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EPFL Lectures on Conformal Field Theory in $D \geqslant 3$ Dimensions

Lecture notes prepared with the help of G. Karananas
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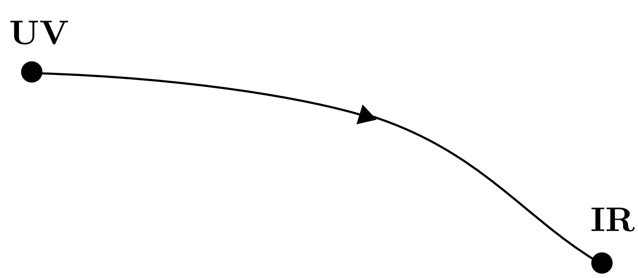
Contents

Lecture 1

Physical Foundations of Conformal Symmetry

1.1 Fixed points

Quantum Field Theory (QFT) is, in most general terms, the study of Renormalization Group (RG) flows, i.e. how the theory evolves from the Ultraviolet (UV) to the Infrared (IR) regimes:



One can ask which IR phases are possible. A priori, there are three possibilities:

A. a theory with a mass gap,

B. a theory with massless particles in the IR,

C. a Scale Invariant (SI) theory with a continuous spectrum.

It is the last class that we will call CFT and will mostly study in these lectures. But first let's look at some examples corresponding to these phases.

Nonabelian Yang-Mills (YM) theory in $D = 4$ dimensions belongs to type A.

$$\mathscr{L} = -\frac{1}{4g^2} F^a_{\mu\nu} F^{a\mu\nu} \ . \tag{1.1.1}$$

The beta function is negative

$$\beta(g) \sim -cg^3 + \mathcal{O}(g^5) \ . \tag{1.1.2}$$

so the theory becomes free at asymptotically high energies (the coupling goes to zero), while at low energies the coupling grows and the theory becomes nonperturbative at the scale

$$\Lambda_{IR} = \Lambda_{UV} \exp(-const./g_{UV}^2). \tag{1.1.3}$$

The low energy spectrum will include the lightest "glueball", a scalar particle of mass $m \sim \Lambda_{IR} > 0$, and some heavier stuff.

QED gives a trivial example of a theory with massless particles in the IR (i.e. type B). For energies $E \ll m_e$, the mass of the electron, we are left with free photons. If the electron were massless, the IR behavior would still be Type B, but it would be reached very slowly (logarithmically)

A less trivial type B example is massless QCD with a small number N_f of fermionic flavors. The UV theory exhibits invariance under $G = SU(N_f)_L \times SU(N_f)_R \times U(1)_B$. In the IR, the chiral symmetry breaks spontaneously $SU(N_f)_L \times SU(N_f)_R \to SU(N_f)_{diag}$, which results in a theory with massless Goldstone bosons.

Examples of type C behavior exist among "fixed point" theories for which the beta function vanishes at some point $g = g_*$, while the theory is still weakly coupled (i.e. we are still inside the perturbative regime):

$$\beta(g_*) = 0 \ , \ g_* \ll 1 \ . \tag{1.1.4}$$

Consider again the YM theory with N_c colors and N_f fermions. The beta function at two loops is given by

$$\beta(g) = -\beta_0 \frac{g^3}{16\pi^2} + \beta_1 \frac{g^5}{(16\pi^2)^2} + \mathcal{O}(g^7) \ , \tag{1.1.5}$$

with

$$\beta_0 = \frac{11}{3} N_c - \frac{2}{3} N_f \quad \text{and} \quad \beta_1 \sim \mathcal{O}(N_c^2, N_c N_f) \ . \tag{1.1.6}$$

Let us choose $N_c, N_f \gg 1$, in such a way that $\beta_0 \sim \mathcal{O}(1)$ (near cancelation). With this choice one can check that $\beta_1 \sim \mathcal{O}(N_c^2)$ (no cancellation) Writing

$$\beta(g) = -\frac{g^3}{16\pi^2} \left(1 - \frac{\beta_1}{\beta_0} \frac{g^2}{16\pi^2} + \ldots \right) \ . \tag{1.1.7}$$

we see that there is a zero at $g = g_* = O(1/(16\pi^2 N_c^2))$. Define λ by

$$\lambda_* = \frac{N_c g_*^2}{16\pi^2} \sim \mathcal{O}(1/N_c) \ , \tag{1.1.8}$$

The effects of the corrections from higher order terms are suppressed by powers of the coupling λ_*, so the perturbative expansion is trustworthy. This IR fixed point is known as the "Banks-Zaks (BZ) fixed point".

For a finite N_c, depending on N_f, there will be the following cases:

1. For N_f small described above, we will have chiral symmetry breaking (Type B)
2. In a range $N_{f,critical} < N_f < \frac{11}{2}N_c$, (so called "conformal window") the theory possesses a Banks-Zaks fixed point (Type C)
3. For $N_f > 11/2N_c$, the theory is not asymptotically free, like massless QED. So IR behavior is asymptotically Type B.

An IR behavior of a generic fixed point theory will be very much unlike standard QFT: the IR spectrum will be continuous and there will be no well-defined particles. When the theory flows, any gauge-invariant operator acquires an anomalous dimension, and when the fixed point is reached, this anomalous dimension "freezes". E.g. the anomalous dimension of $\overline{\psi}\psi$ at the BZ fixed point freezes at

$$\gamma(g_*) = -\frac{g_*^2}{2\pi^2} \neq 0 \ . \tag{1.1.9}$$

This means that this operator, and generically any operator will have non-integer dimensions at the IR fixed point:

$$\Delta = \Delta_{free} + \gamma(g_*) \ . \tag{1.1.10}$$

This in turn implies that the spectrum of this theory will be continuous.

To see why this is so, consider the two-point function of an operator ϕ. Inserting the complete basis of states, we can write:

$$\langle 0|\phi(x)\phi(0)|0\rangle = \int_{F.C.} \frac{d^4p}{(2\pi)^4} e^{-ipx} |\langle 0|\phi|p\rangle|^2 \equiv \int_{F.C.} \frac{d^4p}{(2\pi)^4} e^{-ipx} \rho(p^2) \ , \tag{1.1.11}$$

where $F.C.$ means that the states in healthy theory will have momenta in the Forward Light Cone: $p^2 > 0, p^0 > 0$. The $\rho(p^2)$ is the spectral density for this two point function, in a unitary theory it will be positive.

Now let's consider some examples. For $\Delta_\phi = 1$, it is easy to show that in momentum space the spectral density corresponds to the one of free massless scalar particles (concentrated on the lightcone):

$$\frac{1}{x^2} \to \rho(p^2) = \delta(p^2) \ , \tag{1.1.12}$$

For $\Delta_\phi = 2$, we see that

$$\frac{1}{x^4} \to \rho(p^2) = p^2 \propto \int_{F.C.} d^4q \delta[(p-q)^2]\delta(q^2) \ . \tag{1.1.13}$$

This is not concentrated on the lightcone, but is given by phase space of two massless particle states.

For $\Delta_\phi = 1 + \gamma$ non integer,

$$\frac{1}{(x^2)^{1+\gamma}} \to \rho(p^2) \sim (p^2)^{\gamma - 1} \ . \tag{1.1.14}$$

We see that the spectrum of theory, the set of p^2 values at which $\rho(p^2) \neq 0$, is continuous: it spreads from 0 to $+\infty$. Also there is no way to represent it by a phase space of finitely many massless particles, like for the ϕ^2 operator. Naively we have a fractional number of particles. More correctly, such a theory simply cannot be interpreted in terms of particles. In such a theory there is no S-matrix and the only observables are the correlation functions.

Another example of a Type C theory, which also has an important phenomenological meaning, is $\lambda\phi^4$ in $2 \leqslant D < 4$ dimensions

$$\mathscr{L} = \frac{1}{2}(\partial\phi)^2 + m^2\phi^2 + \lambda\phi^4 \ , \tag{1.1.15}$$

Since λ is dimensionful in $D \neq 4$, let us express m and λ as

$$m^2 = t\mu^2 \quad \text{and} \quad \lambda = \bar{\lambda}\mu^{4-D} \ . \tag{1.1.16}$$

in terms of the renormalization scale μ. The mass dimension of these couplings is positive, which is characteristic of couplings multiplying relevant operators.[1] Let us analyze the structure of the above theory by using the beta functions

$$\begin{aligned} \beta(\bar{\lambda}) &= -(4-D)\bar{\lambda} + c_1 t^2 + c_2\bar{\lambda}^2 + \dots \\ \beta(t) &= -2t + c_3 t^2 + c_4 t\bar{\lambda} + c_5\bar{\lambda}^2 + \dots \ , \end{aligned} \tag{1.1.17}$$

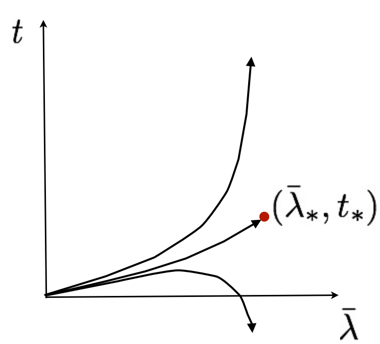
with c_i various computable constants.

We start the RG flow in the UV with $t, \bar{\lambda}$ very close to zero. We see that both couplings start growing, since they are both relevant. In $D = 4$, the beta function of $\bar{\lambda}$ does not have a fixed point in one-loop order. In $D = 4 - \epsilon$ there is a fixed point (λ_*, t_*) where both beta functions vanish:

$$\beta(\bar{\lambda}_*) = \beta(t_*) = 0 \ , \bar{\lambda}_* = O(\epsilon), t_* = O(\epsilon^2) \ . \tag{1.1.18}$$

This is called the Wilson-Fisher fixed point. Perturbative analysis can be trusted only for $\epsilon \ll 1$, but the fixed point actually exists from $D = 4 - \epsilon$ all the way down to $D = 2$.

At the IR fixed point the ϕ^4 will become irrelevant, but ϕ^2 operator will still be relevant. This fact has important consequences for the physics of this theory, as can be seen in the RG flow diagram:



[1]Recall that relevant (irrelevant) operators are those of dimension $\Delta < D$ ($\Delta > D$). Operators of dimension $\Delta = D$ are called marginal.

There is one RG trajectory connecting the origin (free theory) to $(\bar{\lambda}_* = 0, t_* = 0)$ and one has to choose the initial conditions to lie precisely on this trajectory to reach the fixed point. If the initial value of t is slightly different, then we end up with a theory of massive particles, either $t \to \infty$ so that Z_2 symmetry is presevred, or $t \to -\infty$ where Z_2 symmetry is broken. The source of this instability is the fact that ϕ^2 is a relevant operator even at the fixed point.

So we can classify fixed points into two categories:

1. Stable — theories that do not contain relevant scalar operators which are singlets in the sense explained next.
2. Unstable — theories in which there exists relevant scalar operators that are singlets under all global (internal) symmetries. In these theories, the coefficients of these operators cannot be naturally assumed to be small, so fine-tuning is necessary.

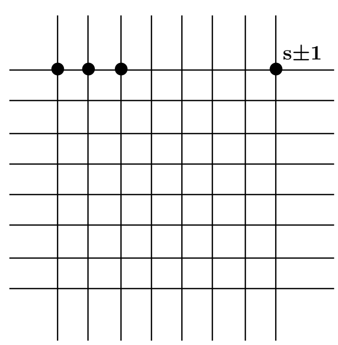
The hierarchy problem of the Standard Model can be seen as the fact that the free scalar theory in $D = 4$ is an unstable scale invariant theory, the operator $|H|^2$ being a relevant operator singlet under all global symmetries.

The Banks-Zaks fixed point is stable. E.g. the operator $\bar{\psi}\psi \equiv \overline{\psi}_L\psi_R + h.c.$ is relevant, but it's not a singlet under the global symmetry $SU(N_f)_L \times SU(N_f)_R$, and therefore does not affect the stability.

Let us consider now a theory from statistical mechanics, the Ising model, which is a microscopic model for ferromagnetism. Its Hamiltonian (i.e. Euclidean action) is given by

$$H = \frac{1}{T} \sum_{<ij>} (1 - s_i s_j) \ , \tag{1.1.19}$$

where T is the temperature. The summation is understood to be over neighboring points of a cubic lattice in R^D and the spins can take the values $s = \pm 1$.



The observables we are interested in are the correlation functions of the spin, that for large distances behaves as

$$\langle s(r)s(0)\rangle \sim e^{-r/\xi(T)} \ , \tag{1.1.20}$$

where $\xi(T)$ is the correlation length. This correlation length is the analogue of inverse mass in particle physics. Indeed, consider the free scalar propagator in Euclidean coordinate space:

$$\int \frac{d^D p}{(2\pi)^4} \frac{e^{ipx}}{p^2 + m^2} \sim \frac{e^{-rm}}{r^{D-2}} \ . \tag{1.1.21}$$

At large distances, $r \gg 1/m$ and we can identify $\xi(T) = 1/m$. As is well known, this model has a critical temperature $T = T_c$, at which $\xi(T_c) = \infty$. This critical theory is scale invariant with continuous spectrum (Type C). In fact this critical theory is identical to the Wilson-Fischer fixed point in D dimensions. This is due to a phenomenon called *Universality*: that in the continuum limit microscopic details of the Lagrangian don't matter and all theories with the same symmetry look the same (up to identification of couplings).

In the Ising model, to reach the critical point we have to fine-tune the temperature to its critical value T_c. This finetuning is the same as the one needed in $\lambda\phi^4$ theory to reach the Wilson-Fisher fixed point.

1.2 Existing techniques

Our goal is to develop methods which allow us to "solve" fixed points. By "solve" we mean compute the observables: the operator dimensions and the correlation functions. In what follows, we give an outline of the methods already developed:

1. Monte-Carlo simulations for lattice models. This method works, but there are numerical difficulties as well as artifacts associated to the lattice and the finite size.

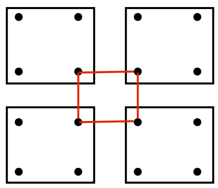
2. High-temperature expansion for lattice models. Consider again the Ising model in (1.1.19) with partition function $Z = \exp[-H]$ and expand the exponential for T larger than T_c. Each term in the expansion will be a finite sum in spins, however the series will start diverging near the critical point. This is the same as the strong coupling expansion in field theory

3. The ϵ-expansion. If we work in $D = 4 - \epsilon$, the loop expansion coincides with the expansion in ϵ and the critical point is weakly coupled. Setting $\epsilon = 1$, one can try to recover the physical case of 3D ferromagnets. However the series obtained this way are badly divergent starting from the second-third term. In order to get results, one has to invent procedures like Borrel summation etc.

Each of the above techniques has systematics which are difficult to control. Any of them by itself would be difficult to believe, but since they more or less mutually agree, there is a coherent picture.

4. The Exact Renormalization Group (ERG).

Consider a lattice theory and do a block-spin transformation, i.e. unite spins into blocks:



The new block variables which are coupled more strongly than the original ones. In terms of those, even though the partition function remains the same, the Hamiltonian is changed since there will appear diagonal, quartic couplings etc. (We assume that the original Hamiltonian had only nearest neighbor couplings)

Every time we repeat this procedure we end up with a new Hamiltonian with different interaction terms. The ERG is a flow in this infinite dimensional space of Hamiltonians. The fixed point is defined as the Hamiltonian which is invariant under this transformation

$$\mathcal{R}[H] = H \ . \tag{1.2.1}$$

Suppose we have found such a Hamiltonian. The natural question to ask is how we can define local operators. We can define them by perturbing this Hamiltonian

$$H + \Delta H \ . \tag{1.2.2}$$

and demanding that the perturbation transform homogeneously under the renormalization group transformation:

$$\mathcal{R}[H + \Delta H] = H + \lambda \Delta H \tag{1.2.3}$$

I.e. we are solving an eigenvalue problem for the ERG transformation linearized around the fixed point hamiltonian. The eigenvector perturbations will be in one-to-one correspondence with the local operators at the fixed point, while the eigenvalue λ will be related to the operator dimension: $\lambda = 2^{4-\Delta}$ (2 because we are changing scale by a factor of 2).

The local operators at the fixed point will be given by a sum of infinitely many terms. E.g. the spin field operator will be give by something like

$$s_i + 3 \text{ spin terms} + 5 \text{ spin terms} + \ldots \tag{1.2.4}$$

while the first Z_2 even operator will be given by

$$s_i s_{i+1} + 2 \text{ spin terms at larger separation} + 4 \text{ spin terms} + \ldots \ , \tag{1.2.5}$$

with all the relative coefficients fixed.

In practice one has to truncate. For example one can allow all couplings within some large distance N. One compute the fixed point Hamiltonian and the local operators and we see that their dimensions are in good agreement with other techniques. Also, one sees that the off-diagonal, quartic, etc couplings are suppressed, e.g. the long distance couplings are suppressed like

$$\frac{1}{n^\gamma} s_i s_{i+n} \ , \tag{1.2.6}$$

so that the fixed-point Hamiltonian remains relatively short ranged.

We now consider the continuous version of the ERG. We start with a scalar theory and we add all operators

$$(\partial\phi)^2 + \sum_i c_i \Lambda^{D-\Delta}\mathcal{O}_i \ , \tag{1.2.7}$$

with infinitely many couplings c_i that can even become non-perturbative. To compensate for the large couplings, we can take a smooth cut-off and integrate out an infinitesimal momentum shell. In this case truncation consists in say including all powers of ϕ but not terms with derivatives. This can be post factum motivated by the fact that the fixed point Hamiltonian remains rather local, so that the expansion in derivatives should work

Limitations of this technique become apparent if we consider a YM theory. All gauge-invariant operators include more derivatives and more powers of the gauge field simultaneously. The situation is better in presence of fermions, since we can consider powers of $\overline{\psi}\psi$. So the QCD conformal window might be amenable to ERG. Another bad case is gravity, since all higher dimensional operators like $R^2, R_{\mu\nu}R^{\mu\nu}, \ldots$, contain more and more derivatives of the metric. So looking for a UV fixed point of gravity with this technique is very dubious.

In summary, the ERG works when we know that a fixed point exists and that the fixed point Hamiltonian is rather local. It may not be suitable to find new fixed points, or to describe fixed points which do not allow for a local description in terms of the original variables.

The above tools are all based on the RG idea: define a theory at the microscopic level and study its behavior in the IR. However, a philosophical problem arises. The fixed points are universal mathematical objects. It is not clear why in order to solve them we have to approach them with a flow. Can't we solve the fixed points by studying them in isolation?

1.3 Towards a nonperturbative definition

What we practically need is a non-perturbative definition of the theory, without any reference to the microscopic level (Lagrangian etc.).

1.3.1 Operator spectrum

The first thing which characterizes any theory is the spectrum of the local operators

$$\mathcal{O}_i \to \Delta_i = \text{scaling dimension} \ . \tag{1.3.1}$$

Once we know the dimension,the two-point function is given by

$$\langle \mathcal{O}_i(x)\mathcal{O}_i(0)\rangle = \frac{c}{|x|^{2\Delta_i}} \ , \tag{1.3.2}$$

where the coefficient $c = 1$ can be chosen as a normalization convention. Since all correlation functions scale like powers, we can consider scale transformations, that can be formally written as

$$x \to \lambda x \ , \quad \mathcal{O}(x) \to \mathcal{O}(\lambda x) = \lambda^{-\Delta}\mathcal{O}(x) \ . \tag{1.3.3}$$

The above guarantees that the two-point functions are invariant

$$\langle \mathcal{O}(\lambda x_1)\mathcal{O}(\lambda x_2)\rangle = \frac{1}{|\lambda x_1 - \lambda x_2|^{2\Delta}} = \lambda^{-2\Delta}\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle \ . \tag{1.3.4}$$

The physical way to understand the scale transformations is as a RG transformation that leaves the Hamiltonian and the correlation functions invariant. This is true provided that the operators are appropriately rescaled.

The above discussion concerned scalar operators, but there will be operators with nonzero spin as well. We consider Lorentz (or rotation) invariant theories, so the operators will come in the irreducible representations of the $SO(D)$ group.

1.3.2 Stress tensor and currents

Among the local operators of the theory, special role will be played by a stress tensor $T_{\mu\nu}$ and conserved currents J_μ associated to global symmetries of a theory. QFT axioms (Wightman axioms) don't require existence of the stress tensor as the energy and momentum density, but only of full energy and momentum charges, and analogously for the conserved currents. Thus the existence of the stress tensor and currents is an extra assumption. It means that the theory preserves some locality.

If the IR fixed point can be reached from a UV theory which has a weakly coupled Lagrangean description (and thus has a stress tensor), then the existence of a stress tensor in the IR is guaranteed. On the other hand, if we reach the critical point starting from a lattice level, the stress tensor existence is not obvious. On the lattice there is no stress tensor, but it may emerge, together with the rotation invariance, in the continuum limit. This is what happens for the Ising model, but it's not guaranteed.

Let us consider a physical example without a stress tensor. Start from a lattice theory with an explicitly non-local Hamiltonian

$$H = \frac{1}{T}\sum_{i,j}\frac{(1 - s_i s_j)}{|i-j|^\gamma} \ , \tag{1.3.5}$$

where the sum is over all pairs of spins (not just neighboring ones). This is called "long-range Ising model", and it is known to have a critical point in $2 \leqslant D < 4$ for any value of $\gamma > 0$. The properties of this critical point depend on γ. For large γ, $\gamma > \gamma_*$, the critical point is identical to that of the usual, short-range Ising model (and thus has a stress tensor). This is because for large γ the function $1/|i-j|^\gamma$ is highly peaked at $i = j$. So we are in the same universality class as the usual Ising model.

For small $\gamma < \gamma_0 (< \gamma_*)$, the resulting theory is simply Gaussian (non-interacting), so it is completely characterized by the two-point function

$$\langle s(r)s(0)\rangle \sim \frac{1}{r^{2\Delta}} \ , \tag{1.3.6}$$

from which the higher-point functions are computed by Wick's theorem. The dimension Δ depends on γ. This theory does not have a stress tensor, i.e. a spin-2 local conserved operator. It can also be described as the AdS dual of a free scalar theory in the bulk with gravity turned off, which also explains that it does not have a stress tensor.

The previous case was already already an example of a theory without a stress tensor, but a boring one (non-interacting). In the intermediate range, $\gamma_* < \gamma < \gamma_0$ the critical point is an interacting (non-Gaussian) theory, which again does not have a stress tensor.

We see that there can be theories which do not have a stress tensor, and they can be of interest for condensed matter physics.

Going back to the theories which do have a stress tensor and currents. Notice that in perturbation theory these two operators don't renormalize, i.e, have zero anomalous dimensions: $\gamma_T = \gamma_J = 0$. This means that at the fixed point the stress tensor and currents have canonical dimensions, i.e.

$$\Delta_T = D \ , \Delta_J = D - 1 \ . \tag{1.3.7}$$

They are also conserved:

$$\partial_\mu T^{\mu\nu} = 0 \quad \text{and} \quad \partial_\mu J^\mu = 0 \ . \tag{1.3.8}$$

Ward Identities

Consider the Ward identities for an operator $\mathcal{O}_1$ in the UV

$$\partial_\mu \langle T^{\mu\nu}(x)\mathcal{O}_1(x_1)\ldots\rangle = \delta(x - x_1)\langle\partial_\mu\mathcal{O}_1(x_1)\ldots\rangle \ , \tag{1.3.9}$$

and suppose that the theory flows from the UV to the IR. It can be shown that the identity also holds in large distances (IR). To see that, start from the integrated Ward identities

$$\int \partial_\mu a(x)\langle T^{\mu\nu}(x)\mathcal{O}_1(x_1)\ldots\rangle = a(x_1)\langle\partial_\mu\mathcal{O}_1(x_1)\ldots\rangle \ , \tag{1.3.10}$$

and choose $a(x)$ to be equal to 1 (0) inside (outside) the sphere. Therefore,

$$\int_{S^{D-1}} d\Sigma \ n_\mu \langle T^{\mu\nu}(x)\mathcal{O}_1(x_1)\ldots\rangle = \langle\partial_\mu\mathcal{O}_1(x_1)\ldots\rangle \ , \tag{1.3.11}$$

where S^{D-1} the sphere in D dimensions. We have the liberty to choose the radius of the sphere to be arbitrarily large, therefore the above relation will hold in the IR.

Since the identity for the correlators of the theory hold in the IR, there is no need to keep the UV tails of the correlators.

Trace of the stress tensor

$T_{\mu\nu}$ can be decomposed into two parts transforming irreducibly under $SO(D)$, the trace and the symmetric traceless part. Generically different $SO(D)$ representations are expected to have different scaling dimensions. But the trace of the stress tensor has the same dimension as the stress tensor itself: $[T^\mu_{\ \mu}] = D$. A resolution of this paradox is that in fact, generically $T^\mu_{\ \mu} = 0$ at fixed points.

In perturbation theory, this can be seen as a consequence of

$$T^\mu_{\ \mu} \sim \beta(g_i)\mathcal{O}_i \ , \tag{1.3.12}$$

where O_i is the operator multiplying the coupling g_i in the Lagrangian. This equation has subtleties associated with it, especially in the multi-field case with broken global symmetries. But roughly it is true. Then, at a fixed point where the beta function vanishes, the trace of the stress tensor is zero.

Another argument which does not rely on perturbation theory is as follows. The RG transformation which rescales globally the coordinates, can be seen as rescaling the metric as

$$x \to x' = \lambda x \ , \quad g_{\mu\nu} \to g'_{\mu\nu} = \lambda^2 g_{\mu\nu} \ , \tag{1.3.13}$$

where $g_{\mu\nu} \equiv \delta_{\mu\nu}$. For $\lambda - 1 = \epsilon \ll 1$ the metric change is small: $\delta g_{\mu\nu} = 2\epsilon\delta_{\mu\nu}$. Let us now use another definition of the stress tensor, as an operator measuring response to changing the metric. So the Hamiltonian changes by

$$\Delta H = \int d^D x T_{\mu\nu} \delta g^{\mu\nu} = \int d^D x T^\mu_{\ \mu} \ . \tag{1.3.14}$$

This must be zero if the theory is SI. Generically this means that the trace of the stress tensor must be a total divergence of a vector operator:

$$T^\mu_{\ \mu} = \partial_\mu K^\mu \ , \quad [K^\mu] = D - 1 \ . \tag{1.3.15}$$

We see that K_μ has canonical dimension. However generically, all vector operators acquire anomalous dimensions apart from those which are conserved. Thus generically $\partial_\mu K^\mu = 0$, and so we conclude again that $T^\mu_{\ \mu}$ must vanish.

One should stress however that rare examples of scale invariant theories with $T^\mu_{\ \mu} \neq 0$ do exist. An example is the theory of elasticity in the Euclidean space

$$\mathscr{L} = a(\partial_\nu u_\mu)^2 + b(\partial_\mu u_\nu) \ , \tag{1.3.16}$$

with a and b the elasticity moduli. Physical cases are $D = 2, 3$. This theory, although interesting and physically motivated, does not have a property called reflection positive, which is the Euclidean analogue of unitarity and which we will discuss in later lectures. If $a = -b$, we get a gauge invariant Maxwell theory which is reflection positive (unitary upon continuation to the Minkowski space). Only in this case and in $D = 4$ the theory has $T^\mu_{\ \mu} = 0$.

Above we argued from genericity. In fact, in 2D and 4D, there are theorems that $T^\mu_{\ \mu} = 0$ in a scale invariant theory. These theorems use unitarity as an assumption (plus extra assumptions).

1.3.3 Conformal invariance

From now on we assume that $T^\mu_{\ \mu} = 0$. As we will see now, this implies that SI of a theory is enhanced to conformal invariance (CI). Hamiltonian is now invariant under metric deformations $\delta g_{\mu\nu} = c(x)\delta_{\mu\nu}$, where $c(x)$ can be an arbitrary function of coordinates, since

$$\Delta H = \int d^D x T_{\mu\nu} \delta g^{\mu\nu} = \int d^D x c(x) T^\mu_{\ \mu} = 0 \ , \tag{1.3.17}$$

Transformations changing the metric this way are also called Weyl transformations. So we are saying that the theory with vanishing $T_{\mu\mu}$ is invariant under infinitesimal Weyl transformations. We note in passing that it will not in general be invariant under finite Weyl transformations, because $T_{\mu\mu}$ will acquire nonzero vev in a curved even-dimensional spacetime (so called Weyl anomaly). Below we only need infinitesimal Weyl transformations and so Weyl anomaly will not play a role.

A general Weyl transformation (even infinitesimal one) changes spacetime geometry from flat to curved, which we don't want to do. There is a subclass of Weyl transformations for which the spacetime remains flat. These are those metric deformations which happen when we apply an infinitesimal coordinate transformation (diffeomorphism) and rewrite the metric in new coordinates:

$$x'^\mu \to x^\mu + \epsilon^\mu(x), \tag{1.3.18}$$

which corresponds to

$$\delta g_{\mu\nu} = \partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu \overset{\text{must be}}{=} c(x)\delta_{\mu\nu} \ , \tag{1.3.19}$$

The last condition is necessary to have a Weyl transformation. Transformations which change the metric this way are called <u>conformal</u>. It turns out that this last equation allows only four classes of solutions in $D \geqslant 3$ dimensions:

$$\begin{array}{ll} \epsilon^\mu = \text{constant} & \text{shift of coordinates, } c(x) = 0 \ , \\ \epsilon^\mu = x^\nu \omega_{[\nu\mu]} & \text{infinitesimal rotation, } c(x) = 0 \ , \\ \epsilon^\mu = \lambda x^\mu & \text{scale transformation,} c(x) = 2\lambda \ , \\ \epsilon^\mu = 2(a\cdot x)x^\mu - x^2 a^\mu & \text{Special Conformal Transformations (SCT)}, c(x) = a\cdot x, \\ \text{with } a^\mu = \text{arbitrary vector.} & \end{array} \tag{1.3.20}$$

In $D = 2$ dimensions, there are more solutions. To see that, we introduce the complex variable

$$z = x_1 + ix_2, \ \bar{z} = x_1 - ix_2 \ , \qquad ds^2 = dz d\bar{z} \ . \tag{1.3.21}$$

If we transform $z \to f(z)$, $\bar{z} \to f(\bar{z})$, with f an analytic function, this corresponds to a conformal rescaling of the metric

$$ds^2 = |f(z)|^2 ds^2 \ . \tag{1.3.22}$$

The solutions for δz are

$$\begin{aligned}
&\delta z = \text{constant} \to \text{shift of coordinates},\\
&\delta z = e^{i\theta} z \to \text{rotation},\\
&\delta z = \lambda z \to \text{scale transformation},\\
&\delta z = cz^2 \to \text{SCT}\ .
\end{aligned} \tag{1.3.23}$$

Higher powers $\delta z \propto z^n$, $n > 2$, corresponds to new conformal transformations which exist only for $D = 2$.

In these lectures we will mostly talk about $D \geqslant 3$. Let us integrate the infinitesimal forms of the conformal transformations to find their finite form. For the SCT, which is the only nontrivial case, we find

$$x'^{\mu} = \beta(x)(x^{\mu} - a^{\mu} x^2), \tag{1.3.24}$$

with

$$\beta(x) = \frac{1}{1 - 2(a \cdot x) + a^2 x^2}\ . \tag{1.3.25}$$

A general conformal transformation $x \to x'$ will be a composition of shifts, rotations, scale transformations and SCT. The group of conformal transformations is a finite-dimensional subgroup of the group *Diff* of diffeomorphisms of $\mathbb{R}^D$. It is in fact the largest finite-dimensional subgroup of *Diff*.

The defining property of conformal transformations, that

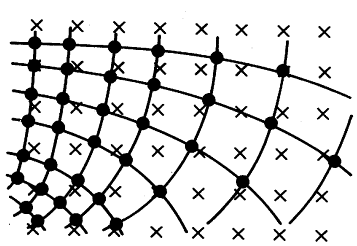
$$g'_{\mu\nu}(x') = c(x)\delta_{\mu\nu}\ , \tag{1.3.26}$$

means that the Jacobian of the coordinate transformation must be proportional to an orthogonal transformation

$$J = \frac{\partial x'^{\mu}}{\partial x^{\nu}} = b(x) O^{\mu}{}_{\nu}(x)\,,\ b(x) = \sqrt{c(x)}\,. \tag{1.3.27}$$

In other words, conformal transformation locally looks like a rotation and a scale transformation.

This allows us physically to think of a conformal transformation as a non-uniform RG transformation. I.e. we start with a uniform lattice which is mapped by a conformal transformation to a nonuniform but still locally orthogonal one. Then we do a block transformation by uniting spins within new cells.



This way of thinking gives another handle on the question why SI implies CI. It simply means that a system invariant under RG transformations will also be invariant under non-uniform RG transformations. Schaefer in 1976 used the Exact RGE and under the assumption that the fixed point Hamiltonian is sufficiently local, showed that it is also invariant under non-uniform ERG transformations expressing conformal invariance. Locality provides a link between SI and CI.

1.3.4 Transformation rule for operators

Invariance of the Hamiltonian is not the full story. Conformal transformations move points around, and correlation functions with insertions at new positions will be related to the old ones. We have to understand first how operators transform under non-uniform RG. We saw earlier that under scale transformations

$$x \to \lambda x \ , \quad \mathcal{O}(x) \to \tilde{\mathcal{O}}(\lambda x) = \lambda^{-\Delta}\mathcal{O}(x) \ . \tag{1.3.28}$$

But in fact $\tilde{\mathcal{O}}$ is the same operator as $\mathcal{O}$ in a theory with the RG-transformed Hamiltonian. Since the Hamiltonian is RG-invariant, we can identify the correlators of $\tilde{\mathcal{O}}$ and $\mathcal{O}$. In other words, the above transformation rule means that the two-point function of $\mathcal{O}$ should satisfy the following covariance property:

$$\langle \mathcal{O}(\lambda x_1)\mathcal{O}(\lambda x_2)\rangle = \lambda^{-2\Delta}\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle \ . \tag{1.3.29}$$

Now, we have to guess how the operators transform under conformal transformations and if their transformation is simple. The simplest assumption is that if an operator is sufficiently local, then it should not feel the effect of the scale factor $b(x)$ in (2.1.2). Then,

$$x \to x', \ \phi(x) \to \tilde{\phi}(x') = b(x)^{-\Delta}\phi(x) \ , \tag{1.3.30}$$

where again the correlators of ϕ and $\tilde{\phi}$ should be identified. If this is the case, $\phi(x)$ is called a primary operator. Even though all operators transform simply under scale transformations, it is not necessary that they follow the above simple rule under conformal transformations with $b(x) \neq const$. For example, if ϕ transforms as a primary, then $\partial\phi$ has a homogeneous and an inhomogeneous part that contains the derivative of the scale function. Such derivative operators are called descendants. As we will see later, all operators in a conformal field theory are either primary or descendants.

Consider now primary operators with intrinsic spin. Their transformation rules should depend on the rotation matrix $O^{\mu}{}_{\nu}(x)$ in (2.1.2). An operator in an irreducible representation R of $SO(D)$ will transform as

$$\phi(x) \to \tilde{\phi}(x') = b(x)^{-\Delta} R[O^{\mu}{}_{\nu}(x)]\phi(x) \ , \tag{1.3.31}$$

where $R[O^{\mu}{}_{\nu}(x)]$ is a representation matrix acting on the indices of $\phi(x)$.

For example, if ϕ_R is a vector, then $R[O^{\mu}{}_{\nu}(x)] = O^{\mu}{}_{\nu}(x)$. So spin one fields will transform as

$$\tilde{V}_{\mu}(x') = b(x)^{-\Delta} O^{\nu}{}_{\mu}(x) V_{\nu}(x) \ . \tag{1.3.32}$$

We will see later on that symmetry currents and the stress tensor will be primary operators (of spin 1 and 2 respectively). Generically, there will be infinitely many primaries for each spin.

Literature

RG, physical foundations of conformal invariance:

- J. L. Cardy, "Scaling and renormalization in statistical physics," Cambridge, UK: Univ. Pr. (1996) 238 p. (Cambridge lecture notes in physics: 3)
- J. Cardy "Conformal invariance," In *Domb, L. (Ed.), Lebowitz, J.l. (Ed.): Phase Transitions and Critical Phenonema, Vol. 11*, 55-126

Derivation of conformal invariance from scale invariance using ERG:

L. Schafer, "Conformal Covariance in the Framework of Wilson's Renormalization Group Approach," J. Phys. A **9**, 377 (1976).

Recent work on scale vs conformal invariance:

M. A. Luty, J. Polchinski and R. Rattazzi, "The a-theorem and the Asymptotics of 4D Quantum Field Theory," arXiv:1204.5221 [hep-th].

Long-range Ising model.

For recent work see M. Picco, "Critical behavior of the Ising model with long range interactions," arXiv:1207.1018 [cond-mat.stat-mech].

Theory of elasticity as an example of scale without conformal invariance:

V. Riva and J. L. Cardy, "Scale and conformal invariance in field theory: A Physical counterexample," Phys. Lett. B **622**, 339 (2005) [hep-th/0504197].

Conformal transformations.

See Ch. 4 of P. Di Francesco, P. Mathieu and D. Senechal, "Conformal field theory," New York, USA: Springer (1997) 890 p

Lecture 2

Conformal kinematics

2.1 Projective null cone

In the previous lecture we motivated the study of Scale Invariant (SI) fixed points. We also argued that the SI of the Renormalization Group (RG) fixed points is in general enhanced to Conformal Invariance (CI) = invariance under coordinate transformations that leave the metric invariant up to a coordinate-dependent scale factor $\lambda(x)$

$$x \to x' \ , \quad \text{such that} \quad dx'^2 = \lambda(x) dx^2 \ . \tag{2.1.1}$$

This means that the Jacobian of the coordinate transformation should have the special form

$$J = \frac{\partial x'^\mu}{\partial x^\nu} = b(x) O^\mu{}_\nu(x) \ , \tag{2.1.2}$$

where $b(x)$ a scale factor and $O^\mu{}_\nu$ a rotation matrix, that both may depend on the coordinates. Then,

$$\lambda(x) = b(x)^2 \tag{2.1.3}$$

We also saw that in $D \geqslant 3$ the group of Conformal Transformations (CT) is finite dimensional and is generated by the Poincaré transformations plus dilatations plus Special Conformal Transformations (SCT).

Then we introduced the concept of primary operators, which transform under CT as

$$\phi(x) \to \tilde{\phi}(x') = \frac{1}{b(x)^\Delta} \phi(x) \ , \tag{2.1.4}$$

if ϕ is scalar, or

$$\phi(x) \to \tilde{\phi}(x') = \frac{1}{b(x)^\Delta} \mathcal{R}[\mathcal{O}^\mu{}_\nu(x)] \phi(x) \ , \tag{2.1.5}$$

if ϕ has intrinsic spin, i.e. belongs to an irreducible representation $\mathcal{R}$ of $SO(D)$. The correlation functions of $\tilde{\phi}$ are the same as those of ϕ ($\tilde{\phi}$ can be thought of as an image of

ϕ under a non-uniform RG transformation). In the future we will sometimes omit the tilde from the start.

Operationally, the above transformation property simply means that the n-point correlation functions of ϕ must satisfy

$$\langle\phi(x')\phi(y')\ldots\rangle = \frac{1}{b(x)^\Delta}\frac{1}{b(y)^\Delta}\ldots\langle\phi(x)\phi(y)\ldots\rangle \ . \tag{2.1.6}$$

This condition is clearly an important constraint on the correlation functions of the theory, and in this lecture we will study its consequences. There are several ways to achieve that, some more pedestrian than others. We will choose a method which gives us the most information in the least possible time.

First of all we have to understand better the conformal algebra. Last time we wrote the formulas for the vector fields that correspond to the generators of the group

$$\begin{aligned} &P_\mu = i\partial_\mu \ \rightarrow \ \text{translations,} \\ &M_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu) \ \rightarrow \ \text{rotations,} \\ &D = ix^\mu\partial_\mu \ \rightarrow \ \text{dilatations,} \\ &K_\mu = i(2x_\mu(x^\nu\partial_\nu) - x^2\partial_\mu) \ \rightarrow \ \text{SCT, .} \end{aligned} \tag{2.1.7}$$

Since we know the precise expressions for the generators, we can now easily compute their commutators. We find some relations which correspond to the Poincare algebra

$$\begin{aligned} &[M_{\mu\nu}, M_{\rho\sigma}] = -i(\delta_{\mu\rho}M_{\nu\sigma} \pm \text{permutations}) \\ &[M_{\mu\nu}, P_\rho] = i(\delta_{\nu\rho}P_\mu - \delta_{\mu\rho}P_\nu) \ , \end{aligned} \tag{2.1.8}$$

The interesting new relations are

$$\begin{aligned} &[D, P_\mu] = -iP_\mu \\ &[D, K_\mu] = iK_\mu \\ &[P_\mu, K_\nu] = 2i(\delta_{\mu\nu}D - M_{\mu\nu}) \ . \end{aligned} \tag{2.1.9}$$

It turns out that this conformal algebra is isomorphic to $SO(D{+}1, 1)$, the algebra of Lorentz transformations in $\mathbb{R}^{D+1,1}$ Minkowski space.

Consider in the latter space the coordinates

$$X^1, \ldots X^D, X^{D+1}, \ X^{D+2} \tag{2.1.10}$$

where X^{D+2} is the timelike direction. We will also use the lightcone coordinates

$$X^+ = X^{D+2} + X^{D+1}, \ X^- = X^{D+2} - X^{D+1} \ . \tag{2.1.11}$$

In terms of the above, the mostly plus metric η_{MN} in $\mathbb{R}^{D+1,1}$ is

$$ds^2 = \sum_{i=1}^{D}(dX^i)^2 - dX^+dX^- \ . \tag{2.1.12}$$

We identify the generators of the conformal algebra with the generators of $SO(D+1,1)$ as follows

$$\begin{aligned} J_{\mu\nu} &= M_{\mu\nu} \ , \\ J_{\mu+} &= P_\mu \ , \\ J_{\mu-} &= K_\mu \\ J_{+-} &= D \ , \end{aligned} \tag{2.1.13}$$

with $\mu, \nu = 1, \ldots, D$. It is understood that $J_{\mu\nu}$ is antisymmetric in the interchange of μ and ν. Then one can check that the conformal algebra commutation relations coincide with those of the $SO(D+1,1)$ algebra:

$$[J_{MN}, J_{RS}] = -i(\eta_{MR} J_{NS} \pm \text{permutations}) \tag{2.1.14}$$

One example is

$$[J_{\mu+}, J_{\nu-}] \propto \delta_{\mu\nu} J_{+-} + \delta_{+-} J_{\mu\nu} \ . \tag{2.1.15}$$

Exercise: Check and fix all the constants in the above identification.

This result means that the conformal group, which acts in a non-trivial way on the D dim space, acts naturally (linearly) on the $\mathbb{R}^{D+1,1}$ space. In the vector representation we can write

$$X^M \to \Lambda^M_N X^N \ , \tag{2.1.16}$$

with Λ^M_N an $SO(D+1,1)$ matrix. If we could somehow get an action on $\mathbb{R}^D$ out of this simple action, then the implications of CI would be easier to understand. To do that however, we have to embed the D dimensional space into the $D+2$ dimensional space, that is to get rid of the two extra coordinates.

To get rid of one of the coordinates let's restrict the attention to the null cone:

$$X^2 = 0 \tag{2.1.17}$$

in the $D+2$ dimensional spacetime . Since this constraint is preserved by the action of the group, we don't lose simplicity.

To get down to D dimensions, we take a generic section of the light-cone:

$$X^+ = f(X^\mu), \tag{2.1.18}$$

The section is parametrized by X^μ which we identify with the $\mathbb{R}^D$ coordinates x^μ

The group $SO(D+1,1)$ acts on the section as follows (see Fig. 1). A point x^μ on the section and defines a lightray. If we apply a Lorentz transformation, this lightray will be mapped into a new one which passes through another point x'^μ. Thus

$$x^\mu \to \text{ light ray} \xrightarrow{\Lambda^M_N \in SO(D+1,1)} \text{light ray}' \to x'^\mu \ , \ x \in \mathbb{R}^D \ . \tag{2.1.19}$$

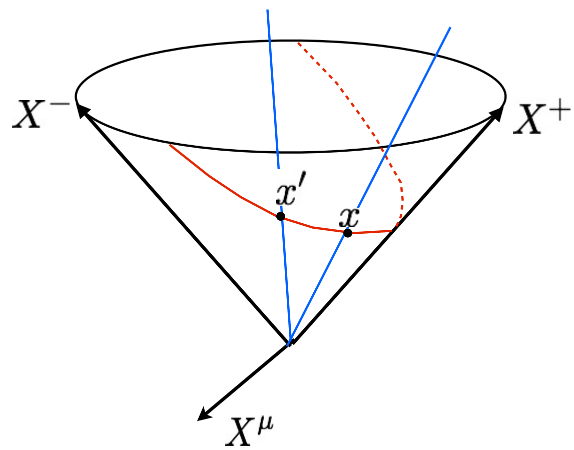


Figure 2.1: Red: section, blue: light ray and light ray$'$.

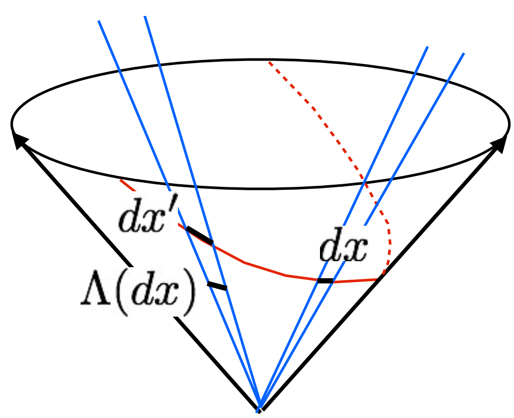


Figure 2.2: How the infinitesimal interval transforms under the defined $SO(D+1,1$ action.

While this defines an action of $SO(D+1,1)$ on $\mathbb{R}^D$, we have to check if this action corresponds to a CT. Consider the metric ds^2 on the section induced from the Minkowski metric in $\mathbb{R}^{D+1,1}$. We have:

$$ds^2 = dx^2 - dX^+ dX^-|_{X^+=f(x), X^-=x^2/X^+} = g_{\mu\nu}(x) dx^\mu dx^\nu \ , \tag{2.1.20}$$

where $g_{\mu\nu}(x)$ is a metric we could compute explicitly in terms of $f(x)$ but we won't need it.

The action of $SO(D+1,1)$ on a point x can be split into two steps, 1) $X \to \Lambda.X$ and 2) then rescale to get back into the section. We want to understand how this action changes the infinitesimal interval length (see Fig.2) The first step is an isometry and does not change ds^2. The second step changes the metric by an x-dependent scale factor. Indeed, assuming that we have to rescale by λ to get back into the section, where λ in general depends on X, we have:

$$(d(\lambda(X)X))^2 = (\lambda dX + X(\nabla\lambda.dX))^2 = \lambda^2 dX^2 \tag{2.1.21}$$

the other terms vanishing by $X^2 = 0$, $X.dX = 0$.

We conclude that the metric transformation is of the form:

$$ds'^2 = c(x) ds^2, \quad c(x) = \lambda(X)^2 \,. \tag{2.1.22}$$

This will exactly agree with the definition of the conformal transformations as long as ds^2 is flat. From the definition of ds^2 it's easy to guess which $f(X)$ achieves this: it is $f(X) = const$, so that $dX^+ = 0$. For simplicity and without loss of generality we take this constant equal to 1. Thus our *Euclidean* section is parametrized as:

$$X = (X^+, X^-, X^\mu) = (1, x^2, x^\mu) \tag{2.1.23}$$

We note in passing that by taking this section and rescaling it in the radial direction by an x-dependent factor we can reproduce any metric which is a Weyl transformation of the flat space metric (for example the metric on the sphere, de Sitter or Anti de Sitter spaces).

We now would like to extend the above action to fields. We thus consider fields $\phi(X)$ defined on the cone. The most natural action of the Lorentz group on such scalar fields is

$$X \to X' \ , \ \phi(X) \to \tilde{\phi}(X') = \phi(X) \ . \tag{2.1.24}$$

The field on the euclidean section will be assumed to coincide with the D-dimensional field:

$$\phi(X)|_{\text{section}} = \phi(x) \tag{2.1.25}$$

, Finally, we will assume that ϕ depends homogeneously on X:

$$\phi(\lambda X) = \lambda^{-\Delta} \phi(X) \ , \tag{2.1.26}$$

Let us show that these conditions imply the correct transformation rule for the fields on $\mathbb{R}^D$

$$\phi(x') = b(x)^{-\Delta} \phi(x) \ . \tag{2.1.27}$$

Indeed, $b(x)$ in this equation is the local expansion factor, and according to Eq. (2.1.22) it must be identfied with $\lambda(X)$, the scale factor in the second phase of $SO(D+1,1)$ action. Since $\phi(X)$ scales homogeneously with λ, we get exactly what we need.

Using this "projective light cone" formalism that we developed, any quantity (e.g. correlation function) which is conformally invariant in the $\mathbb{R}^D$ space, can be lifted to a quantity invariant under $SO(D+1,1)$ in the $D+2$ dimensional space. Basically, this formalism makes CI as simple as Lorentz invariance.

2.2 Simple applications

2.2.1 Primary scalar 2-point function

The expression of the two-point function on the light-cone is

$$\langle\phi(X)\phi(Y)\rangle = \frac{c}{(X\cdot Y)^{\Delta}} \ , \tag{2.2.1}$$

with c a constant and Δ the field's scaling dimension. The above is the most general Lorentz invariant expression consistent with scaling of both $\phi(X)$ and $\phi(Y)$ with degree Δ. Note that $X^2 = Y^2 = 0$ cannot appear. To write the two-point function in the physical space, we project X and Y on the section, i.e.

$$X = (X^+, X^-, X^\mu) = (1, x^2, x^\mu) \text{ and } Y = (Y^+, Y^-, Y^\mu) = (1, y^2, y^\mu) \ . \tag{2.2.2}$$

We get

$$\begin{aligned} X\cdot Y &= X^\mu Y_\mu - \frac{1}{2}(X^+X^- + X^-Y^+) \\ &= x^\mu y_\mu - \frac{1}{2}(x^2+y^2) \\ &= -\frac{1}{2}(x-y)^2 \ . \end{aligned} \tag{2.2.3}$$

The two-point function (2.2.1) is therefore projected to

$$\langle\phi(x)\phi(y)\rangle \propto \frac{1}{(x-y)^{2\Delta}} \tag{2.2.4}$$

That this expression is consistent with the SI of the field $\phi(x)$ is obvious; we have shown that it is also CI. If we wanted to show it in a pedestrian way, without using the projective light cone, we would have to show that

$$|x'-y'|^2 = b(x)b(y)|x-y|^2 \ . \tag{2.2.5}$$

for any CT, which does not look simple. The standard way to proceed is to show first that this holds for the inversion transformation

$$x'^\mu \to \frac{x^\mu}{x^2} \ , \tag{2.2.6}$$

which is a CT. Indeed the Jacobian is given by

$$\frac{\partial x'^\mu}{\partial x^\nu} = \frac{1}{x^2}\left(\delta^{\mu\nu} - \frac{2x^\mu x^\nu}{x^2}\right) \equiv b_{inv}(x) I^{\mu\nu}(x) \ , \tag{2.2.7}$$

where $b_{inv}(x) = 1/x^2$ and $I^{\mu\nu}(x)$ an orthogonal matrix. This can be easily seen if we go to a particular frame where x lies on the x_1 direction. Then the matrix is diagonal

$$I^{\mu\nu}(x) = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \ , \tag{2.2.8}$$

and is clearly an $O(D)$ matrix. But it is not in $SO(D)$. This means that inversion is not in the connected part of the conformal group, i.e. it cannot be obtained by exponentiating a Lie algebra element.

If we apply inversion twice we get back to the connected component. In fact we can reproduce SCT this way:

$$SCT_\alpha = inversion \times translation_\alpha \times inversion \ . \tag{2.2.9}$$

Eq. (2.2.5) is not difficult to verify for inversion:

$$|x' - y'|^2 = \left|\frac{x^\mu}{x^2} - \frac{y^\mu}{y^2}\right|^2 = \frac{|x-y|^2}{x^2 y^2} = b_{inv}(x) b_{inv}(y) |x-y|^2 \ . \tag{2.2.10}$$

And then it holds for SCT and for all other CT's by extension. This way of proof however looks a bit ad hoc, and becomes more and more awkward as we go to fields with spin and higher order correlation functions.

Two more comments about the two-point function of the scalar primary fields. If the fields have different scaling dimensions, $\Delta_1 \neq \Delta_2$, the two-point function vanishes

$$\langle \phi_1(x)\phi_2(y)\rangle = 0 \ . \tag{2.2.11}$$

This is clear from the cone perspective since we cannot construct the analogue of (2.2.1) is $\Delta_1 \neq \Delta_2$.

In a theory with several fields ϕ_i with same scaling dimension Δ, the two-point function is

$$\langle \phi_i(x)\phi_j(y)\rangle = \frac{M_{ij}}{(x-y)^{2\Delta}} \ , \tag{2.2.12}$$

Here the matrix M_{ij} will be positive-definite matrix for a unitary theory (as we will see in future lectures). This means that there exists a field basis such that M_{ij} becomes diagonal

$$\langle \phi_i(x)\phi_j(y)\rangle = \frac{\delta_{ij}}{(x-y)^{2\Delta}} \ . \tag{2.2.13}$$

We will always assume that such a basis is chosen.

2.2.2 Primary scalar three point function

The three-point function of three primary scalar fields with scaling dimensions, $\Delta_1, \Delta_2, \Delta_3$ (could be equal or different) must have the following form on the cone

$$\langle \phi_1(X_1)\phi_2(X_2)\phi_3(X_3)\rangle = \frac{const.}{(X_1X_2)^{\alpha_{123}}(X_1X_3)^{\alpha_{132}}(X_2X_3)^{\alpha_{231}}} \ . \tag{2.2.14}$$

As in the two-point function, the above is the most general Lorentz invariant expression. To make it consistent with scaling, we should impose the constraints

$$\begin{aligned} \alpha_{123} + \alpha_{132} &= \Delta_1 \\ \alpha_{123} + \alpha_{231} &= \Delta_2 \\ \alpha_{132} + \alpha_{231} &= \Delta_3 \ . \end{aligned} \tag{2.2.15}$$

The above is a linear system with three equations for three parameters; it admits a unique solution given by

$$\alpha_{ijk} = \frac{\Delta_i + \Delta_j - \Delta_k}{2} \ . \tag{2.2.16}$$

Thus the three-point function is uniquely determined up to a constant. If we project (2.2.14) on the Euclidean section, we find

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{\lambda_{123}}{(x_{12})^{2\alpha_{123}}(x_{13})^{2\alpha_{132}}(x_{23})^{2\alpha_{231}}} \ , \tag{2.2.17}$$

where λ_{123} a free parameter (which one can call a "coupling constant"), and

$$x_{ij} = x_i - x_j \ . \tag{2.2.18}$$

This formula was first derived by Polyakov in 1970 and is a remarkable result. One can say that it gave birth to CFT. To understand its significance, we should compare it with infinitely many functional forms which would be allowed if we imposed only SI:

$$\sum \frac{const.}{|x_{12}|^a|x_{13}|^b|x_{23}|^c} \ , \ a+b+c = \Delta_1 + \Delta_2 + \Delta_3 \ , \tag{2.2.19}$$

whereas there is only one term consistent with CI.

As a concrete example, the three-point function $\langle\sigma(x)\sigma(y)\epsilon(z)\rangle$ for two spins and energy in the 2-dimensional Ising model at the critical point can be extracted from the exact Onsager's lattice solution. That it agrees with Polyakov's formula is an evidence for the CI of the critical 2D Ising model.

2.2.3 Four point function

We now move on to the four-point function. For simplicity we will only consider four identical fields. Requiring consistency under Lorentz transformations and scaling, we get on the cone

$$\langle \phi(X_1)\phi(X_2)\phi(X_3)\phi(X_4)\rangle = \frac{1}{(X_1\cdot X_2)^{\Delta}(X_3\cdot X_4)^{\Delta}} f(u,v) \ , \tag{2.2.20}$$

Here u and v are *conformally invariant cross-ratios* which on the light-cone are given by Lorentz-invariant expressions

$$u = \frac{(X_1 \cdot X_2)(X_3 \cdot X_4)}{(X_1 \cdot X_3)(X_2 \cdot X_4)} \ , \quad \text{and} \quad v = u|_{2\leftrightarrow 4} = \frac{(X_1 \cdot X_4)(X_2 \cdot X_3)}{(X_1 \cdot X_3)(X_2 \cdot X_4)} \ . \tag{2.2.21}$$

Notice that they have scaling zero in every variable. Since the first term in the RHS takes care of the scaling, any function $f(u,v)$ of u and v can appear as a factor.

Now as we project to the Euclidean section, $X_1 \cdot X_2, X_3 \cdot X_4$ project to x_{12}^2, x_{34}^2 respectively, while u and v become

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad \text{and} \quad v = u|_{2\leftrightarrow 4} = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} \ . \tag{2.2.22}$$

That the four-point function must be given by a simple expression times a function of the conformally invariant cross-ratios is an enormous reduction of the functional freedom, although not as large as for the three point functions where the functional form was completely fixed.

We will later see that $f(u,v)$ is not an independent quantity but is related in a non-trivial way to the three-point function. But this will require dynamics, while here we are just doing kinematics.

For the moment let us notice a functional constraint on $f(u,v)$ which comes from the crossing symmetry of the four-point function. We saw that

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle = \frac{1}{x_{12}^{2\Delta} x_{34}^{2\Delta}} f(u,v) \ . \tag{2.2.23}$$

In the above expression there was no particular reason to group the coordinates that way. If for example we interchange $2 \leftrightarrow 4$, we get

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle = \frac{1}{x_{14}^{2\Delta} x_{23}^{2\Delta}} f(\tilde{u},\tilde{v}) \ , \tag{2.2.24}$$

where now $f(\tilde{u},\tilde{v})$ depends on the conformally invariant cross sections calculated with the interchanged indices. In this case, $2 \leftrightarrow 4$ simply means

$$\tilde{u} = v \ , \quad \tilde{v} = u \ . \tag{2.2.25}$$

Notice that the same function f appears in (2.2.23) and (2.2.24), since the four point function is totally symmetric under permutations. Moreover, (2.2.23) and (2.2.24) must agree:

$$\frac{1}{x_{12}^{2\Delta} x_{34}^{2\Delta}} f(u,v) = \frac{1}{x_{14}^{2\Delta} x_{23}^{2\Delta}} f(v,u) \tag{2.2.26}$$

Multiplying by $x_{14}^{2\Delta} x_{23}^{2\Delta}$. we find that $f(u,v)$ must satisfy:

$$\left(\frac{v}{u}\right)^{\Delta} f(u,v) = f(v,u) \ . \tag{2.2.27}$$

This constraint will play an important role in the last lecture.

2.3 Fields with spin

2.3.1 Extending null cone formalism

So far we only talked about scalar primaries. Let us now consider primaries with spin.

We will consider symmetric traceless primary fields living on the D dimensional space[1]. We will put such a field in correspondence with a fields which lives on the light-cone and is also symmetric and traceless:

$$\phi_{\mu\nu\lambda\ldots}(x) \leftrightarrow \phi_{MNL\ldots}(X) \ . \tag{2.3.1}$$

We notice that the fields on the light-cone have more components that the D dimensional ones. For this correspondence to be useful, we have to eliminate the extra components. Let's first of all impose transversality of the null cone fields

$$X^M \phi_{MNL\ldots}(X) = 0 \ . \tag{2.3.2}$$

This condition eliminates one component per index. We will see below how the remaining one is dealt with.

Then we define $\phi_{\mu\nu\lambda\ldots}(x)$ to be related to $\phi_{MNL\ldots}(X)$ by projection on the Euclidean section

$$\phi_{\mu\nu\lambda\ldots}(x) = \phi_{MNL\ldots}(X) \frac{\partial X^M}{\partial x^\mu} \frac{\partial X^N}{\partial x^\nu} \ldots \ , \tag{2.3.3}$$

where $X^M = (1, x^2, x^\mu)$ is the parametrization of the section, so

$$\frac{\partial X^M}{\partial x^\nu} = (0, 2x_\nu, \delta^\mu_\nu) \ . \tag{2.3.4}$$

Notice that this rule preserves the tracelessness condition: if we start from a traceless tensor, we will end up with a traceless tensor as well. Indeed, to compute the trace of $\phi_{\mu\nu\ldots}$ we have to evaluate the contraction

$$\delta^{\mu\nu} \frac{\partial X^M}{\partial x^\mu} \frac{\partial X^N}{\partial x^\nu} \tag{2.3.5}$$

which can be shown to be equal to

$$\eta^{MN} + X^M \cdot K^N + X^N \cdot K^M \ , \tag{2.3.6}$$

with $K_M = (0, 2, 0)$ an auxiliary vector. Contracted with $\phi_{MN\ldots}$, it will vanish by tracelessness and transversality.

Notice also that anything proportional to X^M projects to zero, since

$$X^2 = 0 \Rightarrow X^M \frac{\partial X^M}{\partial x^\mu} = 0 \ . \tag{2.3.7}$$

[1]Primaries in other representations of $SO(D)$, like antisymmetric tensors of fermions, can also be considered

This means that $\phi_{MNL\ldots}$ is defined up to adding an arbitrary tensor proportional to X^M. This "gauge invariance" reduces the number of degrees of freedom to the needed one.

Let's discuss the transformation properties. Under an $SO(D+1,1)$ transformation, the field on the null cone transforms in the standard Lorentz invariant way:

$$\tilde{\phi}_{MNL\ldots}(X') = \Lambda_M^{M'}\Lambda_N^{N'}\ldots\phi_{M'N'L'\ldots}(X) \ , \tag{2.3.8}$$

Just like for primary scalars, we will impose that the null cone fields are homogeneous in X:

$$\phi_{\ldots}(\lambda X) = \lambda^{-\Delta}\phi_{\ldots}(X) \ , \tag{2.3.9}$$

We claim that the resulting transformations for the fields on the section is what we need:

$$\tilde{\phi}_{\mu\ldots}(x') = \frac{1}{b(x)^{\Delta}}O^{\nu'}{}_{\mu}(x)\cdots\phi_{\nu'\ldots}(x) \ . \tag{2.3.10}$$

Here the line element transforms as

$$dx' = b(x)O(x).dx \tag{2.3.11}$$

To show that (2.3.10) is true, it's enough to show that (we consider spin 1 case for simplicity)

$$\tilde{\phi}(x').dx' = \frac{1}{b(x)^{\Delta-1}}\phi(x).dx \tag{2.3.12}$$

Now, the projection rule implies that

$$\phi(x).dx = \phi(X).dX \tag{2.3.13}$$

When $X \to \Lambda.X$ the scalar product $\phi(X).dX$ is preserved:

$$\phi(Y).dY = \phi(X).dX, \qquad Y = \Lambda.X \tag{2.3.14}$$

To get from Y back into the section we have to rescale: $X' = bY$. When we do it $\phi(Y)$ simply rescales. dY rescales plus gets a contribution proportional to Y if b is not a constant. This extra contribution vanishes when contracted with $\phi(Y)$ because of transversality. The end result is exactly (2.3.12)

2.3.2 Two point function

Let us now see the consequences. Consider the two-point function of a vector field. On the cone we have:

$$\langle\phi_M(X)\phi_N(Y)\rangle = \frac{\eta_{MN} + \alpha\frac{Y_M X_N}{XY}}{(XY)^{\Delta}} \ , \tag{2.3.15}$$

where we once again considered the most general Lorentz invariant form consistent with scaling. Notice that we don't write terms proportional to X_M or Y_N, since they anyway project to zero.

We have to impose transversality which fixes the value of the constant α

$$X^M(\quad) = Y^N(\quad) = 0 \Rightarrow \alpha = -1 \ . \tag{2.3.16}$$

Projecting the two-point function in the physical space, we find:

$$\begin{aligned} &\eta_{MN} \to \delta_{\mu\nu} \\ &Y_M \to -x_\mu + y_\mu \\ &X_N \to x_\nu - y_\nu \\ &X \cdot Y \to -\frac{1}{2}(x-y)^2 \ , \end{aligned} \tag{2.3.17}$$

therefore

$$\frac{\eta_{MN} - \frac{Y_M X_N}{XY}}{(XY)^\Delta} \to \frac{I_{\mu\nu}(x-y)}{(x-y)^{2\Delta}} \ , \tag{2.3.18}$$

with

$$I_{\mu\nu}(x) = \delta_{\mu\nu} - \frac{2x_\mu x_\nu}{x^2} \ . \tag{2.3.19}$$

Notice that in the above expression it is CI that fixes the relative coefficient -2 between the two terms. If we had SI only, this coefficient would be free.

[It would not be so easy to check that the found two point function transforms correctly under CT without using the cone. As usual, it would be sufficient to check how it transforms correctly under inversion. This in turn would be equivalent to the identify

$$I(x)I(x-y)I(y) = I(x'-y') \ , \tag{2.3.20}$$

with $x' = x/x^2$. One can check this by an explicit computation, expanding throughout, but it looks completely accidental this way.]

Similarly, the two point function can be computed for higher spin primary fields. Interestingly, one discovers that, apart from $I_{\mu\nu}$, no new conformally covariant tensors appear which connect two different points. All the two point functions are made of $I_{\mu\nu}$'s connecting different points, and $\delta_{\mu\nu}$'s if the indices μ, ν are associated with the same point. For example, the two-point function for a symmetric traceless field will be

$$\langle \phi_{\mu\nu}(x)\phi_{\lambda\sigma}(y)\rangle = \frac{1}{|x-y|^{2\Delta}}[I_{\mu\lambda}(x-y)I_{\nu\sigma}(x-y) + (\mu \leftrightarrow \nu) + \alpha\delta_{\mu\nu}\delta_{\lambda\sigma}] \ , \tag{2.3.21}$$

where the term with the delta functions (transforming correctly under CT since both indices get multiplied by the same O^μ_ν) was inserted in order to satisfy the tracelessness condition, which fixes the value of α:

$$\alpha = -\frac{2}{D} \ . \tag{2.3.22}$$

To summarize, the two-point functions are completely fixed for higher spin primaries just like for the scalar.

2.3.3 Remark about inversion

In "pedestrian" CFT calculations, not based on the projective null cone formalism, one often checks invariance under the inversion rather than under SCT. However, as we mentioned, inversion is not on the connected part of the conformal group. So is assuming invariance under the inversion an extra assumption?

On the cone, the inversion corresponds to the transformation

$$\begin{aligned} X^{D+1} &\to -X^{D+1} \\ X^{\pm} = X^{D+2} \pm X^{D+1} &\to X^{+} \leftrightarrow X^{-} \ . \end{aligned} \tag{2.3.23}$$

Indeed, if we start with a point $X^M = (1, x^2, x^\mu)$ on the Euclidean section, we end up with

$$(1, x^2, x^\mu) \to (x^2, 1, x^\mu) \xrightarrow{\text{rescale with } x^2} (1, 1/x^2, x^\mu/x^2) \ , \tag{2.3.24}$$

which has inversion in the last component.

Notice that the transformation $X^{D+1} \to -X^{D+1}$ is in $O(D+1,1)$ but not in SO, i.e. it is not in the connected component.

Another transformation in the same class is a simple spatial reflection (parity transformation)

$$X^1 \to -X^1 \ . \tag{2.3.25}$$

The two discrete symmetries parity and inversion are conjugate by $SO(D+1,1)$, which means that if we add to this group one of those, we get the same group. This implies that a CFT invariant under parity will be invariant under inversion and vice versa.

There are CFTs which break parity (and inversion). Correlators in those theories, lifted to the null cone, will involve the $D+2$ dimensional ϵ-tensor, or Γ_{D+3} (the analog of γ_5 matrix) for fermions if $D+2$ is even. Since we only considered scalars and symmetric tensors, these structures did not occur.

2.3.4 Remark on conservation

We have seen that for spin-1 and spin-2 primary fields, the form of the two-point correlation functions is fixed by CI in terms of just one parameter: the dimension of the field. Canonical dimensions

$$\Delta = D - 1 \ , \quad \text{for} \quad l = 1 \ , \quad \Delta = D \ , \quad \text{for} \quad l = 2 \ . \tag{2.3.26}$$

would correspond to the conserved currents and the stress tensor. We expect their two-point functions to be conserved objects. This should happen automatically since there is nothing to be adjusted. And indeed one can check that this is true. E.g. for the currents

$$\partial^\mu \frac{I_{\mu\nu}(x)}{x^{2\Delta}} = 0 \ , \quad \text{for} \quad \Delta = D - 1 \ , \tag{2.3.27}$$

Notice that the null cone formalism is simply a way to compute constraints imposed by CI. For example, current and stress tensor conservation may be more convenient to check in the physical space rather than on the null cone. There is no reason to insist in doing everything on the null cone. The two points of view - null cone and physical space - can be used interchangeably depending what one wants to compute.

2.3.5 Scalar-scalar-spin l

The last correlator that we will study in this lecture is the three-point function of two scalars and one spin l operator. Start with spin one. On the null cone we will have

$$\langle \phi_1(X)\phi_2(Y)\phi_{3M}(Z)\rangle = \text{scalar factor} \times (\text{tensor structure})_M \ , \tag{2.3.28}$$

The scalar factor will be the same as for the scalars

$$\frac{const.}{(XY)^{\alpha_{123}}(YZ)^{\alpha_{231}}(XZ)^{\alpha_{132}}} \ , \tag{2.3.29}$$

where the powers are fixed by the dimensions of the fields in order to get the correct scaling. The tensor structure must then have scaling 0 in all variables, and will also have to be transverse $Z^M(\quad)_M = 0$. Moreover we don't need to include a term proportional to Z_M since it will project to zero. It's then easy to see that the tensor part must be equal to

$$\frac{(YZ)X_M - (XZ)Y_M}{(XZ)^{1/2}(XY)^{1/2}(YZ)^{1/2}} \ , \tag{2.3.30}$$

where the minus sign was fixed from the transversality constraint. We now have to project the tensor part into the physical space, which means we have to multiply by $\partial Z^M/\partial z^\mu$. We find that

$$X_M \to (x-z)_\mu \quad \text{and} \quad Y_M \to (y-z)_\mu \ , \tag{2.3.31}$$

therefore the expression projects into

$$\frac{(x-z)_\mu |y-z|^2 - (y-z)_\mu |x-z|^2}{|x-z||y-z||x-y|} \ , \tag{2.3.32}$$

or in a nicer form

$$\frac{|y-z||x-z|}{|x-y|}\left(\frac{(x-z)_\mu}{|x-z|^2} - \frac{(y-z)_\mu}{|y-z|^2}\right) \equiv R_\mu(x,y|z) \ . \tag{2.3.33}$$

The quantity R_μ transforms correctly under CT, so for a symmetric traceless field with more indices. Again it turns out that this is the only indexed object for three points with this property ($I_{\mu\nu}$ is not useful here since it has two indices at different points). For spin l fields the above is generalized into

$$\langle \phi_1(x)\phi_2(y)\phi_{3\mu\nu\lambda\ldots}(z)\rangle \propto \text{scalar part} \times (R_\mu R_\nu R_\lambda \cdots - \text{traces}) \ . \tag{2.3.34}$$

We see that the three-point function is again completely fixed up to an arbitrary constant. Notice that we cannot use

Suppose that we now look at the three-point functions of two scalars and the current J_μ and the stress tensor $T_{\mu\nu}$

$$\langle \phi_1(x)\phi_2(y)J_\mu(z)\rangle \ , \quad \langle \phi_1(x)\phi_2(y)T_{\mu\nu}(z)\rangle \ . \tag{2.3.35}$$

What if we impose here the conservation condition? For two point functions conservation was automatic, but here it is not so. In fact, these three point functions are conserved if and only if the scalars have equal dimensions, $\Delta_1 = \Delta_2$. Intuitively this happens because the three-point function must satisfy the Ward identities, which relate it to the two-point function. As we have seen, the two-point function is non-zero if and only if $\Delta_1 = \Delta_2$. The conclusion is that the coupling of the stress tensor and conserved currents to two scalar primaries of unequal dimensions must vanish.

2.4 An elementary property of CTs

The following simple property of conformal transformations will be needed in the future: they map circles to circles (including straight lines which are circles with infinite radius).

To prove it, consider a circle in $\mathbb{R}^D$, given by

$$|x - x_0|^2 = R_0^2 \Leftrightarrow x^2 - 2x.x_0 + (x_0^2 - R_0^2) = 0 \tag{2.4.1}$$

It's enough to show that the image under inversion is a circle, i.e.

$$|x/x^2 - x_1|^2 = R_1^2 \Leftrightarrow x^2(x_1^2 - R_1^2) - 2x.x_1 + 1 = 0 \tag{2.4.2}$$

This is true if we fix x_1 and R_1 from the equations

$$(x_1^2 - R_1^2)^{-1} = x_0^2 - R_0^2, \qquad x_1^\mu/(x_1^2 - R_1^2) = x_0 \tag{2.4.3}$$

Q.E.D.

This property of CT is well known in classical geometry. As an example, consider a problem which becomes very easy if one uses it and would be tricky to solve otherwise: Given a circle γ and two point A, B outside of it, construct a circle tangent to γ that passes through A and B.

Idea of solution: Pick a points on the circle γ and apply an inversion with respect to that point. The circle is now mapped onto a straight line. In these coordinates, the problem can be seen to reduce to solving a quadratic equation. Then map back.

In this lecture we learned how to dominate the kinematics of the conformal group. Even though it gets us a long way, by itself it is not enough to solve a theory. In the next lecture we will move to more dynamical issues, in particular the Operator Product Expansion (OPE).

Literature

Projective null cone idea is due to Dirac, and was used by Mack and Salam, Ferrara et al, Siegel and others. More recently it was used by Cornalba et al in its essentially modern form, and then rediscovered by Weinberg who nicely wrote it up:

S. Weinberg, "Six-dimensional Methods for Four-dimensional Conformal Field Theories," Phys. Rev. D **82**, 045031 (2010) [arXiv:1006.3480 [hep-th]].

This formalism is also developed further in:

M. S. Costa, J. Penedones, D. Poland and S. Rychkov, "Spinning Conformal Correlators," JHEP **1111**, 071 (2011) [arXiv:1107.3554 [hep-th]].

For the pedestrian approach to conformal correlators see

H. Osborn and A. C. Petkou, "Implications of conformal invariance in field theories for general dimensions," Annals Phys. **231**, 311 (1994) [hep-th/9307010].

The classic paper by Polyakov which gave birth to CFTs:

A. M. Polyakov, "Conformal symmetry of critical fluctuations," JETP Lett. **12**, 381 (1970) [Pisma Zh. Eksp. Teor. Fiz. **12**, 538 (1970)].

Lecture 3

Radial quantization and OPE

3.1 Radial quantization

In the previous lectures we talked about correlation functions of local operators in the statistical mechanics sense, thinking of them as some sort of average. In this lecture we will develop a parallel point of view, based on Hilbert space and on quantum mechanical evolution. This will be quite useful for understanding all the questions related to the Operator Product Expansion (OPE).

3.1.1 General remarks on quantization

Let us start with general remarks on the Hilbert space construction in Quantum Field Theory (QFT). This procedure is linked to the choice of foliation of the spacetime:

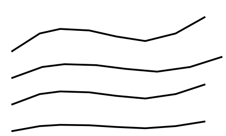


Figure 3.1: D-dimensional spacetime foliated by $D-1$-dimensional surfaces (leafs).

Each leaf of the foliation becomes endowed with its own Hilbert space. We create *in* states $|\Psi_{in}\rangle$ by inserting operators or throwing particles in the past of a given surface. Analogously we deal with *out* states when we inserting operators or measure particles in the future. The overlap of an *in* and *out* states living on the same surface

$$\langle\Psi_{out}|\Psi_{in}\rangle \ . \tag{3.1.1}$$

is equal to the correlation function of operators which create these *in* and *out* states (or to the S-matrix element if we deal with particle scattering).

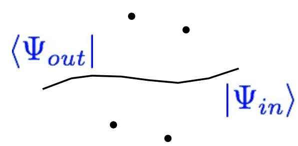


Figure 3.2: Operator insertions in the past and present create Hilbert space states, whose overlap is equal to the correlation function involving these operators.

If the in and out states live on different surfaces, and nothing is inserted in between, there will be a unitary evolution operator U connecting the two Hilbert states and the same correlation function will be equal to:

$$\langle\Psi_{out}|U|\Psi_{in}\rangle \ . \tag{3.1.2}$$

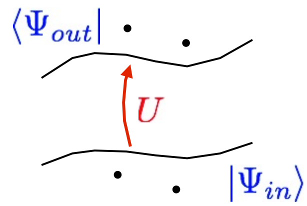


Figure 3.3: If we choose to consider states living on different surfaces, then we must insert the evolution operator to evaluate the correlation function.

The above remarks are very general. In practice one likes to choose foliations that respect the symmetries of the theory. For example, in Poincaré invariant theories, we usually choose to foliate the space by surfaces of equal time. This is convenient, since the Hamiltonian P^0 moves us between the different surfaces. The evolution operator U will be simply given by exponentiating the generator P^0

$$U = e^{iP^0\Delta t} \ . \tag{3.1.3}$$

Also, since all surfaces are related by a symmetry transformation, the Hilbert space is the same on each surface. The states living on these surfaces can be characterized by their energies and momenta

$$P^\mu|k\rangle = k^\mu|k\rangle \ . \tag{3.1.4}$$

Another choice of foliation turns out more convenient in CFT, where we want to describe states created by insertions of local operators. Namely, we will use foliations by S^{D-1} spheres of various radii:

This is called radial quantization. We will assume that the center of the spheres is located at $x = 0$, but of course quantizing with respect to any other point should give the same correlators. Similar arbitrariness is also present in the usual, "equal time", quantization,

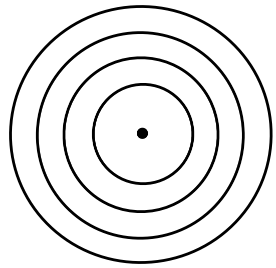


Figure 3.4: Foliation by spheres S^{D-1} all centered at the origin $x = 0$ of the Euclidean D-dimensional space.

since we have to fix a timelike time vector and different choices give rise to different but equivalent quantizations.

The generator that moves us from one surface to the other in radial quantization is the dilatations generator D, and it will play the role of the Hamiltonian:

$$U = e^{iD\Delta\tau} \ , \tag{3.1.5}$$

where $\tau = \log r$. The states living on the spheres will be classified according to their scaling dimension

$$D|\Delta\rangle = i\Delta|\Delta\rangle \ , \tag{3.1.6}$$

and their $SO(D)$ spin l

$$M_{\mu\nu}|\Delta, l\rangle_{\{s\}} = (\Sigma_{\mu\nu})_{\{s\}}^{\{t\}} |\Delta, l\rangle_{\{t\}} \ , \tag{3.1.7}$$

since the only generators that commute with D are the angular momentum operators $M_{\mu\nu}$. The finite dimensional matrices $\Sigma_{\mu\nu}$ acting on the spin indices are non-zero only for fields non-zero spin l.

3.1.2 Algebra action on quantum operators

The next step is to understand how the operators transform. In the previous lectures we discussed how operators transform under finite Conformal Transformations (CT), from the stat.mech. point of view. For example for scalars

$$x \to x' \ , \quad \phi(x) \to \phi(x') = \frac{1}{b(x)^\Delta}\phi(x) \ . \tag{3.1.8}$$

The above relation in terms of the correlation functions is understood as

$$\langle\phi(x')\ldots\phi(y')\rangle = \frac{1}{b(x)^\Delta}\cdots\frac{1}{b(y)^\Delta}\langle\phi(x)\ldots\phi(y)\rangle \ , \tag{3.1.9}$$

or equivalently

$$\langle b(x)^\Delta\phi(x')\ldots b(y)^\Delta\phi(y')\rangle = \langle\phi(x)\ldots\phi(y)\rangle \ . \tag{3.1.10}$$

Let us consider now the infinitesimal form of the transformation

$$x'^{\mu} = x^{\mu} + \epsilon^{\mu}(x) \quad \text{and} \quad b(x) = 1 + \partial_{\mu}\epsilon^{\mu} \ . \tag{3.1.11}$$

Here for each group generator G there will be its own ϵ_{μ}. Expanding the fields in first order in ϵ, we can write

$$b(x)^{\Delta}\phi(x') = \phi(x) + \epsilon(G\phi(x)) \ , \tag{3.1.12}$$

where G should be understood as the action of a generator on the field

$$G\phi(x) = (\Delta\partial_{\mu}\epsilon^{\mu})\phi(x) + \epsilon^{\mu}\partial_{\mu}\phi(x) \ . \tag{3.1.13}$$

Using the above expressions, we can write the transformation of the correlation functions under CT in infinitesimal form

$$\langle G\phi(x_1)\phi(x_2)\phi(x_3)\dots\rangle + \langle \phi(x_1)G\phi(x_2)\phi(x_3)\dots\rangle + \dots = 0 \ . \tag{3.1.14}$$

In QFT, the action of the generators on quantum fields will be given by the same formula but we will write it as the commutator $[G, \phi(x)]$. The vacuum averages of quantum mechanical operators are the correlation functions

$$\langle\phi(x_1)\dots\phi(x_n)\rangle = \langle 0|\phi(x_1)\dots\phi(x_n)|0\rangle \ , \tag{3.1.15}$$

and for consistency the vacuum is supposed to be conformally invariant (CI)

$$G|0\rangle = 0 \ . \tag{3.1.16}$$

Since we know ϵ, we can work out all the generators action on the fields

$$\begin{aligned}
&[P_{\mu}, \mathcal{O}(x)] = -i\partial_{\mu}\mathcal{O}(x) \ , \\
&[D, \mathcal{O}(x)] = -i(\Delta + x^{\mu}\partial_{\mu})\mathcal{O}(x) \ , \\
&[M_{\mu\nu}, \mathcal{O}(x)] = -i(\Sigma_{\mu\nu} + x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu})\mathcal{O}(x) \ , \\
&[K_{\mu}, \mathcal{O}(x)] = -i(2x_{\mu}\Delta + 2x^{\lambda}\Sigma_{\lambda\mu} + 2x_{\mu}(x^{\rho}\partial_{\rho}) - x^2\partial_{\mu})\mathcal{O}(x) \ ,
\end{aligned} \tag{3.1.17}$$

where Δ the scaling dimension of $\mathcal{O}(x)$ and $\Sigma_{\mu\nu}$ the finite dimensional spin matrices we encountered previously. Each commutator has a trivial part coming from ϵ, and an extra part that comes either from the expansion of $b(x)$, or the Σ matrices.

Notice that specializing to the point $x = 0$, we get

$$[K_{\mu}, \mathcal{O}(0)] = 0 \ . \tag{3.1.18}$$

This property is in fact the defining property of primary operators. From this property, together with the natural ones:

$$\begin{aligned}
&[P_{\mu}, \mathcal{O}(x)] = -i\partial_{\mu}\mathcal{O}(x) \ , \\
&[D, \mathcal{O}(0)] = -i\Delta\mathcal{O}(0) \ , \\
&[M_{\mu\nu}, \mathcal{O}(0)] = -i\Sigma_{\mu\nu}\mathcal{O}(0)
\end{aligned} \tag{3.1.19}$$

and the conformal algebra, Eqs. (3.1.17) follow.

3.1.3 Examples of states in radial quantization. State-operator correspondence.

Let us now go back to the radial quantization. We will generate states living on the sphere by inserting operators inside the sphere. Let us consider some simple cases:

1. The vacuum state $|0\rangle$ corresponds to inserting nothing. The dilatation eigenvalue, the "radial quantization energy", is zero for this state.

2. If we insert an operator $\mathcal{O}_\Delta(x=0)$ at the origin, the generated state $|\Delta\rangle = \mathcal{O}_\Delta(0)|0\rangle$ will have energy equal to the scaling dimension Δ. This is in fact easily seen by considering

$$\begin{aligned} D|\Delta\rangle &= D\mathcal{O}_\Delta(0)|0\rangle = [D,\mathcal{O}_\Delta(0)]|0\rangle + \mathcal{O}_\Delta(0)D|0\rangle \\ &= i\Delta\mathcal{O}_\Delta(0)|0\rangle \ = i\Delta|\Delta\rangle. \end{aligned} \tag{3.1.20}$$

3. If we insert an operator $\mathcal{O}_\Delta(x)$ with $x \neq 0$, the resulting state $|\Psi\rangle = \mathcal{O}_\Delta(x)|0\rangle$ is not an eigenstate of the dilatation operator D, since there will be dependence on x^μ as can be seen from eq. (3.1.17). Instead, it is a superposition of states with different energies. This statement becomes clear if we write

$$\begin{aligned} |\Psi\rangle &= \mathcal{O}_\Delta(x)|0\rangle = e^{iPx}\mathcal{O}_\Delta(0)e^{-iPx}|0\rangle \\ &= e^{iPx}|\Delta\rangle = \sum_n \frac{1}{n!}(iPx)^n|\Delta\rangle\ . \end{aligned} \tag{3.1.21}$$

We can see that each time the momentum operator P_μ acts on $|\Delta\rangle$, a state with energy $\Delta+1$ is generated. This is a consequence of the fact that P_μ raises the dimension by 1 as it can be seen from

$$[D,P_\mu] = iP_\mu\ . \tag{3.1.22}$$

Schematically

$$|\Delta\rangle \xrightarrow{P_\mu} |\Delta+1\rangle \xrightarrow{P_\nu} |\Delta+2\rangle\cdots\ . \tag{3.1.23}$$

On the other hand, the operator K_μ lowers the dimension by 1, since

$$[D,K_\mu] = -iK_\mu\ , \tag{3.1.24}$$

thus

$$0 \xleftarrow{K_\mu} |\Delta\rangle \xleftarrow{K_\nu} |\Delta+1\rangle\cdots\ . \tag{3.1.25}$$

This gives rise to the following observation. So far we have been taking the existence of primary operators as an axiom. However, their existence emerges automatically in this algebraic treatment. In fact if we did not have a primary, then we could keep lowering dimensions. Assuming that dimensions are bounded from below (as they are in unitary theories, see below), eventually we must hit zero, and this will give us a primary.

Let us go back to the states generated by inserting a primary operator at the origin. We saw that these states have scaling dimension Δ and are annihilated by K_μ. We can go backwards as well: given a state such that its scaling dimension is Δ which is annihilated by K_μ, we can construct a local primary operators. [This is called state-operator correspondence: states are in one-to-one correspondence with local operators.]

The proof is easy. To construct an operator we must define its correlation functions with other operators. Define them by the equation

$$\langle \phi(x_1)\phi(x_2)\dots \mathcal{O}_\Delta(0)\rangle = \langle 0|\phi(x_1)\phi(x_2)\dots|\Delta\rangle \; . \tag{3.1.26}$$

This definition can be shown to satisfy all the usual transformation properties dictated by CI.

3.1.4 Cylinder interpretation

Consider polar coordinates on $\mathbb{R}^D$

$$\tau = \log r \; , \quad \text{and} \quad \vec{n} \in S^{D-1} \; , \tag{3.1.27}$$

where $0 < r < \infty$ is the radial coordinate. Under dilatations τ shifts:

$$\tau \to \tau + \lambda \quad \text{if} \quad r \to e^\lambda r \; . \tag{3.1.28}$$

In terms of these coordinates, the correlation functions satisfy

$$\langle \phi(r_1, \vec{n}_1)\phi(r_2, \vec{n}_2)\dots\rangle = \frac{1}{r_1^{\Delta_1}}\frac{1}{r_2^{\Delta_2}}\cdots f(\tau_i - \tau_j, \{\vec{n}_i\}) \; , \tag{3.1.29}$$

where the function f can depend only on the differences $\tau_i - \tau_j$ and all the unit vectors $\vec{n}_i$. This is because the factors $1/r_i^{\Delta_i}$ already account for the scaling. What remains must be scale-invariant, so it can only depend on ratios of r_i's, or differences of τ_i's.

This suggests the following picture: start from flat space and go to the $S^{D-1}\times\mathbb{R}$ cylinder parametrized by τ and $\vec{n}$. We can define fields on the cylinder like

$$\phi_{cyl}(\tau, \vec{n}) = r^\Delta \phi_{flat}(r, \vec{n}) \; , \tag{3.1.30}$$

where $\phi_{flat}(r, \vec{n})$ are the fields that live on the flat space. With this identification, the function f is the correlation function of the fields on the cylinder

$$\langle \phi_{cyl}(\tau_1, \vec{n}_1)\phi_{cyl}(\tau_2, \vec{n}_2)\dots\rangle = f(\tau_i - \tau_j, \{\vec{n}_i\}) \; . \tag{3.1.31}$$

This looks natural, since the dynamics on the cylinder should be invariant under translations of τ. Note that the above is more than just rewriting. Actually, the metric of the cylinder is equivalent to the flat space metric by a Weyl transformation

$$ds^2_{flat} = dr^2 + r^2 d\vec{n}^2 \; , \quad \text{and} \quad ds^2_{cyl} = d\tau^2 + d\vec{n}^2 = \frac{1}{r^2} ds^2_{flat} \; . \tag{3.1.32}$$

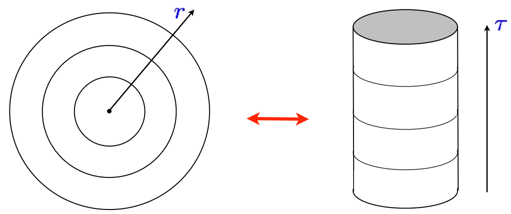


Figure 3.5: Weyl transformation from $\mathbb{R}^D$ to the cylinder $S^{D-1} \times \mathbb{R}$.

It turns out that for any metric equivalent to the flat one by a Weyl transformation, the correlation functions are related by simple rescalling factors

$$\langle \phi(x)\phi(y)\dots\rangle_{g_{\mu\nu}=\Omega(x)^2\delta_{\mu\nu}} = \frac{1}{\Omega(x)^\Delta}\cdots\frac{1}{\Omega(y)^\Delta}\langle\phi(x)\phi(y)\dots\rangle_{flat} \ . \tag{3.1.33}$$

This result might be a bit surprising, since we are considering a finite Weyl transformation. One might wonder if Weyl anomaly could not somehow influence the correlation functions, but one can show that its contribution cancels out between the numerator and denominator (i.e. when one normalizes the correlator dividing by the partition function).

So, the cylinder field ϕ_{cyl} is not just an artificial construct, but it is the very same field ϕ as in flat space, just its correlators are measured in a different geometry. As a concrete illustration of this non-trivial fact, suppose we want to do Monte-Carlo simulations of the 2-D Ising model. We first put the theory on a flat, infinite lattice and adjust the temperature to its critical value. We calculate the correlation function of an operator inserted at two points. Then, we put the model on a cylindrical lattice, using the *same* lattice action, and the *same* temperature, and compute the correlation function of the same operator. It turns out these two correlators will be connected in the way we described above.

3.1.5 N-S quantization

So far we have introduced two quantization pictures. The first one is the radial quantization. In this picture the in and out vacuum states sit at 0 and ∞, so that the relation between them is not entirely obvious. The second one is the cylinder, in which these states are related by the cylinder time reflection $\tau \to -\tau$. This is an advantage. On the other hand, to use the cylinder picture, one has to accept the equivalence relation (3.1.33), while in this course we prefer not to deal with QFT on curved spaces.

We will now present the N-S (North-South pole) quantization picture, that is in between the other two. It is almost as efficient as the cylinder, and has the advantage that it takes place in flat space. Consider the following generator of the Conformal Group (CG)

$$K_1 + P_1 = i(2x_1(x^\mu\partial_\mu) - (1+x^2)\partial_1) \ , \tag{3.1.34}$$

which has two fixed points at $x_1 = \pm 1$, $x_{2,\dots,D} = 0$ which we call N and S. The vector field takes us from one to the other. In this situation we can foliate our space using this generator as a Hamiltonian.

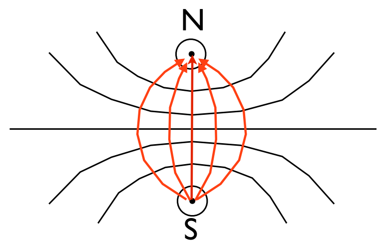


Figure 3.6: NS foliation orthogonal to the $K_1 + P_1$ vector field flow.

Such a quantization is completely equivalent to the radial quantization, because $K_1 + P_1$ is conjugate to D by $SO(D+1,1)$.[1] In fact, we can map the above foliation to the radial one. To do that, we apply a Special Conformal Transformation (SCT) to map the North pole to ∞

$$x_\mu \to \frac{x^\mu - \alpha^\mu x^2}{1 - 2(\alpha x) + \alpha^2 x^2} \ , \ x^\mu = \alpha^\mu \to \infty \ , \tag{3.1.35}$$

followed by a translation to move the South pole to 0.

The reason we introduced this 3rd quantization procedure, is that it will be now very convenient for defining conjugate states.

Let us add some operators below the line $x^1 = 0$. This will generate a state $|\Psi\rangle$ on $x^1 = 0$. If we apply the reflection transformation $\Theta : x^1 \to -x^1$, this state is mapped onto $\langle\Psi|$. If we compute $\langle\Psi|\Psi\rangle$ in a unitary theory, this quantity is the norm of the state and has to be positive

$$\langle\Psi|\Psi\rangle > 0 \ . \tag{3.1.36}$$

This means that $2n$-point functions with operators inserted in a Θ-invariant way will have to be positive in a unitary theory. This property is called reflection positivity.

Conjugation of operators is therefore straightforward in the NS quantization:

$$[\phi(x)]^\dagger = \Theta(\phi(x)) = \phi(\Theta x) \tag{3.1.37}$$

(this is for scalar fields, for tensor fields Θ also changes signs of the 1-components). Notice that this latter relation is in fact completely standard for the QFT Wick-rotated to the Euclidean signature. Indeed,

$$\phi(x,t) = e^{-iHt}\phi(x,0)e^{iHt} \ \to \ \phi(x,t_E) = e^{-Ht_E}\phi(x,0)e^{Ht_E} \tag{3.1.38}$$

and from here

$$\phi(x,t_E)^\dagger = e^{Ht_E}\phi(x,0)^\dagger e^{-Ht_E} = \phi(x,-t_E) \tag{3.1.39}$$

provided that $\phi(x,0)$ is Hermitean.

[1] In the notation of the previous lecture, $K_1 + P_1 \sim J_{1,D+2}$, while $D \sim J_{D+1,D+2}$. These two generator are related by a rotation by $\pi/2$ in the $(D+1,1)$ plane.

3.1.6 Conjugation in radial quantization

To do calculations, we have to go back to the radial quantization. We have to understand how to define the conjugate state in the radial quantization picture. Since the two schemes, radial and N-S, are connected, we simply have to find the image of the reflection operator Θ after a transl.SCT establishing the equivalence. Since this transformation maps $S \to 0$ and $N \to \infty$, and Θ interchanges N and S it is easy to guess that it maps to the inversion:

$$\Theta \to \text{inversion } \mathcal{R} \ . \tag{3.1.40}$$

The above implies that in the radial quantization, the conjugate of $|\Psi\rangle = \phi(x)|0\rangle$ will be

$$\langle\Psi| = \langle 0|[\phi(x)]^\dagger \ , \ [\phi(x)]^\dagger = r^{-2\Delta_\phi}\phi(\mathcal{R}x) \equiv \mathcal{R}[\phi(x)] \ . \tag{3.1.41}$$

This definition has the property that the correlation functions are $\mathcal{R}$-reflection positive (in a unitary theory)

$$\langle 0|[\phi(y)]^\dagger[\phi(x)]^\dagger \dots \phi(x)\phi(y)|\rangle > 0 \ . \tag{3.1.42}$$

From this rule we can establish conjugation properties of algebra generators. In fact the operators K_μ and P_μ are conjugate by applying inversion twice

$$K_\mu = \mathcal{R}P_\mu\mathcal{R} \ , \tag{3.1.43}$$

(remember that we said that SCT can be obtained by inv followed by transl followed by inv?). So we have:

$$(P_\mu|\Psi\rangle)^\dagger = \langle\Psi|K_\mu \ . \tag{3.1.44}$$

i.e.

$$P_\mu = K_\mu^\dagger \ . \tag{3.1.45}$$

3.1.7 Two-point function in radial quantization

Let us use this formalism in a concrete computation. Consider the two-point function of a scalar field

$$\langle\phi(x_2)\phi(x_1)\rangle = \frac{1}{|x_1 - x_2|^{2\Delta}} \ . \tag{3.1.46}$$

Since the action of $\phi(x)$ on the vacuum produces states with energies $E_n = \Delta + n$, the form of the above in radial quantization is expected to be

$$\langle 0|\phi(r_2, \vec{n}_2)\phi(r_1, \vec{n}_1)0|\rangle = \frac{1}{r_1^\Delta}\frac{1}{r_2^\Delta}\sum_n c_n e^{-E_n(\tau_2-\tau_1)} \ , \tag{3.1.47}$$

where $\tau_2 - \tau_1$ is the interval of time on the cylinder and $e^{-(\tau_2-\tau_1)} = r_1/r_2$.

To see that this is indeed true, let us rewrite (3.1.46) as

$$\langle\phi(r_2, \vec{n}_2)\phi(r_1, \vec{n}_1)\rangle = \frac{1}{r_1^\Delta}\frac{1}{r_2^\Delta}\frac{(r_1/r_2)^\Delta}{\left|1 - 2\frac{r_1}{r_2}(\vec{n}_1.\vec{n}_2) + \left(\frac{r_1}{r_2}\right)^2\right|^\Delta} \ . \tag{3.1.48}$$

Expanding the fraction in power of r_1/r_2, we get

$$\frac{1}{\left|1 - 2\frac{r_1}{r_2}(\vec{n}_1\vec{n}_2) + \left(\frac{r_1}{r_2}\right)^2\right|^{\Delta}} = \sum_n c_n \left(\frac{r_1}{r_2}\right)^n \ , \tag{3.1.49}$$

where the coefficients $c_n = \mathcal{P}_n(\vec{n}_1.\vec{n}_2)$ are polynomials in the scalar product. We see that this forms agrees with (3.1.47).

However, can we also calculate the coefficients c_n independently in radial quantization and check the agreement? To accomplish that, let us write

$$\phi(x_1)|0\rangle = e^{iPx_1}|\Delta\rangle \ . \tag{3.1.50}$$

We now have to understand the action of $\phi(x_2)$ on $\langle 0|$. We showed that $[\phi(x)]^\dagger = r^{-2\Delta_\phi}\phi(\mathcal{R}x)$, thus

$$\langle 0|\phi(x_2) = r_2^{-2\Delta_\phi}\langle 0|[\phi(\mathcal{R}x_2)]^\dagger = r_2^{-2\Delta_\phi}\langle\Delta|e^{-iK\mathcal{R}x_2} \ . \tag{3.1.51}$$

The correlation function becomes in this way

$$\langle 0|\phi(x_2)\phi(x_1)|0\rangle = r_2^{-2\Delta_\phi}\langle\Delta|e^{-iK\mathcal{R}x_2}e^{iPx_1}|\Delta\rangle \ . \tag{3.1.52}$$

We can now expand the exponentials and get

$$r_2^{-2\Delta_\phi} = \sum_N \langle\Delta|\frac{(-iK\mathcal{R}x_2)^N}{N!}\frac{(iPx_1)^N}{N!}|\Delta\rangle \ , \tag{3.1.53}$$

where the cross terms with unequal number of powers of K and P will give zero matrix elements (since, as usual, states of unequal energy are orthogonal). We finally get

$$\langle 0|\phi(x_2)\phi(x_1)|0\rangle = \frac{1}{r_1^\Delta r_2^\Delta}\sum_N \langle N, \vec{n}_2|N, \vec{n}_1\rangle \left(\frac{r_1}{r_2}\right)^{\Delta+n} \ , \tag{3.1.54}$$

where we denoted

$$|N, \vec{n}\rangle = \frac{1}{N!}(P_\mu \vec{n}^\mu)^N|\Delta\rangle \ . \tag{3.1.55}$$

We can see that radial quantization predicts that the coefficients c_n must be equal to certain matrix elements. These matrix elements can in turn be evaluated purely algebraically, using conformal algebra. Let us see what happens for $N = 1$. We have to calculate

$$\begin{aligned}\langle\Delta|K_\mu P_\nu|\Delta\rangle &= \langle\Delta|[K_\mu, P_\nu]|\Delta\rangle + \langle\Delta|P_\nu K_\mu|\Delta\rangle \\ &= \langle\Delta|2i(D\delta_{\mu\nu} - M_{\mu\nu})|\Delta\rangle \\ &= \Delta\delta_{\mu\nu}\langle\Delta|\Delta\rangle = \Delta\delta_{\mu\nu} \ ,\end{aligned} \tag{3.1.56}$$

since $K_\mu|\Delta\rangle = 0$, and

$$M_{\mu\nu}|\Delta\rangle = 0 \ , \tag{3.1.57}$$

for spinless fields.

For $N > 1$, we have to calculate an expression with the following structure

$$\langle \Delta | KKK \dots PPP | \Delta \rangle \ . \tag{3.1.58}$$

The evalutation proceeds along the same lines as above. We simply commute the K operators until they hit the state $|\Delta\rangle$ and annihilate it.

Proceeding this way, one can show order by order that the matrix elements agree with what expanding the known two-point function predicts. Of course this is a silly computation, since we know two point function anyway. But it does show that radial quantization works, and in more complex cases we will be able to use to go beyond what is known using other techniques. The first example of this is in the next section.

3.2 Unitarity bounds

A famous result which can be most easily obtained by applying radial quantization to two-point functions is the *Unitarity bounds*: $\Delta \geqslant \Delta_{min}(l)$. By that we mean that the dimension of a symmetric traceless primary field in a unitary theory must be above a minimal allowed value that depends on the spin l of the field

$$\begin{aligned} \Delta_{min}(l) &= l + D - 2 \ , \quad \text{if} \quad l = 1, 2, 3, \dots \quad \text{or} \\ \Delta_{min}(l) &= \frac{D}{2} - 1 \ , \quad \text{if} \quad l = 0 \ . \end{aligned} \tag{3.2.1}$$

Similar bounds exist for other $SO(D)$ representations, like antisymmetric tensors or fermions.

Consider the following matrix

$$A_{\nu\{t\},\mu\{s\}} = {}_{\{t\}}\langle \Delta, l | K_\nu P_\mu | \Delta, l \rangle_{\{s\}} \ , \tag{3.2.2}$$

where $|\Delta, l\rangle$ is a state created by inserting the operator of dimension Δ and spin l, and $\{s\}$ are the spin indeices. This matrix must have only positive eigenvalues in a unitary theory. The proof is by contradiction: suppose than there is a negative eigenvalue $\lambda < 0$ with $\xi_{\mu,\{s\}}$ the corresponding eigenvector. Then consider the state $|\Psi\rangle = \xi_{\mu,\{s\}} P_\mu |\{s\}\rangle$. This state would have a negative norm:

$$\langle \Psi | \Psi \rangle = \xi^\dagger A \xi = \lambda \xi^\dagger \xi < 0 \ . \tag{3.2.3}$$

Now using

$$[K_\nu, P_\mu] \propto i(D\delta_{\mu\nu} - M_{\mu\nu}) \ . \tag{3.2.4}$$

we notice that the eigenvalues of A will get two contributions. The first will be proportional to Δ, whereas the second will be eigenvalues of a Hermitian matrix that depends only on the spin:

$$B_{\nu\{t\},\mu\{s\}} = \langle \{t\} | i M_{\mu\nu} | \{s\} \rangle \ , \tag{3.2.5}$$

The condition that all $\lambda_A \geqslant 0$ will be this equivalent to

$$\Delta \geqslant \lambda_{max}(B) \ , \quad \text{where} \quad \lambda_{max}(B) \text{ is the maximum eigenvalue of } B \ . \tag{3.2.6}$$

The computation of the eigenvalues of B is most easily done by using the analogy with the Spin-Orbit interaction in quantum mechanics. Let us write the action of the operator $M_{\mu\nu}$ on the space of $\{s\}$ and $\{t\}$ in the following way

$$-iM_{\mu\nu} = -\frac{1}{2}(V^{\alpha\beta})_{\mu\nu}(M_{\alpha\beta})_{\{s\},\{t\}} \ , \tag{3.2.7}$$

where the generator $(V^{\alpha b})_{\mu\nu}$ in the vector representation is given by

$$(V^{\alpha\beta})_{\mu\nu} = i(\delta^{\alpha}_{\mu}\delta^{b}_{\nu} - \delta^{\alpha}_{\nu}\delta^{b}_{\mu}) \ . \tag{3.2.8}$$

Let us compare the above with the standard problem in quantum mechanics, where we have to calculate the eigenvalues of

$$L^i \cdot S^i \ . \tag{3.2.9}$$

The operator S^i is the analogue of $M_{\alpha\beta}$, they both act in the space of spin indices. On the other hand L^i is the analogue of $V^{\alpha\beta}$. The coordinate space in which L_i acts is replaced by a vector space in which $V^{\alpha\beta}$ acts.

In QM, the diagonalization of (3.2.9) is easily performed using the identity

$$L^i \cdot S^i = \frac{1}{2}[(L+S)^2 - L^2 - S^2] \ . \tag{3.2.10}$$

Indeed, the operators S^2 and L^2 are Casimirs so their eigenvalues $s(s+1)/2$ and $l(l+1)/2$ are known, while $(L+S)^2$ is the Casimir of the tensor product representation $l \otimes s$ so its eignevalues are $j(j+1)/2$, $j = |l-s|, \dots l+s$.

In our case we have a spin representation l, a vector representation $V_{l=1}$ and a tensor representation R' that occurs for $R' \in V \otimes l$. Thus, the maximal eigenvalue of B will be given by

$$\lambda_{max}(B) = \frac{1}{2}[\mathrm{Cas}(V_{l=1}) + \mathrm{Cas}(l) - \min \mathrm{Cas}(R')] \,. \tag{3.2.11}$$

This can be show equal $D-2+l$ for $l \geqslant 1$, giving the first of the unitarity bounds.

In principle, were we to consider the action with more Ks and Ps, we could get stronger bounds. This indeed happens for $l = 0$ at level 2, where imposing that

$$A_{\mu'\nu',\mu\nu} = \langle \Delta | K_{\mu'} K_{\nu'} P_{\nu} P_{\mu} | \Delta \rangle. \tag{3.2.12}$$

is positive definite, one derives

$$\Delta \geqslant \frac{D}{2} - 1 \tag{3.2.13}$$

which is the second unitarity bound.

It turns out however that higher levels are not needed for spin $l \geqslant 1$, and levels higher than 2 are not needed for scalars, i.e. the constraints we saw above are necessary and sufficient to have unitarity at all levels.

3.3 Operator Product Expansion (OPE)

The idea of OPE is familiar from the usual QFT; it says that we should be able to replace a product of two local operators in the limit where they become very close to each other by a series of operators inserted at the middle point. It is thus no surprise that OPE holds in CFTs. However, in CFTs the OPE acquires additional and very powerful properties thanks to a connection with the radial quantization.

Let us first of all *derive* OPE using radial quantization. Suppose we have two operators inserted inside a sphere:

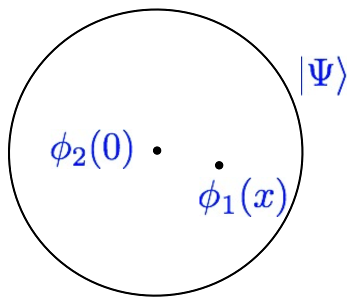


Figure 3.7: Two operators inserted inside a sphere generate a state on the sphere.

They generate a state $\phi_1(x)\phi_2(0)|0\rangle = |\Psi\rangle$ on the surface of the sphere. If we expand this states into eigenstates of energy (i.e. dilatation generator), we get

$$|\Psi\rangle = \sum_n c_n|E_n\rangle \ , \ c_n = c_n(x) \ . \tag{3.3.1}$$

By the state-operator correspondence, the states $E_n\rangle$ will be in one-to-one correspondence either with operators that are primaries (they are located in the origin), or with descendants. This means

$$\phi_1(x)\phi_2(0)|0\rangle = \sum_{\mathcal{O} \text{ primaries}} C_n(x, \partial_y)\mathcal{O}(y)|_{y=0}|0\rangle \ , \tag{3.3.2}$$

where $C_n(x, \partial_y)$ is understood as a power series in ∂_y. We have just proved the existence of an OPE!

Notice that in CFT, the OPE is understood not only in the asymptotic limit as is the case for usual perturbative QFT, but rather as an expansion in a Hilbert space which is complete. This series expansion is therefore convergent.

So far we have only proved the existence of OPE. We can learn more about the coefficients by requiring that they must be consistent with the algebra. Let us focus on a particular term of the series

$$\phi_1(x)\phi_2(0)|0\rangle = \frac{const.}{|x|^k}[\mathcal{O}(0) + \ldots]|0\rangle + \text{ contributions of other primaries,} \tag{3.3.3}$$

where the power k in the denominator will be fixed by scaling, and the ellipses stand for terms with derivatives of $\mathcal{O}$. To fix k, let us act with the operator D on the above expression. We start from the left hand side

$$\begin{aligned} D\phi_1(x)\phi_2(0)|0\rangle &= i(\Delta_1 + x^\mu\partial_\mu)\phi_1(x)\phi_2(0)|0\rangle + i\Delta_2\phi_1(x)\phi_2(0)|0\rangle \ . \\ &\overset{\text{using (3.3.3)}}{=} i(\Delta_1+\Delta_2-k)\frac{const.}{|x|^k}[\mathcal{O}(0)+\ldots]+\ldots \end{aligned} \tag{3.3.4}$$

On the other hand, acting with D on the right hand side of (3.3.3), we see

$$D\frac{const.}{|x|^k}[\mathcal{O}(0)+\ldots]|0\rangle = \frac{const.}{|x|^k}[i\Delta_O\mathcal{O}(0)+\ldots]|0\rangle \ , \tag{3.3.5}$$

where the dilatation operator acts only on $\mathcal{O}$, since the prefactor is simply a c-number. By comparing (3.3.4) with (3.3.5) we conclude that $k = \Delta_1 + \Delta_2 - \Delta_O$.

Let us now consider the descendant terms:

$$\phi_1(x)\phi_2(0)|0\rangle = \frac{const.}{|x|^k}[\mathcal{O}(0) + cx^\mu\partial_\mu O(0)+\ldots]|0\rangle + \text{ contributions of other primaries,} \tag{3.3.6}$$

We claim that the coefficient c can be found by acting on both sides of this equation with K_μ. Once again we consider first the left hand side

$$K_\mu\phi_1(x)\phi_2(0)|0\rangle = (\sim x_\mu + \sim x^2\partial_\mu)\phi_1(x)\phi_2(0)|0\rangle \ , \tag{3.3.7}$$

$$\sim \frac{const.x_\mu}{|x|^k}[\mathcal{O}(0)+\ldots] \tag{3.3.8}$$

where in the first line we used $K_\mu\phi_2(0)=0$, since $\phi_2(0)$ is a primary, and in the second line substituted the OPE. Acting on the right hand side we get:

$$\begin{aligned} &K_\mu\frac{const.}{|x|^k}[\mathcal{O}(0)+cx^\nu\partial_\nu\mathcal{O}(0)+\ldots]|0\rangle \\ &\qquad = \frac{const.}{|x|^k}[K_\mu\mathcal{O}(0)+cK_\mu x^\nu\partial_\nu\mathcal{O}(0)+\ldots]|0\rangle \\ &\qquad \sim \frac{const.}{|x|^k}[c\Delta x^\mu\mathcal{O}(0)+\ldots] \ , \end{aligned} \tag{3.3.9}$$

where we used the fact that K_μ lowers the dimension by 1, so $K_\mu\mathcal{O}(0) = 0$, whereas $K_\mu P_\nu\mathcal{O}(0) \sim \delta_{\mu\nu}\Delta\mathcal{O}(0)$. Matching the coefficients, we see that we can determine c. Analogously, the coefficients of the higher descendants can also be determined recursively.

In a CFT, the consistency of the OPE with CI thus fixes completely the function $C_n(x,\partial_y)$, up to a prefactor number λ_O, one per primary:

$$\sum_O \lambda_O C_\mathcal{O}(x,\partial_y)\mathcal{O}(y)|_{y=0}|0\rangle \ , \quad \text{with} \quad \lambda_O \text{ a free parameter.} \tag{3.3.10}$$

This would not be the case if we had SI only. The above is a great simplification. Now we want to show a more practical way to recover this function $C_n(x,\partial_y)$.

This method is based on the fact that we can use OPE to reduce the n-point function into an infinite sum of $(n-1)$-point functions. To make that explicit, let us consider the very simple case of a three-point function

Using the OPE inside the correlator, we can write:

$$\langle \phi_1(x)\phi_2(0)\Phi(z)\rangle = \sum_O \lambda_O C_{\mathcal{O}}(x,\partial_y)\langle \mathcal{O}(y)|_{y=0}\Phi(z)\rangle \ , \tag{3.3.11}$$

Since we know that the two point function is diagonal, the only surviving term from the infinite sum will be for $O = \Phi$:

$$\langle \phi_1(x)\phi_2(0)\Phi(z)\rangle = \lambda_\Phi C_\Phi(x,\partial_y)\langle \Phi(y)|_{y=0}\Phi(z)\rangle \ , \tag{3.3.12}$$

Moreover, we know both the two and three point functions entering the above equation:

$$\langle \phi_1(x_1)\phi_2(x_2)\Phi(x_3)\rangle = \frac{\lambda_\Phi}{(x_{12})^{\cdots}(x_{13})^{\cdots}(x_{23})^{\cdots}} \tag{3.3.13}$$

$$\langle \Phi(y)\Phi(z)\rangle = \frac{1}{|y-z|^{2\Delta_\Phi}} \ , \tag{3.3.14}$$

Now, expanding both sides around $x=0$ and fixing the coefficients term by term, we can determine the function $C_\Phi(x,\partial_y)$. Since we normalized the three point function using the same coefficient λ_Φ, the function $C_\Phi(x,\partial_y)$ will depend only on the dimensions of involved fields (and the spacetime dimension).

Exercise: Matching the three-point function to the OPE, show that for $\Delta_1 = \Delta_2 = \Delta_\phi$ and $\Delta_\Phi = \Delta$

$$C(x,\partial_y) = \frac{1}{|x|^{2\Delta_\phi - \Delta}}\left[1 + \frac{1}{2}x^\mu\partial_\mu + \alpha x^\mu x^\nu \partial_\mu\partial_\nu + \beta x^2\partial^2 + \dots\right] \ , \tag{3.3.15}$$

where

$$\alpha = \frac{\Delta+2}{8(\Delta+1)}, \quad \text{and} \quad \beta = -\frac{\Delta}{16\left(\Delta - \frac{D}{2}+1\right)(\Delta+1)} \ . \tag{3.3.16}$$

Literature

Radial quantization is usually discussed in courses on 2D CFT or on string theory. See e.g.

Polchinski J., "String theory", Vol.1

where also the argument connecting OPE with the radial quantization is given. A recent paper which discusses these issues in a language similar to these lectures is:

D. Pappadopulo, S. Rychkov, J. Espin and R. Rattazzi, "OPE Convergence in Conformal Field Theory," arXiv:1208.6449 [hep-th].

Concerning infinitesimal action of generators on fields, see

G. Mack and A. Salam, “Finite component field representations of the conformal group,” Annals Phys. **53**, 174 (1969).

For the N-S quantization, see (caution: heavily mathematically written!)

M. Luscher and G. Mack, “Global Conformal Invariance in Quantum Field Theory,” Commun. Math. Phys. **41**, 203 (1975).

For the unitarity bounds, see

G. Mack, “All Unitary Ray Representations of the Conformal Group SU(2,2) with Positive Energy,” Commun. Math. Phys. **55**, 1 (1977).

for the full classification. For a simpler necessary condition explained here see:

S. Minwalla, “Restrictions imposed by superconformal invariance on quantum field theories,” Adv. Theor. Math. Phys. **2**, 781 (1998) [hep-th/9712074].

Lecture 4

Conformal Bootstrap

4.1 Recap

By now we have learned quite a bit about the structure of Conformal Field Theories (CFT). We have seen that:

• Any CFT is characterized by the spectrum of local primary operators, by which we mean the set of pairs $\{\Delta, \mathcal{R}\}$, where Δ is operator's scaling dimension, and $\mathcal{R}$ the representation of the $SO(D)$ under which it transforms. We showed that all other operators are obtained by differentiating the primaries; they are called descendants. We also showed that there is a one-to-one correspondence between the operators $\mathcal{O}_\Delta$ and the states of a radially quantized theory, obtained by inserting the operator at the origin: $|\Delta\rangle = \mathcal{O}_\Delta(0)|0\rangle$. Finally, we showed that (in unitary theories) there exist unitarity bounds for the operators' dimensions

$$\Delta \geqslant \Delta_{min}(\mathcal{R}) \; , \tag{4.1.1}$$

where $\Delta_{min}(\mathcal{R})$ is the lowest allowed vallowed for an operator in the representation $\mathcal{R}$.

• The two-point functions of primaries are fixed. For example in the case of scalars

$$\langle \phi(x)\phi(y)\rangle = \frac{N}{|x-y|^{2\Delta}} \; , \tag{4.1.2}$$

and analogously for higher spins. The normalization constant is usually fixed to $N = 1$, which in the radial quantization language corresponds to the normalization choice $\langle\Delta|\Delta\rangle = 1$. In radial quantization the above correlator can be computed as the matrix element

$$\langle\Delta| e^{-iKy/y^2} e^{iPx} |\Delta\rangle \; . \tag{4.1.3}$$

This matrix element can be evaluated just by using conformal algebra, which also explains why two-point function is fixed.

• The three-point functions of primary operators are fixed up to a constant. For scalars, we saw that they are given by

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{\lambda_{123}}{(x_{12})^{2\alpha_{123}}(x_{13})^{2\alpha_{132}}(x_{23})^{2\alpha_{231}}} \; , \tag{4.1.4}$$

where λ_{123} a free parameter (which is physical; one cannot rescale it away once two-point function normalization has been fixed),

$$x_{ij} = x_i - x_j \ , \tag{4.1.5}$$

and

$$\alpha_{ijk} = \frac{\Delta_i + \Delta_j - \Delta_k}{2} \ . \tag{4.1.6}$$

Analogously the three-point function of two scalar and one spin-l operator is fixed up to a constant. Using the methods of Lecture 2, one can also compute the most general three-point function of three spin-l operators. It turns out that it has a finite number of tensor structures that are consistent with Conformal Symmetry (CS). This means that there is a finite number of constants multiplying these tensors.

• As we discussed, the constant λ_{123} in eq. (4.1.4) is the same as the one appearing in the Conformally Invariant (CI) Operator Product Expansion (OPE)

$$\phi_1(x)\phi_2(0) = \sum_{\mathcal{O} \text{ primaries}} \lambda_{12\mathcal{O}} C_{\mathcal{O}}(x, \partial_y)\mathcal{O}(y)|_{y=0} \ . \tag{4.1.7}$$

Note that the operator $\mathcal{O}(y)$ in the above is inserted at $y = 0$, but it could be inserted in any point between x and 0. Of course then the coefficient functions $C_{\mathcal{O}}(x, \partial_y)$ will have to be changed appropriately. This freedom will turn out to be useful later.

Alternatively, these constants are the coefficients appearing when the state $\phi_1(x)\phi_2(0)|0\rangle$ is expanded into a complete basis of states

$$\phi_1(x)\phi_2(0)|0\rangle = \sum_{\mathcal{O}} \lambda_{12\mathcal{O}} \left[\mathcal{O}(0) + \text{ descendants}\right] \ , \tag{4.1.8}$$

in the radial quantization scheme. It is a non-trivial feature of CFT that the function $C_{\mathcal{O}}(x, \partial_y)$ is completely fixed, i.e. there is one constant $\lambda_{12\mathcal{O}}$ per conformal family. The constants λ_{ijk} are called interchangeably three-point function coefficients, OPE coefficients, or structure constants of the operator algebra.

• With the knowledge of the set {spectrum, OPE coefficients}, which is called *CFT data*, we can compute any $n-$point correlation function of the theory, recursively reducing it to $(n-1)$, $(n-2)$... and finally to three point functions by using the OPE. For example five-point function are reduced to four point functions as in Fig. 4.1. Moreover, this expansion will be convergent. This becomes obvious if we view it as an expansion in the Hilbert space

$$|\Psi_1\rangle = \sum_n c_n |n\rangle \quad \to \quad \langle\Psi_2|\Psi_1\rangle = \sum_n c_n \langle\Psi_2|n\rangle \ . \tag{4.1.9}$$

Here $|\Psi_1\rangle$ and $\langle\Psi_2|$ are the states generated on the dashed sphere by operators inserted inside and outside. Then we expand $|\Psi_1\rangle$ into states generated by operators inserted at the center of the sphere (OPE), and get an expansion for the scalar product $\langle\Psi_2|\Psi_1\rangle$ which describes the correlation function. Expansions of scalar products are convergent for states of finite norm. In our case, the norms $\langle\Psi_i|\Psi_i\rangle$, $i = 1, 2$ are always finite since they correspond to some correlation function of operators inserted in an inversion invariant way.

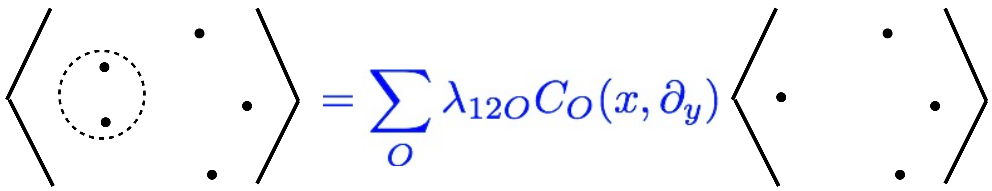
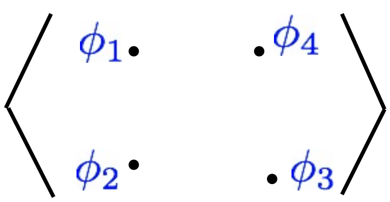


Figure 4.1: Reducing 5point function to a sum of 4-point functions using the OPE.

4.2 Consistency condition on CFT data

As we mentioned, given a CFT data, we can compute all the correlators in a theory. It is therefore natural to ask if any random set of CFT data defines a good theory. The answer is no. At the very least, we have to impose a consistency condition on the CFT data, which comes from studying the four-point functions. Consider a scalar four point function in a generic point configuration:



To compute it via the OPE, we surround two of the operators, say ϕ_1 and ϕ_2 by a sphere and expand into radial quantization states on this sphere. Operationally this means that we are writing:

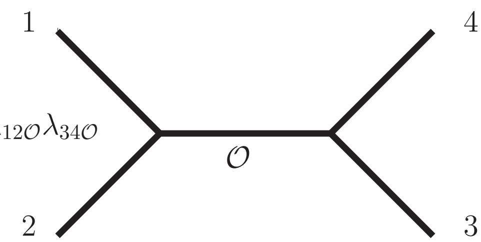
$$\begin{aligned} \phi_1(x_1)\phi_2(x_2) &= \sum_{\mathcal{O}} \lambda_{12\mathcal{O}} C_{\mathcal{O}}(x_{12}, \partial_y)\mathcal{O}(y)\Big|_{y=\frac{x_1+x_2}{2}} \,, \\ \phi_3(x_3)\phi_4(x_4) &= \sum_{\mathcal{O}} \lambda_{34\mathcal{O}} C_{\mathcal{O}}(x_{34}, \partial_z)\mathcal{O}(z)\Big|_{z=\frac{x_3+x_4}{2}} \,. \end{aligned} \tag{4.2.1}$$

Plugging the above into the four-point function, we get

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle = \sum_{\mathcal{O}} \lambda_{12\mathcal{O}}\lambda_{34\mathcal{O}} \big[C_{\mathcal{O}}(x_{12}, \partial_y) C_{\mathcal{O}}(x_{34}, \partial_z)\langle \mathcal{O}(y)\mathcal{O}(z)\rangle\big], \tag{4.2.2}$$

The functiosn in square brackets are completely fixed by conformal symmetry in terms of the dimensions of ϕ_i and of the dimension and spin of $\mathcal{O}$ (since both the functions $C_{\mathcal{O}}$ and the correlator $\langle \mathcal{O}(y)\mathcal{O}(z)\rangle$ are fixed). These functions are called Conformal Partial Waves (CPW). Diagramatically the expansion into conformal partial waves can be written as:

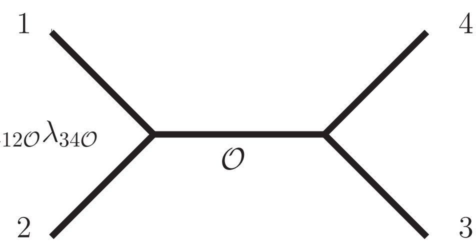
$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle = \sum_{\mathcal{O}} \lambda_{12\mathcal{O}}\lambda_{34\mathcal{O}}$$



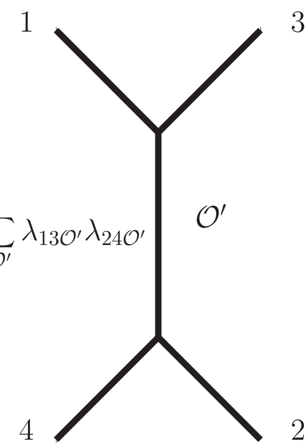
(although one should not confuse CPWs with Feynman diagrams).

Now, notice that we might have chosen to compute the same four point function by choosing a sphere enclosing the operators ϕ_1 and ϕ_4. This means that we would have chosen a different OPE channel, (14)(23) instead of (12)(34). We would have obtained a different CPW expansion, but the result should be the same. As we will see, this is not obvious at all, and gives rise to a non-trivial consistency relation:

$$\sum_{\mathcal{O}} \lambda_{12\mathcal{O}}\lambda_{34\mathcal{O}}$$



$$= \quad \sum_{\mathcal{O}'} \lambda_{13\mathcal{O}'}\lambda_{24\mathcal{O}'}$$



This condition is called in the literature interchangeably as “OPE associativity”, “crossing symmetry” and “conformal bootstrap” condition. It has to be satisfied by any consistent CFT data.

4.3 Conformal Bootstrap

Notice that if we impose OPE associativity on all four-point functions, no new constraints will appear at higher $n-$point functions. This is demonstrated on the five-point function example in Figure 4.2

The above shows that OPE associativity at the four point function level is definitely necessary, and perhaps even sufficient, or very close to being sufficient, condition to have a consistent CFT. We are led to make the following

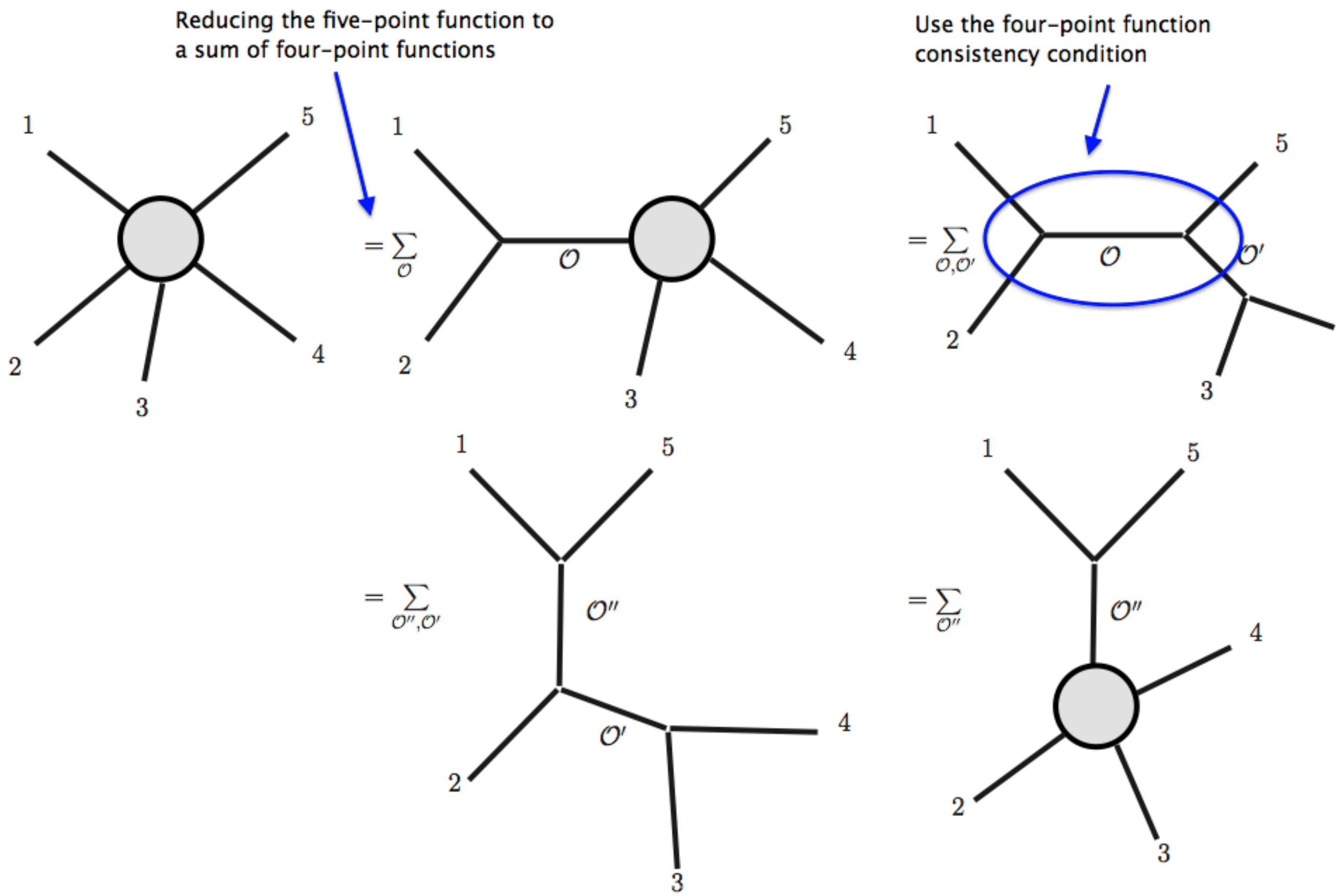


Figure 4.2: For five-point functions, OPE in (12) and (15) channels should give equivalent expansions. This figure demonstrates schematically that this is not an additional constraint but follows from OPE associativity for four point functions.

Definition [Ferrara, Grillo, Gatto 1973, Polyakov 1974, Mack 1977] *A CFT is a set of CFT data which satisfies the OPE associativity for all four-point functions.*

This definition must not be taken too dogmatically. In fact it is known that in $D = 2$ modular invariance provides an extra constraint, but still OPE associativity provides a good zeroth order approximation. The role of modular invariance in $D \geqslant 3$ is much less clear. If extra constraints are discovered in the future, one should not hesitate to update this definition.

But for now, can we use this definition to classify CFTs, in analogy with Lie algebras? (Very ambitious goal!)

More modestly, is it possible to deduce some general properties, beyond those we already discussed, which are valid for any CFT?

Can we try and learn new things about CFTs which we know exist but which are not exactly solvable? An example of such a theory is the three-dimensional Ising model at

$T = T_{crit}$.

This research direction is called “Conformal Bootstrap”.

4.3.1 Success story in $D = 2$

In two-dimensional CFT the conformal bootstrap has been applied in a famous paper by Belavin, Polyakov and Zomolodchikov in 1984. As we discussed in Lecture 1, the two-dimensional conformal algebra has an infinite dimensional extension, which is called Virasoro algebra. The generators are L_n and $\bar{L}_n$. The correspondence with the finite dimensional subalgebra of global conformal transformations studied so far is as follows:

$$\begin{array}{ccc} \underbrace{\ldots, L_{-3}, L_{-2}}_{\text{extra lowering operators}} & , \underbrace{L_{-1}, \quad L_0, \quad L_1}_{K_\mu \ \{D, M_{\mu\nu}\} \ P_\mu}, & \underbrace{L_2, L_3, \ldots}_{\text{extra raising operators}} \\ \ldots & \bar{L}_{-1}, \quad \bar{L}_0, \quad \bar{L}_1 \ , & \ldots \end{array} \tag{4.3.1}$$

The generator L_n raises the state scaling dimension by n units: $L_n|\Delta\rangle = |\Delta + n\rangle$. The notion of a Virasoro primary field is now

$$L_{-n}|\Delta\rangle = 0 \ , \forall \ n = 1, 2, 3, \ldots \ , + \text{ same for } \bar{L}_{-n} \tag{4.3.2}$$

stronger than the condition

$$K_\mu|\Delta\rangle = 0 \Leftrightarrow L_{-1}|\Delta\rangle = 0, \bar{L}_{-1}|\Delta\rangle = 0 \ , \tag{4.3.3}$$

considered so far. The field satisfying the latter condition is called quasiprimary in $D = 2$ CFT literature.

It would take us too long to explain the full 2D story. Very schematically, one first studies the unitarity conditions for the 2D infinite dimensional Virasoro algebra and finds that they are more restricted than in $D \geqslant 3$. In particular, the conditions depends on the value of a certain parameter c appearing in the two-point function of the canonically normalized stress-energy tensor T

$$\langle T(x)T(y)\rangle \sim \frac{c}{|x-y|^4} \ . \tag{4.3.4}$$

This parameter c is called *central charge*.

1. For $c \geqslant 1$, one finds that the unitarity conditions are more or less the same as in the three-dimensional case.

2. For $0 < c < 1$, the condition is much more restrictive. Namely, only a discrete sequence of values for c is allowed

$$c = 1 - \frac{6}{m(m+1)} \ , \ \text{with } m = 3, 4, 5, \ldots \tag{4.3.5}$$

Theories corresponding to such c are called “minimal models”. Moreover, one finds that for $c \geqslant 1$ only a finite discrete set of operator dimensions is allowed to appear:

$$\Delta_{r,s} = \frac{(r + m(r-s))^2 - 1}{2m(m-1)} \ , \text{ where } 1 \leqslant s \leqslant r \leqslant m-1 \ \text{ integers} \ . \tag{4.3.6}$$

Notice that one primary multiplet in two dimensions splits into infinitely many quasiprimary multiplets. So one $2D$ primary is morally equivalent to infinitely many primaries in three dimensions. This is due to the fact that in two dimensions there are new raising operators $L_2, L_3, \ldots$. The condition for an operator to be primary is also strengthen in 2D, so it's not surprising we may have finitely many primaries in 2D, but we will always have infinitely many in $D \geqslant 3$ dimensions.

Once the dimensions are known and since there are finitely many primaries, the OPE associativity equations reduce to a problem of finite dimensional linear algebra. They can be solved in order to find the OPE coefficients.

The simplest minimal model appears for $c = 1/2$ and corresponds to the two-dimensional Ising model at $T = T_{crit}$. The Virasoro primary field content includes just the identity operator $\mathbb{I}$, the spin σ ($\mathbb{Z}_2-$odd) and the energy density ϵ ($\mathbb{Z}_2-$even) fields with dimensions

$$\Delta_{\mathbb{I}} = 0 \ , \ \Delta_\sigma = \frac{1}{8} \ , \text{ and } \Delta_\epsilon = 1 \ . \tag{4.3.7}$$

The OPEs are

$$\begin{aligned} \sigma \times \sigma &= \mathbb{I} + \lambda_{\sigma\sigma\epsilon}\epsilon \\ \epsilon \times \epsilon &= \mathbb{I} + \lambda_{\epsilon\epsilon\epsilon}\epsilon \\ \sigma \times \epsilon &= \lambda_{\sigma\sigma\epsilon}\sigma \ . \end{aligned} \tag{4.3.8}$$

where $\lambda_{\sigma\sigma\epsilon}$ is determined by solving bootstrap equation, while $\lambda_{\epsilon\epsilon\epsilon} = 0$ due to the Kramers-Wannier duality, which is a property of the critical Ising model specific to $D = 2$.

Notice that the above OPE is consistent with the $\mathbb{Z}_2$ symmetry of the model.

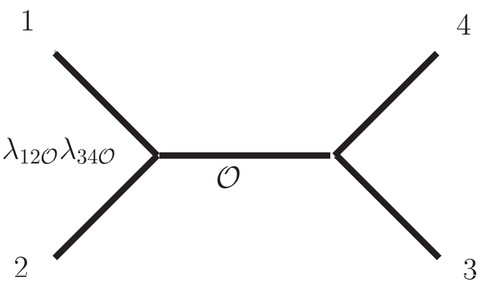
As mentioned for $c < 1$ bootstrap equations are relatively easy because only finitely many fields enter and all dimensions are known. For $c \geqslant 1$, the conformal bootstrap becomes difficult to solve even in $D = 2$. One notable such example when it has been used with success is the Liouville theory.

4.3.2 Tools for the bootstrap in $D \geqslant 3$ dimensions

To do bootstrap in $D \geqslant 3$ we have to revisit some notions which were discussed before, but this time do it more explicitly.

The key role will be played by the so called Conformal Blocks (CB). Let us look for simplicity at the four-point function of fields with the same dimensionality $\Delta_i = d$ (while Δ will be reserved for the dimension of the field appearing in the OPE). We saw that

$$\frac{g(u,v)}{(x_{12})^{2d}(x_34)^{2d}} = \langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle = \sum_{\mathcal{O}} \lambda_{12\mathcal{O}}\lambda_{34\mathcal{O}}$$



Actually, each CPW will have the same transformation properties under the conformal group as the four-point function itself (this follows from conformal invariance of the OPE used to define CPW). In other words, we will have:

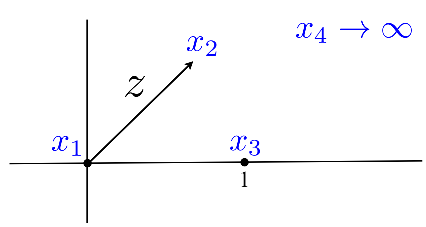
$$\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle = \sum_{\mathcal{O}} \lambda_{12\mathcal{O}}\lambda_{34\mathcal{O}} \frac{G_{\mathcal{O}}(u,v)}{(x_{12})^{2d}(x_{34})^{2d}} \ , \tag{4.3.9}$$

where the quantity $G_{\mathcal{O}}(u,v)$ is called *Conformal Block* (CB). Thus conformal block is simply the most interesting part of the CPW. We have:

$$g(u,v) = \sum_{\mathcal{O}} \lambda_{12\mathcal{O}}\lambda_{34\mathcal{O}} G_{\mathcal{O}}(u,v) \tag{4.3.10}$$

We want to gain some intuition about CBs. Since they depend only on u and v, we can study them by using a conformal transformation to map the four points to some convenient positions. (Remember that conformal transformations leave u and v invariant).

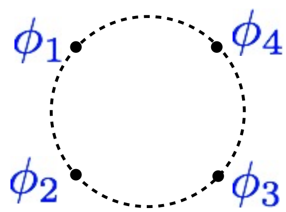
Let us consider the following configuration: map the point $x_4 \to \infty$. Then shift $x_1 \to 0$, and do a rotation followed by a dilatation to put $x_3 = (1, 0, \ldots, 0)$ (all these transformations leave ∞ invariant since there is only one ∞ point). So far we have fixed three points. Let us now do a rotation with respect to the x_1-x_3 axis, to put x_2 somewhere on the plane of the page. We use the z complex coordinate in this plane:



The cross-ratios corresponding to this configuration (and thus to any configuration related to this one by a conformal transformation) are easily evaluated:

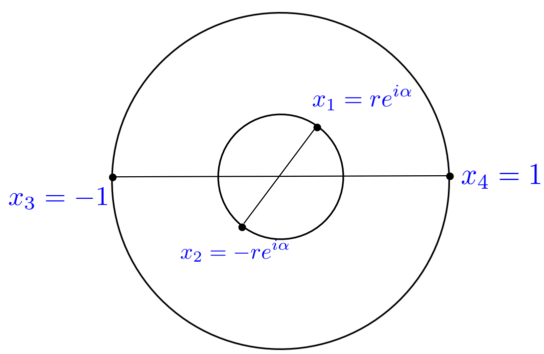
$$u = \left.\frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\right|_{x_4\to\infty} = |z|^2 \ , \quad \text{and analogously} \quad v = |1-z|^2 \ . \tag{4.3.11}$$

In what follows, important role will be played by the special point $z = 1/2$ and its neighborhood. The $z = 1/2$ configuration can be mapped onto four points at the vertices of a square (both have $u = v = 1/4$): The square configuration (and thus $z = 1/2$ configuration



equivalent to it), and its neighborhood will be most natural to study the conformal bootstrap equation. The reason is that this configuration treats symmetrically the (12)(34) with (14)(23) OPE channels, which are the channels compared in the bootstrap equation.

We will now introduce another configuration, which puts the points x_1 and x_2 (and x_3 and x_4) symmetrically with respect to the origin $x = 0$. This configuration is thus



characterized by a complex parameter $\rho = r^{i\alpha}$. This configuration can of course be mapped by a conformal transformation onto the one characterized by z, for a certain value of z. To find the correspondence between ρ and z we just have to make sure that u and v are the same. An explicit computation shows that u and v will be the same if we fix

$$z = \frac{4\rho}{(1+\rho)^2} \Leftrightarrow \rho = \frac{z}{(1+\sqrt{1-z})^2}. \tag{4.3.12}$$

So for $z = 1/2$, we get that $\rho = 3 - 2\sqrt{2} \approx 0.17$.

As discussed in Lecture 3, we can apply a Weyl transformation to map the CFT dynamics from the flat space to the cylinder. The last configuration is then mapped to the one in Fig. 4.3. On the cylinder conformal Block can be computed as:

$$\text{Conformal Block} = \sum \langle 0|\phi_1\phi_2|n\rangle e^{-E_n\tau}\langle n|\phi_3\phi_4|0\rangle \ , \tag{4.3.13}$$

where the sum is over all the descendants of $|\Delta, l\rangle$, $E_n = \Delta + n$, and $\tau = -\log r$ is the cylinder time interval along which we have to propagate exchanged states. The product of

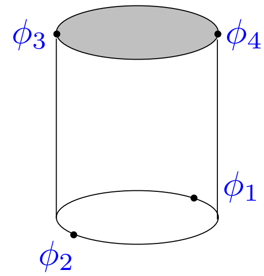


Figure 4.3: The configuration on $S^{D-1} \times \mathbb{R}$ obtained by a Weyl transformation. The pairs of points $\phi_{3,4}$ and $\phi_{1,2}$ are now in antipodal positions on the spheres at the cylinder time 0 and $\log r$. Their positions on their respective spheres are rotated with respect to each other by angle α.

the matrix elements depends only on α, and so we conclude that conformal block must have the form

$$\text{Conformal Block} = \sum_{n=0}^{\infty} A_n(\alpha) r^{\Delta+n} \; . \tag{4.3.14}$$

The coefficients A_n are completely fixed by conformal symmetry. Their precise values can be found, but we will not pursue this problem here. The leading coefficient A_0 is easily determined on physical grounds. The states $\phi_1\phi_2|0\rangle$ and $\phi_3\phi_4|0\rangle$ differ by a rotation by angle α. Therefore, $A_0(\alpha)$ measures how the matrix elements with a spin l state $|\Delta, l\rangle$ change under the rotation by an angle α.

Let us parametrize the state on the cylinder by the unit vector $\vec{n} \in S^{D-1}$ pointing to the point where ϕ_1 is inserted on the sphere. The state $|\Delta, l\rangle$ has internal indices $|\Delta, l\rangle_{\mu_1,\mu_2,\ldots}$ which form a symmetric traceless spin l tensor. The individual matrix elements are:

$$\langle 0|\phi_1\phi_2|\Delta, l\rangle_{\mu_1,\mu_2,\ldots} = const.(\vec{n}_1^{\mu_1} \ldots \vec{n}_1^{\mu_l} - \text{traces}) \; , \tag{4.3.15}$$

since there is only one traceless and symmetric spin l tensor which can be constructed out of a single vector $\vec{n}_1$. Thus, up to normalization, the leading coefficient will be a contraction of two such matrix elements

$$A_0(\alpha) = (n_1^{\mu_1} \ldots \vec{n}_1^{\mu_l} - \text{traces})(n_2^{\mu_1} \ldots \vec{n}_2^{\mu_l} - \text{traces}) = \mathcal{P}(\vec{n}_1.\vec{n}_2) = \mathcal{P}(\cos\alpha) \; , \tag{4.3.16}$$

where $\mathcal{P}$ is a certain polynomial whose coefficients can only depend on the spin l and on the number of spacetime dimensions D.

For $D = 2$ symmetric traceless tensors have only two nonzero components $z, \ldots, z$ and $\bar{z}, \ldots, \bar{z}$, so

$$A_0(\alpha) = (n_1^z n_2^{\bar{z}})^l + c.c. = \cos(l\alpha) \; . \tag{4.3.17}$$

For $D = 3$, the answer is the Legendre polynomials $P_l(\cos\alpha)$

$$A_0(\alpha) = P_l(\cos\alpha) \; . \tag{4.3.18}$$

They are the same Legendre polynomials that appear in the wavefunction of the $m = 0$ spin l state. [**Exercise:** Explain why this is not accidental.] For $D = 4$ we have

$$A_0(\alpha) = \frac{1}{l+1}\frac{\sin[(l+1)\alpha]}{\sin\alpha} \ , \tag{4.3.19}$$

the so-called Chebyshev polynomials. Generalization for any D has the form:

$$A_0(\alpha) = C_l^{(D/2-1)}(\cos\alpha) \ , \tag{4.3.20}$$

where $C_l^{(D/2-1)}(\cos\alpha)$ are the Gegenbauer polynomials.

Here is what the functions $A_0(\alpha)$ look like for different D for $l = 4$. We normalize them so that $A_0(\alpha = 0) = 1$. Then it oscillates and comes back to 1 for $\alpha = \pi$. For odd l we would have $A_0(\alpha = \pi) = -1$.

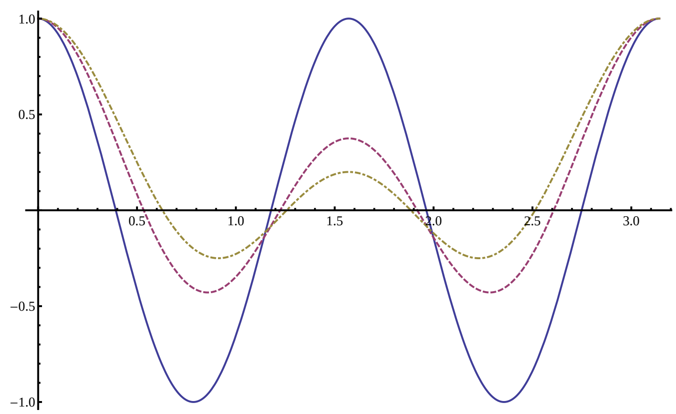


Figure 4.4: $A_0(\alpha)$ for $l = 4$ and $D = 2, 3, 4$ shown by a solid, dashed, and dot-dashed curve.

Note that given the small value of $r \approx 0.17$ for the configuration equivalent to $z = 1/2$ and to the square configuration, leading term in the expansion of the CB, $A_0(\alpha)r^\Delta$, constitutes a very good approximation near this point. The leading correction is $O(r^2) \sim$ few percent, since the term linear to the descendants will not appear if the operators are inserted symmetrically

$$\phi(x)\phi(-x) \sim \mathcal{O}(0) + cx^\mu x^\nu \partial_\mu\partial_\nu\mathcal{O}(0) + \ldots \tag{4.3.21}$$

4.3.3 An application

Now that we understood approximately the CB, let us look at the crossing symmetry equation. Consider the correlator of four identical scalar particles with dimension $\Delta_\phi = d$ (they can for example be the spin fields of the Ising model)

$$\langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{g(u,v)}{(x_{12})^{2d}(x_{34})^{2d}} \ , \tag{4.3.22}$$

with

$$g(u,v) = 1 + \sum_{\mathcal{O}} \lambda_{\mathcal{O}}^2 G_{\mathcal{O}}(u,v) \ , \tag{4.3.23}$$

where 1 is the CB of the unit operator, and the structure of the $G_{\mathcal{O}}(u,v)$ is

$$G_{\mathcal{O}}(u,v) = C_l(\cos\alpha) r^{\Delta}[1 + \mathcal{O}(r^2)] \ . \tag{4.3.24}$$

In order to use the above equation, we have to know the relation between u, v and α, r. As discussed above, this relation takes the form:

$$u = |z|^2 \ , v = |1-z|^2 \ \Rightarrow \rho = re^{i\alpha} = \frac{z}{(1+\sqrt{1-z})^2} \ . \tag{4.3.25}$$

Moreover, in Lecture 1 we saw that the crossing symmetry dictates

$$v^d g(u,v) = u^d g(v,u) \ , \text{ or } (v^d - u^d) + \sum_{\mathcal{O}} \lambda_{\mathcal{O}}^2 [v^d G_{\mathcal{O}}(u,v) - u^d G_{\mathcal{O}}(v,u)] = 0 \ . \tag{4.3.26}$$

This is what the bootstrap equation looks like in the case under consideration. It is simpler than in a generic case, because the same operators occur in both OPE channels (12)(34) and (14)(23). So to compare the channels we can take the difference and collect the OPE coefficients, which gives (4.3.26).

On the one hand, this looks like a non-trivial equation since it is not satisfied term by term, but only in the sum - all operators work together. On the other hand, there is a lot of freedom - we can move the spectrum and adjust the $\lambda_{\mathcal{O}}$s.

A natural first question is as follows: What are the spectra for which we can find $\lambda_{\mathcal{O}}$ such that the crossing is satisfied? Presumably, this will not happen for any spectrum. We will now show it rigorously.

Eq. (4.3.26) has to be satisfied for any u and v, but for this simple demonstration let us look at the points having $0 < z < 1$ real, so that also ρ is real ($\alpha = 0$). Then we get

$$[(1-z)^{2d} - z^{2d}] + \sum_{\mathcal{O}} \lambda_{\mathcal{O}}^2 \{(1-z)^{2d}[\rho(z)]^{\Delta} - z^{2d}[\rho(1-z)]^{\Delta}\} = 0 \ . \tag{4.3.27}$$

In this equation we replaced conformal blocks by their approximate expressions, which can be trusted near $z = 1/2$ where both $\rho(z), \rho(1-z) \simeq 0.17$ and the omitted terms are suppressed by 0.17^2. In what follows we will only work near $z = 1/2$.

Let us Taylor expand near $z = 1/2$. Only odd powers of $z - 1/2$ will appear since the functions are odd around this point. From the first term we get

$$[(1-z)^{2d} - z^{2d}] \simeq -C_d(x + \frac{4}{3}(d-1)(2d-1)x^3 + \mathcal{O}(x^5)) \ , \tag{4.3.28}$$

with $x = z - 1/2$ and $C_d > 0$ a positive constant.

Now, suppose for the sake of the argument that all operators have $\Delta \gg d$. We will now show that such an assumption is inconsistent. Expanding the conformal block terms in (4.3.27), we can neglect the variation of z^{2d} and $(1-z)^{2d}$ factors (since $\Delta \gg d$). We get:

$$[\rho(z)]^\Delta - [\rho(1-z)]^\Delta \simeq B_\Delta \left(x + \frac{4}{3}\Delta^2 x^3 + \dots \right) , \tag{4.3.29}$$

where $B_\Delta > 0$ is another positive constant. Let's change normalization of conformal blocks so that $B_\Delta = 1$ (effectively incorporating this constant into $\lambda^2_{\mathcal{O}}$). Now let's require that (4.3.27) be satisfied term by term in Taylor expansion around $z = 1/2$. We get:

$$\begin{aligned} O(x) &: -C_d + \sum_{\mathcal{O}} \lambda^2_{\mathcal{O}} = 0 \\ O(x^3) &: -C_d \frac{4}{3}(d-1)(2d-1) + \frac{4}{3} \sum_{\mathcal{O}} \lambda^2_{\mathcal{O}} \Delta^2 = 0 . \end{aligned} \tag{4.3.30}$$

The $O(x^5)$ terms etc would give more equations but we won't need them for now.

Now denoting $\min \Delta = \Delta_{min}$, it can be estimated as follows:

$$\Delta^2_{min} \underbrace{\sum_{\mathcal{O}} \lambda^2_{\mathcal{O}}}_{=C_d} \leqslant \sum_{\mathcal{O}} \lambda^2_{\mathcal{O}} \Delta^2 = (d-1)(2d-1)C_d , \tag{4.3.31}$$

where in the first term we used the $O(x)$ expansion equation, the inequality is a triviality, and the last equality is a consequence of the $O(x^3)$ expansion relation. So we conclude:

$$\Delta_{min} \leqslant \sqrt{(d-1)(2d-1)} \sim \mathcal{O}(d) , \tag{4.3.32}$$

which is a contradiction, since we started by assuming $\Delta_{min} \gg d$.

We arrive at the following conclusion: The crossing symmetry requires that the OPE contain operators of low dimension

$$\Delta_{min} \leqslant f(d) . \tag{4.3.33}$$

Here we are just scratching the surface. Clearly, the analysis can be improved:

1. Go out of the $z = 1/2$ real line, in order to be able to distinguish between operators of different spins.

2. Use exact expression for the CBs.

3. Expand to higher order in $x = z - 1/2$.

By doing that, one can find the bound in $D = 4$ dimensions shown in Fig. 4.5 (the original bound is due to Rattazzi, Rychkov, Tonni, Vichi, 2008; the shown plot is from a 2011 paper by Poland, Simmons-Duffin and Vichi)

At this point this course comes to an end. For further reading concerning applications of conformal bootstrap in $D \geqslant 3$ you will have to turn to the original papers listed below.

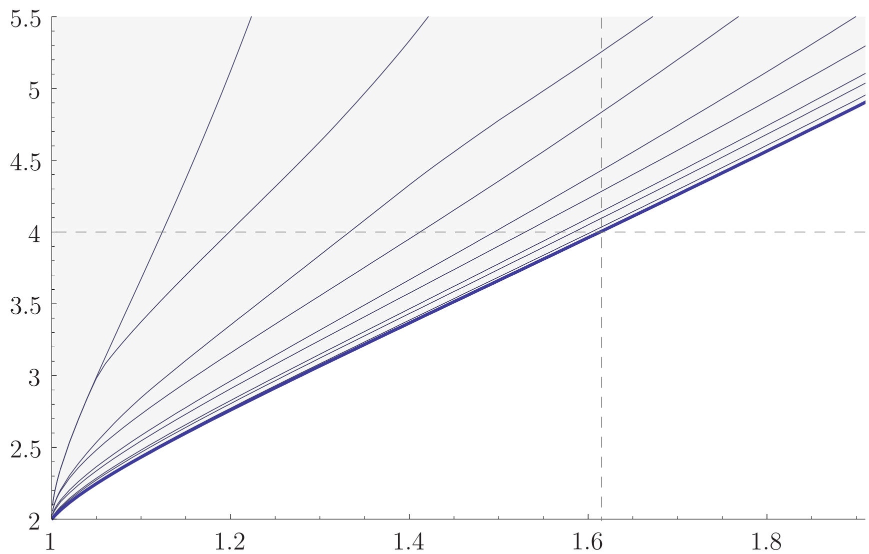


Figure 4.5: Horizontal axis: dimension of a scalar primary ϕ in a $D = 4$ CFT. Vertical axis: The upper bound on the dimension of the first scalar appearing in the OPE $\phi \times \phi$. Thus gray area is excluded. Various curves correspond to expanding to higher and higher order; the bound eventually converges.

Literature

Original papers on conformal bootstrap:

A. M. Polyakov, "Nonhamiltonian approach to conformal quantum field theory," Zh. Eksp. Teor. Fiz. **66**, 23 (1974). (there is an English translation)

S. Ferrara, A. F. Grillo and R. Gatto, "Tensor representations of conformal algebra and conformally covariant operator product expansion," Annals Phys. **76**, 161 (1973).

G. Mack, "Duality in Quantum Field Theory," Nucl. Phys. B **118**, 445 (1977).

Conformal bootstrap in 2D:

A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, "Infinite Conformal Symmetry in Two-Dimensional Quantum Field Theory," Nucl. Phys. B **241**, 333 (1984).

Unitarity bounds in 2D: see the "yellow book" by Di Francesco et al

Conformal blocks in $D \geqslant 3$:

F. A. Dolan and H. Osborn, "Conformal four point functions and the operator product expansion," Nucl. Phys. B **599**, 459 (2001) [hep-th/0011040].

F. A. Dolan and H. Osborn, "Conformal partial waves and the operator product expansion," Nucl. Phys. B **678**, 491 (2004) [hep-th/0309180].

M. S. Costa, J. Penedones, D. Poland and S. Rychkov, "Spinning Conformal Blocks," JHEP **1111**, 154 (2011) [arXiv:1109.6321 [hep-th]].

Bounds from conformal bootstrap in $D = 4$:

R. Rattazzi, V. S. Rychkov, E. Tonni and A. Vichi, "Bounding scalar operator dimensions in 4D CFT," JHEP **0812**, 031 (2008) [arXiv:0807.0004 [hep-th]].

D. Poland, D. Simmons-Duffin and A. Vichi, "Carving Out the Space of 4D CFTs," JHEP **1205**, 110 (2012) [arXiv:1109.5176 [hep-th]].

and papers in between.

Further reading: on applications of conformal bootstrap to 3D Ising model:

S. El-Showk, M. F. Paulos, D. Poland, S. Rychkov, D. Simmons-Duffin and A. Vichi, "Solving the 3D Ising Model with the Conformal Bootstrap," Phys. Rev. D **86**, 025022 (2012) [arXiv:1203.6064 [hep-th]].