Single-Source Shortest Path

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Design and Analysis of Algorithms

Single-Source Shortest Paths

- **Given:** A single source vertex in a weighted, directed graph.
- Want to compute a shortest path for each possible destination.
  - Similar to BFS.
- We will assume either
  - no negative-weight edges, or
  - no reachable negative-weight cycles.
- Algorithm will compute a shortest-path tree.
  - Similar to BFS tree.

General Results (Relaxation)

Lemma 24.1: Let $p = \langle v_1, v_2, \ldots, v_k \rangle$ be a SP from v_1 to v_k. Then, $p_{ij} = \langle v_i, v_{i+1}, \ldots, v_j \rangle$ is a SP from v_i to v_j, where $1 \le i \le j \le k$.

So, we have the optimal-substructure property.
Bellman-Ford's algorithm uses dynamic programming.
Dijkstra's algorithm uses the greedy approach.
Let $\delta(u, v)$ = weight of SP from u to v.

Corollary: Let p = SP from s to v, where $p = s \xrightarrow{p'} u \rightarrow v$. Then, $\delta(s, v) = \delta(s, u) + w(u, v)$.

Lemma 24.10: Let $s \in V$. For all edges $(u,v) \in E$, we have $\delta(s, v) \le \delta(s, u) + w(u,v)$.

- Lemma 24.1 holds because one edge gives the shortest path, so the other edges must give sums that are at least as large.

Relaxation

```
Initialize(G, s)
    for each v ∈ V[G] do
        d[v] := ∞;
        π[v] := NIL
    od;
    d[s] := 0
```

Algorithms keep track of d[v], π[v]. **Initialized** as follows:

```
Relax(u, v, w)
    if d[v] > d[u] + w(u, v) then
        d[v] := d[u] + w(u, v);
        π[v] := u
    fi
```

These values are changed when an edge (u, v) is **relaxed**:

Properties of Relaxation

- d[v], if not ∞, is the length of *some* path from s to v.
- d[v] either stays the same or decreases with time
- Therefore, if d[v] = δ(s, v) at any time, this holds thereafter
- Note that d[v] $\geq$ δ(s, v) always
- After i iterations of relaxing on all (u,v), if the shortest path to v has i edges, then d[v] = δ(s, v).

Properties of Relaxation

Consider any algorithm in which d[v], and π[v] are first initialized by calling Initialize(G, s) [s is the source], and are only changed by calling Relax. We have:

Lemma 24.11: ($\forall$ v:: d[v] $\geq \delta$(s, v)) is an invariant.

Implies d[v] doesn't change once d[v] = δ(s, v).

Proof:

Initialize(G, s) establishes invariant. If call to Relax(u, v, w) changes d[v], then it establishes:

d[v] = d[u] + w(u, v)

$\geq \delta$(s, u) + w(u, v) , invariant holds before call.

$\geq \delta$(s, v) , by Lemma 24.10.

Corollary 24.12: If there is no path from s to v, then d[v] = δ(s, v) = ∞ is an invariant.

- For lemma 24.11, note that initialization makes the invariant true at the beginning.

More Properties

Lemma 24.13: Immediately after relaxing edge (u, v) by calling Relax(u, v, w), we have $d[v] \le d[u] + w(u, v)$.

Lemma 24.14: Let p = SP from s to v, where p = $s \xrightarrow{p'} u \to v$.
If $d[u] = \delta(s, u)$ holds at any time prior to calling Relax(u, v, w), then $d[v] = \delta(s, v)$ holds at all times after the call.

Proof:

After the call we have:

$$\begin{aligned} d[v] &\le d[u] + w(u, v) && \text{, by Lemma 24.13.} \\ &= \delta(s, u) + w(u, v) && \text{, } d[u] = \delta(s, u) \text{ holds.} \\ &= \delta(s, v) && \text{, by corollary to Lemma 24.1.} \end{aligned}$$

By Lemma 24.11, $d[v] \ge \delta(s, v)$, so $d[v] = \delta(s, v)$.

- Lemma 24.13 follows simply from the structure of Relax.
- Lemma 24.14 shows that the shortest path will be found one vertex at a time, if not faster. Thus after a number of iterations of Relax equal to V(G) - 1, all shortest paths will be found.

- Bellman-Ford returns a compact representation of the set of shortest paths from s to all other vertices in the graph reachable from s. This is contained in the predecessor subgraph.

Predecessor Subgraph

Lemma 24.16: Assume given graph G has no negative-weight cycles reachable from s. Let G_π = predecessor subgraph. G_π is always a tree with root s (i.e., this property is an invariant).

Proof:

Two proof obligations:

(1) G_π is acyclic.

(2) There exists a unique path from source s to each vertex in V_π.

Proof of (1):

Suppose there exists a cycle $c = \langle v_0, v_1, \ldots, v_k \rangle$, where $v_0 = v_k$. We have $\pi[v_i] = v_{i-1}$ for $i = 1, 2, \ldots, k$. Assume relaxation of (v_{k-1}, v_k) created the cycle. We show cycle has a negative weight.

Note: Cycle must be reachable from s. (Why?)

Proof of (1) (Continued)

Before call to Relax(v_{k-1}, v_k, w):

$\pi[v_i] = v_{i-1}$ for i = 1, …, k–1. Implies $d[v_i]$ was last updated by "$d[v_i] := d[v_{i-1}] + w(v_{i-1}, v_i)$" for i = 1, …, k–1. [Because Relax updates π.]

Implies $d[v_i] \geq d[v_{i-1}] + w(v_{i-1}, v_i)$ for i = 1, …, k–1. [Lemma 24.13]

Because $\pi[v_k]$ is changed by call, $d[v_k] > d[v_{k-1}] + w(v_{k-1}, v_k)$. Thus,

$$\sum_{i=1}^{k} d[v_i] > \sum_{i=1}^{k} (d[v_{i-1}] + w(v_{i-1}, v_i))$$

$$= \sum_{i=1}^{k} d[v_{i-1}] + \sum_{i=1}^{k} w(v_{i-1}, v_i)$$

Because $\sum_{i=1}^{k} d[v_i] = \sum_{i=1}^{k} d[v_{i-1}]$, $\sum_{i=1}^{k} w(v_{i-1}, v_i) < 0$, i.e., neg.-weight cycle!

Comment on Proof

- $d[v_i] \geq d[v_{i-1}] + w(v_{i-1}, v_i)$ for $i = 1, \ldots, k–1$ because when Relax(v_{i-1} , v_i , w) was called, there was an equality, and $d[v_{i-1}]$ may have gotten smaller by further calls to Relax.
- $d[v_k] > d[v_{k-1}] + w(v_{k-1}, v_k)$ before the last call to Relax because that last call changed $d[v_k]$.

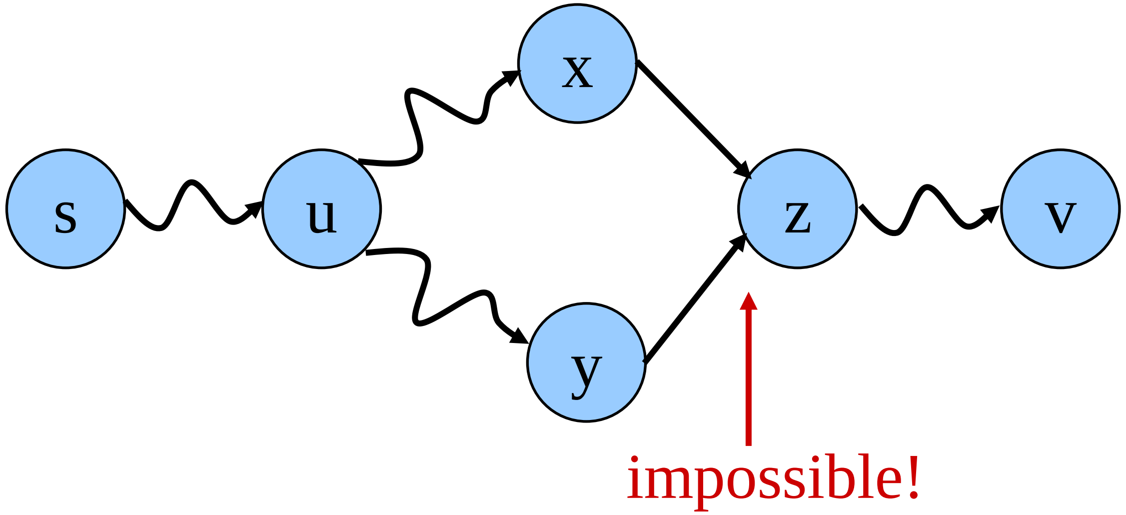
Proof of (2)

($\forall$ v: v $\in$ V_π:: ($\exists$ path from s to v)) is an invariant.

So, for any v in V_π, $\exists$ at least 1 path from s to v.

Show $\leq$ 1 path.

Assume 2 paths.



Lemma 24.17

Lemma 24.17: Same conditions as before. Call Initialize & repeatedly call Relax until $d[v] = \delta(s, v)$ for all v in V. Then, G_π is a shortest-path tree rooted at s.

Proof:

Key Proof Obligation: For all v in V_π, the unique simple path p from s to v in G_π (path exists by Lemma 24.16) is a shortest path from s to v in G.

Let $p = \langle v_0, v_1, \ldots, v_k \rangle$, where $v_0 = s$ and $v_k = v$.

We have $d[v_i] = \delta(s, v_i)$

$d[v_i] \geq d[v_{i-1}] + w(v_{i-1}, v_i)$ (reasoning as before)

Implies $w(v_{i-1}, v_i) \leq \delta(s, v_i) - \delta(s, v_{i-1})$.

Proof (Continued)

$$
\begin{aligned}
& w(p) \\
&= \sum_{i=1}^{k} w(v_{i-1}, v_i) \\
&\le \sum_{i=1}^{k} (\delta(s, v_i) - \delta(s, v_{i-1})) \\
&= \delta(s, v_k) - \delta(s, v_0) \\
&= \delta(s, v_k)
\end{aligned}
$$

So, equality holds and p is a shortest path.

- And note that this shortest path tree will be found after V(G) - 1 iterations of Relax.

Bellman-Ford Algorithm

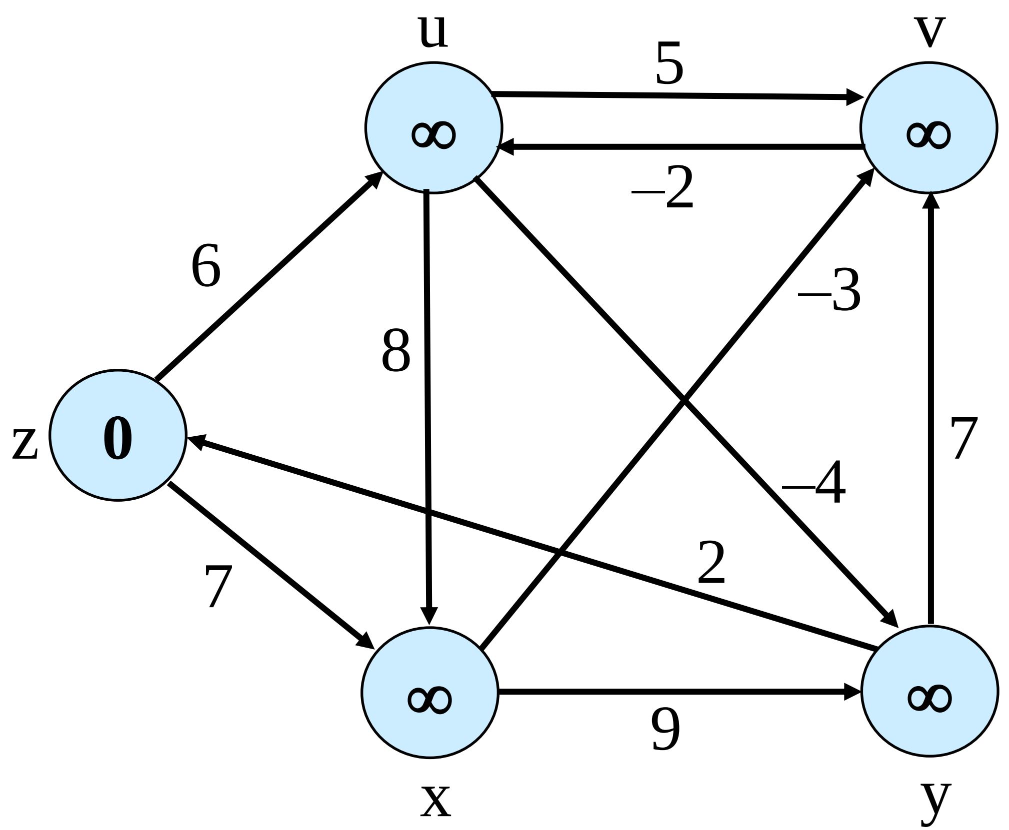
Can have negative-weight edges. Will "detect" reachable negative-weight cycles.

```
Initialize(G, s);
for i := 1 to |V[G]| –1 do
      for each (u, v) in E[G] do
            Relax(u, v, w)
      od
od;
for each (u, v) in E[G] do
      if d[v] > d[u] + w(u, v) then
            return false
      fi
od;
return true
```

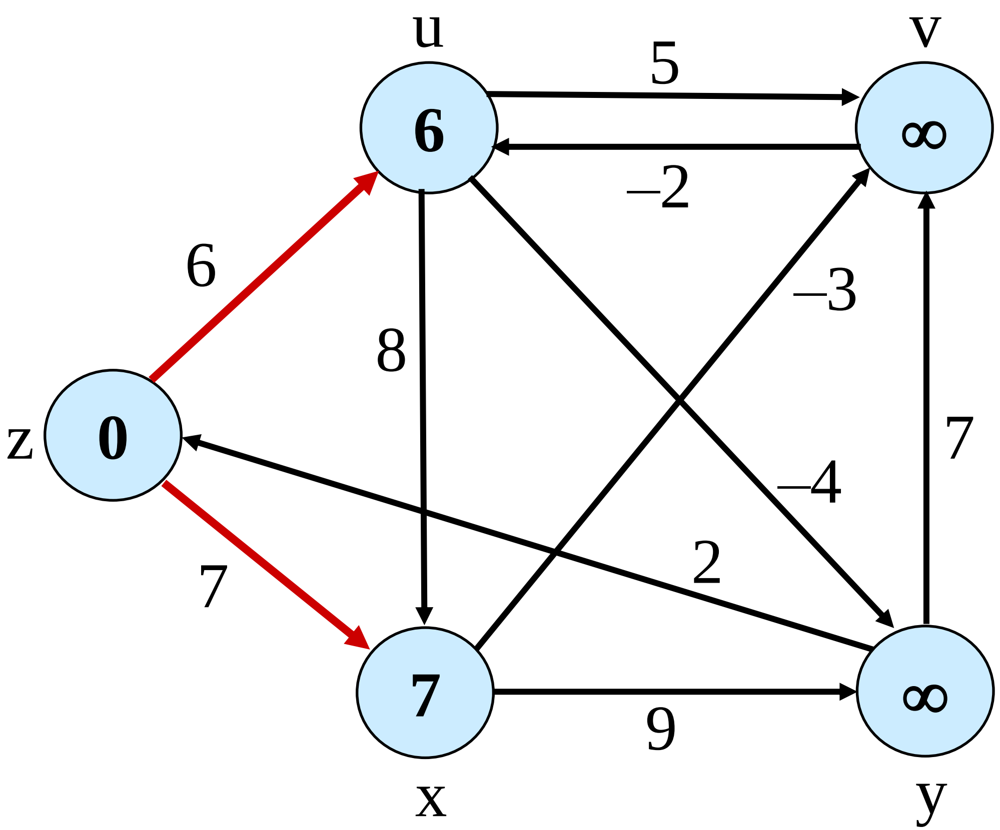
Time Complexity is O(VE).

- So if Bellman-Ford has not converged after V(G) - 1 iterations, then there cannot be a shortest path tree, so there must be a negative weight cycle.

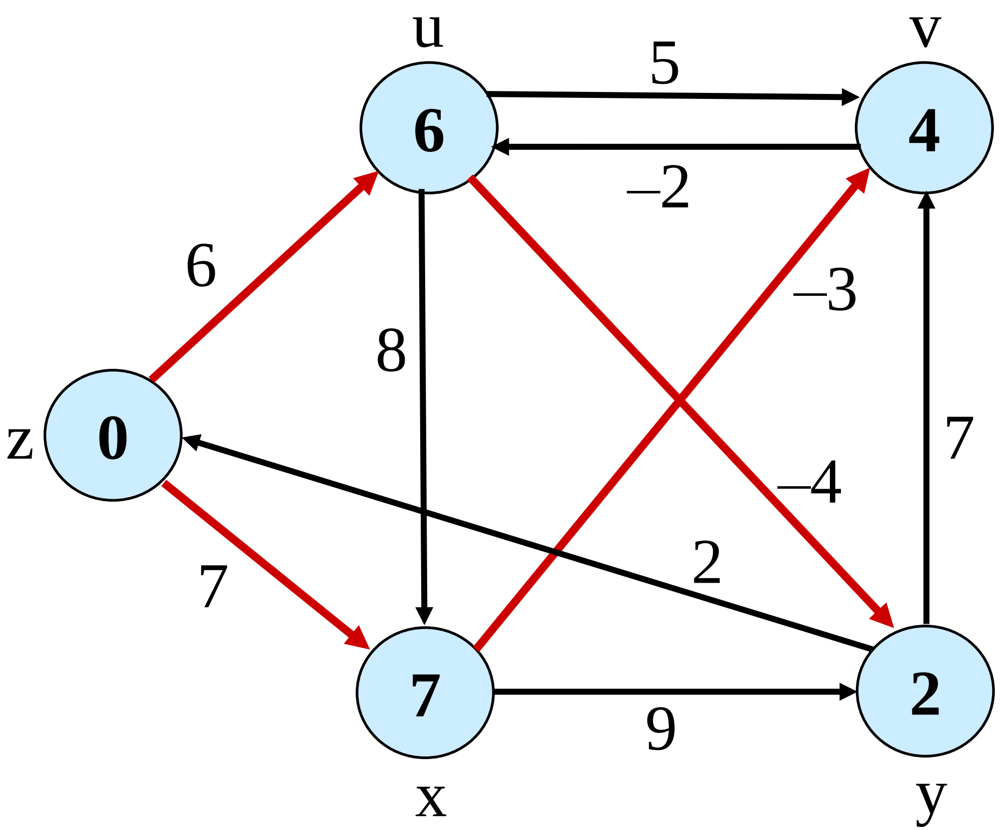
Example



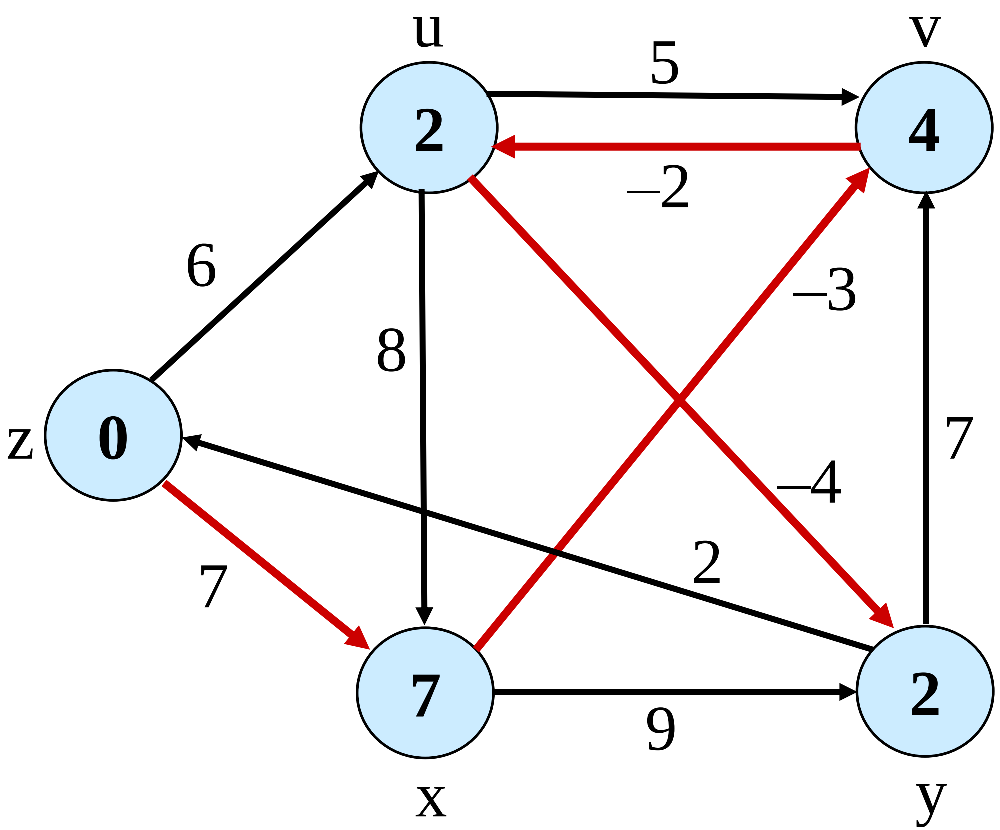
Example



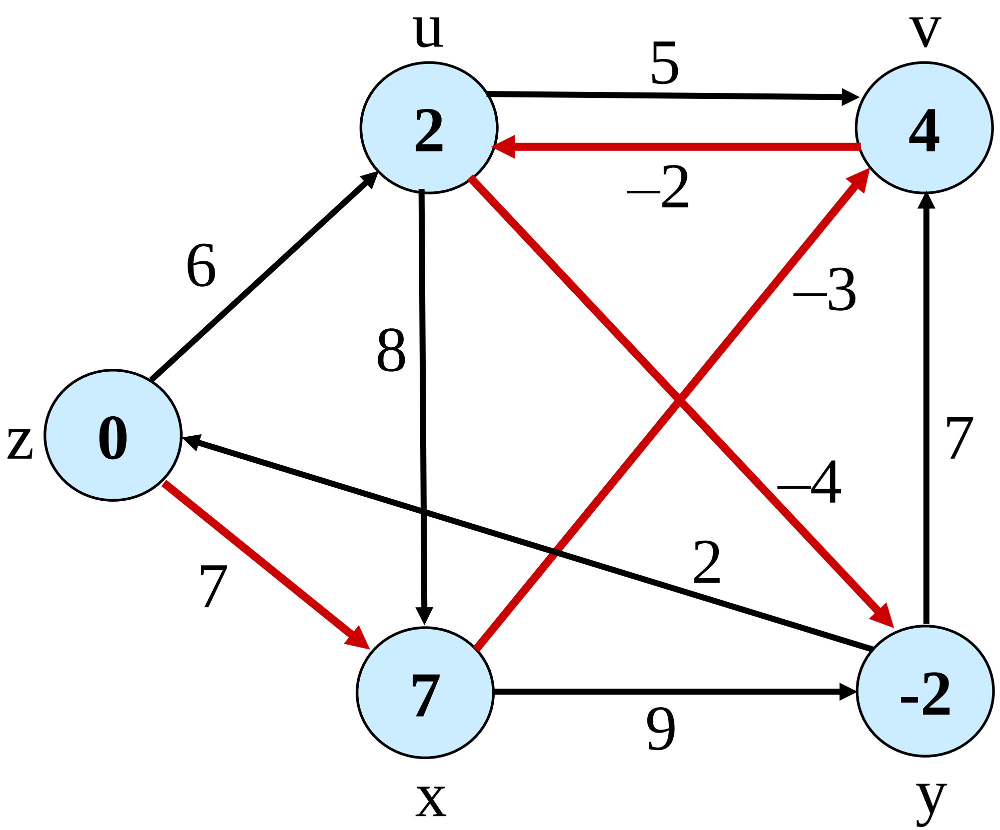
Example



Example



Example



Another Look

Note: This is essentially **dynamic programming**.

Let d(i, j) = cost of the shortest path from s to i that is at most j hops.

$$d(i, j) = \begin{cases} 0 & \textbf{if } i = s \wedge j = 0 \\ \infty & \textbf{if } i \neq s \wedge j = 0 \\ \min(\{d(k, j-1) + w(k, i) : i \in Adj(k)\} \\ \qquad \cup \ \{d(i, j-1)\}) & \textbf{if } j > 0 \end{cases}$$

i →

j ↓	z 1	u 2	v 3	x 4	y 5
0	0	∞	∞	∞	∞
1	0	6	∞	7	∞
2	0	6	4	7	2
3	0	2	4	7	2
4	0	2	4	7	–2

Lemma 24.2

Lemma 24.2: Assuming no negative-weight cycles reachable from s, d[v] = δ(s, v) holds upon termination for all vertices v reachable from s.

Proof:

Consider a SP p, where p = $\langle v_0, v_1, \ldots, v_k \rangle$, where v_0 = s and v_k = v.

Assume $k \le |V| - 1$, otherwise p has a cycle.

Claim: d[v_i] = δ(s, v_i) holds after the i^{th} pass over edges.
Proof follows by induction on i.

By Lemma 24.11, once d[v_i] = δ(s, v_i) holds, it continues to hold.

Correctness

Claim: Algorithm returns the correct value.

(Part of Theorem 24.4. Other parts of the theorem follow easily from earlier results.)

Case 1: There is no reachable negative-weight cycle.

Upon termination, we have for all (u, v):

$d[v] = \delta(s, v)$, by lemma 24.2 (last slide) if v is reachable;
$d[v] = \delta(s, v) = \infty$ otherwise.

$\leq \delta(s, u) + w(u, v)$, by Lemma 24.10.

$= d[u] + w(u, v)$

So, algorithm returns **true**.

Case 2

Case 2: There exists a reachable negative-weight cycle $c = \langle v_0, v_1, \ldots, v_k \rangle$, where $v_0 = v_k$.

We have $\Sigma_{i = 1, \ldots, k}\, w(v_{i-1}, v_i) < 0.$ (*)

Suppose algorithm returns true. Then, $d[v_i] \le d[v_{i-1}] + w(v_{i-1}, v_i)$ for $i = 1, \ldots, k$. (because Relax didn't change any $d[v_i]$). Thus,

$$\Sigma_{i = 1, \ldots, k}\, d[v_i] \le \Sigma_{i = 1, \ldots, k}\, d[v_{i-1}] + \Sigma_{i = 1, \ldots, k}\, w(v_{i-1}, v_i)$$

But, $\Sigma_{i = 1, \ldots, k}\, d[v_i] = \Sigma_{i = 1, \ldots, k}\, d[v_{i-1}]$.

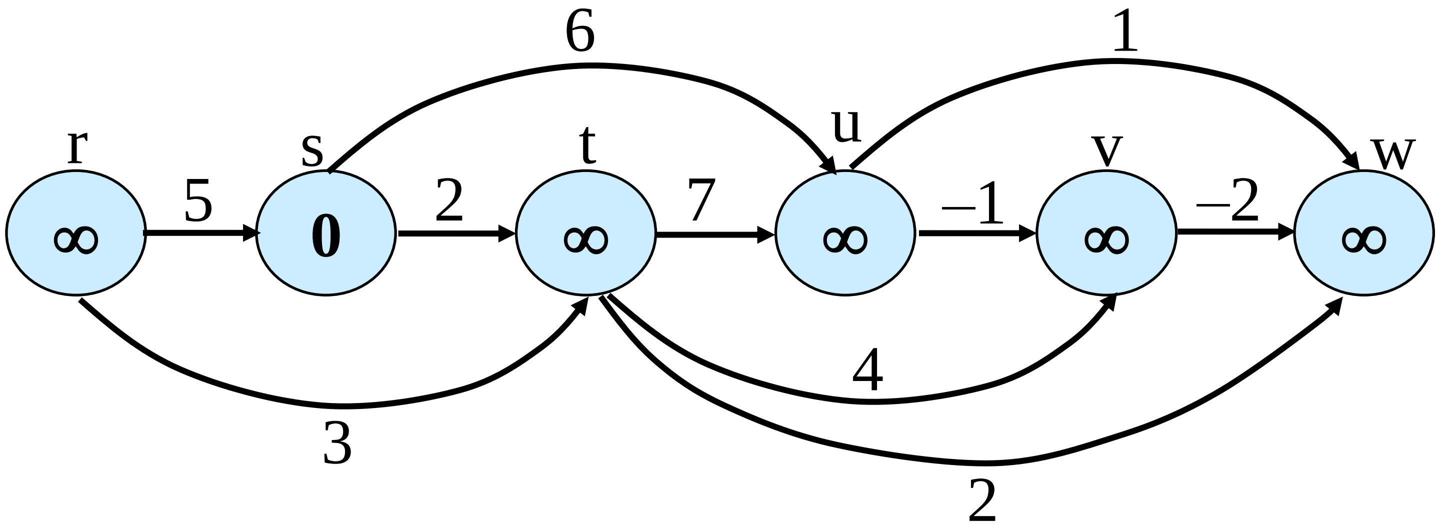
Can show no $d[v_i]$ is infinite. Hence, $0 \le \Sigma_{i = 1, \ldots, k}\, w(v_{i-1}, v_i)$.

Contradicts (*). Thus, algorithm returns **false**.

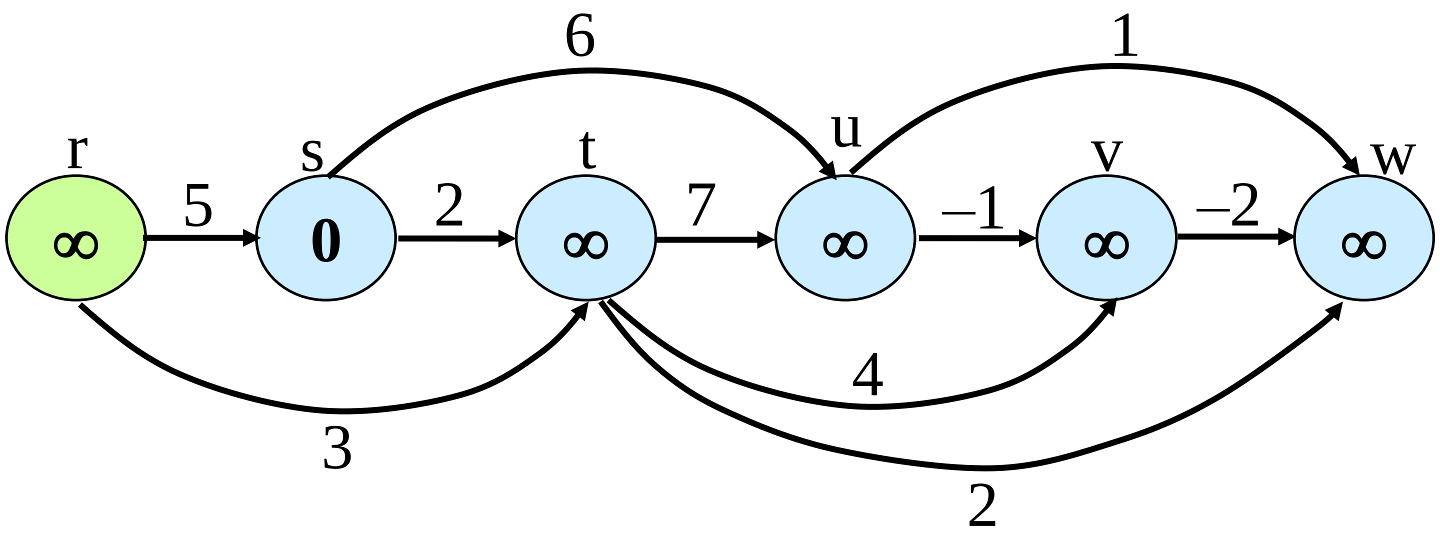
Shortest Paths in DAGs

```
Topologically sort vertices in G;
Initialize(G, s);
for each u in V[G] (in order) do
    for each v in Adj[u] do
        Relax(u, v, w)
    od
od
```

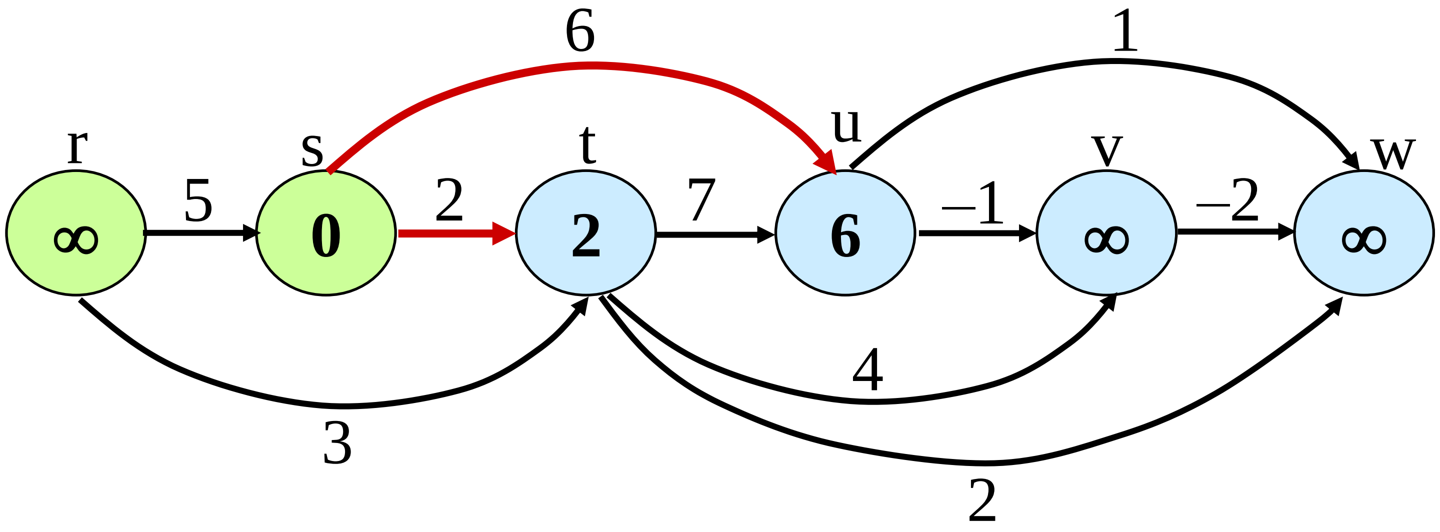
Example



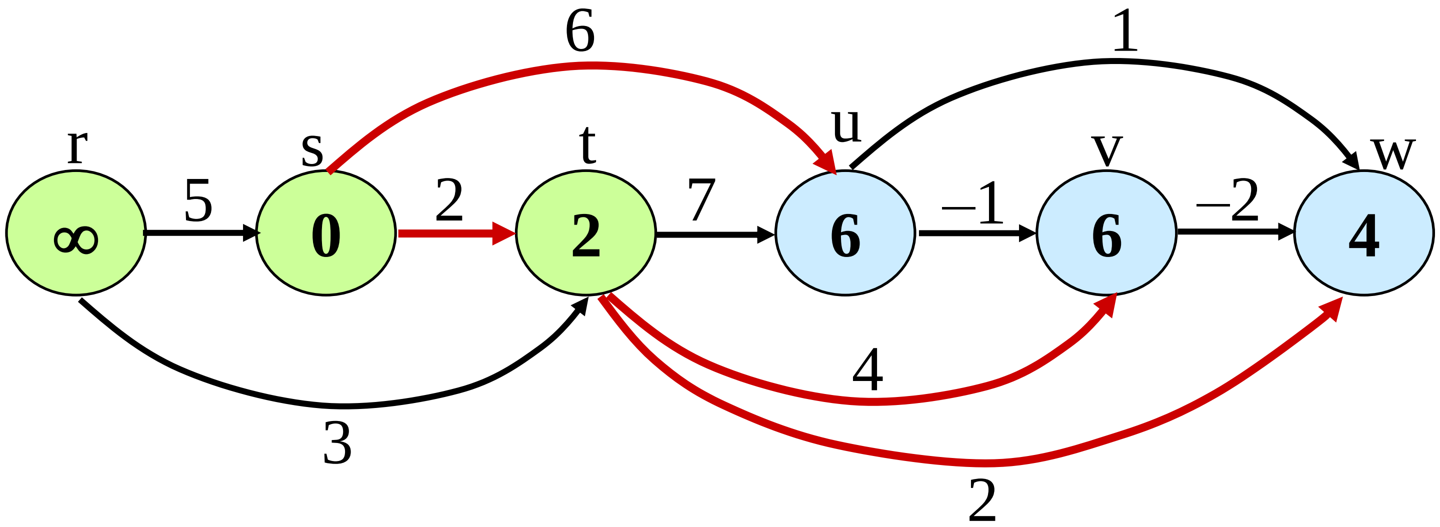
Example



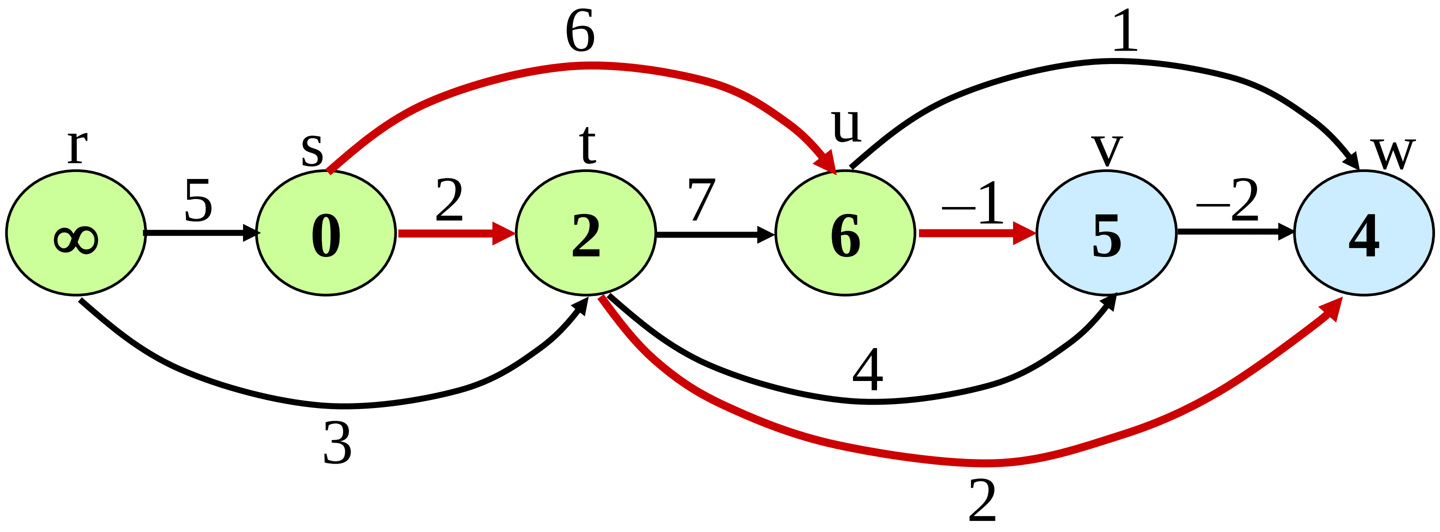
Example



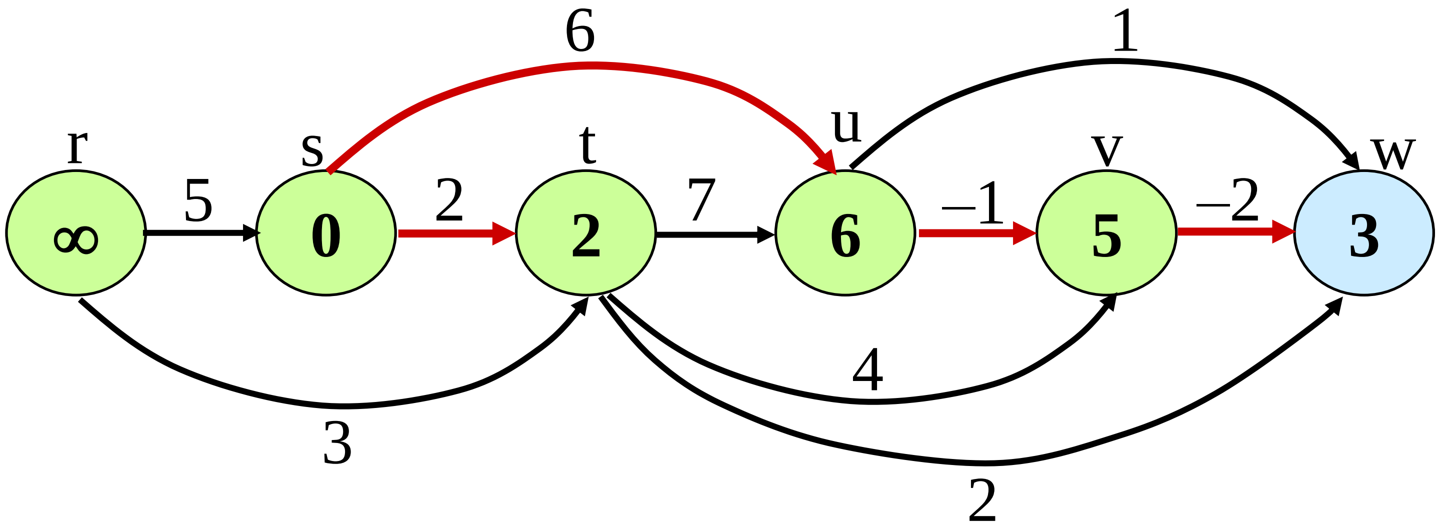
Example



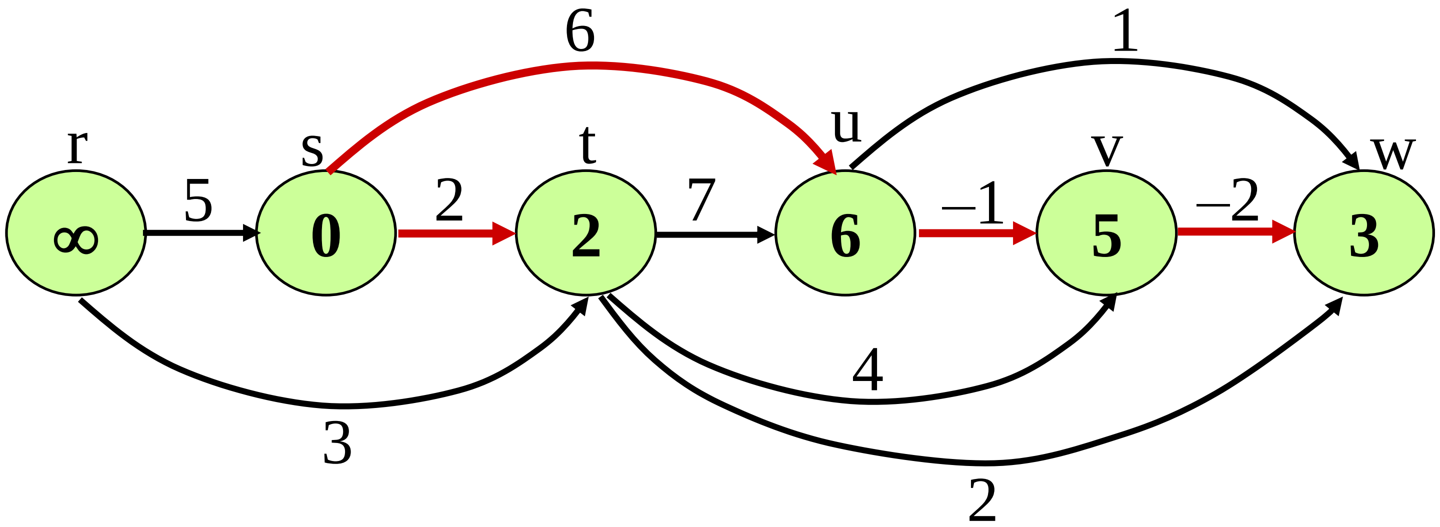
Example



Example



Example



Dijkstra's Algorithm

Assumes **no negative-weight edges**.

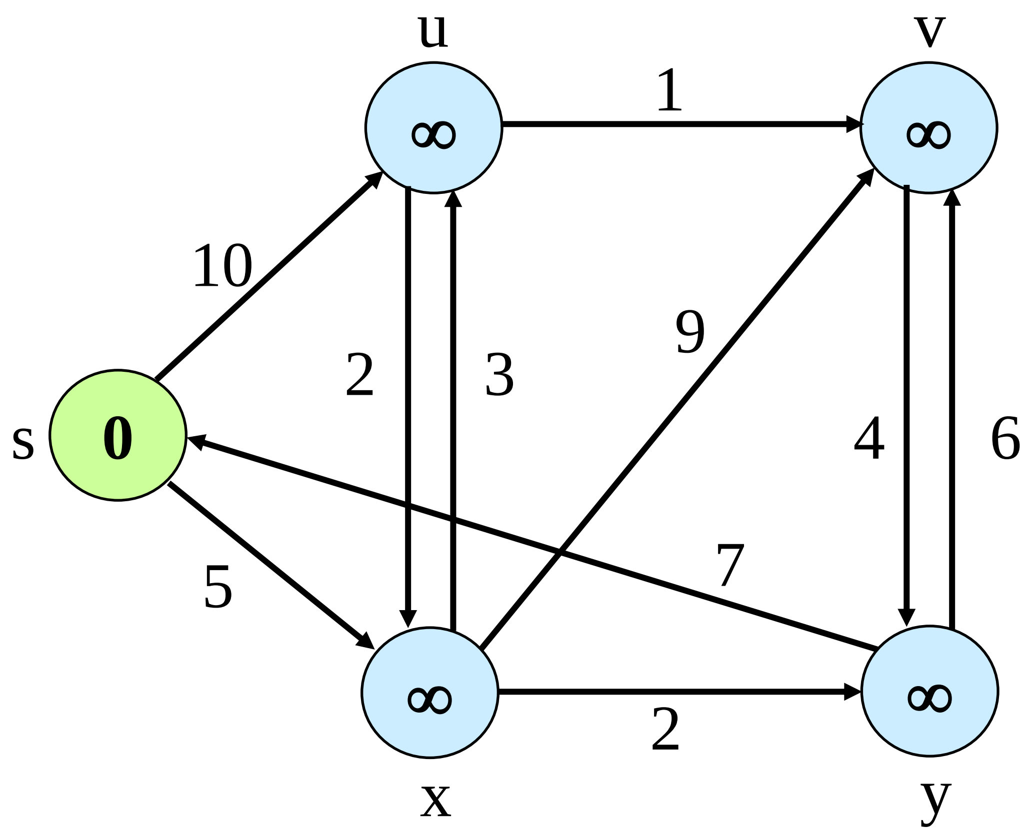
Maintains a set S of vertices whose SP from s has been determined.

Repeatedly selects u in V–S with minimum SP estimate (greedy choice).

Store V–S in priority queue Q.

```
Initialize(G, s);
S := ∅;
Q := V[G];
while Q ≠ ∅ do
    u := Extract-Min(Q);
    S := S ∪ {u};
    for each v ∈ Adj[u] do
        Relax(u, v, w)
    od
od
```

Example



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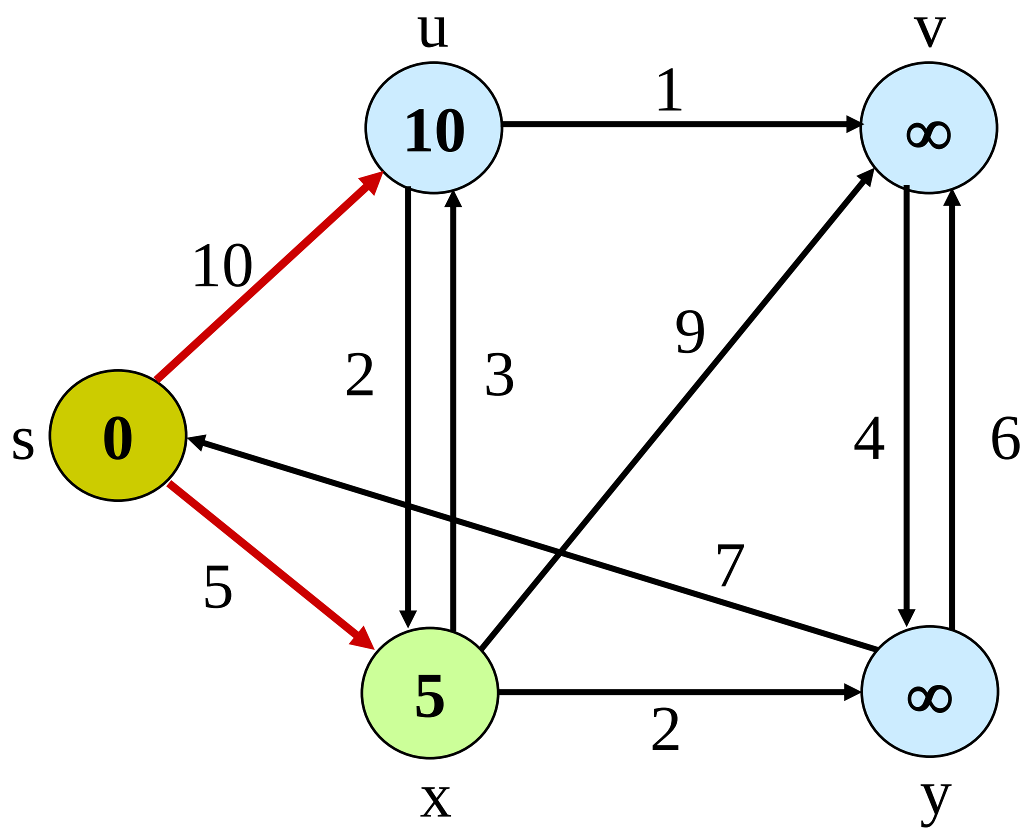

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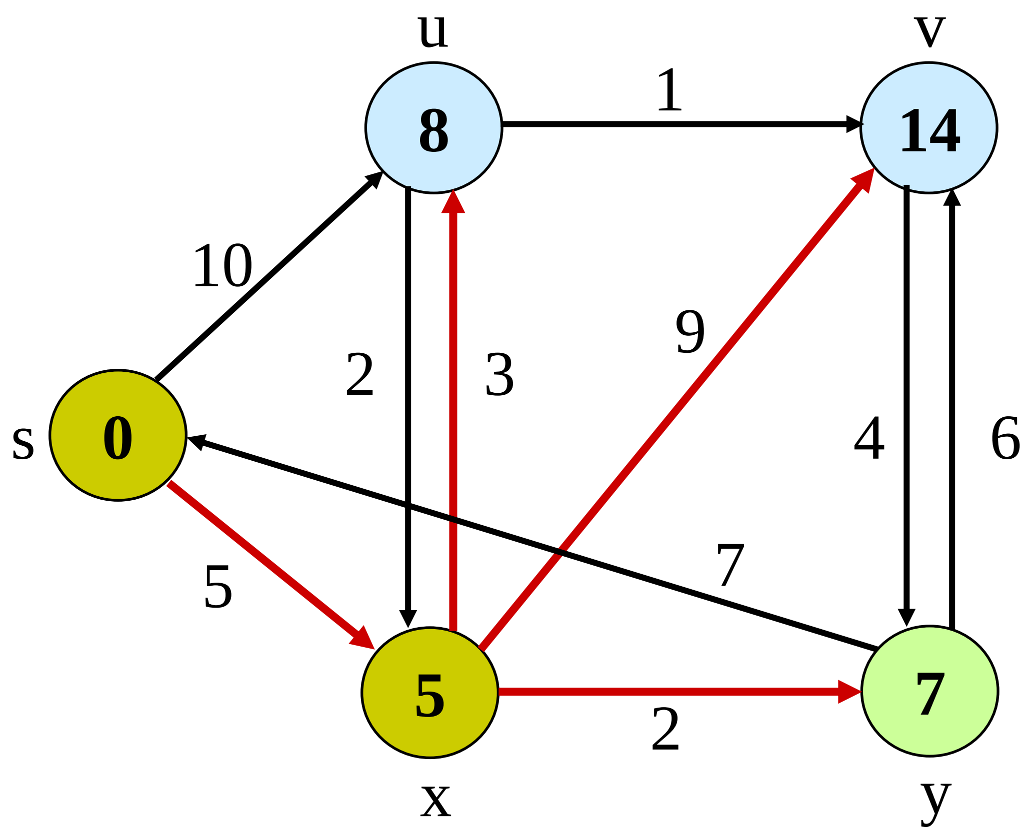

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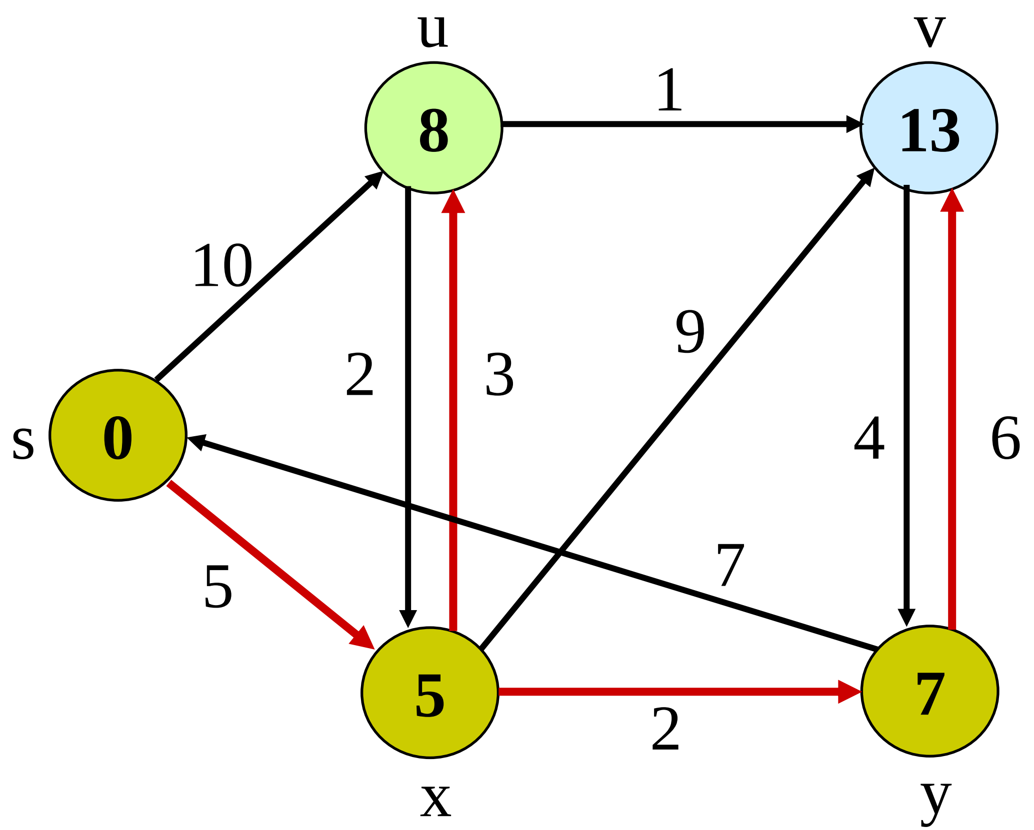
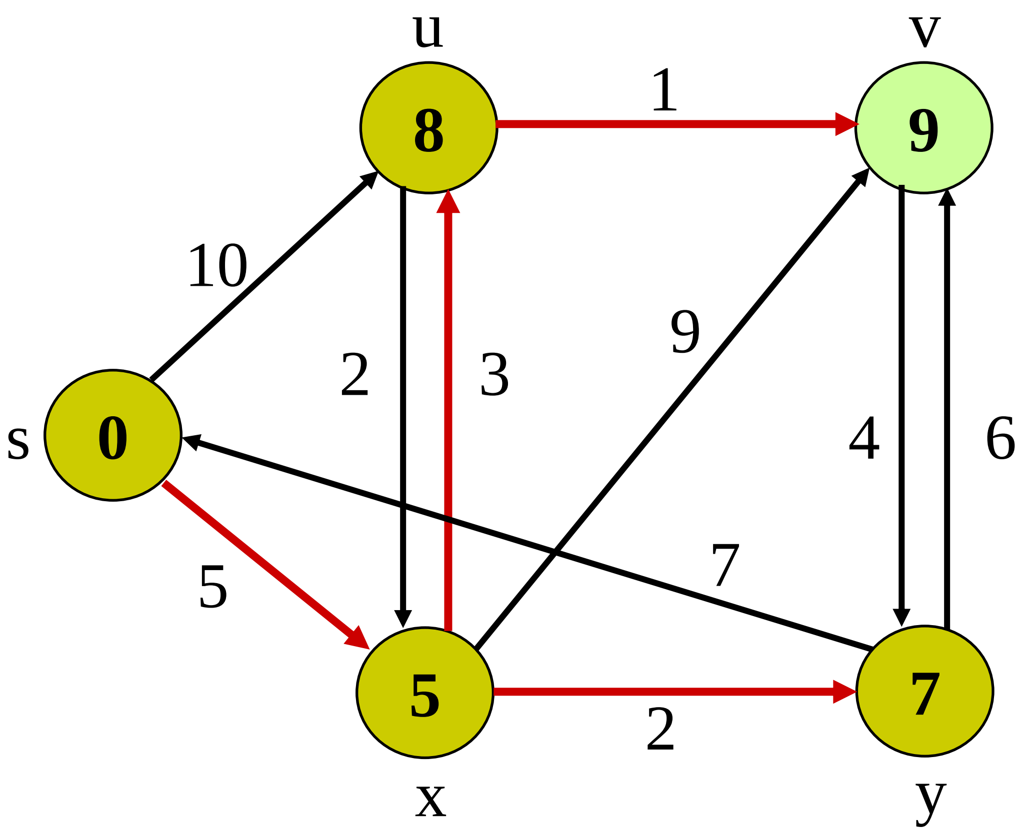
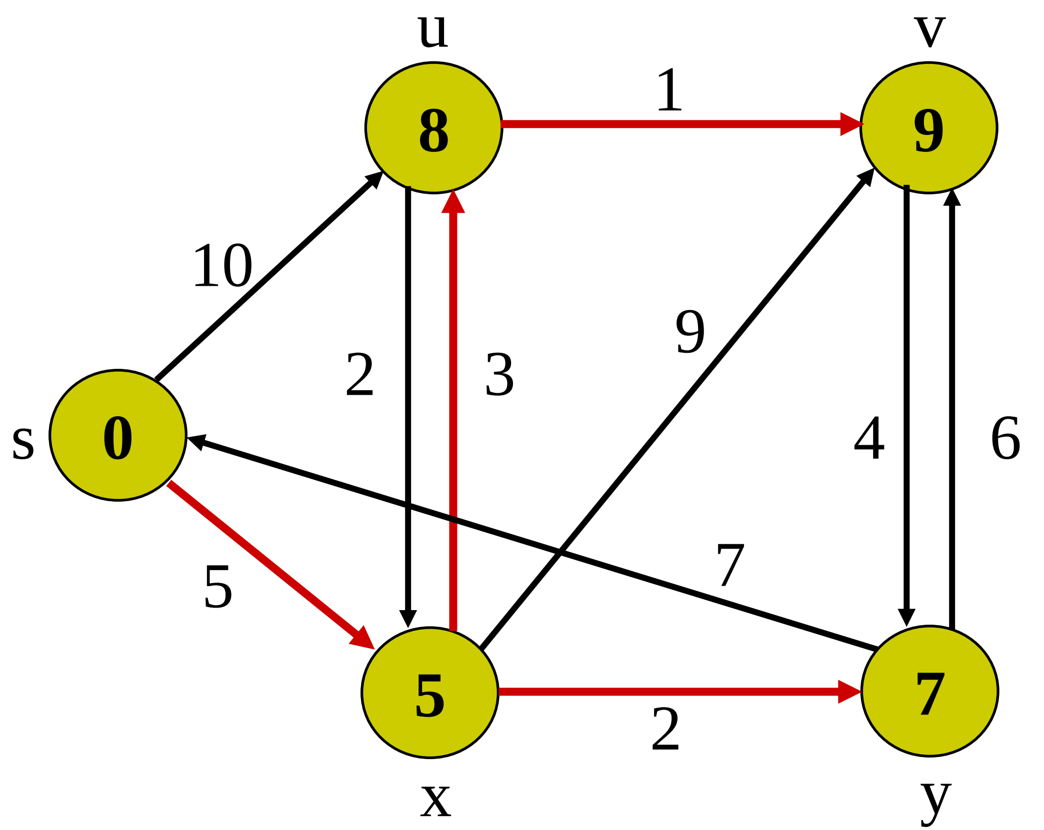


Example



Example



Correctness

Theorem 24.6: Upon termination, $d[u] = \delta(s, u)$ for all u in V (assuming non-negative weights).

Proof:

By Lemma 24.11, once $d[u] = \delta(s, u)$ holds, it continues to hold.

We prove: For each u in V, $d[u] = \delta(s, u)$ when u is inserted in S.

Suppose not. Let u be the first vertex such that $d[u] \neq \delta(s, u)$ when inserted in S.

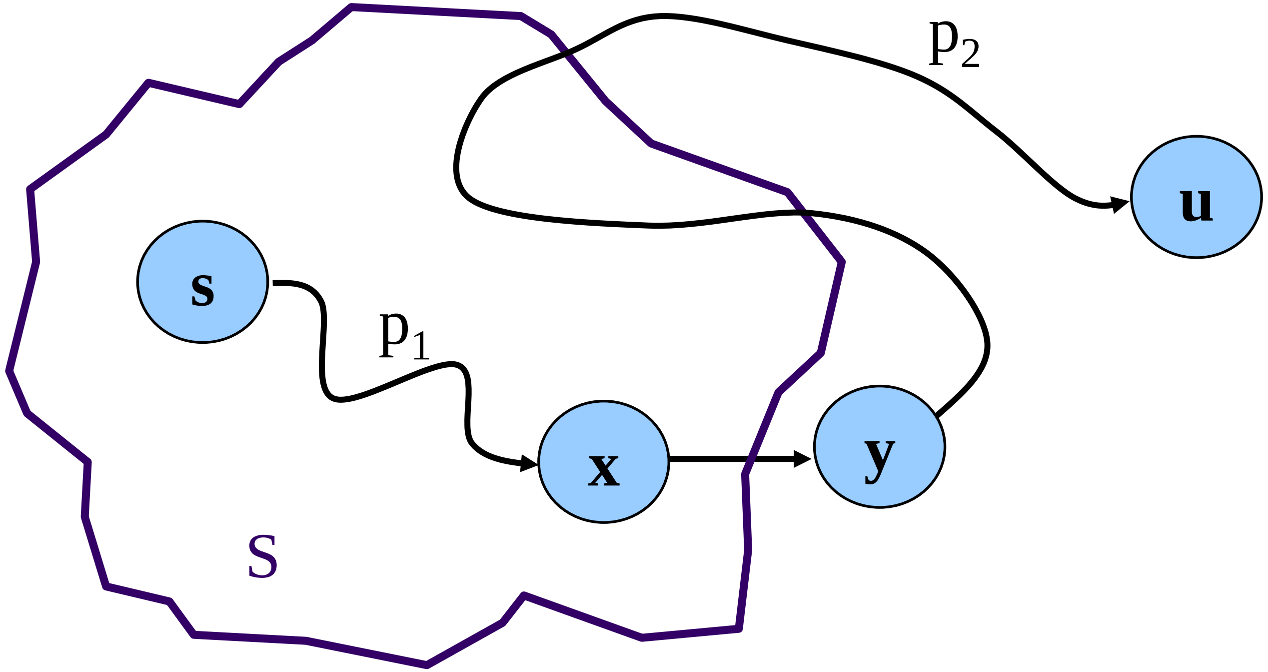
Note that $d[s] = \delta(s, s) = 0$ when s is inserted, so $u \neq s$.

$\Rightarrow$ $S \neq \emptyset$ just before u is inserted (in fact, $s \in S$).

Proof (Continued)

Note that there exists a path from s to u, for otherwise $d[u] = \delta(s, u) = \infty$ by Corollary 24.12.

$\Rightarrow$ there exists a SP from s to u. SP looks like this:



Proof (Continued)

Claim: $d[y] = \delta(s, y)$ when u is inserted into S.

We had $d[x] = \delta(s, x)$ when x was inserted into S.

Edge (x, y) was relaxed at that time.

By Lemma 24.14, this implies the claim.

Now, we have: $d[y] = \delta(s, y)$, by Claim.
$\leq \delta(s, u)$, nonnegative edge weights.
$\leq d[u]$, by Lemma 24.11.

Because u was added to S before y, $d[u] \leq d[y]$.

Thus, $d[y] = \delta(s, y) = \delta(s, u) = d[u]$.

Contradiction.

Complexity

Running time is

$O(V^2)$ using linear array for priority queue.

$O((V + E) \lg V)$ using binary heap.

$O(V \lg V + E)$ using Fibonacci heap.

Acknowledgements

- Cormen, T.H., Leiserson, C.E., Rivest, R.L. and Stein, C., Introduction to algorithms. MIT press, 2009
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