Bounding scalar operator dimensions in 4D CFT

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Abstract

In an arbitrary unitary 4D CFT we consider a scalar operator ϕ, and the operator ϕ^2 defined as the lowest dimension scalar which appears in the OPE $\phi \times \phi$ with a nonzero coefficient. Using general considerations of OPE, conformal block decomposition, and crossing symmetry, we derive a theory-independent inequality $[\phi^2] \leq f([\phi])$ for the dimensions of these two operators. The function $f(d)$ entering this bound is computed numerically. For $d \to 1$ we have $f(d) = 2 + O(\sqrt{d-1})$, which shows that the free theory limit is approached continuously. We perform some checks of our bound. We find that the bound is satisfied by all weakly coupled 4D conformal fixed points that we are able to construct. The Wilson-Fischer fixed points violate the bound by a constant $O(1)$ factor, which must be due to the subtleties of extrapolating to $4-\varepsilon$ dimensions. We use our method to derive an analogous bound in 2D, and check that the Minimal Models satisfy the bound, with the Ising model nearly-saturating it. Derivation of an analogous bound in 3D is currently not feasible because the explicit conformal blocks are not known in odd dimensions. We also discuss the main phenomenological motivation for studying this set of questions: constructing models of dynamical ElectroWeak Symmetry Breaking without flavor problems.

Contents

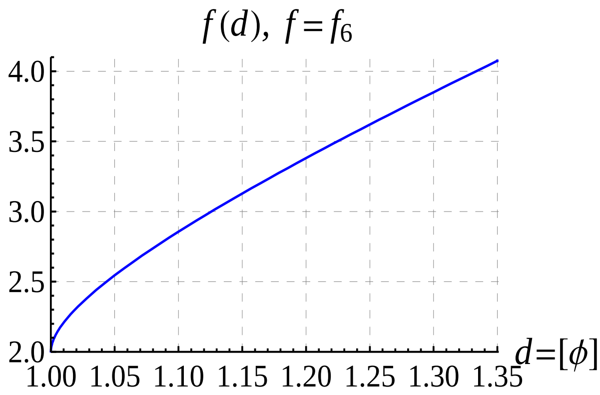


Figure 1: The best current bound (1.4), obtained by the method described in Section 5. The subscript in f_6 refers to the order of derivatives used to compute this bound.

1 The problem and the result

Operator dimensions in unitary Conformal Field Theories (CFT) are subject to important constraints known as unitarity bounds. In the simplest case of a scalar primary operator ϕ, the unitarity bound states that[1]

$$d \equiv [\phi] \geq 1, \tag{1.1}$$

$$d = 1 \Longleftrightarrow \phi \text{ is free.} \tag{1.2}$$

This classic result invites the following question: What happens if $d = 1+\varepsilon$? In particular, is there any sense in which the CFT (or at least its subsector not decoupled from ϕ) should be close to the free scalar theory if d is close to 1? For instance, do all operator dimensions in this subsector approach their free scalar theory values in the limit $d \to 1$? The standard proof of the unitarity bound [1] does not shed light on this question.

In this paper we will show that such continuity indeed holds for the operator 'ϕ^2', by which we mean the lowest dimension *scalar* primary which appears in the OPE of ϕ with itself:

$$\phi(x)\phi(0) \sim (x^2)^{-d}(1 + C|x|^{\Delta_{\min}}\phi^2(0) + \dots)\,, \quad C \neq 0\,. \tag{1.3}$$

In free theory $\Delta_{\min} \equiv [\phi^2] = 2$, and we will show that $\Delta_{\min} \to 2$ in any CFT as $d \to 1$. More precisely, we will show that in any 4D CFT

$$\Delta_{\min} \leq f(d), \tag{1.4}$$

where $f(d)$ is a certain continuous function such that $f(1) = 2$. We will evaluate this function numerically; it is plotted in Fig. 1 for d near 1.

We stress that bound (1.4) applies to the OPE $\phi \times \phi$ of an arbitrary scalar primary ϕ. However, since the function $f(d)$ is monotonically increasing, the bound is strongest for the scalar primary of minimal dimension.

[1] Unless explicitly noted otherwise, all statements of this paper refer to $D = 4$ spacetime dimensions.

Our analysis will use only the most general properties of CFT, such as unitarity, OPE, conformal block decomposition, and crossing symmetry. The resulting bound (1.4) is thus model independent. In particular, it holds independently of the central charge, of the spectrum of whatever other operators which may be present in the CFT, and of the coefficients with which these operators may appear in the OPE (1.3). Our analysis is non-perturbative and does not assume that the CFT may be continuously connected to the free theory.

We do not know of any 4D CFT which comes close to saturating the bound of Fig. 1. We are not claiming that this bound is the best possible, and in fact we do know that it can be somewhat improved using our method and investing some more computer time. In spite of not being the best possible, the curve of Fig. 1 is a valid bound, and represents a necessary condition which should be satisfied in any unitary CFT.

The paper is organized as follows. In Section 2 we explain the phenomenological motivations behind this question, which are related to the naturalness problem of the electroweak scale. In Section 3 we review the necessary CFT techniques. In Section 4 we derive a sum rule for the contributions of all primary fields (with arbitrary spins and dimensions) appearing in the $\phi \times \phi$ OPE. In Section 5 we explain how the sum rule is used to derive the bound (1.4). In Section 6 we check our bound against operator dimensions in various calculable CFTs in $D = 4$ and $4 - \varepsilon$. We also present and similarly check an analogous bound in $D = 2$. In Section 7 we discuss to what extent our result in its current form addresses the phenomenological problem from Section 2. In Section 8 we conclude and indicate future research directions.

2 Phenomenological motivation

The phenomenological motivation of our study is given by one declination of the hierarchy problem, which was lucidly discussed in a paper by Luty and Okui [2] (see also [3]). This section is to a significant extent a review of the discussion in that paper. The bulk of the paper is logically independent of this section, and the reader who is mainly interested in the formal aspects of our result may skip to Section 3.

The issue of mass hierarchies in field theory can be conveniently depicted from a CFT viewpoint. Indeed the basic statement that a given field theory contains two widely separated mass scales $\Lambda_{\rm IR} \ll \Lambda_{\rm UV}$ already implies that the energy dependence of physical quantities at $\Lambda_{\rm IR} \ll E \ll \Lambda_{\rm UV}$ is *small*, corresponding to approximate scale (and conformal) invariance. In the case of perturbative field theories the CFT which approximates the behaviour in the intermediate mass region is just a free one. For instance, in the case of non-SUSY GUT's, $\Lambda_{\rm IR}$ and $\Lambda_{\rm UV}$ are respectively the Fermi and GUT scale, and the CFT which approximates behaviour at intermediate scales is just the free Standard Model. From the CFT viewpoint, the naturalness of the hierarchy $\Lambda_{\rm IR} \ll \Lambda_{\rm UV}$, or equivalently its stability, depends on the dimensionality of the scalar operators describing the perturbations of the CFT Lagrangian around the fixed point. In the language of the RG group, naturalness depends on the relevance of the deformations at the fixed point. If the theory possesses a scalar operator $\mathcal{O}_\Delta$, with dimension $\Delta < 4$, one generically expects[2] UV

[2] See concrete examples in Section 2.1 below.

physics to generate a perturbation

$$\mathcal{L}_{pert} = c\Lambda_{\rm UV}^{4-\Delta}\mathcal{O}_\Delta\,, \tag{2.1}$$

corresponding roughly to an IR scale

$$\Lambda_{\rm IR} = c^{\frac{1}{4-\Delta}}\Lambda_{\rm UV}\,. \tag{2.2}$$

Absence of tuning corresponds to the expectation that c be not much smaller than $O(1)$. If $4-\Delta$ is $O(1)$ (*strongly relevant* operator) a hierarchy between $\Lambda_{\rm IR}$ and $\Lambda_{\rm UV}$ can be maintained only by tuning c to be *hierarchically* smaller than one. This corresponds to an unnatural hierarchy. On the other hand when $4-\Delta$ is close to zero (*weakly relevant* operator) a mass hierarchy is obtained as soon as both $4-\Delta$ and c are just *algebraically* small[3]. For instance for $4-\Delta = c = 0.1$ the mass hierarchy spans 10 orders of magnitude. Therefore for a weakly relevant operator a hierarchy is considered natural. The hierarchy between the confinement and UV scale in Yang-Mills theory is an example in this second class, albeit a limiting one[4]. The only exception to the above classification of naturalness concerns the case in which the strongly relevant operators transform under some global approximate symmetry. In that case it is natural to assume that the corresponding c's be small, even hierarchically small. The stability of the hierarchy depends then on the dimension Δ_S of the scalar singlet (under all global symmetries) of lowest dimension. If $4-\Delta_S \ll 1$ the hierarchy is natural.

According to the above discussion, in the SM the hierarchy between the weak scale and any possible UV scale is unnatural because of the presence of a scalar bilinear in the Higgs field $H^\dagger H$ which is a total singlet with dimension ~ 2. On the other hand in supersymmetric extensions of the SM, such scalar bilinears exist but their coefficient can be naturally chosen to be small. In a general supersymmetric model the weak scale is then naturally generated either by a marginally relevant deformation (dynamical supersymmetry breaking) or simply by adding strongly relevant supersymmetry breaking deformations with small coefficients (soft supersymmetry breaking). Technicolor models are instead similar to the case of YM theory: at the gaussian fixed point there are no gauge invariant scalars of dimension < 4.

As far as the hierarchy is concerned these extensions are clearly preferable to the SM. However as far as flavor physics is concerned the SM has, over its extensions, an advantage which is also a simple consequence of operator dimensionality. In the SM the flavor violating operators of lowest dimensionality, the Yukawa interactions, have dimension $= 4$,

$$\mathcal{L}_Y = y^u_{ij} H\bar{q}_L u_R + y^d_{ij}H^\dagger \bar{q}_L d_R + y^e_{ij}H^\dagger \bar{L}_L e_R \quad ({\rm SM})\,, \tag{2.3}$$

and provide a very accurate description of flavor violating phenomenology. In particular, the common Yukawa origin of masses and mixing angles leads to a critically important suppression of Flavor Changing Neutral Currents (FCNC) and CP violation. This suppression is often called Natural Flavor Conservation or GIM mechanism [4]. Once the hierarchy $v \ll \Lambda_{\rm UV}$ is taken as a fact, no matter how unnatural, extra unwanted sources of flavor violation are automatically suppressed. In particular the leading effects are associated to 4-quark interactions, with dimension

[3] We stole this definition from ref. [3].

[4] This is because the corresponding deformation, the glueball field $G^A_{\mu\nu}G_A^{\mu\nu}$, is marginally relevant: its scaling dimension is $4-ag^2$ and becomes exactly 4 at the gaussian fixed point.

6, and are thus suppressed by $v^2/\Lambda_{\rm UV}^2$. The situation is not as good in supersymmetry, where in addition to the Yukawa interactions flavor is violated by operators of dimension 2 and 3 involving the sfermions. The comparison with technicolor brings us to discuss the motivation for our paper. In technicolor the Higgs field is a techni-fermion bilinear $H = \bar{T}T$ with dimension ~ 3. The SM fermions instead remain elementary, i.e. with dimension 3/2. The Yukawa interactions are therefore irrelevant operators of dimension 6,

$$\mathcal{L}_Y = \frac{y_{ij}}{\Lambda_{\rm F}^2} H\bar{q}q \quad ({\rm TC}), \tag{2.4}$$

and are associated to some new dynamics [5], the flavor dynamics, at a scale $\Lambda_{\rm F}$, which plays the role of our $\Lambda_{\rm UV}$. Very much like in the SM, and as it is found in explicit models [5], we also expect unwanted 4-quark interactions

$$\frac{c_{ijkl}}{\Lambda_{\rm F}^2}\bar{q}_i q_j \bar{q}_k q_l \tag{2.5}$$

suppressed by the same flavor scale. Unlike in the SM, in technicolor the Yukawa interactions are *not* the single most relevant interaction violating flavor. This leads to a tension. On one hand, in order to obtain the right quark masses, $\Lambda_{\rm F}$ should be rather low. On the other hand, the bound from FCNC requires $\Lambda_{\rm F}$ to be generically larger. For instance the top Yukawa implies $\Lambda_{\rm F} \lesssim 10\,{\rm TeV}$. On the other hand the bound from FCNC on operators like Eq. (2.5) is rather strong. Assuming $c_{ijkl} \sim 1$, flavor mixing in the neutral kaon system puts a generic bound ranging from $\Lambda_{\rm F} > 10^3\,{\rm TeV}$, assuming CP conservation and left-left current structure, to $\Lambda_{\rm F} > 10^5\,{\rm TeV}$, with CP violation with left-right current. Of course assuming that c_{ijkl} have a nontrivial structure controlled by flavor breaking selection rules one could in principle obtain a realistic situation. It is however undeniable that the way the SM disposes of extra unwanted sources of flavor violation is more robust and thus preferable. The origin of the problem is the *large* dimension of the Higgs doublet field H. Models of walking technicolor (WTC) [6] partially alleviate it. In WTC, above the weak scale the theory is assumed to be near a non-trivial fixed point, where $H = \bar{T}T$ has a sizeable negative anomalous dimension. WTC is an extremely clever idea, but progress in its realization has been slowed down by the difficulty in dealing with strongly coupled gauge theories in 4D. Most of our understanding of WTC relies on gap equations, a truncation of the Schwinger-Dyson equations for the $\bar{T}T$ self-energy. Although gap equations do not represent a fully defendable approximation, they have produced some interesting results. In case of asymptotically free gauge theory they lead to the result that $H = \bar{T}T$ can have dimension 2 at the quasi-fixed point, but not lower [7]. In this case the Yukawa interactions would correspond to dimension 5 operators, which are more relevant than the unwanted dimension 6 operators in Eq. (2.5). However some tension still remains: the top Yukawa still requires a Flavor scale below the bound from the Kaon system, so that the absence of flavor violation, in our definition, is not robust. It is quite possible that the bound $[H] \geq 2$ obtained with the use of gap equation will not be true in general. Of course the closer $[H]$ is to 1, the higher the flavor scale we can tolerate to reproduce fermion masses, and the more suppressed is the effect of Eq. (2.5). However if $[H]$ gets too close to 1 we get back the SM and the hierarchy problem! More formally, a scalar field of dimension exactly 1 in CFT is a free field and the dimension of its composite $H^\dagger H$ is trivially determined to be 2, that is strongly relevant. By continuity we therefore expect that the hierarchy problem strikes back at some point as $[H]$ approaches 1. However the interesting remark made by Luty and Okui [2] is that, after all, we do not really need $[H]$ extremely close to 1. For instance $[H] = 1.3$ would already be good, in which case the corresponding CFT is not weakly coupled and it could well be that $[H^\dagger H]$ is significantly bigger than $2[H]$ and maybe even close to 4. The motivation of our present work is precisely to find, from prime principles, what is the upper bound on $\Delta_S = [H^\dagger H]$ as $d = [H]$ approaches 1. In simple words this may be phrased as the question: how fast do CFTs, or better a subsector of CFTs, become free as the dimension of a scalar approaches 1?

In the following subsection we would like to make a more quantitative analysis of the tension between flavor and electroweak hierarchy in a scenario where the electroweak symmetry breaking sector sits near a fixed point between the EW scale $\Lambda_{\rm IR}$ and some $\Lambda_{\rm UV}$ at which, or below which, Flavor dynamics must take place.

2.1 Quantitative analysis

Let us normalize fields and couplings in the spirit of Naive Dimensional Analysis (NDA). The Lagrangian will thus be written as

$$\mathcal{L} = \frac{1}{16\pi^2} F(\phi, \lambda, M)\,, \tag{2.6}$$

where fields, couplings and physical mass scales are indicated collectively. With this normalization, Green's functions in the coordinate representation have no factors of π, the couplings λ are loop counting parameters, and the mass scales M correspond to physical masses (as opposed to decay constants).

Our hypothesis is that below some UV scale $\Lambda_{\rm UV}$ the theory splits into the elementary SM without Higgs and a strongly coupled CFT which contains a (composite) Higgs scalar doublet operator H. These two sectors are coupled to each other via weak gauging and the Yukawa interactions. As a warmup exercise consider then the top Yukawa

$$\frac{1}{16\pi^2}\lambda_t H \bar{Q}_L t_R + \text{h.c.} \tag{2.7}$$

and its lowest order correction to the CFT action

$$\begin{aligned}
\Delta\mathcal{L} &= \left(\frac{1}{16\pi^2}\right)^2 \lambda_t^2 \int d^4x d^4y\, H(x)^\dagger H(y)\, \bar{Q}_L t_R(x)\, \bar{t}_R Q_L(y) \\
&\sim \left(\frac{1}{16\pi^2}\right)^2 \lambda_t^2 \int d^4x d^4y\, (H^\dagger H)(x) |x-y|^{\Delta_S - 2d} |x-y|^{-6} \\
&\sim \frac{1}{16\pi^2} \int d^4x\, \lambda_t^2 \Lambda_{\rm UV}^{2+2d-\Delta_S} (H^\dagger H)(x) \\
&\equiv \frac{1}{16\pi^2} \int d^4x\, [\bar{\lambda}_t(\Lambda_{\rm UV})]^2 \Lambda_{\rm UV}^{4-\Delta_S} (H^\dagger H)
\end{aligned} \tag{2.8}$$

where we used the $H^\dagger \times H$ OPE[5] and cut off the d^4y integral at a UV distance $\Lambda_{\rm UV}^{-1}$ (ex.: there exist

[5]Notice that $H^\dagger H$ is *defined* as the scalar $SU(2)$ singlet operator of lowest dimension in the $H^\dagger \times H$ OPE. Weak $SU(2)$ is assumed to be a global symmetry of the CFT. The $SU(2)$ invariance of the fermion propagators realizes the projection on the singlet.

new states with mass $\Lambda_{\rm UV}$). The quantity $\bar\lambda_t(\Lambda_{\rm UV}) = \lambda_t \Lambda_{\rm UV}^{d-1}$ represents the dimensionless running coupling evaluated at the scale $\Lambda_{\rm UV}$. Given our NDA normalization, $\bar\lambda_t^2$ is the loop counting parameter (no extra π's). For $\Delta_S < 4$ Eq. (2.8) represents a relevant deformation of the CFT. Allowing for a fine tuning ϵ_t between our naive estimate of the top loop and the true result, the deformation lagrangian is

$$\mathcal{L}_\Delta = \frac{1}{16\pi^2}\bar\lambda_t(\Lambda_{\rm UV})^2 \epsilon_t \Lambda_{\rm UV}^{4-\Delta_S} H^\dagger H\,, \tag{2.9}$$

corresponding to a physical infrared scale

$$\Lambda_{\rm IR} = [\bar\lambda_t(\Lambda_{\rm UV})^2\epsilon_t]^{\frac{1}{4-\Delta_S}}\Lambda_{\rm UV} \equiv (c_t)^{\frac{1}{4-\Delta_S}}\Lambda_{\rm UV}\,, \tag{2.10}$$

where we made contact with our previous definition of the coefficient c. If the above were the dominant contribution, then a hierarchy would arise for $4-\Delta_S < 1$ provided $\bar\lambda_t(\Lambda_{\rm UV}) < 1$ and/or a mild tuning $\epsilon_t < 1$ exists. However, unlike the normal situation where the Higgs is weakly self-coupled and the top effects dominate, in our scenario the Higgs is strongly self-coupled. Therefore we expect a leading contribution to $\mathcal{L}_\Delta$ to already be present in the CFT independently of the top:

$$\mathcal{L}_\Delta = \frac{c}{16\pi^2}\Lambda_{\rm UV}^{4-\Delta_S} H^\dagger H\,. \tag{2.11}$$

The presence of such an effect basically accounts for the fact that a CFT with a relevant deformation does not even flow to the fixed point unless the deformation parameter is tuned. This is in line with our initial discussion. So we shall work under the assumption that c is somewhat less than 1.

We can describe the generation of the electroweak scale by writing the effective potential for the composite operator vacuum expectation value $\langle (H^\dagger H)\rangle = \mu^{\Delta_S}$. Compatibly with scale invariance there will also be a term

$$V_{\rm CFT} = \frac{a}{16\pi^2}\mu^4 \tag{2.12}$$

where a is a numerical coefficient that depends on the CFT and on the direction of the VEV in operator space. The full effective potential has then the form

$$V_{\rm eff} = \frac{1}{16\pi^2}\left[-\Lambda_{\rm IR}^{4-\Delta_S}\mu^{\Delta_S} + \mu^4\right]\,, \tag{2.13}$$

which is stationary at $\mu \sim \Lambda_{\rm IR}$. Here we put for example $a=1$ and chose a negative sign for the scale breaking contribution. Notice that the vacuum dynamics picture that we just illustrated is analogous to the Randall-Sundrum model [8] with Goldberger-Wise radius stabilization [9] with the identification of $1/\mu$ with the position of the IR brane in conformal coordinates. This fact should not be surprising at all given the equivalence of that model to a deformed CFT [10].

Our analysis of the top sector is however useful to discuss the two basic constraints on this scenario.

The first constraint is the request that $\Lambda_{\rm UV}$ be below the scale where the top Yukawa becomes strong, at which point the SM becomes strongly coupled to the CFT and our picture breaks down. The running top coupling in its standard normalization is $y_t(E) = 4\pi\bar\lambda(E)$. Using the known experimental result $y_t(\Lambda_{\rm IR}) \sim 1$ we thus have

$$y_t(\Lambda_{\rm UV}) = y_t(\Lambda_{\rm IR}) \left(\frac{\Lambda_{\rm UV}}{\Lambda_{\rm IR}}\right)^{d-1} \sim (c)^{-\frac{d-1}{4-\Delta_S}} . \tag{2.14}$$

Perturbativity corresponds to $y_t(\Lambda) \lesssim 4\pi$, that is

$$\frac{d-1}{4-\Delta_S} \ln(1/c) \lesssim \ln(4\pi) \sim \ln 10 . \tag{2.15}$$

Clearly, this bound is better satisfied the closer d is to 1.

A second constraint is presented by the request of robust decoupling of unwanted flavor breaking effects. Assuming any generic interaction among SM states is present at the scale $\Lambda_{\rm UV}$ with strength comparable to $\bar\lambda_t^2(\Lambda)$, we can parameterize flavor violation by

$$\mathcal{L}_{\rm fermion} = \frac{1}{16\pi^2}\left[\bar q \not{D} q + \frac{\bar\lambda_t^2(\Lambda_{\rm UV})}{\Lambda_{\rm UV}^2}(\bar q q)^2\right] , \tag{2.16}$$

which by going to canonical normalization becomes

$$\mathcal{L}_{\rm fermion} = \bar q \not{D} q + \frac{1}{\Lambda_{\rm F}^2}(\bar q q)^2 \tag{2.17}$$

with

$$\Lambda_{\rm F} = \Lambda_{\rm IR}\left(\frac{\Lambda_{\rm UV}}{\Lambda_{\rm IR}}\right)^{2-d} = \Lambda_{\rm IR}(c)^{-\frac{2-d}{4-\Delta_S}} . \tag{2.18}$$

By taking $\Lambda_{\rm IR} = 1\,{\rm TeV}$, the bound from FCNC can be parametrized as $\Lambda_{\rm F} > 10^F\,{\rm TeV}$. Making the conservative assumption that all quark families appear in Eq. (2.16), compatibility with the data requires $\Lambda_{\rm F} \geq 10^3 \div 10^4$ TeV.[6] Thus robust suppression of FCNC corresponds to

$$F > 3 \div 4. \quad ({\rm robust}) . \tag{2.19}$$

On the other hand if only the third family appears in Eq. (2.16), the mixing effects involving the lighter generations are generally suppressed by extra powers of the CKM angles. In that case the bound on $\Lambda_{\rm F}$ is weaker, and $F > 0.5$ is basically enough. In the latter case the detailed structure of the Flavor theory matters. Notice that for conventional walking technicolor models, for which $d \geq 2$, we always have $\Lambda_{\rm F} \leq \Lambda_{\rm IR}$ so that even the weaker bound is somewhat problematic. We are however interested to see to what extent we can neglect this issue by focussing on the robust bound (2.19). From Eq. (2.18) we must have

$$\frac{2-d}{4-\Delta_S} \ln(1/c) \geq F \ln 10 . \tag{2.20}$$

Eqs. (2.15,2.20) together imply

$$\frac{F}{2-d} < \frac{\ln(1/c)}{(4-\Delta_S)\ln 10} < \frac{1}{d-1} \qquad \Longrightarrow \qquad d < 1 + \frac{1}{1+F} , \tag{2.21}$$

[6]We consider the limit $\Lambda_{\rm F} = 10^5$ TeV really an overkill.

which for the robust bound (2.19) requires $d < 1.25$. At the same time the amount of tuning needed to generate the hierarchy between $\Lambda_{\rm IR}$ and the flavor scale $\Lambda_{\rm UV}$ is

$$c = 10^{(4-\Delta_S)F/(d-2)}\,.$$

A reasonable request $c > 0.1$ then reads

$$4 - \Delta_S < \frac{2-d}{F}\,. \tag{2.22}$$

The ultimate goal of our study is thus to find a prime principle upper bound on Δ_S as a function of d. This bound should provide an extra important constraint which together with eqs. (2.21,2.22) may or may not be satisfied. Our main result (1.4) is a step towards this goal, although is not yet a complete solution. The point is that the lowest dimension scalar in the $\phi \times \phi$ OPE, whose dimension Δ appears in (1.4), is not necessarily a singlet. Nonetheless we think our result already represents some interesting piece of information. We postpone a detailed discussion of this connection until Section 7.

3 Necessary CFT techniques

To make the paper self-contained, in this section we will review a few standard CFT concepts and results, concentrating on those which are crucial for understanding our result and its derivation. Our personal preferred list of CFT literature includes [11],[12],[13],[14],[15],[16]. We will mostly work in the Euclidean signature.

3.1 Primary fields and unitarity bounds

In perturbative field theories, classification of local operators is straightforward: we have a certain number of fundamental fields, from which the rest of the operators are obtained by applying derivatives and multiplication. In CFTs, a similar role is played by the *primary fields.* These local operators $\mathcal{O}(x)$ are characterized by the fact that they are annihilated by the Special Conformal Transformation generator K_μ (at $x = 0$). Thus a primary field $\mathcal{O}(x)$ transforms under the little group≡the subgroup of conformal transformations leaving $x = 0$ invariant (this includes Lorentz transformations $M_{\mu\nu}$, dilatations D, and Special Conformal Transformations K_μ) as follows:

$$\begin{aligned} [M_{\mu\nu}, \mathcal{O}(0)] &= \Sigma_{\mu\nu}\mathcal{O}(0)\,, \\ [D, \mathcal{O}(0)] &= i\Delta\,\mathcal{O}(0)\,, \\ [K_\mu, \mathcal{O}(0)] &= 0\,. \end{aligned} \tag{3.1}$$

Here we assume that $\mathcal{O}$ has well-defined quantum numbers: the scaling dimension Δ, and the Lorentz[7] spin $(j, \tilde{j})$ (the matrices $\Sigma_{\mu\nu}$ are the corresponding generators).

Once all the primary operators are known, the rest of the field content is obtained by applying derivatives; the fields obtained in this way are called descendants. The multiplication operation

[7]Or Euclidean rotation, if one is working in the Euclidean.

used to generate composite operators in perturbative field theories has a CFT analogue in the concept of the OPE, which will be discussed in Section 3.3 below. To avoid any possible confusion, we add that this picture applies equally well also to conformal gauge theories, *e.g.* to $\mathcal{N}=4$ Super Yang-Mills, provided that only physical, gauge invariant fields are counted as operators of the theory.

Knowing (3.1), one can determine the transformation rules at any other point x using the conformal algebra commutation relations [11]. In principle, one could also imagine representations where K_μ acts as a nilpotent matrix (type Ib in [11]) rather than zero as in (3.1). However, as proven in [1], only representations of the form (3.1) occur in unitary CFTs. Moreover, unitarity implies important lower bounds on the operator dimensions. We are mostly interested in symmetric traceless fields $\mathcal{O}_{(\mu)}$, $(\mu) \equiv \mu_1 \ldots \mu_l$, which correspond to $j = \tilde{j}$ tensors:

$$\mathcal{O}_{\mu_1 \ldots \mu_l} \equiv \sigma^{\mu_1}_{\alpha_1 \dot{\alpha}_1} \ldots \sigma^{\mu_l}_{\alpha_l \dot{\alpha}_l} \phi^{\alpha_1 \ldots \alpha_l \dot{\alpha}_1 \ldots \dot{\alpha}_l} . \tag{3.2}$$

This is traceless in any pair of μ indices because $\sigma^\mu_{\alpha\dot{\alpha}} \sigma^\mu_{\beta\dot{\beta}} \propto \varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}}$. For such primaries the *unitarity bound* reads [1]:

$$\begin{aligned} &l = 0\colon\ \Delta \geq 1, && \Delta = 1 \text{ only for a free scalar;} \\ &l \geq 1\colon\ \Delta \geq l+2, && \Delta = l+2 \text{ only for a conserved current.} \end{aligned} \tag{3.3}$$

Notice a relative jump of one unit when one passes from $l = 0$ to $l \geq 1$. In particular, a conserved spin-1 current has $\Delta = 3$, while the energy-momentum tensor has $\Delta = 4$. The full list of unitarity bounds, which includes also fields with $j \neq \tilde{j}$, can be found in [1]. Recently [17], some of these bounds were rederived in a very physically transparent way, by weakly coupling a free scalar theory to the CFT and studying the unitarity of the S-matrix generated by exchanges of CFT operators.

3.2 Correlation functions

As is well known, conformal symmetry fixes the coordinate dependence of 2- and 3-point functions of primary fields. For example, for scalar primaries we have:

$$\langle \phi(x)\phi(y) \rangle = \frac{1}{|x-y|^{2\Delta_\phi}} , \tag{3.4}$$

$$\langle \phi(x)\tilde{\phi}(y) \rangle = 0 \qquad (\phi \neq \tilde{\phi}) . \tag{3.5}$$

As it is customary, we normalize ϕ to have a unit coefficient in the RHS of (3.4). Correlators of two fields with unequal dimensions vanish by conformal symmetry. Even if several primaries of the same dimension exist, by properly choosing the basis we can make sure that the nondiagonal correlators (3.5) vanish. Notice that we are working with real fields, corresponding to hermitean operators in the Minkowski space description of the theory.

The 3-point functions are also fixed by conformal symmetry:

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3) \rangle = \frac{\lambda_{123}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3} |x_{23}|^{\Delta_2+\Delta_3-\Delta_1} |x_{13}|^{\Delta_1+\Delta_3-\Delta_2}} ,$$
$$x_{12} \equiv x_1 - x_2 \text{ etc.}$$

The constants λ_{123}, which become unambiguously defined once we normalize the fields via the 2-point functions, are an important characteristic of CFT dynamics. These constants appear as the OPE coefficients (see below), and if they are all known, any n-point function can be reconstructed via the OPE. Thus in a sense finding these constants, together with the spectrum of operator dimensions, is equivalent to solving, or constructing, the theory.

Also the correlator of two scalars and a spin l primary $\mathcal{O}_{(\mu)}$ is fixed up to a constant [14]:

$$\begin{aligned}\langle\phi_1(x_1)\phi_2(x_2)\mathcal{O}_{(\mu)}(x_3)\rangle &= \frac{\lambda_{12\mathcal{O}}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_{\mathcal{O}}+l}|x_{23}|^{\Delta_2+\Delta_{\mathcal{O}}-\Delta_1-l}|x_{13}|^{\Delta_1+\Delta_{\mathcal{O}}-\Delta_2-l}} Z_{\mu_1}\ldots Z_{\mu_l}\,, \qquad (3.6)\\ Z_\mu &= \frac{x_{13}^\mu}{x_{13}^2}-\frac{x_{23}^\mu}{x_{23}^2}\,.\end{aligned}$$

The OPE coefficients $\lambda_{12\mathcal{O}}$ are real, once a real field basis (3.4) (and similarly for higher spin primaries) is chosen. This reality condition follows from the reality of Minkowski-space correlators of hermitean operators at spacelike separation, see Appendix A for a more detailed discussion.

When it comes to 4-point functions, conformal symmetry is no longer sufficient to fix the coordinate dependence completely. In the case of 4 scalar operators, the most general conformally-symmetric expression is

$$\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle = \left(\frac{|x_{24}|}{|x_{14}|}\right)^{\Delta_1-\Delta_2}\left(\frac{|x_{14}|}{|x_{13}|}\right)^{\Delta_3-\Delta_4}\frac{g(u,v)}{|x_{12}|^{\Delta_1+\Delta_2}|x_{34}|^{\Delta_3+\Delta_4}}, \qquad (3.7)$$

where $g(u,v)$ is an arbitrary function of the conformally-invariant cross-ratios:

$$u=\frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2},\qquad v=\frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2}. \qquad (3.8)$$

3.3 Operator Product Expansion

A very powerful property of CFT is the Operator Product Expansion (OPE), which represents a product of two primary operators at finite separation as a sum of local primaries:

$$\begin{aligned}\phi_1(x)\phi_2(0) &= \sum_{\mathcal{O}}\lambda_{12\mathcal{O}}[C_{(\mu)}(x)\mathcal{O}_{(\mu)}(0)+\ldots]\,, \qquad (3.9)\\ C_{(\mu)}(x) &= \frac{1}{|x|^{\Delta_1+\Delta_2-\Delta_{\mathcal{O}}}}\frac{x^{\mu_1}\cdots x^{\mu_l}}{|x|^l}\,.\end{aligned}$$

Here we wrote an OPE appropriate for a pair of scalars $\phi_1\times\phi_2$.

The $\ldots$ in (3.9) stands for an infinite number of terms, less singular in the $x\to 0$ limit, involving the derivatives of the primary $\mathcal{O}_{(\mu)}$ (i.e. its descendants). The coordinate dependence of the coefficients of these descendants is in fact completely fixed by the conformal symmetry, so that $\lambda_{12\mathcal{O}}$ appears as an overall coefficient for the full contribution of $\mathcal{O}_{(\mu)}$ and its descendants. We can write schematically:

$$\phi_1\times\phi_2 = \sum_{\mathcal{O}}\lambda_{12\mathcal{O}} \succ\!\!= \mathcal{O}\,,$$

denoting by $\mathcal{O}$ the contribution of the whole conformal family.

For example, for a scalar operator $\mathcal{O}$ appearing in the OPE $\phi \times \phi$ the first few subleading terms are ([14], p. 125, $\Delta = \Delta_{\mathcal{O}}$)

$$\phi(x)\phi(0) \sim \frac{\lambda_{12\mathcal{O}}}{|x|^{2\Delta_\phi - \Delta}}[1 + \frac{1}{2}x_\mu\partial_\mu + \frac{1}{8}\frac{\Delta+2}{\Delta+1}x_\mu x_\nu \partial_\mu\partial_\nu - \frac{1}{16}\frac{\Delta}{\Delta^2-1}x^2\partial^2 + \ldots]\mathcal{O}(0) \tag{3.10}$$

There are several ways to determine the precise form of the coefficients of the descendants. One, direct, way [12] is to demand that the RHS of (3.9) transform under the conformal algebra in the same way as the known transformation of the LHS. A second way [25] is to require that the full OPE, with the descendant contributions included, correctly sum up to reproduce the 3-point function (3.6) not only in the limit $x_1 \to x_2$, but at finite separation as well. The last, seemingly the most efficient way, is via the so called *shadow field formalism*, which introduces conjugate auxiliary fields of dimension $4 - \Delta$ and uses them to compute "amputated" 3-point functions, which turn out to be related to the OPE coefficient functions [14].

Using the OPE, any n-point function $\langle \phi_1(x)\phi_2(0)\ldots\rangle$ can be reduced to a sum of $(n-1)$-point functions. Applying the OPE recursively, we can reduce any correlator to 3-point functions which are fixed by the symmetry. Of course, this procedure can be carried out in full only if we already know which operators appear in the OPE, and with which coefficients. Consistency of (3.9) and (3.6) in the limit $x_1 \to x_2$ requires that the same constant $\lambda_{12\mathcal{O}}$ appear in both equations. Thus the sum in (3.9) is taken over all primaries $\mathcal{O}_{(\mu)}$ which have non-zero correlators (3.6). It is not difficult to show that the correlator $\langle\phi_1\phi_2\mathcal{O}\rangle$ vanishes if $\mathcal{O}$ has $j \neq \tilde{j}$,[8] and thus such fields do not appear in the OPE of two scalars (see e.g. [20], p.156).

We stress that in CFT, the OPE is *not* an asymptotic expansion but is a bona fide convergent power-series expansion[9]. The region of expected convergence can be understood using the state-operator correspondence in the radial quantization of CFT (see [18], Sections 2.8,2.9 for a lucid discussion in 2D). In this picture, every state $|\Psi\rangle$ defined on a sphere of radius r around the origin can be expanded in a basis of states generated by local operator insertions at the origin acting on the vacuum: $\mathcal{O}(0)|0\rangle$. For example, consider the Euclidean 4-point function $\langle\phi_1(x)\phi_2(0)\phi_3(x_3)\phi_4(x_4)\rangle$, and suppose that

$$0 < |x| < \min\left(|x_3|, |x_4|\right) , \tag{3.11}$$

so that there exists a sphere centered at the origin which contains $0, x$ but not x_3, x_4. Cutting the path integral along the sphere, we represent the 4-point function as a Hilbert-space product

$$\langle\phi_1(x)\phi_2(0)\phi_3(x_3)\phi_4(x_4)\rangle = \langle\Psi_S|\phi_3(x_3)\phi_4(x_4)|0\rangle , \qquad \langle\Psi_S| \equiv \langle 0|\phi_1(x)\phi_2(0) ,$$

where the radial quantization state $|\Psi_S\rangle$ lives on the sphere, and can be expanded in the basis of local operator insertions at $x = 0$. Thus we expect the OPE to converge if (3.11) is satisfied. To

[8]Fields with $j \neq \tilde{j}$ correspond to antisymmetric tensors. The correlator $\left\langle\phi_1(x)\phi_2(-x)\mathcal{O}_{j\neq\tilde{j}}(0)\right\rangle$ must vanish for this particular spacetime configuration, since we cannot construct an antisymmetric tensor out of x_μ. Any other configuration can be reduced to the previous one by a conformal transformation.

[9]In general, it will involve fractional powers depending on the dimensions of the entering fields.

quote [18], the convergence of the OPE is just the usual convergence of a complete set in quantum mechanics. See also [19],[20] for rigorous proofs of OPE convergence in 2D and 4D CFT, based on the same basic idea.

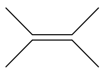
The concept of OPE is also applicable in theories with broken scale invariance, e.g. in asymptotically free perturbative field theories, such as QCD, which are well defined in the UV. These theories can be viewed as a CFT with a relevant deformation associated to a scale Λ_{QCD}. The question of OPE convergence in this case is more subtle. In perturbation theory, the OPE provides an *asymptotic expansion* of correlation functions in the $x \to 0$ limit [21], and is unlikely to be convergent because of non-perturbative ambiguities associated with the renormalons and the choice of the operator basis. It has however been conjectured in [22] that full non-perturbative correlators should satisfy a convergent OPE also in theories with broken scale invariance. This presumably includes QCD, but so far has been proved, by direct inspection, only for free massive scalar (see [24] and Ref. [6] of [22]).

3.4 Conformal blocks

As we mentioned in Section 3.2, conformal invariance implies that a scalar 4-point function must have the form (3.7), where $g(u,v)$ is an arbitrary function of the cross-ratios. Further information about $g(u,v)$ can be extracted using the OPE. Namely, if we apply the OPE to the LHS of (3.7) both in 12 and in 34 channel, we can represent the 4-point function as a sum over primary operators which appear in both OPEs:

$$\langle \phi_1\phi_2\phi_3\phi_4\rangle = \sum_{\mathcal{O}} \lambda_{12\mathcal{O}}\lambda_{34\mathcal{O}}\, \mathtt{CB}_{\mathcal{O}}\,, \tag{3.12}$$

$$\mathtt{CB}_{\mathcal{O}} = \tag{3.13}$$



.

The nondiagonal terms do not contribute to this equation because the 2-point functions of non-identical primaries $\mathcal{O} \neq \mathcal{O}'$ vanish, and so do 2-point functions of any two operators belonging to different conformal families. The functions $\mathtt{CB}_{\mathcal{O}}$, which receive contributions from 2-point functions of the operator $\mathcal{O}$ and its descendants, are called *conformal blocks*. Conformal invariance of the OPE implies that the conformal blocks transform under the conformal group in the same way as $\langle \phi_1\phi_2\phi_3\phi_4\rangle$. Thus they can be written in the form of the RHS of (3.7), with an appropriate function $g_{\mathcal{O}}(u,v)$. In terms of these functions, (3.12) can be rewritten as

$$g(u,v) = \sum_{\mathcal{O}} \lambda_{12\mathcal{O}}\lambda_{34\mathcal{O}}\, g_{\mathcal{O}}(u,v). \tag{3.14}$$

In general, functions $g_{\mathcal{O}}(u,v)$ depend on the spin l and dimension Δ of the operator $\mathcal{O}$, as well as on the dimensions $\Delta_i = [\phi_i]$. Various power-series representations for these functions were known since the 70's, but it seems that simple closed-form expressions were obtained only recently by Dolan and Osborn [25],[26]. In what follows we will heavily use their result, in the particular case when all Δ_i are equal. In this case it takes the form independent of Δ_i:[10]

$$g_{\mathcal{O}}(u,v) \equiv g_{\Delta,l}(u,v) = \frac{(-)^l}{2^l}\frac{z\bar{z}}{z-\bar{z}}\left[\, k_{\Delta+l}(z)k_{\Delta-l-2}(\bar{z}) - (z\leftrightarrow\bar{z})\right],$$
$$k_\beta(x) \equiv x^{\beta/2}{}_2F_1\left(\beta/2,\beta/2,\beta;x\right), \tag{3.15}$$

where the variables $z,\bar{z}$ are related to u,v via

$$u = z\bar{z}, \quad v = (1-z)(1-\bar{z}), \tag{3.16}$$

or, equivalently[11],

$$z,\bar{z} = \frac{1}{2}\left(u-v+1\pm\sqrt{(u-v+1)^2-4u}\right).$$

We will give a brief review of the derivation of Eq. (3.15) in Appendix B. A short comment is here in order about the meaning and range of z and $\bar{z}$ (see Appendix C for a more detailed discussion). With points x_i varying in the 4D Euclidean space, these variables are complex conjugates of each other: $\bar{z} = z^*$. Configurations corresponding to real $z = \bar{z}$ can be characterized as having all 4 points lie on a planar circle. Below we will find it convenient to analytically continue to the Minkowski signature, where z and $\bar{z}$ can be treated as independent real variables. One possible spacetime configuration which realizes this situation (the others being related by a conformal transformation) is shown in Fig. 2. Here we put 3 points along a line in the $T = 0$ Euclidean section:

$$x_1 = (0,0,0,0), \quad x_3 = (1,0,0,0), \quad x_4 = \infty, \tag{3.17}$$

while the 4-th point has been analitically continued to the Minkowski space:

$$x_2 \to x_2^M = (X_1,0,0,T), \quad T = -iX_4. \tag{3.18}$$

One shows (see Appendix C) that in this situation

$$z = X_1 - T, \quad \bar{z} = X_1 + T. \tag{3.19}$$

We do not expect any singularities if x_2^M stays inside the "spacelike diamond" region

$$0 < z,\bar{z} < 1, \tag{3.20}$$

formed by the boundaries of the past and future lightcones of x_1 and x_3. Indeed, one can check that the conformal blocks are real smooth functions in the spacelike diamond.

[10]This happens because for equal Δ_i the coefficients of conformal descendants in the OPE are determined only by Δ and l, see e.g. Eq. (3.10).

[11]Notice that the RHS of (3.15) is invariant under $z \leftrightarrow \bar{z}$.

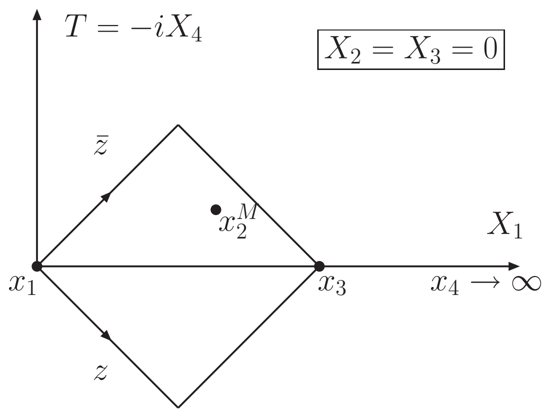


Figure 2: The "spacelike diamond" (3.20) in which the conformal blocks are real and regular, see the text.

4 Crossing symmetry and the sum rule

Let us consider the 4-point function (3.7) with all 4 operators identical: $\phi_i \equiv \phi$. We have:

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \frac{g(u,v)}{|x_{12}|^{2d}|x_{34}|^{2d}}, \quad d = [\phi]. \tag{4.1}$$

The LHS of this equation is invariant under the interchange of any two x_i, and so the RHS should also be invariant, which gives a set of *crossing symmetry* constraints for the function $g(u,v)$. Invariance under $x_1 \leftrightarrow x_2$ and $x_1 \leftrightarrow x_3$ (other permutations do not give additional information) implies:

$$g(u,v) = g(u/v, 1/v) \qquad (x_1 \leftrightarrow x_2), \tag{4.2}$$

$$v^d g(u,v) = u^d g(v,u) \qquad (x_1 \leftrightarrow x_3). \tag{4.3}$$

At the same time, $g(u,v)$ can be expressed via the conformal block decomposition (3.14), which in the considered case takes the form:

$$g(u,v) = 1 + \sum_{\mathcal{O}\in\phi\times\phi} \lambda_{\mathcal{O}}^2\, g_{\mathcal{O}}(u,v)\,. \tag{4.4}$$

Here in the first term we explicitly separated the contribution of the unit operator, present in the $\phi \times \phi$ OPE. Since $\lambda_{\mathcal{O}}$ are real (see Section 3.2), all conformal blocks appear in (4.4) with positive coefficients.

Let us now see under which conditions Eq. (4.4) is consistent with the crossing symmetry. The $x_1 \leftrightarrow x_2$ invariance turns out to be rather trivial. Transformation properties of any conformal block under this crossing depend only on its spin [25]:

$$g_{\Delta,l}(u,v) = (-)^l g_{\Delta,l}(u/v, 1/v).$$

All the operators appearing in the OPE $\phi \times \phi$ have even spin[12]. Thus the first crossing constraint (4.2) will be automatically satisfied for arbitrary coefficients $\lambda^2_{\mathcal{O}}$.

On the other hand, we do get a nontrivial condition by imposing that (4.4) satisfy the second crossing symmetry (4.3). This condition can be conveniently written in the form of the following *sum rule*:

$$\boxed{\begin{aligned} &1 = \sum_{\Delta,l} p_{\Delta,l} F_{d,\Delta,l}(z,\bar{z}), \quad p_{\Delta,l} > 0\,, \\ &F_{d,\Delta,l}(z,\bar{z}) \equiv \frac{v^d g_{\Delta,l}(u,v) - u^d g_{\Delta,l}(v,u)}{u^d - v^d}\,, \end{aligned}} \tag{4.5}$$

where the sum is taken over all Δ, l corresponding to the operators $\mathcal{O} \in \phi \times \phi$, $p_{\Delta,l} = \lambda^2_{\mathcal{O}}$, and u, v are expressed via $z, \bar{z}$ via (3.16). As we will see below, this sum rule contains a great deal of information. It will play a crucial role in the derivation of our bound on the scalar operator dimensions.

Below we will always apply Eq. (4.5) in the spacelike diamond $0 < z, \bar{z} < 1$, see Section 3.4. We will find it convenient to use the coordinates a, b vanishing at the center of the diamond:

$$z = \frac{1}{2} + a + b, \quad \bar{z} = \frac{1}{2} + a - b.$$

The sum rule functions $F_{d,\Delta,l}$ in this diamond:

1. are smooth;
2. are even in both a and b, independently:
$$F_{d,\Delta,l}(\pm a, \pm b) = F_{d,\Delta,l}(a,b)\,; \tag{4.6}$$
3. vanish on its boundary:
$$F_{d,\Delta,l}(\pm 1/2, b) = F_{d,\Delta,l}(a, \pm 1/2) = 0. \tag{4.7}$$

Properties 1,2 are shown in Appendix D. Property 3 trivially follows from the definition of $F_{d,\Delta,l}$, since both terms in the numerator contain factors $z\bar{z}(1-z)(1-\bar{z})$.

A consequence of Property 3 is that the sum rule can never be satisfied with finitely many terms in the RHS.

4.1 The sum rule in the free scalar theory

To get an idea about what one can expect from the sum rule, we will demonstrate how it is satisfied in the free scalar theory. In this case $d = 1$, and only operators of twist $\Delta - l = 2$ are present in the OPE $\phi \times \phi$ [24],[25]. These are the operators

$$\mathcal{O}_{\Delta,l} \propto \phi\, \partial_{\mu_1} \ldots \partial_{\mu_l} \phi + \ldots \qquad (\Delta = l + 2,\ l = 0, 2, 4, \ldots). \tag{4.8}$$

[12] A formal proof of this fact can be given by considering the 3-point function $\langle \phi(x)\phi(-x)\mathcal{O}_{(\mu)}(0)\rangle$. By $x \to -x$ invariance, nonzero value of this correlator is consistent with Eq. (3.6) only if l is even.

The first term shown in (4.8) is traceless by ϕ's equation of motion, but it is not conserved. The extra bilinear in ϕ terms denoted by $\ldots$ make the operator conserved for $l > 0$ (in accord with the unitarity bounds (3.3)), without disturbing the tracelessness. Their exact form can be found e.g. in [23].

In particular, there is of course the dimension 2 scalar

$$\mathcal{O}_{2,0} = \frac{1}{\sqrt{2}}\phi^2\,,$$

where the constant factor is needed for the proper normalization. At spin 2 we have the energy-momentum tensor:

$$\mathcal{O}_{4,2} \propto \phi\partial_\mu\partial_\nu\phi - 2\left[\partial_\mu\phi\partial_\nu\phi - \frac{1}{4}\delta_{\mu\nu}(\partial\phi)^2\right]\,.$$

The operators with $l > 2$ are the conserved higher spin currents of the free scalar theory.

The OPE coefficients of all these operators (or rather their squares) can be found by decomposing the free scalar 4-point function into the corresponding conformal blocks, Eq. (4.4). We have [25],[24]:

$$p_{l+2,l} = 2^{l+1}\frac{(l!)^2}{(2l)!} \qquad (l = 2n)\,. \tag{4.9}$$

Using these coefficients, we show in Fig. 3 how the sum rule (4.5), summed over the first few terms, converges on the diagonal $z = \bar{z}$ of the spacelike diamond. Several facts are worth noticing. First, notice that the convergence is monotonic, i.e. all $F_{d,\Delta,l}$ entering the infinite series are positive. This feature is not limited to the free scalar case and remains true for a wide range of d, Δ, l; it could be used to limit the maximal size of allowed OPE coefficients (see footnote 16).

Second, the convergence is uniform on any subinterval $z \in [\varepsilon, 1-\varepsilon]$, $\varepsilon > 0$, but not on the full interval $[0,1]$, because all the sum rule functions vanish at its ends, see Eq. (4.7). Finally, the convergence is fastest near the middle point $z = 1/2$, corresponding to the center $a = b = 0$ of the spacelike diamond. Below, when we apply the sum rule to the general case $d > 1$, we will focus our attention on a neighborhood of this point.

5 Main results

In this section we will present a derivation of the bound (1.4), based on the sum rule (4.5). We assume that we are given a unitary CFT with a primary scalar operator ϕ of dimension $d > 1$. We consider the 4-point function $\langle\phi\phi\phi\phi\rangle$ and derive the sum rule (4.5), where the sum is over all primary operators appearing in the OPE $\phi\times\phi$. We will use only the most general information about these operators, such as[13]:

[13] The energy-momentum tensor $T_{\mu\nu}$, which is a spin-2 primary of dimension 4, has to appear in the OPE, with a known coefficient [25] $p_{4,2} = 4d/(3\sqrt{c_T})$ depending on the central charge c_T of the theory. However, we are not making any assumptions about the central charge and will not take this constraint into account. It may be worth incorporating such a constraint in the future, since it could make the bound stronger. From the point of view of phenomenology, estimates of the electroweak S-parameter prefer models with small number of degree of freedom, hence small c_T.

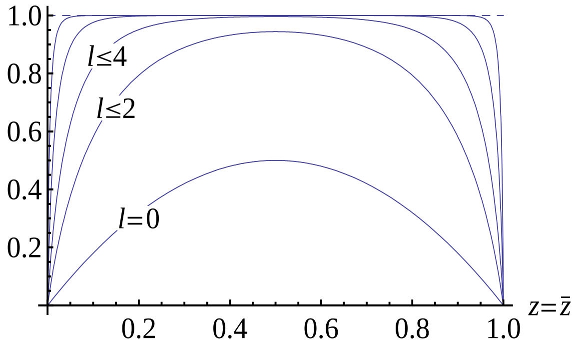


Figure 3: The RHS of the sum rule in the free scalar theory, summed over $l \leq 0, 2, 4, 8, 16$ (from below up) and plotted for $0 \leq z = \bar{z} \leq 1$. The asymptotic approach to 1 (dashed line) is evident. Notice the symmetry with respect to $z = 1/2$, a consequence of (4.6).

1. only the operators satisfying the unitarity bounds (3.3) may appear;
2. their spins l are even;
3. all the coefficients $p_{\Delta,l}$ are non-negative.

We will prove the bound (1.4) by contradiction. Namely, we will show that if only scalar operators of dimension $\Delta > f(d)$ are allowed to appear in the OPE, the sum rule cannot be satisfied no matter what are the dimensions, spins, and OPE coefficients of all the other operators (as long as they satisfy the above assumptions 1,2,3). Thus such a CFT cannot exist! In the process of proving this, we will also derive the value of $f(d)$.

5.1 Why is the bound at all possible?

Let us begin with a very simple example which should convince the reader that some sort of bound should be possible, at least for d sufficiently close to 1.

The argument involves some numerical exploration of functions $F_{d,\Delta,l}$ entering the sum rule (4.5), easily done e.g. with Mathematica. These functions depend on two variables $z, \bar{z}$, but for now it will be enough to explore the case $0 < z = \bar{z} < 1$, which corresponds to the point x_2 lying on the diagonal $x_1 - x_3$ of the spacelike diamond in Fig. 2. We begin by making a series of plots of $F_{d,\Delta,l}$ for $l = 2, 4$ and for Δ satisfying the unitarity bound $\Delta \geq l + 2$ appropriate for these spins (Fig. 4). The scalar case $l = 0$ will be considered below. We take $d = 1$ in these plots.

What we see is that all these functions have a rather similar shape: they start off growing monotonically as z deviates from the symmetric point $z = 1/2$, and after a while decrease sharply as $z \to 0, 1$. These charecteristics become more pronounced as we increase l and/or Δ. We invite the reader to check that, for $d = 1$, these properties are in fact true for all $F_{d,\Delta,l}$ for even $l \geq 2$ and $\Delta \geq l + 2$. By continuity, they are also true for $d = 1 + \varepsilon$ as long as $\varepsilon > 0$ is sufficiently small.[14]

[14]One can check that they are true up to $d \simeq 1.12$. For larger d, $F''_{d,4,2}(z = 1/2)$ becomes negative.

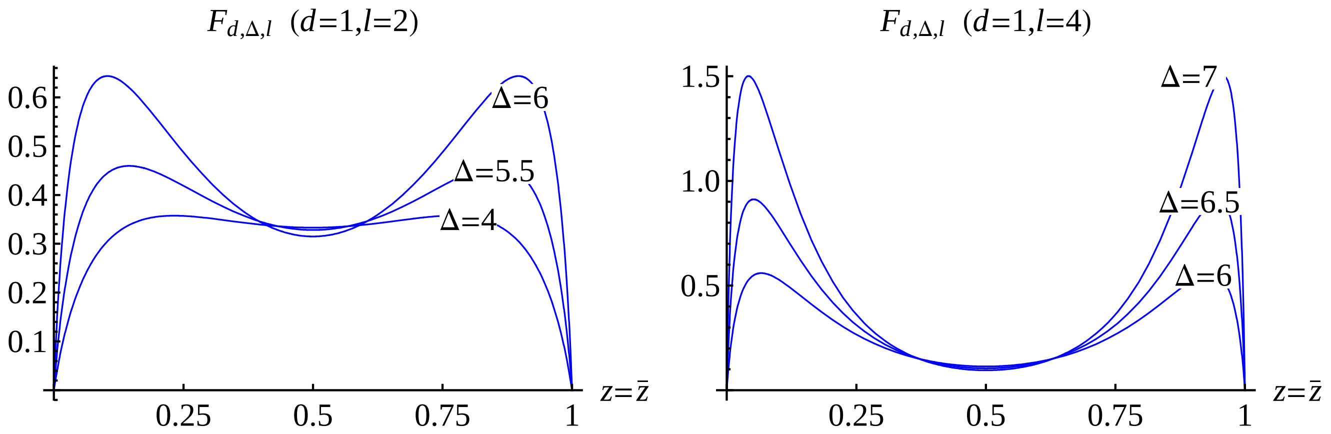


Figure 4: The shape of $F_{d,\Delta,l}$ for $d = 1$, $l = 2, 4$ and several values of Δ satisfying the unitarity bound.

Mathematically, we can express the fact that $F_{d,\Delta,l}$ is downward convex near $z = 1/2$ as:

$$
\begin{aligned}
&F''_{d,\Delta,l} > 0 \quad \text{at } z = \bar{z} = 1/2, \\
&l = 2, 4, 6 \ldots, \quad \Delta \geq l + 2, \\
&1 \leq d \leq 1 + \varepsilon \, .
\end{aligned}
\tag{5.1}
$$

Even before addressing the existence of the bound, let us now ask and answer the following more elementary question: could a CFT without any scalars in the OPE $\phi \times \phi$ exist? Eq. (5.1) immediately implies that the answer is *NO*, at least if d is sufficiently close to 1.

The proof is by contradiction: in such a CFT, the sum rule (4.5) would have to be satisfied with only $l \geq 2$ terms present in the RHS. Applying the second derivative to the both sides of (4.5) and evaluating at $z = \bar{z} = 1/2$, the LHS is identically zero, while in the RHS, by (5.1), we have a sum of positive terms with positive coefficients. This is a clear contradiction, and thus such a CFT does not exist.

To rephrase what we have just seen, the sum rule must contain some terms with negative $F''(z = 1/2)$ to have a chance to be satisfied, and by (5.1) such terms can come only from $l = 0$. Thus, the next natural step is to check the shape of $F_{d,\Delta,l}$ for $l = 0$, which we plot for several $\Delta \geq 2$ in Fig. 5. We see that the second derivative in question is negative at $\Delta = 2$ (it better be since this corresponds to the free scalar theory which surely exists!). By continuity, it is also negative for Δ near 2. However, and this is crucial, it turns *positive* for Δ above certain critical dimension Δ_c between 3 and 4. It is not difficult to check that in fact $\Delta_c \simeq 3.61$ for d near 1.

We arrive at our main conclusion: not only do some scalars have to be present in the OPE, but at least one of them should have $\Delta \leq \Delta_c$! Otherwise such a CFT will be ruled out by the same argument as a CFT without any scalars in the OPE. In other words, we have just established the bound $\Delta_{\min} \leq \Delta_c$ for d near 1.

Admittedly, this first result is extremely crude: for instance, the obtained bound does not approach 2 as $d \to 1$. However, what is important is that it already contains the main idea of the

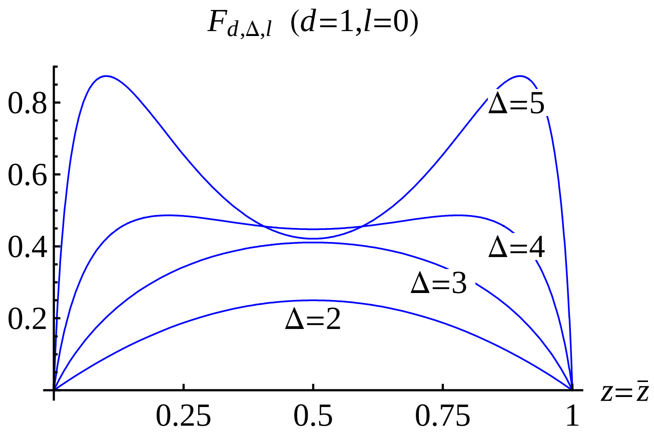


Figure 5: Same as Fig. 4, for $l = 0$.

method which will be developed and used with increasing refinement below. This idea is that we have to look for a differential operator which gives zero acting on the unit function in the LHS of the sum rule, but stays positive when applied to the functions $F_{d,\Delta,l}$ in the RHS.

Now, some readers may find it unappealing that the method as we presented it above seems to be heavily dependent on the numerical evaluation of functions $F_{d,\Delta,l}$ and their derivatives. Do we have an *analytical* proof establishing e.g. the properties (5.1)? – a purist of mathematical rigor may ask.

Partly, the answer is yes, since the asymptotic behavior of $F_{d,\Delta,l}$ for large Δ and/or l is easily accessible to analytical means (see Appendix D). These asymptotics establish Eq. (5.1) in the corresponding limit. On the other hand, we do not have an analytic proof of Eq. (5.1) for finite values of Δ and l. Notice that such a proof must involve controlling hypergeometric functions near $z = 1/2$. No simple general expansions of hypergeometrics exist near this point (apart from the one equivalent to summing up the full series around $z = 0$). Thus we doubt that a simple proof exists.

Nevertheless, and we would like to stress this, the fact that we can establish Eq. (5.1) only numerically (with analytic control of the asymptotic limits) does not make it less mathematically true! The situation can be compared to proving the inequality $e < \pi$. An aesthete may look for a fully analytical proof, but a practically minded person will just evaluate both constants by computer. As long as the numerical precision of the evaluation is high enough, the practical proof is no worse than the aesthete's (and much faster).

To summarize, we should be content that numerical methods allow us to extract from general CFT properties (crossing, unitarity bounds, conformal block decomposition, ...) precious information about operator dimensions which would otherwise simply not be available.

5.2 Geometry of the sum rule

To proceed, it is helpful to develop geometric understanding of the sum rule. Given d and a spectrum $\{\Delta, l\}$ of $\mathcal{O} \in \phi \times \phi$, and allowing for arbitrary positive coefficients $p_{\Delta,l}$, the linear

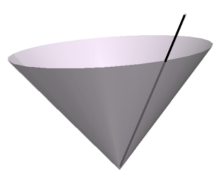


Figure 6: The three geometric situations described in the text. The thick black line denotes the vector corresponding to the function $F \equiv 1$.

combinations in the RHS of (4.5) form, in the language of functional analysis, a *convex cone* $\mathcal{C}$ in the function space $\{F(a,b)\}$. For a fixed spectrum, the sum rule can be satisfied for *some* choice of the coefficients if and only if the unit function $F(a,b) \equiv 1$ belongs to this cone.

Obviously, when we expand the spectrum by allowing more operators to appear in the OPE, the cone gets wider. Let us consider a one-parameter family of spectra:

$$\Sigma(\Delta_{\min}) = \{\Delta, l \;\;|\;\; \Delta \geq \Delta_{\min}\, (l=0), \quad \Delta \geq l+2 \;(l = 2,4,6\ldots)\}\,. \tag{5.2}$$

Thus we include all scalars of dimension $\Delta \geq \Delta_{\min}$, and all higher even spin primaries allowed by the unitarity bounds.

The crucial fact which makes the bound (1.4) possible is that in the limit $\Delta_{\min} \to \infty$ the convex cone generated by the above spectrum does *not* contain the function $F \equiv 1$! In other words, CFTs without any scalars in the OPE $\phi \times \phi$ cannot exist, as we already demonstrated in Section 5.1 for d sufficiently close to 1.

As we lower $\Delta_{\min}$, the spectrum expands, and the cone gets wider. There exists a critical value Δ_c such that for $\Delta_{\min} > \Delta_c$ the cone is not yet wide enough and the function $F \equiv 1$ is still outside, while for $\Delta_{\min} < \Delta_c$ the $F \equiv 1$ function is inside the cone. For $\Delta_{\min} = \Delta_c$ the function belongs to the cone boundary. This geometric picture is illustrated in Fig. 6.

For $\Delta_{\min} > \Delta_c$, the sum rule cannot be satisfied, and a CFT corresponding to the spectrum $\Sigma(\Delta_{\min})$ (or any smaller spectrum) cannot exist. By contradiction, the bound (1.4) with $f(d) = \Delta_c$ must be true in any CFT. The problem thus reduces to determining Δ_c.

Any concrete calculation must introduce a coordinate parametrization of the above function space. We will parametrize the functions by an infinite vector $\left\{F^{(2m,2n)}\right\}$ of even-order mixed derivatives at $a = b = 0$:

$$F^{(2m,2n)} \equiv \partial_a^{2m}\partial_b^{2n} F(a,b)\Big|_{a=b=0}\,. \tag{5.3}$$

Notice that all the odd-order derivatives of the functions entering the sum rule vanish at this point due to the symmetry expressed by Eq. (4.6):

$$F^{(2m+1,2n)} = F^{(2m,2n+1)} = F^{(2m+1,2n+1)} = 0\,.$$

The choice of the $a = b = 0$ point is suggested by this symmetry, and by the fact that it is near this point that the sum rule seems to converge the fastest, at least in the free scalar case, see Fig. 3.

The derivatives (5.3) are relatively fast to evaluate numerically. Presumably, there is also no loss of generality in choosing these coordinates on the function space, since the functions entering the sum rule are analytic in the spacelike diamond.

In terms of the introduced coordinates, the sum rule becomes a sequence of linear equations for the coefficients $p_{\Delta,l} \geq 0$. We have one inhomogeneous equation:

$$1 = \sum p_{\Delta,l} F^{(0,0)}_{d,\Delta,l} \,, \tag{5.4}$$

and an infinite number of homogeneous ones:

$$\begin{aligned} 0 &= \sum p_{\Delta,l} F^{(2,0)}_{d,\Delta,l} \,, \\ 0 &= \sum p_{\Delta,l} F^{(0,2)}_{d,\Delta,l} \,, \\ &\dots \end{aligned} \tag{5.5}$$

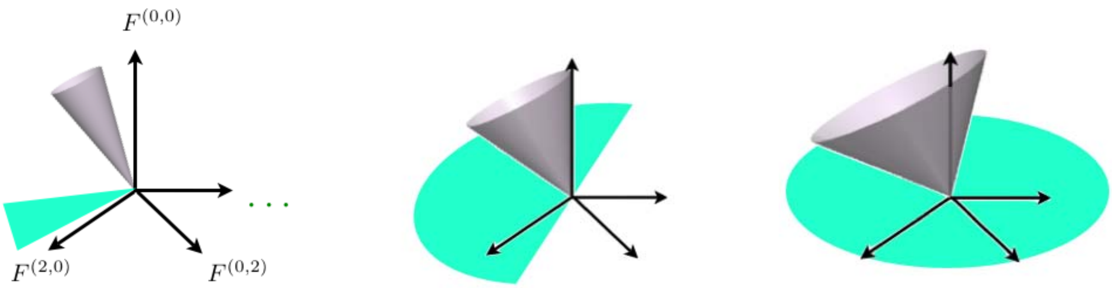
We have to determine if, for a given $\Delta_{\min}$, the above system has a solution or not. It turns out that in the range $d \geq 1$ and $\Delta_{\min} \geq 2$ which is of interest for us, all $F^{(0,0)}_{d,\Delta,l} > 0$ in the RHS of the inhomogeneous equation (5.4). In such a situation, if a nontrivial solution of the homogeneous system (5.5) is found, a solution of the full system (5.4), (5.5) can be obtained by a simple rescaling.[15]

Thus for our purposes it is enough to focus on the existence of nontrivial solutions of the homogeneous system (5.5).[16] Geometrically, this means that we are studying the projection of the convex cone $\mathcal{C}$ on the $F^{(0,0)} = 0$ plane. This *projected cone,* which is by itself a convex cone, may occupy a bigger or smaller portion of the $F^{(0,0)} = 0$ plane, or perhaps all of it. Each of the 3 cases shown in Fig. 6 can be characterized in terms of the shape of the projected cone, see Fig. 7:

- $\Delta_{\min} > \Delta_c$: This case is realized if the opening angle of the projected cone is 'less than π', so that the homogeneous equations have only the trivial solution $p_{\Delta,l} \equiv 0$. In a more technical language, the 'opening angle less than π' condition means that there exists a half-plane strictly containing the projected cone. If we write the boundary of this half-plane as $\Lambda = 0$, the linear functional Λ will be strictly positive on the projected cone (except at its tip).
- $\Delta_{\min} = \Delta_c$: This case is realized if the opening angle of the projected cone is equal to π in at least one direction. In other words, the boundary of the projected cone must contains a linear subspace passing through the origin. This subspace will be spanned by the projections of the vectors saturating the sum rule; the vectors from the "bulk" cannot appear in the sum rule with nonzero coefficients.
- $\Delta_{\min} < \Delta_c$: In this case the projected cone covers the whole plane.

[15] That is, unless the series in the RHS of (5.4) diverge. However, this situation does not occur in practice.

[16] Eq. (5.4) can become useful when studying other questions. E.g., it suggests that arbitrarily large OPE coefficients may not be consistent with the sum rule. It would be interesting to establish a model-independent theoretical upper bound on the OPE coefficients, which could be a rigorous CFT version of the NDA bounds in generic strongly coupled theories. For CFTs weakly coupled to the SM ('unparticle physics' [27]) *experimental* bounds on these OPE coefficients (also known as 'cubic unparticle couplings') have been recently considered in [28].



$\Delta_{\text{min}} > \Delta_c$ $\quad$ $\Delta_{\text{min}} = \Delta_c$ $\quad$ $\Delta_{\text{min}} < \Delta_c$

Figure 7: The shape of the projected cones in each of the 3 alternative cases described in the text.

Using this classification, we are reduced to studying the boundary of the projected cone.

For practical reasons we will have to work with finitely many derivatives, i.e. with a finite-dimensional subspace of the function space or, equivalently, with a finite subset of the homogeneous system (5.5). The above geometric picture applies also within such a subspace. Satisfaction of the sum rule on a subspace gives (in general) weaker but necessary condition, so that we still get a valid bound (1.4) with $f(d) = \Delta_c$. As we expand the subspace by including more and more derivatives, the critical scalar dimension Δ_c will go down, monotonically converging to the optimal value corresponding to the full system.

5.3 Warmup example: $d = 1$

Let us use this philosophy to examine what the sum rule says about the spectrum of operators appearing in the $\phi \times \phi$ OPE when ϕ has dimension $d = 1$. Of course we know that $d = 1$ corresponds to the free scalar, see (1.2), and thus we know everything about this theory. In particular, we know that only twist 2 operators appear in the OPE, see Section 4.1. Our interest here is to derive this result directly from the sum rule. We expect the sum rule based approach to be robust: if we make it work for $d = 1$, chances are it will also give us a nontrivial result for $d > 1$. In contrast, the standard proof of (1.2) is not robust at all: it is based on the fact that the 2-point function of a $d = 1$ scalar is harmonic, and can hardly be generalized to extract any information at $d > 1$.

The attentive reader will notice that our considerations from Section 5.1 were equivalent to retaining only the first equation out of the infinite system (5.5). This truncation did not control well the $d = 1$ limit, since the obtained value of $\Delta_c \simeq 3.61$ was well above the free theory value $\Delta = 2$. The next natural try is to truncate to the first *two* equations in (5.5). As we will see now, this truncation already contains enough information to recover the free theory operator dimensions from $d = 1$.

Following the discussion in Section 5.2, we consider the projected cone—the cone generated by the vectors $F = F_{1,\Delta,l}$ projected into the two-dimensional plane $(F^{(2,0)}, F^{(0,2)})$. For each $l = 0, 2, 4, \ldots$ we get a curve in this plane, starting at the point corresponding to the lowest value of Δ allowed by the unitarity bound (3.3), see Fig. 8. It can be seen from this figure that:

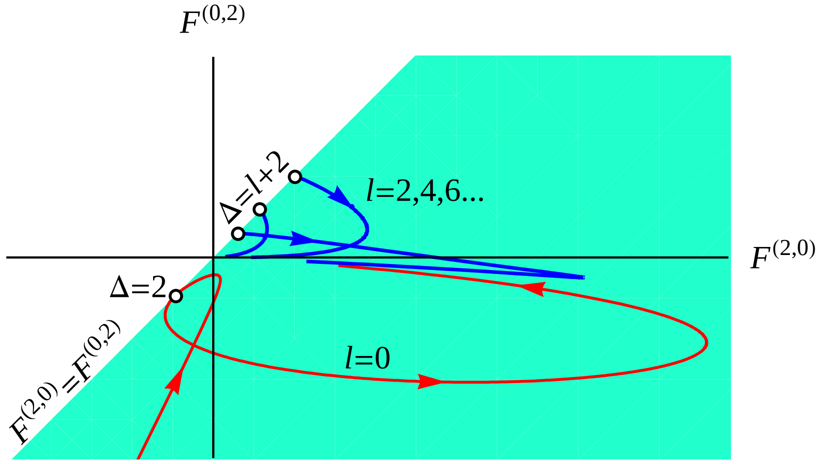


Figure 8: The sum rule terms $F_{1,\Delta,l}$ in the $(F^{(2,0)}, F^{(0,2)})$ plane. The shown curves correspond to $l = 0, 2, 4, 6$. The arrows are in the direction of increasing Δ. The $l = 0$ curve starts at $\Delta = 1$ ($\Delta \lesssim 1.01$ part is outside the plotted range); the $l = 2, 4, 6$ curves—at $\Delta = l + 2$. For large Δ the curves asymptote to the positive $F^{(2,0)}$ axis, see Appendix D. The shaded half-plane is the projected cone for a spectrum which includes the $\Delta = 2$ scalar.

1. the vectors corresponding to the twist 2 operators $\Delta = l + 2$ lie on the line $F^{(2,0)} = F^{(0,2)}$,[17] while all the other vectors lie to the right of this line;
2. the $l = 0$, $\Delta = 2$ vector points in the direction opposite to the higher-spin twist 2 operators.

The boundary of the projected cone is thus given by the line $F^{(2,0)} = F^{(0,2)}$ *if* the spectrum includes the $\Delta = 2$ scalar and at least one higher-spin twist 2 operator (e.g. the energy-momentum tensor). Otherwise the boundary will be formed by two rays forming an angle less than π.[18] By the classification of Section 5.2, it is only in the former case that the sum rule can have a solution. This case corresponds to $\Delta_{\min} = \Delta_c$: the boundary of the projected cone contains a linear subspace passing through the origin. Thus we also have additional information: only the vectors from the boundary, i.e. those of the twist 2 operators, may be present in the sum rule with nonzero coefficients.

The above argument appealed to the geometric intuition. For illustrative purposes we will also give a more formal proof. Taking the difference of the two first equations in (5.5), we get:

$$0 = \sum p_{\Delta,l} \left(F^{(2,0)}_{d,\Delta,l} - F^{(0,2)}_{d,\Delta,l} \right) , \qquad p_{\Delta,l} \geq 0.$$

By property 1 above, for $d = 1$ all the terms in the RHS of this equation are strictly positive unless $\Delta = l + 2$. Thus, only twist 2 operators may appear with nonzero coefficients. End of proof.

It is interesting to note that in Fig. 8 the $l = 0$ curve is tangent to the line $F^{(2,0)} = F^{(0,2)}$ at $\Delta = 2$.[17] Were it not so, we would not be able to exclude the existence of solutions to the sum rule involving scalar operators of $\Delta < 2$.

[17] This fact is easy to check analytically using the definition of $F_{d,\Delta,l}$ at $d = 1$.
[18] We ignore such subtleties as the possibility of a continuous scalar spectrum ending at $\Delta = 2$.

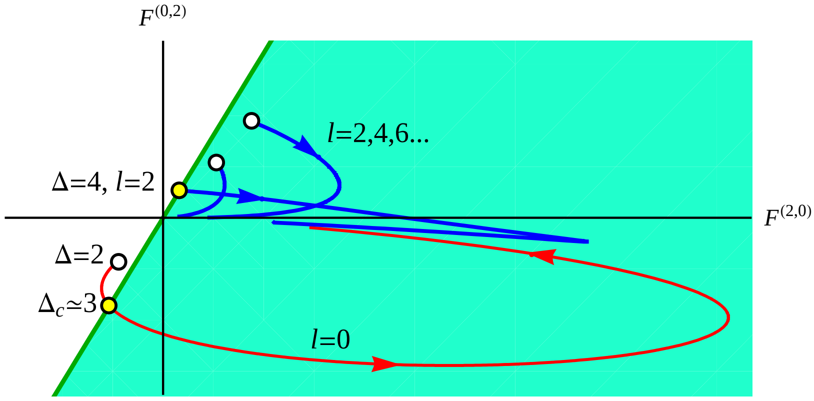


Figure 9: The analogue of Fig. 8 for $d = 1.05$. In this plot we started the $l = 0$ curve at $\Delta = 2$. The green line is the boundary of the projected cone for $\Delta_{\rm min} = \Delta_c \simeq 3$, see Fig. 10. The slope of this line is determined by the energy-momentum tensor vector.

To conclude, we have shown that the spectrum of operators appearing in the sum rule, and hence in the OPE, of a $d = 1$ scalar consists solely of twist 2 fields and that, moreover, a $\Delta = 2$ scalar must be necessarily present in this spectrum.

Isn't it amazing that we managed to find the whole spectrum using only the first two out of the infinitely many equations (5.5)? One may ask if by adding the complete information contained in the sum rule, a stronger result can be proved, namely that the full 4-point function of an arbitrary $d = 1$ scalar is given by the free scalar theory expression. This would constitute an independent proof of the fact that a $d = 1$ scalar is necessarily free. Such a proof can indeed be given [29], but we do not present it here since it is rather unrelated to our main line of reasoning.

5.4 Simplest bound satisfying $f(1) = 2$

We will now present the simplest bound of the form (1.4) which, unlike the bound discussed in Section 5.1, approaches 2 as $d \to 1$. The argument uses the projection on the $(F^{(2,0)}, F^{(0,2)})$ plane similarly to the $d = 1$ case from the previous section. Since that method gave $\Delta_{\rm min} = 2$ for $d = 1$, by continuity we expect that it should give $\Delta_{\rm min} \simeq 2$ for d sufficiently close to 1.

To demonstrate how the procedure works, we pick a d close to 1, say $d = 1.05$, and produce the analogue of the plot in Fig. 8, see Fig. 9. We see several changes with respect to Fig. 8. The energy-momentum tensor determines one part of the projected cone boundary (the green line), while the spins $l = 4, 6, \ldots$ lie in the bulk of the cone. Continuation of the green line to the other side of the origin intersects the $l = 0$ curve at the point corresponding to $\Delta = \Delta_c \simeq 3$. In the terminology of Section 5.2, this gives the critical value of $\Delta_{\rm min}$. Namely, if $\Delta_{\rm min} > \Delta_c$ in the spectrum (5.2), the projected cone will have an angle less than π and the sum rule will have no solutions. On the other hand, for $\Delta_{\rm min} < \Delta_c$ the projected cone covers the full plane, see Fig. 10, and a nontrivial solution to the first two equations of the system (5.5) will exist. For $\Delta_{\rm min} = \Delta_c$ the projected cone covers the half-plane shaded in Fig. 9.

One can check that the same situation is realized for any $d > 1$. In particular, the slope of the

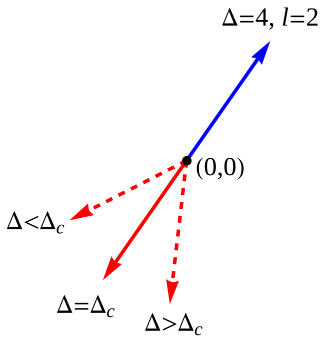


Figure 10: The relative position of the $l = 0$ vectors (red) with respect to the energy-momentum tensor vector (blue, pointing to upper right) determines the shape of the projected cone, see the text. If the cone contains the blue vector and both dashed red vectors, it covers the whole plane by their convex linear combinations.

critical cone boundary, described by the linear equation

$$F^{(2,0)} - \lambda(d) F^{(0,2)} = 0\,, \tag{5.6}$$

is always determined by the energy-momentum tensor:

$$\lambda(d) = \frac{F^{(2,0)}}{F^{(0,2)}}\,, \quad F = F_{d,4,2}. \tag{5.7}$$

Once $\lambda(d)$ is fixed, the critical value of $\Delta_{\min}$ is determined from the intersection of the line (5.6) with the $l = 0$ curve:

$$F^{(2,0)} - \lambda(d) F^{(0,2)} = 0, \quad F = F_{d,\Delta_c,0}\,. \tag{5.8}$$

The $l = 0$, $\Delta > \Delta_c$ points then lie strictly inside the half-plane $F^{(2,0)} - \lambda(d) F^{(0,2)} \geq 0$. For $\Delta_{\min} > \Delta_c$ the cone angle is less than π, and the sum rule has no solution. Thus the bound (1.4) must be valid with $f(d) = \Delta_c$.

In Fig. 11 we plot the corresponding value of $f(d)$ found numerically from Eq. (5.7), (5.8), denoted $f_2(d)$ to reflect the order of derivatives used to derive this bound. As promised, the free field theory value $\Delta = 2$ is approached continuously as $d \to 1$.

The asymptotic behavior of $f_2(d)$ for $d \to 1$ can be determined by expanding the equations defining Δ_c in power series in $d - 1$ and $\Delta_c - 2$. We find:

$$\begin{aligned} f_2(d) &= 2 + \gamma\sqrt{d-1} + O(d-1)\,, \\ \gamma &\equiv [2(K+1)/3]^{1/2} \simeq 2.929, \\ K &\equiv (192 \ln 2 - 133)^{-1}. \end{aligned} \tag{5.9}$$

This asymptotics provides a good approximation for $d - 1 \lesssim 10^{-3}$, see Fig. 11. The square root dependence in (5.9) can be traced to the fact that for $d = 1$ the $l = 0$ curve was *tangent* to the projected cone boundary at $\Delta = 2$. The bound of Fig. 11 will be improved below by taking more derivatives into account, however the square root behavior will persist (albeit with a different coefficient).

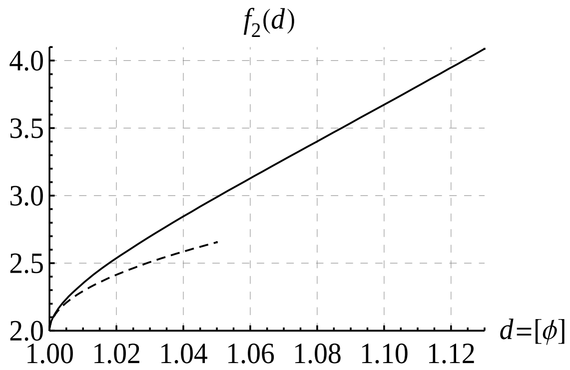


Figure 11: $f_2(d) = \Delta_c$ as determined by solving Eq. (5.8). We plot it only for d rather close to 1 because in any case this bound will be significantly improved below. The dashed line shows the asymptotic behavior (5.9), which becomes a good approximation for $d \lesssim 1.001$.

5.5 Improved bounds: general method

As we already mentioned in Section 5.2, the bound will improve monotonically as we include more and more equations from the infinite system (5.5) in the analysis, i.e. increase the number of derivatives $F^{(2m,2n)}$ that we are controlling. We thus consider a finite basis $\mathcal{B}$, adding several higher-order derivatives to the $F^{(2,0)}$ and $F^{(0,2)}$ included in the previous section:

$$\mathcal{B} = \{F^{(2m,2m)}\} = \{F^{(2,0)}, F^{(0,2)}, \ldots\}. \tag{5.10}$$

According to the discussion in Section 5.2, we have to study the boundary of the projected cone in the finite-dimensional space with coordinates (5.10). The logic in principle is the same as in the previous section. We will have a family of curves corresponding to $l = 0, 2, 4 \ldots$ generating the projected cone. As we lower $\Delta_{\min}$ in the spectrum (5.2), the projected cone grows. For $\Delta_{\min} < \Delta_c$ it will cover the whole space. However, in this many-dimensional situation it is not feasible to look for Δ_c by making plots similar to Fig. 9. We need a more formal approach.

Such an approach uses the language of linear functionals, already encountered in Section 5.2. A linear functional Λ on the finite-dimensional subspace with basis $\mathcal{B}$ is given by

$$\Lambda(F) = \sum_{\mathcal{B}} \lambda_{2m,2n} F^{(2m,2n)}, \tag{5.11}$$

where $\lambda_{2m,2n}$ are some fixed numbers characterizing the functional. They generalize the single parameter $\lambda(d)$ from Section 5.4 to the present situation.

Using linear functionals, the two non-critical cases of Fig. 7 can be distinguished as follows:

$$\begin{aligned} \Delta_{\min} > \Delta_c \iff & \text{there } \mathit{IS} \text{ a functional } \Lambda \text{ such that} \\ & \Lambda(F_{d,\Delta,l}) > 0 \text{ for all } \Delta, l \in \Sigma(\Delta_{\min})\,; \end{aligned} \tag{5.12}$$

$$\begin{aligned} \Delta_{\min} < \Delta_c \iff & \text{there is } \mathit{NO} \text{ functional } \Lambda \text{ such that} \\ & \Lambda(F_{d,\Delta,l}) \geq 0 \text{ for all } \Delta, l \in \Sigma(\Delta_{\min})\,. \end{aligned} \tag{5.13}$$

A numerical procedure which for any given $\Delta_{\min}$ finds such a positive Λ or shows that a non-negative Λ does not exists will be explained below. Assuming that we know how to do this, determination of Δ_c becomes an easy task. First, we bracket Δ_c from above and below by trying out a few values of $\Delta_{\min}$ and checking to which of the two above sets, $\Delta_{\min} > \Delta_c$ or $\Delta_{\min} < \Delta_c$, they belong. Second, we apply the division-in-two algorithm, *i.e.* reduce the length of the bracketing interval by checking its middle point, etc. This achieves exponential precision after a finite number of steps.

We will now explain the numerical procedure. Let us begin with the non-negative functional defined by Eq. (5.13), and comment later about the strictly positive case of Eq. (5.12). Eq. (5.13) can be viewed as a system of *infinitely many* linear inequalities for the coefficients $\lambda_{2m,2n}$. The infinitude is due to three reasons:

- there are infinitely many spins l;
- for each spin l the dimension Δ can be arbitrary large;
- the dimension Δ varies continuously.

To be numerically tractable, this system needs to be truncated to a finite system, removing each of the three infinities. We do it by imposing inequalities in (5.13) not for all $\Delta, l \in \Sigma(\Delta_{\min})$ but only for a 'trial set' such that

- only finite number of spins $l \leq l_{\max}$ are included;
- only dimensions up to a finite $\Delta = \Delta_{\max}$ are included;
- Δ is discretized.

To ensure that we are not losing important information by truncating at $l_{\max}$ and $\Delta_{\max}$, we include into the trial set the vectors corresponding to the large l and large Δ asymptotics of the derivatives. The relevant asymptotics have the form (see Eq. (D.4) in Appendix D):

$$F^{(2m,2n)}_{d,\Delta,l} \sim \frac{const}{(2m+1)(2n+1)} (2\sqrt{2}l\,(1+x))^{2m+1} (2\sqrt{2}l)^{2n+1}, \quad x \equiv \frac{\Delta - l - 2}{l} \geq 0\,, \tag{5.14}$$

where a constant $const > 0$ is independent of m and n. This asymptotics is valid for $l \to \infty$, $x \ll l$ fixed.

Upon truncation to the trial set, Eq. (5.13) becomes a finite system of linear inequalities, a particular case of the *linear programming problem*[19]. It is thus possible to determine if a solution exists (and find it if it does) using one of several existing efficient numerical algorithms (see [30]). In our work we used the classic Simplex Method as realized by the `LinearProgramming` function of MATHEMATICA.

If, using the linear programming, we find that (5.13) truncated to the trial set has no solution, then *a fortiori* the full non-truncated system has no solution, i.e. non-negative Λ does not exist.

[19] A general linear programming problem consists in minimizing a linear function of several variables subject to a set of linear constraints (equalities and inequalities). Our problem is a particular case when all constraints are inequalities and the function to be minimized is absent (or, equivalently, it is constant).

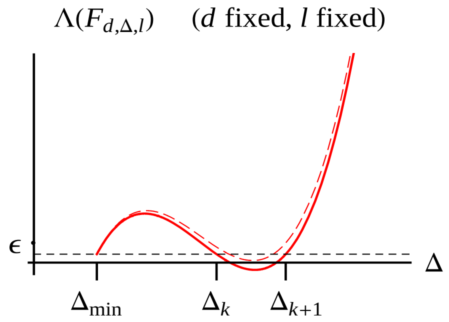


Figure 12: *Solid curve*: schematic typical dependence of the functional $\Lambda(F_{d,\Delta,l})$ on Δ. The functional Λ satisfies (5.15) for all Δ from a discrete trial set, which includes the dimensions Δ_k and $\Delta_{k+1} = \Delta_k + \delta\Delta$. Yet the functional may become slightly negative for $\Delta_k < \Delta < \Delta_{k+1}$. *Dashed curve*: same for a functional corresponding to a smaller $\delta\Delta$. The violation of (5.13) at intermediate values has disappered.

Thus we can safely claim that the considered $\Delta_{\min}$ brackets Δ_c from below: $\Delta_{\min} < \Delta_c$. The accuracy of this bracketing will increase as we include more dimensions in the trial set (e.g. decreasing the discretization step).

Let us now consider bracketing from above, which requires finding a functional satisfying (5.12). First, just as for (5.13), we truncate to a trial set. We'd like to use linear programming methods, however these methods do not work for strict inequalities. Thus we strengthen > 0 in (5.12) to $\geq \varepsilon$, where ε is a fixed small positive number[20]:

$$\Lambda(F_{d,\Delta,l}) \geq \varepsilon\,. \tag{5.15}$$

Then we can use the linear programming to find a solution of the truncated system.

Unlike in the case of bracketing from below, to claim that indeed $\Delta_{\min} > \Delta_c$, we have to check that the found Λ does not violate (5.12) for Δ, l not included in the trial set. This will not happen for $l \geq l_{\max}$ and $\Delta \geq \Delta_{\max}$ if we take these parameters sufficiently large and include the asymptotics. The functional Λ may however become slightly negative at *intermediate* Δ and l, as a consequence of discretizing Δ. This is not unexpected, since the rays (Δ, l) that determine the boundary of the cone and thus the functional are determined with a fuzziness proportional to the discretization step $\delta\Delta$. However, this violation will disappear as $\delta\Delta \to 0$ (for a fixed ε), see Fig. 12. By choosing the discretization step smaller and smaller, we will be able to bracket Δ_c from above with an arbitrary desired precision.

[20] In principle, Eq. (5.15) may become too constraining if all components of the vector $F_{d,\Delta,l}$ are $O(\varepsilon)$, which may happen in the large Δ limit. In this case one simply needs to rescale $F_{d,\Delta,l}$ by a constant factor. Each function $F_{d,\Delta,l}$ determines a ray in the finite-dimensional space, and such a rescaling does not change the content of the original Eq. (5.12). In practice, however, we never had to do such a rescaling.

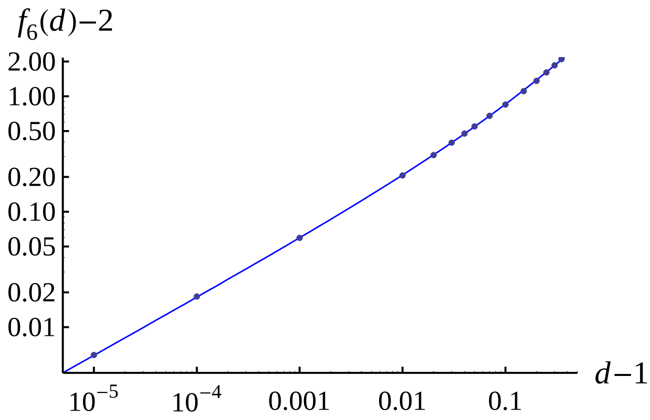


Figure 13: Log-log plot of the maximal allowed anomalous dimension of ϕ^2, $f_6(d)-2$, versus the anomalous dimension of ϕ, $d-1$. The dots correspond to the entries of Table 1, while the curve is the approximation (5.17).

5.6 Best results to date

In our numerical work we explored functionals (5.11) with the leading a-derivative up to $F^{(6,0)}$ and with various choices of subleading derivatives. We now present our best results, which were obtained using the full list of derivatives with $2m+2n \le 6$:

$$\left\{F^{(2m,2n)} \mid (2m,2n) = (6,0),(4,2),(2,4),(0,6),(4,0),(2,2),(0,4),(2,0),(0,2)\right\}. \qquad (5.16)$$

The bound $f(d) \equiv f_6(d)$ corresponding to this choice is plotted in Figs. 1 and 13. Numerical values for several values of d are given in Table 1. For each d we give the bound $f_6(d)$ and the coefficients $\lambda_{2m,2n}$ of the functional used to obtain this bound. These functionals were found using the linear programming method as described in the previous section. However, to check that our bound is true, one does not need to know how we found these functionals; it is enough to check that they indeed satisfy Eq. (5.12) with $\Delta_{\rm min} = f_6(d)$.

We have also computed the bound $f_6(d)$ for other values of d and found that it changes smoothly, interpolating between the points given in Table 1. In the considered interval of d, the anomalous dimension $f_6(d)-2$ turns out to be well approximated (within $\sim 2\%$) by the formula (see Fig. 13):

$$f_6(d) - 2 \simeq 1.79\sqrt{d-1} + 2.9(d-1) \qquad (1 < d < 1.35). \qquad (5.17)$$

While our method would give a bound also for $d > 1.35$, we did not explore this range. The reason is that for $d \gtrsim 1.33$ our bound exceeds $\Delta = 4$ and starts getting not very interesting, taking into account the phenomenological motivations from Section 2.

The coefficients $\lambda_{2m,2n}$ in Table 1 have been rounded with 6 significant digits; we have checked that the resulting slight violation of (5.12) is very small[21]. The anomalous dimensions $f_6(d)-2$ in Table 1 approximate the optimal values, attainable using the subspace (5.16), *from above* to

[21] The rounded functionals violate (5.12) by about 10^{-4} for a few isolated values of Δ, l. This should be compared to the typical $O(1 \div 1000)$ range of $\Lambda(F)$ away from these points.

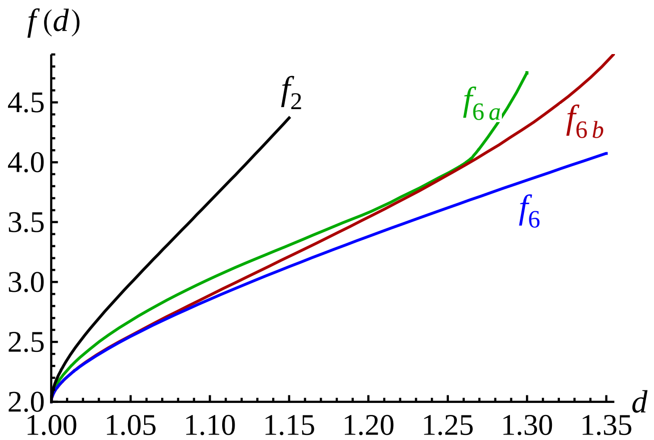


Figure 14: From this plot one can get an idea how the bound monotonically improves as more and more derivatives are taken into acccount in the infinite system (5.5). The black (upper) curve is the simplest bound of Section 5.4, obtained by using only the second derivatives. The next two curves (green and red) correspond to the two sets (5.19). The lowest-lying blue curve is our best current bound obtained using the set (5.16).

within 1% (due to the finite accuracy of the division-in-two algorithm used to bracket Δ_c); they have been rounded *up* with 3 significant digits.

In obtaining these results, we used the trial set in the sense of the previous subsection with $l_{\max} = 10$, $\Delta_{\max} = 20$ $(l = 0)$, $\Delta_{\max} = l + 10$ $(l \geq 2)$. We discretized Δ with step $\delta\Delta = 0.01$, decreased to $\delta\Delta = 0.0025$ around a few critical dimensions where the functional approaches zero, as in Fig. 12 for $\Delta_k < \Delta < \Delta_{k+1}$. To take into account the asymptotic behavior (5.14), we have included into the trial set vectors with

$$F^{(2m,2n)} = \begin{cases} \left[(2m+1)(2n+1)(1+x)^{2n}\right]^{-1}, & 2m+2n = 6\,, \\ 0, & 2m+2n < 6\,, \end{cases} \tag{5.18}$$

obtained from (5.14) by rescaling and taking the $l \to \infty$ limit. The parameter x in (5.18) was varying from $x = 0$ to 10 with $\delta x = 0.01$. We set $\varepsilon = 10^{-4}$ in (5.15).

We expect that including more derivatives in the list (5.16) should somewhat improve the bound, especially for d close to the upper end of the considered interval. We have observed a similar improvement trying out the functionals with the same leading a-derivative $(6,0)$ as in (5.16), but with smaller sets of subleading derivatives:

$$\begin{aligned} &\text{Set 6a: } (2m,2n) = (6,0),(4,0),(2,0),(0,2); \\ &\text{Set 6b: } (2m,2n) = (6,0),(4,0),(2,0),(0,2),(0,4),(0,6)\,. \end{aligned} \tag{5.19}$$

For illustration, we plot the corresponding bounds in Fig. 14, including also the simplest second-derivative bound from Section 5.4.

$d-1$	f_6-2	$\lambda_{6,0}$	$\lambda_{4,2}$	$\lambda_{2,4}$	$\lambda_{0,6}$	$\lambda_{4,0}$	$\lambda_{2,2}$	$\lambda_{0,4}$	$\lambda_{2,0}$	$\lambda_{0,2}$
10^{-5}	0.00573	1.	−0.97747	−1.06327	−0.047622	−116.282	277.34	49.2726	3344.18	−7170.94
10^{-4}	0.0185	1.	0.	−0.00052291	−0.999251	−153.677	48.2058	126.869	7344.28	−8370.35
10^{-3}	0.0593	1.	−1.24135	−0.845225	−0.0234949	−101.866	276.743	36.8807	2626.56	−6384.32
0.01	0.207	1.	−0.738985	0.839453	−1.00868	−104.348	14.8941	88.7308	4989.57	−5288.51
0.02	0.31	1.	−1.04266	1.22926	−1.01672	−87.2012	0.	73.9343	4384.97	−4356.11
0.03	0.397	1.	−0.669013	0.980721	−1.02839	−100.265	−8.59119	82.6461	5012.32	−4772.52
0.04	0.476	1.	−1.14492	1.51543	−1.03388	−77.4978	−20.9209	63.1812	4200.66	−3696.39
0.05	0.548	1.	6.31335	−5.62554	−1.0635	−221.053	−276.119	338.147	11062.1	−4888.67
0.07	0.678	1.	8.20962	−7.39456	−1.07537	−236.106	−425.566	439.223	13130.	−3371.37
0.1	0.849	1.	10.4578	−9.55059	−1.08398	−255.95	−620.004	579.352	16504.5	−1662.47
0.15	1.11	1.	11.981	−10.4519	−1.14201	−261.146	−768.116	649.227	19160.9	−46.5554
0.2	1.36	1.	12.7909	−10.5811	−1.20555	−259.763	−863.149	676.812	20924.6	1544.21
0.25	1.61	1.	14.283	−11.2729	−1.28025	−262.93	−1027.33	746.012	24108.2	3944.39
0.3	1.86	1.	18.0218	−14.0589	−1.36917	−281.038	−1467.79	996.762	32918.8	9435.72
0.35	2.1	1.	24.6535	−19.3331	−1.49588	−292.263	−2367.92	1493.39	51357.2	20484.

Table 1: For several d we give numerical values of $f_6 \equiv f_6(d)$ appearing in the bound (1.4) and the coefficients $\lambda_{2m,2n}$ of a functional satisfying Eq. (5.13) with $\Delta_{\min} = f_6$, see the text. The functionals are normalized via $\lambda_{6,0} = 1$.

6 Comparison to known results

To further test our method, in this section we shall compare our bound to the operator dimensions in calculable CFTs. Several nontrivial tests are offered by exactly solvable CFTs in 2D. We will discuss these examples in subsection 1. On the other hand there are fewer calculable examples in 4D, and they all turn out to satisfy our bound in a somewhat trivial way. The point is that our bound on Δ lies abundantly above the line $\Delta = 2d$, and none of the calculable models is significantly above this line. For instance in supersymmetric gauge theories the operators whose dimension is exactly calculable are chiral. In that case the relation $d = 2r/3$ holds, with r representing the R-charge. Then, given a chiral scalar operator ϕ of dimension d, additivity of R-charge implies that ϕ^2 has dimension $2d$, precisely on the $\Delta = 2d$ line, and thus our bound is trivially satisfied.

In the case of large N theories Green's functions factorize at leading $1/N$ order, implying $\Delta = 2d + O(1/N)$. This relation does not provide a stringent test of our result unless $d - 1 < O(1/N)$, which corresponds to an elementary, free, field at leading order in $1/N$. This situation can potentially be realized in variants of the Belavin-Migdal-Banks-Zaks (BZ) fixed point [31],[32]. The simplest case of a non-Abelian gauge theory with matter consisting of charged fermions obviously does not provide a nontrivial check of our bound. This is because the gauge invariant operator of lowest dimension is $\bar{\psi}\psi$ with dimension already close to 3. In order to have a chance to find a model that nearly saturates our bound we must necessarily add a scalar gauge singlet field ϕ to the theory and look for a new BZ fixed point[22]. Consider then a BZ model based on gauge group $SU(N)$ with $N_{\rm F}$ fermionic flavors in the fundamental coupled to ϕ. The Lagrangian is

$$\mathcal{L} = -\frac{1}{4g^2} G_{\mu\nu} G^{\mu\nu} + \bar{q} \not{D} q + y \phi \bar{q} q + \lambda \phi^4 \,. \tag{6.1}$$

By dialing $N_{\rm F}/N = 11/2 - \epsilon$ with $1/N \lesssim \epsilon \ll 1$ the 1-loop β-function is small and is compensated by the 2-loop contribution which comes with opposite sign ($b > 0$)

$$8\pi^2 \mu \frac{d}{d\mu} \frac{1}{g^2} = \frac{2}{3} N \epsilon - b \frac{g^2 N^2}{8\pi^2} = 0 \,, \qquad \frac{g^2 N}{8\pi^2} \sim \epsilon \,. \tag{6.2}$$

One can then easily check that $\beta(y)$ and $\beta(\lambda)$ possess, already at 1-loop order, non-trivial zeroes satisfying $y^2 \sim \lambda$ and $y^2/8\pi^2 \sim \epsilon/(N N_{\rm F}) \sim \epsilon/N^2$. Notice also that, for such value of y^2, its contribution to the 2-loop gauge β-function is subleading in $1/N$, and therefore does not significantly affect the location of the zero of $\beta(g)$. The anomalous dimensions are given by

$$\gamma_\phi \equiv d - 1 = c_1 \frac{y^2 N^2}{8\pi^2} = a_1 \epsilon \,, \tag{6.3}$$

$$\gamma_{\phi^2} \equiv \Delta - 2 = 2\gamma_\phi + c_2 \frac{\lambda}{8\pi^2} = 2a_1 \epsilon + a_2 \frac{\epsilon}{N^2} \,. \tag{6.4}$$

Again our bound is largely satisfied, just because γ_ϕ arises at leading nontrivial order, at 1-loop. Notice that our bound in the small γ_ϕ region is roughly $\gamma_{\phi^2} \leq c\sqrt{\gamma_\phi}$, $c \simeq 1.79$: in order to saturate it, γ_{ϕ^2} and γ_ϕ should respectively arise at 1- and 2-loop order. This is never going to be the case if ϕ has Yukawa couplings to fermions, but it could in principle be so if ϕ is only coupled via quartic scalar couplings, as these do not lead to wave function renormalization at 1-loop. It is however easy to see that also this option does not help us to produce a nontrivial saturation of our bound. Indeed the fixed point condition necessarily implies that ϕ should enter at most linearly in scalar quartic couplings with charged scalars, otherwise the beta function $\beta(\lambda)$ for the self-coupling would be strictly positive. Then even if a fixed point existed, with such limited, just linear, interaction γ_ϕ and γ_{ϕ^2} would vanish at 1-loop. Saturation of our bound would then require γ_{ϕ^2} = 2-loops and γ_ϕ = 4-loops which seems unlikely to happen. Notice also that ϕ^2 will mix with other invariant bilinears constructed with the charged scalars, so the "dimension of ϕ^2" here means the lowest eigenvalue of an in principle complicated matrix of anomalous dimensions.

One possible conclusion from the above discussion is that in order to saturate our bound, even at γ_ϕ near 0, we necessarily need a theory at small N. In 4D we unfortunately have no other examples to play with. One obvious next try is to consider fixed points in $4 - \epsilon$ dimensions. Even if our bound strictly applies only to 4D theories, the comparison to fixed points in $4 - \epsilon$ is almost compulsory. The result, as we now show, is partly encouraging and partly frustrating. Consider the $O(N)$ theory in $4 - \epsilon$ with Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi_a \partial^\mu \phi_a - \frac{\lambda}{4!} (\phi_a \phi_a)^2 \,. \tag{6.5}$$

[22]See Section 5.1 of [17] for a related discussion.

As was first studied in [33], this model has a fixed point at $\lambda(N+8)/48\pi^2 = \epsilon$. There are two operators playing the role of ϕ^2, the singlet $\mathcal{O}_S = \phi_a\phi_a$ and the symmetric traceless tensor $\mathcal{O}_T = \phi_a\phi_b - (1/N)\delta_{ab}(\phi_c\phi_c)$. The computation of anomalous dimension gives [33]

$$d_\phi = d_{\text{free}} + \gamma_\phi = \Big(1 - \frac{\epsilon}{2}\Big) + \frac{N+2}{4(N+8)^2}\epsilon^2\,, \tag{6.6}$$

$$\Delta_S = \Delta_{\text{free}} + \gamma_S = (2-\epsilon) + \frac{N+2}{N+8}\epsilon\,, \tag{6.7}$$

$$\Delta_T = \Delta_{\text{free}} + \gamma_T = (2-\epsilon) + \frac{2}{N+8}\epsilon\,, \tag{6.8}$$

where in brackets we have indicated the free field scaling dimensions in $D = 4-\epsilon$ dimensions, d_{free} and Δ_{free}, for ϕ and ϕ^2 respectively. In analogy, and "naive continuity", with our study in 4D we should compare the anomalous dimension of the composite and elementary fields. Indeed the anomalous dimension of ϕ arises only at two-loops so that we have $\gamma_{T,S} \propto \sqrt{\gamma_\phi}$ like in our bound! One always has $\gamma_T < \gamma_S$ and the most interesting relation is that between γ_T and γ_ϕ

$$\gamma_T = \frac{4}{\sqrt{N+2}}\sqrt{\gamma_\phi} \equiv c_N\sqrt{\gamma_\phi}\,. \tag{6.9}$$

For $N \geq 3$, $c_N < 1.79$ consistent with our 4D bound. On the other hand $c_1, c_2 > 1.79$, above our bound. It is not clear what to make of this apparent contradiction, given that our bound surely applies only to 4D while here we are discussing a theory in $4-\epsilon$. On one side one would be tempted to argue that our bound smoothly extends to $4-\epsilon$, namely

$$\gamma_{\phi^2} \leq c(\epsilon)\sqrt{\gamma_\phi} \tag{6.10}$$

with $c(\epsilon)$ well behaved near $\epsilon = 0$. If our 4D result is correct then this cannot be the case. Instead it is possible that the relation between γ_{ϕ^2} and γ_ϕ away from $\epsilon = 0$ is more complicated than our result. Indeed we can view our 4D result as a bound on $\gamma_{\phi^2}/\sqrt{\gamma_\phi}$ at $\sqrt{\gamma_\phi} \gg \epsilon$. The full result could be

$$\gamma_{\phi^2} \leq \sqrt{\gamma_\phi}A(\sqrt{\gamma_\phi}/\varepsilon), \tag{6.11}$$

where $A(x)$ is a function which interpolates between our coefficient $c \simeq 1.79$ at $x = \infty$, and a larger coefficient at $x = 0$, with a crossover around $x \sim 1$. For instance:

$$A(x) = c + \frac{\delta c}{x^2+1}, \qquad \delta c > 0, \tag{6.12}$$

could do the job.

6.1 Bounds in 2D CFT and comparison with exact results

A wealth of information accumulated about exactly solvable CFTs in 2D [15] allows for a nontrivial check of our method. Much of our discussion in Sections 3,4,5 carries over to 2D with minimal, simplifying, changes. In 2D CFTs, we must make distinction between the global conformal group

$SL(2,C)$, and the infinite-dimensional Virasoro algebra of local conformal trasnformations, of which $SL(2,C)$ is a finite-dimensional subgroup Virasoro algebra plays crucial role in solving these theories exactly, but it has no analogue in 4D, and the results have to be expressed in $SL(2,C)$ terms to allow for comparison.

When we speak about primaries, descendants, conformal blocks in 2D theories, we must specify with respect to which group we define these concepts, Virasoro or $SL(2,C)$. The former is standard in the 2D CFT literature, while it is the latter that is directly analogous to 4D situation.[23]

Every Virasoro primary is a $SL(2,C)$ primary, but the converse is not true. E.g. the stress tensor in any 2D CFT is a Virasoro descendant of the unit operator. To find $SL(2,C)$ primaries, we need to decompose the sequence of all Virasoro descendants of each Virasoro primary (the so called Verma module) into irreducible $SL(2,C)$ representations. While this is possible in principle, it may not be straightforward in practice. Nevertheless we know that $SL(2,C)$ primaries have dimensions of the form

$$\Delta_{SL(2,C)} = \Delta_{\rm Vir} + n, \quad n = 0 \text{ or } n \geq 2,$$

where $\Delta_{\rm Vir}$ is a Virasoro primary dimension, and n is an integer. This is because the Virasoro operators which are not in $SL(2,C)$ raise the dimension by at least 2 units.

The unitarity bound for bosonic fields in 2D is

$$\Delta \geq l,$$

where $l = 0, 1, 2, \ldots$ is the Lorentz spin. The $SL(2,C)$ conformal blocks in 2D were found in [25],[26][24]; in the same coordinates as before we have

$$g_{\Delta,l}(u,v) = \frac{(-)^l}{2^l}\left[f_{\Delta+l}(z) f_{\Delta-l}(\bar{z}) + (z \leftrightarrow \bar{z})\right]. \tag{6.13}$$

Using the unitarity bound, the known conformal blocks, and the sum rule (4.5), which is valid in any dimension, we can try to answer the same question as in 4D. Namely, for a $SL(2,C)$ scalar primary ϕ of dimension d, what is an upper bound on the dimension $\Delta_{\min}$ of the first scalar operator appearing in the OPE $\phi \times \phi$? I.e. we want a 2D analogue of Eq. (1.4). Since the free scalar is dimensionless in 2D, the region of interest is $d \to d_{\rm free} = 0$.

In Fig. 15 we show such a bound on $\Delta_{\min}$ as a function of d obtained using the second derivatives $F^{(2,0)}$, $F^{(0,2)}$ (thus this bound is analogous to the simple 4D bound from Section 5.4). The dependence looks approximately linear:

$$f(d) = f_2^{(2D)}(d) \simeq 0.53 + 4d \qquad \text{(2D, 2nd derivatives)}.$$

Unlike in 4D, this simplest bound does not approach the canonical value zero as $d \to d_{\rm free} = 0$; we do not know if this has any deep meaning. Improvements of this bound using the method discussed in Section 5.5 are possible; see below.

We will now see how the bound of Fig. 15 checks with the known operator dimensions and OPEs in solvable unitary CFTs in 2D. Our first example is the free scalar in 2D. This CFT contains the so called *vertex operator* primaries given by an exponential of the fundamental field:

$$V_\alpha = e^{i\alpha\phi}, \quad [V_\alpha] = \alpha^2. \tag{6.14}$$

[23] $SL(2,C)$ primaries are sometimes called *quasi-primaries* in the 2D CFT literature.

[24] In contrast, explicit expressions for Virasoro conformal blocks are not known in general.

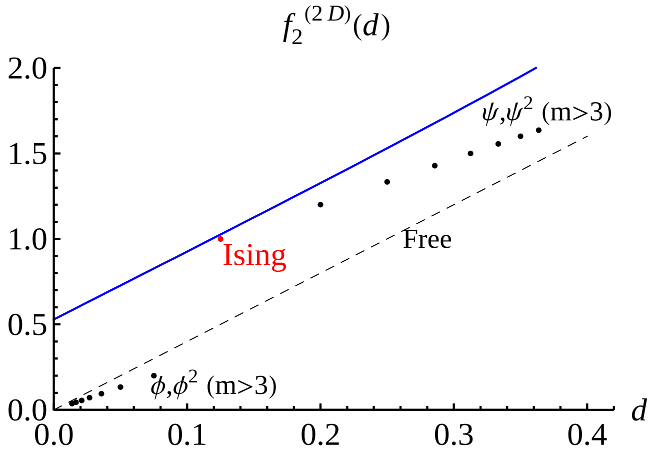


Figure 15: The solid (blue) line represents the simplest upper bound, in an arbitrary 2D CFT, on the dimension $\Delta_{\min}$ of the first scalar in the OPE $\mathcal{O} \times \mathcal{O}$ of a dimension d scalar with itself. The dots show the position of the minimal model OPEs $\phi \times \phi$ and $\psi \times \psi$ (see the text) in this plane. The dashed line corresponds to the free theory OPE (6.14). The bound is respected in all cases.

The basic OPE of V_α with itself has the form:

$$V_\alpha \times V_\alpha = V_{2\alpha} \ .$$

Thus we have $d = \alpha^2$, $\Delta = 4\alpha^2$, which gives the dashed line in Fig. 15,[25] below the bound.

A more interesting example involves the *minimal model* family of exactly solvable 2D CFT. The unitary minimal models (see [15],[34]) are numbered by an integer $m = 3, 4, \ldots$, and describe the universality class of the multicritical Ginzburg-Landau model:

$$\mathcal{L} \sim (\partial\phi)^2 + \lambda\phi^{2m-2} \,. \tag{6.15}$$

For $m = 3$, the Ising model is in the same universality class. The central charge of the model,

$$c = 1 - \frac{6}{m(m-1)},$$

monotonically approaches the free scalar value $c_{\text{free}} = 1$ as $m \to \infty$. Intuitively, as m increases, the potential becomes more and more flat, allows more states near the origin (c grows), and disappears as $m \to \infty$ (free theory).

Minimal models are called so because they have finitely many Virasoro primary fields (the number of $SL(2, C)$ primaries is infinite). All Virasoro primaries are scalar fields $O_{r,s}$ numbered by two integers $1 \leq s \leq r \leq m - 1$, whose dimension is

$$\Delta_{r,s} = \frac{(r + m(r-s))^2 - 1}{2m(m+1)} \,. \tag{6.16}$$

[25]Strictly speaking the bound in Fig. 15 was derived for real fields. However, we can apply it to the real parts which satisfy the OPE $\mathrm{Re}\, V_\alpha \times \mathrm{Re}\, V_\alpha \sim 1 + \mathrm{Re}\, V_{2\alpha}$.

The $O_{1,1}$ is the unit operator ($\Delta_{1,1} = 0$), while the field $\phi \equiv O_{2,2}$ has the smallest dimension among all nontrivial operators:

$$d_\phi = \Delta_{2,2} = \frac{3}{2m(m+1)}. \tag{6.17}$$

This field is identified with the Ginzburg-Landau field in (6.15). For $m = 3$ we have $\Delta_{2,2} = 1/8$, which is the spin field dimension in the Ising model.

It is convenient to extend the Virasoro primary fields to a larger range $1 \leq r \leq m-1$, $1 \leq s \leq m$, subject to the identification

$$(r, s) \leftrightarrow (m - r, m + 1 - s)\,. \tag{6.18}$$

The *fusion rules*, which say which operators appear in the OPE $O_{r_1 s_1} \times O_{r_2 s_2}$ (but do not specify the coefficients) can now be written in a relatively compact form:

$$\begin{aligned} &O_{r_1 s_1} \times O_{r_2 s_2} \sim \sum O_{r,s} \\ &r = |r_1 - r_2| + 1, |r_1 - r_2| + 3, \ldots \min(r_1 + r_2 - 1, 2m - 1 - r_1 - r_2) \\ &s = |s_1 - s_2| + 1, |s_1 - s_2| + 3, \ldots \min(s_1 + s_2 - 1, 2m + 1 - s_1 - s_2) \end{aligned} \tag{6.19}$$

For any m, the fusion rules respect a discrete Z_2 symmetry

$$O_{r,s} \to \pm O_{r,s}, \tag{6.20}$$

where $\pm = (-1)^{s-1}$ for m odd, $(-1)^{r-1}$ for m even (this choice is dictated by consistency with (6.18)). This symmetry corresponds to the $\phi \to -\phi$ symmetry of the Ginzburg-Landau model; in particular $\phi = O_{2,2}$ is odd under (6.20).

We are interested in OPEs of the form $O \times O \sim 1 + \tilde{O} + \ldots$ where both O and $\tilde{O}$ have small dimensions. Two such interesting OPEs are

$$\phi \times \phi \sim 1 + \phi^2 + \ldots \tag{6.21}$$

$$\psi \times \psi \sim 1 + \psi^2 + \ldots, \quad \psi \equiv O_{1,2}, \quad d_\psi = \frac{1}{2} - \frac{3}{2(m+1)}. \tag{6.22}$$

Here ϕ^2 and ψ^2 are just notation for the lowest dimension operators appearing in the RHS. Note that for $m = 3$ we have $\psi \equiv \phi$ via (6.18). Using the fusion rule (6.19) and the operator dimensions (6.16) it is not difficult to make identification:

$$m = 3: \quad \phi^2 \equiv O_{1,3},\ \Delta_{\phi^2} = 1, \quad \text{(Ising)} \tag{6.23}$$

$$\begin{aligned} m > 3: \quad &\phi^2 \equiv O_{3,3},\ \Delta_{\phi^2} = \frac{4}{m(m+1)}\,, \\ &\psi^2 \equiv O_{1,3},\ \Delta_{\psi^2} = 2 - \frac{4}{m+1}. \end{aligned} \tag{6.24}$$

In particular, we see that the $\psi \times \psi$ OPE does not contain ϕ^2, which is precisely the reason why we are considering it[26].

[26] In general, ϕ^2 does not appear in the OPE $O_{r,s} \times O_{r,s}$ for $r = 1$ or $s = 1$. The operators ψ has the lowest dimension among all these fields.

We are now ready for the check. Operator dimensions in both OPEs (6.21) and (6.22) are subject to the bound of Fig. 15, where we marked the corresponding points up to $m = 10$. We see that the bound is respected in all the cases, although the Ising model point lies remarkably close to the boundary. We have tried to improve the bound of Fig. 15 by including more derivatives in the functional, according to the general method described in Section 5.5. Although we have seen some improvement for lower valued of d, there was practically no improvement around the Ising spin dimension $d = 1/8$, so that also the improved bound was respected. One could wonder if the fact that the Ising model (almost) saturates the bound has any special significance.

We have searched for other exactly solvable 2D CFTs which could provide checks of our 2D bound. E.g. some interesting OPEs can be extracted from the WZNW models. However, as far as we could see, none of them come close to saturating the bound.

In conclusion, we would like to mention that some bounds for dimensions of operators appearing in the OPE of two primaries in 2D CFTs were derived in the past by Lewellen [37] and Christe and Ravanini [38]. Those bounds were however of a different nature than our bound (1.4). Roughly, the Lewellen-Christe-Ravanini (LCR) bounds say that *IF* a primary appears in the OPE, its dimension is not bigger than a certain bound. This is of course not the same as our result, which says that a certain primary *MUST* be present in the OPE, with the dimension not bigger than a certain bound.

The methods of LCR are based on studying the monodromy of the conformal blocks near their singularities in the complex plane, as opposed to the more detailed information about the shape and size of conformal blocks at intermediate *regular* points used by us. They have to make a crucial assumption that only a finite number of singularity types exist, which means that the total number N of primaries appearing in the OPE, or at least the total number of such primaries having different $\Delta \bmod 1$, is finite. This assumption is realized in Rational CFTs, but not in general. The LCR bounds become increasingly weak for large N and disappear in the limit $N \to \infty$. Thus it is doubtful that such methods could be useful in our problem, since we would like to be free of any assumptions about the spectrum of higher primaries.

7 Comparison to phenomenology

In this section we will comment on the precise relation of our main result (1.4) with the phenomenological discussion of Section 2. That discussion led to the constraints (2.21), (2.22) on the dimension d of the Higgs field operator H and on the dimension Δ_S of the first *singlet* in the $H \times H^\dagger$ OPE, denoted $H^\dagger H$. Are there any low-dimension non-singlets in this OPE?

The standard considerations related to the ρ-parameter lead us to assume the "custodial" $SO(4) = SU(2)_L \times SU(2)_R$ as the global symmetry of the CFT [27]. The real components h_a of the Higgs field:

$$H = \begin{pmatrix} h_1 + ih_2 \\ h_3 + ih_4 \end{pmatrix},$$

form a multiplet of primary scalars in the fundamental of $SO(4)$. Their basic OPE will in general

[27] Our conclusions would however not be affected by assuming just $SU(2) \times U(1)$.

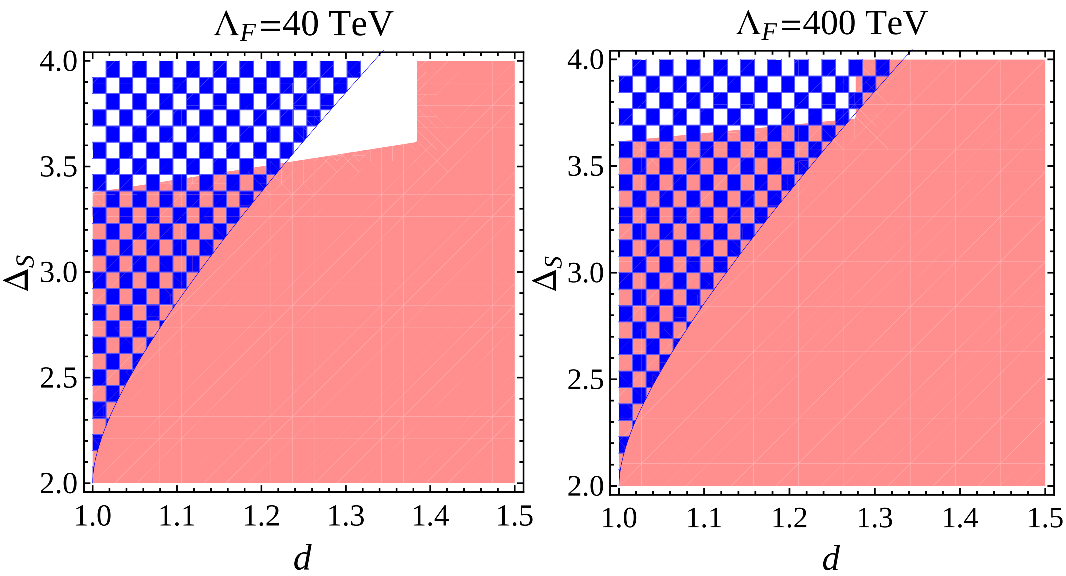


Figure 16: *Solid red*: The region of (d, Δ_S) plane disfavored by phenomenological considerations of Section 2. In this region one or both constraints (2.21), (2.22) are not satisfied. We consider two cases: $\Lambda_{\rm F} = 40$ TeV (left) and $\Lambda_{\rm F} = 400$ TeV (right). *Checkered blue*: the region of the same plane which *would be* excluded by our bound (1.4) with $f = f_6$ if we knew that $\Delta_S < \Delta_T$.

have the structure

$$h_a(x)h_b(0) \sim \frac{1}{|x|^{2d}} \left(\delta_{ab}1 + C_S|x|^{\Delta_S}\delta_{ab}\,(H^\dagger H)(0) + C_T|x|^{\Delta_T}\mathcal{T}_{(ab)}(0) + C_J|x|^2 x^\mu J_\mu^{[ab]}(0) + \ldots\right) . \tag{7.1}$$

Here we indicated the possibility for two symmetry structures in the even-spin sector: along with the $SO(4)$ singlet $H^\dagger H$, a scalar $\mathcal{T}_{ab}$ transforming as a *traceless symmetric* tensor, *i.e.* the $(1,1)$ representation, will in general be present. E.g. in free theory $\mathcal{T}_{ab} = h_a h_b - (1/4)\delta_{ab}\,(h_c h_c)$. On the other hand, in the odd-spin sector the first contribution will be associated with the conserved, dimension 3, $SO(4)$ current $J_\mu^{[ab]}$, which is in the *antisymmetric* tensor representation $(1,0)+(0,1)$. Its coefficient C_J will be related to the normalization of the $SO(4)$ Ward identity.

Clearly (7.1) looks more complicated than (1.3). However, if we choose $a = b = 1$ (say) in (7.1), the current and other odd-spin fields drop out due to antisymmetry, and we get an OPE of the form (1.3). We just need to identify $\phi \equiv h_1$, $\phi^2 \equiv H^\dagger H$ or $\mathcal{T}_{11}$ depending on which of these two scalars has smaller dimension. Eq. (1.4) then implies

$$\min(\Delta_S, \Delta_T) \leq f(d)\,.$$

We get a bound on Δ_S only assuming $\Delta_S < \Delta_T$. This is not satisfactory; in fact the reverse $\Delta_S > \Delta_T$ seems likely (as it happens in the Wilson-Fisher fixed points in $4-\varepsilon$ dimensions, see Section 6). However, we have reasons to believe (or hope) that our bound on $\min(\Delta_S, \Delta_T)$ is close to being saturated by some existing 4D CFT. One reason is that the truncation discussed in Section 5.6 shows clear signs of convergence. The other reason is that in $4-\epsilon$ we do have

$\gamma_{\phi^2} \sim \sqrt{\gamma_\phi}$ as dictated by our bound at small γ_ϕ. Taking this into consideration it makes sense to compare our bound $f(d)$ to the phenomenological constraints on Δ_S. We have done that in Fig. 16 assuming two different constraints on the scale Λ_{F}, a weaker one $\Lambda_{\mathrm{F}} > 40$ TeV and a robust one $\Lambda_{\mathrm{F}} > 400$ TeV. According to the previous reasoning, even for the less likely situation $\Delta_S < \Delta_T$, these plots indicate that there exists space for relaxing the flavor problem. The optimal value for the dimension of the Higgs field turns out to be between 1.2 and 1.3. Is it possible that a situation like this will be realized in nature? Hints of the answer to this question may soon come with the LHC.

In the meantime, we believe that it should be possible to disentangle the contributions of $H^\dagger H$ and $\mathcal{T}$ in (7.1) and obtain a bound on Δ_S free of the assumption that $\Delta_S < \Delta_T$. As the above discussion clearly shows, such a bound cannot be found by considering only *diagonal* $\phi \times \phi$ OPEs where ϕ is a fixed h_a component. One should try to use additional information contained in the *nondiagonal* OPEs with $a \neq b$, something which we did not do in this paper. As it is apparent from (7.1), the global symmetry current and higher odd-spin operators will generally contribute to these nondiagonal OPEs. More work is needed to determine if the contributions of the odd-spin operators can be controlled in a model-independent way.

8 Discussion and Outlook

In this paper we have shown that prime principles of Conformal Field Theory, such as unitarity, OPE, and conformal block decomposition, imply the existence of an upper bound $f(d)$ on the dimension $\Delta_{\min}$ of the first scalar operator ϕ^2 in the OPE of a scalar ϕ of a given dimension d.

We developed a method which allows numerical determination of $f(d)$ with arbitrary desired accuracy. The method is based on the *sum rule*, a function-space identity satisfied by the conformal block decomposition of the 4-point function $\langle\phi\phi\phi\phi\rangle$, which follows from the crossing symmetry constraints. In practical application of the method the sum rule is Taylor-expanded: replaced by finitely many equations for the derivatives. The bound $f(d)$ improves monotonically as more and more derivatives are included; see Figs. 1, 10 for the best current bound obtained using derivatives up to the 6th order, and a sequence of weaker bounds obtained using fewer derivatives.

We have checked that our bound is satisfied, by a large margin, in all weakly coupled 4D CFTs that we are able to construct. We have also derived an analogous bound in 2D and checked it against exact 2D CFT results. Again, the bound is satisfied, and in a less trivial way than in 4D, since the Ising model almost saturates it.

Our results open up several interesting research directions:

1. It should be relatively straightforward to improve our bounds, both in 4D and in 2D, using our method but including more derivatives in the analysis. These improved bounds should monotonically converge to the optimal bound, corresponding to the infinite number of derivatives; we can already see signs of such convergence in Fig. 10.
2. One should search for more examples of CFTs which come close to saturating the bound, especially in 4D.
3. The Dolan-Osborn closed-form expressions for conformal blocks are available only in even dimensions (up to $D = 6$). It is important to find expressions in 3D, comparable in simplicity

to (3.15) and (6.13). Then one could derive an analogous bound in 3D and confront it with the operator dimensions of known 3D CFTs, such as the $O(N)$ universality classes. Although these theories are not exactly solvable, rather precise estimates for critical exponents and operator dimensions have been obtained using ε-expansion, high-temperature expansion, and Monte-Carlo simulations [35]. For example, in 3D Ising model we have $\gamma_\phi = 0.0183(4)$ and $\gamma_{\phi^2} = 0.412(1)$ [36].

4. It would be interesting to understand what is the appropriate extrapolation of our bound to $4-\varepsilon$ dimension. This should explain why the comparison with Wilson-Fischer fixed points in Section 6 was not perfect.

5. A very important but difficult problem is to find a genuine generalization of our bound to the situation when the CFT has a global symmetry. The case of $SO(4)$ symmetry, readily generalized to $SO(N)$, has been discussed in Section 7.

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A Reality property of Euclidean 3-point functions

In this appendix we would like to briefly discuss the reality property of 3-point functions, which was used at some point in our discussion. First of all, one can always choose a basis where each operator corresponds to an hermitian operator in the Minkowski space description. We work in such a basis. The reality properties of the Euclidean n-point functions for such operators are quickly deduced by analitic continuation of the Minkowskian correlators at space-like separation. Consider indeed the 3-point function

$$G^{\alpha_1,\alpha_2,\alpha_3}(x_1,x_2,x_3) \equiv \langle O_1^{\alpha_1}(x_1) O_2^{\alpha_2}(x_2) O_3^{\alpha_3}(x_3)\rangle\,, \tag{A.1}$$

where α_i collectively denote the spin indices (we only consider bosons). When x_{12}, x_{23} and x_{31} are spacelike, the operators commute by causality, thus implying that G is real

$$G^{\alpha_1,\alpha_2,\alpha_3}(x_1,x_2,x_3)^* = \langle O_3^{\alpha_3}(x_3) O_2^{\alpha_2}(x_2) O_1^{\alpha_1}(x_1)\rangle = \langle O_1^{\alpha_1}(x_1) O_2^{\alpha_2}(x_2) O_3^{\alpha_3}(x_3)\rangle = G^{\alpha_1,\alpha_2,\alpha_3}(x_1,x_2,x_3)\,. \tag{A.2}$$

Continuation to the Euclidean then amounts to

$$x^0 \to -i x_E^0 \qquad x^k \to x_E^k \quad (k=1,2,3)\,, \tag{A.3}$$

$$O^\alpha \to (-i)^{n_\alpha} O_E^\alpha\,, \tag{A.4}$$

where n_α is the number of 0 indices in $\{\alpha\}$. Using the above rules, analytically continuing $G^{\alpha_1,\alpha_2,\alpha_3}$ from the spacelike patch we find the Euclidean functions

$$G_E^{\alpha_1,\alpha_2,\alpha_3}(x_{E1},x_{E2},x_{E3}) = (i)^{n_{\alpha_1}+n_{\alpha_2}+n_{\alpha_3}} G^{\alpha_1,\alpha_2,\alpha_3}(-ix^0_{E1},x^k_{E1};-ix^0_{E2},x^k_{E2};-ix^0_{E3},x^k_{E3})\,. \qquad (A.5)$$

Now, by Lorentz and translation invariance $G^{\alpha_1,\alpha_2,\alpha_3}(x_1,x_2,x_3)$ depends on the invariants x_{ij}^2, while the tensor indices are covariantly reproduced by combinations of x_{ij}^μ with the invariant tensors $\eta^{\mu\nu}$ and $\epsilon^{\mu\nu\rho\sigma}$. It is now evident that if $\epsilon^{\mu\nu\rho\sigma}$ does not appear in $G^{\alpha_1,\alpha_2,\alpha_3}$ then the factor $(i)^{n_{\alpha_1}+n_{\alpha_2}+n_{\alpha_3}}$ in eq. (A.5) is exactly compensated by a factor $(-i)^{n_{\alpha_1}+n_{\alpha_2}+n_{\alpha_3}}$ from the coordinate dependence of the tensor structure. In such a situation $G_E^{\alpha_1,\alpha_2,\alpha_3}$ is therefore real. On the other hand, contributions to G_E that are proportional to one (equivalently an odd number) power of $\epsilon^{\mu\nu\rho\sigma}$ are pure imaginary. For the case that interests us in which two operators, say O_1 and O_2 have zero spin and $O_3 \equiv O_3^{\mu_1 \ldots \mu_j}$, the tensor structure of $G^{\mu_1\ldots\mu_j}(x_1,x_2,x_3)$ can only involve $x_{12}^{\mu_k}$, $x_{23}^{\mu_k}$ and $\eta^{\mu\nu}$ and thus the corresponding euclidean function must be real. On the other hand the three point function for vector fields admits in conformal field theory a contribution proportional to $\epsilon^{\mu\nu\rho\sigma}$ which is precisely proportional to the triangle anomaly diagram [39]. Therefore the euclidean 3-point function for vector fields in CFT is in general complex. To conclude we notice that one can simply reproduce the result just discussed by formally assigning the following transformation property to the invariant tensors

$$\eta_{\mu\nu} \to \delta_{\mu\nu} \qquad\qquad \epsilon_{\mu\nu\rho\sigma} \to -i\epsilon_{\mu\nu\rho\sigma}\,. \qquad (A.6)$$

B Closed-form expressions for conformal blocks

The Dolan-Osborn result (3.15) is crucial for us, and we would like to say a few words about how it is derived, following [26][28]. The main idea is that the conformal block is, in a certain sense, a spherical harmonic of the conformal group. In particular, it satisfies an eigenvalue equation

$$\mathcal{D}_{x_1,x_2}\mathtt{CB}_\mathcal{O} = -c_{\Delta,l}\mathtt{CB}_\mathcal{O}\,, \quad c_{\Delta,l} = l(l+2)+\Delta(\Delta-4), \qquad (B.1)$$

where $\mathcal{D}$ is a second-order partial differential operator acting on the coordinates $x_{1,2}$. This operator encodes the action of the quadratic Casimir operator of the conformal group[29],

$$C = \frac{1}{2}M_{\mu\nu}M_{\mu\nu} - D^2 - \frac{1}{2}(P_\mu K_\mu + K_\mu P_\mu) \equiv L_A L_A,$$

on the state $\phi_1(x_1)\phi_2(x_2)|0\rangle$. The defining equation is

$$C\cdot\phi_1(x_1)\phi_2(x_2) \equiv [L_A,[L_A,\phi_1(x_1)\phi_2(x_2)]] = \mathcal{D}_{x_1,x_2}\phi_1(x_1)\phi_2(x_2).$$

On the other hand, $c_{\Delta,l}$ in (B.1) are nothing but the eigenvalues of the Casimir acting on conformal primaries [12]:

$$C\cdot\mathcal{O}_{(\mu)}(0) = -c_{\Delta,l}\mathcal{O}_{(\mu)}(0).$$

[28] The first derivation [25] is a brute force resummation of contributions of all conformal descendants of $\mathcal{O}$ and is not particularly enlightening.

[29] Here and in (B.2) we use the sign conventions of [14] for the conformal generators.

Since C is a Casimir, the same eigenvalue equation is simultaneously satisfied for all descendants of $\mathcal{O}$. Thus Eq. (B.1) is a consequence of the OPE (3.9) and of the definition of conformal blocks.

The explicit form of $\mathcal{D}_{x_1,x_2}$ can be found using the known expressions for the action of conformal generators on scalar primaries:

$$\begin{aligned}
[P_\mu, \phi(x)] &= i\partial_\mu\phi\,, \\
[D, \phi(x)] &= i(\Delta_\phi + x^\mu\partial_\mu)\phi\,, \\
[M_{\mu\nu}, \phi(x)] &= i(x_\mu\partial_\nu - x_\nu\partial_\mu)\phi\,, \\
[K_\mu, \phi(x)] &= i(x^2\partial_\mu - 2x_\mu x\cdot\partial - 2x_\mu\Delta_\phi)\phi\,.
\end{aligned} \tag{B.2}$$

Eq. (B.1) can then be rewritten as a differential equation for $g_{\Delta,l}(u,v)$. The clever change of variables $u, v \to z, \bar{z}$ performed in [26] allows to find explicit solutions. In these variables the differential equation takes the form

$$\begin{aligned}
\mathcal{D}_{z,\bar{z}} g_{\Delta,l} &= \frac{1}{2} c_{\Delta,l} g_{\Delta,l}, \\
\mathcal{D}_{z.\bar{z}} = z^2(1-z)\partial_z^2 + \bar{z}^2(1-\bar{z})\partial_{\bar{z}}^2 &- \left(z^2\partial_z + \bar{z}^2\partial_{\bar{z}}\right) + \frac{2z\bar{z}}{\bar{z}-z}\left[(1-z)\partial_z - (1-\bar{z})\partial_{\bar{z}}\right]\,.
\end{aligned}$$

The OPE fixes the asymptotic behavior:

$$g_{\Delta,l} \sim \frac{(-)^l}{2^l}\frac{(z\bar{z})^{\frac{\Delta-l}{2}}}{z-\bar{z}}(z^{l+1} - \bar{z}^{l+1}), \quad (z, \bar{z} \to 0).$$

With these boundary conditions the solution is unique and is given by (3.15).

C z and $\bar{z}$

Here we comment upon the ranges of the variables z and $\bar{z}$ introduced in Section 3.4. Since z and $\bar{z}$ are functions of the conformally-invariant cross-ratios, we can use conformal symmetry to fix some of the coordinate freedom. For instance, we can put three out of four points along a straight line, and moreover send one of them to infinity, as in Eq. (3.17). After that there still remains freedom to perform rotations leaving this line invariant, so that we can fix

$$x_2 = (X_1, 0, 0, X_4)\,. \tag{C.1}$$

Now it becomes trivial to compute the cross-ratios:

$$\begin{aligned}
u &= x_{12}^2 = X_1^2 + X_4^2\,, \\
v &= x_{23}^2 = (X_1 - 1)^2 + X_4^2\,.
\end{aligned}$$

Moreover, we can easily solve Eq. (3.16) for z and $\bar{z}$:

$$z = X_1 + iX_4, \quad \bar{z} = z^*\,. \tag{C.2}$$

Thus we conclude that in the Euclidean, $\bar{z}$ is always the complex conjugate of z. Moreover, z is real if and only if all four points lie on a circle (which is a conformal image of the straight line in the above parametrization, see Fig. 17).

Finally, Eq. (3.18),(3.19) are obtained from (C.1),(C.2) by Wick-rotating to the Minkowski time.

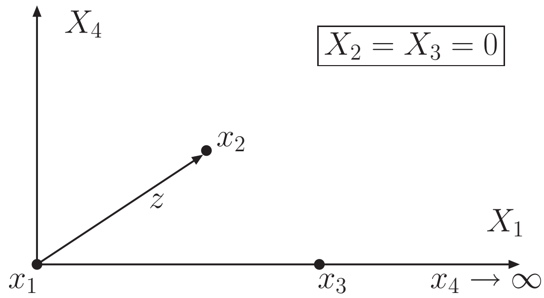

Figure 17: In Euclidean, $\bar{z} = z^*$. For the configuration of points chosen in this figure, $z = X_1 + iX_4$ is the complex coordinate of x_2 in the $X_1 - X_4$ plane.

D Asymptotic behavior

In this Appendix we find large l and Δ asymptotics of derivatives of $F_{d,\Delta,l}$ at $a = b = 0$. It is useful to rewrite the definition of $F_{d,\Delta,l}$ as follows:

$$F_{d,\Delta,l}(a,b) = h_d(a,b)\frac{\tilde{g}_{d,\Delta,l}(a,b) - \tilde{g}_{d,\Delta,l}(-a,b)}{a}\,, \tag{D.1}$$

where we introduced the functions

$$\begin{aligned}\tilde{g}_{d,\Delta,l} &\equiv [(1-z)(1-\bar{z})]^d g_{\Delta,l}\,,\\ h_d(a,b) &\equiv \frac{a}{(z\bar{z})^d - [(1-z)(1-\bar{z})]^d}\,.\end{aligned}$$

These functions are smooth in the spacelike diamond. Moreover, it is not difficult to see that

$$\tilde{g}(a,-b) = \tilde{g}(a,b), \quad h_d(\pm a, \pm b) = h_d(a,b)\,.$$

In particular, from (D.1) we see the property (4.6) from Section 4.

Let us introduce the parameter

$$\delta \equiv \Delta - l - 2.$$

As we will see below, there are three relevant asymptotic limits to consider:

- l large, $\delta = O(1)$;
- l large, δ large, $\delta \ll l^2$;
- δ large, $\delta \gg l^2$.

In all these cases the large asymptotic behavior of derivatives will come from differentiating $g_{\Delta,l}$, which we write in the form

$$g_{\Delta,l} = const(-)^l \frac{z\bar{z}}{b} \left[k_{2l+\delta+2}(z) k_\delta(\bar{z}) - (b \to -b) \right] . \tag{D.2}$$

In this Appendix by *const* we denote various *positive* constants which may depend on d, δ or l but are independent of the derivative order $\partial_a^{2m}\partial_b^{2n}$. These constant factors are irrelevant for controlling the positivity of the linear functionals defined on the cones.

Starting from the following integral representation for the hypergeometric function (see [40])

$${}_2F_1\left(a,b,c;x\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^\infty e^{-bt}(1-e^{-t})^{c-b-1}(1-x\,e^{-t})^{-a}\,dt \quad (\mathrm{Re}\;c > \mathrm{Re}\;b > 0)$$

and using the steepest descent method, we derive the large β asymptotics:

$$\begin{aligned} k_\beta(x) \;&=\; e^{(\beta/2)h(x)} \left[q(x) + O(1/\beta) \right] , \\ h(x) \;&=\; \ln\left(\frac{4(1-\sqrt{1-x}\,)^2}{x} \right), \quad q(x) \;=\; \frac{x}{2\left(1-\sqrt{1-x}\right)\,\sqrt[4]{1-x}} \,. \end{aligned}$$

The leading asymptotic behavior appears when all the derivatives fall on the exponential factors in f_β containing large exponents. Various prefactors appearing in (D.1) and (D.2) are not differentiated in the leading asymptotics. However, the a^{-1} and b^{-1} factors are responsible for changing the order of the needed derivative, as follows:

$$\begin{aligned} F_{d,\Delta,l}^{(2m,2n)} &\sim \frac{const}{2m+1} (g_{\Delta,l})^{(2m+1,2n)} \\ &\sim \frac{const(-)^l}{(2m+1)(2n+1)} (\exp A)^{(2m+1,2n+1)} \,, \\ A &= (l+\delta/2)h(1/2+a+b) + (\delta/2)h(1/2+a-b) \,. \end{aligned}$$

To find the leading asymptotics, we expand A near $a = b = 0$:

$$\begin{aligned} A &= (l+\delta)[h(1/2) + a\,h'(1/2)] + lb\,h'(1/2) + (\delta/2)b^2 h''(1/2) + \ldots, \\ h'(1/2) &= 2\sqrt{2}, \quad h''(1/2) = -2\sqrt{2}. \end{aligned} \tag{D.3}$$

In the case $\delta \ll l^2$ the last term in (D.3) plays no role, and we get:

$$F_{d,\Delta,l}^{(2m,2n)} \sim \frac{const(-)^l}{(2m+1)(2n+1)} [h'(1/2)(l+\delta)]^{2m+1} [h'(1/2)l]^{2n+1}, \quad \delta \ll l^2 \,. \tag{D.4}$$

This asymptotic is applicable for l large, while δ can be small or large, as long as the condition $\delta \ll l^2$ is satisfied; i.e. it covers the first two cases mentioned above. If on the other hand $\delta \gg l^2$, it is the last term in (D.3) which determined the asymptotics of b-derivatives, and we get

$$F^{(2m,2n)}_{d,\Delta,l} \sim const(-)^l \frac{(2n-1)!!}{2m+1}[h'(1/2)\delta]^{2m+1}[h''(1/2)\delta]^n, \quad \delta \gg l^2\,.$$

Because $h''(1/2) < 0$, the last asymptotics changes sign depending on the parity of n.

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