21671-A Homework 2

1] Given

$$A = \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}, \quad x = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$\|Ax\|_2^2 = (\cos\theta - 2\sin\theta)^2 + (4\sin\theta - 3\cos\theta)^2 \equiv f(\theta)$$

Now to find out the value of $\|A_2\|$

We know the formula for doing the l 2 norm of a matrix is

$$\|A\|_2 = \frac{\|Ax\|_2}{\|x\|_2}$$

Let us first find the l_2 norm of x

$$\|x\|_2 = \sqrt{\cos^2\theta + \sin^2\theta} = \sqrt{1} = 1$$

Now letus find the value of l_2 norm of Ax to do that we have to first find the value of θ wherein which $\|Ax\|$ is maximum. To do this we have to differentiate $f(\theta)$

But first let me reduce $f(\theta)$

$$f(\theta) = (\cos\theta - 2\sin\theta)^2 + (4\sin\theta - 3\cos\theta)^2$$

$$= \cos^2\theta - 4\sin\theta\cos\theta + 4\sin^2\theta + 16\sin^2\theta - 24\sin\theta\cos\theta + 9\cos^2\theta$$

$$f(\theta) = 10\cos^2\theta - 28\sin\theta\cos\theta + 20\sin^2\theta$$

We know $\cos^2\theta + \sin^2\theta = 1$ and $\sin 2\theta = 2\sin\theta\cos\theta$

$$f(\theta) = 10 + 10\sin^2\theta - 14\sin 2\theta$$

Let us now differentiate this function

$$\frac{df(\theta)}{d\theta} = \frac{d}{d\theta}(10 + 10\sin^2\theta - 14\sin 2\theta)$$

$$f'(\theta) = \frac{d}{d\theta}(10) + 10\frac{d}{d\theta}(\sin^2\theta) - 14\frac{d}{d\theta}(\sin 2\theta)$$

$$f'(\theta) = 0 + 10(2\sin\theta\cos\theta) - 28\cos 2\theta$$

We know $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$$f'(\theta) = 20\sin\theta\cos\theta - 28(\cos^2\theta - \sin^2\theta)$$

$$f'(\theta) = 20\sin\theta\cos\theta - 28\cos^2\theta + 28\sin^2\theta$$

To find all the values of local maxima and minima we have to equate $f'(\theta) = 0$

$$0 = 20\sin\theta\cos\theta - 28\cos^2\theta + 28\sin^2\theta$$

Let us divide both sides of the equations by $\cos^2\theta$

$0 = 20 \tan\theta - 28 + 28 \tan^2\theta$

$t = \tan\theta$

$0 = 5t - 7 + 7t^2$

$7t^2 + 5t - 7 = 0$

Let us solve the quadratic equation

$a = 7 \quad b = 5 \quad c = -7$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$t = \frac{-5 \pm \sqrt{25 + 4(7)^2}}{14}$$

$$t = \frac{-5 \pm \sqrt{25 + 4(49)}}{14}$$

$$t = \frac{-5 \pm \sqrt{25 + 196}}{14}$$

$$t = \frac{-5 \pm \sqrt{221}}{14}$$

$$t_1 = \frac{-5 + 14.8661}{14} \qquad t_2 = \frac{-5 - 14.8661}{14}$$

$t_1 = 0.7047 \quad , \quad t_2 = -1.419$

Taking $t_1 = 0.7047$

$\tan\theta = 0.7047$

$\theta = \tan^{-1}(0.7047)$

$\theta = 0.6139$

We know $\tan\theta$ repeats twice between 0 and 2π

So,

$\theta = 0.6139, 3.7554$

Taking $t_2 = -1.419$

$\tan\theta = -1.419$

$\theta = \tan^{-1}(-1.419)$

$\theta = -0.9569$

We know that -0.9569 is not between 0 and 2π.
So we have to add π and 2π

$\theta = 2.1847, 5.3263$

Putting $\theta = 0.6139$ in $f(\theta)$

$f(0.6139) = 10 + 10\sin^2(0.6139) - 14\sin(2 \times 0.6139)$

$= 10 + 3.3184 - 13.1845$

$f(0.6139) = 0.1338$

Putting $\theta = 3.7554$ in $f(\theta)$

$f(3.7554) = 10 + 10\sin^2(3.7554) - 14\sin(2 \times 3.7554)$

$f(3.7554) = 0.1339$

Putting $\theta = 2.1847$ in $f(\theta)$

$f(2.1847) = 10 + 10\sin^2(2.1847) - 14\sin(2 \times 2.1847)$

$f(2.1847) = 29.8661$

Putting $\theta = 5.3263$

$f(5.3263) = 10 + 10\sin^2(5.3263) - 14\sin(5.3263)$

$f(5.3263) = 29.8661$

After calculating all the values of $f(\theta)$ we can see that the maximum value is 29.8661

We know

$\|Ax\|_2^2 = f(\theta)$

$\therefore \|Ax\|_2 = \sqrt{f(\theta)}$

$\|Ax\|_2 = \sqrt{29.8661}$

$\|Ax\|_2 = 5.4650$

classmate

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$$\|A\|_2 = \frac{\|Ax\|_2}{\|x\|_2}$$

$$\|A\|_2 = \frac{5.465}{1}$$

$$\|A\|_2 = 5.465$$