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Monte Carlo simulation of Lane–Emden type equations arising in astrophysics

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ABSTRACT

Monte Carlo (MC) method played an essential role in many areas of human activity and has found application in many branches of physical science. This paper proposes a computational technique based on MC algorithms to solve Lane–Emden (LE) type equations. We analyze four LE equations arising in astrophysics: the positive and negative indices of the polytropic gas spheres, the isothermal gas sphere, and the white dwarf equation. We calculated eleven models (i.e., eleven LE equations) of the positive index polytropes, nine for the negative index polytrope, the isothermal gas sphere, and the white dwarf equation. Comparing the MC and numerical/analytical models gives good agreement for the four LE equations under study.

1. Introduction

Lane–Emden (LE) equations play an essential role in the theory of star structure and evolutions, thermodynamics, modeling of galaxy clusters, and many other physics, chemistry, and engineering topics. LE equation can be referred to as a Poisson equation that relates the gradient of the gravitational potential of the self-gravitating, symmetric gas sphere with its density and radius. There is an exact solution of LE equations for only limited values of the polytropic indices, so numerical or analytical techniques might solve these equations.

A preliminary study on the LE equations (polytropic and isothermal) was undertaken by astrophysicists Lane (1870) and Emden (1907). Since this time, a variety of methods have been used to estimate analytical as well as numerical solutions of the LE equations, such as Runge–Kutta type methods, Horedt (1986), Adomian decomposition methods, Shawagfeh (1993), and Wazwaz (2001), Homotopy perturbation methods, Chowdhury and Hashim (2007) and Yildirim and Özis (2007), Series expansion, Ramos (2008), an accelerated power series method Nouh (2004), Variational iteration method, Dehghan and Shakeri (2008), Differential transform method, Mukherjee et al. (2011) and many other methods.

Various novel computing approaches for increasingly complicated problems have emerged fast in recent decades, such as the Genetic Algorithm (e.g., Ge et al. (2008)), Lattice Boltzmann method (e.g., Zhang et al. (2003)), Ant colony Algorithm (e.g., Cao and Guo (2011)), Artificial neural networks (e.g., Morawski and Bejger (2020), Abdel-Salam et al. (2021), Azzam et al. (2021) and Nouh et al. (2021)).

The Monte Carlo method is mainly a series of statistical techniques used to either get solutions of function when it cannot analytically be solved or evaluate the estimated value of a parameter or a function of a specific distribution. In general, any problem can be solved under two techniques of the Monte Carlo method (simulation and integration). In contrast to traditional methods, primarily used to solve problems in specific fields, the Monte Carlo method has been studied as an independent method over the last century as science, technology, and computers have advanced. In the late 1960s, Monte Carlo calculations entered the astrophysics stage, for example, with the works by Auer (1968), Avery and House (1968), and Magnan (1968, 1970). One of the most powerful Monte Carlo algorithms is the Markov chains algorithm (MC). MC method (see, for example, Hestroffer (2012) and Mede and Brandt (2014)) is currently widely utilized in exoplanet research, for astrometric orbits (Tuomi and Kotiranta, 2009; Otor et al., 2016), for visual binary orbits (Mendez et al., 2017) and other topics.

Compared to deterministic or "single point estimate" analysis, Monte Carlo simulation has various benefits. Results demonstrate both what may occur and how probable each possibility is. Analysis may determine precisely which inputs had values combined when specific outcomes occurred. Based on the empirical experiments, more samples generate as the obtained solution gets closer to the exact solution. A few published papers investigate

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the Monte Carlo method to provide solutions to linear and nonlinear differential equations. In 2011, Zhong and Tian (2011) reported a new way of the Monte Carlo method to solve the initial value problem of ordinary differential equations. Akhtar et al. (2015) suggested a new algorithm of the Monte Carlo method that gives accurate solutions to different types of ordinary differential equations. This algorithm efficiently refines the complications in the scheme of Zhong and Tian (2011). Recently, Uslu et al. (2020) discussed a Monte Carlo-based stochastic approach for different systems of the Lotka–Volterra equation. No studies have investigated Monte Carlo integration for accurate numerical solutions for the LE equations.

Although traditional numerical methods like the Runge–Kutta method could be used to solve LE equations, numerous numerical techniques have been proposed to do so, including neural networks (Ahmad et al., 2017; Nouh et al., 2021), genetic algorithms (Ahmad et al., 2016), and the pattern search optimization technique (Lewis et al., 2000). In this study, we introduce four LE equations' numerical solutions utilizing the MC technique. We shall study four LE equations, including the polytropic LE equation, the isothermal LE equation, and the white dwarf equation. The obtained results from the MC approach will be compared to the exact and numerical solutions to demonstrate their efficiency and accuracy. The paper's overall structure builds on four sections, including this introductory section. The second section briefly overviews the derivation of the LE type equations and white dwarfs' equations. The third section has implemented the producers of Monte Carlo integration used for this study. The fourth section analyzes the numerical results. The fifth section gives the conclusion of the paper.

2. Lane-Emden equations

This section briefly derives the LE equations (polytropic and isothermal gas spheres) and the white dwarf equation.

2.1. Polytropic gas sphere

A fundamental equation in the theory of stellar structure is the LE equation of positive polytropic index n. According to the rules of thermodynamics and the mutual attraction of its molecules, the equation represents the temperature variation of a spherical gas cloud. The Polytropic model is a simplified method of equations that relates the pressure and density of the star by the polytropic equation of state

$$P\left(r\right) = K\rho^{(1+1/n)}\left(r\right), \tag{1}$$

with some constant K, radius r, and the polytropic index n.

The polytropic LE equation was the equation of mass continuity and hydrostatic equilibrium as follows

$$\frac{dm\left(r\right)}{dr} = 4\pi r^2\rho\left(r\right), \tag{2}$$

$$\frac{dP\left(r\right)}{dr} = -G\frac{m(r)}{r^2}\rho\left(r\right). \tag{3}$$

Combining Eq. (2) with Eq. (3), then multiply by $\frac{r^2}{\rho(r)}$ and differentiate for r, holds

$$\frac{d}{dr}\left(\frac{r^2}{\rho\left(r\right)}\frac{dP\left(r\right)}{dr}\right) = -G\frac{dm\left(r\right)}{dr}. \tag{4}$$

Substitute Eq. (2) into Eq. (4), and we get

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho\left(r\right)}\frac{dP\left(r\right)}{dr}\right) = -4\pi G\rho\left(r\right). \tag{5}$$

For simplicity, we transform equation (5) into a dimensionless form by defining dimensionless variables θ and r

$$\begin{aligned} &\rho\left(r\right) = \rho_c\left(r\right)\theta^n\left(r\right),\\ &r = \alpha\xi. \end{aligned} \tag{6}$$

Inserting Eq. (6) into Eq. (1) leads to

$$\begin{aligned} &P\left(r\right) = K\rho_c^{1+\frac{1}{n}}\left(r\right)\theta^{n+1}\left(r\right) = P_c\left(r\right)\theta^{n+1}\left(r\right),\\ &P_c\left(r\right) = K\rho_c\left(r\right). \end{aligned} \tag{7}$$

Then Eq. (5) becomes

$$\frac{1}{\alpha^2}\left[\frac{(n+1)K}{4\pi G\rho_c^{1-\frac{1}{n}}\left(r\right)}\right]\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta\left(\xi\right)}{d\xi}\right) = -\theta^n\left(\xi\right), \tag{8}$$

with some constant α^2 (α in centimeters) in squared brackets, so

$$\alpha = \left[\frac{(n+1)K}{4\pi G\rho_c^{1-\frac{1}{n}}\left(r\right)}\right]^{\frac{1}{2}}. \tag{9}$$

Combining Eq. (8) with Eq. (9) yields

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta\left(\xi\right)}{d\xi}\right) = \mp\theta^n\left(\xi\right), \tag{10}$$

The right-hand side of Eq. (10) takes the minus sign when the polytropic indices $-1 < n < \infty$, while the plus sign holds when $-\infty < n < -1$.

Now, the solution of Eq. (10) holds under the initial conditions

$$\frac{d\theta\left(\xi\right)}{d\xi} = 0, \theta\left(\xi\right) = 1 \text{ at } \xi = 0$$

To solve Eq. (10), various analytical and numerical techniques have been applied recently. The equations' singularity causes the primary challenge at $\xi = 0$. There is an exact solution for some values of the polytropic index n, as follows

$$\begin{aligned} &n = 0 \rightarrow \theta\left(\xi\right) = 1 - \frac{\xi^2}{6},\\ &n = 1 \rightarrow \theta\left(\xi\right) = \frac{\sin\left(\xi\right)}{\xi},\\ &n = 5 \rightarrow \theta\left(\xi\right) = \frac{1}{\left(1+\frac{\xi^2}{3}\right)^{\frac{1}{2}}}. \end{aligned} \tag{11}$$

For other values of n, $\theta\left(\xi\right)$ can be computed by numerical or analytical methods.

2.2. The isothermal gas sphere

For an isothermal gas sphere, set $n \rightarrow \infty$ and regard a temperature T as a constant value. Combining Eq. (5) with $P_c\left(r\right) = K\rho_c\left(r\right)$ where $K = \frac{kT}{\mu H}$, gets

$$\frac{K}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho\left(r\right)}\frac{d\rho\left(r\right)}{dr}\right) = -4\pi G\rho\left(r\right). \tag{12}$$

It is convenient to rewrite Eq. (12) as,

$$\frac{K}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\log\rho\left(r\right)\right) = -4\pi G\rho\left(r\right). \tag{13}$$

Set $\rho\left(r\right) = \rho_c\left(r\right)e^{-\varphi}$ and $r = \alpha\xi$. Under the previous assumptions, Eq. (13) becomes,

$$\frac{K}{4\pi G\rho_c\left(\xi\right)}\frac{1}{\alpha^2\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\varphi\left(\xi\right)}{d\xi}\right) = e^{-\varphi(\xi)}, \tag{14}$$

where α can be chosen to have the formulae $\alpha = \left[\frac{K}{4\pi G\rho_c(\xi)}\right]^{\frac{1}{2}}$.

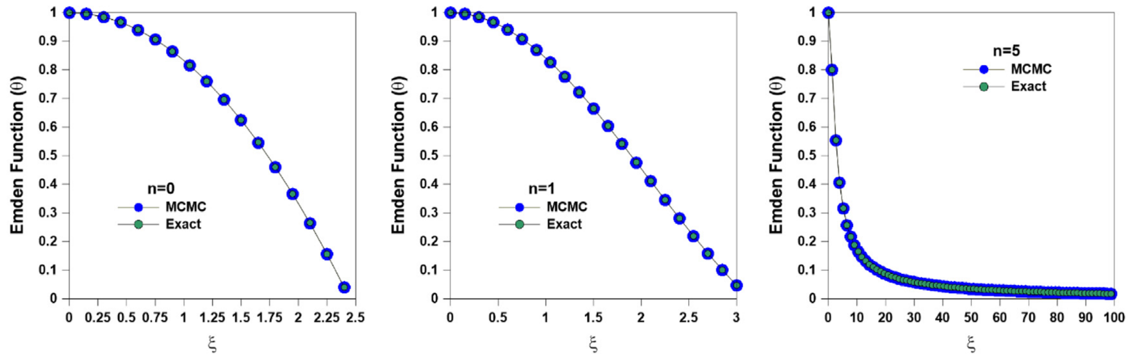

Fig. 1. Comparison between the MC and the exact solutions computed for the polytropic indices $n = 0, 1$ *and* 5.

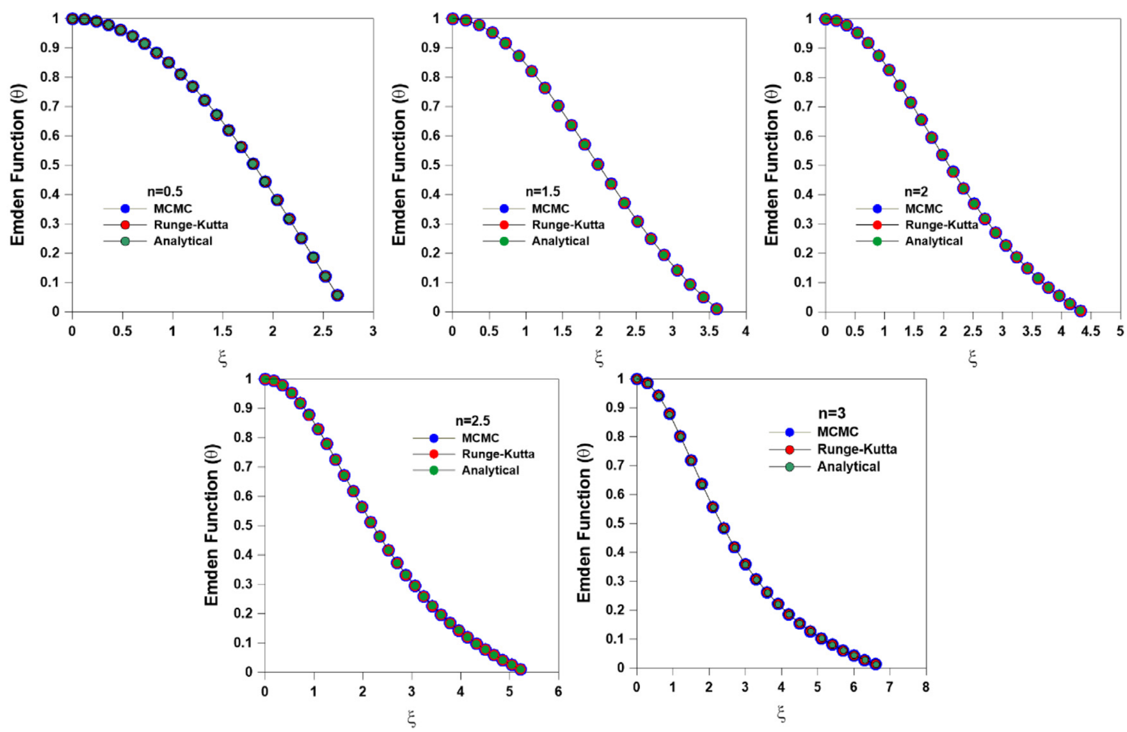

Fig. 2. Comparison between the MC, numerical, and analytical solutions computed for the polytropic indices $n = 0.5, 1.5, 2, 2.5$ *and* 3.

We write Eq. (14) for short, as

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\varphi(\xi)}{d\xi}\right) = e^{-\varphi(\xi)}. \tag{15}$$

Only numerical solutions are held when $\varphi(\xi)$ is subject to the conditions

$$\frac{d\varphi(\xi)}{d\xi} = 0, \varphi(\xi) = 0 \text{ at } \xi = 0.$$

2.3. White dwarf equation

Chandrasekhar (1958) derived an initial value problem ordinary differential equation, namely Chandrasekhar white dwarf equation.

Taking pressure P and density ρ as

$$P(x) = Af(x), \quad \rho(x) = Bx^3, \tag{16}$$

where

$$A = \frac{\pi m_e{}^4 C^5}{3h^3} = 6.02 \times 10^{21},$$

$$B = \frac{8\pi \mu_e m_p m_e{}^3 C^5}{3h^3} = 9.82 \times 10^8, \tag{17}$$

$$f(x) = x\left(2x^2 - 3\right)\left(x^2 + 1\right)^{\frac{1}{2}} + 3\sinh^{-1} x$$

Combining Eq. (16) with Eq. (5) gets

$$\frac{A}{B}\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{x^3}\frac{df(x)}{dr}\right) = -4\pi G B x^3. \tag{18}$$

$$\frac{r^2}{x^3}\frac{df(x)}{dr} = \frac{8x}{\left(x^2+1\right)^{\frac{1}{2}}}\frac{dx}{dr} = 8\frac{d\sqrt{x^2+1}}{dr}. \tag{19}$$

Now, Eq. (18) becomes

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\sqrt{x^2+1}}{dr}\right) = -\frac{\pi G B^2}{2A}x^3. \tag{20}$$

Set $y^2 = x^2 + 1$, $r = \mu\eta$, and $y = y_0\psi$, yields

$$\frac{1}{y_0^2}\left(\frac{2A}{\pi G B^2}\right)\frac{1}{\mu^2\eta^2}\frac{d}{d\eta}\left(\eta^2\frac{d\psi(\eta)}{d\eta}\right) = -\left(\psi^2 - \frac{1}{y_0{}^2}\right)^{\frac{3}{2}}. \tag{21}$$

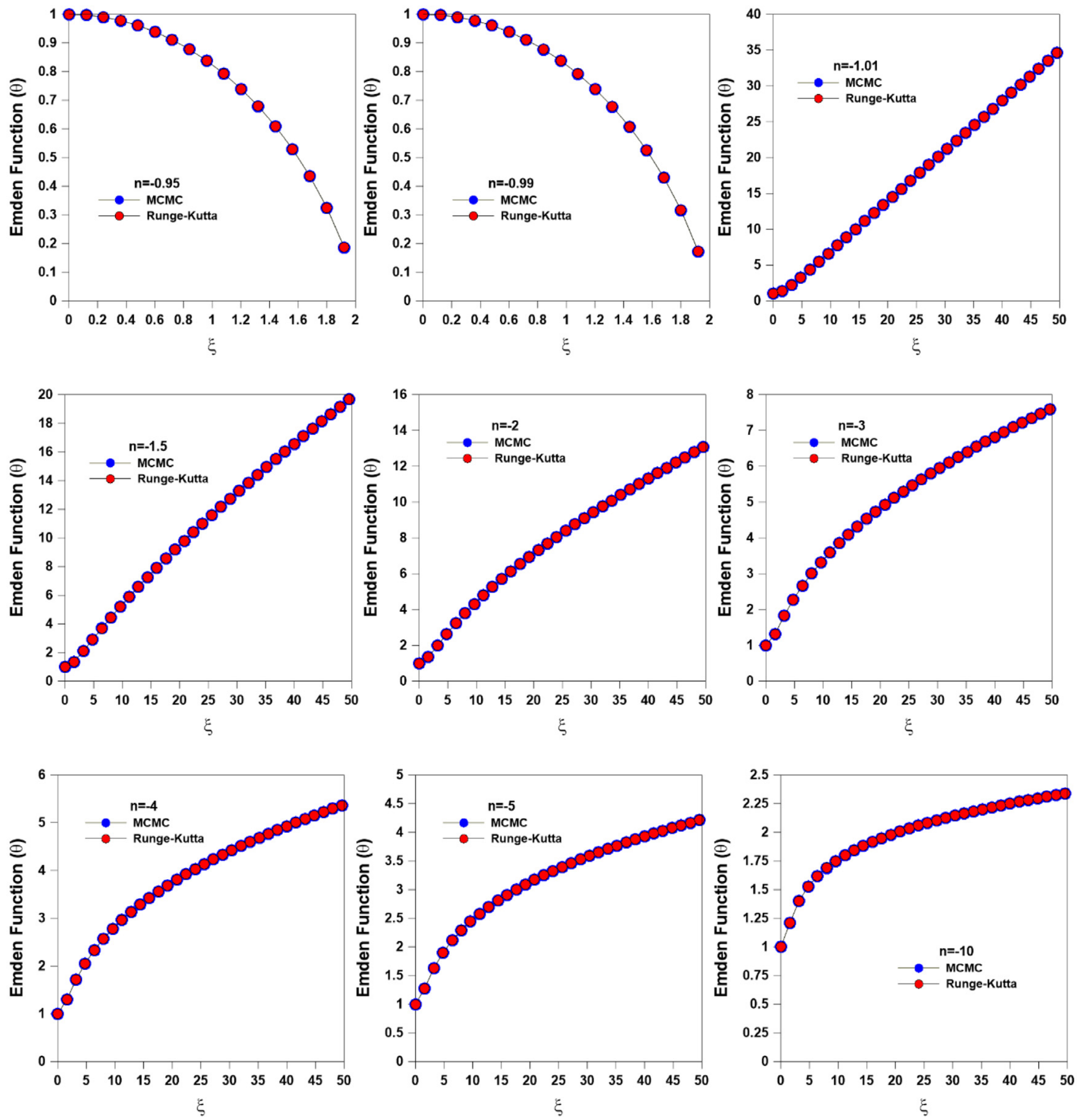

Fig. 3. Comparison between the MC and numerical solutions computed for the polytropic indices $n = -1.01, -1.5, -2, -3, -4, -5$ *and* -10.

Then Eq. (21) can be rewritten in the formulae

$$\frac{1}{\eta^2}\frac{d}{d\eta}\left(\eta^2\frac{d\psi(\eta)}{d\eta}\right) = -\left(\psi^2 - C\right)^{\frac{3}{2}}. \tag{22}$$

With some fixed values of $\mu = \frac{1}{y_0}\left(\frac{2A}{\pi G B^2}\right)^{\frac{1}{2}}$ and $C = \frac{1}{{y_0}^2}$

The existence of solutions $\psi(\eta)$ are held under initial conditions as

$$\frac{d\psi(\eta)}{d\eta} = 0,\ \psi(\eta) = 1 \text{ at } \eta = 0.$$

When η tends to ∞, then $\psi(\eta_\infty) = \sqrt{C}$ and the range of $\psi(\eta)$ becomes

$$\sqrt{C} \le \psi(\eta) \ge 1.$$

The constant C takes values $C = 0.01 - 0.8$..

Eq. (22) is of LE type where $f(y) = \left(y^2 - C\right)^{\frac{3}{2}}$. If $C = 0$, Eq. (22) reduces to LE equation, Eq. (10) of positive polytropic index $n = 3$.

3. The MC method

3.1. The MC integration

The integral of the function $f(x)$ over an interval width $(b - a)$ is given by

$$I = \int_a^b f(x)\,dx = (b - a)\langle f\rangle,$$

where $\langle f\rangle$ is the average value of the function f. We could then calculate the integral if we had some independent technique for estimating the integrand's average value. Assume we have a list of random numbers, x_i, where $i = 1, 2, \ldots, N$ that are evenly distributed between a and b, then the average value of the function is given by

$$\langle f\rangle_N = \frac{1}{N}\sum_{i=1}^{N} f(x_i).$$

In the following subsections, we shall solve an explicit and implicit form of ODEs using the MC method.

3.2. Explicit ODE

Let us assume explicit ODE is written in general form as

$$\frac{dy(x)}{dx} = f(x) \text{ with } y(x_0) = y_0. \tag{23}$$

Firstly, integrate both sides of Eq. (23) concerning x over some specified interval and gets

$$y(x) = y(x_0) + \int_{x_0}^{x} f(t)\,dt \tag{24}$$

Secondly, split the limit of integration into small, discrete chunks, as

$$y(x) = y(x_0) + \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} f(t)\,dt. \quad \text{where } x = x_N \tag{25}$$

Thirdly, the definite integral on the right-hand side can be estimated using the MC method by drawing M samples of x due to uniform distribution $U(x_{i-1}, x_i)$.

$$y(x) = y(x_0) + \sum_{i=1}^{N} \left[\frac{x_i - x_{i-1}}{M} \sum_{j=1}^{M} f(x_j) \right]. \tag{26}$$

Finally, we apply the Markov chains concept (which states that any future generated value depends only on the previous ones in the same sample) to evaluate the value of $y(x)$ for every discrete chunk.

$$y(x_i) = y(x_{i-1}) + \frac{x_i - x_{i-1}}{M} \sum_{j=1}^{M} f(x_j). \tag{27}$$

3.3. Implicit ODE

Set implicit ODE simply in the general form as,

$$\frac{dy(x)}{dx} = f(y(x), x), \text{ with } y(x_0) = y_0. \tag{28}$$

In an explicit ODE, solution $y(x)$ of Eq. (28) can be estimated as,

$$y(x_i) = y(x_{i-1}) + \frac{x_i - x_{i-1}}{M} \sum_{j=1}^{M} f(y(x_{i-1}), x_j). \tag{29}$$

3.4. The MC algorithm for solving LE equations

For simplicity, we generally write the LE equations as

$$\frac{d^2\theta(\xi)}{d\xi^2} = -\frac{2}{\xi^2}\frac{d\theta(\xi)}{d\xi} + f(\theta(\xi), \xi). \tag{30}$$

The solution $\theta(\xi)$ can be integrated with the following steps:

1 Transform equation (30) into a first-order ordinary differential equation, define $\theta(\xi) = y_1(\xi)$ and $\frac{dy_1(\xi)}{d\xi} = y_2(\xi)$ Then Eq. (30) can be reduced to two first order differential equations as,

$$\frac{dy_1(\xi)}{d\xi} = y_2(\xi),$$
$$\frac{dy_2(\xi)}{d\xi} = -\frac{2}{\xi} y_2(\xi) + f(y_1(\xi), \xi) = g(y_2(\xi), y_1(\xi), \xi). \tag{31}$$

2 Split the interval $[0, \xi_f]$, where ξ_f is the first zero of $y_1(\xi)$, into small, discrete chunks with $\Delta\xi = 10^{-3}$.

3 Generate a random sample from a uniform distribution from ξ_i to ξ_{i+1}, with sample size $M = 1e^6$.

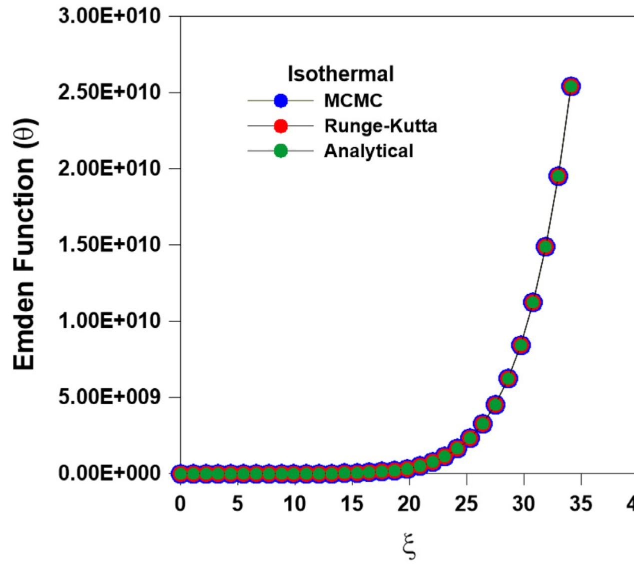

Fig. 4. Comparison between the MC and numerical solutions computed for the isothermal gas sphere.

4 Initialize $\xi \rightarrow 0$, $y_1(\xi) \rightarrow 1$ and $y_2(\xi) \rightarrow 0$.

5 Evaluate $y_2(\xi_{i+1}) = y_2(\xi_i) + \frac{\xi_{i+1} - \xi_i}{M} \sum_{k=1}^{M} g(y_2(\xi_i), y_1(\xi_i), \xi_k)$.

6 Compute solution $y_1(\xi_{i+1}) = y_1(\xi_i) + \Delta\xi \times y_2(\xi_{i+1})$.

7 Set in the second step $\xi_{i+1} = \xi_i + \Delta\xi$..

8 Repeat steps 3-7, until $\xi_{i+1} = \xi_f$.

4. Results

To solve the four LE differential equations (Eqs. (10), (15), and (22)), we elaborated *R* codes that used the MC technique presented in Section 3. For each case, a total of 1 000 000 random samples are used. The four differential equations are solved numerically using the *R* package function rkMethod, based on the Runge–Kutta technique (RK) for solving ordinary differential equations. An odd, spaced interval with increments of 0.001 was chosen. After applying the method to the problem, the predicted MC results were compared to the RK results. We listed in Appendix the numerical values obtained for the Emden functions for each of the four studied equations. In the following subsections, we analyze the behaviors of the results in more detail for each equation.

4.1. Positive index polytropes

For the positive polytrope LE equation with $-1 < n < \infty$ the exact solutions are used to compute the Emden functions for $n = 0, 1$ *and* 5. The numerical or analytical solutions are used for the remaining polytropic indices due to the lack of an exact solution. The numerical results are listed in Tables A.1–A.12. In Fig. 1, we plotted the results for the polytropic indices $n = 0, 1$ *and* 5, the MC, and the exact solutions are displayed in various colors to measure calculation; it is seen that they overlap and cannot be distinguished.

Besides the numerical integration, many methods are proposed to solve LE equations. We shall compare the MC polytropic models with the accelerated power series (APS) models developed by Nouh and Abdel-Salam (2018). Fig. 2 plots the Emden function calculated for the polytropic indices $n = 0.5, 1.5, 2, 2.5$ *and* 3 by the MC, RK, and APS, respectively. The results of the

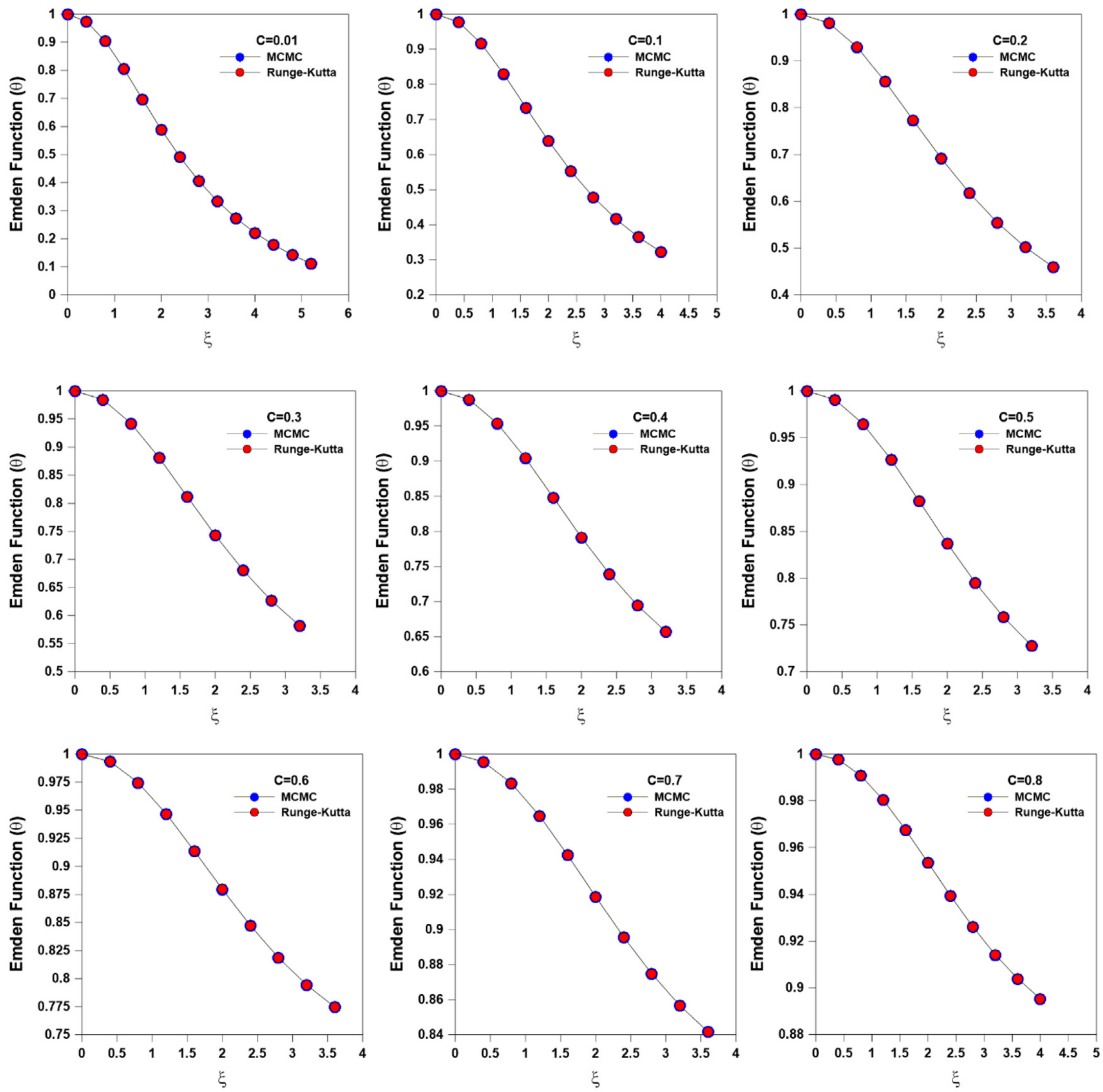

Fig. 5. Comparison between the MC and numerical solutions computed for the white dwarf equations.

three methods are plotted with different colors; however, one can barely distinguish between them.

Another way to confirm the findings is to compare and validate the zeroth of the Emden function obtained using the MC algorithm ($\xi 1$(MC)) with that of the exact and RK solutions ($\xi 1$). The results for various polytropic indices are shown in Table 1. The zeros of the Emden function calculated using the proposed MC technique are shown in the second column, while the zeros computed from the exact or RK solutions are represented in the third column. The relative errors in the fourth column revealed good agreement between the two solutions, with a maximum value of 0.04%.

Table 1
The zeroth of the Emden function of the positive polytrope computed by the MC, exact/RK solutions.

n	$\xi 1$ (MC)	$\xi 1$	Relative error %
0	2.449	2.450	0.040
0.5	2.750	2.750	0
1	3.141	3.142	0.031
1.5	3.654	3.654	0
2	4.352	4.353	0.022
2.5	5.354	5.355	0.018
3	6.896	6.897	0.014
3.5	9.538	9.536	0.020
4	14.973	14.971	0.013
4.5	31.840	31.836	0.012

4.2. Negative index polytropes

The term "negative polytropic index" refers to a process where heat and work are transferred through a system's boundaries concurrently. Such spontaneous processes break the Second Law of thermodynamics. These particular instances are exploited in thermal interaction for some astrophysical applications and chemical energy. There are no exact solutions for the negative polytrope (i.e., LE with $-\infty < n < -1$), so numerical or analytical methods could be used to solve it. Tables A.13–A.19 listed the numerical results for the polytropic indices $n = -1.01, -1.5, -2, -3, -4, -5$ *and* -10.

Fig. 3 plots the distributions of the Emden function with the dimensionless parameter ξ. As obtained for the polytropes with positive indices, there is good agreement between the MC and the RK solutions. Values of the Emden functions for the polytropic indices $n = -0.95$ *and* -0.99 are unity in the center and decrease toward the surface of the gas sphere. The situation for $n < -1$

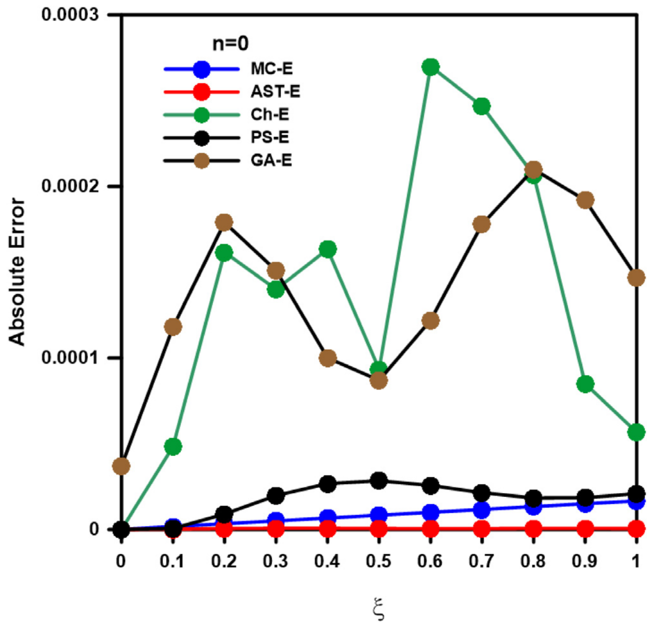

Fig. 6. Comparative study based on values of absolute errors for polytrope with $n = 0$.

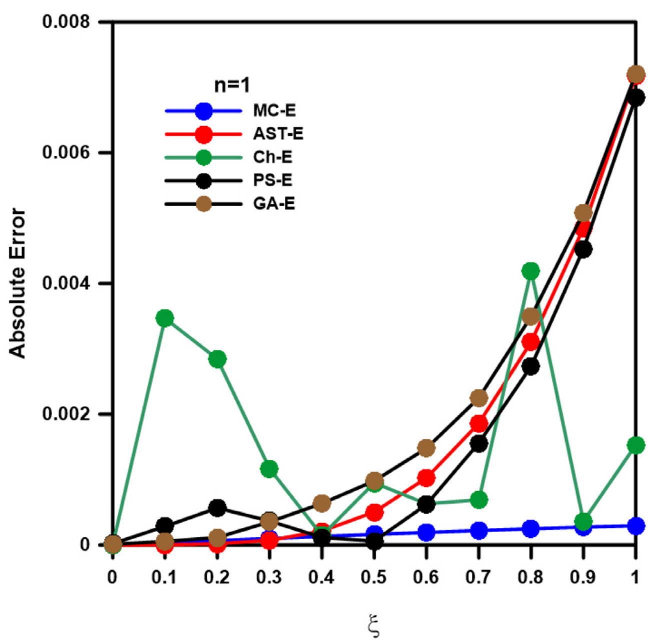

Fig. 7. Comparative study based on values of absolute errors for polytrope with $n = 1$.

is different than that of the polytropes with positive indices; the Emden functions increase monotonically, and there is no zeroth that the Emden function could determine.

4.3. The isothermal gas sphere

The isothermal gas sphere (also called the second type LE equation) is a specific LE equation with $n \to \infty$. It is often used to represent various astrophysical issues, including the star, star cluster, and galaxy formation. This equation has no exact solution and is solved by numerical or analytical methods. The numerical result is presented in Table A.20. In Fig. 4, we compare the MC solution with the RK solution. The calculations of the MC and RK

Table 2
Comparison between different numerical methods and the exact method for $n = 0$.

ξ	Exact	MC	AST-NN	Ch-NN	PS	GA
0	1	1	1.000000	1.0000	1.000001	1.000037
0.1	0.998333	0.998299	0.998337	0.9993	0.998322	0.998451
0.2	0.993333	0.993266	0.993344	0.9901	0.993513	0.993512
0.3	0.98500	0.984900	0.985014	0.9822	0.985395	0.985151
0.4	0.973333	0.973200	0.973347	0.9766	0.973868	0.973433
0.5	0.958333	0.958166	0.958344	0.9602	0.958901	0.958420
0.6	0.94000	0.939800	0.940009	0.9454	0.940514	0.940122
0.7	0.918333	0.918100	0.918343	0.9134	0.918761	0.918511
0.8	0.893333	0.893066	0.893345	0.8892	0.893700	0.893543
0.9	0.865000	0.864700	0.865012	0.8633	0.865370	0.865192
1.0	0.833333	0.833000	0.833344	0.8322	0.833750	0.833480

Table 3
Comparison between different numerical methods and the exact method for $n = 1$.

ξ	Exact	MC	AST-NN	Ch-NN	PS	GA
0	1	1	1.000000	1	0.99998	0.99999
0.1	0.998334	0.998300	0.998337	1.00180	0.99805	0.99828
0.2	0.993346	0.993280	0.993364	0.99050	0.99278	0.99346
0.3	0.985067	0.984968	0.985138	0.98390	0.98470	0.98542
0.4	0.973545	0.973415	0.973757	0.97340	0.97343	0.97418
0.5	0.958851	0.958689	0.959355	0.95980	0.95891	0.95983
0.6	0.941070	0.940879	0.942096	0.94170	0.94169	0.94255
0.7	0.920310	0.920090	0.922172	0.92100	0.92187	0.92256
0.8	0.896695	0.896447	0.899800	0.89250	0.89943	0.90019
0.9	0.870363	0.870090	0.875212	0.87000	0.87489	0.87544
1.0	0.841470	0.841175	0.848656	0.84300	0.84832	0.84868

Table 4
Comparison between different numerical methods and the exact method for $n = 5$.

ξ	Exact	MC	AST-NN	Ch-NN	PS	GA
0	1	1	1.000000	1.00000	0.99998	0.99999
0.1	0.998337	0.998304	0.998342	1.00180	0.99805	0.99828
0.2	0.993399	0.993334	0.993409	0.99050	0.99278	0.99346
0.3	0.985329	0.985234	0.985341	0.98390	0.98470	0.98542
0.4	0.974354	0.974233	0.974364	0.97340	0.97343	0.97418
0.5	0.9607689	0.960624	0.960776	0.95980	0.95891	0.95983
0.6	0.9449111	0.944747	0.944917	0.94170	0.94169	0.94255
0.7	0.9271455	0.926967	0.927151	0.92100	0.92187	0.92256
0.8	0.9078412	0.907652	0.907847	0.89250	0.89943	0.90019
0.9	0.8873565	0.887161	0.887362	0.87000	0.87489	0.87544
1.0	0.8660254	0.865827	0.866030	0.8431	0.84832	0.84868

Emden functions are limited to the upper limit of $\xi = 35$, where this range is that of an isothermal sphere on the brink of gravothermal collapse, Hunter (2001). The comparison indicates good agreement with a maximum relative error of 0.00005.

4.4. White dwarf equation

We solved Eq. (22) for the range of the constant $C = 0.01-0.8$. The numerical results are presented in Tables A.21–A.25. The comparison with the numerical solutions is plotted in Fig. 5, where we obtained an excellent agreement with the RK solutions. Interesting features could be read from the figure; as the parameter C goes higher than zero (for $C = 0$, this case is analog to LE with $n = 3$), dimensionless distance ξ and the Emden function $\theta(\xi)$ have values lesser than that of the polytrope with $n = 3$. Another important notice is that the white dwarf equation has not zeroth as that calculated for the polytropes with positive indices.

4.5. Comparative study

To declare the impact of the proposed MC algorithm on the accuracy of the computed Emden function (u), we calculated the

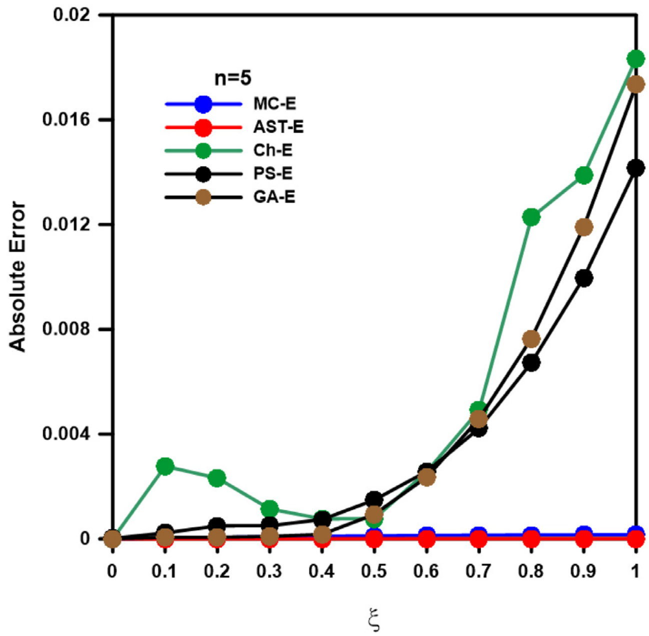

Fig. 8. Comparative study based on values of absolute errors for polytrope with $n = 5$.

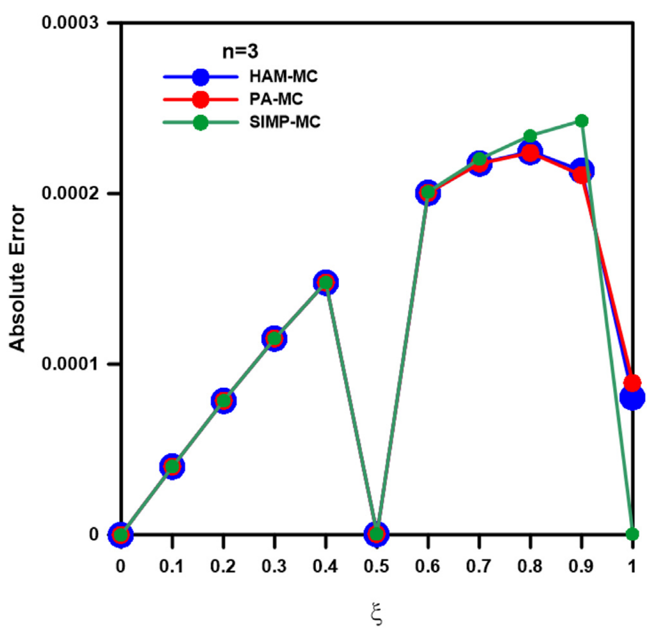

Fig. 9. Comparative study based on values of absolute errors for polytrope with $n = 3$.

absolute errors between the MC solution and some numerical methods. Those methods are ANN based on an active-set algorithm (AST-NN, Ahmad et al., 2017), Chebyshev neural network (Ch-NN, Mall and Chakraverty, 2014), Pattern search optimization technique (PS, Lewis et al., 2000), and Genetic algorithms (GA, Ahmad et al., 2016). We limited the comparison for polytropes with exact solutions only (i.e. polytropes with indexes $n = 0$, $n = 1$, and $n = 5$; Eq. (11)). Comparisons between these solutions are represented in Tables 2–4 and graphical representations for the absolute errors are shown in Figs. 6–8 for the range of the dimensional parameter $\xi = 0 - 1$ For the polytropic index $n = 0$, the comparison revealed that the maximum absolute error between the MC and the exact solution (labeled on the figure as MC-E) is 0.00033, 0.000014 for the AST-NN algorithm (AST-E), 0.0054 for the Ch-NN algorithm (Ch-E), 0.00056 for the PS algorithm (PS-E), and 0.00021 for the Genetic algorithm (GA-E). The comparison for $n = 1$ showed that the maximum absolute error between the MC and the exact solution is 0.00029, 0.0071 for the AST-NN algorithm, 0.0041 for the Ch-NN algorithm, 0.0068 for the PS algorithm, and 0.0072 for the Genetic algorithm. Finally, the comparison for $n = 5$ revealed that the maximum absolute error between the MC and the exact solution is 0.00019, 0.000012 for the AST-NN algorithm, 0.022 for the Ch-NN algorithm, 0.017 for the PS algorithm, and 0.017 for the Genetic algorithm.

Also, we compared the MC solution for the polytropic index with $n = 3$ with the results from different methods. Table 5 shows a comparison of the numerical results applying MC, Homotopy Analysis Method (HAM), Padé approximants (PA) of an order [4; 4], and the numerical solution with the Simpson rule (SIMP); Al-Hayani et al. (2017). Fig. 9 plots the absolute errors between the MC values and the methods mentioned above. The figure shows that the maximum absolute error between the three and the MC methods is about 0.00025.

Table 5
Comparison between different numerical methods and the exact method for $n = 3$.

ξ	MC	HAM	PA	SIMP
0	1	1	1	1
0.1	0.998369	0.998335	0.998335	0.998335
0.2	0.993438	0.993373	0.993373	0.993373
0.3	0.985295	0.985199	0.985199	0.985199
0.4	0.974081	0.973958	0.973958	0.973958
0.5	0.959839	0.959839	0.959839	0.959839
0.6	0.943240	0.943073	0.943073	0.943073
0.7	0.924106	0.923925	0.923922	0.923924
0.8	0.902867	0.902680	0.902672	0.902679
0.9	0.879819	0.879643	0.879617	0.879641
1.0	0.855057	0.855132	0.855057	0.855125

Table A.1
Comparison between Exact and MC solutions of $\theta(\xi)$ with polytrope index $n = 0$.

ξ	θ_{Exact}	θ_{MC}	Relative error
0.000	1.00000e+00	1.00000e+00	0.000000e+00
0.300	0.9850000	0.9849000	1.014802e−04
0.600	0.9400000	0.9398001	0.0002126788
0.900	0.8650000	0.8647001	0.0003466991
1.200	0.7600000	0.7596001	0.0005261556
1.500	0.6250000	0.6245001	0.0007997844
1.800	0.4600000	0.4594001	0.0013040314
2.100	0.2650000	0.2643002	0.002640928
2.400	0.0400000000	0.0392001615	0.019995961
2.440	0.0077333333	0.0069201625	0.105151406

Table A.2
Comparison between Exact and MC solutions of $\theta(\xi)$ with polytrope index $n = 1$.

ξ	θ_{Exact}	θ_{MC}	Relative error
0.000	1.00000e+00	1.00000e+00	0.000000e+00
0.400	0.9735459	0.9734151	1.343466e−04
0.800	0.8966951	0.8964480	0.0002756068
1.200	0.7766992	0.7763629	0.0004330656
1.600	0.6247335	0.6243443	0.0006229333
2.000	0.4546487	0.4542480	0.0008814356
2.400	0.2814430	0.2810717	0.001319296
2.800	0.11963863	0.11933265	0.002557471
3.120	0.0069201845	6.686209e−03	0.033810660
3.130	0.0037036402	3.472125e−03	0.062510190
3.140	0.0005072143	2.781704e−04	0.451572344

Table A.3
Comparison between Exact and MC solutions of $\theta(\xi)$ with polytrope index $n = 5$.

ξ	θ_{Exact}	θ_{MC}	Relative error	ξ	θ_{Exact}	θ_{MC}	Relative error
0.000	1.00000e+00	1.00000e+00	0.000000e+00	30.00	0.05763904	0.05778058	0.002455610
1.000	0.8660254	0.8658271	0.0002289966	35.00	0.04942668	0.04956891	0.002877580
2.000	0.6546537	0.6545291	0.0001903568	40.00	0.04326073	0.04340341	0.003298076
3.000	0.5000000	0.4999688	6.249792e−05	45.00	0.03846154	0.03860452	0.003717582
4.000	0.3973597	0.3973875	7.006908e−05	50.00	0.03462025	0.03476345	0.004136394
5.000	0.3273268	0.3273901	0.0001932484	55.00	0.03147623	0.03161959	0.004554699
6.000	0.2773501	0.2774354	0.0003073958	60.00	0.02885549	0.02899898	0.004972623
11.00	0.1555428	0.1556680	0.0008050144	65.00	0.02663748	0.02678106	0.005390254
15.00	0.114707867	0.114841684	0.001166591	70.00	0.02473601	0.02487967	0.005807653
20.00	0.08627960	0.08641790	0.001602920	75.00	0.02308785	0.02323157	0.006224867
22.00	0.07848671	0.07862602	0.001774929	80.00	0.02164556	0.02178933	0.006641930
24.00	0.07198158	0.07212165	0.001946023	85.00	0.02037284	0.02051665	0.007058869
26.00	0.06647001	0.06661069	0.002116408	90.00	0.01924145	0.01938529	0.007475704
28.00	0.06174094	0.06188210	0.002286234	95.00	0.01822908	0.01837296	0.007892451
29.00	0.059619648	0.059761004	0.002370972	100.0	0.01731791	0.01746181	0.008309124

Table A.4
Comparison between RK and MC solutions of $\theta(\xi)$ with polytrope index $n = 0.5$.

ξ	θ_{RK}	θ_{MC}	Relative error
0.000	1.00000e+00	1.00000e+00	0.000000e+00
0.400	0.9734406	0.9734394	1.221633e−06
0.800	0.8950494	0.8950433	6.816313e−06
1.200	0.7687456	0.7687261	2.537468e−05
1.600	0.6012647	0.6012191	7.583467e−05
2.000	0.4025799	0.4024912	0.0002201448
2.400	0.1869203	0.1867668	0.0008210544
2.740	0.006375151	0.006141292	0.036682865
2.750	0.001350444	0.001113427	0.175510805

Table A.5
Comparison between RK and MC solutions of $\theta(\xi)$ with polytrope index $n = 1.5$.

ξ	θ_{RK}	θ_{MC}	Relative error
0.000	1.00000e+00	1.00000e+00	0.000000e+00
0.500	0.9591043	0.9592605	0.0001628245
1.000	0.8451701	0.8454285	0.0003056639
1.500	0.6811246	0.6814026	0.0004081928
2.000	0.4959369	0.4961598	0.0004494538
2.500	0.3158926	0.3160176	0.0003957444
3.000	0.1588576	0.1588785	1.315799e−04
3.500	0.032615649	0.032552086	0.0019488652
3.600	1.109091e−02	1.101400e−02	0.006934564
3.652	3.566154e−04	2.732765e−04	0.233694247

Table A.6
Comparison between RK and MC solutions of $\theta(\xi)$ with polytrope index $n = 2$.

ξ	θ_{RK}	θ_{MC}	Relative error
0.000	1.00000e+00	1.00000e+00	0.000000e+00
0.500	0.9593532	0.9595062	0.0001594812
1.000	0.8486545	0.8488931	0.0002811817
1.500	0.6953674	0.6956018	0.0003371824
2.000	0.5298365	0.5300028	0.0003138883
2.500	0.3747393	0.3748155	0.0002034021
3.000	0.2418240	0.2418195	1.887474e−05
3.500	0.1339689	0.1339059	0.0004699780
4.000	0.04884002	0.04874116	0.002024153
4.350	3.658919e−04	2.523544e−04	0.310303453

Table A.7
Comparison between RK and MC solutions of $\theta(\xi)$ with polytrope index $n = 2.5$.

ξ	θ_{RK}	θ_{MC}	Relative error
0.000	1.00000e+00	1.00000e+00	0.000000e+00
0.500	0.9595982	0.9597481	1.562349e−04
1.000	0.8519445	0.8521654	0.0002593079
1.500	0.7080725	0.7082719	0.0002816049
2.000	0.5583724	0.5584983	0.0002254515
2.000	0.5583724	0.5584983	0.0002254515
2.500	0.4220076	0.4220525	1.064585e−04
3.000	0.3066750	0.3066553	6.419570e−05
3.500	0.2128325	0.2127699	0.0002938066
4.000	0.1376806	0.1375936	0.0006316995
4.500	0.07755395	0.07745543	0.001270399
5.000	0.02901904	0.02891703	0.003515047
5.350	4.025760e−04	3.010598e−04	0.252166581

Table A.8
Comparison between RK and MC solutions of $\theta(\xi)$ with polytrope index $n = 3$.

ξ	θ_{RK}	θ_{MC}	Relative error
0.000	1.0000e+00	1.0000e+00	0.00000e+00
0.500	0.9598395	0.9599864	0.0001530715
1.000	0.8550579	0.8552628	0.0002396530
1.500	0.7195020	0.7196727	0.0002372486
2.000	0.5828505	0.5829465	0.0001646699
2.500	0.4611265	0.4611503	5.164349e−05
3.000	0.3592264	0.3591976	8.023693e−05
3.500	0.2762625	0.2762013	0.0002215195
4.000	0.2092815	0.2092034	0.0003732390
4.500	0.1550692	0.1549845	0.0005463745
5.000	0.1108197	0.1107347	0.0007669683
5.500	0.07428603	0.07420445	0.001098213
6.000	0.04373784	0.04366168	0.001741298
6.500	0.01786603	0.01779629	0.003903565
6.870	1.143486e−03	1.078780e−03	0.056586346
6.880	7.164872e−04	6.519191e−04	0.090117514
6.890	2.907277e−04	2.262974e−04	0.221617445

5. Conclusion

The present paper introduces an MC solver for LE type equations in astrophysics, physics, and chemistry. We calculated eleven (i.e., eleven LE equations) models of the positive index polytropes (for polytropic indices $n = 0, 0.5, 1.1.5, 2, 2.5, 3, 3.5, 4, 4.5, 4.99$ *and* 5), nine models for the negative index polytrope ($n = -0.95 - 0.99, -1.01, -1.5, -2, -3, -4, -5$ *and* -10), the isothermal gas sphere, and nine models for the white dwarf equation. A total of 1 000 000 random samples are utilized for the MC calculations. The Runge–Kutta approach (RK) for solving ordinary differential equations is used to numerically solve the four differential equations using the R package function rkMethod. We use an oddly spaced spacing with 0.001 increments.

Comparing the Emden function calculated by the MC, the RK, and the APS findings give good agreement for the polytropic indices $n = 0.5, 1.5, 2, 2.5$ *and* 3, respectively. The zeroth of the Emden function obtained using the MC algorithm has been compared with that of the exact and the RK solutions. Relative

Table A.9
Comparison between RK and MC solutions of $\theta(\xi)$ with polytrope index $n = 3.5$.

ξ	θ_{RK}	θ_{MC}	Relative error
0.000	1.00000e+00	1.00000e+00	0.000000e+00
0.500	0.9600772	0.9599264	0.0001570141
1.000	0.8580099	0.8577821	0.0002654604
2.000	0.6041619	0.6039827	0.0002966481
3.000	0.4029444	0.4028763	0.0001688633
4.000	0.2683486	0.2683612	4.671913e−05
5.000	0.1786841	0.1787481	0.0003581215
6.000	0.1166472	0.1167453	0.0008401983
7.000	0.07180062	0.07192273	0.001700682
8.000	0.03805984	0.03820017	0.003687073
9.000	0.011802996	0.011957768	0.013112958
9.500	7.471099e−04	0.0009080641	0.21543566
9.510	5.378502e−04	0.0006989220	0.29947324
9.520	3.290301e−04	0.0004902193	0.48989177
9.530	1.206483e−04	0.0002819546	1.33699652

Table A.10
Comparison between RK and MC solutions of $\theta(\xi)$ with polytrope index $n = 4$.

ξ	θ_{RK}	θ_{MC}	Relative error
0.000	1.00000e+00	1.00000e+00	0.000000e+00
1.000	0.8608141	0.8605968	0.0002523413
2.000	0.6229408	0.6227829	0.0002533719
3.000	0.4400506	0.4399975	0.0001205621
4.000	0.3180423	0.3180612	5.952305e−05
5.000	0.2359226	0.2359864	0.0002706464
6.000	0.1783841	0.1784770	0.0005207540
7.000	0.1363522	0.1364650	0.0008277002
8.000	0.1045040	0.1046314	0.001219358
9.000	0.07961936	0.07975797	0.001740897
10.000	0.05967264	0.05982018	0.002472474
11.000	0.04333999	0.04349488	0.003573840
12.000	0.02972583	0.02988692	0.005418869
13.000	0.01820531	0.01837169	0.009139184
14.000	0.008330434	0.008501414	0.02052474
14.600	0.003054810	0.003228268	0.05678208
14.930	3.339586e−04	0.0005087013	0.5232468
14.940	2.533849e−04	0.0004281658	0.6897839
14.950	1.729190e−04	0.0003477379	1.0109869
14.960	9.256075e−05	0.0002674176	1.8891043

errors revealed good agreement between the two solutions with a maximum value of 0.04%. The calculation for the negative index polytropes gives two interesting features; for $n = -0.95$ and $n = -0.99$, we obtained Emden functions to behave as the positive index polytropes, while for $n < -1$, the Emden functions act as monotonic functions. For the polytropic indexes $n = 0, 1$ *and* 5, we compared the MC results with several numerical approaches and found a good agreement.

CRediT authorship contribution statement

S.H. El-Essawy: Conceptualization, Methodology, Investigation, Software, Validation, Writing – original draft. **M.I. Nouh:** Conceptualization, Methodology, Investigation, Software, Validation, Writing – original draft. **A.A. Soliman:** Formal analysis, Writing – review & editing, Investigation, Data curation. **H.I. Abdel Rahman:** Formal analysis, Writing – review & editing, Investigation, Data curation. **G.A. Abd-Elmougod:** Formal analysis, Writing – review & editing, Investigation, Data curation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Table A.11
Comparison between RK and MC solutions of $\theta(\xi)$ with polytrope index $n = 4.5$.

ξ	θ_{RK}	θ_{MC}	Relative error
0.000	1.00000e+00	1.00000e+00	0.000000e+00
2.000	0.6396537	0.6395138	0.0002186821
4.000	0.3605337	0.3605576	6.636828e−05
6.000	0.2313630	0.2314518	0.0003836721
8.000	0.1617328	0.1618502	0.0007261914
10.000	0.1189406	0.1190736	0.001117632
12.000	0.09015570	0.09029831	0.001581835
14.000	0.06952037	0.06966968	0.002147665
16.000	0.05402041	0.05417468	0.002855762
18.000	0.04195725	0.04211538	0.003768835
20.000	0.03230423	0.03246547	0.004991228
24.000	0.01782304	0.01798900	0.009311670
28.000	0.007479021	0.007648421	0.02265005
30.500	0.002391790	0.002562903	0.07154183
31.000	0.001472806	0.001644230	0.1163928
31.500	0.0005829964	0.0007547220	0.2945569
31.600	0.0004084135	0.0005801983	0.4206150
31.700	2.349320e−04	0.0004067757	0.7314613
31.800	6.254169e−05	0.0002344439	2.7486018

Table A.12
Comparison between RK and MC solutions of $\theta(\xi)$ with polytrope index $n = 4.99$.

ξ	θ_{RK}	θ_{MC}	Relative error
0.000	1.00000e+00	1.00000e+00	0.000000e+00
2.000	0.6543685	0.6542436	0.0001908621
4.000	0.3966708	0.3966986	7.005805e−05
6.000	0.2764896	0.2765749	0.0003086424
8.000	0.2106689	0.2107784	0.0005199808
10.000	0.1696935	0.1698151	0.0007166558
12.000	0.1418680	0.1419964	0.0009052283
14.000	0.1217825	0.1219151	0.001089043
16.000	0.1066194	0.1067548	0.001269925
18.000	0.09477443	0.09491176	0.001448942
20.000	0.08526971	0.08540843	0.001626755
22.000	0.07747604	0.07761579	0.001803793
24.000	0.07097066	0.07111121	0.001980347
26.000	0.06545920	0.06560037	0.002156621
28.000	0.06073046	0.06087213	0.002332762
30.000	0.05662902	0.05677110	0.002508878
35.000	0.04841810	0.04856091	0.002949552
40.000	0.04225369	0.04239699	0.003391392
45.000	0.03745597	0.03759961	0.003834885
50.000	0.03361604	0.03375993	0.004280331
55.000	0.03047326	0.03061734	0.004727931
60.000	0.02785364	0.02799786	0.005177824
65.000	0.02563663	0.02578097	0.005630113
70.000	0.02373607	0.02388050	0.006084878
75.000	0.02208874	0.02223324	0.006542185
80.000	0.02064719	0.02079176	0.007002088
85.000	0.01937514	0.01951977	0.007464636
90.000	0.01824436	0.01838904	0.007929870
95.000	0.01723256	0.01737728	0.008397830
100.00	0.01632191	0.01646666	0.008868552

Data availability

Data will be made available on request.

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Table A.13
Comparison between RK and MC solutions of $\theta(\xi)$ with polytrope index $n = -1.01$.

ξ	θ_{RK}	θ_{MC}	Relative error	ξ	θ_{RK}	θ_{MC}	Relative error
0.000	1.00000e+00	1.00000e+00	0.000000e+00	23.000	16.06488	16.06519	1.907790e−05
1.000	1.159022	1.159321	0.0002581444	24.000	16.76643	16.76673	1.805431e−05
2.000	1.569078	1.569547	0.0002987201	25.000	17.46751	17.46781	1.713843e−05
3.000	2.121349	2.121882	0.0002513584	26.000	18.16817	18.16846	1.631522e−05
4.000	2.746491	2.747036	0.0001985520	27.000	18.86841	18.8687	1.557216e−05
5.000	3.409439	3.409973	0.0001568384	28.000	19.56826	19.56855	1.489877e−05
6.000	4.092713	4.093228	0.0001258860	29.000	20.26774	20.26803	1.428621e−05
7.000	4.787235	4.787728	0.0001029766	30.000	20.96687	20.96716	1.372702e−05
8.000	5.488061	5.488532	8.578283e−05	31.000	21.66566	21.66595	1.321484e−05
9.000	6.192386	6.192836	7.264446e−05	32.000	22.36415	22.36443	1.274422e−05
10.000	6.898565	6.898996	6.242207e−05	33.000	23.06233	23.06261	1.231049e−05
11.000	7.605613	7.606026	5.433329e−05	34.000	23.76023	23.76051	1.190962e−05
12.000	8.312928	8.313325	4.783399e−05	35.000	24.45785	24.45814	1.153811e−05
13.000	9.020142	9.020526	4.253929e−05	36.000	25.15522	25.15551	1.119292e−05
14.000	9.727034	9.727405	3.817210e−05	37.000	25.85235	25.85263	1.087141e−05
15.000	10.43347	10.43383	3.452948e−05	38.000	26.54924	26.54952	1.057124e−05
16.000	11.13938	11.13973	3.146041e−05	39.000	27.24592	27.24620	1.029038e−05
17.000	11.84471	11.84506	2.885078e−05	40.000	27.94238	27.94266	1.002704e−05
18.000	12.54947	12.54980	2.661324e−05	42.000	29.33470	29.33498	9.546672e−06
19.000	13.25365	13.25398	2.468003e−05	46.000	32.11719	32.11747	8.736822e−06
20.000	13.95726	13.95758	2.299800e−05	48.000	33.50747	33.50775	8.391986e−06
21.000	14.66032	14.66064	2.152504e−05	50.000	34.89717	34.89746	8.079390e−06
22.000	15.36285	15.36316					

Table A.14
Comparison between RK and MC solutions of $\theta(\xi)$ with polytrope index $n = -1.5$.

ξ	θ_{RK}	θ_{MC}	Relative error	ξ	θ_{RK}	θ_{MC}	Relative error
0.000	1.00000e+00	1.00000e+00	0.000000e+00	23.000	10.62928	10.62936	7.758030e−06
1.000	1.155613	1.155898	0.0002463248	24.000	11.00046	11.00054	7.271363e−06
2.000	1.532601	1.533004	0.0002633452	25.000	11.36827	11.36834	6.852927e−06
3.000	2.003254	2.003669	0.0002071382	26.000	11.73286	11.73294	6.491663e−06
4.000	2.502164	2.502551	0.0001545821	27.000	12.09441	12.09449	6.178525e−06
5.000	3.003615	3.003963	0.0001159915	28.000	12.45307	12.45314	5.906069e−06
6.000	3.498098	3.498408	8.867979e−05	29.000	12.80895	12.80903	5.668138e−06
7.000	3.982396	3.982672	6.920466e−05	30.000	13.16221	13.16228	5.459613e−06
8.000	4.455784	4.456029	5.506055e−05	31.000	13.51293	13.51300	5.276216e−06
9.000	4.918525	4.918744	4.458176e−05	32.000	13.86124	13.86131	5.114363e−06
10.000	5.371255	5.371452	3.667168e−05	33.000	14.20723	14.20730	4.971033e−06
11.000	5.814722	5.814900	3.059915e−05	34.000	14.55099	14.55106	4.843671e−06
12.000	6.249670	6.249832	2.586713e−05	35.000	14.89261	14.89268	4.730111e−06
13.000	6.676797	6.676945	2.213073e−05	36.000	15.23217	15.23224	4.628509e−06
14.000	7.096737	7.096873	1.914580e−05	37.000	15.56974	15.56981	4.537290e−06
15.000	7.510061	7.510187	1.673635e−05	38.000	15.90540	15.90547	4.455106e−06
16.000	7.917277	7.917394	1.477332e−05	39.000	16.23920	16.23927	4.380800e−06
17.000	8.318837	8.318947	1.316066e−05	40.000	16.57121	16.57128	4.313375e−06
18.000	8.715146	8.715249	1.182582e−05	42.000	17.23008	17.23016	4.195845e−06
19.000	9.106563	9.106661	1.071339e−05	46.000	18.52869	18.52876	4.012420e−06
20.000	9.493410	9.493503	9.780505e−06	48.000	19.16914	19.16921	3.939115e−06
21.000	9.875977	9.876065	8.993683e−06	50.000	19.80410	19.80418	3.874571e−06
22.000	10.25452	10.25461					

Table A.15
Comparison between RK and MC solutions of $\theta(\xi)$ with polytrope index $n = -2$.

ξ	θ_{RK}	θ_{MC}	Relative error	ξ	θ_{RK}	θ_{MC}	Relative error
0.000	1.00000e+00	1.00000e+00	0.000000e+00	23.000	7.836229	7.836236	8.970258e−07
1.000	1.152315	1.152586	0.0002351381	24.000	8.062923	8.062930	7.801629e−07
2.000	1.500526	1.500876	0.0002334171	25.000	8.286250	8.286256	7.033198e−07
3.000	1.907784	1.908113	0.0001724398	26.000	8.506403	8.506408	6.583673e−07
4.000	2.317285	2.317567	0.0001216847	27.000	8.723557	8.723563	6.387954e−07
5.000	2.712219	2.712453	8.639992e−05	28.000	8.937872	8.937878	6.393677e−07
6.000	3.088992	3.089185	6.235126e−05	29.000	9.149493	9.149499	6.558586e−07
7.000	3.448074	3.448232	4.573710e−05	30.000	9.358555	9.358561	6.848487e−07
8.000	3.791064	3.791193	3.402585e−05	31.000	9.565178	9.565185	7.235627e−07
9.000	4.119735	4.119840	2.560709e−05	32.000	9.769476	9.769484	7.697430e−07
10.000	4.435735	4.435821	1.944988e−05	33.000	9.971553	9.971562	8.215521e−07
11.000	4.740509	4.740580	1.488067e−05	34.000	10.17150	10.17151	8.774918e−07
12.000	5.035296	5.035354	1.144913e−05	35.000	10.36942	10.36943	9.363413e−07
13.000	5.321151	5.321199	8.847199e−06	36.000	10.56538	10.56539	9.971055e−07
14.000	5.598975	5.599013	6.859690e−06	37.000	10.75947	10.75948	1.058975e−06

(*continued on next page*)

Table A.15 (*continued*).

ξ	θ_{RK}	θ_{MC}	Relative error	ξ	θ_{RK}	θ_{MC}	Relative error
15.000	5.869535	5.869566	5.333411e−06	38.000	10.95175	10.95176	1.121293e−06
16.000	6.133493	6.133519	4.157427e−06	39.000	11.14230	11.14231	1.183528e−06
17.000	6.391420	6.391441	3.250187e−06	40.000	11.33117	11.33118	1.245252e−06
18.000	6.643812	6.643829	2.550922e−06	42.000	11.70412	11.70414	1.365862e−06
19.000	6.891102	6.891116	2.013802e−06	46.000	12.43233	12.43235	1.589954e−06
20.000	7.133671	7.133682	1.603888e−06	48.000	12.78830	12.78833	1.691613e−06
21.000	7.371855	7.371864	1.294285e−06	50.000	13.13928	13.13931	1.785862e−06
22.000	7.605952	7.605960					

Table A.16
Comparison between RK and MC solutions of $\theta(\xi)$ with polytrope index $n = -3$.

ξ	θ_{RK}	θ_{MC}	Relative error	ξ	θ_{RK}	θ_{MC}	Relative error
0.000	1.00000e+00	1.00000e+00	0.000000e+00	23.000	5.185568	5.185536	6.029996e−06
1.000	1.146205	1.146451	0.0002150764	24.000	5.296815	5.296785	5.697356e−06
2.000	1.448011	1.448282	0.0001869318	25.000	5.405624	5.405595	5.360659e−06
3.000	1.766015	1.766232	0.0001231185	26.000	5.512151	5.512123	5.024180e−06
4.000	2.061805	2.061964	7.740714e−05	27.000	5.616537	5.616511	4.691107e−06
5.000	2.331000	2.331112	4.803012e−05	28.000	5.718910	5.718885	4.363799e−06
6.000	2.576501	2.576576	2.915267e−05	29.000	5.819385	5.819362	4.043973e−06
7.000	2.802106	2.802153	1.679971e−05	30.000	5.918067	5.918045	3.732856e−06
8.000	3.011157	3.011183	8.570528e−06	31.000	6.015052	6.015032	3.431293e−06
9.000	3.206358	3.206367	3.017294e−06	32.000	6.110427	6.110408	3.139839e−06
10.000	3.389851	3.389848	7.543035e−07	33.000	6.204273	6.204255	2.858819e−06
11.000	3.563340	3.563328	3.312759e−06	34.000	6.296662	6.296646	2.588387e−06
12.000	3.728189	3.728170	5.029516e−06	35.000	6.387663	6.387649	2.328562e−06
13.000	3.885500	3.885476	6.153370e−06	36.000	6.477340	6.477326	2.079262e−06
14.000	4.036175	4.036147	6.854634e−06	37.000	6.565749	6.565737	1.840329e−06
15.000	4.180960	4.180930	7.252098e−06	38.000	6.652945	6.652934	1.611547e−06
16.000	4.320479	4.320447	7.429988e−06	39.000	6.738979	6.738969	1.392660e−06
17.000	4.455258	4.455225	7.448851e−06	40.000	6.823897	6.823888	1.183380e−06
18.000	4.585743	4.585710	7.352732e−06	42.000	6.990558	6.990552	7.924104e−07
19.000	4.712320	4.712286	7.173995e−06	46.000	7.312242	7.312241	1.117540e−07
20.000	4.835319	4.835285	6.936606e−06	48.000	7.467794	7.467795	1.830904e−07
21.000	4.955029	4.954996	6.658439e−06	50.000	7.620123	7.620127	4.508943e−07
22.000	5.071705	5.071673					

Table A.17
Comparison between RK and MC solutions of $\theta(\xi)$ with polytrope index $n = -4$.

ξ	θ_{RK}	θ_{MC}	Relative error	ξ	θ_{RK}	θ_{MC}	Relative error
0.000	1.00000e+00	1.00000e+00	0.000000e+00	23.000	3.964045	3.964010	8.814443e−06
1.000	1.140660	1.140885	0.0001976200	24.000	4.031416	4.031382	8.235174e−06
2.000	1.406659	1.406874	0.0001526830	25.000	4.097031	4.097000	7.673072e−06
3.000	1.665324	1.665474	9.014820e−05	26.000	4.161012	4.160983	7.130028e−06
4.000	1.893088	1.893182	4.954760e−05	27.000	4.223467	4.223439	6.607193e−06
5.000	2.092495	2.092547	2.490940e−05	28.000	4.284491	4.284465	6.105170e−06
6.000	2.269121	2.269143	9.820040e−06	29.000	4.344173	4.344148	5.624158e−06
7.000	2.427727	2.427728	4.281099e−07	30.000	4.402590	4.402567	5.164062e−06
8.000	2.571934	2.571920	5.468375e−06	31.000	4.459814	4.459793	4.724572e−06
9.000	2.704447	2.704422	9.154824e−06	32.000	4.515911	4.515892	4.305225e−06
10.000	2.827304	2.827272	1.140705e−05	33.000	4.570940	4.570922	3.905450e−06
11.000	2.942067	2.942030	1.270903e−05	34.000	4.624955	4.624939	3.524605e−06
12.000	3.049948	3.049907	1.337174e−05	35.000	4.678007	4.677993	3.162004e−06
13.000	3.151907	3.151864	1.360011e−05	36.000	4.730142	4.730129	2.816932e−06
14.000	3.248715	3.248671	1.353210e−05	37.000	4.781403	4.781391	2.488667e−06
15.000	3.340998	3.340954	1.326227e−05	38.000	4.831829	4.831818	2.176485e−06
16.000	3.429273	3.429229	1.285638e−05	39.000	4.881456	4.881447	1.879671e−06
17.000	3.513971	3.513927	1.236070e−05	40.000	4.930319	4.930311	1.597526e−06
18.000	3.595453	3.595411	1.180805e−05	42.000	5.025878	5.025873	1.074542e−06
19.000	3.674030	3.673989	1.122191e−05	44.000	5.118735	5.118732	6.023595e−07
20.000	3.749963	3.749924	1.061906e−05	46.000	5.209092	5.209091	1.762041e−07
21.000	3.823483	3.823444	1.001155e−05	48.000	5.297126	5.297127	2.082914e−07
22.000	3.894785	3.894749	9.407985e−06	50.000	5.382995	5.382998	5.550936e−07

Appendix. Numerical results

We list in the following tables the numerical results obtained for the four differential equations, Eqs. (10), (14), and (21), for the positive index polytrope, negative index polytrope, the isothermal gas sphere, and the white dwarf equation, respectively. We listed tables for some selected values; complete tables may be requested from the authors. The designation of the columns is as follows:

Column 1: The dimensionless distance (χ).
Column 2: The Emden function calculated by the exact solution (θ_{Exact}) or RK solution (θ_{RK}).
Column 3: The Emden function calculated by the MC solution (θ_{MC}).
Column 4: The relative error computed for the MC and RK solutions.

Table A.18
Comparison between RK and MC solutions of $\theta(\xi)$ with polytrope index $n = -5$.

ξ	θ_{RK}	θ_{MC}	Relative error	ξ	θ_{RK}	θ_{MC}	Relative error
0.000	1.00000e+00	1.00000e+00	0.000000e+00	23.000	3.278878	3.278846	9.810960e−06
1.000	1.135601	1.135808	0.0001823063	24.000	3.324933	3.324903	9.088937e−06
2.000	1.373127	1.373300	0.0001265409	25.000	3.369666	3.369638	8.397541e−06
3.000	1.589799	1.589906	6.685255e−05	26.000	3.413172	3.413146	7.737074e−06
4.000	1.772978	1.773033	3.081464e−05	27.000	3.455535	3.455510	7.107333e−06
5.000	1.928973	1.928992	9.939410e−06	28.000	3.496829	3.496806	6.507764e−06
6.000	2.064361	2.064356	2.301652e−06	29.000	3.537121	3.537100	5.937573e−06
7.000	2.184023	2.184003	9.543121e−06	30.000	3.576474	3.576455	5.395802e−06
8.000	2.291429	2.291398	1.378873e−05	31.000	3.614942	3.614924	4.881394e−06
9.000	2.389067	2.389029	1.618169e−05	32.000	3.652575	3.652559	4.393229e−06
10.000	2.478760	2.478717	1.740021e−05	33.000	3.689419	3.689405	3.930164e−06
11.000	2.561872	2.561826	1.786125e−05	34.000	3.725516	3.725503	3.491051e−06
12.000	2.639447	2.639400	1.782897e−05	35.000	3.760904	3.760892	3.074753e−06
13.000	2.712298	2.712251	1.747503e−05	36.000	3.795617	3.795607	2.680158e−06
14.000	2.781072	2.781025	1.691324e−05	37.000	3.829690	3.829681	2.306190e−06
15.000	2.846290	2.846244	1.622031e−05	38.000	3.863151	3.863144	1.951805e−06
16.000	2.908376	2.908331	1.544854e−05	39.000	3.896029	3.896023	1.616007e−06
17.000	2.96783	2.967639	1.463393e−05	40.000	3.925142	3.925137	1.328887e−06
18.000	3.024504	3.024462	1.380132e−05	42.000	3.991411	3.991408	7.107817e−0
19.000	3.079088	3.079048	1.296788e−05	44.000	4.052510	4.052509	1.838675e−07
20.000	3.131648	3.131610	1.214540e−05	46.000	4.111796	4.111798	2.890026e−07
21.000	3.182364	3.182328	1.134188e−05	48.000	4.169405	4.169408	7.132958e−07
22.000	3.231395	3.231361	1.056260e−05	50.000	4.225452	4.225457	1.093902e−06

Table A.19
Comparison between RK and MC solutions of $\theta(\xi)$ with polytrope index $n = -10$.

ξ	θ_{RK}	θ_{MC}	Relative error	ξ	θ_{RK}	θ_{MC}	Relative error
0.000	1.00000e+00	1.00000e+00	0.000000e+00	23.000	2.044024	2.044007	7.960936e−06
1.000	1.115647	1.115789	0.0001275456	24.000	2.059288	2.059274	7.068898e−06
2.000	1.268750	1.268821	5.574577e−05	25.000	2.074032	2.074019	6.231806e−06
3.000	1.383919	1.383936	1.224969e−05	26.000	2.088295	2.088283	5.446560e−06
4.000	1.471326	1.471313	8.860233e−06	27.000	2.102112	2.102102	4.710112e−06
5.000	1.540827	1.540798	1.908105e−05	28.000	2.115514	2.115505	4.019506e−06
6.000	1.598300	1.598262	2.385867e−05	29.000	2.128530	2.128523	3.371933e−06
7.000	1.647272	1.647230	2.576803e−05	30.000	2.141184	2.141178	2.764725e−06
8.000	1.689971	1.689927	2.610454e−05	31.000	2.153500	2.153495	2.195354e−06
9.000	1.727871	1.727827	2.556004e−05	32.000	2.165498	2.165494	1.661444e−06
10.000	1.761992	1.761949	2.452301e−05	33.000	2.177196	2.177194	1.160765e−06
11.000	1.793067	1.793025	2.321937e−05	34.000	2.188612	2.188611	6.912254e−07
12.000	1.821633	1.821594	2.178431e−05	35.000	2.199761	2.199761	2.508766e−07
13.000	1.848101	1.848064	2.030003e−05	36.000	2.210657	2.210657	1.621052e−07
14.000	1.872788	1.872753	1.881678e−05	37.000	2.221313	2.221314	5.494215e−07
15.000	1.895942	1.895909	1.736519e−05	38.000	2.231741	2.231743	9.126640e−07
16.000	1.917766	1.917735	1.596349e−05	39.000	2.241952	2.241955	1.253311e−06
17.000	1.938421	1.938392	1.462206e−05	40.000	2.251957	2.251960	1.572748e−06
18.000	1.958042	1.958016	1.334617e−05	42.000	2.271383	2.271388	2.153063e−06
19.000	1.994616	1.994594	1.099696e−05	44.000	2.290089	2.290095	2.662931e−06
20.000	1.994616	1.994594	1.099696e−05	46.000	2.308132	2.308139	3.110478e−06
21.000	2.011743	2.011723	9.922110e−06	48.000	2.325565	2.325573	3.502799e−06
22.000	2.028192	2.028174	8.911012e−06	50.000	2.342434	2.342443	3.846096e−06

Table A.20
Comparison between RK and MC solutions of $\theta(\xi)$ with polytrope index $n \to \infty$.

ξ	θ_{RK}	θ_{MC}	Relative error	ξ	θ_{RK}	θ_{MC}	Relative error
0.000	0.00000e+00	0.00000e+00	0.000000e+00	18.000	5.162940	5.162785	2.997646e−05
0.500	0.04116308	0.04132496	0.003932624	19.000	5.287224	5.287063	3.040928e−05
1.000	0.1588357	0.1591336	0.001875320	20.000	5.403886	5.403721	3.062334e−05
2.000	0.5598280	0.5602713	0.0007917708	21.000	5.513725	5.513556	3.066668e−05
3.000	1.063337	1.063782	0.0004186039	22.000	5.617429	5.617258	3.057622e−05
4.000	1.572233	1.572610	0.0002401128	23.000	5.715594	5.715420	3.038069e−05
5.000	2.044091	2.044382	0.0001425249	24.000	5.808736	5.808561	3.010262e−05
6.000	2.467208	2.467417	8.462947e−05	25.000	5.897309	5.897134	2.975982e−05
7.000	2.842587	2.842724	4.820993e−05	26.000	5.981712	5.981536	2.936645e−05
8.000	3.175394	3.175472	2.426887e−05	27.000	6.062294	6.062118	2.893384e−05
9.000	3.471556	3.471584	7.994843e−06	28.000	6.139367	6.139192	2.847110e−05
10.000	3.736558	3.736545	3.350720e−06	29.000	6.213207	6.213033	2.798560e−05
11.000	3.975117	3.975072	1.140709e−05	30.000	6.284060	6.283888	2.748330e−05
12.000	4.191176	4.191104	1.719919e−05	31.000	6.352149	6.351978	2.696906e−05
13.000	4.387992	4.387898	2.139243e−05	32.000	6.417671	6.417502	2.644681e−05
14.000	4.568251	4.568140	2.443182e−05	33.000	6.480806	6.480638	2.591979e−05
15.000	4.734176	4.734050	2.662318e−05	34.000	6.541714	6.541548	2.539062e−05
16.000	4.887613	4.887475	2.818173e−05	35.000	6.600543	6.600379	2.486147e−05
17.000	5.030102	5.029955	2.926188e−05				

Table A.21
Comparison between RK and MC solutions of $\psi(\eta)$ with constants $C = 0.01 - 0.1$.

$C = 0.01$				$C = 0.1$			
η	θ_{RK}	θ_{MC}	Relative error	η	θ_{RK}	θ_{MC}	Relative error
0.000	1.00000e+00	1.00000e+00	0.000000e+00	0.000	1.00000e+00	1.00000e+00	0.000000e+00
0.500	0.9604334	0.9602827	0.0001568452	0.500	0.9656478	0.9655167	1.357327e−04
1.000	0.8571405	0.8569046	0.0002752544	1.000	0.8754712	0.8752640	0.0002366674
1.500	0.7234100	0.7231662	0.0003370373	1.500	0.7578572	0.7576410	0.0002852371
2.000	0.5885526	0.5883501	0.0003441379	2.000	0.6387512	0.6385709	0.0002822953
2.500	0.4684922	0.4683486	0.0003065139	2.500	0.5330399	0.5329130	0.0002380624
3.000	0.3681576	0.3680721	0.0002323289	3.000	0.4458162	0.4457430	0.0001641943
3.500	0.2867141	0.2866786	1.238057e−04	3.300	0.4022845	0.4022404	1.098144e−04
4.000	0.2212402	0.2212455	2.435932e−05	3.550	0.3705892	0.3705668	6.027593e−05
4.200	0.1987934	0.1988129	9.769646e−05	3.600	0.3647122	0.3646939	4.998560e−05
4.400	0.1781920	0.1782243	0.0001812103	3.650	0.3589814	0.3589672	3.958061e−05
4.600	0.1592579	0.1593020	0.0002769787	3.700	0.3533934	0.3533831	2.906585e−05
4.800	0.1418276	0.1418826	0.0003877678	3.750	0.3479447	0.3479382	1.844580e−05
5.000	0.1257519	0.1258169	0.0005173050	3.800	0.3426318	0.3426291	7.724584e−06
5.100	0.1181791	0.1182489	0.0005906332	3.900	0.3323994	0.3324041	1.400726e−05
5.200	0.1108950	0.1109694	0.0006707111	4.000	0.3226680	0.3226796	3.610706e−05
5.357	0.1000059	0.10008711	0.0008124600	4.069	0.3162301	0.3162464	5.156471e−05

Table A.22
Comparison between RK and MC solutions of $\psi(\eta)$ with constants $C = 0.2 - 0.3$.

$C = 0.2$				$C = 0.3$			
η	θ_{RK}	θ_{MC}	Relative error	η	θ_{RK}	θ_{MC}	Relative error
0.000	1.00000e+00	1.00000e+00	0.000000e+00	0.000	1.00000e+00	1.00000e+00	0.000000e+00
0.500	0.9711551	0.9710448	1.136257e−04	0.500	0.9763417	0.9762509	9.295675e−05
1.000	0.8949289	0.8947525	0.0001971433	1.000	0.91336268	0.9132157	0.000160887
1.500	0.7945548	0.7943683	0.0002346878	1.500	0.8295005	0.8293428	0.0001902091
2.000	0.6921701	0.6920132	0.0002267944	2.000	0.7430857	0.7429509	0.0001813331
2.100	0.6728038	0.6726554	0.0002205926	2.100	0.7266719	0.7265442	0.0001757375
2.200	0.6539855	0.6538461	0.0002130886	2.200	0.7107107	0.7105906	0.0001690950
2.300	0.6357639	0.6356340	0.0002043915	2.300	0.6952487	0.6951364	0.0001615106
2.400	0.6181762	0.6180559	0.0001946114	2.400	0.6803224	0.6802182	0.0001530905
2.500	0.6012488	0.6011382	0.0001838574	2.500	0.6659590	0.6658631	0.0001439404
2.600	0.5849989	0.5848982	0.0001722373	2.600	0.6521774	0.6520899	0.0001341642
2.700	0.5694358	0.5693448	0.0001598560	2.700	0.6389889	0.6389097	0.0001238630
2.800	0.5545614	0.5544800	0.0001468156	2.800	0.6263983	0.6263274	1.131343e−04
2.900	0.5403719	0.5402999	0.0001332138	2.900	0.6144044	0.6143417	1.020706e−04
3.000	0.5268581	0.5267953	0.0001191438	3.000	0.6030012	0.6029465	9.075923e−05
3.200	0.5018003	0.5017551	8.994681e−05	3.100	0.5921780	0.5921310	7.928149e−05
3.400	0.4792399	0.4792113	5.985975e−05	3.200	0.5819203	0.5818809	6.771207e−05
3.500	0.4688382	0.4688173	4.465261e−05	3.300	0.5722101	0.5721780	5.611871e−05
3.600	0.4589869	0.4589734	2.941249e−05	3.400	0.5630267	0.5630016	4.456192e−05
3.700	0.4496568	0.4496504	1.418607e−05	3.500	0.5543459	0.5543276	3.309489e−05
3.727	0.4472230	0.4472185	1.008237e−05	3.580	0.5477452	0.5477320	2.401760e−05

Table A.23
Comparison between RK and MC solutions of $\psi(\eta)$ with constants $C = 0.4 - 0.5$.

$C = 0.4$				$C = 0.5$			
η	θ_{RK}	θ_{MC}	Relative error	η	θ_{RK}	θ_{MC}	Relative error
0.000	1.00000e+00	1.00000e+00	0.000000e+00	0.000	1.00000e+00	1.00000e+00	0.000000e+00
0.500	0.9811835	0.9811111	7.379295e−05	0.500	0.9856505	0.9855952	5.620388e−05
1.000	0.9306870	0.9305681	0.0001277466	1.000	0.9467932	0.9467008	9.761168e−05
1.500	0.8625650	0.8624350	0.0001508094	1.500	0.8935688	0.8934654	0.0001157336
2.000	0.7914187	0.7913053	0.0001431995	2.000	0.8370121	0.8369196	0.0001104989
2.100	0.7778116	0.7777038	0.0001386363	2.100	0.8260864	0.8259979	0.0001071285
2.200	0.7645556	0.7644537	0.0001332529	2.200	0.81541031	0.8153262	.031329e−04
2.300	0.7516939	0.7515983	0.0001271400	2.300	0.8050221	0.8049428	9.858252e−05
2.400	0.7392613	0.7391723	0.0001203888	2.400	0.7949535	0.7948792	9.354869e−05
2.500	0.7272846	0.7272023	0.0001130903	2.500	0.7852301	0.7851609	8.810257e−05
2.600	0.7157837	0.7157083	1.053338e−04	2.600	0.7758717	0.7758078	8.231407e−05
2.700	0.7047717	0.7047032	9.720561e−05	2.700	0.7668926	0.7668342	7.625095e−05
2.800	0.6942560	0.6941944	8.878862e−05	2.800	0.7583024	0.7582493	6.997804e−05
2.900	0.6842384	0.6841836	8.016108e−05	2.900	0.7501059	0.7500582	6.355666e−05
3.000	0.6747162	0.6746680	7.139613e−05	3.000	0.7423039	0.7422615	5.704401e−05
3.100	0.6656822	0.6656406	6.256125e−05	3.100	0.7348934	0.7348563	5.049281e−05
3.200	0.6571257	0.6570904	5.371790e−05	3.200	0.7278680	0.7278360	4.395104e−05
3.300	0.6490323	0.6490032	4.492129e−05	3.300	0.7212182	0.7211912	3.746185e−05
3.400	0.6413845	0.6413613	3.622043e−05	3.400	0.7149314	0.7149092	3.106369e−05
3.500	0.6341614	0.6341439	2.765903e−05	3.500	0.7089920	0.7089744	2.479118e−05
3.524	0.6324884	0.6324722	2.562977e−05	3.532	0.7071617	0.7071456	2.281613e−05

Table A.24
Comparison between RK and MC solutions of $\psi(\eta)$ with constants $C = 0.6 - 0.7$.

$C = 0.6$				$C = 0.7$			
η	θ_{RK}	θ_{MC}	Relative error	η	θ_{RK}	θ_{MC}	Relative error
0.000	1.00000e+00	1.00000e+00	0.000000e+00	0.000	1.00000e+00	1.00000e+00	0.000000e+00
0.500	0.9897043	0.9896644	4.031136e−05	0.500	0.9932918	0.9932657	2.628113e−05
1.000	0.9615379	0.9614701	7.047888e−05	1.000	0.9747213	0.9746760	4.645837e−05
1.500	0.9222581	0.9221802	8.446162e−05	1.500	0.9482583	0.9482045	5.670576e−05
2.000	0.8795931	0.8795210	8.189307e−05	2.000	0.9186996	0.9186477	5.646971e−05
2.100	0.8712368	0.8711674	7.970403e−05	2.100	0.9128021	0.9127516	5.532516e−05
2.200	0.8630349	0.8629684	7.705819e−05	2.200	0.9069769	0.9069281	5.387246e−05
2.300	0.8550190	0.8549557	7.400443e−05	2.300	0.9012477	0.9012007	5.214033e−05
2.400	0.8472164	0.8471566	7.059263e−05	2.400	0.8956356	0.8955907	5.015853e−05
2.500	0.8396500	0.8395938	6.687291e−05	2.500	0.8901590	0.8901163	4.795735e−05
2.600	0.8472164	0.8471566	7.059263e−05	2.600	0.8848337	0.8847933	4.556717e−05
2.700	0.8252958	0.8252473	5.870661e−05	2.700	0.8796728	0.8796349	4.301802e−05
2.800	0.8185334	0.8184889	5.435492e−05	2.800	0.8746870	0.8746517	4.033921e−05
2.900	0.8120585	0.8120179	4.988395e−05	2.900	0.8698847	0.8698520	3.755899e−05
3.000	0.8058748	0.8058383	4.533529e−05	3.000	0.8652719	0.8652419	3.470429e−05
3.100	0.7999836	0.7999510	4.074750e−05	3.100	0.8608526	0.8608252	3.180048e−05
3.200	0.7943828	0.7943541	3.615595e−05	3.200	0.8566287	0.8566040	2.887121e−05
3.300	0.7890683	0.7890433	3.159269e−05	3.300	0.8526004	0.8525783	2.593823e−05
3.400	0.7840330	0.7840117	2.708650e−05	3.400	0.8487660	0.8487465	2.302140e−05
3.500	0.7792678	0.7792502	2.266316e−05	3.500	0.8451223	0.8451053	2.013865e−05
3.603	0.7746296	0.7746155	1.821920e−05	3.600	0.8416643	0.8416497	1.730615e−05
				3.700	0.8383855	0.8383733	1.453868e−05
				3.750	0.8366865	0.8366756	1.307657e−05

Table A.25
Comparison between RK and MC solutions of $\psi(\eta)$ with constant $C = 0.8$.

η	θ_{RK}	θ_{MC}	Relative error	η	θ_{RK}	θ_{MC}	Relative error
0.000	1.00000e+00	1.00000e+00	0.000000e+00	2.100	0.9499440	0.9499123	3.337009e−05
0.100	0.9998511	0.9998481	2.984004e−06	2.200	0.9463841	0.9463530	3.284562e−05
0.200	0.9994054	0.9993994	5.932689e−06	2.300	0.9428509	0.9428205	3.215834e−05
0.300	0.9986665	0.9986576	8.832321e−06	2.400	0.9393579	0.9393284	3.132147e−05
0.400	0.9976403	0.9976287	1.165809e−05	2.500	0.9359173	0.9358889	3.034882e−05
0.500	0.9963349	0.9963206	1.438666e−05	2.600	0.9325403	0.9325130	2.925458e−05
0.600	0.9947606	0.9947437	1.699611e−05	2.700	0.9292364	0.9292103	2.805312e−05
0.700	0.9929295	0.9929102	1.946626e−05	2.800	0.9260143	0.9259895	2.675880e−05
0.800	0.9908555	0.9908340	2.177879e−05	2.900	0.9228813	0.9228579	2.538583e−05
0.900	0.9885541	0.9885305	2.391751e−05	3.000	0.9198434	0.9198214	2.394805e−05
1.000	0.9860421	0.9860166	2.586848e−05	3.100	0.9169057	0.9168851	2.245886e−05
1.100	0.9833374	0.9833102	2.762015e−05	3.200	0.9140719	0.9140528	2.093106e−05
1.200	0.9804587	0.9804301	2.916340e−05	3.300	0.9113449	0.9113272	1.937679e−05
1.300	0.9774255	0.9773957	3.049166e−05	3.400	0.9087264	0.9087102	1.780740e−05
1.400	0.9742574	0.9742266	3.160084e−05	3.500	0.9062172	0.9062025	1.623347e−05
1.500	0.9709744	0.9709428	3.248928e−05	3.600	0.9038173	0.9038041	1.466473e−05
1.600	0.9675963	0.9675642	3.315768e−05	3.700	0.9015257	0.9015139	1.311005e−05
1.700	0.9641427	0.9641103	3.360897e−05	3.800	0.8993406	0.8993302	1.157751e−05
1.800	0.9606326	0.9606001	3.384814e−05	3.900	0.8972594	0.8972504	1.007446e−05
1.900	0.9570845	0.9570521	3.388207e−05	4.045	0.8944189	0.8944118	7.961630e−06
2.000	0.9535161	0.9534839	3.371936e−05				

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