Maximally Recoverable Codes with Hierarchical Locality

Aaditya M Nair, Dr. Lalitha Vadlamani
Signal Processing and Communications Research Center
IIIT Hyderabad

Thesis Defense

Dec 12, 2020

Distributed Storage Systems

- Any large enough storage system needs to be distributed and requires some form of redundancy to ensure reliability.
- Naively, one could store multiple copies of the same data. A common approach is to store two extra copies.
- This approach puts a huge cost at scale. With **two extra copies of all the data**, you can only recover from 2 failures.

Distributed Storage Systems: Erasure Coding

- Alternatively, one could use *erasure coding* to ensure reliability.
- In this approach we divide the data into k data symbols and and add h redundancy symbols that are distinct linear combinations of the data symbols.
- To ensure reliability against two failures, one has to **only add 2 redundancy symbols**.
- This leads to huge savings in storage.

Distributed Storage Systems: Codes with Locality

- Typically since the redundancy symbols depend on all k data symbols, repairing every erasure requires the system to **access all k symbols.**
- Since single erasures are a lot more common than multiple erasures, we can optimise for that scenario.
- We say a code symbol has locality r, if that symbol can be repaired by contacting r other symbols.
- To have faster repairs, we usually have $r << k$.

Codes with Locality

A code C has an (r, ϵ) locality if for every symbol $c_i \in C$, there is a punctured code C_i, such that,

- $c_i \in Supp(C_i)$.
- $d_{min}(C_i) \geq \epsilon$
- $|Supp(C_i)| \leq r + \epsilon - 1$

For an $[n, k, d]$ code with (r, ϵ) locality,

$$d \leq n - k + 1 - (\left\lceil \frac{k}{r} \right\rceil - 1)(\epsilon - 1)$$

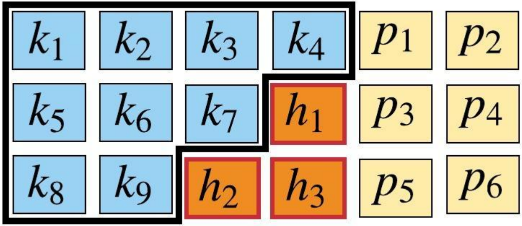
Codes with Locality (alt. definition)

Let C be a systematic $[n, k, d_{min}]$ code. We say that C is an $[k, r, h, \delta]$ local code if the following conditions are satisfied,

- $r|(k+h)$ and $n = k + \frac{k+h}{r}\delta + h$
- There are k data symbols and h global parity symbols where each global parity may depend on all data symbols.
- These $k+h$ symbols are partitioned into $\frac{k+h}{r}$ **local groups** of size r. For each such group, there are δ local parity symbols.

Example

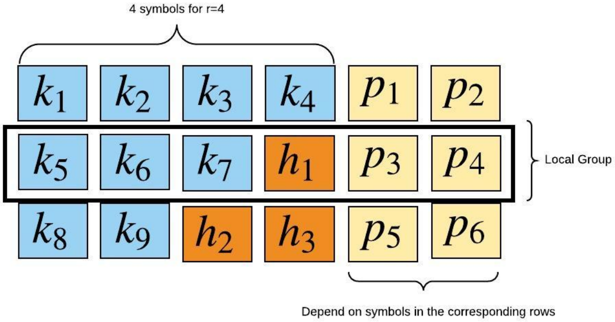
$[k, r, h, \delta]$ code with $k = 9$, $r = 4$, $h = 3$ and $\delta = 2$
where h and n are related as $n = (\frac{k+h}{r})(r + \delta)$



h_1, h_2 and h_3 depend on all k symbols

Example

$[k, r, h, \delta]$ code with $k = 9,\ r = 4,\ h = 3$ and $\delta = 2$



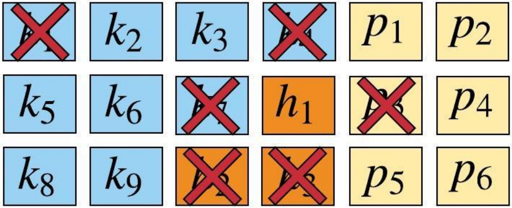
- k data symbols and h global parities are partitioned into $\frac{k+h}{r} = 3$ groups
- There are δ parity symbols for each local group.

Maximal Recoverable Code with Locality

Definition (Maximal Recoverability)

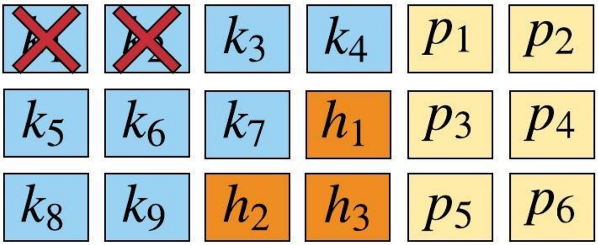
A code is said to be maximally recoverable if it can recover from all the information theoretically recoverable erasure patterns given the locality constraints of the code.

$[k, r, h, \delta]$ local MRC with $k = 9$, $r = 4$, $h = 3$ and $\delta = 2$
Puncture δ symbols per local group.
The resultant is an $[k + h, k]$ MDS code



The problem with LRCs

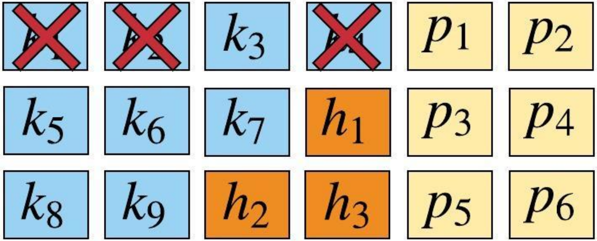
There is an abrupt jump in locality after δ erasures



Can be corrected by contacting r symbols

The problem with LRCs

There is an abrupt jump in locality after δ erasures.



Only corrected by contacting all k symbols

The Solution: Hierarchical Codes

Codes with Hierarchical Locality have multiple levels of locality. They allow for a more controlled increase in locality with the number of erasures.

Codes with Hierarchical Locality

A code C has an $[(r_1, \epsilon_1), (r_2, \epsilon_2)]$ hierarchical locality if for every symbol $c_i \in C$, there is a punctured code C_i, such that,

- $c_i \in Supp(C_i)$.
- $d_{min}(C_i) \geq \epsilon_1$
- $|Supp(C_i)| \leq r_1 + \epsilon_1 - 1$
- C_i is a code with (r_2, ϵ_2) locality

For an $[n, k, d]$ code with $[(r_1, \epsilon_1), (r_2, \epsilon_2)]$ locality,

$$d \leq n - k + 1 - (\left\lceil \frac{k}{r_2} \right\rceil - 1)(\epsilon_2 - 1) - (\left\lceil \frac{k}{r_1} \right\rceil - 1)(\epsilon_1 - \epsilon_2)$$

Codes with Hierarchical Locality (alt. definition)

Easiest to show with an example.

$[k, r_1, r_2, h_1, h_2, \delta]$ code with $k = 9$, $r_1 = 4$, $r_2 = 3$, $h_1 = 3$, $h_2 = 2$ and $\delta = 1$

$n = (\frac{k+h_1}{r_1})(\frac{r_1+h_2}{r_2})(r_2 + \delta) = 24$

c_1	c_2	c_3	c_4		c_9	c_{10}	c_{11}	c_{12}
c_5	c_6	c_7	c_8		c_{13}	c_{14}	c_{15}	c_{16}

c_{17}	c_{18}	c_{19}	c_{20}
c_{21}	c_{22}	c_{23}	c_{24}

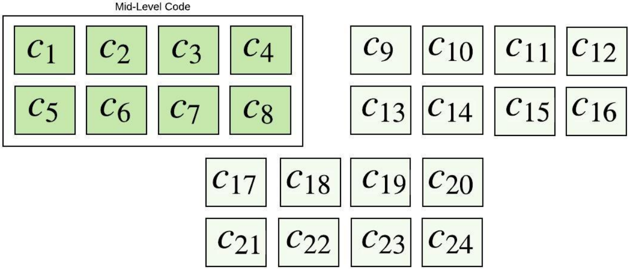
All code symbols satisfy $h_1 = 3$ global parities.

$$\sum_{j=1}^{24} u_j^{(\ell)} c_j = 0, \quad 1 \leq \ell \leq 3$$

Codes with Hierarchical Locality (alt. definition)

Easiest to show with an example

$[k, r_1, r_2, h_1, h_2, \delta]$ code with $k = 9$, $r_1 = 4$, $r_2 = 3$, $h_1 = 3$, $h_2 = 2$ and $\delta = 1$



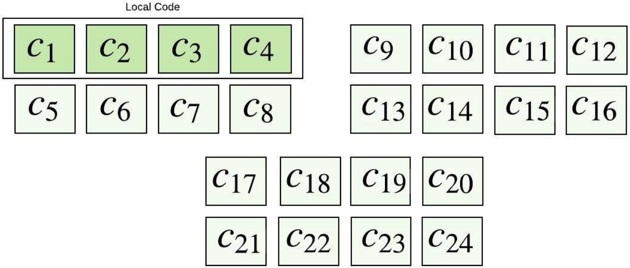
- All symbols are partitioned in $t_1 = \frac{k+h_1}{r_1} = 3$ groups of length $n_1 = \frac{r_1+h_2}{r_2}(r_2 + \delta) = 8$ called mid-level codes.
- Code symbols in a mid-level code satisfy h_2 mid-level parities.

$$\sum_{j=1}^{8} v_j^{(\ell)} c_j = 0, \quad 1 \leq \ell \leq 2 \text{ (\textit{same for the rest of the groups})}$$

Codes with Hierarchical Locality (alt. definition)

Easiest to show with an example

$[k, r_1, r_2, h_1, h_2, \delta]$ code with $k = 9$, $r_1 = 4$, $r_2 = 3$, $h_1 = 3$, $h_2 = 2$ and $\delta = 1$



- n_1 code symbols from the previous step are partitioned into $t_2 = \frac{r_1+h_2}{r_2} = 2$ groups of size $n_2 = r_2 + \delta = 4$.
- Each of these groups satisfy $\delta = 1$ parities.

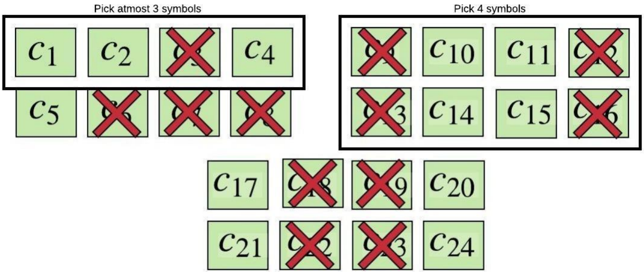
$$\sum_{j=1}^{4} w_j^{(\ell)} c_j = 0, \quad 1 \le \ell \le 1$$

Our Contributions

- Construction of data local hierarchical MRCs from local hierarchical MRCs.
- Definition and constructions for hierarchical local MRCs for all parameters.
- Using Tensor Product Codes to perform the above construction in a smaller field.
- Even smaller field size constructions for the following special cases.
  1. 1 global parity and any number of mid-level parities.
  2. 1 global parity and 1 mid-level parity.
  3. 2 global parities and 1 mid-level parity.

MRCs with Hierarchical Locality

$[k, r_1, r_2, h_1, h_2, \delta]$ code with $k = 9$, $r_1 = 4$, $r_2 = 3$, $h_1 = 3$, $h_2 = 2$ and $\delta = 1$



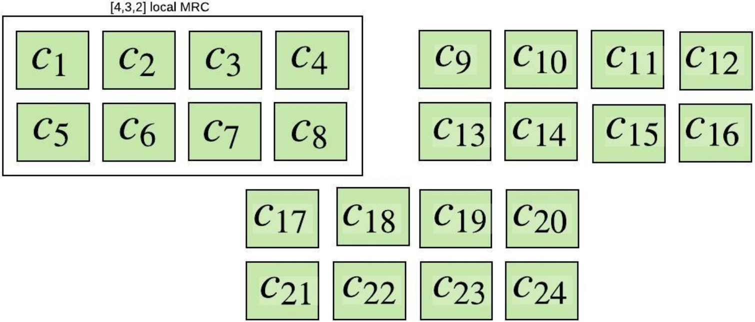
- Pick $k + h_1$ symbols from the code such that,
  - it contains r_1 symbols from each mid-level code
  - it contains at-most r_2 symbols from each local code
- Those $k + h_1$ symbols should form an $[k + h_1, k]$ MDS code.

MRCs with Hierarchical Locality

Lemma

In a $[k, r_1, r_2, h_1, h_2, \delta]$ *hierarchical local MRC, the mid-level codes itself are an* $[r_1, r_2, h_2, \delta]$ *local MRC.*

$[k, r_1, r_2, h_1, h_2, \delta]$ code with $k = 9$, $r_1 = 4$, $r_2 = 3$, $h_1 = 3$, $h_2 = 2$ and $\delta = 1$



Parity Check Matrix

For example code, $[k = 2, r_1 = 2, r_2 = 2, h_1 = 2, h_2 = 2, \delta = 2]$

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & & & & & & & & & & & & \\
0 & \beta & \beta^2 & \beta^3 & & & & & & & & & & & & \\
 & & & & 1 & 1 & 1 & 1 & & & & & & & & \\
 & & & & 0 & \beta & \beta^2 & \beta^3 & & & & & & & & \\
\alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,4} & \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \alpha_{2,4} & & & & & & & & \\
\alpha_{1,1}^q & \alpha_{1,2}^q & \alpha_{1,3}^q & \alpha_{1,4}^q & \alpha_{2,1}^q & \alpha_{2,2}^q & \alpha_{2,3}^q & \alpha_{2,4}^q & & & & & & & & \\
 & & & & & & & & 1 & 1 & 1 & 1 & & & & \\
 & & & & & & & & 0 & \beta & \beta^2 & \beta^3 & & & & \\
 & & & & & & & & & & & & 1 & 1 & 1 & 1 \\
 & & & & & & & & & & & & 0 & \beta & \beta^2 & \beta^3 \\
 & & & & & & & & \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,4} & \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \alpha_{2,4} \\
 & & & & & & & & \alpha_{1,1}^q & \alpha_{1,2}^q & \alpha_{1,3}^q & \alpha_{1,4}^q & \alpha_{2,1}^q & \alpha_{2,2}^q & \alpha_{2,3}^q & \alpha_{2,4}^q \\
\lambda_{1,1,1} & \lambda_{1,1,2} & \lambda_{1,1,3} & \lambda_{1,1,4} & \lambda_{1,2,1} & \lambda_{1,2,2} & \lambda_{1,2,3} & \lambda_{1,2,4} & \lambda_{2,1,1} & \lambda_{2,1,2} & \lambda_{2,1,3} & \lambda_{2,1,4} & \lambda_{2,2,1} & \lambda_{2,2,2} & \lambda_{2,2,3} & \lambda_{2,2,4} \\
\lambda_{1,1,1}^{q^{m_1}} & \lambda_{1,1,2}^{q^{m_1}} & \lambda_{1,1,3}^{q^{m_1}} & \lambda_{1,1,4}^{q^{m_1}} & \lambda_{1,2,1}^{q^{m_1}} & \lambda_{1,2,2}^{q^{m_1}} & \lambda_{1,2,3}^{q^{m_1}} & \lambda_{1,2,4}^{q^{m_1}} & \lambda_{2,1,1}^{q^{m_1}} & \lambda_{2,1,2}^{q^{m_1}} & \lambda_{2,1,3}^{q^{m_1}} & \lambda_{2,1,4}^{q^{m_1}} & \lambda_{2,2,1}^{q^{m_1}} & \lambda_{2,2,2}^{q^{m_1}} & \lambda_{2,2,3}^{q^{m_1}} & \lambda_{2,2,4}^{q^{m_1}}
\end{bmatrix}
$$

- $\mathbb{F}_{q^m}$ is an extension field of $\mathbb{F}_{q^{m_1}}$ which itself is an extension of $\mathbb{F}_q$
- $\mathbb{F}_q =< \beta >$, $\alpha_{i,j} \in \mathbb{F}_{q^{m_1}}$ and $\lambda_{i,j,k} \in \mathbb{F}_{q^m}$

Parity Check Matrix

For example code, $[k = 2, r_1 = 2, r_2 = 2, h_1 = 2, h_2 = 2, \delta = 2]$

$$
\left[\begin{array}{cccccccccccccccc}
1 & 1 & 1 & 1 & & & & & & & & & & & & \\
0 & \beta & \beta^2 & \beta^3 & & & & & & & & & & & & \\
 & & & & 1 & 1 & 1 & 1 & & & & & & & & \\
 & & & & 0 & \beta & \beta^2 & \beta^3 & & & & & & & & \\
\alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,4} & \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \alpha_{2,4} & & & & & & & & \\
\alpha_{1,1}^q & \alpha_{1,2}^q & \alpha_{1,3}^q & \alpha_{1,4}^q & \alpha_{2,1}^q & \alpha_{2,2}^q & \alpha_{2,3}^q & \alpha_{2,4}^q & & & & & & & & \\
 & & & & & & & & 1 & 1 & 1 & 1 & & & & \\
 & & & & & & & & 0 & \beta & \beta^2 & \beta^3 & & & & \\
 & & & & & & & & & & & & 1 & 1 & 1 & 1 \\
 & & & & & & & & & & & & 0 & \beta & \beta^2 & \beta^3 \\
 & & & & & & & & \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,4} & \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \alpha_{2,4} \\
 & & & & & & & & \alpha_{1,1}^q & \alpha_{1,2}^q & \alpha_{1,3}^q & \alpha_{1,4}^q & \alpha_{2,1}^q & \alpha_{2,2}^q & \alpha_{2,3}^q & \alpha_{2,4}^q \\
\lambda_{1,1,1} & \lambda_{1,1,2} & \lambda_{1,1,3} & \lambda_{1,1,4} & \lambda_{1,2,1} & \lambda_{1,2,2} & \lambda_{1,2,3} & \lambda_{1,2,4} & \lambda_{2,1,1} & \lambda_{2,1,2} & \lambda_{2,1,3} & \lambda_{2,1,4} & \lambda_{2,2,1} & \lambda_{2,2,2} & \lambda_{2,2,3} & \lambda_{2,2,4} \\
\lambda_{1,1,1}^{q^{m_1}} & \lambda_{1,1,2}^{q^{m_1}} & \lambda_{1,1,3}^{q^{m_1}} & \lambda_{1,1,4}^{q^{m_1}} & \lambda_{1,2,1}^{q^{m_1}} & \lambda_{1,2,2}^{q^{m_1}} & \lambda_{1,2,3}^{q^{m_1}} & \lambda_{1,2,4}^{q^{m_1}} & \lambda_{2,1,1}^{q^{m_1}} & \lambda_{2,1,2}^{q^{m_1}} & \lambda_{2,1,3}^{q^{m_1}} & \lambda_{2,1,4}^{q^{m_1}} & \lambda_{2,2,1}^{q^{m_1}} & \lambda_{2,2,2}^{q^{m_1}} & \lambda_{2,2,3}^{q^{m_1}} & \lambda_{2,2,4}^{q^{m_1}}
\end{array}\right]
$$

Global parity check conditions, mid-level code, and local parities are highlighted.

Conditions for MRC (mid-level parities)

According to our the previous lemma, each mid-level code is an $[r_1, r_2, h_2, \delta]$ local MRC.

Puncturing $\delta = 2$ coordinates per local group results in an $[r_1 + h_2 = 4, r_1 = 2]$ MDS code.

$$\left[\begin{array}{cccccccccccccccc}
1 & 1 & 1 & 1 & & & & & & & & & & & & \\
0 & \beta & \beta^2 & \beta^3 & & & & & & & & & & & & \\
 & & & & 1 & 1 & 1 & 1 & & & & & & & & \\
 & & & & 0 & \beta & \beta^2 & \beta^3 & & & & & & & & \\
\alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,4} & \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \alpha_{2,4} & & & & & & & & \\
\alpha_{1,1}^q & \alpha_{1,2}^q & \alpha_{1,3}^q & \alpha_{1,4}^q & \alpha_{2,1}^q & \alpha_{2,2}^q & \alpha_{2,3}^q & \alpha_{2,4}^q & & & & & & & & \\
 & & & & & & & & 1 & 1 & 1 & 1 & & & & \\
 & & & & & & & & 0 & \beta & \beta^2 & \beta^3 & & & & \\
 & & & & & & & & & & & & 1 & 1 & 1 & 1 \\
 & & & & & & & & & & & & 0 & \beta & \beta^2 & \beta^3 \\
 & & & & & & & & \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,4} & \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \alpha_{2,4} \\
 & & & & & & & & \alpha_{1,1}^q & \alpha_{1,2}^q & \alpha_{1,3}^q & \alpha_{1,4}^q & \alpha_{2,1}^q & \alpha_{2,2}^q & \alpha_{2,3}^q & \alpha_{2,4}^q \\
\lambda_{1,1,1} & \lambda_{1,1,2} & \lambda_{1,1,3} & \lambda_{1,1,4} & \lambda_{1,2,1} & \lambda_{1,2,2} & \lambda_{1,2,3} & \lambda_{1,2,4} & \lambda_{2,1,1} & \lambda_{2,1,2} & \lambda_{2,1,3} & \lambda_{2,1,4} & \lambda_{2,2,1} & \lambda_{2,2,2} & \lambda_{2,2,3} & \lambda_{2,2,4} \\
\lambda_{1,1,1}^{q^{m_1}} & \lambda_{1,1,2}^{q^{m_1}} & \lambda_{1,1,3}^{q^{m_1}} & \lambda_{1,1,4}^{q^{m_1}} & \lambda_{1,2,1}^{q^{m_1}} & \lambda_{1,2,2}^{q^{m_1}} & \lambda_{1,2,3}^{q^{m_1}} & \lambda_{1,2,4}^{q^{m_1}} & \lambda_{2,1,1}^{q^{m_1}} & \lambda_{2,1,2}^{q^{m_1}} & \lambda_{2,1,3}^{q^{m_1}} & \lambda_{2,1,4}^{q^{m_1}} & \lambda_{2,2,1}^{q^{m_1}} & \lambda_{2,2,2}^{q^{m_1}} & \lambda_{2,2,3}^{q^{m_1}} & \lambda_{2,2,4}^{q^{m_1}}
\end{array}\right]$$

Relevant sub-matrices are highlighted.

Conditions for MRC (mid-level parities)

We consider one such sub-matrix.
We assume that columns 1 and 2 are punctured.

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 0 & \beta & \beta^2 & \beta^3 \\ \hline \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,4} \\ \hline \alpha_{1,1}^q & \alpha_{1,2}^q & \alpha_{1,3}^q & \alpha_{1,4}^q \end{array}\right] \Longrightarrow \begin{bmatrix} M_s & M_{\bar{s}} \\ \alpha_{1,s} & \alpha_{1,\bar{s}} \\ \alpha_{1,s}^q & \alpha_{1,\bar{s}}^q \end{bmatrix}$$

$$\begin{bmatrix} M_s & M_{\bar{s}} \\ \alpha_{1,s} & \alpha_{1,\bar{s}} \\ \alpha_{1,s}^q & \alpha_{1,\bar{s}}^q \end{bmatrix} \Longrightarrow \begin{bmatrix} M_s & M_{\bar{s}} \\ 0 & \alpha_{1,\bar{s}} + \alpha_{1,s} L \\ 0 & (\alpha_{1,\bar{s}} + \alpha_{1,s} L)^q \end{bmatrix}$$

$$L = M_s^{-1} M_{\bar{s}} \text{ } (2 \times 2 \text{ matrix})$$

Since all elements of L are from $\mathbb{F}_q$, $L = L^q$

Conditions for MRC (mid-level parities)

We show one such mid-level code after that operation.

$$\begin{bmatrix} M_s & M_{\bar{s}} & & \\ & & M_s & M_{\bar{s}} \\ 0 & \alpha_{1,\bar{s}} + \alpha_{1,s} L & 0 & \alpha_{2,\bar{s}} + \alpha_{2,s} L \\ 0 & (\alpha_{1,\bar{s}} + \alpha_{1,s} L)^q & 0 & (\alpha_{2,\bar{s}} + \alpha_{2,s} L)^q \end{bmatrix}$$

The punctured sub-matrix,

$$\begin{bmatrix} \alpha_{1,\bar{s}} + \alpha_{1,s} L & \alpha_{2,\bar{s}} + \alpha_{2,s} L \\ (\alpha_{1,\bar{s}} + \alpha_{1,s} L)^q & (\alpha_{2,\bar{s}} + \alpha_{2,s} L)^q \end{bmatrix}$$

should be the PCM for a $[r_1 + h_2 = 4, r_1 = 2]$ MDS code.
Possible because L is a 2×2 matrix.

Conditions for MRC: Some definitions

Definition (k-wise Independence)

A multi-set $S \subseteq \mathbb{F}$ is k-wise independent over $\mathbb{F}$ if for every set $T \subseteq S$ such that $|T| \leq k$, T is linearly independent over $\mathbb{F}$.

Lemma

Let $\mathbb{F}_{q^t}$ be an extension of $\mathbb{F}_q$. Let $a_1, a_2, \ldots, a_n$ be elements of $\mathbb{F}_{q^t}$. The following matrix

$$\begin{bmatrix} a_1 & a_2 & a_3 & \ldots & a_n \\ a_1^q & a_2^q & a_3^q & \ldots & a_n^q \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ a_1^{q^{k-1}} & a_2^{q^{k-1}} & a_3^{q^{k-1}} & \ldots & a_n^{q^{k-1}} \end{bmatrix}$$

is the generator matrix of a $[n, k]$ MDS code if and only if $a_1, a_2, \ldots, a_n$ are k-wise linearly independent over $\mathbb{F}_q$.

Conditions for MRC (mid-level parities)

Using the above lemma, the matrix,

$$\begin{bmatrix} \alpha_{1,\bar{s}} + \alpha_{1,s}L & \alpha_{2,\bar{s}} + \alpha_{2,s}L \\ (\alpha_{1,\bar{s}} + \alpha_{1,s}L)^q & (\alpha_{2,\bar{s}} + \alpha_{2,s}L)^q \end{bmatrix}$$

is the parity check matrix for a $[r_1 + h_2 = 4, r_1 = 2]$ MDS code if the set

$$\Psi = \{\alpha_{1,\bar{s}} + \alpha_{1,s}L,\ \alpha_{2,\bar{s}} + \alpha_{2,s}L\}$$

is $h_2 = 2$ wise independent over $\mathbb{F}_q$.

Condition on $\alpha_{i,j}$

Since,

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 0 & \beta & \beta^2 & \beta^3 \\ \hline \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,4} \\ \hline \alpha_{1,1}^q & \alpha_{1,2}^q & \alpha_{1,3}^q & \alpha_{1,4}^q \end{array}\right] \implies \begin{bmatrix} M_s & M_{\bar{s}} \\ \alpha_{1,s} & \alpha_{1,\bar{s}} \\ \alpha_{1,s}^q & \alpha_{1,\bar{s}}^q \end{bmatrix}$$

$$\Psi = \{\alpha_{1,\bar{s}} + \alpha_{1,s}L,\ \alpha_{2,\bar{s}} + \alpha_{2,s}L\}$$

- Any $\mathbb{F}_q$-linear combination of k elements in Ψ will have at-most $3k$ distinct elements.
- Hence if the set $\{\alpha_{i,j}\}$ is at-least $3h_2 = 6$ wise independent over $\mathbb{F}_q$, then the set Ψ will be $h_2 = 2$ wise independent over $\mathbb{F}_q$.

Conditions for MRC (global parities)

We now consider global parities along with the mid-level codes. After puncturing δ coordinates per local group, puncturing $h_2 = 2$ coordinates per mid-level code results in an MDS code.

$$\left[\begin{array}{cccccccccccccccc}
1 & 1 & 1 & 1 & & & & & & & & & & & & \\
0 & \beta & \beta^2 & \beta^3 & & & & & & & & & & & & \\
& & & & 1 & 1 & 1 & 1 & & & & & & & & \\
& & & & 0 & \beta & \beta^2 & \beta^3 & & & & & & & & \\
\alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,4} & \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \alpha_{2,4} & & & & & & & & \\
\alpha_{1,1}^q & \alpha_{1,2}^q & \alpha_{1,3}^q & \alpha_{1,4}^q & \alpha_{2,1}^q & \alpha_{2,2}^q & \alpha_{2,3}^q & \alpha_{2,4}^q & & & & & & & & \\
& & & & & & & & 1 & 1 & 1 & 1 & & & & \\
& & & & & & & & 0 & \beta & \beta^2 & \beta^3 & & & & \\
& & & & & & & & & & & & 1 & 1 & 1 & 1 \\
& & & & & & & & & & & & 0 & \beta & \beta^2 & \beta^3 \\
& & & & & & & & \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,4} & \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \alpha_{2,4} \\
& & & & & & & & \alpha_{1,1}^q & \alpha_{1,2}^q & \alpha_{1,3}^q & \alpha_{1,4}^q & \alpha_{2,1}^q & \alpha_{2,2}^q & \alpha_{2,3}^q & \alpha_{2,4}^q \\
\lambda_{1,1,1} & \lambda_{1,1,2} & \lambda_{1,1,3} & \lambda_{1,1,4} & \lambda_{1,2,1} & \lambda_{1,2,2} & \lambda_{1,2,3} & \lambda_{1,2,4} & \lambda_{2,1,1} & \lambda_{2,1,2} & \lambda_{2,1,3} & \lambda_{2,1,4} & \lambda_{2,2,1} & \lambda_{2,2,2} & \lambda_{2,2,3} & \lambda_{2,2,4} \\
\lambda_{1,1,1}^{q^{m_1}} & \lambda_{1,1,2}^{q^{m_1}} & \lambda_{1,1,3}^{q^{m_1}} & \lambda_{1,1,4}^{q^{m_1}} & \lambda_{1,2,1}^{q^{m_1}} & \lambda_{1,2,2}^{q^{m_1}} & \lambda_{1,2,3}^{q^{m_1}} & \lambda_{1,2,4}^{q^{m_1}} & \lambda_{2,1,1}^{q^{m_1}} & \lambda_{2,1,2}^{q^{m_1}} & \lambda_{2,1,3}^{q^{m_1}} & \lambda_{2,1,4}^{q^{m_1}} & \lambda_{2,2,1}^{q^{m_1}} & \lambda_{2,2,2}^{q^{m_1}} & \lambda_{2,2,3}^{q^{m_1}} & \lambda_{2,2,4}^{q^{m_1}}
\end{array}\right]$$

Conditions for MRC (global parities)

This time we apply the shortening to global parities as well. We collect the shortened code from the entire mid-level code as in previous steps.

$$\begin{bmatrix} \alpha_{1,\bar{s}} + \alpha_{1,s}L & \alpha_{2,\bar{s}} + \alpha_{2,s}L \\ (\alpha_{1,\bar{s}} + \alpha_{1,s}L)^q & (\alpha_{2,\bar{s}} + \alpha_{2,s}L)^q \\ \lambda_{1,1,\bar{s}} + \lambda_{1,1,s}L & \lambda_{1,2,\bar{s}} + \lambda_{1,2,s}L \\ (\lambda_{1,1,\bar{s}} + \lambda_{1,1,s}L)^{q^{m_1}} & (\lambda_{1,2,\bar{s}} + \lambda_{1,2,s}L)^{q^{m_1}} \end{bmatrix}$$

We perform similar steps as we did for $\alpha_{i,j}$ and arrive at a similar result.

- The set $\{\lambda_{i,j,k}\}$ needs to be at-least $h_1(h_2+1)(\delta+1)$ wise independent over $\mathbb{F}_{q^{m_1}}$

Picking $\alpha_{i,j}$ and $\lambda_{i,j,k}$ (from PCM of codes)

- We pick $\alpha_{i,j}$ and $\lambda_{i,j,k}$ as columns of the PCM of an appropriate code.
- For an $[n, k, d]$ code over $\mathbb{F}_q$, the columns of a PCM are elements in $\mathbb{F}_{q^{n-k}}$ which are $(d-1)$-wise independent over $\mathbb{F}_q$
- Since, $\alpha_{i,j} \in \mathbb{F}_{q^{m_1}}/\mathbb{F}_q$, the value of $n-k$ is m_1.
- Now since $\{\alpha_{i,j}\}$ needs to be 6-wise independent in $\mathbb{F}_q$, the value of d should be 7.
- We need 8 values for $\alpha_{i,j} = \{\alpha_{1,1}, \dots \alpha_{1,4}, \alpha_{2,1}, \dots, \alpha_{2,4}\}$.

Picking $\alpha_{i,j}$ and $\lambda_{i,j,k}$ (BCH codes)

Lemma

There exists $[n = q^t - 1, k, d]$ *BCH codes over* $\mathbb{F}_q$ *, where the parameters are related as*

$$n - k = 1 + \left\lceil \frac{q-1}{q}(d-2) \right\rceil \lceil \log_2(n) \rceil.$$

- We set $t = \lceil log_q(8) \rceil$ to get a PCM with smallest number of columns.
- We already have a value for d and can now calculate the value of k from the above relation.

Similar procedure is followed to get $\{\lambda_{i,j,k}\}$.

More optimisations

- Using Tensor Product Codes to perform the above construction in a smaller field.
- Even smaller field size constructions for the following special cases.
    1. 1 global parity any number of mid-level parities
    2. 1 global parity and 1 mid-level parity
    3. 2 global parities and 1 mid-level parity

Tensor Product Codes

Let C_1 be an $[n, n-\rho]$ linear code in $\mathbb{F}_q$ which can correct e_1 erasures. Also, C_2 is an $[m, m-s]$ code in $\mathbb{F}_{q^\rho}$ that can correct e_2 erasures. An $[nm, nm-s\rho]$ code C in $\mathbb{F}_q$ is called the tensor product code of C_1 and C_2 if

$$\forall x \in C_1 \text{ and } y \in C_2, \quad y \otimes x \in C$$

where $y \otimes x$ is the tensor product of x and y in $\mathbb{F}_q$.

Example

Let C_1 be a $(3,1)$-code in $\mathbb{F}_q$. The PCM,

$$H_1 = \begin{bmatrix} u_{1,1} & u_{2,1} & u_{3,1} \\ u_{1,2} & u_{2,2} & u_{3,2} \end{bmatrix} = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} u_i \in \mathbb{F}_{q^2}$$

Let C_2 be a $(4,1)$-code in $\mathbb{F}_{q^2}$. The PCM here,

$$H_2 = \begin{bmatrix} v_{1,1} & v_{2,1} & v_{3,1} & v_{4,1} \\ v_{1,2} & v_{2,2} & v_{3,2} & v_{4,2} \\ v_{1,3} & v_{2,3} & v_{3,3} & v_{4,3} \end{bmatrix} v_{i,j} \in \mathbb{F}_{q^2}$$

Assume both codes are MDS, therefore $e_1 = 2$ and $e_2 = 3$
Therefore the PCM (H) for the tensor product of C_1 and C_2

$$H = \begin{bmatrix} v_{1,1}u_1 & v_{1,1}u_2 & v_{1,1}u_3 & v_{2,1}u_1 & v_{2,1}u_2 & v_{2,1}u_3 & v_{3,1}u_1 & v_{3,1}u_2 & v_{3,1}u_3 & v_{4,1}u_1 & v_{4,1}u_2 & v_{4,1}u_3 \\ v_{1,2}u_1 & v_{1,2}u_2 & v_{1,2}u_3 & v_{2,2}u_1 & v_{2,2}u_2 & v_{2,2}u_3 & v_{3,2}u_1 & v_{3,2}u_2 & v_{3,2}u_3 & v_{4,2}u_1 & v_{4,2}u_2 & v_{4,2}u_3 \\ v_{1,3}u_1 & v_{1,3}u_2 & v_{1,3}u_3 & v_{2,3}u_1 & v_{2,3}u_2 & v_{2,3}u_3 & v_{3,3}u_1 & v_{3,3}u_2 & v_{3,3}u_3 & v_{4,3}u_1 & v_{4,3}u_2 & v_{4,3}u_3 \end{bmatrix}$$

Tensor Product Codes: Erasure Correction

Theorem

C_1, C_2 and C are as defined above. If the code-words in C are considered to be consisting of m sub-blocks with each sub-block containing n symbols, C will correct all erasure patterns where,

- *At-most e_2 sub-blocks are affected.*
- *At-most e_1 erasures in each affected sub-block.*

$$H = \left[\begin{array}{ccc|ccc|ccc|ccc} v_{1,1}u_1 & v_{1,1}u_2 & v_{1,1}u_3 & v_{2,1}u_1 & v_{2,1}u_2 & v_{2,1}u_3 & v_{3,1}u_1 & v_{3,1}u_2 & v_{3,1}u_3 & v_{4,1}u_1 & v_{4,1}u_2 & v_{4,1}u_3 \\ v_{1,2}u_1 & v_{1,2}u_2 & v_{1,2}u_3 & v_{2,2}u_1 & v_{2,2}u_2 & v_{2,2}u_3 & v_{3,2}u_1 & v_{3,2}u_2 & v_{3,2}u_3 & v_{4,2}u_1 & v_{4,2}u_2 & v_{4,2}u_3 \\ v_{1,3}u_1 & v_{1,3}u_2 & v_{1,3}u_3 & v_{2,3}u_1 & v_{2,3}u_2 & v_{2,3}u_3 & v_{3,3}u_1 & v_{3,3}u_2 & v_{3,3}u_3 & v_{4,3}u_1 & v_{4,3}u_2 & v_{4,3}u_3 \end{array}\right]$$

Here, there are $m = 4$ sub-blocks with $n = 3$ symbols each. This code can recover all erasures where at-most 3 sub-blocks are affected and upto 2 erasures per sub-block

Convention

We say that an $[nm, nm - s\rho]$ tensor product code $C \subseteq \mathbb{F}_q^{m\times n}$ is an $[m, n; e_2, e_1]$ erasure correcting code if it can correct any erasure pattern of the form $\mathbf{E} = (E_1, \ldots, E_m)$ where for $i \in [m]$ and $E_i \subseteq [n]$ and

- $|\{i : E_i \neq \emptyset\}| \leq e_2$.
- for $i \in [m], |E_i| \leq e_1$.

$$H = \left[\begin{array}{ccc|ccc|ccc|ccc} v_{1,1}u_1 & v_{1,1}u_2 & v_{1,1}u_3 & v_{2,1}u_1 & v_{2,1}u_2 & v_{2,1}u_3 & v_{3,1}u_1 & v_{3,1}u_2 & v_{3,1}u_3 & v_{4,1}u_1 & v_{4,1}u_2 & v_{4,1}u_3 \\ v_{1,2}u_1 & v_{1,2}u_2 & v_{1,2}u_3 & v_{2,2}u_1 & v_{2,2}u_2 & v_{2,2}u_3 & v_{3,2}u_1 & v_{3,2}u_2 & v_{3,2}u_3 & v_{4,2}u_1 & v_{4,2}u_2 & v_{4,2}u_3 \\ v_{1,3}u_1 & v_{1,3}u_2 & v_{1,3}u_3 & v_{2,3}u_1 & v_{2,3}u_2 & v_{2,3}u_3 & v_{3,3}u_1 & v_{3,3}u_2 & v_{3,3}u_3 & v_{4,3}u_1 & v_{4,3}u_2 & v_{4,3}u_3 \end{array}\right]$$
$$= \left[\begin{array}{ccc|ccc|ccc|ccc} \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3} & \alpha_{4,1} & \alpha_{4,2} & \alpha_{4,3} \end{array}\right]$$

This code is then a $[4, 3; 3, 2]$ code.

Columns of the Parity Check Matrix

$$\left[\begin{array}{ccc|ccc|ccc|ccc} \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3} & \alpha_{4,1} & \alpha_{4,2} & \alpha_{4,3} \end{array}\right]$$

This code can correct erasures in upto 3 sub-blocks and at-most 2 erasures per sub-block. Therefore for

$$S = \{\alpha_{1,1}, \alpha_{1,2}, \alpha_{1,3}, \alpha_{2,1}, \alpha_{2,2}, \alpha_{2,3}, \alpha_{3,1}, \alpha_{3,2}, \alpha_{3,3}, \alpha_{4,1}, \alpha_{4,2}, \alpha_{4,3}, \}$$

Any set $T \subseteq S$ such that

- There are upto 3 distinct values of i of $\alpha_{i,j}$
- There are upto 2 distinct values of j per i of $\alpha_{i,j}$

Any such subset will be linearly independent in $\mathbb{F}_{q^2}$

Product Construction

Assume the following code, $[k = 8, r_1 = 7, r_2 = 3, h_1 = 6, h_2 = 2, \delta = 2]$
Based on the MRC conditions proved previously,

$$\Psi = \{\alpha_{1,\bar{s}} + \alpha_{1,s}L,\ \alpha_{2,\bar{s}} + \alpha_{2,s}L,\ \alpha_{3,\bar{s}} + \alpha_{3,s}L\}$$

where, $\alpha_{i,s} = \{\alpha_{i,1}, \alpha_{i,2}\}$ and $\alpha_{i,\bar{s}} = \{\alpha_{i,3}, \alpha_{i,4}, \alpha_{i,5}\}$ $\alpha_{i,j} \in \mathbb{F}_{q^{m_1}}$
and L is a 2×3 matrix with elements in $\mathbb{F}_q$.
Ψ is required to be 2-wise independent over $\mathbb{F}_q$.

Product Construction

Assuming $L = \begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix}$ Expand Ψ into individual components,

$$\begin{aligned}\Psi = \{&\alpha_{1,3} + a\alpha_{1,1} + b\alpha_{1,2},\ \alpha_{1,4} + c\alpha_{1,1} + d\alpha_{1,2},\ \alpha_{1,5} + e\alpha_{1,1} + f\alpha_{1,2},\\ &\alpha_{2,3} + a\alpha_{2,1} + b\alpha_{2,2},\ \alpha_{2,4} + c\alpha_{2,1} + d\alpha_{2,2},\ \alpha_{2,5} + e\alpha_{2,1} + f\alpha_{2,2},\\ &\alpha_{3,3} + a\alpha_{3,1} + b\alpha_{3,2},\ \alpha_{3,4} + c\alpha_{3,1} + d\alpha_{3,2},\ \alpha_{3,5} + e\alpha_{3,1} + f\alpha_{3,2}\}\end{aligned}$$

We can pick $h_2 = 2$ elements in two different ways.

- Here, the first index of $\alpha_{i,j}$ remains the same. There are 4 distinct $\alpha_{i,j}$ in that linear combination
- The first index of $\alpha_{i,j}$ are different. In this case, there are 3 distinct $\alpha_{i,j}$ per i.

Product Construction

If we relate this to previous example of the tensor code,

$$\left[\alpha_{1,1} \quad \alpha_{1,2} \quad \alpha_{1,3} \qquad \alpha_{2,1} \quad \alpha_{2,2} \quad \alpha_{2,3} \qquad \alpha_{3,1} \quad \alpha_{3,2} \quad \alpha_{3,3} \qquad \alpha_{4,1} \quad \alpha_{4,2} \quad \alpha_{4,3}\right]$$

(Illustrative example)

If we pick $\alpha_{i,j}$ as these columns we have to ensure that,

- The code can recover from erasures in 2 sub-blocks ($e_2 = 2$).
- The code can recover from 4 erasures per sub-block ($e_1 = 4$).
- There are at-least 3 distinct groups because there are 3 distinct values of i in $\alpha_{i,j}$ ($m = 3$)
- There are 5 columns per group because there are 5 different values of j for every i in $\alpha_{i,j}$. ($n = 5$)

Hence, Ψ is 2-wise independent if $\alpha_{i,j}$ is picked from a $[3, 5; 2, 4]$ code.

Product Construction

More generally,

Theorem

Let C_{TP} be an $[t_2, n_2; h_2, h_2+\delta]$ erasure correcting code with $t_2 > h_2$ and $n_2 > h_2+\delta$ over $\mathbb{F}_q$ with redundancy m_1 and the parity check matrix $H_{TP} = (\alpha_{1,1}, \ldots, \alpha_{t_2,n_2}) \in (\mathbb{F}_{q^{m_1}})^{t_2 n_2}$. Then the set $\{\alpha_{i,j}\}$ chosen as columns of H_{TP} guarantees that the mid-level code is MRC.

Similarly for $\lambda_{i,k,j}$,

Theorem

Let C_{TP} be an $[t_1, t_2 n_2; h_1, (h_1+h_2)(\delta+1)]$ erasure correcting code with $t_1 > h_1$ and $t_2 n_2 > (h_1+h_2)(\delta+1)$ over $\mathbb{F}_{q^{m_1}}$ with redundancy m and the parity check matrix $H_{TP} = (\lambda_{1,1,1}, \ldots, \lambda_{t_1,t_2,n_2}) \in (\mathbb{F}_{q^m})^{t_1 t_2 n_2}$. Then the set $\{\lambda_{i,j,k}\}$ chosen as columns of H_{TP} ensures that global code is MRC.

Optimisation for $h_1 = 1$

Consider the code, $[k = 3, r_1 = 2, r_2 = 2, h_1 = 1, h_2 = 2, \delta = 2]$

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & & & & & & & & & & & & \\
0 & \beta & \beta^2 & \beta^3 & & & & & & & & & & & & \\
& & & & 1 & 1 & 1 & 1 & & & & & & & & \\
& & & & 0 & \beta & \beta^2 & \beta^3 & & & & & & & & \\
\alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,4} & \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \alpha_{2,4} & & & & & & & & \\
\alpha_{1,1}^q & \alpha_{1,2}^q & \alpha_{1,3}^q & \alpha_{1,4}^q & \alpha_{2,1}^q & \alpha_{2,2}^q & \alpha_{2,3}^q & \alpha_{2,4}^q & & & & & & & & \\
& & & & & & & & 1 & 1 & 1 & 1 & & & & \\
& & & & & & & & 0 & \beta & \beta^2 & \beta^3 & & & & \\
& & & & & & & & & & & & 1 & 1 & 1 & 1 \\
& & & & & & & & & & & & 0 & \beta & \beta^2 & \beta^3 \\
& & & & & & & & \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,4} & \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \alpha_{2,4} \\
& & & & & & & & \alpha_{1,1}^q & \alpha_{1,2}^q & \alpha_{1,3}^q & \alpha_{1,4}^q & \alpha_{2,1}^q & \alpha_{2,2}^q & \alpha_{2,3}^q & \alpha_{2,4}^q \\
\lambda_{1,1,1} & \lambda_{1,1,2} & \lambda_{1,1,3} & \lambda_{1,1,4} & \lambda_{1,2,1} & \lambda_{1,2,2} & \lambda_{1,2,3} & \lambda_{1,2,4} & \lambda_{2,1,1} & \lambda_{2,1,2} & \lambda_{2,1,3} & \lambda_{2,1,4} & \lambda_{2,2,1} & \lambda_{2,2,2} & \lambda_{2,2,3} & \lambda_{2,2,4}
\end{bmatrix}
$$

standard construction. $\lambda_{i,j,k} \in \mathbb{F}_{q^m}$

Optimisation for $h_1 = 1$

Consider the code, $[k = 3, r_1 = 2, r_2 = 2, h_1 = 1, h_2 = 2, \delta = 2]$

$$\begin{bmatrix}
1 & 1 & 1 & 1 & & & & & & & & & & & & \\
0 & \beta & \beta^2 & \beta^3 & & & & & & & & & & & & \\
 & & & & 1 & 1 & 1 & 1 & & & & & & & & \\
 & & & & 0 & \beta & \beta^2 & \beta^3 & & & & & & & & \\
\alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,4} & \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \alpha_{2,4} & & & & & & & & \\
\alpha_{1,1}^q & \alpha_{1,2}^q & \alpha_{1,3}^q & \alpha_{1,4}^q & \alpha_{2,1}^q & \alpha_{2,2}^q & \alpha_{2,3}^q & \alpha_{2,4}^q & & & & & & & & \\
 & & & & & & & & 1 & 1 & 1 & 1 & & & & \\
 & & & & & & & & 0 & \beta & \beta^2 & \beta^3 & & & & \\
 & & & & & & & & & & & & 1 & 1 & 1 & 1 \\
 & & & & & & & & & & & & 0 & \beta & \beta^2 & \beta^3 \\
 & & & & & & & & \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,4} & \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \alpha_{2,4} \\
 & & & & & & & & \alpha_{1,1}^q & \alpha_{1,2}^q & \alpha_{1,3}^q & \alpha_{1,4}^q & \alpha_{2,1}^q & \alpha_{2,2}^q & \alpha_{2,3}^q & \alpha_{2,4}^q \\
\alpha_{1,1}^{q^2} & \alpha_{1,2}^{q^2} & \alpha_{1,3}^{q^2} & \alpha_{1,4}^{q^2} & \alpha_{2,1}^{q^2} & \alpha_{2,2}^{q^2} & \alpha_{2,3}^{q^2} & \alpha_{2,4}^{q^2} & \alpha_{1,1}^{q^2} & \alpha_{1,2}^{q^2} & \alpha_{1,3}^{q^2} & \alpha_{1,4}^{q^2} & \alpha_{2,1}^{q^2} & \alpha_{2,2}^{q^2} & \alpha_{2,3}^{q^2} & \alpha_{2,4}^{q^2}
\end{bmatrix}$$

Optimised construction. $\alpha_{i,j} \in \mathbb{F}_{q^{m_1}}$

All the MRC conditions can be proved for this construction as well.

Optimisation for $h_1 = 1$ and $h_2 = 1$

Consider the code, $[k = 5, r_1 = 3, r_2 = 2, h_1 = 1, h_2 = 1, \delta = 2]$

$$
\begin{bmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & & & & & & & & & & & & \\
\alpha_1^2 & \alpha_2^2 & \alpha_3^2 & \alpha_4^2 & & & & & & & & & & & & \\
& & & & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & & & & & & & & \\
& & & & \alpha_1^2 & \alpha_2^2 & \alpha_3^2 & \alpha_4^2 & & & & & & & & \\
\lambda_1 & \lambda_1 & \lambda_1 & \lambda_1 & \lambda_2 & \lambda_2 & \lambda_2 & \lambda_2 & & & & & & & & \\
& & & & & & & & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & & & & \\
& & & & & & & & \alpha_1^2 & \alpha_2^2 & \alpha_3^2 & \alpha_4^2 & & & & \\
& & & & & & & & & & & & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\
& & & & & & & & & & & & \alpha_1^2 & \alpha_2^2 & \alpha_3^2 & \alpha_4^2 \\
& & & & & & & & \lambda_1 & \lambda_1 & \lambda_1 & \lambda_1 & \lambda_2 & \lambda_2 & \lambda_2 & \lambda_2 \\
\alpha_1^3 & \alpha_2^3 & \alpha_3^3 & \alpha_4^3 & \alpha_1^3 & \alpha_2^3 & \alpha_3^3 & \alpha_4^3 & \alpha_1^3 & \alpha_2^3 & \alpha_3^3 & \alpha_4^3 & \alpha_1^3 & \alpha_2^3 & \alpha_3^3 & \alpha_4^3
\end{bmatrix}
$$

- q is a prime power such that there exists a subgroup G of $\mathbb{F}_q^*$ of size at-least 4 and with at-least 2 cosets.
- $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in G$ and $\alpha_i \neq \alpha_j$.
- $\lambda_1, \lambda_2 \in \mathbb{F}_q^*$ be elements from distinct cosets of G.

Optimisation for $h_1 = 2$ and $h_2 = 1$

Consider the code, $[k = 4, r_1 = 3, r_2 = 2, h_1 = 2, h_2 = 1, \delta = 2]$

$$\begin{bmatrix}
\frac{1}{\alpha_1-\beta_1} & \frac{1}{\alpha_2-\beta_1} & \frac{1}{\alpha_3-\beta_1} & \frac{1}{\alpha_4-\beta_1} & & & & & & & & & & & & \\
\frac{1}{\alpha_1-\beta_2} & \frac{1}{\alpha_2-\beta_2} & \frac{1}{\alpha_3-\beta_2} & \frac{1}{\alpha_4-\beta_2} & & & & & & & & & & & & \\
& & & & \frac{1}{\alpha_1-\beta_1} & \frac{1}{\alpha_2-\beta_1} & \frac{1}{\alpha_3-\beta_1} & \frac{1}{\alpha_4-\beta_1} & & & & & & & & \\
& & & & \frac{1}{\alpha_1-\beta_2} & \frac{1}{\alpha_2-\beta_2} & \frac{1}{\alpha_3-\beta_2} & \frac{1}{\alpha_4-\beta_2} & & & & & & & & \\
\frac{1}{\alpha_1-\beta_3} & \frac{1}{\alpha_2-\beta_3} & \frac{1}{\alpha_3-\beta_3} & \frac{1}{\alpha_4-\beta_3} & \frac{1}{\alpha_1-\beta_3} & \frac{1}{\alpha_2-\beta_3} & \frac{1}{\alpha_3-\beta_3} & \frac{1}{\alpha_4-\beta_3} & & & & & & & & \\
& & & & & & & & \frac{1}{\alpha_1-\beta_1} & \frac{1}{\alpha_2-\beta_1} & \frac{1}{\alpha_3-\beta_1} & \frac{1}{\alpha_4-\beta_1} & & & & \\
& & & & & & & & \frac{1}{\alpha_1-\beta_2} & \frac{1}{\alpha_2-\beta_2} & \frac{1}{\alpha_3-\beta_2} & \frac{1}{\alpha_4-\beta_2} & & & & \\
& & & & & & & & & & & & \frac{1}{\alpha_1-\beta_1} & \frac{1}{\alpha_2-\beta_1} & \frac{1}{\alpha_3-\beta_1} & \frac{1}{\alpha_4-\beta_1} \\
& & & & & & & & & & & & \frac{1}{\alpha_1-\beta_2} & \frac{1}{\alpha_2-\beta_2} & \frac{1}{\alpha_3-\beta_2} & \frac{1}{\alpha_4-\beta_2} \\
& & & & & & & & \frac{1}{\alpha_1-\beta_3} & \frac{1}{\alpha_2-\beta_3} & \frac{1}{\alpha_3-\beta_3} & \frac{1}{\alpha_4-\beta_3} & \frac{1}{\alpha_1-\beta_3} & \frac{1}{\alpha_2-\beta_3} & \frac{1}{\alpha_3-\beta_3} & \frac{1}{\alpha_4-\beta_3} \\
\frac{\lambda_1}{\alpha_1-\beta_4} & \frac{\lambda_1}{\alpha_2-\beta_4} & \frac{\lambda_1}{\alpha_3-\beta_4} & \frac{\lambda_1}{\alpha_4-\beta_4} & \frac{\lambda_2}{\alpha_1-\beta_4} & \frac{\lambda_2}{\alpha_2-\beta_4} & \frac{\lambda_2}{\alpha_3-\beta_4} & \frac{\lambda_2}{\alpha_4-\beta_4} & \frac{\lambda_3}{\alpha_1-\beta_4} & \frac{\lambda_3}{\alpha_2-\beta_4} & \frac{\lambda_3}{\alpha_3-\beta_4} & \frac{\lambda_3}{\alpha_4-\beta_4} & \frac{\lambda_4}{\alpha_1-\beta_4} & \frac{\lambda_4}{\alpha_2-\beta_4} & \frac{\lambda_4}{\alpha_3-\beta_4} & \frac{\lambda_4}{\alpha_4-\beta_4} \\
\frac{\mu_1}{\alpha_1-\beta_5} & \frac{\mu_1}{\alpha_2-\beta_5} & \frac{\mu_1}{\alpha_3-\beta_5} & \frac{\mu_1}{\alpha_4-\beta_5} & \frac{\mu_2}{\alpha_1-\beta_5} & \frac{\mu_2}{\alpha_2-\beta_5} & \frac{\mu_2}{\alpha_3-\beta_5} & \frac{\mu_2}{\alpha_4-\beta_5} & \frac{\mu_3}{\alpha_1-\beta_5} & \frac{\mu_3}{\alpha_2-\beta_5} & \frac{\mu_3}{\alpha_3-\beta_5} & \frac{\mu_3}{\alpha_4-\beta_5} & \frac{\mu_4}{\alpha_1-\beta_5} & \frac{\mu_4}{\alpha_2-\beta_5} & \frac{\mu_4}{\alpha_3-\beta_5} & \frac{\mu_4}{\alpha_4-\beta_5}
\end{bmatrix}$$

Optimisation for $h_1 = 2$ and $h_2 = 1$: Conditions

- $q_0 \geq 15$ is a prime power.
- There exists a subgroup G of $\mathbb{F}_{q_0}^*$ of size at least 6 with at-least 4 cosets.
- $\mathbb{F}_q$ is an extension field of $\mathbb{F}_{q_0}$.
- $\mu_1, \ldots, \mu_4$ are picked from distinct cosets of G.
- Choose distinct $\beta_3, \beta_4, \beta_5 \in \mathbb{F}_{q_0}$.
- Pick $\alpha_1, \ldots \alpha_4 \in \mathbb{F}_{q_0}$ such that, $\frac{\alpha_i - \beta_4}{\alpha_i - \beta_5}, \frac{\alpha_i - \beta_3}{\alpha_i - \beta_5} \in G$.
- Pick distinct $\beta_1, \beta_2 \in \mathbb{F}_{q_0} \setminus \{\alpha_1, \ldots, \alpha_4, \beta_3, \beta_4, \beta_5\}$.
- $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{F}_q$ are picked 4 wise-independent over $\mathbb{F}_{q_0}$.

Future Work

- We define only two levels of locality. The work can be extended to any y levels.
- We haven't figured out the bounds in which our codes work.
- Even though we have reduced the number of symbols required, we still haven't optimised the repair process itself.

References

1. B. Sasidharan, G. K. Agarwal, and P. V. Kumar, "Codes with hierarchical locality," in Information Theory (ISIT), 2015 IEEE International Symposium on, pp. 1257–1261, IEEE, 2015.
2. R. Gabrys, E. Yaakobi, M. Blaum, and P. H. Siegel, "Constructions of partial mds codes over small fields," in Information Theory (ISIT), 2017 IEEE International Symposium on, pp. 1–5, IEEE, 2017.
3. P. Gopalan, C. Huang, B. Jenkins, and S. Yekhanin, "Explicit maximally recoverable codes with locality.," IEEE Trans. Information Theory, vol. 60, no. 9, pp. 5245–5256, 2014.
4. G. M. Kamath, N. Prakash, V. Lalitha, and P. V. Kumar, "Codes with local regeneration and erasure correction," IEEE Transactions on Information Theory, vol. 60, no. 8, pp. 4637–4660, 2014.
5. J. Wolf. On codes derivable from the tensor product of check matrices. IEEE Transactions on Information Theory, 11(2):281–284, 1965.
6. S. Gopi, V. Guruswami, and S. Yekhanin. Maximally recoverable lrcs: A field size lower bound and constructions for few heavy parities. In Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms, pages 2154–2170. SIAM, 2019
7. R. Roth, Introduction to coding theory. Cambridge University Press, 2006

Thanks!

Email: aaditya.mnair@research.iiit.ac.in