21671-A HOMEWORK 2

1] Given

$$A = \begin{bmatrix} 1 & -2 \\ -3 & 4 \end{bmatrix}, \quad x = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$\|Ax\|_2^2 = (\cos\theta - 2\sin\theta)^2 + (4\sin\theta - 3\cos\theta)^2 \equiv f(\theta)$$

Now to find out the value of $\|A_2\|$

We know the formula for doing the l 2 norm of a matrix is

$$\|A\|_2 = \frac{\|Ax\|_2}{\|x\|_2}$$

Let us first find the l_2 norm of x

$$\|x\|_2 = \sqrt{\cos^2\theta + \sin^2\theta} = \sqrt{1} = 1$$

Now let us find the value of l 2 norm of Ax to do that we have to first find the value of θ wherein which $\|Ax\|$ is maximum. To do this we have to differentiate $f(\theta)$

But first let me reduce $f(\theta)$

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$f(\theta) = (\cos\theta - 2\sin\theta)^2 + (4\sin\theta - 3\cos\theta)^2$

$= \cos^2\theta - 4\sin\theta\cos\theta + 4\sin^2\theta + 16\sin^2\theta - 24\sin\theta\cos\theta$
$+ 9\cos^2\theta$

$f(\theta) = 10\cos^2\theta - 28\sin\theta\cos\theta + 20\sin^2\theta$

We know $\cos^2\theta + \sin^2\theta = 1$ and $\sin 2\theta = 2\sin\theta\cos\theta$

$f(\theta) = 10 + 10\sin^2\theta - 14\sin 2\theta$

Let us now differentiate this function

$$\frac{df(\theta)}{d\theta} = \frac{d}{d\theta}(10 + 10\sin^2\theta - 14\sin 2\theta)$$

$$f'(\theta) = \frac{d(10)}{d\theta} + 10\frac{d}{d\theta}(\sin^2\theta) - 14\frac{d(\sin 2\theta)}{d\theta}$$

$f'(\theta) = 0 + 10(2\sin\theta\cos\theta) - 28\cos 2\theta$
We know $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$f'(\theta) = 20\sin\theta\cos\theta - 28(\cos^2\theta - \sin^2\theta)$

$f'(\theta) = 20\sin\theta\cos\theta - 28\cos^2\theta + 28\sin^2\theta$

To find all the values of local maxima and minima we have to equate $f'(\theta) = 0$

$0 = 20\sin\theta\cos\theta - 28\cos^2\theta + 28\sin^2\theta$

Let us divide both sides of the equations by $\cos^2\theta$

$$0 = 20\tan\theta - 28 + 28\tan^2\theta$$

$$t = \tan\theta$$

$$0 = 5t - 7 + 7t^2$$

$$7t^2 + 5t - 7 = 0$$

Let us solve the quadratic equation

$a = 7 \quad b = 5 \quad c = -7$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$t = \frac{-5 \pm \sqrt{25 + 4(7)^2}}{14}$$

$$t = \frac{-5 \pm \sqrt{25 + 4(49)}}{14}$$

$$t = \frac{-5 \pm \sqrt{25 + 196}}{14}$$

$$t = \frac{-5 \pm \sqrt{221}}{14}$$

$$t_1 = \frac{-5 + 14.8661}{14} \qquad t_2 = \frac{-5 - 14.8661}{14}$$

$$t_1 = 0.7047 \quad , \quad t_2 = -1.419$$

Taking $t_1 = 0.7047$

$\tan\theta = 0.7047$
$\theta = \tan^{-1}(0.7047)$
$\theta = 0.6139$

We know $\tan\theta$ repeats twice between 0 and 2π
So,
$\theta = 0.6139, 3.7554$

Taking $t_2 = -1.419$

$\tan\theta = -1.419$
$\theta = \tan^{-1}(-1.419)$
$\theta = -0.9569$

We know that -0.9569 is not between 0 and 2π.
So we have to add π and 2π

$\theta = 2.1847, 5.3263$

Putting $\theta = 0.6139$ in $f(\theta)$

$f(0.6139) = 10 + 10\sin^2(0.6139) - 14\sin(2\times 0.6139)$

$= 10 + 3.3184 - 13.1845$

$f(0.6139) = 0.1338$

Putting $\theta = 3.7554$ in $f(\theta)$

$f(3.7554) = 10 + 10\sin^2(3.7554) - 14\sin(2 \times 3.7554)$

$f(3.7554) = 0.1339$

Putting $\theta = 2.1847$ in $f(\theta)$

$f(2.1847) = 10 + 10\sin^2(2.1847) - 14\sin(2 \times 2.1847)$

$f(2.1847) = 29.8661$

Putting $\theta = 5.3263$

$f(5.3263) = 10 + 10\sin^2(5.3263) - 14\sin(5.3263)$

$f(5.3263) = 29.8661$

After calculating all the values of $f(\theta)$ we can see that the maximum value is 29.8661

We know

$\|Ax\|_2^2 = f(\theta)$

$\therefore \|Ax\|_2 = \sqrt{f(\theta)}$

$\|Ax\|_2 = \sqrt{29.8661}$

$\|Ax\|_2 = 5.4650$

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$$\|A\|_2 = \frac{\|Ax\|_2}{\|x\|_2}$$

$$\|A\|_2 = \frac{5.465}{1}$$

$$\|A\|_2 = 5.465$$

The unit vector can be found by plugging in the value of θ into the vector $\vec{x}$.

$$\therefore \vec{x} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \begin{bmatrix} \cos(5.3263) \\ \sin(5.3263) \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} 0.5761 \\ -0.8174 \end{bmatrix}$$

Question_2

September 27, 2024

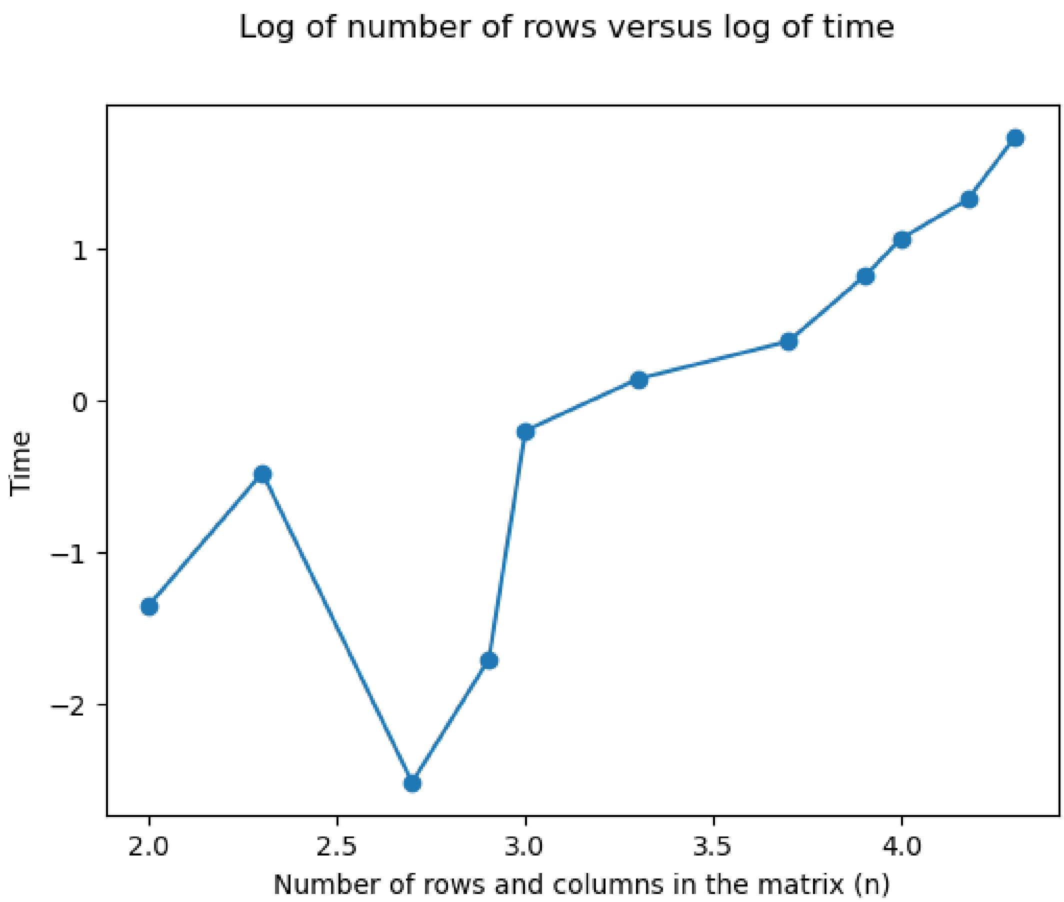
[5]:
```
from numpy.random import rand
from random import seed
from numpy import matmul
from time import time
from numpy.linalg import solve, norm
from numpy import matmul
import matplotlib.pyplot as plt
from math import log10
%matplotlib inline
```

[2]:
```
def solve_and_measure(n):
    A = rand(n, n)
    x = rand(n, 1)
    b = matmul(A, x)
    start_time = time()
    x_hat = solve(A, b)
    solve_time = time()-start_time
    x_norm = norm(x, 2)
    relative_error = norm(x-x_hat, 2)/x_norm
    residual_norm = norm(b-matmul(A, x_hat), 2)/norm(b, 2)
    return [solve_time, x_norm, relative_error, residual_norm]
n = [100, 200, 500, 800, 1000, 2000, 5000, 8000, 10000, 15000, 20000]
results = {"n":[], "solve_time":[], "x_norm":[], "relative_error":[],␣
↪"residual_norm":[]}
for i in n:
    a = solve_and_measure(i)
    results["n"].append(i)
    results["solve_time"].append(a[0])
    results["x_norm"].append(a[1])
    results["relative_error"].append(a[2])
    results["residual_norm"].append(a[3])
```

[6]:
```
fig, ax = plt.subplots()
a = results["solve_time"]
log_n = []
log_a = []
for i in range(len(n)):
    log_n.append(log10(n[i]))
```

```
    log_a.append(log10(a[i]))
ax.plot(log_n, log_a, 'o-')
fig.suptitle('Log of number of rows versus log of time')
ax.set_xlabel("Number of rows and columns in the matrix (n)")
ax.set_ylabel("Time")
```

[6]: Text(0, 0.5, 'Time')



From this graph it can be seen that after the number of rows and columns passes 500 it can be seen that the amount of time it takes to compute x_hat actually increases. I can also see that the amount of time it takes to compute x_hat for a matrix that is of size 200x200 is more than the amount of time it takes for both the 100x100 and the 500x500