MFDS Assignments - 1

Group Number - 21

Solution for Question 2

Q2 i) LU Decomposition, Vector space & LT

i) Find the LU decomposition of matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ a & a & a \\ b & b & a \end{bmatrix}$ when it exists for which real numbers a and b does it exist?

$\Rightarrow$ $A = LU$

$$\begin{bmatrix} 1 & 0 & 1 \\ a & a & a \\ b & b & a \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ c & 1 & 0 \\ d & e & 1 \end{bmatrix} \begin{bmatrix} f & g & h \\ 0 & i & j \\ 0 & 0 & k \end{bmatrix}$$

$$\therefore \begin{bmatrix} 1 & 0 & 1 \\ a & a & a \\ b & b & a \end{bmatrix} = \begin{bmatrix} f & g & h \\ cf & cg+i & ch+j \\ df & dg+ei & dh+ej+k \end{bmatrix}$$

let's equate both the sides we get.

$\boxed{f = 1 \quad , g = 0 \quad , h = 1}$

$cf = a$ ie. $c.(1) = a$ $\therefore \boxed{c = a}$

$df = b$ ie. $d\,(1) = b$ $\therefore \boxed{d = b}$

$cg + i = a$ ie $c(0) + i = a$ $\therefore \boxed{i = a}$

$dg + ei = b$ ie $d(0) + e.a = b$ $\boxed{\therefore e = b/a}$

$ch + j = a$ ie. $a(1) + j = a$ $\boxed{\therefore j = 0}$

$dh + ej + k = a$ ie $b(1) + \frac{b}{a}(0) + k = a$ $\therefore \boxed{k = a - b}$

$\therefore A = LU$ becomes

$$\begin{bmatrix} 1 & 0 & 1 \\ a & a & a \\ b & b & a \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & b/a & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & a & 0 \\ 0 & 0 & a-b \end{bmatrix}$$

As we have converted the given matrix in LU format, Using Doolittle Method,

its known that in Doolittle method matrix 'L' turns out to be the Matrix of multipliers

$\therefore m_{21} = a \quad m_{31} = b \quad m_{32} = b/a$

multiplier m_{32} depends on 'a' as if $a = 0$ then $m_{32} = \infty$

$\therefore a \neq 0$

Thus, 'a' can be any real number other than zero and 'b' can be any real number

ii) find the diamention of the vector space spanned by the vector $\{[1,1,-2,0,1],[1,2,0,-4,1],[0,1,3,-3,2],[2,3,0,-2,0]\}$ and find the basis of vector

$\Rightarrow$ given vectors can be Represented in matrix format.

$$A = \begin{bmatrix} 1 & 1 & -2 & 0 & 1 \\ 1 & 2 & 0 & -4 & 1 \\ 0 & 1 & 3 & -3 & 2 \\ 2 & 3 & 0 & -2 & 0 \end{bmatrix}$$

Now Applying the elementory Row transformation

$R_2 \to R_2 - R_1 \quad \& \quad R_4 \to R_4 - 2R_1$

$$= \begin{bmatrix} 1 & 1 & -2 & 0 & 1 \\ 0 & 1 & +2 & -4 & 0 \\ 0 & 1 & 3 & -3 & 2 \\ 0 & 1 & 4 & -2 & -2 \end{bmatrix}$$

$R_3 \to R_3 - R_2$ $\qquad$ $R_4 \to R_4 - R_2$

$$\equiv \begin{bmatrix} 1 & 1 & -2 & 0 & 1 \\ 0 & 1 & 2 & -4 & 0 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 2 & -2 \end{bmatrix}$$

$R_4 \to R_4 - 2R_3$

$$\equiv \begin{bmatrix} 1 & 1 & -2 & 0 & 1 \\ 0 & 1 & 2 & -4 & 0 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & -6 \end{bmatrix}$$

Here we have 4 linerly independent vectors, so these 4 vectos will span the vector space of A.

Thus, the basis are $\{[1,1,-2,0,1], [0,1,2,-4,0], [0,0,1,1,2], [0,0,0,0,-6]\}$

Diamention of vector space is 4

Q2 iii) Suppose that A is a matrix such that the complete solution to $Ax = \begin{bmatrix} 1 \\ 4 \\ 1 \\ 1 \end{bmatrix}$ is of the form

$$x = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \quad c \in R$$

a) what can be said about the columns of matrix A
b) find the diamention of null space and rank of matrix A

$\Rightarrow$ a) from eq^n $Ax = \begin{bmatrix} 1 \\ 4 \\ 1 \\ 1 \end{bmatrix}$ we can say that

$A_{4\times m} \times x_{m\times 1} = \begin{bmatrix} 1 \\ 4 \\ 1 \\ 1 \end{bmatrix}$ As for matrix multiplication rows of A should match with colums of x at complete solution,

$$x = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \quad c \in R$$

we can see that x has 3 ~~columns~~ Rows & 1 column, ie. (3×1)

$\therefore$ diamention of $x = (m\times 1) = 3\times 1$ thus $m=3$

$\therefore$ diamention of $a = 4\times m = 4\times 3$

$\boxed{\therefore \text{ matrix } A \text{ has '3' columns.}}$

$\Rightarrow$ b) the complete solution is a combination of perticular solution & Special solution

$x = x_p + x_s$ — x_p: perticular solution
x_s = special solution
x = complete solution.

Number of special solution will give us the diamention of null space

thus, $\boxed{\dim(\text{null space}) = 1}$

By Rank-Nullity Theorem, we have

$$\text{Rank}(A) + \text{Nullity}(A) = \text{Columns}(A)$$

we know that Nullity (A) is 1 and A has 3 columns.

so,

$$\text{Rank}(A) = 3 - 1 = 2$$

$$\boxed{\therefore \text{Rank}(A) = 2}$$