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Conformal Partial Waves and the Operator Product Expansion

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By solving the two variable differential equations which arise from finding the eigenfunctions for the Casimir operator for $O(d,2)$ succinct expressions are found for the functions, conformal partial waves, representing the contribution of an operator of arbitrary scale dimension Δ and spin ℓ together with its descendants to conformal four point functions for $d=4$, recovering old results, and also for $d=6$. The results are expressed in terms of ordinary hypergeometric functions of variables x,z which are simply related to the usual conformal invariants. An expression for the conformal partial wave amplitude valid for any dimension is also found in terms of a sum over two variable symmetric Jack polynomials which is used to derive relations for the conformal partial waves.

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1. Introduction

The operator product expansion is a crucial part of the theoretical structure of conformal field theories. Using the operator product expansion to analyse conformal correlation functions determines the spectrum of operators, their scale dimensions and spins, which are present in the theory. In essence the procedure is analogous to the the partial wave expansion of scattering amplitudes in terms of Legendre polynomials and their generalisations which applied to experimental data reveals resonant states of various masses and spins. There are two critical differences in the application to conformal amplitudes, firstly that the symmetry group is non compact so that the expansion functions are not finite order polynomials and secondly that for four point functions we have to deal with functions of two variables, corresponding to the two conformal invariants in this case.

Results [1,2,3] obtained for the contribution of a single conformal primary operator and its descendants to the four point correlation function of just scalar fields have been rather complicated, involving in general multiple series and integral representations. Nevertheless we recently found a quite simple formula for the result for conformal theories in four dimensions which just involves ordinary hypergeometric functions [4]. These expressions have been used to analyse the correlation functions for $\mathcal{N} = 4$ superconformal chiral primary operators and reveal anomalous dimensions for arbitrary spin both perturbatively and in ther large N limit using results from the AdS/CFT correspondence [5].

Nevertheless conformal fields theories are potentially of interest in 3,5 and also 6 dimensions. In the last case there are isolated non trivial field theories with $(2,0)$ superconformal symmetry which have no free field limit but for which large N results have been obtained, [6]. The results obtained in [4] depend on using new variables x, z which are simply related to the usual conformal invariants but the method used depends on solving a non trivial recurrence relation which is not easy to generalise. Here we adopt an alternative approach by considering the equations obtained by looking for the eigenfunctions, with appropriate boundary conditions, of the d-dimensional conformal group $O(d,2)$ Casimir operator.

As has been known for a long time [7,8] it is convenient to use homogeneous coordinates η^A on the projective null cone $\eta^2 = g_{AB}\eta^A\eta^B = 0$ with $\eta^A \sim \lambda\eta^A$ and the $d{+}2$-dimensional metric $g_{AB} = \mathrm{diag.}(-1, 1, \dots, 1, -1)$. Conformal transformations act on η linearly and the corresponding generators are

$$L_{AB} = \eta_A \frac{\partial}{\partial\eta^B} - \eta_B \frac{\partial}{\partial\eta^A} \, . \tag{1.1}$$

Quantum fields are then extended to homogeneous functions of η, for a scalar field ϕ of scale dimension Δ, $\phi(\lambda\eta) = \lambda^{-\Delta}\phi(\eta)$.

The four point correlation function for such scalar fields may then in general be expressed as

$$\begin{aligned}
&\big\langle \phi_1(\eta_1)\,\phi_2(\eta_2)\,\phi_3(\eta_3)\,\phi_4(\eta_4)\big\rangle \\
&\quad = \frac{1}{(\eta_1{\cdot}\eta_2)^{\frac12(\Delta_1+\Delta_2)}\,(\eta_3{\cdot}\eta_4)^{\frac12(\Delta_3+\Delta_4)}} \left(\frac{\eta_2{\cdot}\eta_4}{\eta_1{\cdot}\eta_4}\right)^{\frac12\Delta_{12}} \left(\frac{\eta_1{\cdot}\eta_4}{\eta_1{\cdot}\eta_3}\right)^{\frac12\Delta_{34}} F(u,v)\,,
\end{aligned} \tag{1.2}$$

where

$$\Delta_{ij} = \Delta_i - \Delta_j\,, \tag{1.3}$$

and u,v are two conformal invariants

$$u = \frac{\eta_1{\cdot}\eta_2\,\eta_3{\cdot}\eta_4}{\eta_1{\cdot}\eta_3\,\eta_2{\cdot}\eta_4}\,, \qquad v = \frac{\eta_1{\cdot}\eta_4\,\eta_2{\cdot}\eta_3}{\eta_1{\cdot}\eta_3\,\eta_2{\cdot}\eta_4}\,. \tag{1.4}$$

For the contribution of a single operator of scale dimension Δ and spin ℓ in the operator product expansion of $\phi_1(\eta_1)\,\phi_2(\eta_2)$ to the four point function, (1.2) then requires a function $F = G^{(\ell)}_\Delta$ so that

$$\begin{aligned}
L^2\big\langle \phi_1(\eta_1)\,\phi_2(\eta_2)\,\phi_3(\eta_3)\,\phi_4(\eta_4)\big\rangle &= -\,C_{\Delta,\ell}\big\langle \phi_1(\eta_1)\,\phi_2(\eta_2)\,\phi_3(\eta_3)\,\phi_4(\eta_4)\big\rangle\,, \\
C_{\Delta,\ell} &= \Delta(\Delta-d) + \ell(\ell+d-2)\,,
\end{aligned} \tag{1.5}$$

where

$$L^2 = \tfrac12 L^{AB}L_{AB}\,, \qquad L_{AB} = L_{1AB} + L_{2AB}\,, \tag{1.6}$$

for L_{iAB} the generator formed from η_i as in (1.1) (conformal invariance of course requires $\sum_i L_{iAB}\big\langle \phi_1(\eta_1)\,\phi_2(\eta_2)\,\phi_3(\eta_3)\,\phi_4(\eta_4)\big\rangle = 0$). $G^{(\ell)}_\Delta(u,v)$ are here referred to as conformal partial waves and (1.5) gives a corresponding eigenvalue equation for them,

$$\begin{aligned}
L^2 G^{(\ell)}_\Delta - (\Delta_{12}-\Delta_{34})\bigg((1+u-v)\Big(u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}\Big) - (1-u-v)\frac{\partial}{\partial v}\bigg) G^{(\ell)}_\Delta - \tfrac12\Delta_{12}\Delta_{34} G^{(\ell)}_\Delta& \\
= -C_{\Delta,\ell} G^{(\ell)}_\Delta\,,&
\end{aligned} \tag{1.7}$$

where for any $F(u,v)$,

$$\begin{aligned}
\tfrac12 L^2 F = &-\big((1-v)^2 - u(1+v)\big)\frac{\partial}{\partial v}v\frac{\partial}{\partial v}F - (1-u+v)u\frac{\partial}{\partial u}u\frac{\partial}{\partial u}F \\
&+ 2(1+u-v)uv\frac{\partial^2}{\partial u\partial v}F + d\,u\frac{\partial}{\partial u}F\,.
\end{aligned} \tag{1.8}$$

For $u \to 0$, $G^{(\ell)}_\Delta$ is independent of d and, with a convenient normalisation [1,4], it has the form

$$G^{(\ell)}_\Delta(u,v) \sim u^{\frac12(\Delta-\ell)}\big(-\tfrac12(1-v)\big)^\ell F\big(\tfrac12(\Delta+\ell-\Delta_{12}), \tfrac12(\Delta+\ell+\Delta_{34}); \Delta+\ell; 1-v\big)\,, \tag{1.9}$$

where F is a hypergeometric function. The leading behaviour as $u \to 0$ and then $v \to 1$ is thus simply determined by Δ, ℓ. For $\ell = 0$ explicit solutions as double power series in u and $1 - v$ are known, see [4], but there is no apparent concise generalisation for $\ell > 0$.

The aim here is then to solve (1.7) in as simple a form as possible. In the next section we show how this is possible in four and six dimensions as products of ordinary hypergeometric functions while in the following section for general dimension we obtain an expansion in terms of symmetric Jack polynomials. This is used to obtain various relations for conformal partial waves valid in any dimension.

2. Solutions in Four and Six Dimensions

It is now apparent that finding expressions for conformal partial waves valid for any ℓ is assisted by introducing, as in [4], new variables x, z, such that

$$u = xz \,, \qquad v = (1-x)(1-z) \,. \tag{2.1}$$

These may be interpretated [9] as the eigenvalues of a 2×2 spinor matrix formed from x_1, x_2, x_3, x_4. Manifestly all results must be symmetric under $x \leftrightarrow z$. The eigenvalue equation (1.7) then becomes

$$2D_\varepsilon G_\Delta^{(\ell)} = C_{\Delta,\ell} G_\Delta^{(\ell)} \,, \tag{2.2}$$

where D_ε is a symmetric differential operator

$$\begin{aligned} D_\varepsilon = {}& x^2(1-x)\frac{\partial^2}{\partial x^2} + z^2(1-z)\frac{\partial^2}{\partial z^2} + c\Big(x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z}\Big) - (a+b+1)\Big(x^2\frac{\partial}{\partial x} + z^2\frac{\partial}{\partial z}\Big) \\ & - ab(x+z) + \varepsilon\,\frac{xz}{x-z}\Big((1-x)\frac{\partial}{\partial x} - (1-z)\frac{\partial}{\partial z}\Big) \,, \end{aligned} \tag{2.3}$$

depending on parameters a, b, c, ε which for application in (2.2) should take the values

$$a = -\tfrac{1}{2}\Delta_{12} \,, \qquad b = \tfrac{1}{2}\Delta_{34} \,, \qquad c = 0 \,, \qquad \varepsilon = d - 2 \,. \tag{2.4}$$

In general we then seek eigenfunctions of D_ε, $F^{(\varepsilon)}_{\lambda_1\lambda_2}(a,b;c;x,z)$, which are symmetric in x, z with the boundary behaviour, suppressing unnecessary arguments,

$$F_{\lambda_1\lambda_2}(x,z) \underset{z\to 0, x\to 0}{\sim} x^{\lambda_1} z^{\lambda_2} \,, \quad \lambda_1 - \lambda_2 = 0, 1, 2, \ldots \,, \tag{2.5}$$

where the limit $z \to 0$ is taken first[1]. The corresponding eigenvalues are then

$$\lambda_1(\lambda_1 + c - 1) + \lambda_2(\lambda_2 + c - 1 - \varepsilon) \,. \tag{2.6}$$

[1] Since $D_\varepsilon (xz)^p = (xz)^p (D_\varepsilon|_{a\to a+p, b\to b+p, c\to c+2p} + 2p(p-1+c-\frac{1}{2}\varepsilon))$, and consequently $(xz)^p F^{(\varepsilon)}_{\lambda_1\lambda_2}(a+p, b+p; c+2p; x,z) = F^{(\varepsilon)}_{\lambda_1+p\,\lambda_2+p}(a,b;c;x,z)$, the parameter c is superfluous but it is convenient to keep it here.

To match (1.9)

$$\lambda_1 = \tfrac{1}{2}(\Delta + \ell)\,, \qquad \lambda_2 = \tfrac{1}{2}(\Delta - \ell)\,, \tag{2.7}$$

and then (2.6) gives the result for $C_{\Delta,\ell}$ in (1.5).

When $\varepsilon = 0$ (2.3) is essentially the sum of two hypergeometric operators and we may make use of

$$\begin{aligned} x\Big(x(1-x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \big(c-(a+b+1)x\big)\frac{\mathrm{d}}{\mathrm{d}x} - ab\Big)& x^p F(p+a,p+b;2p+c;x) \\ &= p(p+c-1)x^p F(p+a,p+b;2p+c;x)\,. \end{aligned} \tag{2.8}$$

The eigenfunctions of D_0 satisfying (2.5) are then

$$x^{\lambda_1} z^{\lambda_2} F(\lambda_1+a,\lambda_1+b;2\lambda_1+c;x)F(\lambda_2+a,\lambda_2+b;2\lambda_2+c;z) + x \leftrightarrow z\,, \tag{2.9}$$

with the expected eigenvalue from (2.6). With a, b, c as in (2.4) this solution is applicable to two dimensional conformal field theories where in Euclidean space z and $x = z^*$ are the usual complex coordinates.

For $\varepsilon = 2$ the solutions are almost as simple making use of

$$D_2 \frac{1}{x-z} = \frac{1}{x-z}\Big(D_0\big|_{\substack{a\to a-1,b\to b-1\\ c\to c-2}} - c + 2\Big)\,. \tag{2.10}$$

Using the known eigenfunctions of D_0 the solutions with the required leading behaviour for $x, z \sim 0$ are then

$$\frac{1}{x-z}\Big(x^{\lambda_1+1} z^{\lambda_2} F(\lambda_1+a,\lambda_1+b;2\lambda_1+c;x)F(\lambda_2+a-1,\lambda_2+b-1;2\lambda_2+c-2;z) - x \leftrightarrow z\Big)\,, \tag{2.11}$$

with the eigenvalue given by (2.6). The solution given by (2.11) with (2.4) and λ_1, λ_2 as in (2.7) is identical with that in [4] for $d = 4$.

For $\varepsilon = 4$ the results are less simple but they can be derived in a similar fashion by using

$$\begin{aligned} &D_4 \frac{1}{(x-z)^3} \\ &= \frac{1}{(x-z)^3}\bigg(D_0\big|_{\substack{a\to a-3,b\to b-3\\ c\to c-6}} - \frac{2xz}{x-z}\Big((1-x)\frac{\partial}{\partial x} - (1-z)\frac{\partial}{\partial z}\Big) - 3c + 12\bigg)\,. \end{aligned} \tag{2.12}$$

To obtain the solutions in this case we introduce

$$F^{\pm}_{p,q}(x,z) = x^{p+3} z^{q+3} F(p+a,p+b;2p+c;x)F(q+a,q+b;2q+c;x) \pm x \leftrightarrow z\,, \tag{2.13}$$

and write the eigenfunctions satisfying

$$D_4 F = CF\,, \tag{2.14}$$

in the form

$$F(x,z) = \sum_{p,q} a_{pq}\, \frac{F^-_{pq}(x,z)}{(x-z)^3}\,. \tag{2.15}$$

Substituting (2.15) in (2.14) gives

$$\begin{aligned} & \sum_{p,q} a_{pq}\Big(p(p+c-3)+q(q+c-3)+3c-12-C\Big)\Big(\frac{1}{x}-\frac{1}{z}\Big)F^-_{pq}(x,z) \\ & = -2\sum_{p,q} a_{pq}\Big((1-x)\frac{\partial}{\partial x}-(1-z)\frac{\partial}{\partial z}\Big)F^-_{pq}(x,z)\,. \end{aligned} \tag{2.16}$$

To solve (2.16) we use

$$\begin{aligned} \Big(\frac{1}{x}-\frac{1}{z}\Big)F^-_{p\,q}(x,z) &= F^+_{p-1\,q}(x,z) - F^+_{p\,q-1}(x,z) + A_{pq}F^+_{p\,q}(x,z) + B_p F^+_{p+1\,q}(x,z) - B_q F^+_{p\,q+1}(x,z)\,, \\ A_{pq} &= (p-q)(p+q+c-1)\hat{A}_{pq}\,, \quad \hat{A}_{pq} = \frac{2(2a-c)(2b-c)}{(2p+c)(2p+c-2)(2q+c)(2q+c-2)}\,, \\ B_p &= \frac{(p+a)(p+b)(p+c-a)(p+c-b)}{(2p+c-1)(2p+c)^2(2p+c+1)}\,, \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} & \Big((1-x)\frac{\partial}{\partial x}-(1-z)\frac{\partial}{\partial z}\Big)F^-_{pq}(x,z) \\ & \quad = p\,F^+_{p-1\,q}(x,z) - q\,F^+_{p\,q-1}(x,z) - \tfrac{1}{2}(c-8)A_{pq}\,F^+_{p\,q}(x,z) \\ & \qquad - (p+c-4)B_p\,F^+_{p+1\,q}(x,z) + (q+c-4)B_q\,F^+_{p\,q+1}(x,z)\,, \end{aligned} \tag{2.18}$$

which follow from identities for ordinary hypergeometric functions, to obtain recurrence relations for a_{pq} involving $a_{pq}, a_{p\pm1q}$ and $a_{pq\pm1}$. For the right value of C this may be solved for only 5 terms non zero. Choosing a solution with a normalisation determined by (2.5) we have the following non zero terms

$$\begin{aligned} a_{\lambda_1\lambda_2-3} &= 1\,, \qquad a_{\lambda_1-1\lambda_2-2} = -\frac{\lambda_1-\lambda_2+3}{\lambda_1-\lambda_2+1}\,, \\ a_{\lambda_1+1\lambda_2-2} &= -\,\frac{\lambda_1+\lambda_2+c-4}{\lambda_1+\lambda_2+c-2}\,B_{\lambda_1}\,, \\ a_{\lambda_1\lambda_2-1} &= \frac{(\lambda_1+\lambda_2+c-4)(\lambda_1-\lambda_2+3)}{(\lambda_1+\lambda_2+c-2)(\lambda_1-\lambda_2+1)}\,B_{\lambda_2-2}\,, \\ a_{\lambda_1\lambda_2-2} &= -\,(\lambda_1+\lambda_2+c-4)(\lambda_1-\lambda_2+3)\hat{A}_{\lambda_1\,\lambda_2-2}\,, \end{aligned} \tag{2.19}$$

with C given by (2.6) for $\varepsilon = 4$.

Using (2.7) the result, if $\Delta > 2$, for the six-dimensional conformal partial wave function satisfying (1.7) and (1.9) is then

$$\begin{aligned} G_\Delta^{(\ell)} = {} & \mathcal{F}_{00} - \frac{\ell+3}{\ell+1}\,\mathcal{F}_{-11} \\ & - \frac{\Delta-4}{\Delta-2}\,\frac{(\Delta+\ell-\Delta_{12})(\Delta+\ell+\Delta_{12})(\Delta+\ell+\Delta_{34})(\Delta+\ell-\Delta_{34})}{16(\Delta+\ell-1)(\Delta+\ell)^2(\Delta+\ell+1)}\,\mathcal{F}_{11} \\ & + \frac{(\Delta-4)(\ell+3)}{(\Delta-2)(\ell+1)} \\ & \times \frac{(\Delta-\ell-\Delta_{12}-4)(\Delta-\ell+\Delta_{12}-4)(\Delta-\ell+\Delta_{34}-4)(\Delta-\ell-\Delta_{34}-4)}{16(\Delta-\ell-5)(\Delta-\ell-4)^2(\Delta-\ell-3)}\,\mathcal{F}_{02} \\ & + 2(\Delta-4)(\ell+3)\,\frac{\Delta_{12}\,\Delta_{34}}{(\Delta+\ell)(\Delta+\ell-2)(\Delta+\ell-4)(\Delta+\ell-6)}\,\mathcal{F}_{01}\,, \end{aligned} \tag{2.20}$$

where

$$\begin{aligned} \mathcal{F}_{nm}(x,z) = {} & \frac{(xz)^{\frac{1}{2}(\Delta-\ell)}}{(x-z)^3}\Big\{(-\tfrac{1}{2}x)^\ell\,x^{n+3}z^m \\ & \times F\big(\tfrac{1}{2}(\Delta+\ell-\Delta_{12})+n,\tfrac{1}{2}(\Delta+\ell+\Delta_{34})+n;\Delta+\ell+2n;x\big) \\ & \times F\big(\tfrac{1}{2}(\Delta-\ell-\Delta_{12})-3+m,\tfrac{1}{2}(\Delta-\ell+\Delta_{34})-3+m;\Delta-\ell-6+2m;z\big) \\ & \qquad - x\leftrightarrow z\Big\}\,. \end{aligned} \tag{2.21}$$

This may perhaps be relevant in the analysis of six dimensional conformal theories.

3. Solution in terms of Jack Polynomials

The differential operator D_ε, defined in (2.3), is closely related to a general class of differential operators acting on symmetric functions of several variables, see [10,11]. We describe here an approach for finding the eigenfunctions of the operator D_ε which is a modification of the method described in [11] and which is potentially applicable for any ε. To do this we introduce a simpler second order symmetric operator

$$D_\varepsilon^J = x^2\frac{\partial^2}{\partial x^2} + z^2\frac{\partial^2}{\partial z^2} + \varepsilon\,\frac{1}{x-z}\Big(x^2\frac{\partial}{\partial x} - z^2\frac{\partial}{\partial z}\Big)\,, \tag{3.1}$$

so that

$$\begin{aligned} D_\varepsilon = {} & -\tfrac{1}{2}\Big[D_\varepsilon^J, x^2\frac{\partial}{\partial x} + z^2\frac{\partial}{\partial z}\Big] + D_\varepsilon^J \\ & + (c-\varepsilon)\Big(x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z}\Big) - (a+b-\tfrac{1}{2}\varepsilon)\Big(x^2\frac{\partial}{\partial x} + z^2\frac{\partial}{\partial z}\Big) - ab(x+z)\,. \end{aligned} \tag{3.2}$$

The symmetric eigenfunctions of D^J_ε satisfy[2],

$$D^J_\varepsilon P_{\lambda_1\lambda_2}(x,z) = \big(\lambda_1(\lambda_1-1+\varepsilon)+\lambda_2(\lambda_2-1)\big)P_{\lambda_1\lambda_2}(x,z)\,, \quad \lambda_1\ge\lambda_2\,, \tag{3.3}$$

where $P_{\lambda_1\lambda_2}(x,z)$ is a homogeneous polynomial of degree $\lambda_1+\lambda_2$ in x,z, and for $z\to 0$ $P_{\lambda_1\lambda_2}(x,z)\propto x^{\lambda_1}z^{\lambda_2}$. For λ_1,λ_2 integers, $P_{\lambda_1\lambda_2}(x,z)$ are just the Jack polynomials [12] in two variables. It is easy to see from the form of D^J_ε in (3.1) that we can take

$$(xz)^f P_{\lambda_1\lambda_2}(x,z) = P_{\lambda_1+f\,\lambda_2+f}(x,z)\,, \tag{3.4}$$

so that the polynomials $P_{\lambda_1\lambda_2}(x,z)$ may be extended to arbitrary λ_1,λ_2 with $\lambda_1-\lambda_2 = 0,1,2\ldots$. In the two variable case there is an explicit solution [11] involving Gegenbauer polynomials

$$P_{\lambda_1\lambda_2}(x,z) = \frac{\lambda_-!}{(\varepsilon)_{\lambda_-}}\,(xz)^{\frac12(\lambda_1+\lambda_2)}C^{\frac12\epsilon}_{\lambda_-}(\sigma)\,, \quad \lambda_- = \lambda_1-\lambda_2\,, \quad \sigma = \frac{x+z}{2(xz)^{\frac12}}\,, \tag{3.5}$$

where we have chosen a somewhat arbitrary normalisation so that $P_{\lambda_1\lambda_2}(1,1)=1$. In this case

$$P_{\lambda_1\lambda_2}(x,z) \underset{z\to 0}{\sim} \frac{(\frac12\varepsilon)_{\lambda_-}}{(\varepsilon)_{\lambda_-}}\,x^{\lambda_1}z^{\lambda_2}\,. \tag{3.6}$$

The following are special cases which may be verified directly (note $D^J_2\frac{1}{x-z} = \frac{1}{x-z}D^J_0$),

$$\begin{aligned}
P_{\lambda_1\lambda_2}(x,z)\big|_{\varepsilon=0} &= \tfrac12\big(x^{\lambda_1}z^{\lambda_2}+x^{\lambda_2}z^{\lambda_1}\big)\,,\\
P_{\lambda_1\lambda_2}(x,z)\big|_{\varepsilon=2} &= \frac{1}{\lambda_-+1}\bigg(\frac{x^{\lambda_1+1}z^{\lambda_2}-x^{\lambda_2}z^{\lambda_1+1}}{x-z}\bigg) = \frac{(xz)^{\lambda_2}}{\lambda_-+1}\bigg(\frac{x^{\lambda_-+1}-z^{\lambda_-+1}}{x-z}\bigg)\,,\\
P_{\lambda_1\lambda_2}(x,z)\big|_{\varepsilon=4} &= \frac{6}{\lambda_-+2}\bigg(\frac{x^{\lambda_1+3}z^{\lambda_2}-x^{\lambda_2}z^{\lambda_1+3}}{(\lambda_-+3)(x-z)^3} - \frac{x^{\lambda_1+2}z^{\lambda_2+1}-x^{\lambda_2+1}z^{\lambda_1+2}}{(\lambda_-+1)(x-z)^3}\bigg).
\end{aligned} \tag{3.7}$$

The Jack polynomials play the role of an extension of simple powers of x for single variable functions when the series expansion of multi-variable symmetric functions is considered. For subsequent use there are important recurrence relations which may be verified by using standard identities [13] for $\sigma C^{\frac12\epsilon}_n(\sigma)$ and $(1-\sigma^2)C^{\frac12\epsilon}_n{}'(\sigma)$ as well as its defining differential equation,

$$\begin{aligned}
(x+z)P_{\lambda_1\lambda_2}(x,z) &= \frac{\lambda_-+\varepsilon}{\lambda_-+\frac12\varepsilon}\,P_{\lambda_1+1\,\lambda_2}(x,z) + \frac{\lambda_-}{\lambda_-+\frac12\varepsilon}\,P_{\lambda_1\,\lambda_2+1}(x,z)\,,\\
\Big(x^2\frac{\partial}{\partial x}+z^2\frac{\partial}{\partial z}\Big)P_{\lambda_1\lambda_2}(x,z) &= \frac{\lambda_1(\lambda_-+\varepsilon)}{\lambda_-+\frac12\varepsilon}\,P_{\lambda_1+1\,\lambda_2}(x,z) + \frac{(\lambda_2-\frac12\varepsilon)\lambda_-}{\lambda_-+\frac12\varepsilon}\,P_{\lambda_1\,\lambda_2+1}(x,z)\,.
\end{aligned} \tag{3.8}$$

[2] $P_{\lambda_1\lambda_2}(x,z)$ are also eigenfunctions of the commuting operator $x\frac{\partial}{\partial x}+z\frac{\partial}{\partial z}$, with eigenvalue $\lambda_1+\lambda_2$.

These are valid for any $\lambda_1 - \lambda_2 = 0, 1, 2, \ldots$, for $\lambda_1 = \lambda_2$ the second term on the right hand side is absent ($P_{\lambda\lambda}(x,z) = (xz)^\lambda$, $P_{\lambda+1\,\lambda}(x,z) = (xz)^\lambda \frac{1}{2}(x+z)$). Using (3.8) with (3.2) and (3.3) we may find,

$$\begin{aligned} D_\varepsilon P_{\lambda_1\lambda_2} =& \big(\lambda_1(\lambda_1+c-1) + \lambda_2(\lambda_2+c-1-\varepsilon)\big) P_{\lambda_1\lambda_2} \\ &- \frac{\lambda_- + \varepsilon}{\lambda_- + \frac{1}{2}\varepsilon}\, (\lambda_- + a)(\lambda_- + b)\, P_{\lambda_1+1\,\lambda_2} \\ &- \frac{\lambda_-}{\lambda_- + \frac{1}{2}\varepsilon}\, (\lambda_- - \tfrac{1}{2}\varepsilon + a)(\lambda_- - \tfrac{1}{2}\varepsilon + b)\, P_{\lambda_1\,\lambda_2+1}\,. \end{aligned} \tag{3.9}$$

We then seek a solution for the eigenfunction for D_ε, with the limiting behaviour given by (2.5), in the form

$$F_{\lambda_1\lambda_2} = \sum_{m,n\geq 0} r_{mn}\, P_{\lambda_1+m\,\lambda_2+n}\,. \tag{3.10}$$

The necessary recurrence relation is simplified by taking

$$r_{mn} = (\lambda_1+a)_m(\lambda_1+b)_m(\lambda_2 - \tfrac{1}{2}\varepsilon + a)_n(\lambda_2 - \tfrac{1}{2}\varepsilon + b)_n\, \hat r_{mn}\,, \tag{3.11}$$

and then, with the eigenvalue given by (2.6), we obtain

$$\begin{aligned} &\big(m(2\lambda_1+c-1+m) + n(2\lambda_2+c-1-\varepsilon+n)\big)\hat r_{mn} \\ &\quad = \frac{\lambda_- + m - n - 1 + \varepsilon}{\lambda_- + m - n - 1 + \frac{1}{2}\varepsilon}\, \hat r_{m-1\,n} + \frac{\lambda_- + m - n + 1}{\lambda_- + m - n + 1 + \frac{1}{2}\varepsilon}\, \hat r_{m\,n-1}\,. \end{aligned} \tag{3.12}$$

This may be solved iteratively starting from $\hat r_{00}$. The general solution is rather involved so we consider first some special cases. It is easy to see that

$$\hat r_{m0} = \frac{(\varepsilon)_{\lambda_-+m}}{m!(2\lambda_1+c)_m(\frac{1}{2}\varepsilon)_{\lambda_-+m}}\,, \tag{3.13}$$

choosing $\hat r_{00}$ so that from (3.6) we have

$$F_{\lambda_1\lambda_2}(x,z) \underset{z\to 0}{\sim} x^{\lambda_1} z^{\lambda_2}\, F(\lambda_1+a, \lambda_1+b; 2\lambda_1+c; x)\,. \tag{3.14}$$

For $\varepsilon = 0, 2$ respectively the solutions of (3.12) are quite simple

$$\hat r_{mn} = \frac{2}{m!\,n!(2\lambda_1+c)_m(2\lambda_2+c)_n}\,, \quad \hat r_{mn} = \frac{\lambda_- + 1 + m - n}{m!\,n!(2\lambda_1+c)_m(2\lambda_2+c-2)_n}\,, \tag{3.15}$$

where we match (3.13) for $n = 0$. Substituting in (3.10) and using the relevant results from (3.7) we easily recover the solutions (2.9) and (2.11). For $\varepsilon = 4$ the result is

$$\begin{aligned} \hat r_{mn} =& \frac{1}{m!\,n!(2\lambda_1+c)_m(2\lambda_2+c-4)_n}\, \frac{\lambda_- + 2 + m - n}{(\lambda_1+\lambda_2+c-2)(\lambda_-+1)} \\ &\times \tfrac{1}{6}\Big((\lambda_1+\lambda_2+c-2)\big((\lambda_-+1)(\lambda_-+3+m-n) - 2n\big) - 2mn\Big)\,. \end{aligned} \tag{3.16}$$

With the corresponding result for $P_{\lambda_1\lambda_2}$ from (3.7) we may also obtain the solution given by (2.15) and (2.19) if we recognise that $\hat{r}_{m-1\,n} - \hat{r}_{m\,n-1}$ contains a factor $\lambda_- + m - n + 2$ so that the summand is expressible as a linear combination of five expressions in which the dependence on m, n factorises in an appropriate fashion.

In the above results for $\varepsilon = 0, 2, 4$ the summations over m, n are unrestricted so that they involve $P_{\lambda_1\lambda_2}$ for $\lambda_1 < \lambda_2$. However for ε even we may note the symmetry

$$P_{\lambda_1\,\lambda_2} = P_{\lambda_2 - \frac{1}{2}\varepsilon\,\lambda_1 + \frac{1}{2}\varepsilon}\,. \tag{3.17}$$

In consequence in (3.10) we may take instead of the results given by (3.11) and (3.15) or (3.16),

$$\begin{aligned} \hat{r}_{mn} &\to \hat{r}_{mn} + \hat{r}_{n-\lambda_- - \frac{1}{2}\varepsilon\, m+\lambda_- + \frac{1}{2}\varepsilon}\,, \qquad \varepsilon > 0\,, \\ \hat{r}_{mn} &\to \begin{cases} \hat{r}_{mn} + \hat{r}_{n-\lambda_-\, m+\lambda_-}\,, & \lambda_- + m - n > 0\,, \\ \hat{r}_{mn}\,, & \lambda_- + m - n = 0\,, \end{cases} \qquad \varepsilon = 0\,, \end{aligned} \tag{3.18}$$

with the summation now restricted to $\lambda_- + m - n \geq 0$. Except for $\varepsilon = 0$ we should note that $\hat{r}_{mn} = 0$ if $\lambda_- + m - n + \frac{1}{2}\varepsilon = 0$ and contributions for $-\varepsilon < \lambda_- + m - n < 0$, for which both contributions in (3.18) do not satisfy $\lambda_- + m - n \geq 0$, cancel. When $\varepsilon = 2$ the resulting expansion is in terms of Schur polynomials as considered by Heslop and Howe [9] for the four-dimensional operator product expansion.

In general the recurrence relations (3.12) can be solved in the form

$$\begin{aligned} \hat{r}_{mn} = {} & \frac{(\lambda_- + m)!\,(\varepsilon)_{\lambda_- + m - n}}{m!\,n!\,(\lambda_- + m - n)!\,(2\lambda_1 + c)_m (2\lambda_2 + c - \frac{1}{2}\varepsilon)_n (\frac{1}{2}\varepsilon)_{\lambda_- + m + 1}} \\ & \times (\lambda_- + m - n + \tfrac{1}{2}\varepsilon)\, \frac{(-\lambda_1 - \lambda_2 - c + 1 + \frac{1}{2}\varepsilon)_{\lambda_-} (\varepsilon)_{\lambda_-}}{(-\lambda_1 - \lambda_2 - c + 1 + \varepsilon)_{\lambda_-} (\frac{1}{2}\varepsilon)_{\lambda_-}} \\ & \times {}_4F_3\left(\begin{matrix} -\lambda_-, -\lambda_- - m + n, \lambda_1 + \lambda_2 + c - 1, \frac{1}{2}\varepsilon \\ -\lambda_- - m, 2\lambda_2 + c - \frac{1}{2}\varepsilon + n,\ \varepsilon \end{matrix} ; 1 \right), \end{aligned} \tag{3.19}$$

which matches (3.13) for $n = 0$. The terminating ${}_4F_3$ function in (3.19) has $\lambda_- + 1$ terms in its expansion, explicit straightforward results may be obtained for $\lambda_- = 0, 1$. The solution (3.19) of the recurrence relation (3.12) was obtained initially by adapting the results of Koornwinder and Sprinkhuizen-Kuyper [14], but an independent proof is given in appendix A.

4. Compact Case

The above discussion is an extension of known results for two variable orthogonal polynomials. These also arise [15] in the expansion of four point correlation functions when the fields belong to tensor representations of $O(n)$. We here describe some analogous results to make the comparison clear. The relevant differential operation representing the Casimir operator, or Laplacian, acting on symmetric functions of two variables has the form,

$$\begin{aligned}\tilde{D}_\varepsilon = &-x(1-x)\frac{\partial^2}{\partial x^2} - z(1-z)\frac{\partial^2}{\partial z^2} - (a+1)\Big(\frac{\partial}{\partial x} + \frac{\partial}{\partial z}\Big) + (a+b+2)\Big(x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z}\Big)\\ &-\varepsilon\,\frac{1}{x-z}\Big(x(1-x)\frac{\partial}{\partial x} - z(1-z)\frac{\partial}{\partial z}\Big) + \tfrac{1}{2}(a+b)(a+b+\varepsilon+2)\,,\end{aligned} \tag{4.1}$$

for a, b integers and $\varepsilon = n-4$. This may be be written in the form

$$\begin{aligned}\tilde{D}_\varepsilon = &-\frac{1}{w}\Big(\frac{\partial}{\partial x}w\,x(1-x)\frac{\partial}{\partial x}\Big) + \frac{\partial}{\partial z}w\,z(1-z)\frac{\partial}{\partial z}\Big) + \tfrac{1}{2}(a+b)(a+b+\varepsilon+2)\,,\\ &w = x^a z^a (1-x)^b (1-z)^b (x-z)^\varepsilon\,,\end{aligned} \tag{4.2}$$

so that the symmetric eigenfunctions are orthogonal, if $\varepsilon > 0$, integrated over $0 < z < x < 1$. For $\varepsilon = 2, 4$ solutions were obtained by Vretare [16] in terms of Jacobi polynomials which are similar to (2.11) and (2.15) with (2.19).

For general ε we may obtain and expansion in terms of Jack polynomials as in section 3. Instead of (3.2) we may now write

$$\begin{aligned}\tilde{D}_\varepsilon = &\tfrac{1}{2}\Big[D^J_\varepsilon, \frac{\partial}{\partial x} + \frac{\partial}{\partial z}\Big] + D^J_\varepsilon\\ &-(a+1)\Big(\frac{\partial}{\partial x} + \frac{\partial}{\partial z}\Big) + (a+b+2)\Big(x\frac{\partial}{\partial x} + z\frac{\partial}{\partial z}\Big) + \tfrac{1}{2}(a+b)(a+b+\varepsilon+2)\,.\end{aligned} \tag{4.3}$$

The relevant equation for Jack polynomials is then

$$\Big(\frac{\partial}{\partial x} + \frac{\partial}{\partial z}\Big)P_{\lambda_1\lambda_2}(x,z) = \frac{(\lambda_1 + \frac{1}{2}\varepsilon)\lambda_-}{\lambda_- + \frac{1}{2}\varepsilon}\,P_{\lambda_1-1\,\lambda_2}(x,z) + \frac{\lambda_2(\lambda_- + \varepsilon)}{\lambda_- + \frac{1}{2}\varepsilon}\,P_{\lambda_1\,\lambda_2-1}(x,z)\,. \tag{4.4}$$

The polynomial eigenfunctions have the form

$$\tilde{F}_{\lambda_1\lambda_2} = \sum_{m=0}^{\lambda_1}\sum_{n=0, n\le m}^{\lambda_2} s_{mn}\,P_{mn}\,, \tag{4.5}$$

and for the series to truncate the eigenvalue of (4.3) must be

$$\big(\lambda_1 + \tfrac{1}{2}(a+b)\big)\big(\lambda_1 + \tfrac{1}{2}(a+b) + 1 + \varepsilon\big) + \big(\lambda_2 + \tfrac{1}{2}(a+b)\big)\big(\lambda_2 + \tfrac{1}{2}(a+b) + 1\big)\,. \tag{4.6}$$

If we write

$$s_{mn} = \frac{1}{(1+\frac{1}{2}\varepsilon)_m n!\,(a+1+\frac{1}{2}\varepsilon)_m (a+1)_n}\,\hat{s}_{mn}\,, \tag{4.7}$$

the necessary recurrence relation simplifies to

$$\begin{aligned}\big((\lambda_1-m)(\lambda_1+a+b+1+\varepsilon+m)+(\lambda_2-n)(\lambda_2+a+b+1+n)\big)\hat{s}_{mn}&\\ =-\frac{m-n-1+\varepsilon}{m-n-1+\frac{1}{2}\varepsilon}\,\hat{s}_{m\,n+1}-\frac{m-n+1}{m-n+1+\frac{1}{2}\varepsilon}\,\hat{s}_{m+1\,n}\,.&\end{aligned}\tag{4.8}$$

In this case we may solve for $\hat{s}_{mn}$ starting just from $\hat{s}_{\lambda_1\lambda_2}$. The recurrence relation (4.8) is related to (3.12) by $m\to\lambda_2-n$, $n\to\lambda_1-m$, $\lambda_1\leftrightarrow-\lambda_2$, $c\to-(a+b)$ so that we can then use the same solution with $\hat{s}_{mn}\propto\hat{r}_{\lambda_2-n\,\lambda_1-m}$ (also in (3.11) and (4.7) we may let $a\to-a$, $b\to 0$). This gives

$$\begin{aligned}\hat{s}_{mn}=(-1)^{m+n}&\frac{(\lambda_1-n)!\,(\lambda_1+a+b+1+\frac{1}{2}\varepsilon)_m(\lambda_2+a+b+1)_n(\varepsilon)_{m-n}}{(\lambda_1-m)!\,(\lambda_2-n)!\,(m-n)!\,(\frac{1}{2}\varepsilon)_{\lambda_1-n+1}}\\ &\times(m-n+\tfrac{1}{2}\varepsilon)\,(\tfrac{1}{2}\varepsilon)_{\lambda_1}\lambda_2!\\ &\qquad\times{}_4F_3\left(\begin{matrix}-\lambda_-,n-m,-\lambda_1-\lambda_2-a-b-1,\frac{1}{2}\varepsilon\\ -\lambda_1+n,-\lambda_1-a-b-\frac{1}{2}\varepsilon-m,\,\varepsilon\end{matrix};1\right),\end{aligned}\tag{4.9}$$

where we have chosen a normalisation such that $\hat{s}_{00}=1$.

5. Recurrence Relations for Conformal Amplitudes

With the aid of the solution obtained in section 3 we may determine recurrence relations for conformal amplitudes which are valid for general dimension. We first exhibit

$$\begin{aligned}\Big(\frac{x+z}{xz}-1\Big)F_{\lambda_1\lambda_2}(x,z)&=F_{\lambda_1\lambda_2-1}(x,z)+\frac{\lambda_-(\lambda_--1+\varepsilon)}{(\lambda_-+\frac{1}{2}\varepsilon)(\lambda_--1+\frac{1}{2}\varepsilon)}F_{\lambda_1-1\lambda_2}(x,z)\\ &-\tfrac{1}{4}\big((2\lambda_1+c)(2\lambda_1+c-2)+(2\lambda_2+c-\varepsilon)(2\lambda_2+c-2-\varepsilon)\\ &\qquad\qquad-\varepsilon(\varepsilon-2)\big)\hat{A}_{\lambda_1\lambda_2-\frac{1}{2}\varepsilon}\,F_{\lambda_1\lambda_2}(x,z)\\ &+\frac{(\lambda_1+\lambda_2+c-1)(\lambda_1+\lambda_2+c-\varepsilon)}{(\lambda_1+\lambda_2+c-\frac{1}{2}\varepsilon)(\lambda_1+\lambda_2+c-\frac{1}{2}\varepsilon)}\,B_{\lambda_1}F_{\lambda_1+1\lambda_2}(x,z)\\ &+\frac{\lambda_-(\lambda_--1+\varepsilon)}{(\lambda_-+\frac{1}{2}\varepsilon)(\lambda_--1+\frac{1}{2}\varepsilon)}\,\frac{(\lambda_1+\lambda_2+c-1)(\lambda_1+\lambda_2+c-\varepsilon)}{(\lambda_1+\lambda_2+c-\frac{1}{2}\varepsilon)(\lambda_1+\lambda_2+c-\frac{1}{2}\varepsilon)}\,B_{\lambda_2-\frac{1}{2}\varepsilon}F_{\lambda_1\lambda_2+1}(x,z)\,,\end{aligned}\tag{5.1}$$

with B_p and $\hat{A}_{pq}$ as in (2.19). This result reduces to the special case for $(1-v)G^{(\ell)}_{\Delta}(u,v)/u$ given in [5] when $\varepsilon=2$.

More intricately we have

$$
\begin{aligned}
&\Big(\frac{1}{xz} - \frac{x+z}{2\,xz} + \frac{1}{4}\Big) F_{\lambda_1\lambda_2}(x,z) \\
&= F_{\lambda_1-1\lambda_2-1}(x,z) - A_{\lambda_1}\, F_{\lambda_1\lambda_2-1}(x,z) - \frac{\lambda_-(\lambda_- -1+\varepsilon)}{(\lambda_- + \frac12\varepsilon)(\lambda_- - 1 + \frac12\varepsilon)} A_{\lambda_2-\frac12\varepsilon}\, F_{\lambda_1-1\lambda_2}(x,z) \\
&\quad + B_{\lambda_1}\, F_{\lambda_1+1\lambda_2-1}(x,z) + \frac{\lambda_-(\lambda_- -1)(\lambda_- -1+\varepsilon)(\lambda_- -2+\varepsilon)}{(\lambda_- + \frac12\varepsilon)(\lambda_- -1+\frac12\varepsilon)^2(\lambda_- -2+\frac12\varepsilon)} B_{\lambda_2-\frac12\varepsilon}\, F_{\lambda_1-1\lambda_2+1}(x,z) \\
&\quad + \bigg(A_{\lambda_1} A_{\lambda_2-\frac12\varepsilon} \\
&\quad - \frac{\varepsilon(\varepsilon-2)}{32(\lambda_- -1+\frac12\varepsilon)(\lambda_- +1+\frac12\varepsilon)(\lambda_1+\lambda_2+c-\frac12\varepsilon)(\lambda_1+\lambda_2+c-2-\frac12\varepsilon)} \\
&\quad \times \Big(\tfrac12(2\lambda_1+c)(2\lambda_1+c-2) + \tfrac12(2\lambda_2+c-\varepsilon)(2\lambda_2+c-2-\varepsilon) - (2a-c)^2 - (2b-c)^2 \\
&\qquad\qquad + (2a-c)(2b-c)(A_{\lambda_1} + A_{\lambda_2-\frac12\varepsilon}) - 16\, A_{\lambda_1} A_{\lambda_2-\frac12\varepsilon} \Big)\bigg) F_{\lambda_1\lambda_2}(x,z) \\
&\quad - \frac{(\lambda_1+\lambda_2+c-1)(\lambda_1+\lambda_2+c-\varepsilon)}{(\lambda_1+\lambda_2+c-\frac12\varepsilon)(\lambda_1+\lambda_2+c-1-\frac12\varepsilon)} \bigg(B_{\lambda_1} A_{\lambda_2-\frac12\varepsilon}\, F_{\lambda_1+1\lambda_2}(x,z) \\
&\qquad\qquad + \frac{\lambda_-(\lambda_- -1+\varepsilon)}{(\lambda_- +\frac12\varepsilon)(\lambda_- -1+\frac12\varepsilon)} B_{\lambda_2-\frac12\varepsilon} A_{\lambda_1}\, F_{\lambda_1\lambda_2+1}(x,z) \bigg) \\
&\quad + \frac{(\lambda_1+\lambda_2+c)(\lambda_1+\lambda_2+c-1)(\lambda_1+\lambda_2+c+1-\varepsilon)(\lambda_1+\lambda_2+c-\varepsilon)}{(\lambda_1+\lambda_2+c-1-\frac12\varepsilon)(\lambda_1+\lambda_2+c-\frac12\varepsilon)^2(\lambda_1+\lambda_2+c+1-\frac12\varepsilon)} \\
&\qquad\qquad \times B_{\lambda_1} B_{\lambda_2-\frac12\varepsilon}\, F_{\lambda_1+1\lambda_2+1}(x,z)\,,
\end{aligned} \tag{5.2}
$$

where we here define

$$
A_p = \frac{(2a-c)(2b-c)}{2(2p+c)(2p+c-2)}\,. \tag{5.3}
$$

Again this result reduces to the special case for $(1+v)G^{(\ell)}_{\Delta}(u,v)/u$ given in [5] when $\varepsilon = 2$.

Both (5.1) and (5.2) can be proved using (3.4) and the first result in (3.8) in the expansion in terms of Jack polynomials together with various results for the expansion coefficients which are listed in the appendix.

The formula obtained for the solution when $\varepsilon = 4$ is also a special case of a relation between the results for ε and $\varepsilon + 2$ which takes the form, for a normalisation given by (3.14),

$$
\begin{aligned}
\frac{(x-z)^2}{(xz)^2} F^{(\varepsilon+2)}_{\lambda_1\lambda_2}(x,z) &= F^{(\varepsilon)}_{\lambda_1\,\lambda_2-2}(x,z) - \frac{(\lambda_- +\varepsilon)(\lambda_- +1+\varepsilon)}{(\lambda_- +\frac12\varepsilon)(\lambda_- +1+\frac12\varepsilon)} F^{(\varepsilon)}_{\lambda_1-1\,\lambda_2-1}(x,z) \\
&\quad - \frac{(\lambda_1+\lambda_2+c-1-\varepsilon)(\lambda_1+\lambda_2+c-2-\varepsilon)}{(\lambda_1+\lambda_2+c-1-\frac12\varepsilon)(\lambda_1+\lambda_2+c-2-\frac12\varepsilon)}
\end{aligned} \tag{5.4}
$$

$$\times\left(B_{\lambda_1}F^{(\varepsilon)}_{\lambda_1+1\,\lambda_2-1}(x,z)-\frac{(\lambda_-+\varepsilon)(\lambda_-+1+\varepsilon)}{(\lambda_-+\frac{1}{2}\varepsilon)(\lambda_-+1+\frac{1}{2}\varepsilon)}B_{\lambda_2-1-\frac{1}{2}\varepsilon}\,F^{(\varepsilon)}_{\lambda_1\,\lambda_2}(x,z)\right)$$
$$-(\lambda_1+\lambda_2+c-2-\varepsilon)(\lambda_-+1+\varepsilon)\hat{A}_{\lambda_1\,\lambda_2-1-\frac{1}{2}\varepsilon}\,F^{(\varepsilon)}_{\lambda_1\,\lambda_2-1}(x,z)\,.$$

This may be obtained from an analogous relation for $P_{\lambda_1\lambda_2}$,

$$\frac{(x-z)^2}{(xz)^2}P^{(\varepsilon+2)}_{\lambda_1\lambda_2}(x,z)=\frac{2(1+\varepsilon)}{\lambda_-+1+\frac{1}{2}\varepsilon}\Big(P^{(\varepsilon)}_{\lambda_1\,\lambda_2-2}(x,z)-P^{(\varepsilon)}_{\lambda_1-1\,\lambda_2-1}(x,z)\Big)\,, \tag{5.5}$$

which follows from standard identities for $C_n^{\frac{1}{2}\varepsilon}(\sigma)$.

6. Conclusion

We have endeavoured to obtain detailed expressions for conformal partial wave amplitudes which may be relevant in the operator product expansion analysis of conformal four point functions. The results for two and four dimensions, which confirm those obtained earlier in [4], are as simple as may reasonably be expected. For six dimensions the results obtained here do not have any apparent further simplifications, except in possible special cases. However for general dimensions, in particular for three, the results given as an expansion in terms of Jack polynomials, while related to similar results in the mathematical literature, still leave some things to be desired which might be expected to hold in more explicit expressions. In particular we may note the symmetry properties of the four point function (1.2) under interchange of fields which is reflected in corresponding properties of the conformal amplitudes. To exhibit these we define

$$x'=\frac{x}{x-1}\,,\qquad z'=\frac{z}{z-1}\,, \tag{6.1}$$

and then, from the form of the operator in (2.3), we have

$$\big((1-x)(1-z)\big)^{-b}D_{\varepsilon,x',z'}\big((1-x)(1-z)\big)^{b}=D_{\varepsilon,x,z}\big|_{a\to c-a}\,, \tag{6.2}$$

and also correspondingly for $a\leftrightarrow b$. Hence we have

$$(-1)^{\lambda_1+\lambda_2}F^{(\varepsilon)}_{\lambda_1\lambda_2}(a,b;c;x',z')=\big((1-x)(1-z)\big)^{b}F^{(\varepsilon)}_{\lambda_1\lambda_2}(c-a,b;c;x,z)\,, \tag{6.3}$$

which corresponds in (1.2) to taking $\phi_1(\eta_1)\leftrightarrow\phi_2(\eta_2)$. The analogous result obtained when $a\leftrightarrow b$ is also interpretated as corresponding to $\phi_3(\eta_3)\leftrightarrow\phi_4(\eta_4)$. This symmetry is easily demonstrated in the explicit solutions for $d=2,4,6$ but is necessarily hidden in the Jack polynomial expansion.

On the other hand we may note that Jack polynomials have a ready extension to superspace [17] so that there should be corresponding version of superconformal partial wave amplitudes.

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Appendix A. Solution of Recurrence Relation

The ${}_4F_3$ function appearing in (3.19) is balanced or Saalschützian, i.e. we have

$$ {}_4F_3\Big(\begin{matrix} a,b,c,d \\ e,f,g \end{matrix};1\Big)\,, \qquad a+b+c+d+1 = e+f+g\,, \tag{A.1}$$

with one of a,b,c,d a negative integer. In this case there are important contiguous relations, see section 3.7 in [18]. The crucial one has the form

$$\begin{aligned} & e(f-1)(e-a)(g-a)\bigg({}_4F_3\Big(\begin{matrix} a{-}1,b,c,d \\ e,f{-}1,g \end{matrix};1\Big) - {}_4F_3\Big(\begin{matrix} a,b,c,d \\ e,f,g \end{matrix};1\Big)\bigg) \\ & + a(e-b)(e-c)(e-d)\bigg({}_4F_3\Big(\begin{matrix} a{+}1,b,c,d \\ e{+}1,f,g \end{matrix};1\Big) - {}_4F_3\Big(\begin{matrix} a,b,c,d \\ e,f,g \end{matrix};1\Big)\bigg) \\ & \quad = -(e-a)bcd\, {}_4F_3\Big(\begin{matrix} a,b,c,d \\ e,f,g \end{matrix};1\Big)\,. \end{aligned} \tag{A.2}$$

If we define

$$ F_{mn} = {}_4F_3\bigg(\begin{matrix} -\lambda_-, -\lambda_- - m + n, \lambda_1+\lambda_2+c-1, \tfrac{1}{2}\varepsilon \\ -\lambda_- - m, 2\lambda_2 + c - \tfrac{1}{2}\varepsilon + n,\, \varepsilon \end{matrix};1\bigg)\,, \tag{A.3}$$

then (A.2) gives

$$\begin{aligned} & n(\lambda_- + m)(\lambda_- + m - n + \varepsilon)(2\lambda_2 + c - 1 - \tfrac{1}{2}\varepsilon + n)F_{m\,n-1} \\ & + m(\lambda_- + m + \tfrac{1}{2}\varepsilon)(\lambda_- + m - n)(2\lambda_1 + c - 1 + m)F_{m-1\,n} \\ & = (\lambda_- + m)(\lambda_- + m - n + \tfrac{1}{2}\varepsilon)\big(m(2\lambda_1 + c - 1 + m) + n(2\lambda_2 + c - 1 - \varepsilon + n)\big)F_{mn}\,. \end{aligned} \tag{A.4}$$

It is then straightforward to verify that (3.19) satisfies (3.12). It is evident that F_{m0} reduces to a ${}_3F_2$ function and we have

$$ F_{m0} = \frac{(-\lambda_1 - \lambda_2 - c + 1 + \varepsilon)_{\lambda_-}\,(\tfrac{1}{2}\varepsilon)_{\lambda_-}}{(-\lambda_1 - \lambda_2 - c + 1 + \tfrac{1}{2}\varepsilon)_{\lambda_-}\,(\varepsilon)_{\lambda_-}}\,. \tag{A.5}$$

We may also use contiguous relations to obtain formulae involving changes in λ_1, λ_2 by ± 1 and also ε by 2. These were used to prove the various relations given in section 5 but are rather long winded to write down. We here present them for $\bar{r}^{(\varepsilon)}_{\lambda_1\lambda_2,mn}$ which is a solution of (3.12) with $\bar{r}^{(\varepsilon)}_{\lambda_1\lambda_2,00} = 1$, $\hat{r}_{mn} = (\varepsilon)_{\lambda_-} \bar{r}^{(\varepsilon)}_{\lambda_1\lambda_2,mn}/(\frac{1}{2}\varepsilon)_{\lambda_-}$,

$$
\begin{aligned}
&(m+1)(\lambda_- - n - 1 + \tfrac{1}{2}\varepsilon)\, \bar{r}^{(\varepsilon)}_{\lambda_1-1\lambda_2,m+1n} \\
&+ (n+1)(\lambda_- + m + 1 + \tfrac{1}{2}\varepsilon)\, \bar{r}^{(\varepsilon)}_{\lambda_1\lambda_2-1,mn+1} \\
&- 2(\lambda_- + \tfrac{1}{2}\varepsilon)(\lambda_1+\lambda_2+c-1-\varepsilon) \\
&\qquad\qquad \times \frac{(2\lambda_1+2m+c)(2\lambda_2+2n+c-\varepsilon)}{(2\lambda_1+c-2)(2\lambda_1+c)(2\lambda_2+c-2-\varepsilon)(2\lambda_2+c-\varepsilon)}\, \bar{r}^{(\varepsilon)}_{\lambda_1\lambda_2,mn} \\
&+ \frac{(\lambda_1+\lambda_2+c-1-\varepsilon)(\lambda_1+\lambda_2+c-\varepsilon)}{(\lambda_1+\lambda_2+c-1-\frac{1}{2}\varepsilon)(\lambda_1+\lambda_2+c-\frac{1}{2}\varepsilon)}(\lambda_1+\lambda_2+c+m-\tfrac{1}{2}\varepsilon) \\
&\qquad\qquad \times \frac{2\lambda_2+n+c-\varepsilon}{(2\lambda_2+c-1-\varepsilon)(2\lambda_2+c-\varepsilon)^2(2\lambda_2+c+1-\varepsilon)}\, \bar{r}^{(\varepsilon)}_{\lambda_1\lambda_2+1,mn-1} \\
&- \frac{(\lambda_1+\lambda_2+c-1-\varepsilon)(\lambda_1+\lambda_2+c-\varepsilon)}{(\lambda_1+\lambda_2+c-1-\frac{1}{2}\varepsilon)(\lambda_1+\lambda_2+c-\frac{1}{2}\varepsilon)}(\lambda_1+\lambda_2+c+n-\tfrac{1}{2}\varepsilon) \\
&\qquad\qquad \times \frac{2\lambda_1+m+c}{(2\lambda_1+c-1)(2\lambda_1+c)^2(2\lambda_1+c+1)}\, \bar{r}^{(\varepsilon)}_{\lambda_1+1\lambda_2,m-1n} = 0\,.
\end{aligned}
\tag{A.6}
$$

For a change in ε,

$$
\begin{aligned}
&(m+1)(2\lambda_1+c-2)\, \bar{r}^{(\varepsilon)}_{\lambda_1-1\lambda_2,m+1n} \\
&- (\lambda_- + m + 1 + \tfrac{1}{2}\varepsilon)(2\lambda_2+c-2-\varepsilon)\, \bar{r}^{(\varepsilon)}_{\lambda_1\lambda_2-1,mn+1} \\
&+ \frac{(\lambda_1+\lambda_2+c-1-\varepsilon)(\lambda_1+\lambda_2+c-\varepsilon)}{(\lambda_1+\lambda_2+c-1-\frac{1}{2}\varepsilon)(\lambda_1+\lambda_2+c-\frac{1}{2}\varepsilon)}(\lambda_1+\lambda_2+c+m-\tfrac{1}{2}\varepsilon) \\
&\qquad\qquad \times \frac{1}{(2\lambda_2+c-1-\varepsilon)(2\lambda_2+c-\varepsilon)(2\lambda_2+c+1-\varepsilon)}\, \bar{r}^{(\varepsilon)}_{\lambda_1\lambda_2+1,mn-1} \\
&- \frac{(\lambda_1+\lambda_2+c-1-\varepsilon)(\lambda_1+\lambda_2+c-\varepsilon)}{(\lambda_1+\lambda_2+c-1-\frac{1}{2}\varepsilon)(\lambda_1+\lambda_2+c-\frac{1}{2}\varepsilon)} \\
&\qquad\qquad \times \frac{2\lambda_1+m+c}{(2\lambda_1+c-1)(2\lambda_1+c)(2\lambda_1+c+1)}\, \bar{r}^{(\varepsilon)}_{\lambda_1+1\lambda_2,m-1n} \\
&+ (\lambda_- + \tfrac{1}{2}\varepsilon)\frac{\lambda_- + m - n + \frac{1}{2}\varepsilon}{\lambda_- + m - n - 1 + \frac{1}{2}\varepsilon}(2\lambda_2+2n+c-\varepsilon)\, \bar{r}^{(\varepsilon+2)}_{\lambda_1\lambda_2+1,mn+1} = 0\,.
\end{aligned}
\tag{A.7}
$$

There is also an associated independent relation obtained from (A.7) by $\lambda_1 \to \lambda_2 - \frac{1}{2}\varepsilon$, $\lambda_2 \to \lambda_1 + \frac{1}{2}\varepsilon$, $m \leftrightarrow n$ and then letting $\bar{r}^{(\varepsilon)}_{\lambda_2-\frac{1}{2}\varepsilon\,\lambda_1+\frac{1}{2}\varepsilon,nm} \to \bar{r}^{(\varepsilon)}_{\lambda_1\lambda_2,mn}$ ((A.6) is unchanged under this transformation).

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