Maximally Recoverable Codes with Hierarchical Locality

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Codes with Locality

A code C has an (r, ϵ) locality if for every symbol $c_i \in C$, there is a punctured code C_i, such that,

- $c_i \in Supp(C_i)$.
- $d_{min}(C_i) \geq \epsilon$
- $|Supp(C_i)| \leq r + \epsilon - 1$

For an $[n, k, d]$ code with (r, ϵ) locality,

$$d \leq n - k + 1 - (\left\lceil \frac{k}{r} \right\rceil - 1)(\epsilon - 1)$$

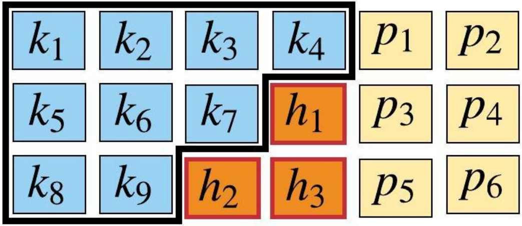
Codes with Locality (alt. definition)

Let C be a systematic $[n, k, d_{min}]$ code. We say that C is an $[k, r, h, \delta]$ local code if the following conditions are satisfied,

- $r|(k+h)$ and $n = k + \frac{k+h}{r}\delta + h$
- There are k data symbols and h global parity symbols where each global parity may depend on all data symbols.
- These $k+h$ symbols are partitioned into $\frac{k+h}{r}$ **local groups** of size r. For each such group, there are δ local parity symbols.

Example

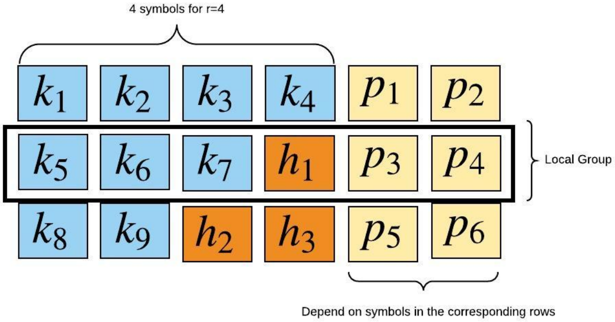
$[k, r, h, \delta]$ code with $k = 9$, $r = 4$, $h = 3$ and $\delta = 2$
where h and n are related as $n = (\frac{k+h}{r})(r + \delta)$



h_1, h_2 and h_3 depend on all k symbols

Example

$[k, r, h, \delta]$ code with $k = 9,\ r = 4,\ h = 3$ and $\delta = 2$



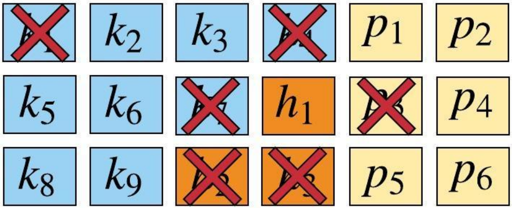
- k data symbols and h global parities are partitioned into $\frac{k+h}{r} = 3$ groups
- There are δ parity symbols for each local group.

Maximal Recoverable Code with Locality

Definition (Maximal Recoverability)

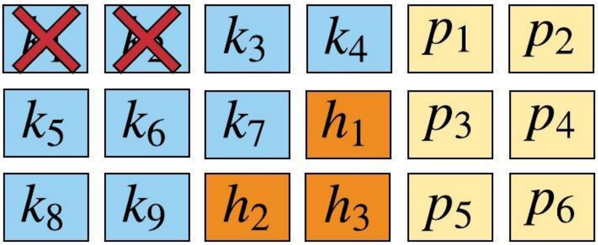
A code is said to be maximally recoverable if it can recover from all the information theoretically recoverable erasure patterns given the locality constraints of the code.

$[k, r, h, \delta]$ local MRC with $k = 9$, $r = 4$, $h = 3$ and $\delta = 2$
Puncture δ symbols per local group.
The resultant is an $[k + h, k]$ MDS code



The problem with LRCs

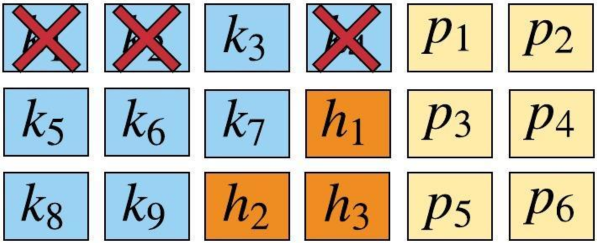
There is an abrupt jump in locality after δ erasures



Can be corrected by contacting r symbols

The problem with LRCs

There is an abrupt jump in locality after δ erasures.



Only corrected by contacting all k symbols

The Solution: Hierarchical Codes

Codes with Hierarchical Locality have multiple levels of locality. They allow for a more controlled increase in locality with the number of erasures.

Codes with Hierarchical Locality

A code C has an $[(r_1, \epsilon_1), (r_2, \epsilon_2)]$ hierarchical locality if for every symbol $c_i \in C$, there is a punctured code C_i, such that,

- $c_i \in Supp(C_i)$.
- $d_{min}(C_i) \geq \epsilon_1$
- $|Supp(C_i)| \leq r_1 + \epsilon_1 - 1$
- C_i is a code with (r_2, ϵ_2) locality

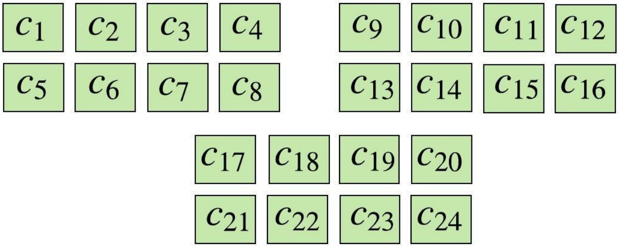
For an $[n, k, d]$ code with $[(r_1, \epsilon_1), (r_2, \epsilon_2)]$ locality,

$$d \leq n - k + 1 - (\left\lceil \frac{k}{r_2} \right\rceil - 1)(\epsilon_2 - 1) - (\left\lceil \frac{k}{r_1} \right\rceil - 1)(\epsilon_1 - \epsilon_2)$$

Codes with Hierarchical Locality (alt. definition)

Easiest to show with an example.

$[k, r_1, r_2, h_1, h_2, \delta]$ code with $k = 9$, $r_1 = 4$, $r_2 = 3$, $h_1 = 3$, $h_2 = 2$ and $\delta = 1$



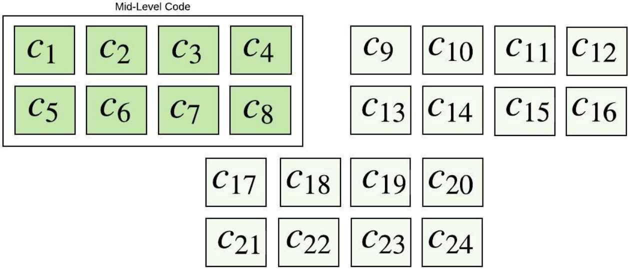
All code symbols satisfy $h_1 = 3$ global parities.

$$\sum_{j=1}^{24} u_j^{(\ell)} c_j = 0, \quad 1 \le \ell \le 3$$

Codes with Hierarchical Locality (alt. definition)

Easiest to show with an example

$[k, r_1, r_2, h_1, h_2, \delta]$ code with $k = 9$, $r_1 = 4$, $r_2 = 3$, $h_1 = 3$, $h_2 = 2$ and $\delta = 1$



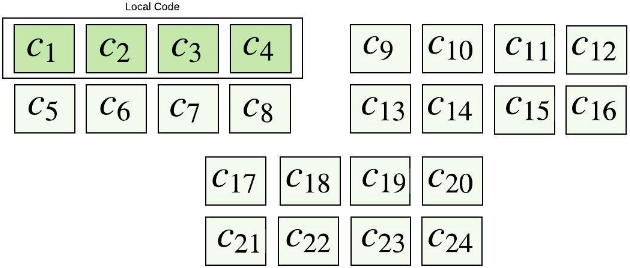
- All symbols are partitioned in $t_1 = \frac{k+h_1}{r_1} = 3$ groups of length $n_1 = \frac{r_1+h_2}{r_2}(r_2 + \delta) = 8$ called mid-level codes.
- Code symbols in a mid-level code satisfy h_2 mid-level parities.

$$\sum_{j=1}^{8} v_j^{(\ell)} c_j = 0, \quad 1 \leq \ell \leq 2 \ (\textit{same for the rest of the groups})$$

Codes with Hierarchical Locality (alt. definition)

Easiest to show with an example

$[k, r_1, r_2, h_1, h_2, \delta]$ code with $k = 9$, $r_1 = 4$, $r_2 = 3$, $h_1 = 3$, $h_2 = 2$ and $\delta = 1$



- n_1 code symbols from the previous step are partitioned into $t_2 = \frac{r_1+h_2}{r_2} = 2$ groups of size $n_2 = r_2 + \delta = 4$.
- Each of these groups satisfy $\delta = 1$ parities.

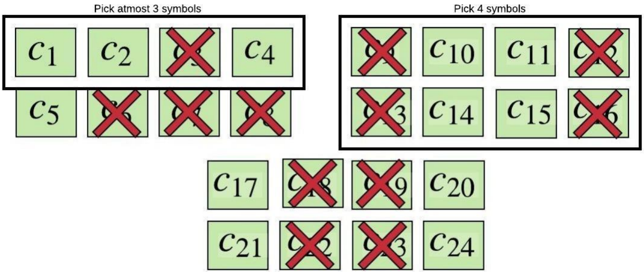
$$\sum_{j=1}^{4} w_j^{(\ell)} c_j = 0, \quad 1 \leq \ell \leq 1$$

Our Contributions

- Construction of data local hierarchical MRCs from local hierarchical MRCs.
- Definition and constructions for hierarchical local MRCs for all parameters.
- Using Tensor Product Codes to perform the above construction in a smaller field.
- Even smaller field size constructions for the following special cases.
    1. 1 global parity
    2. 1 global parity and 1 mid-level parity
    3. 2 global parities and 1 mid-level parity

MRCs with Hierarchical Locality

$[k, r_1, r_2, h_1, h_2, \delta]$ code with $k = 9$, $r_1 = 4$, $r_2 = 3$, $h_1 = 3$, $h_2 = 2$ and $\delta = 1$



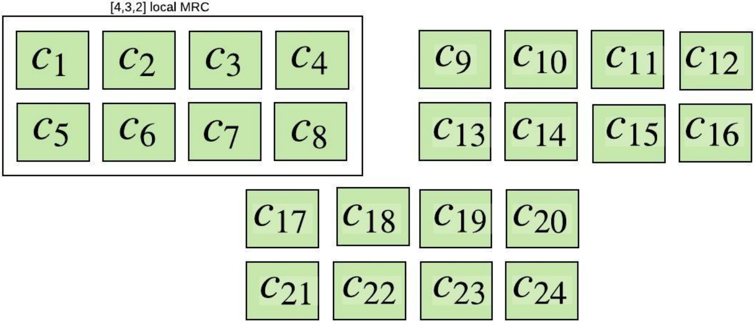
- Pick $k + h_1$ symbols from the code such that,
  - it contains r_1 symbols from each mid-level code
  - it contains at-most r_2 symbols from each local code
- Those $k + h_1$ symbols should form an $[k + h_1, k]$ MDS code.

MRCs with Hierarchical Locality

Lemma

In a $[k, r_1, r_2, h_1, h_2, \delta]$ hierarchical local MRC, the mid-level codes itself are an $[r_1, r_2, h_2, \delta]$ local MRC.

$[k, r_1, r_2, h_1, h_2, \delta]$ code with $k = 9$, $r_1 = 4$, $r_2 = 3$, $h_1 = 3$, $h_2 = 2$ and $\delta = 1$



Parity Check Matrix

For example code, $[k = 2, r_1 = 2, r_2 = 2, h_1 = 2, h_2 = 2, \delta = 2]$

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & & & & & & & & & & & & \\
0 & \beta & \beta^2 & \beta^3 & & & & & & & & & & & & \\
& & & & 1 & 1 & 1 & 1 & & & & & & & & \\
& & & & 0 & \beta & \beta^2 & \beta^3 & & & & & & & & \\
\alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,4} & \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \alpha_{2,4} & & & & & & & & \\
\alpha_{1,1}^q & \alpha_{1,2}^q & \alpha_{1,3}^q & \alpha_{1,4}^q & \alpha_{2,1}^q & \alpha_{2,2}^q & \alpha_{2,3}^q & \alpha_{2,4}^q & & & & & & & & \\
& & & & & & & & 1 & 1 & 1 & 1 & & & & \\
& & & & & & & & 0 & \beta & \beta^2 & \beta^3 & & & & \\
& & & & & & & & & & & & 1 & 1 & 1 & 1 \\
& & & & & & & & & & & & 0 & \beta & \beta^2 & \beta^3 \\
& & & & & & & & \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,4} & \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \alpha_{2,4} \\
& & & & & & & & \alpha_{1,1}^q & \alpha_{1,2}^q & \alpha_{1,3}^q & \alpha_{1,4}^q & \alpha_{2,1}^q & \alpha_{2,2}^q & \alpha_{2,3}^q & \alpha_{2,4}^q \\
\lambda_{1,1,1} & \lambda_{1,1,2} & \lambda_{1,1,3} & \lambda_{1,1,4} & \lambda_{1,2,1} & \lambda_{1,2,2} & \lambda_{1,2,3} & \lambda_{1,2,4} & \lambda_{2,1,1} & \lambda_{2,1,2} & \lambda_{2,1,3} & \lambda_{2,1,4} & \lambda_{2,2,1} & \lambda_{2,2,2} & \lambda_{2,2,3} & \lambda_{2,2,4} \\
\lambda_{1,1,1}^{q^{m_1}} & \lambda_{1,1,2}^{q^{m_1}} & \lambda_{1,1,3}^{q^{m_1}} & \lambda_{1,1,4}^{q^{m_1}} & \lambda_{1,2,1}^{q^{m_1}} & \lambda_{1,2,2}^{q^{m_1}} & \lambda_{1,2,3}^{q^{m_1}} & \lambda_{1,2,4}^{q^{m_1}} & \lambda_{2,1,1}^{q^{m_1}} & \lambda_{2,1,2}^{q^{m_1}} & \lambda_{2,1,3}^{q^{m_1}} & \lambda_{2,1,4}^{q^{m_1}} & \lambda_{2,2,1}^{q^{m_1}} & \lambda_{2,2,2}^{q^{m_1}} & \lambda_{2,2,3}^{q^{m_1}} & \lambda_{2,2,4}^{q^{m_1}}
\end{bmatrix}
$$

- $\mathbb{F}_{q^m}$ is an extension field of $\mathbb{F}_{q^{m_1}}$ which itself is an extension of $\mathbb{F}_q$
- $\mathbb{F}_q =< \beta >$, $\alpha_{i,j} \in \mathbb{F}_{q^{m_1}}$ and $\lambda_{i,j,k} \in \mathbb{F}_{q^m}$

Parity Check Matrix

For example code, $[k=2, r_1=2, r_2=2, h_1=2, h_2=2, \delta=2]$

$$\begin{bmatrix}
1 & 1 & 1 & 1 & & & & & & & & & & & & \\
0 & \beta & \beta^2 & \beta^3 & & & & & & & & & & & & \\
& & & & 1 & 1 & 1 & 1 & & & & & & & & \\
& & & & 0 & \beta & \beta^2 & \beta^3 & & & & & & & & \\
\alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,4} & \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \alpha_{2,4} & & & & & & & & \\
\alpha_{1,1}^q & \alpha_{1,2}^q & \alpha_{1,3}^q & \alpha_{1,4}^q & \alpha_{2,1}^q & \alpha_{2,2}^q & \alpha_{2,3}^q & \alpha_{2,4}^q & & & & & & & & \\
& & & & & & & & 1 & 1 & 1 & 1 & & & & \\
& & & & & & & & 0 & \beta & \beta^2 & \beta^3 & & & & \\
& & & & & & & & & & & & 1 & 1 & 1 & 1 \\
& & & & & & & & & & & & 0 & \beta & \beta^2 & \beta^3 \\
& & & & & & & & \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,4} & \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \alpha_{2,4} \\
& & & & & & & & \alpha_{1,1}^q & \alpha_{1,2}^q & \alpha_{1,3}^q & \alpha_{1,4}^q & \alpha_{2,1}^q & \alpha_{2,2}^q & \alpha_{2,3}^q & \alpha_{2,4}^q \\
\lambda_{1,1,1} & \lambda_{1,1,2} & \lambda_{1,1,3} & \lambda_{1,1,4} & \lambda_{1,2,1} & \lambda_{1,2,2} & \lambda_{1,2,3} & \lambda_{1,2,4} & \lambda_{2,1,1} & \lambda_{2,1,2} & \lambda_{2,1,3} & \lambda_{2,1,4} & \lambda_{2,2,1} & \lambda_{2,2,2} & \lambda_{2,2,3} & \lambda_{2,2,4} \\
\lambda_{1,1,1}^{q^{m_1}} & \lambda_{1,1,2}^{q^{m_1}} & \lambda_{1,1,3}^{q^{m_1}} & \lambda_{1,1,4}^{q^{m_1}} & \lambda_{1,2,1}^{q^{m_1}} & \lambda_{1,2,2}^{q^{m_1}} & \lambda_{1,2,3}^{q^{m_1}} & \lambda_{1,2,4}^{q^{m_1}} & \lambda_{2,1,1}^{q^{m_1}} & \lambda_{2,1,2}^{q^{m_1}} & \lambda_{2,1,3}^{q^{m_1}} & \lambda_{2,1,4}^{q^{m_1}} & \lambda_{2,2,1}^{q^{m_1}} & \lambda_{2,2,2}^{q^{m_1}} & \lambda_{2,2,3}^{q^{m_1}} & \lambda_{2,2,4}^{q^{m_1}}
\end{bmatrix}$$

Global parity check conditions, mid-level code, and local parities are highlighted.

Conditions for MRC (mid-level parities)

According to our the previous lemma, each mid-level code is an $[r_1, r_2, h_2, \delta]$ local MRC.

Puncturing $\delta = 2$ coordinates per local group results in an $[r_1 + h_2 = 4, r_1 = 2]$ MDS code.

$$
\left[\begin{array}{cccccccccccccccc}
1 & 1 & 1 & 1 & & & & & & & & & & & & \\
0 & \beta & \beta^2 & \beta^3 & & & & & & & & & & & & \\
 & & & & 1 & 1 & 1 & 1 & & & & & & & & \\
 & & & & 0 & \beta & \beta^2 & \beta^3 & & & & & & & & \\
\alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,4} & \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \alpha_{2,4} & & & & & & & & \\
\alpha_{1,1}^q & \alpha_{1,2}^q & \alpha_{1,3}^q & \alpha_{1,4}^q & \alpha_{2,1}^q & \alpha_{2,2}^q & \alpha_{2,3}^q & \alpha_{2,4}^q & & & & & & & & \\
 & & & & & & & & 1 & 1 & 1 & 1 & & & & \\
 & & & & & & & & 0 & \beta & \beta^2 & \beta^3 & & & & \\
 & & & & & & & & & & & & 1 & 1 & 1 & 1 \\
 & & & & & & & & & & & & 0 & \beta & \beta^2 & \beta^3 \\
 & & & & & & & & \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,4} & \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \alpha_{2,4} \\
 & & & & & & & & \alpha_{1,1}^q & \alpha_{1,2}^q & \alpha_{1,3}^q & \alpha_{1,4}^q & \alpha_{2,1}^q & \alpha_{2,2}^q & \alpha_{2,3}^q & \alpha_{2,4}^q \\
\lambda_{1,1,1} & \lambda_{1,1,2} & \lambda_{1,1,3} & \lambda_{1,1,4} & \lambda_{1,2,1} & \lambda_{1,2,2} & \lambda_{1,2,3} & \lambda_{1,2,4} & \lambda_{2,1,1} & \lambda_{2,1,2} & \lambda_{2,1,3} & \lambda_{2,1,4} & \lambda_{2,2,1} & \lambda_{2,2,2} & \lambda_{2,2,3} & \lambda_{2,2,4} \\
\lambda_{1,1,1}^{q^{m_1}} & \lambda_{1,1,2}^{q^{m_1}} & \lambda_{1,1,3}^{q^{m_1}} & \lambda_{1,1,4}^{q^{m_1}} & \lambda_{1,2,1}^{q^{m_1}} & \lambda_{1,2,2}^{q^{m_1}} & \lambda_{1,2,3}^{q^{m_1}} & \lambda_{1,2,4}^{q^{m_1}} & \lambda_{2,1,1}^{q^{m_1}} & \lambda_{2,1,2}^{q^{m_1}} & \lambda_{2,1,3}^{q^{m_1}} & \lambda_{2,1,4}^{q^{m_1}} & \lambda_{2,2,1}^{q^{m_1}} & \lambda_{2,2,2}^{q^{m_1}} & \lambda_{2,2,3}^{q^{m_1}} & \lambda_{2,2,4}^{q^{m_1}}
\end{array}\right]
$$

Relevant sub-matrices are highlighted.

Conditions for MRC (mid-level parities)

We consider one such sub-matrix.
We assume that columns 1 and 2 are punctured.

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 0 & \beta & \beta^2 & \beta^3 \\ \hline \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,4} \\ \hline \alpha_{1,1}^q & \alpha_{1,2}^q & \alpha_{1,3}^q & \alpha_{1,4}^q \end{array}\right] \Longrightarrow \begin{bmatrix} M_s & M_{\bar{s}} \\ \alpha_{1,s} & \alpha_{1,\bar{s}} \\ \alpha_{1,s}^q & \alpha_{1,\bar{s}}^q \end{bmatrix}$$

$$\begin{bmatrix} M_s & M_{\bar{s}} \\ \alpha_{1,s} & \alpha_{1,\bar{s}} \\ \alpha_{1,s}^q & \alpha_{1,\bar{s}}^q \end{bmatrix} \Longrightarrow \begin{bmatrix} M_s & M_{\bar{s}} \\ 0 & \alpha_{1,\bar{s}} + \alpha_{1,s} L \\ 0 & (\alpha_{1,\bar{s}} + \alpha_{1,s} L)^q \end{bmatrix}$$

$$L = M_s^{-1} M_{\bar{s}} \text{ } (2 \times 2 \text{ matrix})$$

Since all elements of L are from $\mathbb{F}_q$, $L = L^q$

Conditions for MRC (mid-level parities)

We show one such mid-level code after that operation.

$$\begin{bmatrix} M_s & M_{\bar{s}} & & \\ & & M_s & M_{\bar{s}} \\ 0 & \alpha_{1,\bar{s}} + \alpha_{1,s} L & 0 & \alpha_{2,\bar{s}} + \alpha_{2,s} L \\ 0 & (\alpha_{1,\bar{s}} + \alpha_{1,s} L)^q & 0 & (\alpha_{2,\bar{s}} + \alpha_{2,s} L)^q \end{bmatrix}$$

The punctured sub-matrix,

$$\begin{bmatrix} \alpha_{1,\bar{s}} + \alpha_{1,s} L & \alpha_{2,\bar{s}} + \alpha_{2,s} L \\ (\alpha_{1,\bar{s}} + \alpha_{1,s} L)^q & (\alpha_{2,\bar{s}} + \alpha_{2,s} L)^q \end{bmatrix}$$

should be the PCM for a $[r_1 + h_2 = 4, r_1 = 2]$ MDS code.
Possible because L is a 2×2 matrix.

Conditions for MRC: Some definitions

Definition (k-wise Independence)

A multi-set $S \subseteq \mathbb{F}$ is k-wise independent over $\mathbb{F}$ if for every set $T \subseteq S$ such that $|T| \leq k$, T is linearly independent over $\mathbb{F}$.

Lemma

Let $\mathbb{F}_{q^t}$ be an extension of $\mathbb{F}_q$. Let $a_1, a_2, \ldots, a_n$ be elements of $\mathbb{F}_{q^t}$. The following matrix

$$\begin{bmatrix} a_1 & a_2 & a_3 & \ldots & a_n \\ a_1^q & a_2^q & a_3^q & \ldots & a_n^q \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ a_1^{q^{k-1}} & a_2^{q^{k-1}} & a_3^{q^{k-1}} & \ldots & a_n^{q^{k-1}} \end{bmatrix}$$

is the generator matrix of a $[n, k]$ MDS code if and only if $a_1, a_2, \ldots, a_n$ are k-wise linearly independent over $\mathbb{F}_q$.

Conditions for MRC (mid-level parities)

Using the above lemma, the matrix,

$$\begin{bmatrix} \alpha_{1,\bar{s}} + \alpha_{1,s}L & \alpha_{2,\bar{s}} + \alpha_{2,s}L \\ (\alpha_{1,\bar{s}} + \alpha_{1,s}L)^q & (\alpha_{2,\bar{s}} + \alpha_{2,s}L)^q \end{bmatrix}$$

is the parity check matrix for a $[r_1 + h_2 = 4, r_1 = 2]$ MDS code if the set

$$\Psi = \{\alpha_{1,\bar{s}} + \alpha_{1,s}L,\ \alpha_{2,\bar{s}} + \alpha_{2,s}L\}$$

is $h_2 = 2$ wise independent over $\mathbb{F}_q$.

Condition on $\alpha_{i,j}$

$$\Psi = \{\alpha_{1,\bar{s}} + \alpha_{1,s}L,\ \alpha_{2,\bar{s}} + \alpha_{2,s}L\}$$

- Any $\mathbb{F}_q$-linear combination of k elements in Ψ will have at-most $3k$ distinct elements.
- Hence if the set $\{\alpha_{i,j}\}$ is at-least $3h_2 = 6$ wise independent over $\mathbb{F}_q$, then the set Ψ will be $h_2 = 2$ wise independent over $\mathbb{F}_q$.

Conditions for MRC (global parities)

We now consider global parities along with the mid-level codes. After puncturing δ coordinates per local group, puncturing $h_2 = 2$ coordinates per mid-level code results in an MDS code.

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & & & & & & & & & & & & \\
0 & \beta & \beta^2 & \beta^3 & & & & & & & & & & & & \\
 & & & & 1 & 1 & 1 & 1 & & & & & & & & \\
 & & & & 0 & \beta & \beta^2 & \beta^3 & & & & & & & & \\
\alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,4} & \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \alpha_{2,4} & & & & & & & & \\
\alpha_{1,1}^q & \alpha_{1,2}^q & \alpha_{1,3}^q & \alpha_{1,4}^q & \alpha_{2,1}^q & \alpha_{2,2}^q & \alpha_{2,3}^q & \alpha_{2,4}^q & & & & & & & & \\
 & & & & & & & & 1 & 1 & 1 & 1 & & & & \\
 & & & & & & & & 0 & \beta & \beta^2 & \beta^3 & & & & \\
 & & & & & & & & & & & & 1 & 1 & 1 & 1 \\
 & & & & & & & & & & & & 0 & \beta & \beta^2 & \beta^3 \\
 & & & & & & & & \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,4} & \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \alpha_{2,4} \\
 & & & & & & & & \alpha_{1,1}^q & \alpha_{1,2}^q & \alpha_{1,3}^q & \alpha_{1,4}^q & \alpha_{2,1}^q & \alpha_{2,2}^q & \alpha_{2,3}^q & \alpha_{2,4}^q \\
\lambda_{1,1,1} & \lambda_{1,1,2} & \lambda_{1,1,3} & \lambda_{1,1,4} & \lambda_{1,2,1} & \lambda_{1,2,2} & \lambda_{1,2,3} & \lambda_{1,2,4} & \lambda_{2,1,1} & \lambda_{2,1,2} & \lambda_{2,1,3} & \lambda_{2,1,4} & \lambda_{2,2,1} & \lambda_{2,2,2} & \lambda_{2,2,3} & \lambda_{2,2,4} \\
\lambda_{1,1,1}^{q^{m_1}} & \lambda_{1,1,2}^{q^{m_1}} & \lambda_{1,1,3}^{q^{m_1}} & \lambda_{1,1,4}^{q^{m_1}} & \lambda_{1,2,1}^{q^{m_1}} & \lambda_{1,2,2}^{q^{m_1}} & \lambda_{1,2,3}^{q^{m_1}} & \lambda_{1,2,4}^{q^{m_1}} & \lambda_{2,1,1}^{q^{m_1}} & \lambda_{2,1,2}^{q^{m_1}} & \lambda_{2,1,3}^{q^{m_1}} & \lambda_{2,1,4}^{q^{m_1}} & \lambda_{2,2,1}^{q^{m_1}} & \lambda_{2,2,2}^{q^{m_1}} & \lambda_{2,2,3}^{q^{m_1}} & \lambda_{2,2,4}^{q^{m_1}}
\end{bmatrix}
$$

Conditions for MRC (global parities)

This time we apply the shortening to global parities as well. We collect the shortened code from the entire mid-level code as in previous steps.

$$\begin{bmatrix} \alpha_{1,\bar{s}} + \alpha_{1,s}L & \alpha_{2,\bar{s}} + \alpha_{2,s}L \\ (\alpha_{1,\bar{s}} + \alpha_{1,s}L)^q & (\alpha_{2,\bar{s}} + \alpha_{2,s}L)^q \\ \lambda_{1,1,\bar{s}} + \lambda_{1,1,s}L & \lambda_{1,2,\bar{s}} + \lambda_{1,2,s}L \\ (\lambda_{1,1,\bar{s}} + \lambda_{1,1,s}L)^{q^{m_1}} & (\lambda_{1,2,\bar{s}} + \lambda_{1,2,s}L)^{q^{m_1}} \end{bmatrix}$$

We perform similar steps as we did for $\alpha_{i,j}$ and arrive at a similar result.

- The set $\{\lambda_{i,j,k}\}$ needs to be at-least $h_1(h_2+1)(\delta+1)$ wise independent over $\mathbb{F}_{q^{m_1}}$

Picking $\alpha_{i,j}$ and $\lambda_{i,j,k}$ (from PCM of codes)

- We pick $\alpha_{i,j}$ and $\lambda_{i,j,k}$ as columns of the PCM of an appropriate code.
- For an $[n, k, d]$ code over $\mathbb{F}_q$, the columns of a PCM are elements in $\mathbb{F}_{q^{n-k}}$ which are $(d-1)$-wise independent over $\mathbb{F}_q$
- Since, $\alpha_{i,j} \in \mathbb{F}_{q^{m_1}}/\mathbb{F}_q$, the value of $n-k$ is m_1.
- Now since $\{\alpha_{i,j}\}$ needs to be 6-wise independent in $\mathbb{F}_q$, the value of d should be 7.

Picking $\alpha_{i,j}$ and $\lambda_{i,j,k}$ (BCH codes)

Lemma

There exists $[n = q^t - 1, k, d]$ BCH codes over $\mathbb{F}_q$, where the parameters are related as

$$n - k = 1 + \left\lceil \frac{q-1}{q}(d-2) \right\rceil \lceil \log_2(n) \rceil.$$

- We need 8 values for $\alpha_{i,j}$.
- We set $t = \lceil log_q(8) \rceil$ to get a PCM with smallest number of columns.

Similar procedure is followed to get $\{\lambda_{i,j,k}\}$.

More optimisations

- Using Tensor Product Codes to perform the above construction in a smaller field.
- Even smaller field size constructions for the following special cases.
  1. 1 global parity any number of mid-level parities
  2. 1 global parity and 1 mid-level parity
  3. 2 global parities and 1 mid-level parity

Tensor Product Codes

Let C_1 be an $[n, n-\rho]$ linear code in $\mathbb{F}_q$ which can correct e_1 erasures. Also, C_2 is an $[m, m-s]$ code in $\mathbb{F}_{q^\rho}$ that can correct e_2 erasures. An $[nm, nm-s\rho]$ code C in $\mathbb{F}_q$ is called the tensor product code of C_1 and C_2 if

$$\forall x \in C_1 \text{ and } y \in C_2, \quad y \otimes x \in C$$

where $y \otimes x$ is the tensor product of x and y in $\mathbb{F}_q$.

Example

Let C_1 be a $(3,1)$-code in $\mathbb{F}_q$. The PCM,

$$H_1 = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} u_i \in \mathbb{F}_{q^2}$$

Let C_2 be a $(4,1)$-code in $\mathbb{F}_{q^2}$. The PCM here,

$$H_2 = \begin{bmatrix} v_{1,1} & v_{2,1} & v_{3,1} & v_{4,1} \\ v_{1,2} & v_{2,2} & v_{3,2} & v_{4,2} \\ v_{1,3} & v_{2,3} & v_{3,3} & v_{4,3} \end{bmatrix} v_{i,j} \in \mathbb{F}_{q^2}$$

Assume both codes are MDS, therefore $e_1 = 2$ and $e_2 = 3$
Therefore the PCM (H) for the tensor product of C_1 and C_2

$$H = \begin{bmatrix} v_{1,1}u_1 & v_{1,1}u_2 & v_{1,1}u_3 & v_{2,1}u_1 & v_{2,1}u_2 & v_{2,1}u_3 & v_{3,1}u_1 & v_{3,1}u_2 & v_{3,1}u_3 & v_{4,1}u_1 & v_{4,1}u_2 & v_{4,1}u_3 \\ v_{1,2}u_1 & v_{1,2}u_2 & v_{1,2}u_3 & v_{2,2}u_1 & v_{2,2}u_2 & v_{2,2}u_3 & v_{3,2}u_1 & v_{3,2}u_2 & v_{3,2}u_3 & v_{4,2}u_1 & v_{4,2}u_2 & v_{4,2}u_3 \\ v_{1,3}u_1 & v_{1,3}u_2 & v_{1,3}u_3 & v_{2,3}u_1 & v_{2,3}u_2 & v_{2,3}u_3 & v_{3,3}u_1 & v_{3,3}u_2 & v_{3,3}u_3 & v_{4,3}u_1 & v_{4,3}u_2 & v_{4,3}u_3 \end{bmatrix}$$

Tensor Product Codes: Erasure Correction

Theorem

C_1, C_2 and C are as defined above. If the code-words in C are considered to be consisting of m sub-blocks with each sub-block containing n symbols, C will correct all erasure patterns where,

- *At-most e_2 sub-blocks are affected.*
- *At-most e_1 erasures in each affected sub-block.*

$$H = \left[\begin{array}{ccc|ccc|ccc|ccc} v_{1,1}u_1 & v_{1,1}u_2 & v_{1,1}u_3 & v_{2,1}u_1 & v_{2,1}u_2 & v_{2,1}u_3 & v_{3,1}u_1 & v_{3,1}u_2 & v_{3,1}u_3 & v_{4,1}u_1 & v_{4,1}u_2 & v_{4,1}u_3 \\ v_{1,2}u_1 & v_{1,2}u_2 & v_{1,2}u_3 & v_{2,2}u_1 & v_{2,2}u_2 & v_{2,2}u_3 & v_{3,2}u_1 & v_{3,2}u_2 & v_{3,2}u_3 & v_{4,2}u_1 & v_{4,2}u_2 & v_{4,2}u_3 \\ v_{1,3}u_1 & v_{1,3}u_2 & v_{1,3}u_3 & v_{2,3}u_1 & v_{2,3}u_2 & v_{2,3}u_3 & v_{3,3}u_1 & v_{3,3}u_2 & v_{3,3}u_3 & v_{4,3}u_1 & v_{4,3}u_2 & v_{4,3}u_3 \end{array}\right]$$

Here, there are $m = 4$ sub-blocks with $n = 3$ symbols each. This code can recover all erasures where at-most 3 sub-blocks are affected and upto 2 erasures per sub-block

Convention

We say that an $[nm, nm - s\rho]$ tensor product code $C \subseteq \mathbb{F}_q^{m \times n}$ is an $[m, n; e_2, e_1]$ erasure correcting code if it can correct any erasure pattern of the form $\mathbf{E} = (E_1, \ldots, E_m)$ where for $i \in [m]$ and $E_i \subseteq [n]$ and

- $|\{i : E_i \neq \emptyset\}| \leq e_2$.
- for $i \in [m], |E_i| \leq e_1$.

$$H = \left[\begin{array}{ccc|ccc|ccc|ccc} v_{1,1}u_1 & v_{1,1}u_2 & v_{1,1}u_3 & v_{2,1}u_1 & v_{2,1}u_2 & v_{2,1}u_3 & v_{3,1}u_1 & v_{3,1}u_2 & v_{3,1}u_3 & v_{4,1}u_1 & v_{4,1}u_2 & v_{4,1}u_3 \\ v_{1,2}u_1 & v_{1,2}u_2 & v_{1,2}u_3 & v_{2,2}u_1 & v_{2,2}u_2 & v_{2,2}u_3 & v_{3,2}u_1 & v_{3,2}u_2 & v_{3,2}u_3 & v_{4,2}u_1 & v_{4,2}u_2 & v_{4,2}u_3 \\ v_{1,3}u_1 & v_{1,3}u_2 & v_{1,3}u_3 & v_{2,3}u_1 & v_{2,3}u_2 & v_{2,3}u_3 & v_{3,3}u_1 & v_{3,3}u_2 & v_{3,3}u_3 & v_{4,3}u_1 & v_{4,3}u_2 & v_{4,3}u_3 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc|ccc|ccc} \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \alpha_{3,1} & \alpha_{3,2} & \alpha_{3,3} & \alpha_{4,1} & \alpha_{4,2} & \alpha_{4,3} \end{array}\right]$$

This code is then a $[4, 3; 3, 2]$ code.

Product Construction

Assume the following code, $[k = 8, r_1 = 7, r_2 = 3, h_1 = 6, h_2 = 2, \delta = 2]$
Based on the MRC conditions proved previously,

$$\Psi = \{\alpha_{1,\bar{s}} + \alpha_{1,s}L,\ \ \alpha_{2,\bar{s}} + \alpha_{2,s}L,\ \ \alpha_{3,\bar{s}} + \alpha_{3,s}L\}$$

where, $\alpha_{i,s} = \{\alpha_{i,1}, \alpha_{i,2}\}$ and $\alpha_{i,\bar{s}} = \{\alpha_{i,1'}, \alpha_{i,2'}, \alpha_{i,3'}\}$ $\alpha_{i,j} \in \mathbb{F}_{q^{m_1}}$
and L is a 2×3 matrix with elements in $\mathbb{F}_q$.
Ψ is required to be 2-wise independent over $\mathbb{F}_q$.

Product Construction

Assuming $L = \begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix}$ Expand Ψ into individual components,

$$\begin{aligned}\Psi = \{&\alpha_{1,1'} + a\alpha_{1,1} + b\alpha_{1,2},\ \alpha_{1,2'} + c\alpha_{1,1} + d\alpha_{1,2},\ \alpha_{1,3'} + e\alpha_{1,1} + f\alpha_{1,2},\\ &\alpha_{2,1'} + a\alpha_{2,1} + b\alpha_{2,2},\ \alpha_{2,2'} + c\alpha_{2,1} + d\alpha_{2,2},\ \alpha_{2,3'} + e\alpha_{2,1} + f\alpha_{2,2},\\ &\alpha_{3,1'} + a\alpha_{3,1} + b\alpha_{3,2},\ \alpha_{3,2'} + c\alpha_{3,1} + d\alpha_{3,2},\ \alpha_{3,3'} + e\alpha_{3,1} + f\alpha_{3,2}\}\end{aligned}$$

We can pick $h_2 = 2$ elements in two different ways.

- Here, the first index of $\alpha_{i,j}$ remains the same. There are 4 distinct $\alpha_{i,j}$ in that linear combination
- The first index of $\alpha_{i,j}$ are different. In this case, there are 3 distinct $\alpha_{i,j}$ per i.

Product Construction

If we relate this to previous example of the tensor code,

$$\left[\alpha_{1,1} \quad \alpha_{1,2} \quad \alpha_{1,3} \qquad \alpha_{2,1} \quad \alpha_{2,2} \quad \alpha_{2,3} \qquad \alpha_{3,1} \quad \alpha_{3,2} \quad \alpha_{3,3} \qquad \alpha_{4,1} \quad \alpha_{4,2} \quad \alpha_{4,3}\right]$$

(Illustrative example)

If we pick $\alpha_{i,j}$ as these columns we have to ensure that,

- The code can recover from erasures in 2 sub-blocks ($e_2 = 2$).
- The code can recover from 4 erasures per sub-block ($e_1 = 4$).
- There are at-least 3 distinct groups because there are 3 distinct values of i in $\alpha_{i,j}$ ($m = 3$)
- There are 5 columns per group because there are 5 different values of j for every i in $\alpha_{i,j}$. ($n = 5$)

Hence, Ψ is 2-wise independent if $\alpha_{i,j}$ is picked from a $[3, 5; 2, 4]$ code.

Product Construction

More generally,

Theorem

Let C_{TP} be an $[t_2, n_2; h_2, h_2 + \delta]$ erasure correcting code with $t_2 > h_2$ and $n_2 > h_2 + \delta$ over $\mathbb{F}_q$ with redundancy m_1 and the parity check matrix $H_{TP} = (\alpha_{1,1}, \ldots, \alpha_{t_2,n_2}) \in (\mathbb{F}_{q^{m_1}})^{t_2 n_2}$. Then the set $\{\alpha_{i,j}\}$ chosen as columns of H_{TP} guarantees that the mid-level code is MRC.

Similarly for $\lambda_{i,k,j}$,

Theorem

Let C_{TP} be an $[t_1, t_2 n_2; h_1, (h_1 + h_2)(\delta + 1)]$ erasure correcting code with $t_1 > h_1$ and $t_2 n_2 > (h_1 + h_2)(\delta + 1)$ over $\mathbb{F}_{q^{m_1}}$ with redundancy m and the parity check matrix $H_{TP} = (\lambda_{1,1,1}, \ldots, \lambda_{t_1,t_2,n_2}) \in (\mathbb{F}_{q^m})^{t_1 t_2 n_2}$. Then the set $\{\lambda_{i,j,k}\}$ chosen as columns of H_{TP} ensures that global code is MRC.

Optimisation for $h_1 = 1$

Consider the code, $[k = 3, r_1 = 2, r_2 = 2, h_1 = 1, h_2 = 2, \delta = 2]$

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & & & & & & & & & & & & \\
0 & \beta & \beta^2 & \beta^3 & & & & & & & & & & & & \\
 & & & & 1 & 1 & 1 & 1 & & & & & & & & \\
 & & & & 0 & \beta & \beta^2 & \beta^3 & & & & & & & & \\
\alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,4} & \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \alpha_{2,4} & & & & & & & & \\
\alpha_{1,1}^q & \alpha_{1,2}^q & \alpha_{1,3}^q & \alpha_{1,4}^q & \alpha_{2,1}^q & \alpha_{2,2}^q & \alpha_{2,3}^q & \alpha_{2,4}^q & & & & & & & & \\
 & & & & & & & & 1 & 1 & 1 & 1 & & & & \\
 & & & & & & & & 0 & \beta & \beta^2 & \beta^3 & & & & \\
 & & & & & & & & & & & & 1 & 1 & 1 & 1 \\
 & & & & & & & & & & & & 0 & \beta & \beta^2 & \beta^3 \\
 & & & & & & & & \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,4} & \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \alpha_{2,4} \\
 & & & & & & & & \alpha_{1,1}^q & \alpha_{1,2}^q & \alpha_{1,3}^q & \alpha_{1,4}^q & \alpha_{2,1}^q & \alpha_{2,2}^q & \alpha_{2,3}^q & \alpha_{2,4}^q \\
\lambda_{1,1,1} & \lambda_{1,1,2} & \lambda_{1,1,3} & \lambda_{1,1,4} & \lambda_{1,2,1} & \lambda_{1,2,2} & \lambda_{1,2,3} & \lambda_{1,2,4} & \lambda_{2,1,1} & \lambda_{2,1,2} & \lambda_{2,1,3} & \lambda_{2,1,4} & \lambda_{2,2,1} & \lambda_{2,2,2} & \lambda_{2,2,3} & \lambda_{2,2,4}
\end{bmatrix}
$$

standard construction. $\lambda_{i,j,k} \in \mathbb{F}_{q^m}$

Optimisation for $h_1 = 1$

Consider the code, $[k = 3, r_1 = 2, r_2 = 2, h_1 = 1, h_2 = 2, \delta = 2]$

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & & & & & & & & & & & & \\
0 & \beta & \beta^2 & \beta^3 & & & & & & & & & & & & \\
 & & & & 1 & 1 & 1 & 1 & & & & & & & & \\
 & & & & 0 & \beta & \beta^2 & \beta^3 & & & & & & & & \\
\alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,4} & \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \alpha_{2,4} & & & & & & & & \\
\alpha_{1,1}^q & \alpha_{1,2}^q & \alpha_{1,3}^q & \alpha_{1,4}^q & \alpha_{2,1}^q & \alpha_{2,2}^q & \alpha_{2,3}^q & \alpha_{2,4}^q & & & & & & & & \\
 & & & & & & & & 1 & 1 & 1 & 1 & & & & \\
 & & & & & & & & 0 & \beta & \beta^2 & \beta^3 & & & & \\
 & & & & & & & & & & & & 1 & 1 & 1 & 1 \\
 & & & & & & & & & & & & 0 & \beta & \beta^2 & \beta^3 \\
 & & & & & & & & \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \alpha_{1,4} & \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \alpha_{2,4} \\
 & & & & & & & & \alpha_{1,1}^q & \alpha_{1,2}^q & \alpha_{1,3}^q & \alpha_{1,4}^q & \alpha_{2,1}^q & \alpha_{2,2}^q & \alpha_{2,3}^q & \alpha_{2,4}^q \\
\alpha_{1,1}^{q^2} & \alpha_{1,2}^{q^2} & \alpha_{1,3}^{q^2} & \alpha_{1,4}^{q^2} & \alpha_{2,1}^{q^2} & \alpha_{2,2}^{q^2} & \alpha_{2,3}^{q^2} & \alpha_{2,4}^{q^2} & \alpha_{1,1}^{q^2} & \alpha_{1,2}^{q^2} & \alpha_{1,3}^{q^2} & \alpha_{1,4}^{q^2} & \alpha_{2,1}^{q^2} & \alpha_{2,2}^{q^2} & \alpha_{2,3}^{q^2} & \alpha_{2,4}^{q^2}
\end{bmatrix}
$$

Optimised construction. $\alpha_{i,j} \in \mathbb{F}_{q^{m_1}}$

All the MRC conditions can be proved for this construction as well.

Optimisation for $h_1 = 1$ and $h_2 = 1$

Consider the code, $[k = 5, r_1 = 3, r_2 = 2, h_1 = 1, h_2 = 1, \delta = 2]$

$$\left[\begin{array}{cccccccccccccccc}
\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & & & & & & & & & & & & \\
\alpha_1^2 & \alpha_2^2 & \alpha_3^2 & \alpha_4^2 & & & & & & & & & & & & \\
& & & & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & & & & & & & & \\
& & & & \alpha_1^2 & \alpha_2^2 & \alpha_3^2 & \alpha_4^2 & & & & & & & & \\
\lambda_1 & \lambda_1 & \lambda_1 & \lambda_1 & \lambda_2 & \lambda_2 & \lambda_2 & \lambda_2 & & & & & & & & \\
& & & & & & & & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & & & & \\
& & & & & & & & \alpha_1^2 & \alpha_2^2 & \alpha_3^2 & \alpha_4^2 & & & & \\
& & & & & & & & & & & & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\
& & & & & & & & & & & & \alpha_1^2 & \alpha_2^2 & \alpha_3^2 & \alpha_4^2 \\
& & & & & & & & \lambda_1 & \lambda_1 & \lambda_1 & \lambda_1 & \lambda_2 & \lambda_2 & \lambda_2 & \lambda_2 \\
\alpha_1^3 & \alpha_2^3 & \alpha_3^3 & \alpha_4^3 & \alpha_1^3 & \alpha_2^3 & \alpha_3^3 & \alpha_4^3 & \alpha_1^3 & \alpha_2^3 & \alpha_3^3 & \alpha_4^3 & \alpha_1^3 & \alpha_2^3 & \alpha_3^3 & \alpha_4^3
\end{array}\right]$$

- q is a prime power such that there exists a subgroup G of $\mathbb{F}_q^*$ of size at-least 4 and with at-least 2 cosets.
- $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in G$ and $\alpha_i \neq \alpha_j$.
- $\lambda_1, \lambda_2 \in \mathbb{F}_q^*$ be elements from distinct cosets of G.

Optimisation for $h_1 = 2$ and $h_2 = 1$

Consider the code, $[k = 4, r_1 = 3, r_2 = 2, h_1 = 2, h_2 = 1, \delta = 2]$

$$
\begin{bmatrix}
\frac{1}{\alpha_1-\beta_1} & \frac{1}{\alpha_2-\beta_1} & \frac{1}{\alpha_3-\beta_1} & \frac{1}{\alpha_4-\beta_1} & & & & & & & & & & & & \\
\frac{1}{\alpha_1-\beta_2} & \frac{1}{\alpha_2-\beta_2} & \frac{1}{\alpha_3-\beta_2} & \frac{1}{\alpha_4-\beta_2} & & & & & & & & & & & & \\
& & & & \frac{1}{\alpha_1-\beta_1} & \frac{1}{\alpha_2-\beta_1} & \frac{1}{\alpha_3-\beta_1} & \frac{1}{\alpha_4-\beta_1} & & & & & & & & \\
& & & & \frac{1}{\alpha_1-\beta_2} & \frac{1}{\alpha_2-\beta_2} & \frac{1}{\alpha_3-\beta_2} & \frac{1}{\alpha_4-\beta_2} & & & & & & & & \\
\frac{1}{\alpha_1-\beta_3} & \frac{1}{\alpha_2-\beta_3} & \frac{1}{\alpha_3-\beta_3} & \frac{1}{\alpha_4-\beta_3} & \frac{1}{\alpha_1-\beta_3} & \frac{1}{\alpha_2-\beta_3} & \frac{1}{\alpha_3-\beta_3} & \frac{1}{\alpha_4-\beta_3} & & & & & & & & \\
& & & & & & & & \frac{1}{\alpha_1-\beta_1} & \frac{1}{\alpha_2-\beta_1} & \frac{1}{\alpha_3-\beta_1} & \frac{1}{\alpha_4-\beta_1} & & & & \\
& & & & & & & & \frac{1}{\alpha_1-\beta_2} & \frac{1}{\alpha_2-\beta_2} & \frac{1}{\alpha_3-\beta_2} & \frac{1}{\alpha_4-\beta_2} & & & & \\
& & & & & & & & & & & & \frac{1}{\alpha_1-\beta_1} & \frac{1}{\alpha_2-\beta_1} & \frac{1}{\alpha_3-\beta_1} & \frac{1}{\alpha_4-\beta_1} \\
& & & & & & & & & & & & \frac{1}{\alpha_1-\beta_2} & \frac{1}{\alpha_2-\beta_2} & \frac{1}{\alpha_3-\beta_2} & \frac{1}{\alpha_4-\beta_2} \\
& & & & & & & & \frac{1}{\alpha_1-\beta_3} & \frac{1}{\alpha_2-\beta_3} & \frac{1}{\alpha_3-\beta_3} & \frac{1}{\alpha_4-\beta_3} & \frac{1}{\alpha_1-\beta_3} & \frac{1}{\alpha_2-\beta_3} & \frac{1}{\alpha_3-\beta_3} & \frac{1}{\alpha_4-\beta_3} \\
\frac{\lambda_1}{\alpha_1-\beta_4} & \frac{\lambda_1}{\alpha_2-\beta_4} & \frac{\lambda_1}{\alpha_3-\beta_4} & \frac{\lambda_1}{\alpha_4-\beta_4} & \frac{\lambda_2}{\alpha_1-\beta_4} & \frac{\lambda_2}{\alpha_2-\beta_4} & \frac{\lambda_2}{\alpha_3-\beta_4} & \frac{\lambda_2}{\alpha_4-\beta_4} & \frac{\lambda_3}{\alpha_1-\beta_4} & \frac{\lambda_3}{\alpha_2-\beta_4} & \frac{\lambda_3}{\alpha_3-\beta_4} & \frac{\lambda_3}{\alpha_4-\beta_4} & \frac{\lambda_4}{\alpha_1-\beta_4} & \frac{\lambda_4}{\alpha_2-\beta_4} & \frac{\lambda_4}{\alpha_3-\beta_4} & \frac{\lambda_4}{\alpha_4-\beta_4} \\
\frac{\mu_1}{\alpha_1-\beta_5} & \frac{\mu_1}{\alpha_2-\beta_5} & \frac{\mu_1}{\alpha_3-\beta_5} & \frac{\mu_1}{\alpha_4-\beta_5} & \frac{\mu_2}{\alpha_1-\beta_5} & \frac{\mu_2}{\alpha_2-\beta_5} & \frac{\mu_2}{\alpha_3-\beta_5} & \frac{\mu_2}{\alpha_4-\beta_5} & \frac{\mu_3}{\alpha_1-\beta_5} & \frac{\mu_3}{\alpha_2-\beta_5} & \frac{\mu_3}{\alpha_3-\beta_5} & \frac{\mu_3}{\alpha_4-\beta_5} & \frac{\mu_4}{\alpha_1-\beta_5} & \frac{\mu_4}{\alpha_2-\beta_5} & \frac{\mu_4}{\alpha_3-\beta_5} & \frac{\mu_4}{\alpha_4-\beta_5}
\end{bmatrix}
$$

Optimisation for $h_1 = 2$ and $h_2 = 1$: Conditions

- $q_0 \geq 15$ is a prime power.
- There exists a subgroup G of $\mathbb{F}_{q_0}^*$ of size at least 6 with at-least 4 cosets.
- $\mathbb{F}_q$ is an extension field of $\mathbb{F}_{q_0}$.
- $\mu_1, \ldots, \mu_4$ are picked from distinct cosets of G.
- Choose distinct $\beta_3, \beta_4, \beta_5 \in \mathbb{F}_{q_0}$.
- Pick $\alpha_1, \ldots \alpha_4 \in \mathbb{F}_{q_0}$ such that, $\frac{\alpha_i - \beta_4}{\alpha_i - \beta_5}, \frac{\alpha_i - \beta_3}{\alpha_i - \beta_5} \in G$.
- Pick distinct $\beta_1, \beta_2 \in \mathbb{F}_{q_0} \setminus \{\alpha_1, \ldots, \alpha_4, \beta_3, \beta_4, \beta_5\}$.
- $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{F}_q$ are picked 4 wise-independent over $\mathbb{F}_{q_0}$.

Future Work

- We define only two levels of locality. The work can be extended to any y levels.
- We haven't figured out the bounds in which our codes work.
- Even though we have reduced the number of symbols required, we still haven't optimised the repair process itself.

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Thanks!

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