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# DETAILED EFFECTS OF LIMB DARKENING UPON LIGHT AND VELOCITY CURVES OF CLOSE BINARY SYSTEMS

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*ABSTRACT. Theoretical expressions are established for light and velocity curves exhibited by the components of close binary systems, whose form is one of equilibrium under the instantaneous centrifugal and tidal forces, and whose apparent surface brightness varies from center to limb in accordance with a generalized cosine law of the form*

$$J(\gamma) = J(0)\ (1 - u_1 - u_2 - u_3 - \dots + u_1\cos\gamma + u_2\cos^2\gamma + u_3\cos^3\gamma + \dots),$$

*where $\gamma$ denotes the angle of foreshortening and $u_1$, $u_2$, $u_3$, ... are the various coefficients of limb darkening. These expressions, which hold good for the variation in light or radial velocity between minima as well as within eclipses (of any type), are exact when the rotation of the respective components is slow enough for quantities of the order of squares and higher powers of the rotational distortion to be negligible, and when the tidal action of the companion can be regarded as arising from a mass-point. All results are expressed in terms of the associated alpha-functions and the $\mathcal{J}$-integrals employed in the writer's previous investigations.*

*In the second part of the present paper the numerical values of the coefficients $u_1$, $u_2$, $u_3$, ... appropriate for integrated light are deduced from the theory of radiative equilibrium of stellar atmospheres, and the photometric consequences are investigated. The light changes exhibited by the rotation of distorted stars are found to be none too sensitive to the distribution of brightness near the limb of the star's apparent disks. On the whole, the deviations of the actual distribution of brightness from a simple cosine law are found to magnify the photometric effects of the second and third harmonic by one and four per cent, respectively, and to diminish the coefficient of the fourth harmonic by about two per cent. The differences between a quadratic and a bi-quadratic law of limb darkening of the above form turn out to lead to altogether negligible consequences.*

In two previous investigations by the writer[1] the theoretical light and radial velocity curves of close binary systems were investigated to the degree of accuracy to which quantities of the order of squares and higher powers of the rotational distortion remain negligible, and the mutual tidal action of the components can be regarded as being due to mass-points. Throughout these investigations, the apparent surface brightness $J$ of the constituent components was assumed to vary from center to limb in accordance with the well-known cosine law of the form

$$J(\gamma) = J(0)\ (1 - u - u\cos\gamma), \tag{1}$$

where $\gamma$ denotes, as usual, the angle of foreshortening and $u$, the "coefficient of limb darkening". A formula of this form has so far been used exclusively in all investigations dealing with the eclipsing stars, mainly on grounds of simplicity, and also because it appeared to represent the center-to-limb variation of brightness over most of the apparent disk of the sun with tolerable accuracy[2]. Furthermore, some theoretical justification for such a formula was provided in 1906 by K. Schwarzschild[3], who proved that if the actual radiation field in an ordinary stellar atmosphere is replaced by two anti-parallel streams, an approximate solution of the respective equation in radiative transfer can indeed be made consistent with a "law" of the form *(1)*. The subsequent three decades which elapsed since the days of Schwarzschild's pioneer investigation have witnessed a great effort, by many workers[4], in the search for better approximations, until a rigorous solution of the equation of transfer in the plane problem was eventually established by Wiener and Hopf[5]. A comparison of this solution with a simple cosine law (see Fig. 1) reveals at a glance the weakness of the latter near the limb of a star, and stimulates therefore an inquiry as to what extent the deviations of the actual variation of $J(\gamma)$ from simple cosine law may affect the theoretical light or velocity curves of close binary systems.

It may indeed be objected that, in all previous investigations, the effects of gravity darkening were probably considered to greater detail than those of limb darkening, because of the inaccuracy inherent to the adopted law *(1)*. The previously derived expressions for theoretical light or velocity curves of close binaries appear therefore to be in need of a refinement by resorting to a more appropriate law of limb darkening which, though less simple, corresponds more closely to reality. The limitations of the "linear" law of darkening, and of all deductions based upon it, have recently been emphasized by Münch and Chandrasekhar in the preceding Circular of this series. In the present paper such refinements will be given.

We shall presume, quite generally, that the actual law of limb darkening can be expanded in a series of the form

$$J(\gamma) = J(0)\,(1 - u_1 - u_2 - u_3 - \ldots + u_1\cos\gamma + u_2\cos^2\gamma + u_3\cos^3\gamma + \ldots), \qquad (2)$$

and, in what follows, shall investigate the photometric as well as spectroscopic consequences of the individual terms on the right-hand side of *(2)*. The respective analysis will be found to follow lines so parallel to those developed in earlier investigations by the writer that frequent appeal to these memoirs - which, will hereafter consistently be referred to as paper I and II, respectively - will facilitate our present proposition to such an extent as to enable us to write down the results in an almost telegraphic style.

In the second part of this paper, we shall fall back to the physical theory of stellar limb darkening for the purpose of determining the numerical values of the coefficients $u_1, u_2, u_3, \ldots$ in total radiation, and discuss the significance of the results in more general terms. In particular, we shall find that the light changes exhibited by close binary systems out of eclipses are, on the whole, not too sensitive to the distribution of brightness near the limb of their components. The deviations of the actual distribution of brightness from *(1)* will be found to magnify the photometric effects of the second and third tidal harmonic distortion by one and four per cent, respectively, and to diminish the coefficient of the fourth harmonic by about two per cent.

## PART I

As is well known, the light $\mathfrak{L}$ of a component of a binary system as seen by a distant observer can be generally expressed as

$$\mathfrak{L} = \int_S J(\gamma)\cos\gamma\, d\sigma\,, \qquad (3)$$

where $d\sigma$ denotes the respective surface element and $S$, the range of integration which is to be extended over the whole visible area. Let, furthermore, the law of limb darkening $J(\gamma)$ be expressible as a series of the form[6]

$$J(\gamma) = J(0)\,\{1 + \sum_{i=0}^{n} u_i(\cos^i\gamma - 1)\}\,, \qquad (2.1)$$

where $J(\gamma)$ denotes the intensity of radiation emerging from a stellar atmosphere in the direction parallel to the line of sight (which, in distorted stars, will in turn vary proportionally to local gravity), and $n$ is an arbitrary integer. The reader may notice that, for $n = 0$, equation *(2.1)* corresponds to uniformly bright disks; for $n = 1$, to a conventional darkening at limb of the form *(1)*. The main object of our inquiry in the present paper will be the consequences of *(2.1)* when $n > 1$.

Without the loss of generality, let us consider temporarily our star to be a sphere of radius $r_1$. Then, out of eclipses, its full light will be equal to

$$\pi\int_0^{r_1} J\sin^{-1}(r/r_1)\,dr^2 = \pi r_1^2\, J(0)\;\; 1 - \sum_{i=0}^{n}\frac{iu_i}{i+2} \qquad (4)$$

If we now divide *(3)* by *(4)* and denote by $\pi r^2 J(0)\mathfrak{L}^{(j)}$ the fractional light of a disk whose surface brightness falls off from center to limb as a j-th power of $\cos\gamma$, the fractional light $\mathcal{L}$ of our component may evidently be expressed as a weighted mean of the individual $\mathcal{L}^{(j)}$'s of the form

$$\mathcal{L} = \sum_{j=0}^{n} C^{(j)}\mathcal{L}^{(j)}, \tag{5}$$

where

$$C^{(0)} = \frac{1 - u_1 - u_2 - \ldots - u_n}{1 - \sum_{i=1}^{n} \frac{iu_i}{2+i}} \tag{6.0}$$

while, for $j > 0$,

$$C^{(j)} = \frac{2}{2+j} \frac{u_j}{1 - \sum_{i=1}^{n} \frac{iu_i}{2+i}} \tag{6.1}$$

If our star were distorted, the foregoing equations *(5)-(6)* would continue to hold good; it is only important to remember that the unit in terms of which $\mathcal{L}$ is expressed is the luminosity of the respective component in its undistorted state.

The reader may easily verify that, in the case of an ordinary limb darkening ($n = 1$), equation *(5)* reduces to the well-known formula

$$\mathcal{L} = \frac{3(1-u_1)}{3 - u_1}\mathcal{L}^{(0)} + \frac{2u_1}{3-u_1}\mathcal{L}^{(1)}; \tag{7.0}$$

if $n = 2$, it yields

$$\mathcal{L} = \frac{6(1 - u_1 - u_2)}{6 - 2u_1 - 3u_2}\mathcal{L}^{(0)} + \frac{4u_1}{6 - 2u_1 - 3u_2}\mathcal{L}^{(1)} + \frac{3u_2}{6 - 2u_1 - 3u_2}\mathcal{L}^{(2)}. \tag{7.1}$$

If $n = 3$, .

$$\mathcal{L} = \frac{30(1 - u_1 - u_2 - u_3)}{30 - 10u_1 - 15u_2 - 18u_3}\mathcal{L}^{(0)} + \frac{20u_1}{30 - 10u_1 - 15u_2 - 18u_3}\mathcal{L}^{(1)}$$
$$+ \frac{15u_2}{30 - 10u_1 - 15u_2 - 18u_3}\mathcal{L}^{(2)} + \frac{12u_3}{30 - 10u_1 - 15u_2 - 18u_3}\mathcal{L}^{(3)}, \tag{7.2}$$

and if $n = 4$,

$$\mathcal{L} = \frac{30(1 - u_1 - u_2 - u_3 - u_4)}{30 - 10u_1 - 15u_2 - 18u_3 - 20u_4}\mathcal{L}^{(0)} + \frac{1}{30 - 10u_1 - 15u_2 - 18u_3 - 20u_4} \times$$
$$\{20u_1\mathcal{L}^{(1)} + 15u_2\mathcal{L}^{(2)} + 12u_3\mathcal{L}^{(3)} + 10u_4\mathcal{L}^{(4)}\}. \tag{7.3}$$

The explicit forms of equation *(5)* for $n > 4$, if needed, may be worked out by the reader without any difficulty. It may be noticed in passing that, if all the $\mathcal{L}^{(j)}$'s are equal to unity, so is the resultant light $\mathcal{L}$; for the C's have evidently been normalized so as to render

$$\sum_{j=0}^{n} C^{(j)} = 1.$$

Now equation *(5)* or its particular forms *(7.0)-(7.1)* hold good regardless of what the limits of integration S in *(3)* may be, and can therefore also be employed to express the loss of light suffered by the component undergoing eclipse (when S is extended over the eclipsed area instead of over the whole disk). The way in which the loss of light caused by eclipses of limb and gravity darkened components of close binary systems are evaluated has been amply discussed in paper I, to which the reader is referred for details of procedure. With the aid of the theory developed in this latter investigation, the loss of light f(k, p) ( k being the ratio of the radii of both components; p , the geometrical depth of the eclipse) of stars whose limb darkening is governed by *(2)* can be written down at once: namely if, consistently with equation *(5)*, we set

$$f(k,p) = \sum_{j=0}^{n} C^{(j)} f^{(j)}(k,p) , \qquad (8)$$

and if, furthermore,

$$f^{(j)} = \frac{2}{j+2} \{f_*^{(j)} + f_1^{(j)} + f_2^{(j)}\} , \qquad (9)$$

where $f_*$ , $f_1$ , and $f_2$ have the same meaning as in equations *(66)* of paper I, then the reader should experience no difficulty in proving that

$$\begin{aligned} f_*^{(j)}(k,p) = \; & \tfrac{1}{3} \{\tfrac{1}{2}[\beta_2 - 2(j+2)][3(n_0^2 - n_1^2)\, \alpha^0_{j+2} + 3(n_2^2 - n_1^2)\, \alpha^2_j + 6n_0 n_2 \alpha^1_{j+1} + 2P_2(n_1)\alpha^0_j] \\ & \qquad + (j+1)[2P_2(n_0)\alpha^0_j + 3n_0 n_2 \alpha^1_{j-1}]\} \, v_1^{(2)} \\ & \{\tfrac{1}{2}[\beta_2 - 2(j+2)][3l_0^2\alpha^0_{j+2} + 6l_0 l_2 \alpha^1_{1+j} + 3l_2^2\alpha^2_j - \alpha^0_j] \\ & \qquad + (j+1)[2P_2(l_0)\alpha^0_j + 3l_0 l_2 \alpha^1_{j-1}]\} \, w_1^{(2)} \\ & - \{\tfrac{1}{2}[\beta_3 - 3(j+2) + 1][5l_0^3\alpha^0_{j+3} + 15l_0^2 l_2 \alpha^1_{j+2} + 15 l_0 l_2^2 \alpha^2_{j+1} + 5l_2^3\alpha^3_j - 3l_0\alpha^0_{j+1} - 3l_2\alpha^1_j] \\ & + (j+1)[(l_0\alpha^0_{j+1} + 2l_2\alpha^1_j)\, P_3'(l_0) - \tfrac{3}{2} l_0(\alpha^0_{j+1} - 5l_2^2\alpha^2_{j-1} + \alpha^0_{j-1})]\} \, w_1^{(3)} \\ & - \{\tfrac{1}{8}[\beta_4 - 4(j+2) + 2][35l_0^4\alpha^0_{j+4} + 140l_0^3 l_2 \alpha^1_{j+3} + 210 l_0^2 l_2^2 \alpha^2_{j+2} + 140 l_0 l_2^3 \alpha^3_{j+1} \\ & \qquad + 35 l_2^4 \alpha^4_j - 30 l_0^2 \alpha^0_{j+2} - 60 l_0 l_2 \alpha^1_{j+1} - 30 l_2^2 \alpha^2_j + 3\alpha^0_j] \\ & + \tfrac{1}{2}(j+1)[2l_0 P_4'(l_0)\alpha^0_{j+2} + 15l_2^2(7l_0 - 1)\alpha^2_j - 2P_3'(l_0)\alpha^0_j + 5l_0 l_2(21 l_0^0 - 6)\alpha^1_{j+1} \\ & \qquad + 35 l_0 l_2^3 \alpha^3_{j-1} - 15 l_0 l_2 \alpha^1_{j-1}]\} \, w_1^{(4)} + \dots \end{aligned} \qquad (10)$$

$$\begin{aligned} f_1^{(j)}(k,p) = \; & \tfrac{1}{3} \{3n_0^2 \mathcal{J}^0_{-1,j+2} + 3n_1^2 \mathcal{J}^0_{1,j} + 3n_2^2 \mathcal{J}^2_{-1,j} + 6n_0 n_2 \mathcal{J}^1_{-1,j+1} - \mathcal{J}^0_{-1,j}\} \, v_1^{(2)} \\ & - \{3l_0^2 \mathcal{J}^0_{-1,j+2} + 6l_0 l_2 \mathcal{J}^1_{-1,j+1} + 3l_2^2 \mathcal{J}^2_{-1,j} - \mathcal{J}^0_{-1,j}\} \, w_1^{(2)} \end{aligned}$$

$$- \{5l_0^3\, \mathcal{A}^0_{-1,j+3} + 15l_0^2 l_2\, \mathcal{A}^1_{-1,j+2} + 15l_0 l_2^2\, \mathcal{A}^2_{-1,j+1} + 5l_2^3\, \mathcal{A}^3_{-1,j} - 3l_0\, \mathcal{A}^0_{-1,j+1} - 3l_2\, \mathcal{A}^1_{-1,j}\}\, w_1^{(3)} \tag{11}$$

$$- \tfrac{1}{4}\{35l_0^4\, \mathcal{A}^0_{-1,j+4} + 140l_0^3 l_2\, \mathcal{A}^1_{-1,j+2} + 210l_0^2 l_2^2\, \mathcal{A}^2_{-1,j+2} + 140l_0 l_2^3\, \mathcal{A}^3_{-1,j+1} + 35l_2^4\, \mathcal{A}^4_{-1,j} - 30l_2^2\, \mathcal{A}^0_{-1,j+2} - 60l_0 l_2\, \mathcal{A}^1_{-1,j+1} - 30l_2^2\, \mathcal{A}^2_{-1,j} + 3\, \mathcal{A}^0_{-1,j}\}\, w_1^{(4)} + \ldots$$

and

$$(r_1/r_2)^2 f_2^{(j)}(k,p) = -\tfrac{1}{3}\{3n_1^2\, I^0_{1,j} + 3n_2^2\, I^2_{-1,j} - I^0_{-1,j}\}\, v^{(2)} + \{3l_2^2\, I^2_{-1,j} - I^0_{-1,j}\}\, w_2^{(2)} + \{5l_2^3\, I^3_{-1,j} - 3l_2\, I^1_{-1,j}\}\, w_2^{(3)} + \tfrac{1}{4}\{35l_2^4\, I^4_{-1,j} - 30l_2^2\, I^2_{-1,j} + 3\, I^0_{-1,j}\}\, w_2^{(4)} + \ldots \tag{12}$$

where, apart from

$$\beta_k = \{\frac{2k+1}{\Delta_k} + 1 - k\}\,\tau, \qquad k = 2, 3, 4$$

all symbols and abbreviations have exactly the same meanings as in paper I and their explanations need not therefore be repeated.

The foregoing equations *(10)-(12)* are exact as long as the axial rotation of the component undergoing eclipse is so slow that squares and higher powers of its rotational distortion can be ignored, and as long as the tidal action of the secondary may be regarded as arising from a mass-point. The reader may also notice that $f^{(0)}$ and $f^{(1)}$ as defined by the above equations are identical with $f^U$ and $(2/3)f^D$ of paper I, respectively. With the introduction of limb darkening of higher order the old notation using letters for superscripts would now become ambiguous and had to be given up.

The effects of higher-order darkening upon radial velocity curves of components of close binary systems can be written down with equal ease. If we adhere strictly to the notations used in paper II, the g-functions (analogous to the f's of the photometric problem) readily take the forms

$$\begin{aligned} g_*^{(j)}(k,p) = {} & \{\tfrac{1}{2}[\beta_2 - 2(j+2) - 1][(n_0^2 - n_1^2)\alpha_{j+2} + (n_2^2 - n_1^2)\alpha_j^3 + 2n_0 n_2 \alpha_{j+1}^2 + \tfrac{2}{3}P_2(n_1)\alpha_j^1] \\ & + (j+1)[\tfrac{2}{3}P_2(n_0)\alpha_j^1 + n_0 n_2 \alpha_{j-1}^2]\}\, n_1 v_1^{(2)} \\ & - \{[\beta_2 - 2(j+2) - 1][n_0(\alpha_{j+1}^0 - \alpha_{j+1}^2 - \alpha_{j+3}^0) + n_2(\alpha_j^1 - \alpha_j^3 - \alpha_{j+2}^1)] \\ & + (j+1)[n_0(\alpha_{j-1}^0 - \alpha_{j-1}^2 - \alpha_{j+1}^0)]\}\, n_1 n_2\, v_1^{(2)} \\ & - \{\tfrac{1}{2}[\beta_2 - 2(j+2) - 1][3l_0^2\alpha_{j+2}^1 + 6l_0 l_2 \alpha_{j+1}^2 + 3l_2^2\alpha_j^3 - \alpha_{j+1}^0] \\ & + (j+1)[2P_2(l_0)\alpha_j^1 + 3l_0 l_2 \alpha_{j-1}^2]\}\, n_1 w_1^{(2)} \end{aligned} \tag{13}$$

$$
\begin{aligned}
&- \{\tfrac{1}{2}[\beta_3 - 3(j+2)]\,[5l_0^3 a_{j+3}^1 + 15l_0^2 l_2 a_{j+2}^2 + 15l_0 l_2^2 a_{j+1}^3 + 5l^3 a^4 - 3l_0 a_{j+1}^1 - 3l_2 a_j^2] \\
&\qquad + (j+1)[(l_0 a_{j+1}^1 + 2l_2 a_j^2)\, P_3'(l_0) - \tfrac{3}{2} l_0 (a_{j+1}^1 - 5l_2^2 a_{j-1}^3 \\
&\qquad\qquad + a_{j-1}^1)]\}\, n_1 w_1^{(3)} \\
&- \{\tfrac{1}{8}[\beta_4 - 4(j+2) + 1]\,[35l_0^4 a_{j+4}^1 + 140 l_0^3 l_2 a_{j+3}^2 + 210 l_0^2 l_2^2 a_{j+2}^3 \\
&\quad + 140 l_0 l_2^3 a_{j+1}^4 + 35 l_2^4 a_j^5 - 30 l_0^2 a_{j+2}^1 - 60 l_0 l_2 a_{j+1}^2 - 30 l_2^2 a_j^3 + 3a_j^1] \\
&\quad + \tfrac{1}{2}(j+1)[2l_0\, P_4'(l_0)\, a_{j+2}^1 + 15 l_2^2\, (7l_0^2 - 1) a_j^3 - 2P_3'(l_0)\, a_j^1 + 5 l_0 l_2 (21 l_0^2 a_{j+1}^2 \\
&\qquad - 6a_{j+1}^2 + 7l_2^2 a_{j-1}^4 - 3a_{j-1}^2)]\}\, w_1^{(4)} + \dots + a_j^1\,,
\end{aligned}
$$

$$
\begin{aligned}
g_1^{(j)}(k,p) \;=\;& \{n_0^2\, \mathcal{J}_{-1,j+2}^1 + n_1^2\, \mathcal{J}_{1,j}^1 + n_2^2\, \mathcal{J}_{-1,j}^3 + 2n_0 n_2\, \mathcal{J}_{-1,j+1}^2 - \tfrac{1}{3}\, \mathcal{J}_{-1,j}^1\}\, n_1 v_1^{(2)} \\
&- 2\{n_0\, \mathcal{J}_{1,j+1}^0 + n_2\, \mathcal{J}_{1,j}^1\}\, n_1 n_2 v_1^{(2)} \\
&- \{3l_0^2\, \mathcal{J}_{-1,j+2}^1 + 6 l_0 l_2\, \mathcal{J}_{-1,j+1}^1 + 3l^2\, \mathcal{J}_{-1,j}^2 - \mathcal{J}_{-1,j}^0\}\, n_1 w_1^{(2)} \\
&- \{5l_0^3\, \mathcal{J}_{-1,j+3}^1 + 15 l_0^2 l_2\, \mathcal{J}_{-1,j+2}^2 + 15 l_0 l_2^2\, \mathcal{J}_{-1,j+1}^3 + 5 l_2^3\, \mathcal{J}_{-1,j}^4 \\
&\qquad - 3l_0 \mathcal{J}_{-1,j+1}^1 - 3l_2\, \mathcal{J}_{-1,j}^2\}\, n_1 w_1^{(3)} \\
&- \tfrac{1}{4}\{35 l_0^4\, \mathcal{J}_{-1,j+4}^1 + 140 l_0^3 l_2\, \mathcal{J}_{-1,j+3}^2 + 210 l_0^2 l_2^2\, \mathcal{J}_{-1,j+2}^3 + 140 l_0 l_2^3\, \mathcal{J}_{-1,j+1}^4 \\
&\quad + 35 l_2^4\, \mathcal{J}_{-1,j}^5 - 30 l_0^2\, \mathcal{J}_{-1,j+2}^1 - 60 l_0 l_2\, \mathcal{J}_{-1,j+1}^2 - 30 l_2^2\, \mathcal{J}_{-1,j}^3 + 3\, \mathcal{J}_{-1,j}^1\}\, n_1 w_1^{(4)} + \dots
\end{aligned}
\tag{14}
$$

and

$$
\begin{aligned}
g_2^{(j)}(k,p) \;=\;& -\{n_1^2\, \mathcal{K}_{1,j}^1 + n_2^2\, \mathcal{K}_{-1,j}^3 - \tfrac{1}{3}\, \mathcal{K}_{-1,j}^1\}\, n_1 v_2^{(2)} + \frac{2n_1 n_2^2}{\pi r_1^2 r_2}\, v_2^{(2)}\, J_{1,j}^1 \\
&+ \{3l_2^2\, \mathcal{K}_{-1,j}^3 - \mathcal{K}_{-1,j}^1\}\, n_1 w_2^{(2)} + \{5 l_2^3\, \mathcal{K}_{-1,j}^4 - 3l_2\, \mathcal{K}_{-1,j}^2\}\, n_1 w_2^{(3)} \\
&+ \tfrac{1}{4}\{35 l_2^4\, \mathcal{K}_{-1,j}^5 - 30 l_2^2\, \mathcal{K}_{-1,j}^3 + 3\, \mathcal{K}_{-1,j}^1\}\, n_1 w_2^{(4)} + \dots
\end{aligned}
\tag{15}
$$

which permit us to specify the theoretical velocity curves of distorted components of close binary systems within eclipses to the same accuracy as was previously done with their light curves.

If the eclipse becomes total ($p = -1$) and the light intercepted by the eclipsing component becomes equal to the whole luminosity of the eclipsed star, it can be shown (cf. again papers I and II) that all "boundary integrals" $I_{\beta,\gamma}^m$, $\mathcal{J}_{\beta,\gamma}^m$, and $\mathcal{K}_{\beta,\gamma}^m$ vanish, and so do all associated alpha-functions $a_n^m$

whose order m is odd; while if m is even,

$$\alpha_{2\nu}^{2\mu}(k,-1) = \frac{\nu!\Gamma(\mu+\frac{1}{2})}{\sqrt{\pi}(\mu+\nu+1)!}$$

or $\left.\begin{matrix}\mu\\ \nu\end{matrix}\right] = 0,1,2,\ldots$

$$\alpha_{2\nu-1}^{2\mu}(k,-1) = \frac{\Gamma(\mu+\frac{1}{2})\Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}\Gamma(\mu+\nu+\frac{3}{2})}$$

depending on whether their index n is even or odd. In consequence,

$$f_1^{(j)}(j,-1) = f_2^{(j)}(k,-1) = 0,$$

$$g_1^{(j)}(k,-1) = g_2^{(j)}(k,-1) = 0,$$

the only non-vanishing f's or g's being those marked with asterisks. If we insert in them the above constants for the $\alpha$'s at $p = -1$ we find after some rearrangements that, out of eclipses,

$$f_*^{(j)}(k,-1) = \frac{(j+1)(4+\beta_2)}{j+4}\{\tfrac{1}{3}v_1^{(2)}P_2(n_0) - w_1^{(2)}P_2(l_0)\}$$

$$- \frac{j(j+2)(10+\beta_3)}{(j+3)(j+5)} w_1^{(3)}P_3(l_0) \tag{16}$$

$$- \frac{(j+1)(j+1)(18+\beta_4)}{(j+4)(j+6)} w_1^{(4)}P_4(l_0) - \ldots$$

whereas

$$g_*^{(j)}(k,-1) = -\frac{(j+2)(\beta_2-j)}{(j+3)(j+5)} w_1^{(2)}m_0P_2'(l_0)$$

$$- \frac{(j+1)(6+\beta_3-j)}{(j+4)(j+6)} w_1^{(3)}m_0P_3'(l_0) \tag{17}$$

$$- \frac{j(j+2)(14+\beta_4-j)}{(j+3)(j+5)(j+7)} w_1^{(4)}m_0P_4'(l_0) - \ldots$$

The variation of light $\delta\mathfrak{L}$ of limb and gravity darkened components of close binary systems between minima can then be expressed as

$$\delta\mathfrak{L} = \sum_{j=0}^{n} C^{(j)}f_*^{(j)}(k,-1). \tag{18}$$

while the contribution $\delta V$, to their observed radial velocity, arising from their axial rotation with an angular velocity $\omega_1$ turns out to be given by

$$\delta V = \omega_1 R_1 \sum_{j=0}^{n} C^{(j)} g_*^{(j)}(k,-1), \tag{19}$$

where $R_1$ stands for the mean radius of the respective rotating star. Particular cases of the foregoing relations for $n = 0$ (uniform disks) and $n = 1$ (ordinary limb darkening) have already been given in papers I and II; in the present investigation we have merely generalized them for any value of n - i.e., for any law of limb darkening in which $J(\gamma)$ falls off from center to limb as a linear combination of any integral power of cos $\gamma$.

The reader may notice that, unlike *(16)*, the right-hand side of equation *(17)* contains no term due to the rotational distortion; for the aspect of a rotational ellipsoid does not change, to a distant observer, in the course of a revolution irrespective of what its limb darkening may be. In consequence, the non-orbital contributions to radial velocities of rotating components of close binary systems are due to their tidal distortion alone. Furthermore, it should be observed that, in equation *(18)*, $f_*^{(j)}$ will contain no term varying as $P_3(l_0)$ if $j = 0$ (uniform disks), nor one varying as $P_4(l_0)$ if $j = 1$. For any law of darkening characterized by $j > 1$ all three harmonics are, however, bound to be present. As to the $g_*^{(j)}$ -function, none of the three terms on the right-hand side of equation *(17)* will ordinarily vanish except for the last one when $j = 0$ - provided, of course, that the $\beta$'s are not integers. Otherwise this will evidently happen if $\beta_2 = j$, $\beta_3 = j-6$, or $\beta_4 = j-14$. For total radiation ($\tau = 1$) and centrally condensed stars $\beta_2 = 4$, $\beta_3 = 5$, and $\beta_4 = 6$, in which case the effects upon radial velocity of the second, third, or fourth harmonic distortion of components in close binary systems should vanish if their apparent surface brightness were to vary from center to limb as $\cos^j\gamma$, where $j = 6$, 11, or 20, respectively. In reality, however, there is but little likelihood that the actual values of $\beta$, relevant to observations carried out in more or less limited spectral ranges, would come out to be integers. Hence, it is very probable that all three harmonics of non-orbital origin (due to the tides) will ordinarily be present in the observed radial-velocity curves of close binary systems.

## PART II

Thus far the coefficients $u_i$ in equation *(2)* were left unrestricted - save for the obvious requirement that $J(\gamma)$ is to remain positive over the whole visible hemisphere, and that $dJ/d\gamma$ remains negative. If these coefficients were really arbitrary, the problem of predicting theoretical light or velocity curves of close eclipsing systems would be largely indeterminate; for although it should be possible in principle to infer the u's simultaneously with other elements from the observed data, it would no doubt be exceedingly difficult to do so in practice. Fortunately, a great deal about the actual behavior of $J(\gamma)$ can be deduced from the theory of radiative transfer of light in stellar atmospheres.

As is well known, the variation of surface brightness over the apparent disk of a star is due to the temperature-gradient in its outer layers of finite optical depth. In the great majority of stars constituting binary systems the extent of these layers (i.e., of their atmospheres) appears to be so small a fraction of the total size of the star that their curvature can be neglected and the atmosphere itself regarded as stratified in plane parallel layers. If this is so, then the intensity $I(\tau,\gamma)$ of total radiation at an optical depth $\tau$ in radiative equilibrium is known to satisfy an integro-differential equation of the form

$$\cos\gamma \frac{dI(\tau,\gamma)}{d\tau} = I(\tau,\gamma) - \tfrac{1}{2}\int_{-1}^{+1} I(\tau,\gamma)\, d(\cos\gamma), \tag{20}$$

and our "law of darkening" $J(\gamma)$ which should govern the angular intensity distribution of light emerging from an atmosphere in the direction making an angle $\gamma$ with the line of sight is evidently identical with $I(0,\gamma)$.

Equation *(20)* has attracted the attention of a great many investigators ever since it was first derived by Schwarzschild, until its rigorous solution has finally been established by Wiener and Hopf[5]. This solution turned out to be expressible in terms of a certain definite integral which defined evaluation in a finite number of terms; but its value was recently tabulated by Chandrasekhar[7] for ten discrete values of $\cos\gamma$ by quadratures. The center-to-limb variation of surface brightness $J(\gamma)/J(0)$ corresponding to this rigorous solution is reproduced in column *(3)* of the following Table I. If we expand (by numerical means) this solution in a Taylor series in the neighborhood of $\cos\gamma = 1$, we find that the following expression

$$J(\gamma)/J(0) = 0.3726 + 0.6500 \cos\gamma - 0.0226 \cos^2\gamma \ldots, \tag{21.0}$$

tabulated in column *(4)* of Table I, approximates the exact solution of *(20)* within the limits of deviations given in the subsequent column *(5)*. A glance at these deviations reveals that an approximation of the exact solution by *(22.0)* is entirely satisfactory, except over the last three per cent of the star's fractional radius (column 2) where the residuals become appreciable. They can be further diminished if the quadratic term on the right-hand side of equation *(21.0)* is split up in two and *(21.0)* replaced by

TABLE I

| $\cos\gamma$ | $\sin\gamma$ | $J(\gamma)/J(0)$ (exact) | $J(\gamma)/J(0)$ (quadratic) | O–C | $J(\gamma)/J(0)$ (quartic) | O–C | $J(\gamma)/J(0)$ (linear) | O–C |
|---|---|---|---|---|---|---|---|---|
| 1.0 | 0.0000 | 1.0000 | 1.0000 | 0.0000 | 1.0000 | 0.0000 | 1.0000 | 0.0000 |
| 0.9 | 0.4359 | 0.9391 | 0.9393 | -0.0002 | 0.9393 | -0.0002 | 0.9400 | -0.0009 |
| 0.8 | 0.6000 | 0.8779 | 0.8781 | -0.0002 | 0.8783 | -0.0004 | 0.8800 | -0.0021 |
| 0.7 | 0.7142 | 0.8164 | 0.8165 | -0.0001 | 0.8168 | -0.0004 | 0.8200 | -0.0036 |
| 0.6 | 0.8000 | 0.7546 | 0.7545 | 0.0001 | 0.7548 | -0.0002 | 0.7600 | -0.0054 |
| 0.5 | 0.8660 | 0.6922 | 0.6920 | 0.0002 | 0.6921 | 0.0001 | 0.7000 | -0.0078 |
| 0.4 | 0.9165 | 0.6291 | 0.6290 | 0.0001 | 0.6286 | 0.0005 | 0.6400 | -0.0109 |
| 0.3 | 0.9540 | 0.5649 | 0.5656 | -0.0007 | 0.5641 | 0.0008 | 0.5800 | -0.0151 |
| 0.2 | 0.9798 | 0.4988 | 0.5017 | -0.0029 | 0.4984 | 0.0004 | 0.5200 | -0.0212 |
| 0.1 | 0.9950 | 0.4290 | 0.4374 | -0.0084 | 0.4312 | -0.0022 | 0.4600 | -0.0310 |
| 0.0 | 1.0000 | 0.3439 | 0.3726 | -0.0287 | 0.3622 | -0.0183 | 0.4000 | -0.0561 |

They can be further diminished if the quadratic term on the right-hand side of equation *(21.0)* is split up in two and *(21.0)* replaced by

$$J(\gamma)/J(0) = 0.3622 + 0.6998 \cos\gamma - 0.1053 \cos^2\gamma + 0.0578 \cos^3\gamma - 0.0145 \cos^4\gamma + \ldots, \tag{21.1}$$

which is similarly tabulated in column *(6)* of Table I, with the respective deviations from the exact solution (column 7). If one compares *(21.0)* or *(21.1)* with Milne's well-known solution[8]

$$\frac{J(\gamma)}{J(0)} = \tfrac{2}{5} + \tfrac{3}{5}\cos\gamma, \tag{21.2}$$

tabulated also in column *(8)* of Table I for the sake of comparison, and inspects its deviations from the exact solution as listed in the ultimate column *(9)* or shown graphically on Figure 2, the superiority of the present equations *(21.0)* or *(21.1)* becomes apparent at once.

As an illustration of the bearing of the above results on the study of close binary systems, let us follow up the photometric consequences of closely approximate laws of limb darkening of the form *(21.0)* or *(21.1)* on the light curves of distorted eclipsing systems between minima. If, in equation *(18)*, we collect the coefficients of Legendre polynomials of equal orders, the variation of light $\delta\mathfrak{L}$ accompanying the revolution of rotationally and tidally distorted components can, to the order of accuracy we have been working, be rewritten as

$$\begin{aligned}\delta\mathfrak{L} = {} & X_2(n)\{1 + \tfrac{1}{4}\beta_2\}\{\tfrac{1}{3} v_1^{(2)} P_2(n_0) - w_1^{(2)} P_2(l_0)\} \\ & - X_3(n)\{1 + \tfrac{1}{10}\beta_3\} w_1^{(3)} P_3(l_0) \\ & - X_4(n)\{1 + \tfrac{1}{18}\beta_4\} w_1^{(4)} P_4(l_0) - \ldots,\end{aligned} \tag{22}$$

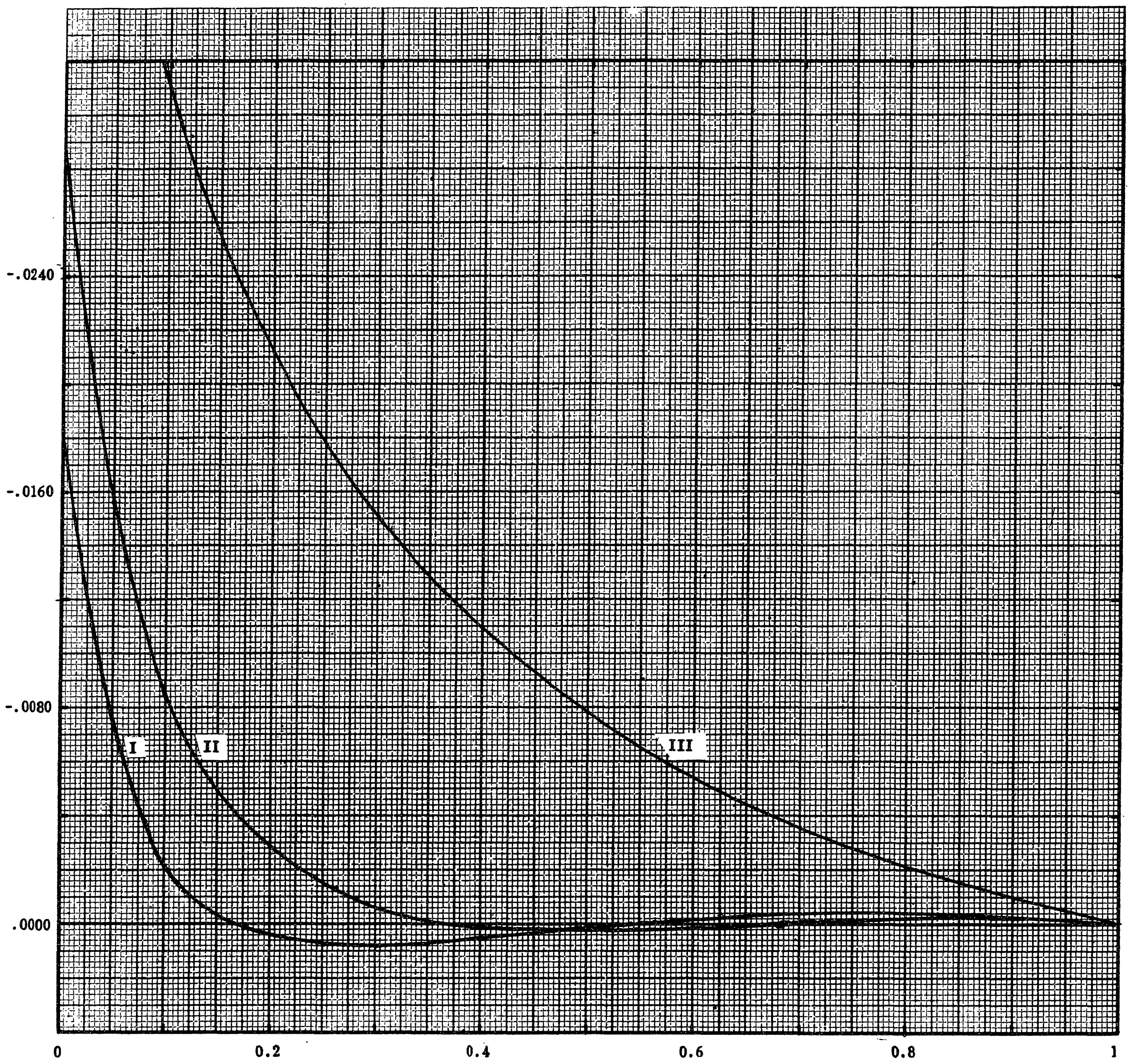


FIGURE I. Deviations of approximate laws of limb darkening as defined by equations *(22.0)*, *(22.1)* and *(22.2)* from the exact law *(21)*, are plotted against cos $\gamma$. Curves I, II, and III show graphically the errors inherent to equations *(22.1)*, *(22.0)*, and *(22.2)*, respectively.

where the $X_j(n)$'s denote the photometric effects of limb darkening. If $n = 0$ (uniform disks), all $X_j$'s are naturally equal to unity. For an ordinary limb darkening ($n = 1$)

$$X_2(1) = \frac{15+u_1}{5(3-u_1)}, \quad X_3(1) = \frac{5u_1}{2(3-u_1)}, \quad X_4(1) = \frac{9(u_1-1)}{4(3-u_1)}. \tag{23}$$

For $n = 2$,

$$X_2(2) = \frac{2}{5}\,\frac{15 + u_1}{6 - 2u_1 - 3u_2}, \tag{24.0}$$

$$X_3(2) = \frac{1}{7}\,\frac{35u_1 + 48u_2}{6 - 2u_1 - 3u_2}, \tag{24.1}$$

$$X_4(2) = \frac{9}{8}\,\frac{4(u_1-1)+7u_2}{6 - 2u_1 - 3u_2}; \tag{24.2}$$

for $n = 3$,

$$X_2(3) = \frac{2}{7}\,\frac{7(15+u_1) - 9u_3}{30 - 10u_1 - 15u_2 - 18u_3}, \tag{25.0}$$

$$X_3(3) = \frac{5}{14}\,\frac{70u_1+96u_2+105u_3}{30 - 10u_1 - 15u_2 - 18u_3}, \tag{25.1}$$

$$X_4(3) = \frac{3}{56}\,\frac{420(u_1-1)+735u_2+932u_3}{30 - 10u_1 - 15u_2 - 18u_3}; \tag{25.2}$$

while for $n = 4$,

$$X_2(4) = \frac{1}{7}\,\frac{210+14u_1-18u_3-35u_4}{30 - 10u_1 - 15u_2 - 18u_3 - 20u_4}, \tag{26.0}$$

$$X_3(4) = \frac{5}{42}\,\frac{210u_1+288u_2+315u_3+320u_4}{30 - 10u_1 - 15u_2 - 18u_3 - 20u_4}, \tag{26.1}$$

and

$$X_4(4) = \frac{3}{56}\,\frac{420(u_1-1)+735u_2+932u_3+1050u_4}{30 - 10u_1 - 15u_2 - 18u_3 - 20u_4}. \tag{26.2}$$

The forms of these functions for $n > 4$ will not be needed in the present paper. The above results corresponding to $n = 1$ have already been deduced before by several writers; but all others are new.

When $n = 1$, the value of the only coefficient $u_1$ in equation *(2)* is, in accordance with *(21.2)*, equal to three-fifths for total radiation; and the numerical values of $X_j(1)$ appropriate for $u = 0.6$ are given in the first column of the following Table II. When $n = 2$, equation *(21.0)* leads to $u_1 = 0.6500$ and $u_2 = -0.0226$, for which the $X_j(2)$'s take the values listed in the second column of Table II. The third column of the same table ultimately gives the $X_j(4)$'s computed with

$$u_1 = 0.6998, \quad u_3 = 0.0576,$$
$$u_2 = -0.1053, \quad u_4 = -0.0145,$$

as defined by equation *(21.1)*.

TABLE II

| | $n = 1$ | $n = 2$ | $n = 4$ |
|---|---|---|---|
| $X_2(n)$ | 1.3 | 1.3130 | 1.3142 |
| $X_3(n)$ | 0.625 | 0.6492 | 0.6500 |
| $X_4(n)$ | -0.375 | -0.3677 | -0.3709 |

Although the numerical data collected in Table II pertain to the effects observable in total radiation alone[9], their inspection reveals what is probably the general situation to be expected in the case of observations carried out in any reasonable spectral range (that is, between λλ 4000-7000A) as well. A glance at the results discloses that, on the whole, *the photometric effects of rotation of distorted stars are not very sensitive to the behavior of* $J(\gamma)$ *in the neighborhood of* $\gamma = 90°$. The deviations of the actual distribution of brightness near the limb from a simple cosine law are found to magnify the effects of the second and third harmonic distortion by one and four per cent, respectively, and to diminish the coefficient of the fourth harmonic by about two per cent. The differences between a quadratic and a quartic approximation to the actual law of darkening, such as represented by equations *(21.0)* and *(21.1)* prove to have altogether negligible consequences. This all leads us to conclude that, for the time being and probably for several years to come, an interpretation of even the most precisely observed light curves should not call for the retention of more than the quadratic term in equation *(2)*; while, in ordinary cases, a failure to consider this quadratic term should produce barely sensible effects.

1 Proc. Amer. Phil. Soc., 85, 399, 1942; 89, 517, 1945. In what follows these two investigations will consistently be referred to as paper I and II, respectively. Unless specifically stated to the contrary, the notations used in the present paper will be made strictly consistent with I and II.

2 Cf., however, the investigation by Chalonge and Kourganoff, Annales d'Astrophysique, 9, 69, 1946, concerning the limits of accuracy of such a representation.

3 Nachr. K. Ges. Wiss. Göttingen, No. 41, 1906.

4 For their comprehensive review cf., for instance, Milne's article in Handb. d. Astroph., Bd. III, Erste Hälfte, Berlin 1930.

5 Berlin Berichte, Math.-Phys. Klasse, p.696, 1931.

6 For its justification cf. Gratton, Mem. Soc. Astro. Ital., 10, 309, 1937, or (more fully) Chandrasekhar, Ap.J., 99, 180, 1944.

7 Ap.J., 99, 180, 1944 (Table 2).

8 M.N., 81, 361, 1921.

9 For the latest discussion of the theory of stellar limb darkening in selective radiation cf. S. Chandrasekhar, Ap.J., 101, 328, 1945. It may be also mentioned that the distribution of brightness over the apparent disk of the Sun has recently been analyzed for terms varying as higher powers of $\cos\gamma$ by Chalonge and Kourganoff (Annales d'Astrophysique, 9, 69, 1946). Tableau 2 of this paper contains quantities equivalent to our coefficients $u_1$ and $u_2$ for light of frequencies ranging from 3230A to 20970A.

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