b - 2021)$$

is positive and has exactly three positive divisors?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

8. In regular hexagon $ABCDEF$, diagonals $\overline{AC}$ and $\overline{BF}$ intersect at a point G. If the area of $\triangle ABG$ is 2, what is the area of hexagon $ABCDEF$?

(A) 18 **(B)** 24 **(C)** 27 **(D)** 32 **(E)** 36

9. Let a and b be positive integers. If a is divisible by 15 but not 20, and b is divisible by 20 but not 15, how many possible distinct two-digit values of $a + b$ are there?

(A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

10. In a plane, eight rays emanate from a point P such that every two adjacent rays form an acute angle with measure 45°. Next, a finite line segment in the plane is drawn. If the line segment shares at least one point with exactly n of the rays, what is the sum of all possible values of n? (If the line segment passes through P, then $n = 8$.)

(A) 13 **(B)** 14 **(C)** 17 **(D)** 18 **(E)** 23

11. For certain real numbers x and y, the first three terms of a geometric sequence are $x - 2$, $2y$, and $x + 2$ (in that order), and the sum of those three terms is equal to 4. What is the fifth term of the geometric sequence?

(A) $\frac{144}{7}$ **(B)** $\frac{64}{3}$ **(C)** $\frac{196}{9}$ **(D)** $\frac{256}{11}$ **(E)** $\frac{128}{5}$

12. A circle has radius 8 and a diameter $\overline{AB}$. A distinct circle ω in the same plane intersects the circle at point A and another point C, and point D lies on ω such that $\overline{CD}$ is a diameter of ω. If $ABCD$ is a parallelogram, what is the perimeter of $ABCD$?

(A) 16 **(B)** 24 **(C)** 32 **(D)** 36 **(E)** 48

13. Let x and y be real numbers such that

$$|x - |y - x|| = 1,$$
$$|y - |x - y|| = 2.$$

What is the largest possible value of $x + y$?

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

14. In a motel, there are 12 rooms arranged in a row, and there are some visitors who want to stay at the motel. A visitor may rent one room for 5 dollars, or two adjacent rooms for 4 dollars each. At most one visitor may rent a given room at a time, and no two visitors may rent rooms adjacent to each other. If the leftmost and rightmost rooms must be rented, what is the maximum dollar amount that the motel can earn?

(A) 30 **(B)** 31 **(C)** 32 **(D)** 33 **(E)** 34

15. For all positive integers n, let $f(n) = 1 + 2 + 3 + \cdots + n$, and let

$$g(n) = f(1) \times f(2) \times f(3) \times \cdots \times f(n).$$

How many integers k (between 1 and 100, inclusive) are there such that $f(k)$ and $g(k)$ have the same last digit?

(A) 10 **(B)** 12 **(C)** 20 **(D)** 21 **(E)** 22

16. Four people are in a tournament where every person duels each other person exactly once. Every duel results in one person winning and the other losing (no ties). After the tournament, each person counts the number of wins they have and adds one, and then their numbers are multiplied. What is the minimum possible resulting product?

(A) 15 **(B)** 16 **(C)** 18 **(D)** 24 **(E)** 27

17. Two rays emanating from a point form a 60° angle. Two circles ω_1 and ω_2 of radius 4 are such that ω_1 is tangent to both rays, and ω_2 is tangent to only one of the rays, on the same side of the ray as ω_1. If there exists a circle of radius 3, externally tangent to ω_2 and tangent to both rays, what is the distance between the centers of ω_1 and ω_2?

(A) $2\sqrt{6}$ **(B)** $3\sqrt{3}$ **(C)** $4\sqrt{2}$ **(D)** 6 **(E)** $4\sqrt{3}$

18. How many positive divisors does the number $2^{19} - 2^9 - 1$ have?

(A) 16 **(B)** 18 **(C)** 24 **(D)** 32 **(E)** 36

19. In trapezoid $ABCD$ with $\overline{AB} \parallel \overline{CD}$, $AB = 4$, and $AD = BC = 5$, let the angle bisector of $\angle ADC$ intersect the diagonal $\overline{AC}$ at a point P. If line BP intersects the side $\overline{CD}$ at a point Q such that $CQ = 8$, what is the area of trapezoid $ABCD$?

(A) 24 **(B)** 28 **(C)** 32 **(D)** 36 **(E)** 40

20. Bill and Ben each have 2 fair coins. Each turn, they flip all their coins at the same time, if they have any. If a coin lands heads, then the other person gets that coin. If a coin lands tails, then that coin stays with the same person. What is the probability that after Bill and Ben take exactly 3 turns, they each end up with 2 coins?

(A) $\frac{1}{4}$ **(B)** $\frac{5}{16}$ **(C)** $\frac{3}{8}$ **(D)** $\frac{7}{16}$ **(E)** $\frac{1}{2}$

21. What is the sum of the digits of the smallest positive integer n such that

$$\sqrt{5n-1} - \sqrt{5n-2} + \sqrt{5n-3} - \sqrt{5n-4}$$

is strictly less than 0.05?

(A) 8 **(B)** 9 **(C)** 10 **(D)** 11 **(E)** 12

22. How many positive integers n (greater than 1) are there such that when the number $57^n + 64^n$ is divided by $n!$, the remainder is equal to 1?

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

23. There are 15 people in a room, where everyone shakes hands with a positive number of other people in the room exactly once. If exactly 6 people shook 1 hand, exactly 5 people shook between 2 and 4 hands, inclusive, exactly 1 person shook 8 hands, and exactly 1 person shook 14 hands, what is the least possible total number of handshakes?

(A) 24 **(B)** 25 **(C)** 26 **(D)** 27 **(E)** 28

24. In $\triangle ABC$ with $AB = 3$ and $AC = 6$, let D be the intersection of the angle bisector of $\angle BAC$ and $\overline{BC}$, and let M be the midpoint of $\overline{AC}$. Let the circumcircle of $\triangle DMC$ intersect line AD again at a point P, distinct from D. If $DM = 2$, what is CP^2?

(A) $\frac{54}{5}$ **(B)** 12 **(C)** $\frac{66}{5}$ **(D)** $\frac{72}{5}$ **(E)** $\frac{78}{5}$

25. Let x and y be distinct real numbers chosen at random from the interval $[-100, 100]$, excluding 0. What is the probability that

$$\left\lfloor \frac{|x|}{|y|} \right\rfloor \geq \left\lfloor \frac{|x+y|}{|x-y|} \right\rfloor,$$

where $\lfloor r \rfloor$ denotes the greatest integer less than or equal to a real number r?

(A) $\frac{1}{2}$ **(B)** $\frac{9}{16}$ **(C)** $\frac{7}{12}$ **(D)** $\frac{5}{8}$ **(E)** $\frac{2}{3}$

2021 DMC 10B

DO NOT OPEN UNTIL FRIDAY, July 2, 2021

Questions and complaints about problems and solutions for this exam should be sent by private message to:

DeToasty3.

The 2022 DIME may or may not be held. It would be a 15-question, 3-hour, integer-answer exam if it was to be held. You may or may not be invited to participate because this contest may or may not exist. *A complete listing of our previous publications may be found at our web site:*

`https://detoasty3.github.io/dmc.html`

****Try Administering This Exam On An Earlier Date. Oh Wait, You Can't.****

1. All the information needed to administer this exam is not contained in the non-existent DMC 10 Teacher's Manual.
2. YOU must not verify on the non-existent DMC 10 COMPETITION CERTIFICATION FORM that you followed all rules associated with the administration of the exam.
3. Send **DeToasty3**, **nikenissan**, **pog**, and **vsamc** a private message submitting your answers to the DMC 10. AoPS is the only way to submit your answers.
4. The publication, reproduction or communication of the problems or solutions of this exam during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, e-mail, World Wide Web or media of any type during this period is a violation of the competition rules.

The **2021 De Mathematics Competitions**
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