ANNALS OF PHYSICS: 53, 174–202 (1969)

Finite-Component Field Representations of the Conformal Group

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We review work done on realization of broken symmetry under the conformal group of space-time in the framework of finite-component field theory. Topics discussed include: Most general transformation law of fields over Minkowski space. Consistent formulation of an orderly broken conformal symmetry in the framework of Lagrangian field theory; algebra of currents and their divergences; Manifestly conformally covariant fields and their couplings.

I. INTRODUCTION

The conformal symmetry of space time as a possible generalization of Poincaré symmetry has provided a recurrent theme in particle physics.[1] The problems associated with conformal symmetry are (i) its physical interpretation and (ii) the problems arising from its broken character and the precise manner of descent to Poincaré invariance.

In this paper we wish to concentrate on (ii) and present work done on realization of conformal symmetry—and in particular of the algebra associated with the group—using field operators which satisfy Lagrangian equations of motion. The fields may be defined over Minkowski space–time manifold x_μ or over a projective six-dimensional manifold η_A related to x_μ. We believe this approach to conformal symmetry offers the best hope of exploiting the symmetry physically in contrast to approaches based on a group theoretic treatment of state vector spaces associated with the group.[2] This is essentially because in the latter approach it would be much more difficult to see how to break the symmetry down to Poincaré invariance.

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[1] A historical survey may be found in Refs. (*1–3*).

[2] Unitary irreducible representations of $SU(2, 2)$ ($\approx O(2, 4)$) have been constructed in Refs. (*4–10*). The problem of reduction with respect to the Poincaré subgroup has been considered in Refs. (*11–15*) for few special cases.

The plan of the paper is as follows: In Section II we give the most general transformation law of fields (defined over Minkowski space x_μ) for conformal symmetry, using the theory of induced representations.

In Section III we enumerate the Lagrangians (for particles of spin $\leqslant 1$) which are conformally invariant and describe some modes of symmetry breaking—in particular the physically interesting case of conformal symmetry breaking to the extent of breaking dilatation invariance only. A current algebra formulation of broken dilatation symmetry has recently been found useful and gave rise to sucessful predictions for multimeson production (*16*, *17*). Here we extend this scheme to the full conformal group, considering it as a broken symmetry in the same sense as chiral $SU(3) \otimes SU(3)$. For an explicit Lagrangian model see Appendix.

Even where the formalism of Sections II and III is conformally covariant, it is not manifestly so. In Section IV we treat the manifestly covariant formulation of wave equations for quantized fields defined over a six-dimensional projective space.

In Section V we review an attempt to understand $V - A$ or $V + A$ weak interaction theory as a conformally invariant theory of fundamental interactions. Not considered in this paper are representations of the conformal group which give rise to infinite component fields. They will be discussed elsewhere (*18*).

The reader who is mainly interested in a discussion of possible physical interpretations of the conformal group is referred to Refs. (*19–25*). Mention should also be made of work of Ref. (*26*) which is somewhat different in spirit.

Without further discussion of the usefulness and consistency of possible different physical interpretations, we adopt for the purpose of the present paper the following point of view, following essentially Kastrup (*2*, *24*)

1. Exact dilatation symmetry would imply that to every physical system in any given state *sub specie aeternitatis* and for arbitrary $\rho > 0$ another one exists which is realizable in nature and differs from the first one only in that every physical observable is changed by a factor ρ^l; where l is the dimension of length of the observable in question.

2. Special conformal transformations may be interpreted as space–time dependent dilatations.

3. We consider only infinitesimal transformations. Then no problem arises with causality; in particular the local equal time commutation relations (C.R.) of fields will be invariant. (Section III.)

II. TRANSFORMATION LAW OF FIELDS

The conformal group of space time consists of coordinate transformations as follows:

1. Dilatations

$$x'_\mu = \rho x_\mu , \qquad \rho > 0$$

2. Special conformal transformations

$$x'_\mu = \sigma^{-1}(x)(x_\mu - c_\mu x^2) \tag{II.1}$$

where

$$\sigma(x) = 1 - 2cx + c^2x^2$$

3. Inhomogeneous Lorentz transformations

The generators D of dilatations, K_μ of special conformal transformations, and P_μ , $M_{\mu\nu}$ of the Poincaré group admit of the following commutation relations (C.R.):

$$\begin{array}{ll} [D, P_\mu] = -iP_\mu & [D, M_{\mu\nu}] = 0 \\ [D, K_\mu] = iK_\mu & [K_\mu , K_\nu] = 0 \\ [K_\mu , P_\nu] = -2i(g_{\mu\nu}D + M_{\mu\nu}) & [K_\rho , M_{\mu\nu}] = i(g_{\rho\mu}K_\nu - g_{\rho\nu}K_\mu) \end{array} \tag{II.2}$$

plus those of the Poincaré algebra. Parity must satisfy

$$\Pi D\Pi^{-1} = D, \qquad \Pi K_\mu \Pi^{-1} = \pm K_\mu , \qquad \Pi P_\mu \Pi^{-1} = \pm P_\mu \tag{II.3}$$

where the $+$ sign stands for $\mu = 0$, and the $-$ sign for $\mu = 1, 2, 3$.

The C.R. (II.2) can be brought into a form which exhibits the $O(2,4)$ structure of the conformal group explicitly by defining, for $\mu, \nu = 0 \cdots 3$,

$$J_{\mu\nu} = M_{\mu\nu} , \qquad J_{65} = D, \qquad J_{5\mu} = \tfrac{1}{2}(P_\mu - K_\mu), \qquad J_{6\mu} = \tfrac{1}{2}(P_\mu + K_\mu).$$

Then

$$[J_{\mathrm{KL}} , J_{\mathrm{MN}}] = i(g_{\mathrm{KN}}J_{\mathrm{LM}} + g_{\mathrm{LM}}J_{\mathrm{KN}} - g_{\mathrm{KM}}J_{\mathrm{LN}} - g_{\mathrm{LN}}J_{\mathrm{KM}}) \tag{II.4}$$

where

$$g_{AA} = (+---, -+), \qquad A = 0, 1, 2, 3, 5, 6.$$

Note that the special conformal transformations do not take momentum eigenstates into momentum eigenstates, as $[K_\mu , P_\nu]$ does not commute with the momenta P_ρ . We also notice the relation (*27*)

$$e^{i\alpha D}P^2e^{-i\alpha D} = e^{2\alpha}P^2.$$

Because of this relation, exact dilatation symmetry (with an integrable generator D that takes one-particle states into one-particle states) *implies that the mass spectrum* *is either continuous or all masses are zero.* This clearly implies that exact dilatation symmetry is physically unacceptable and one will therefore have to make assumptions on the dynamics which specify how the conformal symmetry is broken. A theory of this type, which is in a sense analogous to the $SU(3) \otimes SU(3)$ current algebra with PCAC will be presented in the next section.

First we have to define, however, what we mean by an (infinitesimal) dilatation or special conformal transformation, as we want it to transform a physical system into another one that is realizable in nature (and not, e.g., a proton with mass m into some nonexistent particle with mass $\rho^{-1}m$, for arbitrary $\rho > 0$).

To do this we postulate that there exist interpolating fields to every particle which transform according to a representation of the conformal algebra, i.e.

$$(T(g)\,\varphi)_\alpha\,(x) = S_{\alpha\beta}(g, x)\,\varphi_\beta(g^{-1}x) \qquad \text{for infinitesimal} \quad g \in o\,(2, 4) \tag{II.5}$$

where g acts on the coordinates x as indicated in Eq. (II.1).

It follows from Eq. (II.5) and the multiplication law for the representation matrices $T(g)$ that

$S(g, 0)$ *must be a representation of the stability subgroup of* $x = 0$.

It is seen from Eq. (II.1) that this subgroup (the little group in physical usage) which leaves $x = 0$ invariant is given by special conformal transformations, dilatations and homogeneous Lorentz transformations. From the C.R. Eq. (II.2) one finds that the Lie algebra of this subgroup is isomorphic to a Poincaré algebra + dilatations, i.e. we have

$$(SO(3,1) \otimes \{D\}) \mathbin{\circledS} T_4\,. \tag{II.6}$$

The 4-dimensional translation subgroup T_4 corresponds to the special conformal transformations, and $SO(3,1)$ is the spin part of the Lorentz group.

Given any representation $S(g, 0)$ of the little group (II.6) we can now determine, in accordance with the standard theory of induced representations (*28*), the complete action of the generators of the conformal group on the field $\varphi(x)$ as follows:

Let $\Sigma_{\mu\nu}$, Δ, κ_μ be the infinitesimal generators of the little group (II.6) corresponding to Lorentz transformations, dilatations, and special conformal transformations, respectively. They satisfy

$$\begin{aligned} [\kappa_\mu\,, \kappa_\nu] &= 0, \qquad [\Delta, \kappa_\mu] = +i\kappa_\mu \\ [\kappa_\rho\,, \Sigma_{\mu\nu}] &= i(g_{\rho\mu}\kappa_\nu - g_{\rho\nu}\kappa_\mu), \\ [\Sigma_{\rho\sigma}\,, \Sigma_{\mu\nu}] &= i(g_{\sigma\mu}\Sigma_{\rho\nu} - g_{\rho\mu}\Sigma_{\sigma\nu} - g_{\sigma\nu}\Sigma_{\rho\mu} + g_{\rho\nu}\Sigma_{\sigma\mu}) \end{aligned} \tag{II.7}$$

Choose the basis in index space in such a way that space time translations do not act on the indices, i.e.

$$P_\mu \varphi_\alpha(x) = i \frac{\partial}{\partial x^\mu} \varphi_\alpha(x) \tag{II.8}$$

It follows that for every element X of the conformal algebra

$$X\varphi(x) = \exp(-iP_\mu x^\mu)\, X' \varphi(0)$$

where

$$\begin{aligned} X' &= \exp(+iP_\mu x^\mu)\, X \exp(-iP_\mu x^\mu) \\ &= \sum_{n=0}^{\infty} \frac{i^n}{n!} x^{\nu_1} \cdots x^{\nu_n} [P_{\nu_1}, [\cdots [P_{\nu_n}, X] \cdots]]]. \end{aligned} \tag{II.9}$$

The important point is that the sum on the *RHS.* of Eq. (II.9) is actually finite.[3] From the C.R. Eq. (II.2) it is found by inspection that there are at most three non-vanishing terms in this sum. Evaluating the finite multiple commutators, e.g., for $X = K_\mu$, we get

$$\exp(+iP_\nu x^\nu) K_\mu \exp(-iP_\nu x^\nu) = K_\mu - 2x^\nu (g_{\mu\nu} D + M_{\mu\nu}) + 2x_\mu x^\nu P_\nu - x^2 P_\mu .$$

From this we now deduce the action of K_μ, D, $M_{\mu\nu}$ on $\varphi(x)$, since the action on $\varphi(0)$ is known by hypothesis; e.g. $K_\mu \varphi(0) = \kappa_\mu \varphi(0)$. The final results are

$$\begin{aligned} P_\mu \varphi(x) &= i\partial_\mu \varphi(x) \\ M_{\mu\nu}\varphi(x) &= \{i(x_\mu \partial_\nu - x_\nu \partial_\mu) + \Sigma_{\mu\nu}\}\, \varphi(x) \\ D\varphi(x) &= \{ix_\nu \partial^\nu + \Delta\}\, \varphi(x) \\ K_\mu \varphi(x) &= \{i(2x_\mu x_\nu \partial^\nu - x^2 \partial_\mu - 2ix^\nu [g_{\mu\nu} \Delta + \Sigma_{\mu\nu}]) + \kappa_\mu\}\, \varphi(x) \end{aligned} \tag{II.10}$$

where the matrices $\Sigma_{\mu\nu}$, Δ, κ_μ satisfy the C.R. (II.7).

We have thus shown that all field theoretically admissible representations of the conformal algebra are induced by a representation of the algebra of the little group (II.6). Since this algebra has two nontrivial ideals (=invariant subalgebras) $\{D\} \otimes T_4 \supset T_4$ there arise the following types of representations:

I. finite-dimensional representations of the little group
   (a) $\kappa_\mu = 0$
   (b) $\kappa_\mu \neq 0$ but nilpotent

[3] The finiteness of this sum follows also from a general theorem of O'Raifertaigh's (*29*): Let **G** be any finite dimensional Lie algebra containing the Poincaré algebra $P = SO(3,1) \otimes T$. Then there exists a finite number n_0 such that

$$[T, [T, [\ldots T, X] \ldots]]] = 0 \text{ for } n \geqslant n_0 \text{ and any } X \in \mathbf{G}.$$

II. infinite-dimensional representations of the little group.
Regarding these representations and their physical uses, the following remarks are in order:

(1) For case (Ia) $\Delta = il\mathbf{1}$ by Schur's lemma if the $\Sigma_{\mu\nu}$ form an irreducible representation of the homogeneous Lorentz algebra.

(2) In Section III it will be shown that the notion of a (broken) conformal symmetry admits of a perfectly consistent formulation in the framework of ordinary Lagrangian field theory. This theory makes use of finite-dimensional representations of the little group (II.6), with $\kappa_\mu = 0$ (type Ia). All generators will be hermitian.

(3) For case (Ib) the conclusion that all the κ_μ must be nilpotent follows from the well-known fact that in any finite-dimensional representation of the Poincaré algebra the generators of translations are nilpotent *(30)*.[4]

(4) The possibility of using representations of type (Ib) for physical purposes could be interesting because it may give rise to spin multiplets. The problem is that the representations induced in this way are apparently not fully reducible.

Further discussion on this possibility will be given in Section V, where Hepner's work on the use of these representations will be reviewed *(31)*.

(5) The possible use of infinite-dimensional representations of the little group will not be discussed in this paper. This would lead to the consideration of infinite-component field theories which will be discussed elsewhere *(18)*.

III. LAGRANGIAN FIELD THEORY; APPLICATION TO STRONG AND ELECTROMAGNETIC INTERACTION

In the present section we shall show that the idea of an orderly broken symmetry under the conformal group of space time admits of a perfectly consistent formulation in the framework of ordinary Lagrangian field theory. The considerations presented here are an extension of an unpublished note by one of the present authors *(32)*. For simplicity we assume fields with spin $\leqslant 1$ and minimal couplings.[5] We shall show that:

(1) There exist local conformal currents $\mathscr{K}_{\nu\mu}$ and a dilatation current $\mathscr{D}_\mu$ such that the corresponding generators K_μ and D are hermitian and have C.R. with the particle fields as given in Eq. (II.10), with $\Delta = il\mathbf{1}$, $\kappa_\mu = 0$ (type Ia). *This is true independently of whether the action* $\int \mathscr{L}\, d^4x$ *is invariant or not.*

[4] Recall that nilpotency of a matrix κ means that $\kappa^m = 0$ for a suitable positive integer m.

[5] Minimal couplings are those involving no derivatives of fermion fields and up to first order derivatives of boson fields; they have the virtue of not affecting the canonical C.R. of the fields.

(2) The kinetic energy term without mass is conformal invariant.[6] The same is true of all non-derivative couplings with dimensionless coupling constants and all couplings arising from (Yang–Mills type) (*34*) gauge field theories. This includes electromagnetism. It also includes weak interactions mediated by an intermediate boson, if this boson is associated with a gauge field associated with some internal group (e.g., for hadrons, the Cabbibo $SU(2)$ subgroup of one of the $SU(3)$ ideals of chiral $SU(3) \otimes SU(3)$, or the $U(2) \otimes U(2)$ group considered for leptons by Ward and Salam (*36*), which includes both *EM* and weak interactions.)

(3) Besides the exact symmetry limit corresponding to massless particles only, the possibility also exists of a spontaneous breakdown of conformal symmetry. There, all particles can be massive except for $I = 0$, $J^P = 0^+$ massless Goldstone boson. An example of a corresponding Lagrangian (the σ model of Gell–Mann and Lévy) is discussed in the Appendix.

Abstracting from Lagrangian field theory, a current algebra type scheme may be set up. It is composed of the C.R. of the currents with the particle fields, Eqs. (III.1) and (III.6), and the relation (III.17) between the divergences of the currents. In addition, an algebraic relation between the divergence of the dilatation current and the axial vector currents of chiral $SU(3) \otimes SU(3)$ has been proposed elsewhere (*16*) and is given in Eqs. (A.4) and (A.5) of the Appendix. Equation (III.17) expresses the idea that the breaking of conformal symmetry is minimal in the sense that there is only as much breaking of the conformal symmetry as is induced by the breaking of dilatation symmetry alone.

1. *The conformal currents*

According to Eq. (II.10) we want to transform the interpolating fields as follows

$$\begin{aligned}
&\text{(a)} \quad D\varphi(x) = i(-l + x_\nu \partial^\nu)\,\varphi(x) \\
&\text{(b)} \quad K_\mu\varphi(x) = i(-2lx_\mu + 2x_\mu x_\nu \partial^\nu - x^2\partial_\mu - 2ix^\nu \Sigma_{\mu\nu})\,\varphi(x) \qquad \text{(III.1)} \\
&\text{(c)} \quad M_{\mu\nu}\varphi(x) = i(x_\mu\partial_\nu - x_\nu\partial_\mu - i\Sigma_{\mu\nu})\,\varphi(x)
\end{aligned}$$

If φ is the electromagnetic vector potential, we may postulate this transformation law only up to a gauge transformation. Eq. (III.1) is to be understood in this sense in the following. The well known (*35*), (*27*) reason for this is that, for a massless

[6] The conformal invariance of the free Maxwell equations and the free massless Klein–Gordon, Dirac- and Weyl- equations has been known for a long time, cf. Refs. (*1–3*).

A conformal invariant spinor equation with a nonlinear interaction $(\bar\psi\psi)^{4/3}$ has been considered by Gürsey (*1*)

Conformal invariant wave equations in Minkowski space for massless free particles with arbitrary spin have been considered in refs. (*12, 15,* and *33*).

particle, the vector potential is not a manifestly Lorentz covariant field in the ordinary sense. Through the last equation, $\Sigma_{\mu\nu}$ is defined in terms of the spin of the particle:

$$\begin{array}{ll} \text{when acting on a spin 0 field} & \Sigma_{\mu\nu} = 0 \\ \text{when acting on a spin } \tfrac{1}{2} \text{ field} & \Sigma_{\mu\nu} = \dfrac{i}{4}[\gamma_\mu, \gamma_\nu] \\ \text{when acting on a vector field} & (\Sigma_{\mu\nu}A)_\rho = i(g_{\mu\rho}A_\nu - g_{\nu\rho}A_\mu) \end{array}$$

The present theory does not lead to multiplets of particles with different spin. We fix the values of l to be

$$\begin{array}{ll} l = -1 & \text{for sclar and vector fields} \\ l = -\tfrac{3}{2} & \text{for spin } \tfrac{1}{2} \text{ fields} \end{array} \tag{III.2}$$

and $l = -1 - j$ for $(0, j) \oplus (j, 0)$ fields for arbitrary spin j, cf. ref. *15*). This choice is necessary in order to obtain acceptable currents because only then the canonical equal time commutation relations of the fields are invariant.[7] The values of l in (III.2) agree with the actual dimension of length of the fields in question. Note that (III.1c) is of the form $\varphi'(x) = \rho^l \varphi(\rho^{-1}x)$ under $x_\mu \to \rho x_\mu$; $\rho = 1 + \epsilon$ so that the fields transform under dilatations according to their actual dimension of length.

We can now write down the following local currents:

$$\begin{aligned} \mathscr{D}_\nu(x) &= x^\rho T_{\nu\rho} - \sum_r l_r \frac{\partial \mathscr{L}}{\partial \partial_\nu \varphi_r} \varphi_r \\ \mathscr{K}_{\nu\mu}(x) &= 2x^\rho x_\mu T_{\nu\rho} - x^2 T_{\nu\mu} - \sum_r \left(\frac{\partial \mathscr{L}}{\partial \partial^\nu \varphi_r} (2l_r x_\mu + 2ix^\rho \Sigma_{\mu\rho}) \varphi_r \right) - \sum_{\substack{\text{spin } 0 \\ \text{fields}}} g_{\nu\mu} \phi^+ \phi \end{aligned} \tag{III.3}$$

For the free field case with exact symmetry, similar currents have been written down by McLennan (*33*) and Wess (*27*).

The angular momentum current has the usual form. The energy momentum tensor is defined by

$$T_{\nu\rho}(x) = -g_{\nu\rho}\mathscr{L} + \sum_r \frac{\partial \mathscr{L}}{\partial \partial^\nu \varphi_r} \partial_\rho \varphi_r$$

[7] Eq. (III.2) also follows from the requirement that the free massless wave equation be fully conformal invariant.

It is convenient to choose the kinetic energy term in the Lagrangian as:

$$\mathscr{L}_0 = \sum_{\text{spin}\frac{1}{2}} \bar{\psi}(\tfrac{1}{2}\gamma_\mu \overleftrightarrow{\partial^\mu} - M)\,\psi + \sum_{\text{spin}\,0} \tfrac{1}{2}(\partial_\mu\varphi^+\partial^\mu\varphi - \mu^2\varphi^+\varphi)$$

$$- \sum_{\text{spin}\,1} \{\tfrac{1}{4}(\partial_\mu B_\nu - \partial_\nu B_\mu)^2 + \tfrac{1}{2}m^2 B_\nu B^\nu\} \tag{III.4}$$

Since the zeroth components of the currents, $T_{0\mu}$, $\mathscr{D}_0$, $\mathscr{K}_{0\mu}$ and $\mathscr{M}_{0\nu\lambda}$ are all hermitian, the corresponding generators, formally defined as the space integrals of the zeroth component of these currents, are also hermitian.

One checks by using the canonical equal time C.R. of the fields

$$[\phi(x), \pi(x')]_\pm = i\delta^3(\mathbf{x} - \mathbf{x}') \qquad \text{for} \quad x_0 = x_0' \tag{III.5}$$

that

$$\begin{aligned} D\varphi(x) &\equiv -\int_v d^3x[\mathscr{D}_0(x'), \varphi(x)]_{x_0=x_0'} \\ K_\mu\varphi(x) &\equiv -\int_v d^3x[\mathscr{K}_{0\mu}(x'), \varphi(x)]_{x_0=x_0'} \end{aligned} \tag{III.6}$$

satisfy Eqs. (III.1). This is independent of whether the Lagrangian is conformal invariant or not. (The integration in (III.4) goes over some volume v including $\mathbf{x} = \mathbf{x}'$.)

2. *Divergence of the Currents.*

We now turn to the discussion of the properties of the divergence of the dilatation current $\mathscr{D}_\nu$ and special conformal currents $\mathscr{K}_{\nu\mu}$. The dynamical information of a symmetry (exact or broken) defined by the transformation law of the fields, lies in the properties of the divergence of the corresponding currents.

From Eq. (III.3) it is seen that the dilatation current $\mathscr{D}_\nu(x)$ depends on x explicitly and can therefore not be coupled to the field of a vector particle. However, using energy momentum conservation $\partial^\nu T_{\nu\rho} = 0$, we find for its divergence

$$\partial^\nu \mathscr{D}_\nu(x) = T_\nu{}^\nu - \partial_\nu \left\{\sum_r l_r \frac{\partial\mathscr{L}}{\partial\partial_\nu\varphi_r}\,\varphi_r\right\} \tag{III.7}$$

From this we see that:

The divergence of the dilatation current is a local
$I = 0$, $J^P = 0^+$ *field.*

Using the Euler Lagrange equation of motion, (III.7) can be rewritten in the form (*37*)

$$\partial^\nu \mathscr{D}_\nu(x) = -4\mathscr{L} - \sum_r \left(l_r \frac{\partial \mathscr{L}}{\partial \varphi_r} \varphi_r + (l_r - 1) \frac{\partial \mathscr{L}}{\partial \partial_\mu \varphi_r} \partial_\mu \varphi_r\right) \tag{III.8}$$

From Eq. (III.8) it is seen that all pieces of the Lagrangian that do not involve constants with nonzero dimension of length give zero contribution. To see this, notice that the RHS of Eq. (III.8) vanishes, since it is simply the Euler equation for a homogenous function (all terms having the same dimensionality). This establishes that *the kinetic energy term without mass is dilatation invariant, and so are all couplings with a dimensionless coupling constant.*

Next we turn to the divergence of the conformal currents. Using energy momentum conservation we find from Eq. (III.3)

$$\begin{aligned}\partial^\nu \mathscr{K}_{\nu\mu}(x) = {} & 2x_\mu \left(T^\nu{}_\nu - \partial_\nu \left\{\sum_r l_r \frac{\partial \mathscr{L}}{\partial \partial_\nu \varphi_r} \varphi_r\right\}\right) \\ & - \sum_r \left\{2l_r \frac{\partial \mathscr{L}}{\partial \partial^\mu \varphi_r} \varphi_r + 2i \frac{\partial \mathscr{L}}{\partial \partial_\nu \varphi_r} \Sigma_{\mu\nu} \varphi_r\right\} + \sum_{\text{spin } 0} \partial_\mu(\varphi^+ \varphi) \\ & + 2x^\rho \left\{T_{\mu\rho} - T_{\rho\mu} + 2i\, \partial_\nu \left(\sum_r \frac{\partial \mathscr{L}}{\partial \partial_\nu \varphi_r} \Sigma_{\mu\rho} \varphi_r\right)\right\}\end{aligned} \tag{III.9}$$

The last line is equal to $2x^\rho \partial^\nu \mathscr{M}_{\nu\mu\rho}$ and vanishes by angular momentum conservation. Hence

$$\partial^\nu \mathscr{K}_{\nu\mu} = 2x_\mu \partial^\nu \mathscr{D}_\nu + V_\mu \tag{III.10}$$

where V_μ is a local vector field defined by the second line on the RHS of Eq. (III.9). From Eq. (III.4) one checks by explicit calculation that the kinetic energy term without mass gives zero contribution to V_μ. Since it gives no contribution to $\partial^\nu \mathscr{D}_\nu$ either, we see that the kinetic energy term without mass is fully conformal invariant. Furthermore, from (III.10) the condition that an interaction Lagrangean $\mathscr{L}_I$ be fully conformal invariant is found to be the following

(1) It is dilatation invariant, i.e., has a dimensionless coupling constant

$$(2) \quad \sum_r \left(l_r \frac{\partial \mathscr{L}_I}{\partial \partial^\mu \varphi_r} \varphi_r + i \frac{\partial \mathscr{L}_I}{\partial \partial_\nu \varphi_r} \Sigma_{\mu\nu} \varphi_r\right) = 0; \quad \mu = 0 \cdots 3 \tag{III.11}$$

Condition (2) is independent from condition (1) as is seen from the example $\mathscr{L}_I = gA^\mu \pi(\partial_\mu \sigma)$ which satisfies (1) but not (2). An example which satisfies neither (1) nor (2) is the derivative pion nucleon coupling $\overline{N}\gamma_5\gamma_\mu N \partial^\mu \pi$. For nonderivative couplings, condition (2) is trivially satisfied.

3. *Yang–Mills theory*

Let us now turn to the question of the conformal invariance of the coupling of vector gauge fields in a Yang–Mills type gauge field theory. Let A be the internal n-parameter symmetry algebra, and $B_\mu{}^a$, $a = 1 \cdots n$, the corresponding gauge vector fields. Under an infinitesimal transformation with constant infinitesimal parameter ϵ^a, all fields transform according to

$$\Phi'_A = \Phi_A + \delta\Phi_A , \qquad \delta\Phi_A = i\epsilon^a T^B_{a,A} \Phi_B \tag{III.12}$$

where the matrices $\mathbf{T}_a$ form a hermitian representation of the algebra $\mathbf{A}$. Hermiticity reads

$$(T^B_{a,A})^* = T^A_{a,B} \tag{III.13}$$

As is well known (*34, 38*), all couplings of the vector fields $B_\mu{}^a$ are completely determined from the postulates of a Yang–Mills type theory and are obtained by the substitution

$$\partial_\mu \Phi_A \to \partial_\mu \Phi_A - igT^B_{a,A} B_\mu{}^a \Phi_B \tag{III.14}$$

Here, g is a dimensionless real coupling constant. To test for conformal invariance we see that condition (1) above is always fulfilled, while condition (2) is also satisfied because the only derivative couplings are the couplings of mesons, which have the form

$$\mathscr{L}_I = -\frac{i}{2} g\{(\partial^\mu \Phi^A)^* (T^B_{a,A})^* \Phi_B - (\partial^\mu \Phi^A)\, T^B_{a,A} \Phi^*_B\}\, B_\mu{}^a \tag{III.15}$$

for spin 0 fields Φ_A, and

$$\mathscr{L}_I = \frac{i}{2} g\{(\partial^\mu \varphi^{\nu A})^* (T^B_{a,A})^* (\varphi_{\nu B} B_\mu{}^a - \varphi_{\mu B} B_\nu{}^a) - h.c.\} \tag{III.16}$$

for spin 1 fields $\varphi_A{}^\nu$. Inserting into Eq. (III.11) and making use of Eqs. (III.13) and (III.2) one finds that condition (2) is indeed satisfied.

It is now tempting to speculate that in physics there is only as much breaking of conformal symmetry as is induced by the breaking of dilatation symmetry. In other words there should be a remainder of conformal symmetry in the sense that all couplings satisfy condition (2) above. This is equivalent to the algebraic condition

$$\partial^\nu \mathscr{K}_{\nu\mu} = 2x_\mu \partial^\nu \mathscr{D}_\nu \tag{III.17}$$

The virtue of this restriction is that it still allows for a breaking of the symmetry by the mass terms in the Lagrangian.

IV. MANIFESTLY $O(2, 4)$ COVARIANT FIELD TRANSFORMATION LAW

General experience from the history of elementary particle physics may lead one to the opinion that "the only good covariance is a manifest covariance". This is the motivation for the present section. It is mainly pedagogical in character and much of the material presented may be found in the literature for special cases, but is presented here in a unified way. Manifestly conformal invariant free field equations were first discussed by Dirac (*39*). Invariant interactions were given by Kastrup (*40*).

The problems to be solved are the following:

(1) Write down manifestly conformal invariant transformation laws for fields.

(2) Determine the relation between the old fields $\varphi_\alpha(x)$ occurring in the transformation law Eq. (II.10) and the new fields which are transformed manifestly covariantly.

(3) Write down manifestly invariant free field equations and interactions.

The new fields will be multispinor functions on the four-dimensional surface in a five-dimensional projective space rather than Minkowski space. Their physical interpretation will nevertheless be guaranteed by correspondence with ordinary fields $\varphi_\alpha(x)$ over Minkowski space discussed in Section II. This correspondence also allows one to consider questions of unitarity and quantization by reference to Minkowski space.

1. *Manifestly Covariant Transformation Law for Fields*

A manifestly $O(2, 4)$ covariant transformation law may be written down for multispinor functions $\chi_\beta(\eta)$ defined on the five-dimensional hypersurfaces of R^6 given by

$$\eta_B \eta^B = L^2 \tag{IV.1}$$

and satisfying $\chi_\beta(\eta) = \chi_\beta(-\eta)$.

Summation over B is over 0, 1, 2, 3, 5, 6 with metric $(+ - - -; - +)$. There are three essentially different surfaces, corresponding to $L^2 = \pm 1, 0$.

Suppose that s_{AB} is any finite dimensional representation of the algebra of $O(2, 4)$ ($\approx SU(2, 2)$) acting on the indices of $\chi_\beta(\eta)$ only. Then a manifestly covariant transformation law including an orbital part is given by (cf. Eq. (II.4))

$$\delta\chi(\eta) \equiv -i\epsilon^{AB} J_{AB} \chi(\eta) = -i\epsilon^{AB}(L_{AB} + s_{AB})\, \chi(\eta) \tag{IV.2}$$

where

$$L_{AB} = i(\eta_A \partial_B - \eta_B \partial_A), \qquad \partial_B \equiv \frac{\partial}{\partial \eta^B}$$

Clearly L_{AB} and s_{AB} commute with each other and satisfy the C.R. of $O(2,4)$ separately. It is important to notice that L_{AB} is a well defined operator when acting on functions that are only defined on the hypersurface (IV.1) (*39*). This is so because L_{AB} corresponds to an infinitesimal coordinate transformation which is a pseudo-rotation of the hypersurface (V.1) into itself.

The cone

$$\eta_B \eta^B = 0 \tag{IV.3}$$

is also left invariant by the coordinate transformation $\eta_B \to \lambda\eta_B$, $\lambda > 0$. Moreover, this transformation commutes with the $O(2,4)$ rotations. Therefore we may require the fields to be homogeneous functions on the cone (IV.3)

$$\chi(\lambda\eta) = \lambda^n \chi(\eta), \qquad \text{i.e.} \quad \eta^B \partial_B \chi(\eta) = n\chi(\eta). \tag{IV.4}$$

These homogeneous functions then depend arbitrarily only on 4 of the 5 coordinates which determine a point on the cone (V.3), i.e., just as many as there are coordinates in Minkowski space.[8] We shall restrict our attention to this case in the following.

2. *Mathematical preliminaries.*

Before proceeding, we need to know some mathematical lemmas.

LEMMA 1. *By a suitable choice of basis, each finite dimensional representation of the algebra* (II.7) *by matrices $\Sigma_{\mu\nu}$, Δ, κ_μ can be brought into the form*

$$\Sigma_{\mu\nu} = \begin{pmatrix} \boxed{*} & & & 0 \\ & \boxed{*} & & \\ & & \ddots & \\ 0 & & & \boxed{*} \end{pmatrix} \qquad \Delta = -i \begin{pmatrix} \lambda_0 \mathbf{1} & & & \\ & \lambda_1 \mathbf{1} & & \\ & & \ddots & \\ & & & \lambda_m \mathbf{1} \end{pmatrix}$$

(a) (b)

$$\kappa_\mu = \begin{pmatrix} \boxed{0} & & & 0 \\ & \boxed{0} & & \\ & & \ddots & \\ & * & & \boxed{0} \end{pmatrix} \tag{IV.5}$$

(c)

Note in particular that all κ_μ are nilpotent, as $(\kappa_\mu)^m = 0$, and their top rows are identically zero.

[8] A subtle point here is that the points where at least one coordinate x_μ is infinite may also correspond to finite coordinates on the cone. This is however not important here as we consider only infinitesimal transformations.

Proof. Finite dimensional representations of $SL(2C)$ algebra are fully reducible, whence (a). As $[\Sigma_{\mu\nu}, \Delta] = 0$, (b) follows by Schur's lemma. Without loss of generality, $\operatorname{Re}\lambda_0 \geqslant \operatorname{Re}\lambda_1 \geqslant \cdots \geqslant \operatorname{Re}\lambda_m$. By C.R. (II.7), each κ_μ is a stepoperator lowering the eigenvalue of $i\Delta$ by 1. This implies (c).

We also need some properties of finite dimensional representations of $O(2, 4)$ algebra. We make use of its isomorphy with $SU(2, 2)$ algebra and the well known Weyl's unitary trick to arrive at

LEMMA 2. *All finite dimensional representations of $O(2, 4)$ algebra (without parity) by matrices $s_{AB} = -s_{BA}$ can be obtained by reducing out tensor products of the following two inequivalent fundamental 4-dimensional representation $\Delta^{(+)}$ and $\Delta^{(-)}$ given by matrices γ_{AB} as follows:*

$$\begin{aligned} \Delta^{(+)}:\ \gamma_{\mu\nu} &= \frac{i}{4}[\gamma_\mu, \gamma_\nu]; & \gamma_{\mu 5} &= -\frac{i}{2}\gamma_\mu\gamma_5 \\ \gamma_{56} &= -\tfrac{1}{2}\gamma_5; & \gamma_{\mu 6} &= \tfrac{1}{2}\gamma_\mu \\ \Delta^{(-)}:\ \dot\gamma_{\mu\nu} &= \frac{i}{4}[\gamma_\mu, \gamma_\nu]; & \dot\gamma_{\mu 5} &= \frac{i}{2}\gamma_\mu\gamma_5 \\ \dot\gamma_{56} &= \tfrac{1}{2}\gamma_5; & \dot\gamma_{\mu 6} &= \tfrac{1}{2}\gamma_\mu \end{aligned} \tag{IV.6}$$

γ_μ are Dirac matrices defined by $\{\gamma_\mu, \gamma_\nu\}_+ = 2g_{\mu\nu}$, and $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$. All matrices satisfy $A\gamma_{BC}A = \gamma_{BC}^+$, where $A = \gamma_0$ for the usual choice of Dirac matrices with γ_0 and $i\gamma_j$ hermitian.

Parity transforms $\Delta^{(+)}$ into $\Delta^{(-)}$ and vice versa. The two representations $\Delta^{(+)}$ and $\Delta^{(-)}$ are conjugate to each other.

This is a weak form of a standard result (*41*).

The simplest nontrivial representation of $O(2, 4)$ with parity is eightdimensional and unique up to a choice of basis. It is given by $\Delta = \Delta^{(+)} \oplus \Delta^{(-)}$. It is convenient to carry out a basis transformation such that

$$\begin{aligned} \gamma_{\mu\nu} &= \frac{i}{4}[\gamma_\mu, \gamma_\nu], & \gamma_{\mu 5} &= -\frac{i}{2}\gamma_\mu\tau_2 \\ \gamma_{56} &= -\frac{i}{2}\tau_3 & \gamma_{\mu 6} &= \tfrac{1}{2}\gamma_\mu\tau_1 \\ && \Pi = \gamma_0 && \end{aligned} \tag{IV.7}$$

Parity is represented by γ_0 here. τ_i are Pauli matrices. In this form the eight dimensional representation has been given by Murai (*42*). For this representation a Clifford algebra exists such that (*39*, *40*)

$$\frac{i}{4}[\beta_A, \beta_B] = \gamma_{AB}; \qquad \{\beta_A, \beta_B\} = 2g_{AB} \tag{IV.8}$$

The matrices β_A transform as a six-vector in the sense that

$$s(\alpha)\,\beta^A s^{-1}(\alpha) = \alpha^A{}_B \beta^B, \qquad \text{i.e.} \quad [\beta_A, \gamma_{BC}] = i(g_{AB}\beta_C - g_{AC}\beta_B)$$

The real numbers $\alpha^A{}_B$ are the parameters characterizing an element of $O(2, 4)$ group. This transformation law will be important for constructing invariant couplings. Explicitly the β_A may be given by

$$\beta_\mu = \gamma_\mu \tau_3\,; \qquad \beta_5 = i\tau_1\,; \qquad \beta_6 = \tau_2 \tag{IV.9}$$

There is also a conformal pseudoscalar β_7

$$\beta_7 = -i\beta_0\beta_1\beta_2\beta_3\beta_5\beta_6 = i\gamma_5\tau_3 \qquad \Pi\beta_7\Pi = -\beta_7\,.$$

All matrices satisfy

$$A\gamma_{BC}A = \gamma_{BC}^{+} \qquad \text{and} \qquad A\beta_B A = -\beta_B{}^{+}, \qquad \text{where} \qquad A = \gamma_0\tau_1 \tag{IV.10}$$

We further need the following consequence of lemma 2:

LEMMA 3. *Suppose we are given a representation of $SU(2, 2)$ algebra by matrices s_{AB} acting on a finite dimensional vector space $\mathscr{H}$. Evidently this provides at the same time a representation of the subalgebra* (II.6)*f by matrices*

$$\kappa_\mu = s_{6\mu} - s_{5\mu}\,, \qquad \Sigma_{\mu\nu} = s_{\mu\nu}\,; \qquad \Delta = s_{65}\,. \tag{IV.11}$$

We assert that there exists a projection operator E such that

(1) $\kappa_\mu E \neq 0,$ (IV.12)

and

(2) $\kappa'_\mu \equiv E\kappa_\mu E = 0, \qquad \Sigma'_{\mu\nu} \equiv E\Sigma_{\mu\nu}E, \qquad \Delta' \equiv E\Delta E - in\,\mathbf{1}$

form a new representation of the algebra of the little group (II.6) *acting on $\mathscr{H}' = E\mathscr{H}$.*

n may be any c-number.

Moreover, every representation of the algebra of the little group with $\kappa_\mu = 0$ may be obtained in this way from a suitable representation of $SU(2, 2)$ algebra.

A constructive proof will be given below.[9] In the basis of Lemma 1, $\mathscr{H}'$ is spanned by vectors of the form

$$\begin{pmatrix} * \\ \hline 0 \\ \vdots \\ 0 \end{pmatrix}.$$

[9] Note that this construction does not amount to restriction to an invariant subspace under the full little group, as obviously $[E, \kappa_\mu] \neq 0$. Instead we are here exploiting the fact that our little group is not semisimple, and its finite dimensional representations are in general not completely reducible. $\kappa_\mu E \neq 0$ will be important later for the construction of invariant wave-equations.

It follows from (IV.5) that κ_μ maps these into the orthogonal complement $\mathscr{H}'^{\perp}$ of $\mathscr{H}'$. Consequently $E\kappa_\mu E = 0$. However, none of the matrices $\Sigma_{\mu\nu}$, Δ, κ_μ makes a transition $\mathscr{H}'^{\perp} \to \mathscr{H}'$, as $\mathscr{H}'$ and $\mathscr{H}'^{\perp}$ are separately invariant under $\Sigma_{\mu\nu}$ and Δ by virtue of Lemma 1, and κ_μ has the form (IV.5). Thus $\mathscr{H}'$ forms a representation space for the algebra $\{\Sigma'_{\mu\nu}, \Delta'\}$, extended to one of (II.7) by adjoining $\kappa'_\mu = E\kappa_\mu E = 0$. This completes the demonstration of the first part of the lemma.

It is useful to reformulate in matrix language. With a choice of basis as in Lemma 1, the matrices $(\Sigma'_{\mu\nu})_{\alpha\beta}$, $(\Delta' + \text{in}\,\mathbf{1})_{\alpha\beta}$, $(\kappa'_\mu)_{\alpha\beta} = 0$ are submatrices of $\Sigma_{\mu\nu}$, Δ, κ_μ with $\alpha, \beta \in I_{\text{phys}}$. I_{phys} is the set of indices α such that the basisvectors $\mathbf{e}_\alpha \in \mathscr{H}'$, where

$$\mathscr{H}' = \mathscr{H} - \bigcup_\mu \kappa_\mu \mathscr{H}$$

Finally, addition of a multiple of the identity to Δ' does obviously not change the C.R.

The second part of the Lemma 3 may be proved by explicit construction.

Consider the multispinor which transforms as direct product of 4-dimensional fundamental representations $\Delta^{(+)}$ and $\Delta^{(-)}$

$$\delta\xi_{\alpha_1\ldots\alpha_j;\dot\beta_1\ldots\dot\beta_k} = -i\epsilon^{AB}\left\{\sum_{i=1}^{j}(\gamma_{AB})_{\alpha_i\alpha_i'}\xi_{\alpha_1\ldots\alpha_i'\ldots\alpha_j;\dot\beta_1\ldots\dot\beta_k}\right.$$

$$\left.+\sum_{i=1}^{k}(\dot\gamma_{AB})_{\dot\beta_i\dot\beta_i'}\xi_{\alpha_1\ldots\alpha_j;\dot\beta_1\ldots\dot\beta_i'\ldots\dot\beta_k}\right\}; \qquad \alpha_i, \dot\beta_i = 1 \cdots 4 \tag{IV.13}$$

As a necessary condition for irreducibility we may impose symmetry properties in indices of the same kind, and tracelessness

$$\delta^{\alpha_i}_{\dot\beta_i}\xi^{\dot\beta_1\dot\beta_2\ldots}_{\alpha_1\alpha_2\ldots} = 0.$$

The multispinor with upper dotted indices is obtained by applying the index raising operator $C\gamma_0$, with $C\gamma_\mu C = -\bar\gamma_\mu$.

Since $[\Sigma'_{\mu\nu}, \Delta'] = 0$ and $\kappa'_\mu = 0$ we may restrict our attention to an irreducible representation of $SL(2C)$ which may be characterized by a pair $(j/2, k/2)$ in the usual way.

The desired representation of $SU(2, 2)$ acts on multispinors

$$\xi_{\{\alpha_1\ldots\alpha_j\};\{\dot\beta_1\ldots\dot\beta_k\}} \equiv \xi_{\{\alpha\},\{\beta\}} \tag{IV.14}$$

which are totally symmetric in each kind of indices and have j undotted and k dotted indices. Each index takes 4 values. The dot-notation has been adopted in anticipation of connection which will be made with Lorentz group.

We now adopt a basis for the Dirac matrices as follows

$$\gamma_0 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}; \qquad \gamma_i = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix}; \qquad i\gamma_5 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \tag{IV.15}$$

Then projection onto $\mathscr{H}'$ amounts to considering only those multispinor components with

$$\alpha_i = 3, 4; \qquad \dot{\beta}_i = 1, 2$$

for all undotted indices α_i and dotted indices $\dot{\beta}_i$, respectively. Using (IV.11), (IV.13), (IV.6) it is readily established that

$$\delta\xi_{\{\alpha\},\{\dot{\beta}\}} \equiv -i\epsilon^{\mu}(\kappa_\mu\xi)_{\{\alpha\},\{\dot{\beta}\}} = 0.$$

Moreover, under infinitesimal Lorentz transformations these multispinor components are found to transform precisely like the familiar Lorentz multispinor with j undotted and k dotted indices, i.e. according to the representation $(j/2, k/2)$ as desired. Finally, $\Delta' = -il\mathbf{1}$ by Schur's lemma. This completes the proof.

Remark: *$\mathscr{H}'$ may also be characterized as the eigenspace belonging to the highest eigenvalue of is_{65}.*

Provided s_{AB} is irreducible, this follows from the fact that κ_μ is a step operator which lowers the eigenvalue of is_{65} by one unit. Irreducibility is true for traceless multispinors of the form (IV.14) but will not be proved here.

We now proceed to the proof and statement of our last lemma. Suppose that we are given a field representation of the conformal algebra of the form (II.10) induced by finite dimensional representation κ_μ, $\Sigma_{\mu\nu}$, Δ of the algebra of the little group (II.6). As we have seen, there arise in this way two types of representations, $\kappa_\mu = 0$ (type Ia) and $\kappa_\mu \neq 0$ (type Ib). Consider the latter case, i.e. $\kappa_\mu \neq 0$. By virtue of Lemma 1 we may assume that the four matrices κ_μ are in lower triangular form. Then there will be a set I_{phys} of values of the index α, defined by the requirement

$$(\kappa_\mu)_{\alpha\beta} = 0 \qquad \text{for} \qquad \alpha \in I_{\text{phys}} \text{ and all } \beta \text{ and } \mu.$$

Consider those components $\varphi_\alpha(x)$ of the fields with $\alpha \in I_{\text{phys}}$. Anticipating later applications we shall refer to these as "physical components". Lemma 1 asserts that such components always exist. Furthermore, from Lemma 1 and proof of Lemma 3, part 1, we know that

$$(\Sigma_{\mu\nu})_{\alpha\beta} = 0, \quad \Delta_{\alpha\beta} = 0 \qquad \text{for} \quad \alpha \in I_{\text{phys}}, \quad \beta \notin I_{\text{phys}}.$$

By Schur's Lemma we may assume, without loss of generality, that

$$\Delta_{\alpha\beta} = -i\lambda_0\delta_{\alpha\beta}; \qquad \alpha, \beta \in I_{\text{phys}}$$

(otherwise there is an invariant subspace on which this is true). Upon inserting these relations into the general transformation law (II.10) it is found that the "physical components" of the field transform under infinitesimal transformations according to the law (III.1)—i.e. a transformation of type Ia—in the sense that e.g. for special conformal transformations

$$\delta\varphi_\alpha(x) = -i\epsilon^\mu \sum_{\beta\in I_{\rm phys}} \{i(-2lx_\mu + 2x_\mu x_\nu\,\partial^\nu - x^2\,\partial_\mu)\,\delta_{\alpha\beta} - 2ix^\nu(\Sigma_{\mu\nu})_{\alpha\beta}\}\,\varphi_\beta(x); \qquad \text{for} \quad \alpha\in I_{\rm phys}\,.$$

An important point here is, of course, that the physical components of the transformed field depend only on the physical components of the old field.

We may summarize our findings in the following

LEMMA 4. *Suppose we are given a representation of the conformal algebra of the form* (II.10), *induced by any finitedimensional representation* κ_μ, $\Sigma_{\mu\nu}$, Δ *of the algebra of the little group* (II.6). *Choose a basis in indexspace as in Lemma 1. Then a suitable* (*nonempty*) *part* $\{\varphi_\alpha(x),\ \alpha\in I_{\rm phys}\}$ *of the components of the field may be selected such that a representation of the form* (III.1) (*i.e. of type* Ia) *is induced on these components.* $I_{\rm phys}$ *is defined by the requirement that*

$$(\kappa_\mu)_{\alpha\beta} = 0 \qquad \textit{for} \quad \alpha\in I_{\rm phys}\ \textit{and all}\ \beta\ \textit{and}\ \mu. \tag{IV.16}$$

3. *Relation between manifestly covariant fields and fields over Minkowski space.*

After the preparations of the preceding subsection we are finally able to prove the main result of this section.

THEOREM. *Suppose we are given a manifestly conformal covariant* (*multispinor*) *field* $\chi(\eta)$, *defined on the cone* $\eta^2 = 0$ *in* R^6, *and satisfying a homogeneity condition*

$$\eta^{\rm B}\,\partial_{\rm B}\chi(\eta) = n\chi(\eta) \qquad \textit{where} \qquad \partial_{\rm B} = \frac{\partial}{\partial\eta^{\rm B}} \tag{IV.17}$$

Suppose it transforms under infinitesimal conformal transformations according to

$$\delta\chi(\eta) = -i\epsilon^{\rm AB}(L_{\rm AB} + s_{\rm AB})\,\chi(\eta), \qquad L_{\rm AB} = i(\eta_{\rm A}\partial_{\rm B} - \eta_{\rm B}\partial_{\rm A})$$

where $s_{\rm AB}$ *is some finite dimensional irreducible representation of* $O(2,4)$ *algebra acting on indexspace* $\mathscr{H}$. *Let* λ_0 *be the highest eigenvalue of* is_{65}. *Consider then*

$$\varphi(x) \equiv \kappa^{-n}{\rm EV}\chi(\eta) \tag{IV.18}$$

where

$$x_\mu = \eta_\mu/\kappa; \qquad \kappa = \eta_5 + \eta_6; \qquad V = \exp -i(s_{6\mu} + s_{5\mu})$$

and E is the projection operator onto the eigenspace $\mathscr{H}'$ *belonging to eigenvalue* λ_0 *of* is_{65}. *We assert*

(1) $\delta\varphi(x) = \kappa^{-n}\mathrm{EV}\, \delta\chi(\eta)$ *is given by Eq.* (III.1) *and* (II.3)*f with* $l = n + \lambda_0$ *and real.*

(2) *every field* $\varphi(x)$ *over Minkowskispace with transformation law* (III.1) *may be obtained in this way from a suitable manifestly covariant field* $\chi(\eta)$ *as described above, provided the matrices* $\Sigma_{\mu\nu}$ *in* (III.1) *form an irreducible representation of* $SL(2C)$. *A suitable* $SU(2, 2)$ *representation* s_{AB} *is given in terms of the completely symmetric multispinors described in Lemma* 3.

The extension to reducible $SL(2C)$ representations is obvious. The most important case are the familiar Bargmann Wigner Lorentz-multispinors $\xi_{\{\gamma_1 \cdots \gamma_{2j}\}}$, $\gamma_i = 1 \cdots 4$ which transform like a product of $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ representations. The conformal extension is given by $\xi_{\{\gamma_1 \cdots \gamma_{2j}\}}$ where γ_i now run from $1 \cdots 8$. It transforms under $SU(2, 2)$ like a product of eight-dimensional representations (IV.7).

Proof. The proof of the first part of the theorem will proceed in three steps:

(1) coordinate transformation

(2) x-dependent basis transformation in index space to transform away the intrinsic part of the translation operator, i.e. ensure Eq. (II.8)

(3) Project out unphysical components.

Step 1 has first been carried out by Dirac (*39*). For spin $\frac{1}{2}$ case, step 2 has been described by Hepner (*30*). The present discussion covers arbitrary spin.

Step 1. We introduce a new set of six real independent variables x_μ, κ, η^2 related to η_{A} by

$$x_\mu = \eta_\mu / \kappa, \qquad \kappa = \eta_5 + \eta_6, \qquad \eta^2 = \eta_{\mathrm{A}}\eta^{\mathrm{A}}$$

Let us now reexpress the functions $\chi_\alpha(\eta)$ as functions of the new variables

$$\chi_\alpha(\eta) \equiv \chi'_\alpha(x_\mu, \kappa, \eta^2)$$

Making use of the chain rule

$$\begin{aligned}\frac{\partial}{\partial \eta^{\mathrm{A}}} &= \left(\frac{\partial x^\mu}{\partial \eta^{\mathrm{A}}}\right)\frac{\partial}{\partial x_\mu} + \left(\frac{\partial \kappa}{\partial \eta^{\mathrm{A}}}\right)\frac{\partial}{\partial \kappa} + \left(\frac{\partial \eta^2}{\partial \eta^{\mathrm{A}}}\right)\frac{\partial}{\partial \eta^2} \\ &= \frac{1}{\kappa}\left(g_{\mathrm{A}}{}^{\mu} - [g_{\mathrm{A}}{}^{6} - g_{\mathrm{A}}{}^{5}]\, x^\mu\right)\frac{\partial}{\partial x^\mu} + (g_{\mathrm{A}}{}^{6} - g_{\mathrm{A}}{}^{5})\frac{\partial}{\partial \kappa} + 2\eta_{\mathrm{A}}\frac{\partial}{\partial \eta^2}\end{aligned}$$

we may reexpress $L_{AB} \equiv i(\eta_A \partial_B - \eta_B \partial_A)$ as a differential operator in the new variables. The differential operator $\partial/\partial\eta^2$ does not enter this expression for L_{AB}. L_{AB} is therefore a well defined differential operator acting on functions defined on the cone $\eta^2 = 0$ only. We may then restrict our attention to this cone. We find

$$\eta^A \partial_A = \kappa \frac{\partial}{\partial \kappa} \qquad \text{on the cone } \eta^2 = 0.$$

Hence the homogeneity condition Eq. (IV.4) on $\chi(\eta)$ reads now

$$\left(\kappa \frac{\partial}{\partial \kappa} - n\right) \chi'(x_\mu, \kappa, 0) = 0 \tag{IV.20}$$

Inserting this into the expression for L_{AB} we find finally the action of L_{AB} on the function χ':

$$\begin{aligned} L_{\mu\nu} &= i\left(x_\mu \frac{\partial}{\partial x^\nu} - x_\nu \frac{\partial}{\partial x^\mu}\right) \\ L_{6\mu} + L_{5\mu} &= i \frac{\partial}{\partial x^\mu} \\ L_{6\mu} - L_{5\mu} &= i\left(-2n x_\mu + 2x_\mu x_\nu \frac{\partial}{\partial x_\nu} - x^2 \frac{\partial}{\partial x^\mu}\right) \\ L_{65} &= i\left(-n + x^\nu \frac{\partial}{\partial x^\nu}\right) \end{aligned} \tag{IV.21}$$

Let us now introduce

$$\tilde{\varphi}_\alpha(x) = \kappa^{-n} \chi'_\alpha(x, \kappa, 0) \equiv (\eta_5 + \eta_6)^{-n} \chi_\alpha(\eta) \tag{IV.22}$$

This new field does not depend on κ because of Eq. (IV.20), and is therefore a function of x only as stated. Furthermore a conformal transformation (IV.2) of $\chi(\eta)$ induces on $\tilde{\varphi}(x)$ a transformation

$$\delta\tilde{\varphi}(x) = -i\epsilon^{AB}(L_{AB} + s_{AB})\, \tilde{\varphi}(x) \tag{IV.23}$$

where L_{AB} is given by Eq. (IV.21). To see this, note that L_{AB} does no longer contain any differentiation $\partial/\partial\kappa$ and the proportionality factor κ^{-n} in Eq. (IV.22) therefore commutes with L_{AB}. In particular we have from (IV.21) and (IV.23)

$$P_\mu \tilde{\varphi}(x) = \left(i \frac{\partial}{\partial x^\mu} + \pi_\mu\right) \qquad \text{where} \qquad \pi_\mu = s_{6\mu} + s_{5\mu}$$

Step 2. Define *(11, 30)*

$$\varphi_\alpha(x) = V_{\alpha\beta} \tilde{\varphi}_\beta(x) \qquad \text{where} \quad V = \exp -ix^\nu \pi_\nu \tag{IV.24}$$

V is always a finite polynomial in x because the π_μ are also nilpotent. It could therefore be worked out explicitly in each case. In practice such straightforward but sometimes tedious calculations can usually be avoided by using translation invariance and the fact that $V = 1$ at $x = 0$.

Because all π_μ commute, V has an inverse given by

$$V^{-1} = \exp(+ix^\nu \pi_\nu)$$

Furthermore

$$V\left(i\frac{\partial}{\partial x^\mu}\right)V^{-1} = i\frac{\partial}{\partial x^\mu} - \pi_\mu$$

Using Eqs. (IV.21) and the C.R. of the matrices s_{AB} as given by (II.4) one may check that the components of the field φ in the new basis do indeed transform according to Eq. (II.10) with

$$\Sigma_{\mu\nu} = s_{\mu\nu}\,; \qquad \kappa_\mu = s_{6\mu} - s_{5\mu}\,; \qquad \Delta = s_{65} - in$$

n is given by Eq. (IV.4). The remaining matrices have disappeared from the transformation law.

Step 3. After having arrived at a field which transforms according to Eq. (II.10) it remains to project out unphysical components such that the term involving κ_μ in Eq. (II.10) goes away. This problem has been solved by Lemma 4 in the preceding subsection.

This completes the proof of the first part of the theorem. The second part is now a consequence of the first, and of Lemmas 3 and 4.

4. *Invariant wave equations and interactions.*

With manifestly covariant fields it is straightforward to write down manifestly invariant wave equations and interactions. The following examples are due to Kastrup (*40*).

All fields are to correspond to fields over Minkowski space which transform according to Eq. (II.10) with $\kappa_\mu = 0$, as do the fields employed in Section III.

The spin 0 field (scalar or pseudoscalar) corresponds to a conformal scalar $A(\eta)$ with degree of homogeneity $n = -1$, i.e.

$$\eta^B \partial_B A(\eta) = -A(\eta), \qquad s_{AB}A(\eta) \equiv 0.$$

The free wave equation is

$$\Box_6 A(\eta) = 0 \qquad \text{where} \quad \Box_6 = \partial^B \partial_B\,. \tag{IV.25}$$

As discussed by Dirac (*39*), this is a well defined equation for $A(\eta)$ defined on the cone $\eta^2 = 0$ only, if and only if $n = -1$ as we assume. By Eq. (IV.18)f, $l = n = -1$ in agreement with the discussion in Section III. The scalar field in Minkowski space is then given by

$$a(x) = (\eta_5 + \eta_6)\, A(\eta)$$

and satisfies

$$\Box_x a(x) = 0.$$

The spin ½ field is an 8-component spinor $\chi(\eta)$ of degree of homogeneity $n = -2$

$$\eta^B \partial_B \chi(\eta) = -2\chi(\eta), \qquad s_{AB} = \gamma_{AB} \text{ given by Eq. (IV.7).} \tag{IV.26}$$

The adjoint is defined by

$$\bar{\chi} = \chi^* \gamma_0 \tau_1 \tag{IV.27}$$

cf. Eq. (IV.10).

The corresponding 8-spinor over Minkowski space is again given by Eqs. (IV.18) which reads

$$\psi(x) = (\eta_5 + \eta_6)^{+2}\, (1 + i x^\mu \gamma_\mu \tau^-)\, \chi(\eta) \quad \text{where} \quad \tau^- = \tfrac{1}{2}(\tau_1 - i\tau_2) \tag{IV.28}$$

Its physical components are those satisfying

$$(1 + \tau_3)\, \psi(x) = 0. \tag{IV.29}$$

In the basis where τ_3 has the usual diagonal form, these are just the lowest four components. From Eqs. (IV.7) and (IV.18)f one finds $l = n + \frac{1}{2} = -\frac{3}{2}$, as was assumed in Section III because of unitarity requirements.

The free wave equation is

$$-2i(\gamma^{AB} L_{AB} + 2)\, \chi(\eta) = 0; \qquad L_{AB} \equiv i(\eta_A \partial_B - \eta_B \partial_A). \tag{IV.30}$$

This amounts to diagonalizing the second-order Casimir operator $\frac{1}{2} J^{AB} J_{AB}$ ((Hepner and Murai) (*30*, *42*)).

The spin 1 gauge fields are 6-vectors $A_B(\eta)$ of degree of homogeneity $n = -1$.

$$\eta^B \partial_B A_C(\eta) = -A_C(\eta); \qquad (s_{AB} A)_C = i(g_{AC} A_B - g_{BC} A_A) \tag{IV.31}$$

satisfying the subsidiary condition

$$\eta^B A_B(\eta) = 0. \tag{IV.32}$$

If we impose in addition the generalized Lorentz condition

$$\partial^{C} A_{C}(\eta) = 0$$

then the admissible gauge transformations are, for the electromagnetic potential,

$$A_{C}(\eta) \to A_{C}(\eta) + \partial_{C} S(\eta) \tag{IV.33}$$

where the gauge function S must be specified on a whole neighbourhood of the cone $\eta^2 = 0$, and satisfy there

$$\eta^{B} \partial_{B} S(\eta) = 0; \qquad \Box_6 S(\eta) = 0.$$

The free field equation is then

$$\Box_6 A_{C}(\eta) = 0. \tag{IV.34}$$

Again, the choice of $n = -1$ makes this into a well-defined equation for $A_{C}(\eta)$ defined on the cone $\eta^2 = 0$ only.

The corresponding field $a_{B}(x)$ is again given by Eqs. (IV.18)

For the first four components this takes the explicit form

$$a_\mu(x) = (\eta_5 + \eta_6)[A_\mu(\eta) - x_\mu\{A_5(\eta) - A_6(\eta)\}] \qquad \mu = 0 \cdots 3 \tag{IV.35}$$

and the subsidiary condition Eq. (IV.32) reads

$$a_6(x) - a_5(x) = 0 \tag{IV.36}$$

This can be seen in the following way: For $x_\mu = 0$, we have $\eta_\mu = 0$ and $\eta_5 = \eta_6$. Therefore Eq. (IV.36) is clearly true at this point. Now Eq. (IV.32) is conformal invariant and therefore, in particular, translation invariant. However, by construction, all $a_{B}(x)$ transform under translations according to Eq. (II.10). Thus, by translation invariance the validity of Eq. (IV.36) for arbitrary x_μ follows from its validity at $x_\mu = 0$.

The first four components of $a_{B}(x)$ are the physical ones as they satisfy Eq. (IV.16) by virtue of the subsidiary condition Eq. (IV.36):

$$(\kappa_\mu a)_\nu(x) = (\{s_{6\mu} - s_{5\mu}\}a)_\nu = -ig_{\mu\nu}(a_6 - a_5) = 0 \qquad \nu = 0 \cdots 3 \tag{IV.37}$$

Invariant couplings. Following Kastrup, it is easy to see that a conformal invariant coupling between a pseudoscalar field $A(\eta)$ and the spin $\frac{1}{2}$ field $\chi(\eta)$ is given by the following wave equation:

$$\begin{aligned} \Box_6 A &= -g\eta^{C} \bar{\chi} \beta_{C} \beta_7 \chi \\ -2i(\gamma^{AB} L_{AB} + 2)\chi &= g\eta^{C} \beta_{C} \beta_7 \chi A \end{aligned} \tag{IV.38}$$

and the coupling of the electromagnetic field to the spin $\frac{1}{2}$ field is given by

$$[2\gamma^{AB}\{\eta_A(\partial_B - iqA_B) - \eta_B(\partial_A - iqA_A)\} - 4i]\chi = 0$$
$$\Box_6 A_C = qj_C(\eta) \tag{IV.39}$$

where

$$j_C(\eta) = 2\eta^B\bar{\chi}\gamma_{BC}\chi.$$

The β-matrices are given by Eqs. (IV.8)f.

As is seen, the electromagnetic coupling is obtained by making the gauge-invariant substitution $\partial_C \to \partial_C - iqA_C$. In this form it can be immediately generalized to arbitrary sets of gauge fields $A_C{}^a(\eta)$. Let $\mathbf{T}_a$ be the representation matrices of the relevant group as discussed in Section III; then the general rule is to substitute

$$\partial_C \to \partial_C - iqA_C{}^a\mathbf{T}_a \tag{IV.40}$$

in the free field equations. Summation over a is understood. In this way one obtains couplings which are both conformal invariant and gauge invariant.

Finally there also exists a quadrilinear conformal invariant spin 0 boson coupling. A corresponding wave equation would be, e.g.,

$$\Box_6 A(\eta) = g[A(\eta)]^3. \tag{IV.41}$$

Of course all couplings mentioned above can occur simultaneously.

A point which we wish to stress is the following: Not all manifestly covariant looking couplings are actually acceptable. They must in addition satisfy the following requirements:

(1) The wave equations must have homogeneous functions as solutions. This requires that all terms in a certain wave equation must have the same degree of homogeneity. The degree of homogeneity of such a term is calculated by counting each explicit coordinate η with $+1$, each derivative $\partial/\partial\eta$ with -1, and each field with the appropriate number n indicating its degree of homogeneity (e.g. $n = -1$ for scalar and vector, and -2 for 8-spinors χ in the cases discussed above.)

(2) The interaction terms must not couple unphysical field components to physical ones.

The couplings given above satisfy this condition, while e.g. a coupling $\bar{\chi}\chi A$ would violate (2).

Condition (2) is crucial and may not be given up without destroying the physical interpretation of the theory. It *must* be satisfied at least for a suitable choice of the electromagnetic gauge (IV.33). A sufficient condition for this is invariance of the wave equation under a new type of additive gauge transformations which affect only the unphysical components. For the spin $\frac{1}{2}$ field such gauge transformations have the form

$$\chi'(\eta) = \chi(\eta) + \eta^A \beta_A \zeta(\eta), \qquad \eta^A \partial_A \zeta = (n-1)\zeta,$$

where ζ is an arbitrary complex 8-spinor function like χ, and satisfies the indicated homogeneity condition. β_A is given by Eq. (IV.8) or (IV.9). Invariance of Eqs. (IV.38), (IV.39) follows immediately from $(\eta^A \beta_A)^2 = 0$, and Eqs. (IV.10) and (IV.32).

There is also an easy way of checking by explicit computation whether condition (2) is satisfied, without going through the tedious transformations of Section (IV.3). Because of translation invariance, it is sufficient to check that the condition is satisfied at $x_\mu = 0$. This corresponds to $\eta_\mu = 0$ and $\eta_6 = \eta_5$. At this point the boost operator V in Eq. (IV.18) is simply unity: $V = 1$.

By using this trick it is also easy to verify that the wave equations (IV.38) and (IV.39) do indeed correspond to the Dirac, Klein-Gordon and Maxwell equations for the physical field components in Minkowski space, with minimal electromagnetic interaction and nonderivative pseudoscalar pion-nucleon interaction.

In Section III an alternative characterization of all (physically acceptable) conformal invariant couplings for spin $\leqslant 1$ has been given. If all vector mesons are assumed to be gauge fields, then the manifestly conformal invariant couplings given above, and their obvious generalization to the case of several fields of the same spin, exhaust all possibilities for spin $\leqslant 1$. This may be checked by enumerating all possibilities, as there are only a few types of couplings with dimensionless coupling constant, and the gauge field couplings are fixed in their form.

V. REPRESENTATIONS INDUCED BY REPRESENTATIONS OF THE LITTLE GROUP WITH $\kappa_\mu \neq 0$.

In this Section we review an attempt to understand $V - A$ or $V + A$ weak interaction theory as a conformally invariant theory of fundamental interactions. This is at the same time as yet the only application which representations with $\kappa_\mu \neq 0$ have found. To the best of our knowledge, the idea goes back to Hepner (*30*). In the sequel we follow Hepner in part.

Assume we are working with a four component spinor (quark, μ or e-field). If we postulate that a four fermion interaction be invariant under conformal transformations (II.10) with (IV.6), then there exist two possible couplings

$$g\bar{\psi}_1\psi_2\bar{\psi}_3\psi_4 \tag{V.1a}$$
$$g\bar{\psi}_1\gamma_\mu(1 \pm i\gamma_5)\,\psi_2\bar{\psi}_3\gamma^\mu(1 \pm i\gamma_5)\,\psi_4 \tag{V.1b}$$

where we have $(1 \pm i\gamma_5)$ depending on whether $\kappa_\mu = \mp \frac{1}{2}\gamma_\mu(1 \pm i\gamma_5)$, i.e. which of the two four-dimensional representations $\Delta^{(\pm)}$ we use for the little group. (These couplings correspond to $\bar{\chi}_1\chi_2\bar{\chi}_3\chi_4$ and $\eta_A\bar{\chi}_1\gamma^{AB}\chi_2\eta^C\bar{\chi}_3\gamma_{CB}\chi_4$ in the six dimensional language of Section IV.) The most general conformal invariant linear field equation with interaction (V.1b) is determined to be

$$\left\{i\gamma_\mu \tfrac{1}{2}(1 \pm i\gamma_5)\,\partial^\mu + \frac{i}{2}(n-2)\,\gamma_5 - m\right\}\psi = g\gamma_\mu(1 \pm i\gamma_5)\,\psi\bar{\psi}\gamma^\mu(1 \pm i\gamma_5)\psi \qquad \text{(V.2)}$$

Here $\Delta = n \mp \frac{1}{2}\gamma_5$, this relates n to Δ which appears in the transformation law (II.10). It turns out that it must satisfy $n^3 - n + 2 = 0$ in order that Eq. (V.2) be actually invariant; therefore $n \neq 2$.

While the interaction term is precisely what we want, the LHS of Eq. (V.2) is not. For one thing the term $\frac{1}{2}i(n-2)\gamma_5 \neq 0$ prevents us from having a conserved hermitian probability current. Secondly, those components of ψ which are not coupled do not contribute to the kinetic energy in Eq. (V.2) either. Therefore, if we want to keep all four components of our spinor field as physical we have no choice but to add suitable symmetry breaking terms to the kinetic energy in Eq. (V.2) such that we arrive at the usual Dirac equation with weak interaction:

$$(i\gamma_\mu\partial^\mu - m)\psi = g\gamma_\mu(1 \pm i\gamma_5)\,\psi\bar{\psi}\gamma^\mu(1 \pm i\gamma_5)\psi \qquad \text{(V.3)}$$

In other words, the interaction term in Eq. (V.3) is conformal invariant, but the kinetic energy term is not. Of course, we thereby loose the existence of hermitian dilatation and conformal currents, as these rely on the invariance of the canonical equal time C.R. of the fields with their conjugate momenta, and hence of the kinetic energy (apart from mass). This is in contrast with the theory of Section III which is different from the one presently discussed both in physical spirit and mathematical detail.

APPENDIX: THE σ-MODEL AS AN ILLUSTRATION OF IDEAS IN SECTION III

Consider the Lagrangian of the σ-model of Gell-Mann and Lévy (*43*)

$$\begin{aligned}\mathscr{L} &= \bar{N}(i\not\partial - M)\,N + ig\bar{N}\gamma_5\boldsymbol{\tau}N\boldsymbol{\pi} + \tfrac{1}{2}(\partial_\mu\boldsymbol{\pi}\,\partial^\mu\boldsymbol{\pi} - \mu^2\pi^2)\\ &\quad + \frac{1}{2}\left(\partial_\mu\,\sigma\partial^\mu\sigma - \left\{\mu^2 + \frac{2\lambda}{f^2}\right\}\sigma^2\right) - \lambda\left(\{\pi^2+\sigma^2\} - \frac{2}{f}\,\sigma\{\pi^2+\sigma^2\}\right). \qquad \text{(A.1)}\end{aligned}$$

Here N is the nucleon field, π is the pion field, and σ is the field of a $I = 0$, $J^P = 0^+$ meson. $f = g/2M$.

Let us choose the free parameter λ to be

$$\lambda = g^2 \frac{\mu^2}{4M^2}$$

Reexpressing the Lagrangian in terms of the field $\sigma' = \sigma - (2f)^{-1}$, and calculating the dilatation current and conformal currents from Eq. (III.3), one finds for their divergences

$$\text{(a) } \partial^\mu \mathscr{D}_\mu = g^{-1} M m_\sigma^2 \sigma(x), \qquad \text{(b) } \partial^\nu{}_\nu \mathscr{K}_\mu = 2x_\mu \partial^\nu \mathscr{D}_\nu \,. \tag{A.2}$$

The last equation follows direct from Eq. (III.10)ff since the present Lagrangian does not involve any derivative couplings. m_σ is the (bare) σ-mass. *We see that in the limit of a massless boson σ both currents are conserved, and we have a spontaneous breakdown of conformal symmetry.*

With the usual definition of the axial vector current $\mathscr{A}_j{}^\mu$ for this model, one finds that generally, also for $m_\sigma \neq 0$

$$[F_i{}^5(x_0), \partial_\mu \mathscr{A}_j{}^\mu(x)] = -i\delta_{ij}\tfrac{1}{3}(\partial^\mu \mathscr{D}_\mu - \tfrac{3}{4}\mu^2 f^{-2}) \qquad \text{for} \quad i, j = 1, 2, 3$$

where

$$F_i{}^5 = \int d^3x \mathscr{A}_i{}^0. \tag{A.3}$$

Elsewhere it has been proposed to generalize this formula to chiral $SU(3) \otimes SU(3)$ in the following form (*16*):

$$\partial^\mu \mathscr{D}_\mu = \alpha_0 u_0(x) + \alpha_8 u_8(x) - \langle 0 \mid \{\alpha_0 u_0 + \alpha_8 u_8\} \mid 0 \rangle \tag{A.4}$$

with

$$\alpha_0 + \frac{1}{\sqrt{2}} \alpha_8 = -3 \sqrt{\frac{3}{2}}$$

α_8/α_0 is a measure of the breaking of the eightfold way. The u_i must satisfy the C.R. of (integrated) scalar densities with vector and axial vector currents as proposed by Gell-Mann ($i, j, k = 0 \cdots 8$) (*44*)

$$\begin{aligned} [F_i{}^5, v_j(x)]_{\text{eq.t.}} &= i\, d_{ijk} u_k(x) \\ [F_i{}^5, u_j(x)]_{\text{eq.t.}} &= -i\, d_{ijk} v_k(x) \end{aligned} \tag{A.5}$$

with

$$v_j(x) = \partial^\mu \mathscr{A}_\mu{}^j(x) \qquad \text{for} \quad j = 1, 2, 3.$$

The matrix elements of $u_0 + 1/\sqrt{2}\, u_8$ are known in current algebra calculations as "σ-terms." A method to calculate them on the basis of Eq. (A.4) and Eq. (III.1) has been outlined in Ref. (*16*).

Finally we wish to point out that the "σ-meson" plays two different roles in this model. Firstly it is a manifestation of the breaking of dilatation symmetry, Eq. (A.2a). Secondly, it provides an attractive πN force which is necessary to cancel most of the big s-wave repulsion inherent in the nonderivative πN coupling, which is not observed experimentally. Recall that requirement of the validity of Eq. (A.2b) does not allow for a derivative πN coupling (Section III). There seems to be experimental evidence for the existence of a scalar, isoscalar meson with a mass around 700 MeV (*45*).

RECEIVED: January 2, 1969

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