Conformal Four Point Functions and the Operator Product Expansion

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Various aspects of the four point function for scalar fields in conformally invariant theories are analysed. This depends on an arbitrary function of two conformal invariants u,v. A recurrence relation for the function corresponding to the contribution of an arbitrary spin field in the operator product expansion to the four point function is derived. This is solved explicitly in two and four dimensions in terms of ordinary hypergeometric functions of variables z,x which are simply related to u,v. The operator product expansion analysis is applied to the explicit expressions for the four point function found for free scalar, fermion and vector field theories in four dimensions. The results for four point functions obtained by using the AdS/CFT correspondence are also analysed in terms of functions related to those appearing in the operator product discussion.

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1. Introduction

Much work has been undertaken in the last few years based on the AdS/CFT correspondence in terms of understanding non trivial conformal field theories in four, and also three and six, dimensions. In particular this has been applied to $\mathcal{N}=4$ supersymmetric SU(N) gauge theories when supergravity on AdS₅ determines the large N limit of the associated conformal field theory which is defined on the boundary.

The correlation functions of operators on the boundary in the AdS/CFT correspondence are then determined to leading order in 1/N by tree graphs with vertices given by the supergravity theory and with appropriate boundary/bulk and bulk/bulk propagators determined by the Green functions on the AdS space. The form of the two and three point functions are determined by conformal invariance while the four point function depends on an arbitrary function of two conformal invariants.

The four point function is of particular interest since it is constrained by the operator product expansion for any two fields. For $\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle$, which depends on a function of two conformal invariant cross ratios u, v, the operator product expansion for $\phi_1(x_1)\phi_2(x_2)$ should provide in principle an expansion for the four point function which is convergent in some region when $x_1 \approx x_2$. Nevertheless this requires knowing the form of the contribution, as a function of u, v, corresponding to operators of arbitrary spin, including all their derivatives, which is analogous to determining an explicit expression for a partial wave expansion. In d dimensions, when, with a Euclidean metric, the conformal group is O(d+1,1), then for scalar fields we are concerned just with the contribution of fields belonging to $(\ell, 0, ...)$ representations of O(d), corresponding to fields $O^{(\ell)}_{\mu_1...\mu_\ell}$ which are totally symmetric traceless rank ℓ tensors. Although conformal partial wave expansions were obtained long ago [1,2,3] they have not been in a form which is easy to apply to disentangling the contributions of different operators to the four point functions found through the AdS/CFT correspondence. This has necessitated approximate calculations of just the leading terms in the power expansion for the contribution of operators with non zero spin in the operator product expansion [4]. More recently [5] results have been obtained for the contributions of a conserved vector current and the energy momentum tensor, which correspond to $\ell = 1, 2$ operators with dimensions d - 1, d, to the scalar four point function. We follow a similar approach to find a recurrence relation for the contribution of operators for any ℓ . This recurrence relation may be solved when d=2, when the result follows from the special simplifications arising from the use of complex coordinates, and also when d=4. In both cases the result is expressible in terms of products of ordinary hypergeometric functions. In two dimensions these depend on $\eta, \bar{\eta}$ which are related to the factors of u, v found by using complex coordinates, $u = \eta \bar{\eta}, v =$ $(1-\eta)(1-\bar{\eta})$ while in four dimensions we similarly define $u=zx,\ v=(1-z)(1-x)$. The permutation symmetries of the four point function may be translated into transformations of z, x in addition to the requirement of invariance under $z \leftrightarrow x$.

The structure of this paper is thus that in section 2 we review the operator product expansion and obtain the recurrence relation for the contribution of a quasi-primary operator of spin ℓ in d dimensions to the scalar four point function in terms of the results for $\ell-1, \ell-2$ by using the corresponding relation for Gegenbaur polynomials. In section 3 we obtain compact explicit solutions for arbitrary ℓ when d=2,4. In section 4 we briefly use these results to obtain the complete contribution arising from the energy momentum tensor, for which $\ell=2$. The overall normalisation is determined by Ward identities. In section 5 we analyse the form of the integrals arising in the AdS/CFT correspondence to the scalar four point function in some simple cases. The results are related to a two variable function H introduced by us earlier [6] which is related to the functions arising in the operator product analysis. In section 6 we consider the simple expressions for the four point function arising from free conformal field theories and analyse their expansion in terms of the conformal partial wave expressions obtained here for appropriate scale dimension Δ and spin ℓ . The results satisfy the positivity conditions required by unitarity. Some mathematical details are deferred to various appendices. In appendix A we construct the derivative operators which appear in the operator product expansion for $\ell=1,2$ while in appendix B we find, by direct calculation, the action of these differential operators to give the contribution of an $\ell=1$ operator to the operator product expansion. The result is in accord with the less direct discussion of section 2. In appendix C we describe some results for a function H which is obtained by the AdS/CFT integrals of section 5. This is obtained explicitly in various cases of interest. In appendix D we sketch some details of the derivation of the four point function for free vector theories.

2. Operator Product Expansion Analysis

For scalar operators ϕ_i of scale dimension Δ_i the contribution of a spin ℓ operator $O^{(\ell)}_{\mu_1...\mu_{\ell}}$ of dimension Δ to the operator product expansion, including all derivatives or descendants, may be written as

$$\phi_1(x_1)\phi_2(x_2) \sim C_{\phi_1\phi_2O^{(\ell)}} \frac{1}{r_{12}^{\frac{1}{2}(\Delta_1 + \Delta_2 - \Delta)}} C^{(\ell)}(x_{12}, \partial_{x_2})_{\mu_1\dots\mu_\ell} O^{(\ell)}_{\mu_1\dots\mu_\ell}, \qquad (2.1)$$

where

$$x_{ij} = x_i - x_j$$
, $r_{ij} = (x_i - x_j)^2$. (2.2)

The derivative operator $C^{(\ell)}(s,\partial)$ is determined by the form of the associated three and

two point functions. For the former, conformal invariance requires

$$\langle \phi_{1}(x_{1})\phi_{2}(x_{2}) O^{(\ell)}(x_{3}) \cdot \mathcal{C} \rangle$$

$$= C_{\phi_{1}\phi_{2}O^{(\ell)}} \frac{1}{r_{12}^{\frac{1}{2}(\Delta_{1}+\Delta_{2}-\Delta+\ell)} r_{13}^{\frac{1}{2}(\Delta+\Delta_{12}-\ell)} r_{23}^{\frac{1}{2}(\Delta-\Delta_{12}-\ell)}} Z_{\mu_{1}} \dots Z_{\mu_{\ell}} C_{\mu_{1}\dots\mu_{\ell}},$$
(2.3)

where

$$Z_{\mu} = \frac{x_{13\mu}}{r_{13}} - \frac{x_{12\mu}}{r_{12}}, \qquad Z^2 = \frac{r_{12}}{r_{13} r_{23}},$$
 (2.4)

transforms as a conformal vector at x_3 and $\mathcal{C}_{\mu_1...\mu_\ell}$ is an arbitrary symmetric traceless tensor, $O^{(\ell)} \cdot \mathcal{C} = O^{(\ell)}_{\mu_1...\mu_\ell} \mathcal{C}_{\mu_1...\mu_\ell}$, and

$$\Delta_{ij} = \Delta_i - \Delta_j \,. \tag{2.5}$$

The two point function for $O^{(\ell)}$ is given by

$$\langle O^{(\ell)}(x_1) \cdot \mathcal{C} O^{(\ell)}(x_2) \cdot \mathcal{C}' \rangle = \frac{1}{r_{12}^{\Delta}} \mathcal{C}_{\mu_1 \dots \mu_\ell} I_{\mu_1 \nu_1}(x_{12}) \dots I_{\mu_\ell \nu_\ell}(x_{12}) \mathcal{C}'_{\nu_1 \dots \nu_\ell}, \qquad (2.6)$$

where

$$I_{\mu\nu}(x) = \delta_{\mu\nu} - 2\frac{x_{\mu}x_{\nu}}{x^2},$$
 (2.7)

is the inversion tensor. For completeness we assume the normalisation of the scalar fields is determined by

$$\langle \phi_i(x_1)\phi_j(x_2)\rangle = \delta_{ij} \frac{1}{r_{12}^{\Delta_i}}.$$
 (2.8)

As a consequence of (2.3) and (2.6) we must therefore require in (2.1) $C^{(\ell)}(s,\partial) = C^{\frac{1}{2}(\Delta+\Delta_{12}-\ell),\frac{1}{2}(\Delta-\Delta_{12}-\ell)}(s,\partial)$ where

$$C^{a,b}(x_{12}, \partial_{x_2})_{\mu_1 \dots \mu_\ell} \frac{1}{r_{23}^S} I_{\mu_1 \nu_1}(x_{23}) \dots I_{\mu_\ell \nu_\ell}(x_{23}) C_{\nu_1 \dots \nu_\ell}$$

$$= \frac{1}{r_{13}^a r_{23}^b} Z_{\mu_1} \dots Z_{\mu_\ell} C_{\mu_1 \dots \mu_\ell},$$
(2.9)

and

$$S = a + b + \ell. \tag{2.10}$$

We construct $C^{a,b}(s,\partial)$ explicitly in appendix A for $\ell=1,2$ where it is given in terms of the known results for $\ell=0$. The generalisation to arbitrary ℓ is evident but is not needed here.

If (2.1) is applied in the four point function then the corresponding contribution has the form

$$\langle \phi_{1}(x_{1})\phi_{2}(x_{2})\phi_{3}(x_{3})\phi_{4}(x_{4})\rangle$$

$$\sim \frac{1}{r_{12}^{\frac{1}{2}(\Delta_{1}+\Delta_{2})}r_{34}^{\frac{1}{2}(\Delta_{3}+\Delta_{4})}} \left(\frac{r_{24}}{r_{14}}\right)^{\frac{1}{2}\Delta_{12}} \left(\frac{r_{14}}{r_{13}}\right)^{\frac{1}{2}\Delta_{34}} C_{\phi_{1}\phi_{2}O^{(\ell)}} C_{\phi_{3}\phi_{4}O^{(\ell)}}$$

$$\times u^{\frac{1}{2}(\Delta-\ell)} G^{(\ell)} \left(\frac{1}{2}(\Delta-\Delta_{12}-\ell), \frac{1}{2}(\Delta+\Delta_{34}-\ell), \Delta; u, v\right),$$
(2.11)

which depends on the two conformal invariants

$$u = \frac{r_{12} r_{34}}{r_{13} r_{24}}, \qquad v = \frac{r_{14} r_{23}}{r_{13} r_{24}}. \tag{2.12}$$

The functions $G^{(\ell)}$ are determined by

$$C^{a,b}(x_{12},\partial_{x_2})_{\mu_1\dots\mu_\ell} \frac{1}{r_{23}^e r_{24}^f} Y_{\mu_1}\dots Y_{\mu_\ell} = \frac{1}{r_{14}^a r_{24}^b} \left(\frac{r_{14}}{r_{13}}\right)^e G^{(\ell)}(b,e,S;u,v), \qquad (2.13)$$

where

$$Y_{\mu} = \frac{x_{32\mu}}{r_{23}} - \frac{x_{42\mu}}{r_{24}}, \qquad (2.14)$$

and, in addition to (2.10), we have

$$S = e + f + \ell. \tag{2.15}$$

We have undertaken a direct evaluation of (2.13) in appendix B for $\ell = 1$ but for a discussion of arbitrary ℓ we establish a recurrence relation which may be used iteratively to determine $G^{(\ell)}$ from $G^{(0)}$. We start from the integral representation

$$\mathcal{C}_{\mu_{1}\dots\mu_{\ell}} \frac{1}{r_{23}^{e} r_{24}^{f}} Y_{\mu_{1}} \dots Y_{\mu_{\ell}}
= r_{34}^{\frac{1}{2}d-S} \mathcal{C}_{\mu_{1}\dots\mu_{\ell}} N_{\ell,e,f} \int d^{d}x \frac{1}{(x_{2}-x)^{2S}(x_{3}-x)^{2(\frac{1}{2}d-f-\ell)}(x_{4}-x)^{2(\frac{1}{2}d-e-\ell)}}
\times I_{\mu_{1}\nu_{1}}(x_{2}-x) \dots I_{\mu_{\ell}\nu_{\ell}}(x_{2}-x) X'_{\nu_{1}} \dots X'_{\nu_{\ell}},$$
(2.16)

with

$$X' = \frac{x_3 - x}{(x_3 - x)^2} - \frac{x_4 - x}{(x_4 - x)^2}.$$
 (2.17)

The general structure of (2.16) follows from conformal invariance assuming (2.15) and is a generalisation of the well known result when $\ell = 0$ [7]. To obtain the overall constant in (2.16) we may define

$$y = \frac{x - x_2}{(x - x_2)^2}, \qquad y_i = \frac{x_i - x_2}{(x_i - x_2)^2}, \quad i = 2, 3, \qquad Y' = \frac{y_3 - y}{(y_3 - y)^2} - \frac{y_4 - y}{(y_4 - y)^2}, \quad (2.18)$$

so that, for $\alpha_2 + \alpha_3 + \alpha_4 + \ell = d$,

$$\mathcal{C}_{\mu_{1}...\mu_{\ell}} \int d^{d}x \frac{1}{(x_{2}-x)^{2\alpha_{2}}(x_{3}-x)^{2\alpha_{3}}(x_{4}-x)^{2\alpha_{4}}} I_{\mu_{1}\nu_{1}}(x_{2}-x) \dots I_{\mu_{\ell}\nu_{\ell}}(x_{2}-x) X'_{\nu_{1}} \dots X'_{\nu_{\ell}}$$

$$= y_{3}^{2\alpha_{3}} y_{4}^{2\alpha_{4}} \mathcal{C}_{\mu_{1}...\mu_{\ell}} \int d^{d}y \frac{1}{(y_{3}-y)^{2\alpha_{3}}(y_{4}-y)^{2\alpha_{4}}} Y'_{\mu_{1}} \dots Y'_{\mu_{\ell}}$$

$$= \pi^{\frac{1}{2}d} \prod_{r=0}^{\ell-1} (d-1-\alpha_{2}+r) \prod_{i=1}^{3} \frac{\Gamma(\frac{1}{2}d-\alpha_{i})}{\Gamma(\alpha_{i}+\ell)}$$

$$\times \mathcal{C}_{\mu_{1}...\mu_{\ell}}(y_{3}-y_{4})_{\mu_{1}} \dots (y_{3}-y_{4})_{\mu_{\ell}} \frac{y_{3}^{2\alpha_{3}} y_{4}^{2\alpha_{4}}}{(y_{3}-y_{4})^{2(\frac{1}{2}d-\alpha_{2})}}.$$
(2.19)

It is easy to see that (2.19) is in accord with (2.16) if we take

$$N_{\ell,e,f} = \frac{1}{\pi^{\frac{1}{2}d}} \frac{1}{(d-1-S)_{\ell}} \frac{\Gamma(\frac{1}{2}d-e)\Gamma(\frac{1}{2}d-f)\Gamma(S+\ell)}{\Gamma(e+\ell)\Gamma(f+\ell)\Gamma(\frac{1}{2}d-S)},$$
 (2.20)

for

$$(\gamma)_n = \frac{\Gamma(\gamma + n)}{\Gamma(\gamma)} \,. \tag{2.21}$$

We may now use (2.16) in (2.13) and apply the definition (2.9) to give

$$C^{a,b}(x_{12},\partial_{x_2})_{\mu_1\dots\mu_\ell} \frac{1}{(x_2-x)^{2S}} I_{\mu_1\nu_1}(x_2-x)\dots I_{\mu_\ell\nu_\ell}(x_2-x)X'_{\nu_1}\dots X'_{\nu_\ell}$$

$$= \frac{1}{(x_1-x)^{2a}(x_2-x)^{2b}} X_{\mu_1}\dots X_{\mu_\ell} \mathcal{E}^{(\ell)}_{\mu_1\dots\mu_\ell,\nu_1\dots\nu_\ell} X'_{\nu_1}\dots X'_{\nu_\ell},$$
(2.22)

where $\mathcal{E}^{(\ell)}_{\mu_1...\mu_\ell,\nu_1...\nu_\ell}$ is the projector onto symmetric traceless rank ℓ tensors, and

$$X = \frac{x_1 - x}{(x_1 - x)^2} - \frac{x_2 - x}{(x_2 - x)^2}.$$
 (2.23)

The contraction in (2.22) may further be evaluated as

$$X_{\mu_{1}} \dots X_{\mu_{\ell}} \mathcal{E}^{(\ell)}_{\mu_{1} \dots \mu_{\ell}, \nu_{1} \dots \nu_{\ell}} X'_{\nu_{1}} \dots X'_{\nu_{\ell}} = \frac{\ell!}{2^{\ell} (\frac{1}{2}d - 1)_{\ell}} (X^{2}X'^{2})^{\frac{1}{2}\ell} C_{\ell}^{\frac{1}{2}d - 1}(t),$$

$$t = \frac{X \cdot X'}{(X^{2}X'^{2})^{\frac{1}{2}}}.$$

$$(2.24)$$

 $C_n^{\lambda}(t)$ are Gegenbaur polynomials of order ℓ (for $\lambda = \frac{1}{2}$ these are just Legendre polynomials). In consequence (2.13) gives

$$r_{34}^{\frac{1}{2}d-S} \frac{\ell!}{2^{\ell}(\frac{1}{2}d-1)_{\ell}} N_{\ell,e,f} \int d^{d}x \frac{\left(X^{2}X'^{2}\right)^{\frac{1}{2}\ell} C_{\ell}^{\frac{1}{2}d-1}(t)}{(x_{1}-x)^{2a}(x_{2}-x)^{2b}(x_{3}-x)^{2(\frac{1}{2}d-f-\ell)}(x_{4}-x)^{2(\frac{1}{2}d-e-\ell)}}$$

$$= \frac{1}{r_{14}^{a} r_{24}^{b}} \left(\frac{r_{14}}{r_{13}}\right)^{e} \left(G^{(\ell)}(b,e,S;u,v)\right)$$

$$+ K_{b,e,S}^{(\ell)} u^{\frac{1}{2}d-S} G^{(\ell)}(\frac{1}{2}d+b-S,\frac{1}{2}d+e-S,d-S;u,v)\right). (2.25)$$

The last term on the right hand side of (2.25) is a so called shadow term which is non analytic at u = 0.1 This term may be neglected in our analysis of $G^{(\ell)}$. To evaluate the left hand side of (2.25) we may note that

$$X^{2} = \frac{r_{12}}{(x_{1} - x)^{2}(x_{2} - x)^{2}}, \qquad X^{2} = \frac{r_{34}}{(x_{3} - x)^{2}(x_{4} - x)^{2}}, \qquad (2.26)$$

and

$$2X \cdot X' = -\frac{r_{13}}{(x_1 - x)^2 (x_3 - x)^2} + \frac{r_{23}}{(x_2 - x)^2 (x_3 - x)^2} - \frac{r_{24}}{(x_2 - x)^2 (x_4 - x)^2} + \frac{r_{14}}{(x_1 - x)^2 (x_4 - x)^2},$$
(2.27)

so that the integral in (2.25) is reducible to linear combinations of the form

$$\int d^d x \, \frac{1}{(x_1 - x)^{2\alpha_1} (x_2 - x)^{2\alpha_2} (x_3 - x)^{2\alpha_3} (x_4 - x)^{2\alpha_4}} \,, \quad \sum_i \alpha_i = d \,, \tag{2.28}$$

which are expressible in terms of functions of the conformal invariants u, v defined in (2.12). A useful recurrence relation may be obtained by using the recurrence relation for Gegenbaur polynomials,

$$\ell C_{\ell}^{\lambda}(t) = 2(\lambda + \ell - 1)t C_{\ell-1}^{\lambda}(t) - (2\lambda + \ell - 2)C_{\ell-2}^{\lambda}(t).$$
 (2.29)

Substituting this into (2.25) and with (2.20) we may then obtain using (2.26) and (2.27) a corresponding relation for $G^{(\ell)}$,

$$G^{(\ell)}(b, e, S; u, v) = \frac{1}{2} \frac{S + \ell - 1}{d - S + \ell - 2} \left\{ \frac{\frac{1}{2}d - e - 1}{f + \ell - 1} \left(vG^{(\ell - 1)}(b + 1, e + 1, S; u, v) - G^{(\ell - 1)}(b, e + 1, S; u, v) \right) + \frac{\frac{1}{2}d - f - 1}{e + \ell - 1} \left(G^{(\ell - 1)}(b, e, S; u, v) - G^{(\ell - 1)}(b + 1, e, S; u, v) \right) \right\} - \frac{1}{4} \frac{(S + \ell - 1)(S + \ell - 2)}{(d - S + \ell - 2)(d - S + \ell - 3)} \frac{(\frac{1}{2}d - e - 1)(\frac{1}{2}d - f - 1)}{(f + \ell - 1)(e + \ell - 1)} \frac{(\ell - 1)(d + \ell - 4)}{(\frac{1}{2}d + \ell - 2)(\frac{1}{2}d + \ell - 3)} \times uG^{(\ell - 2)}(b + 1, e + 1, S; u, v),$$

$$(2.30)$$

$$\begin{split} K_{b,e,S}^{(\ell)} &= \frac{\Gamma(\frac{1}{2}d+b-S+\ell)\Gamma(\frac{1}{2}d+e-S+\ell)\Gamma(S+\ell)\Gamma(S+\ell-1)}{\Gamma(b+\ell)\Gamma(e+\ell)} \\ &\times \frac{\Gamma(\frac{1}{2}d-b)\Gamma(\frac{1}{2}d-e)\Gamma(S-\frac{1}{2}d)\Gamma(d-S-1)}{\Gamma(S-b)\Gamma(S-e)\Gamma(\frac{1}{2}d-S)\Gamma(S-1)} \,. \end{split}$$

¹ Both terms in (2.25) satisfy the recurrence relation derived for $G^{(\ell)}$ below. Using this result and the evaluation of (2.25) for $\ell = 0$ determines

where f is determined by (2.15).

The starting point in the iteration $G^{(0)}$, corresponding to a scalar field in the operator product expansion, has been obtained by various authors. In the short distance limit $x_1 \to x_2$ we have $u \to 0$, $v \to 1$ and it is given as a power series in u, 1 - v,

$$G^{(0)}(b, e, S; u, v) = G(b, e, S + 1 - \frac{1}{2}d, S; u, 1 - v), \qquad (2.31)$$

where

$$G(\alpha, \beta, \gamma, \delta; u, 1 - v) = G(\beta, \alpha, \gamma, \delta; u, 1 - v)$$

$$= \sum_{m,n=0} \frac{(\delta - \alpha)_m (\delta - \beta)_m}{m! (\gamma)_m} \frac{(\alpha)_{m+n} (\beta)_{m+n}}{n! (\delta)_{2m+n}} u^m (1 - v)^n$$

$$= \frac{\Gamma(\delta) \Gamma(\delta - \alpha - \beta)}{\Gamma(\delta - \alpha) \Gamma(\delta - \beta)} F_4(\alpha, \beta, \gamma, \alpha + \beta + 1 - \delta; u, v)$$

$$+ \frac{\Gamma(\delta) \Gamma(\alpha + \beta - \delta)}{\Gamma(\alpha) \Gamma(\beta)} v^{\delta - \alpha - \beta} F_4(\delta - \alpha, \delta - \beta, \gamma, \delta - \alpha - \beta + 1; u, v),$$

$$(2.32)$$

with F_4 one of the well documented Appell functions, ref.[8], p. 1080-1084. Two important symmetry relations may be obtained from the associated result demonstrated for G in [6] and checking consistency with (2.30). These correspond to letting $x_1 \leftrightarrow x_2$, $a \leftrightarrow b$ and also $x_3 \leftrightarrow x_4$, $e \leftrightarrow f$, and using (2.10) and (2.15) we have

$$G^{(\ell)}(b, e, S; u, v) = (-1)^{\ell} v^{-e} G^{(\ell)}(a, e, S; u', v'),$$

$$= (-1)^{\ell} v^{-b} G^{(\ell)}(b, f, S; u', v'), \qquad u' = u/v, \ v' = 1/v.$$
(2.33)

Although (2.30) is rather involved it may be solved directly when u = 0 when the last term is absent. From (2.31) and (2.32) it is evident that $G^{(0)}(b, e, S; 0, v) = F(b, e; S; 1-v)$, an ordinary hypergeometric function and by using standard hypergeometric identities

$$G^{(\ell)}(b, e, S; 0, v) = \left(-\frac{1}{2}(1 - v)\right)^{\ell} F(b + \ell, e + \ell; S + \ell; 1 - v). \tag{2.34}$$

Other general results for arbitrary d are hard to find but for v = 1 we may determine the leading behaviour for small u,

$$G^{(\ell)}(b, e, S; u, 1) \sim \begin{cases} \frac{1}{2^{\ell}} (-1)^{\frac{1}{2}\ell} g_{\ell} u^{\frac{1}{2}\ell}, & \ell \text{ even,} \\ \frac{1}{2^{\ell+1}} (-1)^{\frac{1}{2}(\ell-1)} g_{\ell+1} \frac{(\ell + \frac{1}{2}d - 1)(a - b)(e - f)}{(S - d - \ell + 2)(S + \ell)} u^{\frac{1}{2}(\ell+1)}, & \ell \text{ odd,} \end{cases}$$

$$(2.35)$$

where from (2.30) $g_{\ell} = (\ell - 1)(\ell + d - 4)g_{\ell-2}/(\ell + \frac{1}{2}d - 2)(\ell + \frac{1}{2}d - 3)$ and starting from $g_0 = 1$,

$$g_{\ell} = \frac{\ell! \left(\frac{1}{2}d - 1\right)_{\frac{1}{2}\ell}}{\left(\frac{1}{2}\ell\right)! \left(\frac{1}{2}d - 1\right)_{\ell}}.$$
 (2.36)

3. Solutions in Two and Four Dimensions

We show here how the recursion relation (2.30) may be solved explicitly in two and four dimensions.

For d=2, $G^{(\ell)}$ may be found directly using the simplifications obtained through using complex coordinates z, \bar{z} in this case. Letting $x^z \equiv z$ and, from $x^2 = z\bar{z}$, $x_z = \frac{1}{2}\bar{z}$ then the inversion tensor defined in (2.7) is, in this complex basis, $I_{z\bar{z}}(x) = I_{\bar{z}z}(x) = 0$ and $I_{zz}(x) = -\frac{1}{2}\bar{z}/z$, $I_{\bar{z}\bar{z}}(x) = -\frac{1}{2}z/\bar{z}$. Since (2.4) reduces to

$$Z^z = -\frac{\bar{z}_{12}}{\bar{z}_{13}\,\bar{z}_{23}}\,,\qquad Z^{\bar{z}} = -\frac{z_{12}}{z_{13}\,z_{23}}$$
 (3.1)

the equation (2.22) becomes

$$C^{a,b}(x_{12}, \partial_{x_2})^{z...z} \frac{1}{z_{23}^{S+\ell} \bar{z}_{23}^{S-\ell}} = \frac{z_{12}^{\ell}}{z_{13}^{a+\ell} z_{23}^{b+\ell}} \frac{1}{\bar{z}_{13}^{a} \bar{z}_{23}^{b}},$$
(3.2)

together with its conjugate. The differential operator therefore factorises in the form[9]

$$C^{a,b}(x_{12},\partial_{x_2})^{z\dots z} = z_{12}^{\ell} {}_{1}F_{1}(a+\ell,S+\ell;z_{12}\partial_{z_2}){}_{1}F_{1}(a,S-\ell;\bar{z}_{12}\partial_{\bar{z}_2}), \qquad (3.3)$$

where ${}_{1}F_{1}(\alpha,\beta;x) = \sum_{n} (\alpha)_{n}/n!(\beta)_{n} x^{n}$. Since

$$_{1}F_{1}(a;S;z_{12}\partial_{z_{2}})\frac{1}{z_{23}^{e}z_{24}^{f}} = \frac{1}{z_{14}^{a}z_{24}^{b}} \left(\frac{z_{14}}{z_{13}}\right)^{e} F(b,e;S;\eta), \quad S = a+b=e+f,$$
 (3.4)

where

$$\eta = \frac{z_{12} \, z_{34}}{z_{13} \, z_{24}} \,, \tag{3.5}$$

the definition (2.13) gives

$$G^{(0)}(b, e, S; u, v) = F(b, e; S; \eta) F(b, e; S; \bar{\eta}),$$

$$G^{(\ell)}(b, e, S; u, v) = (-\frac{1}{2}\eta)^{\ell} F(b + \ell, e + \ell; S + \ell; \eta) F(b, e; S - \ell; \bar{\eta})$$
+ conjugate, $\ell > 0$, (3.6)

for

$$u = \eta \bar{\eta}, \qquad v = (1 - \eta)(1 - \bar{\eta}).$$
 (3.7)

For $G^{(0)}$ the validity of the result (3.6) depends, from (2.31), on the factorisation formula shown in [6] for the function G defined in (2.32). This is obtained from a similar reduction formula for F_4 and takes the form

$$G(\alpha, \beta, \gamma, \gamma; u, 1 - v) = F(\alpha, \beta; \gamma; x) F(\alpha, \beta; \gamma; z).$$
(3.8)

where

$$u = xz$$
, $v = (1-x)(1-z)$. (3.9)

We have verified that $G^{(\ell)}$ defined then through (2.30) for $\ell = 1, 2, \ldots$ agrees with (3.6). The calculation is very similar to the d = 4 case which we describe next.

When d = 4 the result for $G^{(0)}$ given by (2.31) and (2.32) may also be simplified by use of a reduction formula extending (3.8) which was also obtained in [6],

$$G(\alpha, \beta, \gamma, \gamma + 1; u, 1 - v) = \frac{1}{z - x} \left(zF(\alpha - 1, \beta - 1; \gamma - 1; x)F(\alpha, \beta; \gamma + 1; z) - xF(\alpha, \beta; \gamma + 1; x)F(\alpha - 1, \beta - 1; \gamma - 1; z) \right).$$

$$(3.10)$$

The solution for any ℓ which follows from this is then

$$G^{(\ell)}(b, e, S; u, v) = \frac{1}{z - x} \Big((-\frac{1}{2}z)^{\ell} z F(b - 1, e - 1; S - 2 - \ell; x) F(b + \ell, e + \ell; S + \ell; z) - (-\frac{1}{2}x)^{\ell} x F(b - 1, e - 1; S - 2 - \ell; z) F(b + \ell, e + \ell; S + \ell; x) \Big).$$

$$(3.11)$$

The verification of (3.11) from (2.30) depends on

$$zF(b-1,e-1;S-2-\ell;x)F(b+\ell,e+\ell;S+\ell;z)$$

$$= -\frac{S+\ell-1}{S-\ell-2} \left\{ \frac{e-1}{f+\ell-1} \left((1-z)(1-x)F(b,e;S-1-\ell;x)F(b+\ell,e+\ell;S+\ell-1;z) - F(b-1,e;S-1-\ell;x)F(b+\ell-1,e+\ell;S+\ell-1;z) \right) \right.$$

$$\left. -F(b-1,e;S-1-\ell;x)F(b+\ell-1,e+\ell;S+\ell-1;z) \right)$$

$$\left. + \frac{f-1}{e+\ell-1} \left(F(b-1,e-1;S-1-\ell;x)F(b+\ell-1,e+\ell-1;S+\ell-1;z) - F(b,e-1;S-1-\ell;x)F(b+\ell-1,e+\ell-1;S+\ell-1;z) \right) \right\}$$

$$\left. -\frac{(S+\ell-1)(S+\ell-2)}{(S-\ell-2)(S-\ell-1)} \frac{(e-1)(f-1)}{(f+\ell-1)(e+\ell-1)} \right.$$

$$\left. \times xF(b,e;S-\ell;x)F(b+\ell-1,e+\ell-1;S+\ell-2;z) \right. \tag{3.12}$$

This corresponds exactly to the form of (2.30) for d=4 except if $\ell=1$ when the last term, which matches the last term in (3.12), is missing. However in this case this piece times z is symmetric under $z \leftrightarrow x$ and it is then cancelled by the other term in (3.11) which is obtained from the analogous result to (3.12) for $z \leftrightarrow x$. The justification of (3.12) depends

on standard hypergeometric identities, ref.[8], p. 1071, in particular we use,

$$(S+\ell-1)\big((1-z)F(b+\ell,e+\ell;S+\ell-1;z) - F(b+\ell-1,e+\ell;S+\ell-1;z)\big)$$

$$= -(f+\ell-1)zF(b+\ell,e+\ell;S+\ell;z),$$

$$(S+\ell-1)\big(F(b+\ell,e+\ell-1;S+\ell-1;z) - F(b+\ell-1,e+\ell-1;S+\ell-1;z)\big)$$

$$= (e+\ell-1)zF(b+\ell,e+\ell;S+\ell;z),$$

$$(e-1)(1-x)F(b,e;S-1-\ell;x) + (f-1)F(b+\ell,e+\ell-1;S+\ell-1;z)$$

$$= (S-\ell-2)F(b-1,e-1;S-2-\ell;x).$$
(3.13)

For consistency we may check that (3.11) satisfies the consistency relations (2.33). It is crucial to recognise that (3.9) is invariant under $x \leftrightarrow z$, under which of course (3.11) is invariant. For the transformations corresponding to $x_1 \leftrightarrow x_2$ or $x_3 \leftrightarrow x_4$ we choose

$$x \to x' = \frac{x}{x-1}, \quad z \to z' = \frac{z}{z-1} \quad \Rightarrow \quad u \to u' = \frac{u}{v}, \quad v \to v' = \frac{1}{v}.$$
 (3.14)

Using standard hypergeometric results, ref. [8], p. 1069, with (2.10), (2.15) we have

$$F(b-1, e-1; S-2-\ell; x) = (1-x)^{-e+1} F(a-1, e-1; S-2-\ell; x')$$

$$= (1-x)^{-b+1} F(b-1, f-1; S-2-\ell; x'),$$

$$F(b+\ell, e+\ell; S+\ell; z) = (1-z)^{-e-\ell} F(a+\ell, e+\ell; S+\ell; z')$$

$$= (1-z)^{-b-\ell} F(b+\ell, f+\ell; S+\ell; z').$$
(3.15)

which in (3.11), with z - x = -(z' - x')v, are sufficient to verify (2.33).

Both results (3.6) and (3.11) are of course compatible with $G^{(0)}(0,0,0;u,v) = 1$, representing the contribution of the identity operator in the operator product expansion.

4. Energy Momentum Tensor

It is of interest to specialise the general discussion to the particular case of the energy momentum tensor $T_{\mu\nu}$ which is an $\ell=2$ operator with dimension d satisfying the conservation equation $\partial_{\mu}T_{\mu\nu}=0$. In this case the normalisation is not fixed by (2.6) but instead through Ward identities. For the canonically normalised energy momentum tensor, but with the scalar field ϕ still normalised as in (2.8), the $\langle \phi \phi T \rangle$ three point function obtained from (2.3) becomes [10]

$$\langle \phi(x_1)\phi(x_2) T_{\mu\nu}(x_3) \rangle = -\frac{1}{S_d} \frac{\Delta d}{d-1} \frac{1}{r_{12}^{\Delta - \frac{1}{2}d} r_{13}^{\frac{1}{2}d} r_{23}^{\frac{1}{2}d}} \left(\frac{Z_{\mu}Z_{\nu}}{Z^2} - \frac{1}{d} \delta_{\mu\nu} \right), \tag{4.1}$$

for $S_d = 2\pi^{\frac{1}{2}d}/\Gamma(\frac{1}{2}d)$. The energy momentum tensor two point function is also of the form given by (2.6) and can be written

$$\langle T_{\mu_1\mu_2}(x_1)T_{\nu_1\nu_2}(x_2)\rangle = \frac{C_T}{S_d^2} \frac{1}{r_{12}^d} \left(\frac{1}{2} \left(I_{\mu_1\nu_1}(x_{12})I_{\mu_2\nu_2}(x_{12}) + I_{\mu_1\nu_2}(x_{12})I_{\mu_2\nu_1}(x_{12}) \right) - \frac{1}{d} \,\delta_{\mu_1\mu_2}\delta_{\nu_1\nu_2} \right), \tag{4.2}$$

with the coefficient $C_T > 0$ depending on the particular conformal field theory.² The contribution to the four point function is then from (2.11)

$$\langle \phi(x_1)\phi(x_2)\phi'(x_3)\phi'(x_4)\rangle \sim \frac{1}{r_{12}^{\Delta} r_{34}^{\Delta'}} \frac{\Delta \Delta' d^2}{C_T(d-1)^2} u^{\frac{1}{2}(d-2)} G^{(2)}(\frac{1}{2}(d-2), \frac{1}{2}(d-2), d; u, v).$$

$$(4.3)$$

Using (3.6) and (3.11) we can give complete expressions in two and four dimensions. If d = 2 then in (4.3) we have

$$G^{(2)}(0,0,2;u,v) = -3\left(1 + \frac{1}{\eta}(1 - \frac{1}{2}\eta)\ln(1 - \eta)\right) + \text{conjugate},$$
 (4.4)

while for d = 4,

$$uG^{(2)}(1,1,4;u,v) = -45 \frac{x\left(1 - \frac{1}{2}z + \frac{1}{z}\left(1 - z + \frac{1}{6}z^2\right)\ln(1-z)\right) - z \leftrightarrow x}{z - x}.$$
 (4.5)

5. AdS/CFT Integrals

The metric on AdS_{d+1} defines a Weyl equivalence class of metrics on the boundary S^d while the isometry group SO(d+1,1) becomes the conformal group on the boundary. This is at the root of the AdS/CFT correspondence. Much work [11,12,13,14,15,16,17,18] has been undertaken investigating the structure of the conformally covariant correlation functions for boundary points x_i obtained in terms Feynman graphs for propagators on AdS_{d+1} linking x_i [19]. For vertices defined by IIB supergravity this is relevant to the large N limit of $\mathcal{N}=4$ SYM. The two and three point functions are dictated by conformal invariance while the four point function involves functions of the two invariants u, v, as defined in (2.12), which may be matched to the operator product expansion.

For d=2 with $T(z)=-2\pi T_{zz}(x)$ (4.1) and (4.2) reduce to the well known results $\langle \phi(x_1)\phi(x_2)T(z_3)\rangle = \frac{1}{2}\Delta(z_{12}/z_{13}z_{23})^2|z_{12}|^{-2\Delta}$ and $\langle T(z_1)T(z_2)\rangle = C_T/(4z_{12}^4)$ so that $\frac{1}{2}C_T=c$ the Virasoro central charge.

We show here how the simplest graphs lead to integrals which may be reduced to the functions G defined earlier in (2.32) and discussed here and in [6]. With the usual metric on AdS_{d+1} and coordinates $z = (z_0, x), x \in \mathbb{R}^d, z_0 \in \mathbb{R}_+$,

$$ds^{2} = \frac{1}{z_{0}^{2}} (dz_{0}^{2} + dx_{\mu} dx_{\mu}), \qquad (5.1)$$

the boundary corresponds to $z_0 = 0$ together with the point at infinity $z_0 = \infty$. The bulk/boundary propagator is then [19]

$$K_{\Delta}(z, x') = \frac{\Gamma(\Delta)}{2\pi^{\frac{1}{2}d}\Gamma(\Delta + 1 - \frac{1}{2}d)} \tilde{K}_{\Delta}(z, x'), \qquad \tilde{K}_{\Delta}(z, x') = \left(\frac{z_0}{z_0^2 + (x - x')^2}\right)^{\Delta}. \quad (5.2)$$

We are initially interested in integrals defining conformal N-point functions of the form, as defined in [13],

$$D_{\Delta_1...\Delta_N}(x_1,...,x_N) = \frac{1}{\pi^{\frac{1}{2}d}} \int d^{d+1}z \frac{1}{z_0^{d+1}} \prod_{i=1}^N \tilde{K}_{\Delta_i}(z,x_i).$$
 (5.3)

Using standard integral representations for $(z_0^2 + (x - x_i)^2)^{-\Delta_i}$ the z-integration may be undertaken giving

$$D_{\Delta_1...\Delta_N}(x_1,...x_N) = \frac{\Gamma(\Sigma - \frac{1}{2}d)}{2\prod_i \Gamma(\Delta_i)} \int_0^\infty \prod_{i=1}^N d\lambda_i \, \lambda_i^{\Delta_i - 1} \, \frac{1}{\Lambda^{\Sigma}} e^{-\frac{1}{\Lambda} \sum_{i < j} \lambda_i \lambda_j r_{ij}} \,, \tag{5.4}$$

where $\Lambda = \sum_{i} \lambda_i$ and

$$\Sigma = \frac{1}{2} \sum_{i=1}^{N} \Delta_i \,. \tag{5.5}$$

A crucial observation of Symanzik [20], recounted in [6], is that for integrals of the form (5.4), subject to (5.5), then Λ may be modified, without changing the integral, to the form $\sum_{i} \kappa_{i} \lambda_{i}$ for any κ_{i} , not all zero, satisfying $\kappa_{i} \geq 0$. In particular we may choose $\Lambda = \lambda_{N}$ and the integral may then be directly written in terms of conformally invariant cross ratios like u, v. For N = 3 we have

$$D_{\Delta_1 \Delta_2 \Delta_3}(x_1, x_2, x_3) = \frac{\Gamma(\Sigma - \frac{1}{2}d)}{2\Gamma(\Delta_1)\Gamma(\Delta_2)\Gamma(\Delta_3)} \frac{\Gamma(\Sigma - \Delta_1)\Gamma(\Sigma - \Delta_2)\Gamma(\Sigma - \Delta_3)}{r_{23}^{\Sigma - \Delta_1} r_{13}^{\Sigma - \Delta_2} r_{12}^{\Sigma - \Delta_3}}.$$
 (5.6)

The result for N=4 may also be expressed in term of a function of the conformal invariants u, v in the form

$$D_{\Delta_{1}\Delta_{2}\Delta_{3}\Delta_{4}}(x_{1}, x_{2}, x_{3}, x_{4}) = \frac{\Gamma(\Sigma - \frac{1}{2}d)}{2\Gamma(\Delta_{1})\Gamma(\Delta_{2})\Gamma(\Delta_{3})\Gamma(\Delta_{4})} \frac{r_{14}^{\Sigma - \Delta_{1} - \Delta_{4}} r_{34}^{\Sigma - \Delta_{3} - \Delta_{4}}}{r_{13}^{\Sigma - \Delta_{4}} r_{24}^{\Delta_{2}}} \times \overline{D_{\Delta_{1}\Delta_{2}\Delta_{3}\Delta_{4}}(u, v)}.$$
(5.7)

Using Symanzik's procedure for evaluating the integrals we have

$$\overline{D}_{\Delta_1 \Delta_2 \Delta_3 \Delta_4}(u, v) = H(\Delta_2, \Sigma - \Delta_4, \Delta_1 + \Delta_2 - \Sigma + 1, \Delta_1 + \Delta_2; u, v), \qquad (5.8)$$

where the function H, defined in [6], is directly related to the function G given by the power series in u, 1 - v in (2.32) through

$$H(\alpha, \beta, \gamma, \delta; u, v) = H(\beta, \alpha, \gamma, \delta; u, v)$$

$$= \frac{\Gamma(1 - \gamma)}{\Gamma(\delta)} \Gamma(\alpha) \Gamma(\beta) \Gamma(\delta - \alpha) \Gamma(\delta - \beta) G(\alpha, \beta, \gamma, \delta; u, 1 - v)$$

$$+ \frac{\Gamma(\gamma - 1)}{\Gamma(\delta - 2\gamma + 2)} \Gamma(\alpha - \gamma + 1) \Gamma(\beta - \gamma + 1) \Gamma(\delta - \gamma - \alpha + 1) \Gamma(\delta - \gamma - \beta + 1)$$

$$\times u^{1 - \gamma} G(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma, \delta - 2\gamma + 2; u, 1 - v).$$
(5.9)

The symmetry properties of the integral (5.4), for N=4, are reflected in various identities obeyed by H which are listed in appendix C.

Besides (5.7) we may also consider the scalar exchange contribution to the four point function which is given by

$$S^{\Delta}_{\Delta_1 \Delta_2 \Delta_3 \Delta_4}(x_1, x_2, x_3, x_4) = \frac{1}{\pi^{\frac{1}{2}d}} \int d^{d+1}z \frac{1}{z_0^{d+1}} \int d^{d+1}w \frac{1}{w_0^{d+1}} \tilde{K}_{\Delta_1}(z, x_1) \tilde{K}_{\Delta_2}(z, x_2) \times G_{\Delta}(z, w) \tilde{K}_{\Delta_3}(w, x_3) \tilde{K}_{\Delta_4}(w, x_4),$$

$$(5.10)$$

for G_{Δ} the scalar Green function on AdS_{d+1} ,

$$(-\nabla^2 + \Delta(\Delta - d))G_{\Delta}(z, w) = z_0^{d+1}\delta^{d+1}(z - w).$$
 (5.11)

The explicit form for G_{Δ} is unnecessary here except for

$$G_{\Delta}(z,w) \sim z_0^{\Delta} \frac{1}{2\pi^{\frac{1}{2}d}} \frac{\Gamma(\Delta)}{\Gamma(\Delta - \frac{1}{2}d + 1)} \tilde{K}_{\Delta}(w,x) \quad \text{as} \quad z_0 \to 0.$$
 (5.12)

In consequence

$$R(z; x_3, x_4) = \int d^{d+1}w \frac{1}{w_0^{d+1}} G_{\Delta}(z, w) \tilde{K}_{\Delta_3}(w, x_3) \tilde{K}_{\Delta_4}(w, x_4), \qquad (5.13)$$

satisfies

$$(-\nabla^2 + \Delta(\Delta - d))R(z; x_3, x_4) = \tilde{K}_{\Delta_3}(z, x_3)\tilde{K}_{\Delta_4}(z, x_4), \qquad (5.14)$$

with the boundary condition, obtained by using (5.12) inside the integral

$$R(z; x_3, x_4) \sim z_0^{\Delta} \frac{\Gamma(\Delta)}{2\Gamma(\Delta - \frac{1}{2}d + 1)} D_{\Delta\Delta_3\Delta_4}(x, x_3, x_4) \quad \text{as} \quad z_0 \to 0.$$
 (5.15)

To solve (5.14) with (5.15) we make use of

$$(-\nabla^2 + (\Delta_3 + \Delta_4)(\Delta_3 + \Delta_4 - d)) (\tilde{K}_{\Delta_3}(z, x_3)\tilde{K}_{\Delta_4}(z, x_4))$$

$$= 4\Delta_3 \Delta_4 \, \tilde{K}_{\Delta_3 + 1}(z, x_3) \tilde{K}_{\Delta_4 + 1}(z, x_4) \, r_{34} \, .$$
(5.16)

With the aid of (5.16) we may then write a series solution for R as

$$R(z; x_{3}, x_{4}) = -\frac{1}{4} \sum_{s=0} \frac{(\Delta_{3})_{s}(\Delta_{4})_{s}}{(\frac{1}{2}(\Delta_{3} + \Delta_{4} - \Delta))_{s+1}(\frac{1}{2}(\Delta_{3} + \Delta_{4} + \Delta - d))_{s+1}} \times \tilde{K}_{\Delta_{3}+s}(z, x_{3})\tilde{K}_{\Delta_{4}+s}(z, x_{4}) r_{34}^{s} + \frac{1}{4}\Gamma(\frac{1}{2}(\Delta_{3} + \Delta_{4} + \Delta - d)) \frac{\Gamma(\frac{1}{2}(\Delta_{3} + \Delta_{4} - \Delta))\Gamma(\frac{1}{2}(\Delta + \Delta_{34}))\Gamma(\frac{1}{2}(\Delta - \Delta_{34}))}{\Gamma(\Delta - \frac{1}{2}d + 1)\Gamma(\Delta_{3})\Gamma(\Delta_{4})} \times \sum_{s=0} \frac{(\frac{1}{2}(\Delta + \Delta_{34}))_{s}(\frac{1}{2}(\Delta - \Delta_{34}))_{s}}{s!(\Delta - \frac{1}{2}d + 1)_{s}} \times \tilde{K}_{\frac{1}{2}(\Delta + \Delta_{34})+s}(z, x_{3})\tilde{K}_{\frac{1}{2}(\Delta - \Delta_{34})+s}(z, x_{4}) r_{34}^{\frac{1}{2}(\Delta - \Delta_{3} - \Delta_{4})+s}.$$
 (5.17)

The series in (5.17) are convergent for sufficiently small z_0 or r_{34} . The first term in (5.17) satisfies the inhomogeneous equation (5.14) while the second term obeys the corresponding homogeneous equation but reproduces the required boundary behaviour (5.15) after using (5.6). This term generates the dominant contribution as $z_0 \to 0$ if $\Delta < \Delta_3 + \Delta_4$ which is necessary for the derivation of (5.15) to be valid. If $\Delta_3 + \Delta_4 - \Delta = 2n$, n = 1, 2, ... the two series cancel except for a finite number of terms and we get

$$R(z; x_3, x_4) = \frac{1}{4}(n-1)! \sum_{s=-n}^{-1} \frac{(\Delta_3)_s(\Delta_4)_s}{(n+s)!(\Delta - \frac{1}{2}d + n)_{s+1}} \tilde{K}_{\Delta_3 + s}(z, x_3) \tilde{K}_{\Delta_4 + s}(z, x_4) r_{34}^s,$$
(5.18)

which coincides with the solution obtained in [12]. Using (5.17) in (5.10) we may obtain from (5.7)

$$S^{\Delta}_{\Delta_{1}\Delta_{2}\Delta_{3}\Delta_{4}}(x_{1}, x_{2}, x_{3}, x_{4}) = \frac{1}{8\Gamma(\Delta_{1})\Gamma(\Delta_{2})\Gamma(\Delta_{3})\Gamma(\Delta_{4})} \frac{r_{14}^{\Sigma-\Delta_{1}-\Delta_{4}} r_{34}^{\Sigma-\Delta_{3}-\Delta_{4}}}{r_{13}^{\Sigma-\Delta_{4}} r_{24}^{\Delta_{2}}} \times S^{\Delta}_{\Delta_{1}\Delta_{2}\Delta_{3}\Delta_{4}}(u, v),$$
(5.19)

where

$$S_{\Delta_{1}\Delta_{2}\Delta_{3}\Delta_{4}}^{\Delta}(u,v) = -\sum_{s=0} \frac{\Gamma(\Sigma - \frac{1}{2}d + s)}{(\frac{1}{2}(\Delta_{3} + \Delta_{4} - \Delta))_{s+1}(\frac{1}{2}(\Delta_{3} + \Delta_{4} + \Delta - d))_{s+1}} \times H(\Delta_{2}, \Sigma - \Delta_{4}, \Delta_{1} + \Delta_{2} - \Sigma + 1 - s, \Delta_{1} + \Delta_{2}; u, v) + \frac{\Gamma(\frac{1}{2}(\Delta_{3} + \Delta_{4} + \Delta - d))\Gamma(\frac{1}{2}(\Delta_{3} + \Delta_{4} - \Delta))}{\Gamma(\Delta - \frac{1}{2}d + 1)} \times \sum_{s=0} \frac{\Gamma(\frac{1}{2}(\Delta_{1} + \Delta_{2} + \Delta - d) + s)}{s!(\Delta - \frac{1}{2}d + 1)_{s}} \times H(\Delta_{2}, \Sigma - \Delta_{4}, \frac{1}{2}(\Delta_{1} + \Delta_{2} - \Delta) + 1 - s, \Delta_{1} + \Delta_{2}; u, v).$$
(5.20)

As in (5.18) it is easy to verify that this reduces to a finite sum if $\Delta_3 + \Delta_4 - \Delta = 2n$. For general Δ we may use (5.9) and (2.32) to rewrite (5.20) as the sum of three terms with different leading powers in u [21],

$$u^{\frac{1}{2}(\Delta_{1}+\Delta_{2})}\mathcal{S}^{\Delta}_{\Delta_{1}\Delta_{2}\Delta_{3}\Delta_{4}}(u,v)$$

$$=\Gamma(\frac{1}{2}(\Delta_{1}+\Delta_{2}-\Delta))\Gamma(\frac{1}{2}(\Delta_{3}+\Delta_{4}-\Delta))\Gamma(\frac{1}{2}(\Delta_{1}+\Delta_{2}+\Delta-d))\Gamma(\frac{1}{2}(\Delta_{3}+\Delta_{4}+\Delta-d))$$

$$\times\frac{1}{\Gamma(\Delta)\Gamma(\Delta+1-\frac{1}{2}d)}\Gamma(\frac{1}{2}(\Delta+\Delta_{12}))\Gamma(\frac{1}{2}(\Delta-\Delta_{12}))\Gamma(\frac{1}{2}(\Delta+\Delta_{34}))\Gamma(\frac{1}{2}(\Delta-\Delta_{34}))$$

$$\times u^{\frac{1}{2}\Delta}G(\frac{1}{2}\Delta-\frac{1}{2}\Delta_{12},\frac{1}{2}\Delta+\frac{1}{2}\Delta_{34},\Delta+1-\frac{1}{2}d,\Delta;u,1-v)$$

$$-u^{\frac{1}{2}(\Delta_{3}+\Delta_{4})}\mathcal{G}^{\Delta}_{\Delta_{1}\Delta_{2}\Delta_{3}\Delta_{4}}(u,v)-u^{\frac{1}{2}(\Delta_{1}+\Delta_{2})}\mathcal{G}^{\Delta}_{\Delta_{4}\Delta_{3}\Delta_{2}\Delta_{1}}(u,v),$$
(5.21)

for

$$\mathcal{G}_{\Delta_{1}\Delta_{2}\Delta_{3}\Delta_{4}}^{\Delta}(u,v) = \frac{\Gamma(\Sigma - \frac{1}{2}d)}{\Gamma(\Delta_{3} + \Delta_{4})} \Gamma(\Sigma - \Delta_{3} - \Delta_{4}) \Gamma(\Delta_{3}) \Gamma(\Delta_{4}) \Gamma(\Sigma - \Delta_{1}) \Gamma(\Sigma - \Delta_{2})
\times \sum_{m,n=0} \frac{(\Sigma - \Delta_{2})_{m}(\Delta_{4})_{m} G_{m}^{\Delta}}{m!(\Sigma - \Delta_{1} - \Delta_{2} + 1)_{m}} \frac{(\Delta_{3})_{m+n}(\Sigma - \Delta_{1})_{m+n}}{(\Delta_{3} + \Delta_{4})_{2m+n}} u^{m} (1 - v)^{n},
G_{m}^{\Delta} = \sum_{s=0}^{m} (-1)^{s} \frac{m!}{(m-s)!} \frac{(\Sigma - \frac{1}{2}d)_{s}}{(\frac{1}{2}(\Delta_{3} + \Delta_{4} - \Delta))_{s+1}(\frac{1}{2}(\Delta_{3} + \Delta_{4} + \Delta - d))_{s+1}}.$$
(5.22)

To achieve the desired form for $\mathcal{G}^{\Delta}_{\Delta_4\Delta_3\Delta_2\Delta_1}$ requires non trivial relations for ${}_3F_2$ functions with argument 1. The first term in (5.21) matches exactly the contribution of a scalar operator with dimension Δ in the operator product expansion. From (5.22) $\mathcal{G}^{\Delta}_{\Delta_1\Delta_2\Delta_3\Delta_4}(u,v) = v^{-\Delta_3}\mathcal{G}^{\Delta}_{\Delta_2\Delta_1\Delta_3\Delta_4}(u/v,1/v) = v^{-\Sigma+\Delta_1}\mathcal{G}^{\Delta}_{\Delta_1\Delta_2\Delta_4\Delta_3}(u/v,1/v)$. Furthermore $u^{\frac{1}{2}(\Delta_1+\Delta_2)}\mathcal{S}^{\Delta}_{\Delta_1\Delta_2\Delta_3\Delta_4}(u,v) = u^{\frac{1}{2}(\Delta_3+\Delta_4)}\mathcal{S}^{\Delta}_{\Delta_4\Delta_4\Delta_2\Delta_1}(u,v)$ as well as satisfying relations for $\Delta_1 \leftrightarrow \Delta_2$ or $\Delta_3 \leftrightarrow \Delta_4$ and $u \to u/v, v \to 1/v$ which follow from the preceding relations for $\mathcal{G}^{\Delta}_{\Delta_1\Delta_2\Delta_3\Delta_4}$. The result given by (5.21) and (5.22) provides a representation

valid for $u \sim 0$, $v \sim 1$ and corresponds essentially with that given by Liu [15]. In principle (5.20) may be used to find a form appropriate for $u \sim 1$, $v \sim 0$ but, for the general case, the results are significantly more complicated [18]. The last line in (5.21) corresponds to the contribution of operators with dimensions $\Delta_3 + \Delta_4 + 2n$, $\Delta_1 + \Delta_2 + 2n$, $n = 0, 1, 2, \ldots$

6. Results for Free Fields

With the explicit formula (3.11) for the contribution for arbitrary spin operators to the four point function in four dimensions then it is natural to consider the associated partial wave expansion for some simple conformally invariant expressions for the four point function. We consider here some examples arising in free field theories and verify consistency with the expected form of the operator product expansion.

For the general case considered here we relax the normalisation assumption of section 2 and consider a quasi-primary scalar field ϕ , with scale dimension Δ_{ϕ} , for which the two point function has the form

$$\langle \phi(x_1)\phi(x_2)\rangle = \frac{N_\phi}{r_{12}^{\Delta_\phi}}.$$
 (6.1)

The corresponding four point function is then taken to be

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = N_{\phi}^2 \frac{1}{r_{12}^{\Delta_{\phi}} r_{34}^{\Delta_{\phi}}} (1 + g_{\phi}(u, v)),$$
 (6.2)

where $g_{\phi}(u, v)$ satisfies the symmetry conditions

$$g_{\phi}(u,v) = g_{\phi}(u/v, 1/v), \qquad 1 + g_{\phi}(u,v) = \left(\frac{u}{v}\right)^{\Delta_{\phi}} \left(1 + g_{\phi}(v,u)\right),$$
 (6.3)

and we may assume $g_{\phi}(0, v) = 0$. The 1 in (6.2) is then the leading singularity as $r_{12} \to 0$ and represents just the contribution of the identity operator in the operator product expansion of $\phi(x_1)\phi(x_2)$. By virtue of (2.11) $g_{\phi}(u, v)$ should be expanded as

$$g_{\phi}(u,v) = \sum_{\Delta,\ell} c_{\Delta,\ell} u^{\frac{1}{2}(\Delta-\ell)} G^{(\ell)} \left(\frac{1}{2}(\Delta-\ell), \frac{1}{2}(\Delta-\ell), \Delta; u, v \right), \tag{6.4}$$

with $\Delta > 0$. The set of Δ, ℓ which are necessary in (6.4) determines the spectrum of operators which contribute to the operator product expansion of $\phi(x_1)\phi(x_2)$. As a consequence of (2.33) and the first relation in (6.3) ℓ must be even.

For free field theories $g_{\phi}(u, v)$ is analytic in u, 1 - v which corresponds to requiring that any Δ contributing to the sum in (6.4) is an even integer as well. To apply the explicit result (3.11) we first define, with the definitions (3.9),

$$h_{\phi}(z,x) = \frac{z-x}{u} g_{\phi}(u,v), \qquad (6.5)$$

where

$$h_{\phi}(z,x) = -h_{\phi}(x,z) = -h_{\phi}(z',x'), \qquad x' = \frac{x}{x-1}, \quad z' = \frac{z}{z-1}.$$
 (6.6)

Writing $\ell = 2m$, $\Delta - \ell = 2(t+1)$, then, with (3.11), (6.4) becomes

$$h_{\phi}(z,x) = \sum_{t=0} \left(H_t(z) \, x^t F(t,t;2t;x) - H_t(x) \, z^t F(t,t;2t;z) \right),\tag{6.7}$$

where, with $c_{\Delta,\ell} \to c_{mt}$ for m,t integers,

$$H_t(z) = \sum_{m=0}^{\infty} c_{mt} \frac{1}{2^{2m}} z^{2m+t+1} F(2m+t+1, 2m+t+1; 4m+2t+2; z).$$
 (6.8)

By virtue of standard hypergeometric identities this satisfies

$$H_t(z) = (-1)^{t+1} H_t(z')$$
. (6.9)

Furthermore if we compare with (4.3), with $\Delta = \Delta' = \Delta_{\phi}$, d = 4, the contribution of the energy momentum tensor with $\ell = 2$ corresponds to

$$c_{10} = \frac{16\Delta_{\phi}^2}{9C_T} \,. \tag{6.10}$$

To obtain an algorithm for determining c_{mt} it is convenient to write firstly

$$h_{\phi}(z,x) = h_0(z) - h_0(x) + \hat{h}_{\phi}(z,x), \qquad (6.11)$$

where $\hat{h}_{\phi}(z,x)$ is O(zx) and $h_0(z) = -h_0(z')$, from (6.6). We also expand $\hat{h}_{\phi}(z,x)$ in powers of x,

$$\hat{h}_{\phi}(z,x) = \sum_{n=1} \hat{h}_n(z)x^n.$$
(6.12)

The essential equation (6.7) may then be decomposed into independent equations for each power in x. For the terms of $O(x^0)$ we get

$$h_0(z) = H_0(z),$$
 (6.13)

which determines c_{m0} . For the terms $O(x^n)$, n > 0, we have

$$\hat{h}_{n}(z) - \sum_{t=1}^{n-1} \frac{\left((t)_{n-t}\right)^{2}}{(n-t)!(2t)_{n-t}} H_{t}(z)$$

$$+ \sum_{t=1}^{n-1} \sum_{m=0}^{\left[\frac{1}{2}(n-t-1)\right]} c_{mt} \frac{1}{2^{2m}} \frac{\left((2m+t+1)_{n-2m-t-1}\right)^{2}}{(n-2m-t-1)!(4m+2t+1)_{n-2m-t-1}} z^{t} F(t,t;2t;z) = H_{n}(z),$$
(6.14)

which expresses c_{mn} recursively in terms of $c_{mn'}$, n' < n. For n = 1 (6.14) becomes just $\hat{h}_1(z) = H_1(z)$ determining c_{m1} . With further manipulation this can be written as

$$\hat{h}^{(n)}(z) + \sum_{m=0}^{\left[\frac{1}{2}n\right]-1} c_{mt} \frac{1}{2^{2m}} z^t F(t, t; 2t; z) \bigg|_{t=n-1-2m} = H_n(z), \qquad (6.15)$$

where, as also in (6.14), $\left[\frac{1}{2}n\right]$ denotes the integer part and

$$\hat{h}^{(n)}(z) = \hat{h}_n(z) - \sum_{t=1}^{n-1} \frac{\left((t)_{n-t}\right)^2}{(n-t)!(2t)_{n-t}} \hat{h}^{(t)}(z) = \sum_{r=0}^{n-1} (-1)^r \frac{\left((n-r)_r\right)^2}{r!(2n-r-1)_r} \hat{h}_{n-r}(z) . \quad (6.16)$$

It is crucial that the left hand side of (6.15) is compatible with (6.9). For the terms involving $z^t F(t, t; 2t; z)$ this is automatic with the restriction t = n - 1 - 2m. The condition $\hat{h}^{(n)}(z) = (-1)^{n+1} \hat{h}^{(n)}(z')$ follows from (6.16) together with (6.6) which, with (6.11) and (6.12), implies $\hat{h}_n(z) = -\sum_{r=1}^n (-1)^r (r)_{n-r} \hat{h}_r(z')/(n-r)!$. It is also necessary, although less evident, that the left hand side of (6.15) is $O(z^{n+1})$ to match the leading term on the right hand side.

The simplest case is that for a free scalar field, $\phi \to \varphi$ satisfying $\partial^2 \varphi = 0$ and $\Delta_{\varphi} = 1$. With canonical normalisation in (6.1) we have $N_{\varphi} = 1/4\pi^2$. For this case it is easy to calculate that in (6.2)

$$g_{\varphi}(u,v) = u + \frac{u}{v}, \qquad (6.17)$$

so that in (6.11)

$$h_0(z) = z + \frac{z}{1-z}, \qquad \hat{h}_{\varphi}(z,x) = 0.$$
 (6.18)

The only equation to solve is then (6.13). By using algebraic manipulation programmes we have verified that

$$c_{m0} = 2^{2m+1} \frac{\left((2m)!\right)^2}{(4m)!}, \tag{6.19}$$

is compatible with direct calculations for the first 20 terms. Although a formal proof of (6.19) is doubtless possible we have not invested the effort necessary to achieve it. Reassuringly $c_{m0} > 0$ as required by unitarity. Furthermore $c_{10} = \frac{4}{3}$ and then (6.10) gives $C_T = \frac{4}{3}$ which is the correct value for the free scalar theory in four dimensions [10]. Since $c_{mt} = 0, t \geq 1$ only operators with twist $(\Delta - \ell)$ two are present in the operator product expansion. This is also as expected since they are just $\varphi \partial_{\mu_1} \dots \partial_{\mu_\ell} \varphi$ in free field theory.

A further example in free scalar theory arises for $\phi \to \frac{1}{2}\varphi^2$, where we have $\Delta_{\frac{1}{2}\varphi^2} = 2$ and in (6.1) $N_{\frac{1}{2}\varphi^2} = 1/32\pi^4$. In the corresponding four point expression (6.2) we then have

$$g_{\frac{1}{2}\varphi^2}(u,v) = u^2 + \frac{u^2}{v^2} + C\left(u + \frac{u}{v} + \frac{u^2}{v}\right) = g_C(u,v) + C\left(u + \frac{u}{v}\right),$$
 (6.20)

for

$$g_C(u,v) = u^2 + \frac{u^2}{v^2} + C\frac{u^2}{v}$$
 (6.21)

For a single scalar field C=4, more generally for N_s free scalar fields $C=4/N_s$. It is convenient then to allow for arbitrary values of the parameter C. The terms which are O(u) give in this case

$$h_0(z) = C\left(z + \frac{z}{1-z}\right). \tag{6.22}$$

Applying (6.5) and (6.11) to $g_C(u, v)$ as given in (6.21) gives

$$\hat{h}_C(z,x) = (z-x)zx\left(1 + \frac{1}{(1-z)^2(1-x)^2} + \frac{C}{(1-z)(1-x)}\right). \tag{6.23}$$

The expansion in powers of x is straightforward and from (6.16) we find

$$\hat{h}_C^{(n)}(z) = \begin{cases} \frac{(n-1)!^2}{(2n-2)!} \left(z^2 + z'^2 + \left(n(n-1) - C \right) (z+z') \right), & n \text{ odd,} \\ -\frac{(n-1)!^2}{(2n-2)!} \left(z^2 - z'^2 + \left(n(n-1) + C \right) (z-z') \right), & n \text{ even,} \end{cases}$$

$$(6.24)$$

for z' = z/(z-1). The required symmetry under $z \leftrightarrow z'$ is evident. The result for c_{m0} obtained from (6.13) is clearly C times that given by (6.19), which if C = 4 gives the correct value of c_{10} for the same value of C_T . A similar approach for solving (6.15) to that leading to (6.19) suggests, for any C,

$$c_{mt} = 2^{2m} \frac{(2m+t-1)! (2m+t)! ((t-1)!)^2}{(4m+2t-1)! (2t-2)!} (2(2m+1)(m+t) + (-1)^{t+1}C), \quad t \ge 1.$$
(6.25)

This is positive for -2 < C < 4. If C = 4 then $c_{02} = 0$ which reflects the fact that the only potential operator with $\ell = 0$, $\Delta = 6$, $\varphi^2 \partial \varphi \partial \varphi$ is a descendant of φ^4 .

For the case of a free fermion field ψ we may consider $\phi \to \bar{\psi}\psi$, which is a scalar operator with $\Delta_{\bar{\psi}\psi} = 3$. the basic two point function is

$$\langle \psi(x_1)\bar{\psi}(x_2)\rangle = \frac{\gamma \cdot x_{12}}{2\pi^2 r_{12}^2},$$
 (6.26)

from which in (6.1) $N_{\bar{\psi}\psi} = 1/\pi^4$. In (6.2)

$$g_{\bar{\psi}\psi}(u,v) = u^3 + \frac{u^3}{v^3} + \frac{1}{4} \left(u(v-1-u) + \frac{u}{v^2} (1-u-v) + \frac{u^3}{v^2} (u-1-v) \right). \tag{6.27}$$

From (6.5) and (6.11),

$$h_0(z) = \frac{1}{4} \frac{z^3}{(1-z)^2} (2-z), \qquad (6.28)$$

and

$$\hat{h}_{\bar{\psi}\psi}(z,x) = (z-x)z^2x^2\left(1 + \frac{1}{(1-z)^3(1-x)^3} - \frac{1}{4(1-z)^2(1-x)} - \frac{1}{4(1-z)(1-x)^2}\right). \tag{6.29}$$

From (6.12) and (6.16) in this case

$$\hat{h}^{(n)}(z) = \begin{cases} -\frac{(n-1)!n!}{8(2n-3)!} \left(4(z^3 + z'^3) + \left(n(n-1) - 3 \right)(z^2 + z'^2) \right), & n \text{ odd,} \\ \frac{(n-1)!n!}{8(2n-3)!} \left(4(z^3 - z'^3) + \left(n(n-1) - 1 \right)(z^2 - z'^2) \right), & n \text{ even.} \end{cases}$$
(6.30)

In a similar fashion as previously we have found for t=0

$$c_{00} = 0,$$
 $c_{m0} = 2^{2m-2} \frac{(2m)!(2m+1)!}{(4m-1)!}, \quad m = 1, 2, \dots,$ (6.31)

and for t = 1, 2, ...,

$$c_{mt} = 2^{2m-3} \frac{(2m+t+1)! (2m+t)! (t-1)! t!}{(4m+2t-1)! (2t-3)!} (2(2m+1)(m+t) + (-1)^{t+1}).$$
 (6.32)

The operators for t=0 are of the form $\bar{\psi}\gamma_{(\mu_1}\partial_{\mu_2}\dots\partial_{\mu_\ell)}\psi$ and the leading operator which contributes is the $\ell=2$ energy momentum tensor. In this theory $c_{10}=2$ so that (6.10) gives $C_T=8$, as expected for free Dirac fermions. From (6.32) we may note that $c_{m1}=0$ which follows from (6.29) since the leading term is $O(z^2x^2)$. This may be explained since the relevant operators are $\bar{\psi}\partial^r\psi\,\bar{\psi}\partial^s\psi$ with $\Delta=6+r+s$ and $r+s\geq\ell$ so that the minimal t for operators of this form is t=2.

We may also consider the case of free vector fields when we may take $\phi \to \frac{1}{4}F^2 = \frac{1}{4}F_{\mu\nu}F_{\mu\nu}$, the lowest dimension scalar operator with $\Delta_{\frac{1}{4}F^2} = 4$. The essential two point function is

$$\langle F_{\mu\nu}(x_1)F_{\sigma\rho}(x_2)\rangle = \frac{1}{\pi^2 r_{12}^2} \left(I_{\mu\sigma}(x_{12})I_{\nu\rho}(x_{12}) - I_{\mu\rho}(x_{12})I_{\nu\sigma}(x_{12}) \right), \tag{6.33}$$

where the inversion tensor is given by (2.7). It is easy to find in (6.1) $N_{\frac{1}{4}F^2} = 3/\pi^4$. In this case we may calculate in (6.2) (the derivation is sketched in appendix D),

$$g_{\frac{1}{4}F^2}(u,v) = u^4 + \frac{u^4}{v^4} + \frac{2}{9} \left(u(v-1-u)^2 + \frac{u}{v^3} (1-u-v)^2 + \frac{u^4}{v^3} (u-1-v)^2 - u^2 - \frac{u^2}{v^2} - \frac{u^4}{v^2} \right). \eqno(6.34)$$

From (6.5) and (6.11),

$$h_0(z) = \frac{2}{9} \left(z^3 + \frac{z^3}{(1-z)^3} \right). \tag{6.35}$$

and

$$\hat{h}_{\frac{1}{4}F^{2}}(z,x) = (z-x)z^{3}x^{3} \left(1 + \frac{1}{(1-z)^{4}(1-x)^{4}} + \frac{2}{9} \left(\frac{1}{(1-z)(1-x)^{3}} + \frac{1}{(1-z)^{2}(1-x)^{2}} + \frac{1}{(1-z)^{3}(1-x)}\right)\right).$$
(6.36)

As previously we may calculate

$$\hat{h}^{(n)}(z) = \begin{cases} (n-2)\frac{(n-1)!(n+1)!}{72(2n-3)!} \left(9(z^4+z'^4) + \left(n(n-1)-8\right)(z^3+z'^3)\right), & n \text{ odd,} \\ -(n-2)\frac{(n-1)!(n+1)!}{72(2n-3)!} \left(9(z^4-z'^4) + \left(n(n-1)-4\right)(z^3-z'^3)\right), & n \text{ even.} \end{cases}$$
(6.37)

Just as before we then find

$$c_{00} = 0$$
, $c_{m0} = 2^{2m-1}(2m-1)\frac{(2m)!(2m+2)!}{9(4m-1)!}$, $m = 1, 2, \dots$, (6.38)

and for t = 1, 2, ...,

$$c_{mt} = (t-2)2^{2m-4} \frac{(2m+t+2)! (2m+t)! (t-1)! (t+1)!}{9(4m+2t-1)! (2t-3)!} (2m+t-1) \times ((2m+1)(m+t)+(-1)^{t+1}).$$
(6.39)

For t=0 the relevant operators are $F_{(\mu_1|\nu|}\partial_{\mu_2}\dots\partial_{\mu_{\ell-1}}F_{\mu_{\ell})\nu}$, with traces subtracted. For $\ell=2$, giving the energy momentum tensor, we have $c_{10}=16/9$ so that (6.10) gives $C_T=16$, in accord with the required result for free vector theories. It should also be noted that the operator $\frac{1}{4}F^2$ does not appear in the operator product expansion of $\frac{1}{4}F^2(x_1)\frac{1}{4}F^2(x_2)$. We also have in (6.39) $c_{m1}=c_{m2}=0$ which has a similar explanation as for the fermion case, the F^4 operators which are present have $\Delta \geq \ell + 8$.

Finally we consider an example based of a four point function formed by scalar operators $T^{ij} = T^{ji}$ and its conjugate \bar{T}_{ij} which, with i, j = 1, 2 $SU(2)_R$ indices, have dimensions $\Delta_T = \Delta_{\bar{T}} = 2$ and are the lowest dimension scalar operators in a $\mathcal{N} = 2$ supersymmetric theory [22]. The relevant four point function has the general form

$$\langle T^{i_1j_2}(x_1)\bar{T}_{i_2j_2}(x_2)T^{i_3j_3}(x_3)\bar{T}_{i_4j_4}(x_4)\rangle$$

$$= \delta^{(i_1}{}_{i_2}\delta^{j_1)}{}_{j_2}\delta^{(i_3}{}_{i_4}\delta^{j_3)}{}_{j_4}\frac{1}{r_{12}^2r_{34}^2}a(u,v) + \delta^{(i_1}{}_{i_4}\delta^{j_1)}{}_{j_4}\delta^{(i_3}{}_{i_2}\delta^{j_3)}{}_{j_2}\frac{1}{r_{14}^2r_{23}^2}b(u,v)$$

$$+ \delta^{(i_1}{}_{(i_2}\delta^{j_1)}{}_{(i_4}\delta^{(i_3}{}_{j_4)}\delta^{j_3)}{}_{j_2})\frac{1}{r_{12}r_{23}r_{34}r_{14}}c(u,v), \qquad (6.40)$$

with a(u,v) = b(v,u), c(u,v) = c(v,u). In the free case a,b,c are constants and setting, with suitable normalisation a = b = 1, the conformal invariant four point functions for $T\bar{T}$ projected on R = 0, 1, 2 representations are

$$A_0(u,v) = 1 + \frac{1}{3}\frac{u^2}{v^2} + \frac{1}{2}c\frac{u}{v}, \quad A_1(u,v) = \frac{u^2}{v^2} + \frac{u}{v}, \quad A_2(u,v) = \frac{u^2}{v^2}.$$
 (6.41)

For this case there is no symmetry under $u \leftrightarrow u/v, v \leftrightarrow 1/v$ but it is straightforward to decompose each term arising in (6.41) into even and odd pieces. For the even pieces, $u^2/v^2 + u^2$ and u/v + u, the relevant expansion coefficients are given by (6.25), for C = 0, and (6.19).³ For the odd pieces only odd values of ℓ contribute and, with the same definition of t as previously we can then write

$$\frac{u^2}{v^2} - u^2 = -\sum_{m,t} d_{mt} u^{t+1} G^{(2m+1)}(t+1,t+1,2t+2m+3;u,v), \qquad (6.42)$$

where the negative sign is a consequence of the fact that the operators occurring in the operator product expansion are anti-hermitian for odd ℓ in this example. As before we determine

$$d_{mt} = 2^{2m+2} \frac{(2m+t+1)! (2m+t)! ((t-1)!)^{2}}{(4m+2t+1)! (2t-2)!} (2m+2t+1)(m+1).$$
 (6.43)

A similar equation to (6.42) may be written for u/v - v but in this case only t = 0 contributues. This equation may be reduced to

$$\frac{z}{1-z} - z = \sum_{m=0}^{\infty} d_m \frac{1}{2^{2m+1}} z^{2m+2} F(2m+2, 2m+2; 4m+4; z), \qquad (6.44)$$

which determines

$$d_m = 2^{2m+2} \frac{\left((2m+1)!\right)^2}{(4m+2)!}.$$
(6.45)

$$c_{mt}^{(n)} = 2(2m+1)(m+t) \left(\frac{2^m(t-1)!}{(n-1)!(n-2)!}\right)^2 \frac{(2m+t-1)!(2m+t)!(2m+t+n-2)!(t+n-3)!}{(4m+2t-1)!(2m+t-n+2)!(2t-2)!(t-n+1)!}.$$

This is zero for $t = 0, 1, ..., n - 2, n \ge 2$. This formula is also correct for n = 1 when only t = 0 contributes.

³ Perhaps we may note that for $(u/v)^n + u^n$ the expansion coefficients $c_{mt}^{(n)}$ are given by

7. Conclusion

A crucial result of this paper is that it is possible to fine a simple closed form expression for the contribution of an arbitrary spin operator to the four point function. For simplicity, taking $\Delta_1 = \Delta_2$, $\Delta_3 = \Delta_4$, the result from (2.11) and (3.11) is

$$u^{\frac{1}{2}(\Delta-\ell)}G^{(\ell)}(\frac{1}{2}(\Delta-\ell), \frac{1}{2}(\Delta-\ell), \Delta; u, v)$$

$$= \frac{(zx)^{\frac{1}{2}(\Delta-\ell)}}{z-x} \left((-\frac{1}{2}z)^{\ell} z F(\frac{1}{2}(\Delta+\ell), \frac{1}{2}(\Delta+\ell); \Delta+\ell; z) \right) \times F(\frac{1}{2}(\Delta-\ell-2), \frac{1}{2}(\Delta-\ell-2); \Delta-\ell-2; x) - z \leftrightarrow x),$$
(7.1)

for u=zx, v=(1-z)(1-x), as in (3.9). In the previous section we have shown how this may be applied to identify the relevant operators in some simple cases based on free field theories. The variables z, x play an essential role in the expression (7.1) and it would be desirable to find a more direct justification of this result in which the significance of such a parameterisation of u, v was perhaps more transparent. Of course individual contributions of the form (7.1) do not have the required form for $v \sim 0$, $u \sim 1$ and this constrains the contributions of different operators although, as yet, there is no organising principle as in two dimensions. The critical unitarity constraint that the invariant function of u, v which describes the four point function in conformal field theories should be expandible in terms of contributions of the form (7.1), for suitable Δ and ℓ and with positive coefficients in appropriate cases, should become easier to analyse with the explicit expressions for $G^{(\ell)}$ obtained here and exhibited in (7.1). These results should allow further extension of the analysis of four point functions obtained through the AdS/CFT correspondence in terms of the operator product expansion in the large N limit of $\mathcal{N}=4$ supersymmetric gauge theories [23,5].

To see the simplifications obtained by using the variables z, x in another context we mention also some recent results [22] found through the use of superconformal Ward identities, using the harmonic superspace formalism, for the four point function exhibited in (6.40). Although T^{ij}, \bar{T}_{ij} are scalar fields which are the lowest components of $\mathcal{N}=2$ hypermultiplets the superconformal Ward identity lead to constraints on the dependance of a, b, c on the invariants u, v,

$$\frac{\partial}{\partial u}c = \frac{v}{u}\frac{\partial}{\partial v}a - \frac{\partial}{\partial v}b - \left(1 - \frac{1}{v} + \frac{u}{v}\right)\frac{\partial}{\partial u}b,
\frac{\partial}{\partial v}c = \frac{u}{v}\frac{\partial}{\partial u}b - \frac{\partial}{\partial u}a - \left(1 - \frac{1}{u} + \frac{v}{u}\right)\frac{\partial}{\partial v}a.$$
(7.2)

Rewriting these equations in terms of z, x gives

$$\frac{\partial}{\partial x}c = \frac{1-z}{z}\frac{\partial}{\partial x}a + \frac{z}{1-z}\frac{\partial}{\partial x}b, \qquad \frac{\partial}{\partial z}c = \frac{1-x}{x}\frac{\partial}{\partial z}a + \frac{x}{1-x}\frac{\partial}{\partial z}b.$$
 (7.3)

The solutions are then clearly

$$c - \frac{1-z}{z}a - \frac{z}{1-z}b = f(z), \qquad c - \frac{1-x}{x}a - \frac{x}{1-x}b = f(x),$$
 (7.4)

where we have imposed the essential symmetry under $z \leftrightarrow x$. Eliminating c or a gives

$$\frac{1}{u}a - \frac{1}{v}b = \frac{f(z) - f(x)}{z - x}, \qquad \frac{1}{v}c - \frac{1 - v - u}{v^2}b = \frac{\frac{z}{1 - z}f(z) - \frac{x}{1 - x}f(x)}{z - x}, \tag{7.5}$$

which are equivalent to the solutions found in [22]. To satisfy the symmetry under $u \leftrightarrow v$ we must have f(z) = f(1-z). For the free case discussed in section 6 f(z) = 2+c-1/z(1-z). Eden et al [22] have argued that there are no higher order corrections in the interacting theory. It would be interesting to see the implications in the context of the operator product expansion.

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Appendix A. Differential Operators for the Operator Product Expansion

We describe here how to construct differential operators satisfying (2.9) for $\ell = 1, 2$. These cases have been considered previously for a = b and S = d - 1, d respectively in [24,5] and results for any ℓ were given in [25].

For $\ell=1$ and S=a+b+1 we use the definition of Z in (2.4) to write

$$\frac{1}{r_{13}^{a} r_{23}^{b}} Z_{\nu} = \frac{1}{S} \frac{1}{B(a+1,b+1)} \int_{0}^{1} d\alpha \frac{\alpha^{a-1} (1-\alpha)^{b-1}}{(\alpha r_{13} + (1-\alpha)r_{23})^{S}} (b\alpha x_{13} - a(1-\alpha) x_{23})_{\nu}$$

$$= \frac{1}{S} \frac{1}{B(a+1,b+1)} \int_{0}^{1} d\alpha \alpha^{a-1} (1-\alpha)^{b-1} \sum_{n=0}^{\infty} \frac{1}{n!} (A \frac{1}{4} \partial_{x_{2}}^{2})^{n}$$

$$\times \frac{1}{y^{2S}} \left(\frac{S-1}{(S+1-\frac{1}{2}d)_{n}} \alpha (1-\alpha) x_{12\nu} + \frac{1}{(S-\frac{1}{2}d)_{n}} (b\alpha - a(1-\alpha)) y_{\nu} \right), \tag{A.1}$$

where

$$y = x_{23} + \alpha x_{12}, \qquad A = -\alpha (1 - \alpha) r_{12},$$
 (A.2)

so that $\alpha r_{13} + (1 - \alpha)r_{23} = y^2 + A$, and we have used, for $\ell = 0, 1$,

$$\left(\frac{1}{4}\partial^2\right)^n \frac{1}{y^{2S}} y_{\nu_1} \dots y_{\nu_\ell} \mathcal{C}_{\nu_1 \dots \nu_\ell} = (S)_n (S + 1 - \frac{1}{2}d - \ell)_n \frac{1}{y^{2(S+n)}} y_{\nu_1} \dots y_{\nu_\ell} \mathcal{C}_{\nu_1 \dots \nu_\ell} .$$
 (A.3)

We now employ

$$\frac{S-1}{S} \frac{1}{y^{2S}} x_{12\nu} = \frac{1}{y^{2S}} x_{12\mu} I_{\mu\nu}(y) - \frac{1}{S} \frac{\mathrm{d}}{\mathrm{d}\alpha} \frac{1}{y^{2S}} y_{\nu} , \qquad (A.4)$$

and, after integrating by parts, the relation

$$\frac{1}{(S+1-\frac{1}{2}d)_n} \left((a+n)(1-\alpha) - (b+n)\alpha \right) + \frac{1}{(S-\frac{1}{2}d)_n} \left(b\alpha - a(1-\alpha) \right)
= -n \frac{1}{(S-\frac{1}{2}d)_{n+1}} \left((a+1-\frac{1}{2}d)\alpha - (b+1-\frac{1}{2}d)(1-\alpha) \right),$$
(A.5)

to obtain

$$\frac{1}{r_{13}^{a} r_{23}^{b}} Z_{\nu} = \frac{1}{B(a+1,b+1)} \int_{0}^{1} d\alpha \, \alpha^{a} (1-\alpha)^{b} \\
\times \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{(S+1-\frac{1}{2}d)_{n}} \left(A_{4}^{\frac{1}{2}} \partial_{x_{2}}^{2}\right)^{n} \frac{1}{y^{2S}} x_{12\mu} I_{\mu\nu}(y) \\
+ \frac{r_{12}}{B(a+1,b+1)} \int_{0}^{1} d\alpha \, \alpha^{a} (1-\alpha)^{b} \left((a+1-\frac{1}{2}d)\alpha - (b+1-\frac{1}{2}d)(1-\alpha)\right) \\
\times \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{(S+1-\frac{1}{2}d)_{n+1}} \left(A_{4}^{\frac{1}{2}} \partial_{x_{2}}^{2}\right)^{n} \frac{1}{y^{2(S+1)}} y_{\nu} . \tag{A.6}$$

Since

$$\partial_{\mu} \frac{1}{y^{2S}} I_{\mu\nu}(y) = 2(S - d + 1) \frac{1}{y^{2(S+1)}} y_{\nu}, \qquad (A.7)$$

we may therefore write for the $\ell = 1$ case

$$C^{a,b}(s,\partial)_{\mu} = \frac{1}{B(a+1,b+1)} \int_{0}^{1} d\alpha \, \alpha^{a} (1-\alpha)^{b} e^{\alpha s \cdot \partial} \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\alpha (1-\alpha) \frac{1}{4} s^{2} \partial^{2}\right)^{n}$$

$$\times \left(\frac{1}{(S+1-\frac{1}{2}d)_{n}} s_{\mu} \right)$$

$$+ \frac{s^{2}}{(S+1-\frac{1}{2}d)_{n+1}} \frac{(a+1-\frac{1}{2}d)\alpha - (b+1-\frac{1}{2}d)(1-\alpha)}{2(S-d+1)} \, \partial_{\mu}\right).$$
(A.8)

For $\ell=2$, S=a+b+2 the calculation is similar although more tedious. Following the same route as led to (A.1) we have, using (A.3) for $\ell=0,1,2$ and for $\mathcal{C}_{\nu_1\nu_2}$ an arbitrary symmetric traceless tensor,

$$\frac{1}{r_{13}^{a} r_{23}^{b}} Z_{\nu_{1}} Z_{\nu_{2}} C_{\nu_{1}\nu_{2}}
= \frac{1}{S(S+1)} \frac{1}{B(a+2,b+2)} \int_{0}^{1} d\alpha \, \alpha^{a-1} (1-\alpha)^{b-1} \sum_{n=0}^{\infty} \frac{1}{n!} \left(A \frac{1}{4} \partial_{x_{2}}^{2} \right)^{n} \frac{1}{y^{2S}} C_{\nu_{1}\nu_{2}}
\times \left(\frac{S(S-1)}{(S+1-\frac{1}{2}d)_{n}} \alpha^{2} (1-\alpha)^{2} x_{12\nu_{1}} x_{12\nu_{2}} \right)
+ \frac{2(S-1)}{(S-\frac{1}{2}d)_{n}} \alpha (1-\alpha) \left((b+1)\alpha - (a+1)(1-\alpha) \right) x_{12\nu_{1}} y_{\nu_{2}}
+ \frac{1}{(S-1-\frac{1}{2}d)_{n}} \left(b(b+1)\alpha^{2} + a(a+1)(1-\alpha)^{2} - 2(b+1)(a+1)\alpha(1-\alpha) \right) y_{\nu_{1}} y_{\nu_{2}} \right),$$
(A.9)

We may now write

$$\frac{S-1}{S+1} \frac{1}{y^{2S}} x_{12\nu_1} x_{12\nu_1} = \frac{1}{y^{2S}} x_{12\mu_1} x_{12\mu_1} I_{\mu_1\nu_1}(y) I_{\mu_2\nu_2}(y)
- \frac{S-1}{S(S+1)} \frac{\mathrm{d}}{\mathrm{d}\alpha} \frac{1}{y^{2S}} (x_{12\nu_1} y_{\nu_2} + x_{12\nu_2} y_{\nu_1})
- \frac{1}{S(S+1)} \frac{\mathrm{d}^2}{\mathrm{d}\alpha^2} \frac{1}{y^{2S}} y_{\nu_1} y_{\nu_2} - \frac{2}{S+1} \frac{r_{12}}{y^{2(S+1)}} y_{\nu_1} y_{\nu_2}.$$
(A.10)

The resulting expression has three pieces, the first of which comes from the first line of (A.10) in (A.9) and is readily seen to be

$$\frac{1}{B(a+2,b+2)} \int_{0}^{1} d\alpha \, \alpha^{a+1} (1-\alpha)^{b+1} \times \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{(S+1-\frac{1}{2}d)_{n}} \left(A \, \frac{1}{4} \partial_{x_{2}}^{2}\right)^{n} \frac{1}{y^{2S}} x_{12\mu_{1}} x_{12\mu_{1}} I_{\mu_{1}\nu_{1}}(y) I_{\mu_{2}\nu_{2}}(y) \mathcal{C}_{\nu_{1}\nu_{2}}. \tag{A.11}$$

After integrating by parts the remaining terms in (A.10) then in addition to (A.11) we have

$$\frac{S-1}{S+1} \frac{2r_{12}}{B(a+2,b+2)} \int_{0}^{1} d\alpha \, \alpha^{a+1} (1-\alpha)^{b+1} \left((a+1-\frac{1}{2}d)\alpha - (b+1-\frac{1}{2}d)(1-\alpha) \right) \\
\times \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{(S+1-\frac{1}{2}d)_{n+1}} \left(A \frac{1}{4} \partial_{x_{2}}^{2} \right)^{n} \frac{1}{y^{2(S+1)}} x_{12\nu_{1}} y_{\nu_{2}} C_{\nu_{1}\nu_{2}} \\
+ \frac{1}{S+1} \frac{r_{12}}{B(a+2,b+2)} \int_{0}^{1} d\alpha \, \alpha^{a} (1-\alpha)^{b} \\
\times \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{(S-\frac{1}{2}d)_{n+2}} J - \frac{2}{(S+1-\frac{1}{2}d)_{n}} \alpha (1-\alpha) \right) \left(A \frac{1}{4} \partial_{x_{2}}^{2} \right)^{n} \frac{1}{y^{2(S+1)}} y_{\nu_{1}} y_{\nu_{2}} C_{\nu_{1}\nu_{2}}, \\
J = (a+1-\frac{1}{2}d) \left((S+b-\frac{1}{2}d)n+2(S-\frac{1}{2}d)(b+1) \right) \alpha^{2} \\
+ (b+1-\frac{1}{2}d) \left((S+a-\frac{1}{2}d)n+2(S-\frac{1}{2}d)(a+1) \right) (1-\alpha)^{2} \\
-2 \left(((S-\frac{1}{2}d)(S-\frac{1}{2}d-1)-(a+1)(b+1))n \right) \\
+ (S-\frac{1}{2}d) \left((S-\frac{1}{2}d-1)(S+1)-2(a+1)(b+1) \right) \alpha (1-\alpha). \tag{A.12}$$

We may now use, similarly to (A.4),

$$\frac{S-1}{S+1} \frac{1}{y^{2(S+1)}} x_{12\nu_1} y_{\nu_2} C_{\nu_1 \nu_2} = \frac{1}{y^{2(S+1)}} x_{12\mu_1} I_{\mu_1 \nu_1}(y) y_{\nu_2} C_{\nu_1 \nu_2}
- \frac{1}{S+1} \frac{\mathrm{d}}{\mathrm{d}\alpha} \frac{1}{y^{2(S+1)}} y_{\nu_1} y_{\nu_2} C_{\nu_1 \nu_2},$$
(A.13)

so that the first term in (A.12) gives a contribution,

$$\frac{r_{12}}{B(a+2,b+2)} \int_0^1 d\alpha \,\alpha^{a+1} (1-\alpha)^{b+1} \left((a+1-\frac{1}{2}d)\alpha - (b+1-\frac{1}{2}d)(1-\alpha) \right) \\
\times \sum_{n=0}^\infty \frac{1}{n!} \frac{1}{(S+1-\frac{1}{2}d)_{n+1}} \left(A \,\frac{1}{4} \partial_{x_2}^{\ 2} \right)^n \frac{1}{y^{2(S+1)}} x_{12\mu_1} I_{\mu_1\nu_1}(y) y_{\nu_2} \mathcal{C}_{\nu_1\nu_2} . \tag{A.14}$$

After another integration by parts the final contribution becomes

$$\frac{r_{12}^{2}}{B(a+2,b+2)} \int_{0}^{1} d\alpha \, \alpha^{a+1} (1-\alpha)^{b+1} \times \left((a+1-\frac{1}{2}d)(a+2-\frac{1}{2}d)\alpha^{2} + (b+1-\frac{1}{2}d)(b+2-\frac{1}{2}d)(1-\alpha)^{2} - 2(a+1-\frac{1}{2}d)(b+1-\frac{1}{2}d)\alpha(1-\alpha) \right) \times \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{(S+1-\frac{1}{2}d)_{n+2}} \left(A \frac{1}{4} \partial_{x_{2}}^{2} \right)^{n} \frac{1}{y^{2(S+2)}} y_{\nu_{1}} y_{\nu_{2}} C_{\nu_{1}\nu_{2}} . \tag{A.15}$$

Just as with (A.7) we may now use

$$\partial_{\mu_2} \frac{1}{y^{2S}} I_{\mu_1 \nu_1}(y) I_{\mu_2 \nu_2}(y) \mathcal{C}_{\nu_1 \nu_2} = 2(S-d) \frac{1}{y^{2(S+1)}} I_{\mu_1 \nu_1}(y) y_{\nu_2} \mathcal{C}_{\nu_1 \nu_2} ,$$

$$\partial_{\mu_1} \partial_{\mu_2} \frac{1}{y^{2S}} I_{\mu_1 \nu_1}(y) I_{\mu_2 \nu_2}(y) \mathcal{C}_{\nu_1 \nu_2} = 2(S-d)(S-d+1) \frac{1}{y^{2(S+1)}} y_{\nu_1} y_{\nu_2} \mathcal{C}_{\nu_1 \nu_2} ,$$
(A.16)

in (A.14) and (A.15) to write the sum of (A.11), (A.14) and (A.15) in the form

$$C^{a,b}(x_{12}, \partial_{x_2})_{\mu_1 \mu_2} \frac{1}{r_{23}^S} I_{\mu_1 \nu_1}(x_{23}) I_{\mu_2 \nu_2}(x_{23}) \mathcal{C}_{\nu_1 \nu_2}, \qquad (A.17)$$

defining $C^{a,b}(s,\partial)$ for $\ell=2$.

Both this result and (A.8) may be expressed in terms of the differential operators

$$C_{\kappa}^{a,b}(s,\partial) = \frac{1}{B(a,b)} \int_{0}^{1} d\alpha \,\alpha^{a-1} (1-\alpha)^{b-1} e^{\alpha s \cdot \partial} \sum_{n=0}^{\infty} \frac{1}{n!(\kappa)_{n}} \left(-\alpha (1-\alpha) \frac{1}{4} s^{2} \partial^{2}\right)^{n}. \quad (A.18)$$

For any ℓ , neglecting terms $O(s^2)$, $C^{a,b}(s,\partial)\cdot C$ has the form $C^{a+\ell,b+\ell}_{S+1-\frac{1}{2}d}(s,\partial)s_{\mu_1}\dots s_{\mu_\ell}C_{\mu_1\dots\mu_\ell}$.

Appendix B. Calculation for $\ell = 1$

Here we consider (2.13) for $\ell = 1$,

$$C^{a,b}(x_{12}, \partial_{x_2})_{\mu} \frac{1}{r_{23}^e r_{24}^f} Y_{\mu} = \frac{1}{r_{14}^a r_{24}^b} \left(\frac{r_{14}}{r_{13}}\right)^e G^{(1)}(b, e, S; u, v), \qquad (B.1)$$

with $C^{a,b}(x_{12}, \partial_{x_2})_{\mu}$ given by (A.8), and evaluate directly $G^{(1)}(b, e, S; u, v)$ following a similar route to that described for $\ell = 0$ in [26] and sketched in [6]. Using an integral representation similar to (A.1) the first term (A.8) leads to a contribution,

$$\frac{1}{S} \frac{1}{B(a+1,b+1)B(e+1,f+1)} \int_{0}^{1} d\alpha \, \alpha^{a} (1-\alpha)^{b} \int_{0}^{1} d\beta \, \beta^{e-1} (1-\beta)^{f-1} \\
\times \sum_{m,n=0} \frac{1}{m!n!} \frac{(S)_{n}}{(S+1-\frac{1}{2}d)_{m}} A^{m} B^{n} \left(\frac{1}{4}z^{2} \partial_{z}^{2}\right)^{m} \\
\times \frac{1}{z^{2(S+n)}} x_{12} \cdot \left((S-1)\beta(1-\beta)x_{34} + \left(e(1-\beta) - f\beta\right)z\right), \tag{B.2}$$

for

$$z = x_{24} + \alpha x_{12} - \beta x_{34}, \qquad A = -\alpha (1 - \alpha) r_{12}, \quad B = -\beta (1 - \beta) r_{34}.$$
 (B.3)

Carrying out the differentiation according to (A.3) and using

$$2(S+m+n-1)\frac{1}{z^{2(S+m+n)}}x_{12}\cdot z = -\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{1}{z^{2(S+m+n-1)}},$$
 (B.4)

allows (B.2) to be rewritten as

$$\frac{S-1}{S} \frac{x_{12} \cdot x_{34}}{B(a+1,b+1)B(e+1,f+1)} \int_{0}^{1} d\alpha \, \alpha^{a} (1-\alpha)^{b} \int_{0}^{1} d\beta \, \beta^{e} (1-\beta)^{f}$$

$$\times \sum_{m,n=0} \frac{1}{m!n!} \frac{(S)_{m+n}(S+1-\frac{1}{2}d)_{m+n}}{(S+1-\frac{1}{2}d)_{m}(S+1-\frac{1}{2}d)_{n}} \frac{A^{m}B^{n}}{z^{2(S+m+n)}}$$

$$+ \frac{1}{2S(S-1)} \frac{1}{B(a+1,b+1)B(e+1,f+1)} \int_{0}^{1} d\alpha \, \alpha^{a-1} (1-\alpha)^{b-1} \int_{0}^{1} d\beta \, \beta^{e-1} (1-\beta)^{f-1}$$

$$\times (a(1-\alpha)-b\alpha) \left(e(1-\beta)-f\beta\right)$$

$$\times \sum_{m,n=0} \frac{1}{m!n!} \frac{(S-1)_{m+n}(S-\frac{1}{2}d)_{m+n}}{(S-\frac{1}{2}d)_{m}(S-\frac{1}{2}d)_{n}} \frac{A^{m}B^{n}}{z^{2(S-1+m+n)}}$$

$$+ \frac{1}{2S} \frac{r_{12}}{B(a+1,b+1)B(e+1,f+1)} \int_{0}^{1} d\alpha \, \alpha^{a} (1-\alpha)^{b} \int_{0}^{1} d\beta \, \beta^{e-1} (1-\beta)^{f-1}$$

$$\times \left((a-\frac{1}{2}d+1)\alpha - (b-\frac{1}{2}d+1)(1-\alpha)\right) \left(e(1-\beta)-f\beta\right)$$

$$\times \sum_{m,n=0} \frac{1}{m!n!} \frac{(S)_{m+n}(S-\frac{1}{2}d)_{m+n+1}}{(S-\frac{1}{2}d)_{m+2}(S-\frac{1}{2}d)_{n}} \frac{A^{m}B^{n}}{z^{2(S+m+n)}}. \tag{B.5}$$

The term in (A.8) involving ∂_{μ} also gives a contribution

$$-\frac{1}{2S}\frac{r_{12}}{B(a+1,b+1)B(e+1,f+1)}\int_{0}^{1}\mathrm{d}\alpha\,\alpha^{a}(1-\alpha)^{b}\int_{0}^{1}\mathrm{d}\beta\,\beta^{e-1}(1-\beta)^{f-1} \\ \times \left(\left(a-\frac{1}{2}d+1\right)\alpha-\left(b-\frac{1}{2}d+1\right)(1-\alpha)\right)\left(e(1-\beta)-f\beta\right) \\ \times \sum_{m,n=0}^{\infty}\frac{1}{m!n!}\frac{(S)_{m+n}(S+1-\frac{1}{2}d)_{m+n}}{(S+1-\frac{1}{2}d)_{m+1}}\frac{A^{m}B^{n}}{z^{2(S+m+n)}} \\ +\frac{1}{2}\frac{e-f}{S-d+1}\frac{r_{12}\,r_{34}}{B(a+1,b+1)B(e+1,f+1)}\int_{0}^{1}\mathrm{d}\alpha\,\alpha^{a}(1-\alpha)^{b}\int_{0}^{1}\mathrm{d}\beta\,\beta^{e}(1-\beta)^{f} \\ \times \left(\left(a-\frac{1}{2}d+1\right)\alpha-\left(b-\frac{1}{2}d+1\right)(1-\alpha)\right) \\ \times \sum_{m,n=0}^{\infty}\frac{1}{m!n!}\frac{(S+1)_{m+n}(S+2-\frac{1}{2}d)_{m+n}}{(S+1-\frac{1}{2}d)_{m+1}(S+2-\frac{1}{2}d)_{n}}\frac{A^{m}B^{n}}{z^{2(S+1+m+n)}}.$$

To obtain the form shown in (B.1) requires three critical steps. First, writing $z^2 = A + B + C$ where

$$C = \alpha(1-\beta)r_{14} + \beta(1-\alpha)r_{23} + \alpha\beta r_{13} + (1-\alpha)(1-\beta)r_{24},$$
(B.7)

we use

$$\sum_{m,n=0} \frac{1}{m!n!} \frac{(\lambda)_{m+n}(\kappa)_{m+n}}{(\kappa)_m(\kappa)_n} \frac{A^m B^n}{z^{2(\lambda+m+n)}} = \frac{1}{C^{\lambda}} \sum_{m=0} \frac{(\lambda)_{2m}}{m!(\kappa)_m} \left(\frac{AB}{C^2}\right)^m. \tag{B.8}$$

Secondly we require the α, β integrals to be of the form

$$\int_{0}^{1} d\alpha \, \alpha^{a-1} (1-\alpha)^{b-1} \frac{1}{C^{a+b}} = B(a,b) \, \frac{1}{s^{a} \, t^{b}} \,,$$

$$s = \beta r_{13} + (1-\beta)r_{14} \,, \ t = \beta r_{23} + (1-\beta)r_{24} \,,$$

$$\int_{0}^{1} d\beta \, \beta^{e-1} (1-\beta)^{f-1} \frac{1}{C^{e+f}} = B(e,f) \, \frac{1}{\hat{s}^{e} \, \hat{t}^{f}} \,,$$

$$\hat{s} = \alpha r_{13} + (1-\alpha)r_{23} \,, \ \hat{t} = \alpha r_{14} + (1-\alpha)r_{24} \,,$$
(B.9)

and then finally, for $\lambda = a + b = e + f$,

$$\int_{0}^{1} d\beta \, \beta^{e-1} (1-\beta)^{f-1} \, \frac{1}{s^{a} \, t^{b}} = B(e, f) \, \frac{1}{r_{14}^{a} \, r_{24}^{b}} \left(\frac{r_{14}}{r_{13}}\right)^{e} F(b, e; \lambda; 1-v) \,,$$

$$\int_{0}^{1} d\alpha \, \alpha^{a-1} (1-\alpha)^{b-1} \, \frac{1}{\hat{s}^{e} \, \hat{t}^{f}} = B(a, b) \, \frac{1}{r_{14}^{a} \, r_{24}^{b}} \left(\frac{r_{14}}{r_{13}}\right)^{e} F(b, e; \lambda; 1-v) \,. \tag{B.10}$$

To apply these results to (B.5) and (B.6) requires some manipulation. We may immediately apply (B.8) to the first two terms in (B.5). In the first term we write

$$2\alpha\beta x_{12} \cdot x_{34} = -C + r_{24} + \alpha(r_{14} - r_{24}) + \beta(r_{23} - r_{24}), \qquad (B.11)$$

and the -C piece may be combined with the second term in (B.5), using $a(1-\alpha) - b\alpha = (S-1)(1-\alpha) - b$, $e(1-\beta) - f\beta = (S-1)(1-\beta) - f$, so that either the α or β integration may be carried out using (B.9). After further algebra these two terms become

$$\frac{1}{r_{14}^{a}r_{24}^{b}} \left(\frac{r_{14}}{r_{13}}\right)^{e} \frac{S(S-1)}{2ae} \left(G(b,e,S-\frac{1}{2}d,S-1;u,1-v)\right) \\ -G(b+1,e,S+1-\frac{1}{2}d,S;u,1-v)\right) \\ -\frac{1}{2}(S-1) \frac{r_{12}r_{34}}{B(a+1,b+1)B(e+1,f+1)} \int_{0}^{1} d\alpha \, \alpha^{a} (1-\alpha)^{b+1} \int_{0}^{1} d\beta \, \beta^{e} (1-\beta)^{f+1} \\ \times (S-d+1) \frac{1}{C^{S+1}} \sum_{m=0}^{\infty} \frac{1}{m!} \frac{(S+1)_{2m}}{(S-\frac{1}{2}d)_{m+2}} \left(\frac{AB}{C^{2}}\right)^{m} \\ +r_{12}r_{34} \frac{S-1}{2a} \frac{a-\frac{1}{2}d+1}{B(e+1,f+1)} \int_{0}^{1} d\beta \, \beta^{e} (1-\beta)^{f+1} \frac{1}{s^{a+1}t^{b+1}} \\ \times \sum_{m=0}^{\infty} \frac{1}{m!} \frac{(a)_{m+1}(b+1)_{m}}{(S-\frac{1}{2}d)_{m+2}} \left(\frac{r_{12}r_{34}\beta(1-\beta)}{st}\right)^{m} \\ +r_{12}r_{34} \frac{S-1}{2e} \frac{e-\frac{1}{2}d+1}{B(a+1,b+1)} \int_{0}^{1} d\alpha \, \alpha^{a} (1-\alpha)^{b+1} \frac{1}{\hat{s}^{a+1}\hat{t}^{b+1}} \\ \times \sum_{m=0}^{\infty} \frac{1}{m!} \frac{(e)_{m+1}(f+1)_{m}}{(S-\frac{1}{2}d)_{m+2}} \left(\frac{r_{12}r_{34}\alpha(1-\alpha)}{s\hat{t}}\right)^{m}, \tag{B.12}$$

with G defined by the series in (2.32). Combining the last term in (B.5) with (B.6) leads to

$$\frac{1}{2} \frac{S-1}{S-d+1} \frac{r_{12} r_{34}}{B(a+1,b+1)B(e+1,f+1)} \int_{0}^{1} d\alpha \, \alpha^{a} (1-\alpha)^{b} \int_{0}^{1} d\beta \, \beta^{e} (1-\beta)^{f} \\
\times \left((a - \frac{1}{2}d+1)\alpha - (b - \frac{1}{2}d+1)(1-\alpha) \right) \left((e - \frac{1}{2}d+1)\beta - (f - \frac{1}{2}d+1)(1-\beta) \right) \\
\times \sum_{m,n=0} \frac{1}{m!n!} \frac{(S+1)_{m+n} (S - \frac{1}{2}d)_{m+n+2}}{(S - \frac{1}{2}d)_{m+2}} \frac{A^{m} B^{n}}{z^{2(S+1+m+n)}}.$$
(B.13)

This is symmetric and the summation formula (B.8) may now be applied and the result combined with the corresponding term in (B.12) using

$$\left((a - \frac{1}{2}d + 1)\alpha - (b - \frac{1}{2}d + 1)(1 - \alpha) \right) \left((e - \frac{1}{2}d + 1)\beta - (f - \frac{1}{2}d + 1)(1 - \beta) \right)
- (S - d + 1)^{2}(1 - \alpha)(1 - \beta)
= (a - \frac{1}{2}d + 1)(e - \frac{1}{2}d + 1)
- (S - d + 1) \left((a - \frac{1}{2}d + 1)(1 - \beta) + (e - \frac{1}{2}d + 1)(1 - \alpha) \right).$$
(B.14)

The integrals arising from the terms in the last line involving $1 - \alpha$, $1 - \beta$ then cancel the remaining α , β integrals in (B.12) leaving just the first term in (B.14) for which the associated integral may be evaluated giving

$$\frac{1}{r_{14}^{a} r_{24}^{b}} \left(\frac{r_{14}}{r_{13}}\right)^{e} \frac{1}{2} (S-1) \frac{(a-\frac{1}{2}d+1)(e-\frac{1}{2}d+1)}{(S-d+1)(S-\frac{1}{2}d)(S-\frac{1}{2}d+1)} \times uG(b+1,e+1,S-\frac{1}{2}d+2,S+1;u,1-v).$$
(B.15)

In consequence from (B.12) and (B.15) we have finally altogether

$$G^{(1)}(b, e, S; u, v) = \frac{S(S-1)}{2ae} \left(G(b, e, S - \frac{1}{2}d, S - 1; u, 1 - v) - G(b+1, e, S+1 - \frac{1}{2}d, S; u, 1 - v) \right)$$

$$+ \frac{(S-1)(a - \frac{1}{2}d+1)(e - \frac{1}{2}d+1)}{2(S-d+1)(S - \frac{1}{2}d)(S - \frac{1}{2}d+1)} uG(b+1, e+1, S - \frac{1}{2}d+2, S+1; u, 1 - v) .$$
(B.16)

The result (B.16) is different from that which is given by (2.30) for $\ell = 1$ together with (2.31). To show the equivalence of the two expressions it is sufficient to use the following

relations for the function G defined by (2.32),

$$\alpha vG(\alpha+1,\beta+1,\gamma,\delta+1;u,1-v) + (\delta-\alpha-\beta)G(\alpha,\beta+1,\gamma,\delta+1;u,1-v) - (\delta-\beta)G(\alpha,\beta,\gamma,\delta+1;u,1-v)$$

$$= \frac{\alpha(\delta-\beta)(\delta-\alpha-\beta)}{\gamma(\delta+1)} uG(\alpha+1,\beta+1,\gamma+1,\delta+2;u,1-v),$$

$$\beta G(\alpha,\beta+1,\gamma,\delta+1;u,1-v) + (\delta-\beta)G(\alpha,\beta,\gamma,\delta+1;u,1-v) - \delta G(\alpha,\beta,\gamma-1,\delta;u,1-v)$$

$$= -(\delta-\alpha-\gamma+1) \frac{\alpha\beta(\delta-\beta)}{(\gamma-1)\gamma(\delta+1)} uG(\alpha+1,\beta+1,\gamma+1,\delta+2;u,1-v),$$

$$\alpha G(\alpha+1,\beta,\gamma,\delta+1;u,1-v) - \beta G(\alpha,\beta+1,\gamma,\delta+1;u,1-v) - (\alpha-\beta)G(\alpha,\beta,\gamma,\delta+1;u,1-v)$$

$$= -(\alpha-\beta) \frac{\alpha\beta}{\gamma(\delta+1)} uG(\alpha+1,\beta+1,\gamma+1,\delta+2;u,1-v).$$
(B.17)

These may be obtained from relations given in [6] which express $G(\alpha, \beta, \gamma, \delta; u, 1 - v)$ in terms of $G(\alpha', \beta', \gamma', \delta'; u, 1 - v)$ with $\delta' - \gamma' = \delta - \gamma + 1$.

Appendix C. Identities for H

From the definition (5.9) and properties of the function G the following relations were obtained in [6],

$$H(\alpha, \beta, \gamma, \delta; u, v) = v^{-\alpha} H(\alpha, \delta - \beta, \gamma, \delta; u/v, 1/v)$$
(C.1a)

$$= v^{\delta - \alpha - \beta} H(\delta - \alpha, \delta - \beta, \gamma, \delta; u, v)$$
 (C.1b)

$$= H(\alpha, \beta, \alpha + \beta - \delta + 1, \alpha + \beta - \gamma + 1; v, u)$$
 (C.1c)

 $= u^{1-\gamma}H(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma, \delta - 2\gamma + 2; u, v). \quad (C.1d)$

In terms of the result (5.8) for the four point function these give

$$\overline{D}_{\Delta_{1} \Delta_{2} \Delta_{3} \Delta_{4}}(u, v) = \overline{D}_{\Sigma - \Delta_{3} \Sigma - \Delta_{4} \Sigma - \Delta_{1} \Sigma - \Delta_{2}}(u, v)$$

$$= v^{-\Delta_{2}} \overline{D}_{\Delta_{1} \Delta_{2} \Delta_{4} \Delta_{3}}(u/v, 1/v)$$

$$= v^{\Delta_{4} - \Sigma} \overline{D}_{\Delta_{2} \Delta_{1} \Delta_{3} \Delta_{4}}(u/v, 1/v)$$

$$= v^{\Delta_{1} + \Delta_{4} - \Sigma} \overline{D}_{\Delta_{2} \Delta_{1} \Delta_{4} \Delta_{3}}(u, v)$$

$$= \overline{D}_{\Delta_{3} \Delta_{2} \Delta_{1} \Delta_{4}}(v, u)$$

$$= u^{\Delta_{3} + \Delta_{4} - \Sigma} \overline{D}_{\Delta_{4} \Delta_{2} \Delta_{2} \Delta_{1}}(u, v),$$
(C.2)

which reflect the symmetry under the interchanges $x_i, \Delta_i \leftrightarrow x_j, \Delta_j$ for various i, j.

Results obtained in [6] for G also imply

$$(\alpha - \beta)H(\alpha, \beta, \gamma, \delta; u, v) = H(\alpha + 1, \beta, \gamma, \delta + 1; u, v) - H(\alpha, \beta + 1, \gamma, \delta + 1; u, v),$$

$$(\delta - \alpha - \beta)H(\alpha, \beta, \gamma, \delta; u, v) = H(\alpha, \beta, \gamma, \delta + 1; u, v) - vH(\alpha + 1, \beta + 1, \gamma, \delta + 1; u, v),$$

$$(1 - \gamma)H(\alpha, \beta, \gamma, \delta; u, v) = H(\alpha, \beta, \gamma - 1, \delta; u, v) - uH(\alpha + 1, \beta + 1, \gamma + 1, \delta + 2; u, v),$$

$$(\delta - \gamma - \beta + 1)H(\alpha, \beta, \gamma, \delta; u, v) = H(\alpha, \beta, \gamma - 1, \delta; u, v) + H(\alpha + 1, \beta, \gamma, \delta + 1; u, v)$$

$$+ H(\alpha, \beta, \gamma, \delta + 1; u, v).$$
(C.3)

The first three relations imply

$$(\Delta_{2} + \Delta_{4} - \Sigma)\overline{D}_{\Delta_{1} \Delta_{2} \Delta_{3} \Delta_{4}}(u, v) = \overline{D}_{\Delta_{1} \Delta_{2} + 1 \Delta_{3} \Delta_{4} + 1}(u, v) - \overline{D}_{\Delta_{1} + 1 \Delta_{2} \Delta_{3} + 1 \Delta_{4}}(u, v),$$

$$(\Delta_{1} + \Delta_{4} - \Sigma)\overline{D}_{\Delta_{1} \Delta_{2} \Delta_{3} \Delta_{4}}(u, v) = \overline{D}_{\Delta_{1} + 1 \Delta_{2} \Delta_{3} \Delta_{4} + 1}(u, v) - v\overline{D}_{\Delta_{1} \Delta_{2} + 1 \Delta_{3} + 1 \Delta_{4}}(u, v),$$

$$(\Delta_{3} + \Delta_{4} - \Sigma)\overline{D}_{\Delta_{1} \Delta_{2} \Delta_{3} \Delta_{4}}(u, v) = \overline{D}_{\Delta_{1} \Delta_{2} \Delta_{3} + 1 \Delta_{4} + 1}(u, v) - u\overline{D}_{\Delta_{1} + 1 \Delta_{2} + 1 \Delta_{3} \Delta_{4}}(u, v),$$

$$(C.4)$$

which are equivalent to results obtained in [13]. The last relation in (C.3) also gives

$$\Delta_{4}\overline{D}_{\Delta_{1} \Delta_{2} \Delta_{3} \Delta_{4}}(u,v) = \overline{D}_{\Delta_{1} \Delta_{2} \Delta_{3}+1 \Delta_{4}+1}(u,v) + \overline{D}_{\Delta_{1} \Delta_{2}+1 \Delta_{3} \Delta_{4}+1}(u,v)
+ \overline{D}_{\Delta_{1}+1 \Delta_{2} \Delta_{3} \Delta_{4}+1}(u,v).$$
(C.5)

We also have

$$\partial_v H(\alpha, \beta, \gamma, \delta; u, v) = -H(\alpha + 1, \beta + 1, \gamma, \delta + 1; u, v),$$

$$\partial_u H(\alpha, \beta, \gamma, \delta; u, v) = -H(\alpha + 1, \beta + 1, \gamma + 1, \delta + 2; u, v),$$
(C.6)

or

$$\partial_{v}\overline{D}_{\Delta_{1}\,\Delta_{2}\,\Delta_{3}\,\Delta_{4}}(u,v) = -\overline{D}_{\Delta_{1}\,\Delta_{2}+1\,\Delta_{3}+1\,\Delta_{4}}(u,v)$$

$$\partial_{u}\overline{D}_{\Delta_{1}\,\Delta_{2}\,\Delta_{3}\,\Delta_{4}}(u,v) = -\overline{D}_{\Delta_{1}+1\,\Delta_{2}+1\,\Delta_{3}\,\Delta_{4}}(u,v).$$
(C.7)

When γ is an integer the definition (5.9) gives $\ln u$ terms as well as an expansion in powers in u, 1-v. For $\gamma=n=1,2,\ldots$ we easily find the terms involving $\ln u$ as

$$H(\alpha, \beta, n, \delta; u, v)_{\text{log.terms}} = \ln u \frac{(-1)^n}{(n-1)!} \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\delta - \alpha)\Gamma(\delta - \beta)}{\Gamma(\delta)} G(\alpha, \beta, n, \delta; u, 1 - v).$$
(C.8)

If $\gamma = n = 0, -1, -2, ...$ then the leading log term is $u^{1-n} \ln u$ and the corresponding formula may be obtained from (C.8) using (C.1d). By virtue of (C.1c) there are similarly terms involving $\ln v$ in an expansion for $v \sim 0$ when $\alpha + \beta - \delta$ is an integer. For $\gamma = 1$, which is of relevance in (5.7) when $\Delta_1 + \Delta_2 = \Delta_3 + \Delta_4$, a complete formula for the additional

power terms as well as the log. terms displayed in (C.8) is given in [6]. For $\gamma = 0$ the corresponding formula is

$$H(\alpha, \beta, 0, \delta; u, v) = \frac{1}{\Gamma(\delta)} \Gamma(\alpha) \Gamma(\beta) \Gamma(\delta - \alpha) \Gamma(\delta - \beta) \left\{ F(\alpha, \beta; \delta; 1 - v) - \sum_{m=1, n=0} \frac{(\delta - \alpha)_m (\delta - \beta)_m}{(m-1)! \, m!} \frac{(\alpha)_{m+n} (\beta)_{m+n}}{n! \, (\delta)_{2m+n}} (-\ln u + f_{mn}) u^m (1 - v)^n \right\},$$

$$f_{mn} = \psi(1+m) + \psi(m) + 2\psi(\delta + 2m+n) - \psi(\delta - \alpha + m) - \psi(\delta - \beta + m) - \psi(\alpha + m+n) - \psi(\beta + m+n). \tag{C.9}$$

With the aid of the above relations we may determine $\overline{D}_{n_1n_2n_3n_4}(u,v)$ for $n_i=1,2,\ldots,$ $\sum_i n_i$ even, in terms of $\overline{D}_{1111}(u,v)$. For example if $n_i=n=1,2,\ldots$ then, from (5.8), $\overline{D}_{nnn}(u,v)=H(n,n,1,2n;u,v)$, where

$$H(n, n, 1, 2n; u, v) = v^{-n}H(n, n, 1, 2n; u/v, 1/v) = H(n, n, 1, 2n; v, u),$$
(C.10)

and using (C.6) and (C.1b, d) we may obtain the recurrence relation

$$H(n+1, n+1, 1, 2n+2; u, v) = \partial_u u \partial_u H(n, n, 1, 2n; u, v) = \partial_v v \partial_v H(n, n, 1, 2n; u, v).$$
(C.11)

For the starting point \overline{D}_{1111} , with the definitions in (3.9) for z, x (note that $(z-x)^2 = 1 + u^2 + v^2 - 2u - 2v - 2uv$), we have

$$\overline{D}_{1111}(u,v) = H(1,1,1,2;u,v) = \frac{1}{z-x}\Phi(z,x), \qquad (C.12)$$

with

$$\Phi(z,x) = \ln zx \, \ln \frac{1-z}{1-x} - 2\text{Li}_2(x) + 2\text{Li}_2(z) \,, \tag{C.13}$$

where Li₂ is the dilogarithm function. The result (C.12) with (C.13) was derived in [6] and is equivalent to results [27] known for some time for the integral in (2.28) with $\alpha_i = 1$, d = 4. The function Φ in (C.13) satisfies the following critical identities

$$\Phi(z,x) = -\Phi(x,z) = -\Phi(1-z,1-x) = -\Phi(z',x'), \qquad x' = \frac{x}{x-1}, \quad z' = \frac{z}{z-1},$$
(C.14)

which depend on standard results for the dilogarithm function. The results (C.14) are required in order to satisfy (C.10) for n = 1.

For n=2 we may use

$$\partial_u \Phi(z, x) = \frac{1}{z - x} \left(\frac{1 - u - v}{u} \ln v + 2 \ln u \right), \quad \partial_v \Phi(z, x) = \frac{1}{z - x} \left(\frac{1 - u - v}{v} \ln u + 2 \ln v \right), \tag{C.15}$$

to obtain

$$H(2,2,1,4;u,v) = \left(\frac{12uv}{(z-x)^5} + \frac{1+u+v}{(z-x)^3}\right)\Phi(z,x) + \frac{6}{(z-x)^4}\left((1+u-v)v\ln v + (1-u+v)u\ln u\right) + \frac{2}{(z-x)^2}\left(\ln uv + 1\right).$$
(C.16)

Also for n = 3 we have

$$H(3,3,1,6;u,v) = \left(\frac{1680u^{2}v^{2}}{(z-x)^{9}} + \left(\frac{240uv}{(z-x)^{7}} + \frac{24}{(z-x)^{5}}\right)(1+u+v) + \frac{4}{(z-x)^{3}}\right)\Phi(z,x)$$

$$+ \left(\left(\frac{840u}{(z-x)^{8}} + \frac{100}{(z-x)^{6}}\right)v^{2}(1+u-v) + \frac{480uv}{(z-x)^{6}}\right)$$

$$+ \frac{1}{(z-x)^{4}}\left(12(1+u) + 76v\right)\ln v + u \leftrightarrow v$$

$$+ \frac{260uv}{(z-x)^{6}} + \frac{26}{(z-x)^{4}}(1+u+v). \tag{C.17}$$

For recent applications [28] it is necessary to know $D_{\Delta_1\Delta_2\Delta_3\Delta_4}$ for other small integer values of Δ_i . To this end we first determine

$$H(1,1,1,3;u,v) = v^{-1}H(1,2,1,3;u/v,1/v) = vH(2,2,1,3;u,v)$$

= $vH(2,2,2,4;v,u) = H(1,1,0,2;v,u)$, (C.18)

which follow from (C.1,) and correspond to $\overline{D}_{2112}(u,v) = v^{-1}\overline{D}_{2121}(u/v,1/v) = v\overline{D}_{1221}(u,v) = v\overline{D}_{2211}(v,u) = \overline{D}_{1122}(v,u)$. From (C.6) we may find

$$H(1,1,1,3;u,v) = -v\frac{1+u-v}{(z-x)^3}\Phi(z,x) - \frac{1}{(z-x)^2}((1-u-v)\ln u + 2v\ln v). \quad (C.19)$$

Similarly for $\overline{D}_{2233}(u,v)$ it is necessary to determine

$$H(2,2,1,5;u,v) = v^{-2}H(2,3,1,5;u/v,1/v) = vH(3,3,1,5;u,v)$$

= $vH(3,3,2,6;v,u) = H(2,2,0,4;v,u)$, (C.20)

and explicitly we have

H(2,2,1,5;u,v)

$$= -\left(\frac{60uv^{2}}{(z-x)^{7}}(1+u-v) + \frac{6v}{(z-x)^{5}}(v(1+u-v)+4u) + \frac{4v}{(z-x)^{3}}\right)\Phi(z,x)$$

$$-\left(\frac{120uv^{2}}{(z-x)^{6}} + \frac{2v}{(z-x)^{4}}(6(1+u)+5v)\right)\ln v$$

$$-\left(\frac{60uv}{(z-x)^{6}}(1-u-v) + \frac{2}{(z-x)^{4}}(1-u+v-9uv-2v^{2}) - \frac{1}{(z-x)^{2}}\right)\ln u$$

$$-\frac{10v}{(z-x)^{4}}(1+u-v) - \frac{2}{(z-x)^{2}}.$$
(C.21)

Appendix D. Vector Four Point Function

We describe here a few details concerning the derivation of the result (6.34) where maintaining manifest conformal invariance simplifies the calculation (a related calculation is described by Herzog [4]). The disconnected graphs contributing to $\langle \frac{1}{4}F^2(x_1)\frac{1}{4}F^2(x_2)\frac{1}{4}F^2(x_3)\frac{1}{4}F^2(x_4)\rangle$ are of course straightforward while the connected graphs, using (6.33), give

$$\frac{1}{\pi^8} \frac{1}{(r_{12}r_{24}r_{34}r_{13})^2} \frac{1}{2} \left(\left(\operatorname{tr}(I(x_{12})I(x_{24})I(x_{43})I(x_{31})) \right)^2 - \operatorname{tr}(I(x_{12})I(x_{24})I(x_{43})I(x_{31})I(x_{12})I(x_{24})I(x_{43})I(x_{31})) \right), \tag{D.1}$$

together with two other permutations. To evaluate the traces of the inversion tensors in (D.1) we use [10]

$$I(x_{1i})I(x_{ij})I(x_{j1}) = I(X_{1(ij)}), X_{1(ij)} = \frac{x_{i1}}{r_{1i}} - \frac{x_{j1}}{r_{1j}}, X_{1(ij)}^2 = \frac{r_{ij}}{r_{1i} r_{2j}}, (D.2)$$

where $X_{1(ij)}$ transforms as a conformal vector at x_1 . With (D.2)

$$\operatorname{tr}(I(x_{12})I(x_{24})I(x_{43})I(x_{31})) = \operatorname{tr}(I(X_{1(24)})I(X_{1(43)})) = 4 \frac{(X_{1(24)} \cdot X_{1(43)})^2}{X_{1(24)}^2 X_{1(43)}^2}.$$
(D.3)

Similarly

$$\operatorname{tr}(I(x_{12})I(x_{24})I(x_{43})I(x_{31})I(x_{12})I(x_{24})I(x_{43})I(x_{31}))$$

$$= \operatorname{tr}(I(X_{1(24)})I(X_{1(43)})I(X_{1(24)})I(X_{1(43)}))$$

$$= \operatorname{tr}(I(X_{1(43)})I(X_{1(24)})I(X_{1(43)})) - \frac{2}{X_{1(24)}^{2}}X_{1(24)} \cdot I(X_{1(43)})I(X_{1(24)})I(X_{1(43)}) \cdot X_{1(24)}$$

$$= 4\left(1 - 2\frac{(X_{1(24)} \cdot X_{1(43)})^{2}}{X_{1(24)}^{2}X_{1(43)}^{2}}\right)^{2}. \tag{D.4}$$

Since

$$2X_{1(24)} \cdot X_{1(43)} = \frac{r_{23}}{r_{12} r_{14}} \left(1 - \frac{1}{v} - \frac{u}{v} \right) \tag{D.5}$$

we have

$$\left(\operatorname{tr}(I(x_{12})I(x_{24})I(x_{43})I(x_{31}))\right)^2 - \operatorname{tr}\left(I(x_{12})I(x_{24})I(x_{43})I(x_{31})I(x_{12})I(x_{24})I(x_{43})I(x_{31})\right)$$

$$= 16 \frac{\left(X_{1(24)} \cdot X_{1(43)}\right)^2}{X_{1(24)}^2 X_{1(43)}^2} - 4 = 4 \frac{1}{u} (v - 1 - u)^2 - 4,$$
(D.6)

and (D.1) becomes

$$\frac{1}{\pi^8} \frac{2}{(r_{12} r_{34})^4} \left(u(v - 1 - u)^2 - u^2 \right). \tag{D.7}$$

The other two contributions may be obtained similarly or by applying permutations to the results (D.7).

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