Topics in conformal Regge theory

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Outline

Introduction to Regge theory

Overview

- Regge theory has been a useful tool in understanding the high energy behavior of scattering amplitudes.
- With the advent of the AdS/CFT correspondence, it is an interesting question to ask if there is a generalization of Regge theory in the context of conformal field theories.
- The goal of this thesis is to explore various aspects of conformal Regge theory and its applications.

Kinematics of S matrix

- Consider the scattering matrix of four identical particles denoted by S.
- Constrained by symmetry to depend only on the Mandelstam variables,

$$\begin{aligned} &s = (p_1 + p_2)^2, \quad t = (p_1 + p_3)^2, \quad u = (p_1 + p_4)^2 \\ &s + t + u = 4m^2. \end{aligned}$$

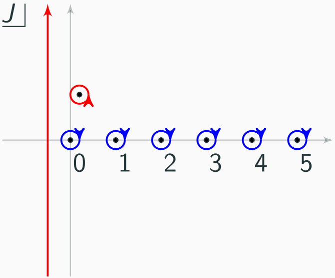
- Partial wave expansion in the s channel with $\cos\theta = 1 + \frac{2t}{s-4m^2}$,

$$S(s,t) = \sum_{J=0}^{\infty} (2J+1) S_J(s) P_J(\cos\theta).$$

- Particles exchanges in the s channel with mass m show up as poles in the s variable.

High energy scattering and Regge limit

- Regge limit corresponds to high energy scattering with fixed momentum transfer
- In terms of the Mandelstam variables, $s \gg |t|$.
- While individual partial wave amplitudes grow with energy, the sum over partial waves is bounded.
- The partial wave coefficients conspire to give a finite amplitude in the Regge limit.
- This can be shown using Sommerfeld-Watson transform.



Reggeized amplitude

- The sum over spins J, can be replaced by a contour integral in the complex angular momentum plane.

$$\sum_{J=0,even}^{\infty} \rightarrow \frac{1}{2\pi i} \oint_{\mathcal{C}} dJ \frac{e^{i\pi J}}{1 - e^{\pi i J}}.$$

- Since individual partial wave grows as s^{J-1}, the sum over spin can be handled using contour deformation.
- Thus, an effective particle with a complex spin dominates the scattering amplitude, shown in red in the figure.
- Scattering amplitude takes a particularly simple form in the Regge limit

$$S(s,t) \sim (-s)^{\alpha(t)} \quad \text{with} \quad \alpha(t) = \alpha(0) + \alpha' t.$$

- The Regge trajectory $\alpha(t)$ is a function of the momentum transfer t.

Kinematics of conformal field thoeries

- Kinematics of the conformal field theories is constrained by the conformal group $SO(d+1,1)$.
- The discussion simplifies by working in the embedding space formalism, wherein the action of the conformal group is linear.

$$X^A = \left(\frac{1+x^2}{2}, \frac{1-x^2}{2}, x^\mu\right) \quad X^2 = 0.$$

- Correlation functions of two and three scalar operators are essentially fixed,

$$\langle \phi(X)\,\phi(Y)\rangle = \frac{\delta_{\Delta_1\Delta_2}}{(X\cdot Y)^{\Delta_1}}$$

$$\langle \phi(X)\,\phi(Y)\,\phi(Z)\rangle = \frac{c}{(X\cdot Y)^{\alpha_{123}}\,(Y\cdot Z)^{\alpha_{231}}\,(Z\cdot X)^{\alpha_{132}}},$$

$$\alpha_{ijk} = (\Delta_i + \Delta_j - \Delta_k)/2.$$

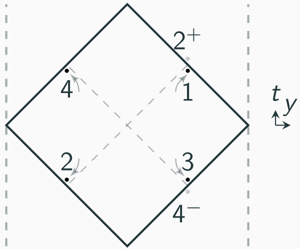
Regge limit in CFT

- The four point function is not completely fixed by conformal symmetry. It is fixed upto a function of the cross ratios u and v.

$$u = z\bar{z} = \frac{X_{12}^2 X_{34}^2}{X_{13}^2 X_{24}^2}, \quad v = (1-z)(1-\bar{z}) = \frac{X_{14}^2 X_{23}^2}{X_{13}^2 X_{24}^2}.$$

- One can consider a Lorentzian configuration of the cross ratios which acts like a high energy scattering, called conformal Regge limit.
- In terms of the cross ratios, it is given by

$$z \to 0, \quad \bar{z} \circlearrowleft 1, \quad \bar{z} \to 0, \quad z/\bar{z} \to \text{fixed}.$$



Double discontinuity

- An important object in this analysis is a certain linear combination of the correlator, called 'double discontinuity', say in the s channel,

$$\mathrm{dDisc}_s\, A\left(x_i\right) = -\frac{1}{2}\,\mathrm{Disc}_{12}\,\mathrm{Disc}_{34}\, A\left(x_i\right),$$
$$\mathrm{Disc}_{jk}\, A\left(x_i\right) = A\left(x_i\right)\big|_{x_{jk}^2 \to x_{jk}^2 e^{i\pi}} - A\left(x_i\right)\big|_{x_{jk}^2 \to x_{jk}^2 e^{-i\pi}}\,. \quad (1)$$

- The double discontinuity is analogous to the imaginary part of the scattering amplitude.
- Positive semidefinite, $\mathrm{dDisc}_{s,t,u}\, A\left(x_i\right) \geq 0$.
- dDisc is sufficient to reconstruct the full correlator using dispersion relations,

$$A\left(u,v\right) = \int_{u',v'} K\left(u,v,u',v'\right) \mathrm{dDisc}\, A\left(u',v'\right). \quad (2)$$

- Admits a nice interpretation in terms of double commutator $\langle [\phi,\phi]\,[\phi,\phi]\rangle$

Optical theorem in AdS

Optical theorem in S matrix

- Optical theorem relates the imaginary part of the forward scattering amplitude to the total cross section.

$$\mathrm{Im}\left[\mathcal{A}\left(s, t=0\right)\right] \propto \sigma_{tot}\left(s\right).$$

- derivation follows from the unitarity of the S matrix, $S^\dagger S = 1$.
- Using $S = \mathbb{I} + iT$,

$$\mathrm{Im}\, T = T - T^\dagger = iT^\dagger T. \tag{3}$$

- For Feynman diagrams, the optical theorem follows from "cutting the diagram".

Motivation in CFT

- AdS/CFT correspondence relates the scattering in AdS to the correlation functions in CFT
- We discuss the optical theorem in CFT by generalizing the ideas from the S matrix.
- Potentially useful for constraining the CFT data which is difficult to compute using string theory methods
- Find the implications of full nonlinear unitarity on the CFT data
- Part of a larger program of unitarity methods in AdS/CFT

Double trace operators and large N expansion

- Consider a conformal field theory with large central charge c and tunable parameter λ.
- For instance, in $\mathcal{N} = 4$ super Yang-Mills theory, $\lambda = g_{YM}^2 N_c, c \propto N^2$ and N_c denotes the number of colors
- Single trace operator is an operator formed by tracing over several elementary fields $\mathcal{O} = \mathrm{Tr}\,(\Phi_1 \Phi_2 \cdots \Phi_n)$.
- Double trace operators are schematically of the form $[\mathcal{O}_1 \mathcal{O}_2]_{n,l} = \mathcal{O}_1 \Box^n \partial_{\mu_1} \partial_{\mu_2} \cdots \partial_{\mu_l} \mathcal{O}_2$

Derivation using CFT techniques

- Correlation function of four operators

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle = A^{1234} = T^{1234} \mathcal{A}^{1234}(u, v)$$

$$T^{1234} = \frac{1}{(y_{12})^{\Delta_1+\Delta_2} (y_{34})^{\Delta_3+\Delta_4}} \left(\frac{y_{14}^2}{y_{24}^2}\right)^{\Delta_{21}/2} \left(\frac{y_{14}^2}{y_{13}^2}\right)^{\Delta_{34}/2}$$

- Using the conformal block decomposition in the t channel,

$$A(y_i) = \sum_{\mathcal{O}} \langle \mathcal{O}_3 \mathcal{O}_2 | \mathcal{O} | \mathcal{O}_1 \mathcal{O}_4 \rangle, \tag{4}$$

where there sum is over the single and double trace operators and $|\mathcal{O}|$ denotes the projector.

Derivation using CFT techniques

- A correlator can be expanded in terms of 'partial waves',

$$A\left(y_i\right)=\sum_{\rho\in SO(d)}\int_{d/2}^{d/2+i\infty}\frac{d\Delta}{2\pi i}I\left(\Delta,\rho\right)\Psi_{\mathcal{O}_{\Delta,\rho}}\left(y_i\right).$$

- I is called 'OPE function' which has poles at the position of the operators in the OPE.
- Inspired by the S matrix optical theorem, we write a proposal for the CFT analgoue,

$$\begin{aligned}&\mathrm{dDisc}_t\ A_{\text{1-loop}}(y_i)\Big|_{\text{d.t.}}\\&=-\frac{1}{2}\sum_{\mathcal{O}_5,\mathcal{O}_6}\int dy_5 dy_6\ \mathrm{Disc}_{23}\,A^{3652}_{\text{tree}}(y_k)\,\mathbf{S}_5\mathbf{S}_6\ \mathrm{Disc}_{14}\,A^{1564}_{\text{tree}}(y_k)\Big|_{[\mathcal{O}_5\mathcal{O}_6]}.\end{aligned}$$

- We also show that this proposal follows from "Conglomeration" techniques,

$$|[\mathcal{O}_5\mathcal{O}_6]_{n,\ell}\rangle=\int dy_5 dy_6\,|\mathcal{O}_5(y_5)\mathcal{O}_6(y_6)\rangle\langle S[\mathcal{O}_5](y_5)S[\mathcal{O}_6](y_6)[\mathcal{O}_5\mathcal{O}_6]_{n,\ell}\rangle\,.$$

Simplification in the impact parameter space

- CFT optical theorem simplifies in the 'impact parameter space'
- Impact parameter space concerns two variables S, L where S corresponds to the center of mass energy and L corresponds to the impact parameter
- In AdS, the impact parameter space is related to the geodesic distance between the two points along the transverse directions

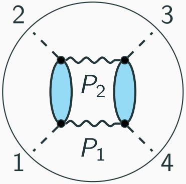
$$\begin{aligned} A(x, \bar{x}) &= \int_{M^+} dp d\bar{p}\, e^{-2ix\cdot y + 2i\bar{x}\cdot\bar{y}}\, A(p, \bar{p}) \, . \\ S = |p||\bar{p}|, &\quad S \cosh L = -p \cdot \bar{p}, \end{aligned}$$

- In the impact parameter space, the optical theorem takes the form

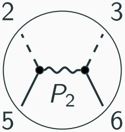
$$-\operatorname{Re} \mathcal{B}_{\text{1-loop}} = \frac{1}{2} \sum_{\mathcal{O}_5 \mathcal{O}_6} \mathcal{B}^{3652}_{\text{tree}}(-\bar{p}, -p)^* \, \mathcal{B}^{1564}_{\text{tree}}(p, \bar{p}) \, . \qquad (5)$$

Pictorial representation

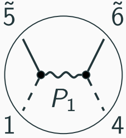
dDisc_t



$\sim \sum_{\mathcal{O}_5,\mathcal{O}_6} \int \mathrm{Disc}_{23}$



Disc_{14}



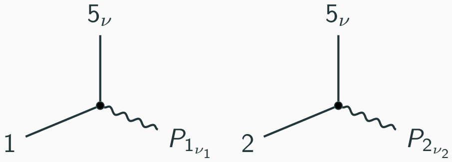
- The double discontinuity of the one loop correlator is related to the tree level correlator.
- $P_{1,2}$ denote the pomeron exchanges.
- Given dDisc, we can reconstruct the one loop correlator.

Constraints on CFT data

- Using this result, we can write a one loop correlator in the Regge limit in terms of the tree level result.
- While the states being exchanged are possibly in a complicated representation of the rotation group, they can be packaged into a single scalar function, called ‘vertex function’,

$$\mathcal{B}_{1-loop}(S,L) \approx \int d\nu d\nu_1 d\nu_2 \, V(\nu,\nu_1,\nu_2)^2 \, S^{j(\nu_1)+j(\nu_2)-2}\Omega_{i\nu}(L) .$$

- Here, $\Omega_{i\nu}$ denotes the harmonic function on the impact parameter space. Vertex function $V(\nu,\nu_1,\nu_2)$ can be described pictorially by



Flat space limit

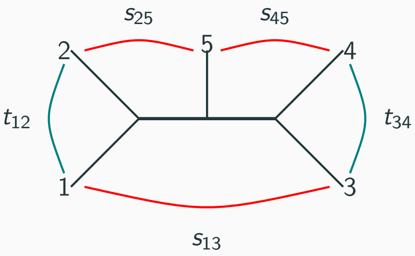
- AdS/CFT admits a flat space limit.
- In the impact parameter space, it can be achieved by taking the radius of AdS, R, to infinity using the following identification,

$$S = \frac{R^2 s}{4} \qquad\qquad L = \frac{b}{R}.$$

  where s denotes the center of mass energy and b denotes the impact parameter, in the flat space.
- Flat space vertex function can be calculated using string theory methods.
- Requiring that the flat space vertex function matches with the CFT vertex function, we can constrain the CFT data.

Conformal multi-Regge theory

S matrix multi Regge limit

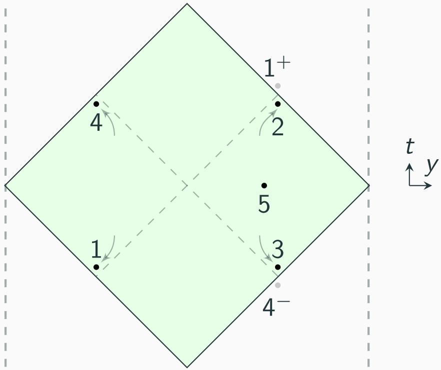


- Mandelstam invariants: $s_{i,j} = -\left(k_{i+1} + \cdots + k_j\right)^2$

$$\begin{aligned} s \text{ type}: &\quad s_{r+1,r+3}\,, \\ t \text{ type}: &\quad s_{1,r+2}\,, \\ \omega \text{ type (Toller angles)}: &\quad \frac{s_{p-2,p}s_{p-1,p+1}}{s_{p-2,p+1}}\,, \end{aligned}$$

  where labels are $r = 1, \cdots, n-3$, $p = 4, \cdots, n-1$.
- Regge limit: s type variables $\to \infty$ with the rest of the variables fixed

Regge limit proposal in CFT



- Our proposal for the conformal multi-Regge limit.
- Note that the point 5 is fixed away from the plane with the other points, generically.

Checks of the proposal

- It reduces to the four point Regge limit when the operator at the point 5 is the identity.
- Cross ratios for the five point functions are chosed to be,

$$u_1 = \frac{x_{12}^2 x_{35}^2}{x_{13}^2 x_{25}^2}\,, \qquad u_{i+1} = u_i\big|_{x_i \to x_{i+1}}\,. \tag{6}$$

  Euclidean OPE limit corresponds to $u_1, u_3, 1-u_2, 1-u_4, 1-u_5 \to 0$ while keeping three geometric angles ξ_1, ξ_2, ξ_3 fixed.
- Our proposal reproduces this while being a Lorentzian limit and crossing several lightcones.
- This serves as a nontrivial check of consistency of the proposal.
- It generalizes the notion that the Regge limit is OPE limit on the secondary sheet of the correlator.

Structure of the conformal blocks

- Five point conformal block is a complicated function of five cross ratios, $G_{J_1,J_2,\ell} \approx \langle 1,2,R_1\rangle\langle R_1,5,R_2\rangle^{\ell}\langle 4,3,R_2\rangle$.
- We consider the block in the Euclidean OPE limit.
- It is useful as a first step towards the derivation of Euclidean inversion formula.
- We use the following cross ratios,

$$\xi_1 = \frac{1-u_5}{2\sqrt{u_1}}\,, \qquad \xi_2 = \frac{1-u_4}{2\sqrt{u_3}}\,, \qquad \xi_3 = \frac{u_2-1}{2\sqrt{u_1}\sqrt{u_3}}\,,$$

- We show that there exists a basis of three point functions $J_1 - J_2 - \ell$ that dramatically simplifies the structure of the block

$$\begin{aligned} G_{\Delta_1,J_1,\Delta_2,J_2,\ell} \approx\, & u_1^{\frac{\Delta_1}{2}} u_3^{\frac{\Delta_2}{2}} \left(\left(1-\xi_1^2\right)\left(1-\xi_2^2\right)\right)^{\ell/2} \\ & \times C^{h-1+\ell}_{J_1-\ell}(\xi_1)\, C^{h-1+\ell}_{J_2-\ell}(\xi_2)\, C^{h-3/2}_{\ell}\left(\frac{\xi_3+\xi_1\xi_2}{\sqrt{(1-\xi_1^2)(1-\xi_2^2)}}\right) \end{aligned}$$

Relation to Mellin amplitudes

- Mellin amplitudes for the correlator are defined as

$$\langle \mathcal{O}(x_1) \dots \mathcal{O}(x_n) \rangle = \int [d\delta_{ij}] \, \mathcal{M}(\delta_{ij}) \prod_{1 \le i < j \le n} \frac{\Gamma(\delta_{ij})}{(x_{ij}^2)^{\delta_{ij}}} \, ,$$

- After the following identifications, the proposed Regge limit gives rise to a limit of Mellin amplitude that generalizes the S-matrix Regge limit.

$$t_{12} = 2\Delta_\phi - 2\delta_{12} \, , \qquad t_{34} = 2\Delta_\phi - 2\delta_{34} \, ,$$
$$s_{13} = \Delta_\phi + 2\delta_{13} \, , \qquad s_{25} = -2\delta_{25} \, , \qquad s_{45} = -2\delta_{45} \, ,$$

- Note that the precise analytic continuations in the cross ratios are crucial for this to work.
- This is another nontrivial check of the proposal.

Regge limit of the conformal blocks

- We consider the Regge limit of the conformal blocks in Mellin space to arrive at

$$\mathcal{M}\left(s_{ij}, t_{ij}\right) \approx s_{25}^{J_1-\ell} s_{45}^{J_2-\ell} s_{13}^{\ell} f\left(t_{12}, t_{34}\right).$$

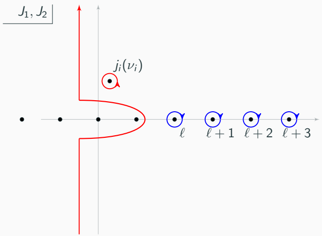
- We write the recursion relation representation for f and analytically solve it for $J_1 = J_2 \geq \ell$.
- Conformal correlator in Mellin space can be expanded in terms of partial waves in Mellin space as,

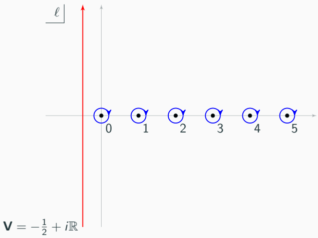
$$\mathcal{M}(s_{ij}, t_{ij}) = \sum_{\ell=0}^{\infty} \sum_{J_1, J_2=\ell}^{\infty} \int \frac{d\nu_1}{2\pi i} \frac{d\nu_2}{2\pi i}\, b_{J_1,J_2,\ell}(\nu_1, \nu_2) \mathcal{M}_{J_1,J_2,\ell}(s_{ij}, t_{ij}).$$

- The OPE function is denoted by b,

$$b_{J_1,J_2,\ell}\left(\nu_1, \nu_2\right) \approx \frac{P^{\ell}_{\nu_1,\nu_2,J_1,J_2}}{\left(\nu_1^2 - \left(\Delta_1 - h\right)^2\right)\left(\nu_2^2 - \left(\Delta_2 - h\right)^2\right)}.$$

Sommerfeld Watson transform





- Integration contours for the spin quantum numbers J_1, J_2, as well as ℓ
- Blue: Euclidean contour, Red: Regge contour
- We are assuming that there are no dynamical poles in the ℓ plane.
- This assumption is motivated by the analogous claim in the flat space limit.
- It would be interesting to study the validity of this assumption in the future, using Steinmann relation for CFTs.

Reggeized correlator

- In the Regge limit, the correlator is dominated by the Regge pole.
- This leads to an integral representation for the correlator in the Regge limit in terms of the CFT data,

$$\mathcal{M} = \int \frac{d\nu_1}{2\pi i} \frac{d\nu_2}{2\pi i} \, s_{25}^{j_1(\nu_1)} s_{45}^{j_2(\nu_2)} \times \int_{\mathbf{V}} \frac{d\ell}{2\pi i} \, \frac{b_{j_1(\nu_1), j_2(\nu_2), \ell}(\nu_1, \nu_2) \, f_{\nu_1, \nu_2, \ell}(t_{ij}) \, \eta^\ell}{\sin(\pi\ell) \sin\big(\pi(j_1(\nu_1) - \ell)\big) \sin\big(\pi(j_2(\nu_2) - \ell)\big)} \,,$$

Conclusions

Conclusions

- We studied two aspect of conformal Regge theory.
- First, we studied the optical theorem in AdS/CFT in the Regge limit.
- Abstractly, we showed that the one loop correlator can be obtained by squaring the tree level correlator.
- In the future, it would be interesting to study eikonalization of the CFT amplitude in the Regge limit at finite λ, by studying the higher loop correlators.
- Second, we studied the generalization of the conformal Regge limit to higher point correlation functions.
- In the future, it would be interesting to study the dispersion relation and polynomial boundedness of the higher point correlators in CFTs.

Thank you!