Discrete Methods in CS

V. Adamchik

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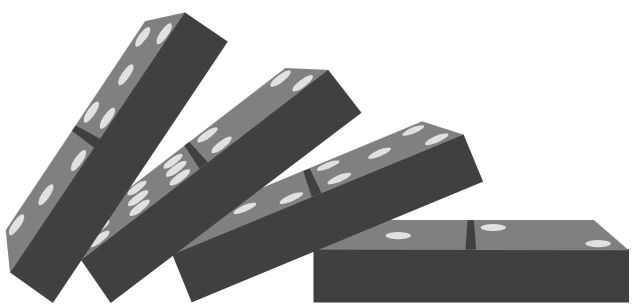
Induction Proofs

Domino Effect

Line up any number of dominos in a row;

knock the first one over and they will all fall.



Dominoes Numbered 1 to n

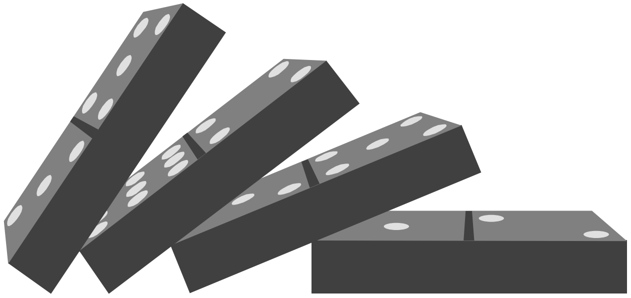
If we set them up in a row, then each one will knock over the next.

Let D_k be a statement "the k^{th} domino falls"

Here are the rules:

1. D_1
2. $D_k \rightarrow D_{k+1}$ (if k^{th} falls, then $(k+1)^{st}$ falls)

$D_1 \rightarrow D_2 \rightarrow D_3 \rightarrow \ldots$ All Dominoes Fall



Proof by
Mathematical Induction

In formal notation. Let P(k) be a predicate, then

$$[P(0) \wedge \forall k \in \mathbb{N}, (P(k) \rightarrow P(k+1)] \Longrightarrow \forall n \in \mathbb{N}, \ P(n)$$

Instead of attacking a problem directly,

we only explain how to *get* a proof for P(k+1) out of P(k).

We assume P(k) and prove P(k+1).

In *general*, mathematical induction is used to prove statements that assert that *P(n)* is true for positive integers n.

The Principle of Induction

Let $P(n)$ be a statement (predicate) for each integer $n \geq a \geq 0$.
The principle of induction is a way of proving that $P(n)$ is true for $\forall n \geq a$.

$$[P(a) \land \forall k \geq a, (P(k) \to P(k+1)] \Longrightarrow \forall n \geq a,\ P(n)$$

Base case(s): Show that $P(a)$ holds.

Induction Hypothesis (IH): Assume that $P(k)$ holds for some $k \geq a$.

Induction Step (IS): Show that $P(k)$ implies $P(k+1)$.
The induction step not necessarily should start with *k*.
You can change the step from $k - 1$ to k.
However, you cannot make a step from $k - 1$ to $k + 1$.

Then we conclude that $P(n)$ is true for all integers $n \geq a$.

Example 1

Prove that $2^n > 2 \cdot n$ for $n > 2$.

Let P(k) be the predicate "$\forall k > 2,\ 2^k > 2 \cdot k$"

Base case: $k = 3$, $2^k = 2^3 = 8 > 6 = 2 \cdot k$. True

IH: assume P(k) is true for some k: $2^k > 2 \cdot k$

IS: we need to prove that P(k+1) is also true: $2^{k+1} > 2 \cdot (k+1)$

Proof:

$2^{k+1} = 2 \cdot 2^k >$ (by the IH)

$2 \cdot (2 \cdot k) > 2 \cdot (k + 1)$

The last inequality holds since,

$4 \cdot k > 2 \cdot k + 2$

$2 \cdot k > 2$

$k > 1$

Hence, by mathematical induction *P(n)* is true for all integers *n > 2*.

A Template for Induction Proofs

State that the proof uses induction.
If there are several variables, indicate which variable you will be using.

Define a statement P(k), this is the IH.

Prove initial case(s).

Prove that P(k) implies P(k+1), this is the IS.

State the induction principle allows you to conclude that P(n) is true for all nonnegative n.

The trick to using the Principle of Induction is to spot how to use P(k) to prove P(k+1).

Example 2

Prove: $1 + 2 + 3 + \ldots + n = n(n + 1)/2$, for all integers $n \geq 1$.

Let $P(k)$ = "$\forall k \geq 1$, $1 + 2 + 3 + \ldots + k = k(k + 1)/2$."

Base step (case): $k = 1$. Verify $1 = 1(1 + 1)/2$. True.

IH: Assume $P(k)$ is true for some fixed $k \geq 1$.

IS: Prove $P(k + 1)$ is also true.

Consider the LHS of $P(k + 1)$:

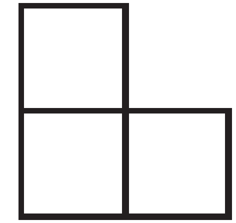
$$\underbrace{1 + 2 + 3 + \ldots + k} + (k + 1) = \text{(by the IH)}$$

$$k(k + 1)/2 + (k + 1) =$$

$$k(k + 1)/2 + 2(k + 1)/2 = (k + 2)(k + 1)/2$$

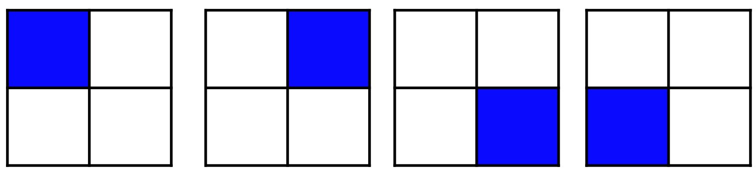
Hence, by mathematical induction $P(n)$ is true for all integers $n \geq 1$.

The Tiling problem



A checkerboard is a $2^n \times 2^n$ grid of equally-sized squares. A tromino covers three squares of a checkerboard, it resembles the letter L in shape. If a checkerboard has one square removed, can it be tiled with trominos?

Base Cases:

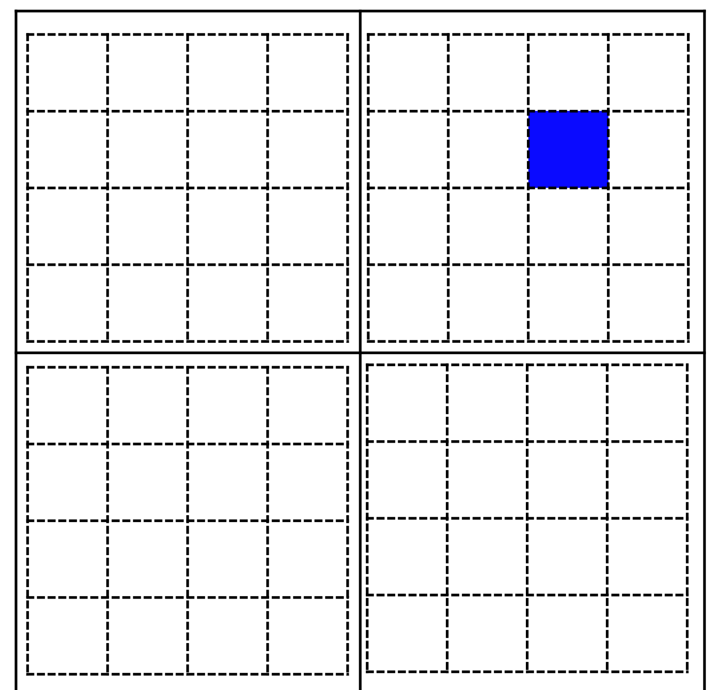


IH: The claim is true for a $2^k \times 2^k$ grid

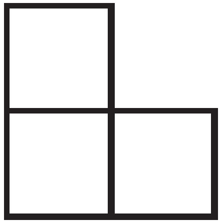
IS: Prove it for a $2^{k+1} \times 2^{k+1}$ grid .

Split the board in four quadrants.

Only one of those four quadrants
can be tiled (by the IH).



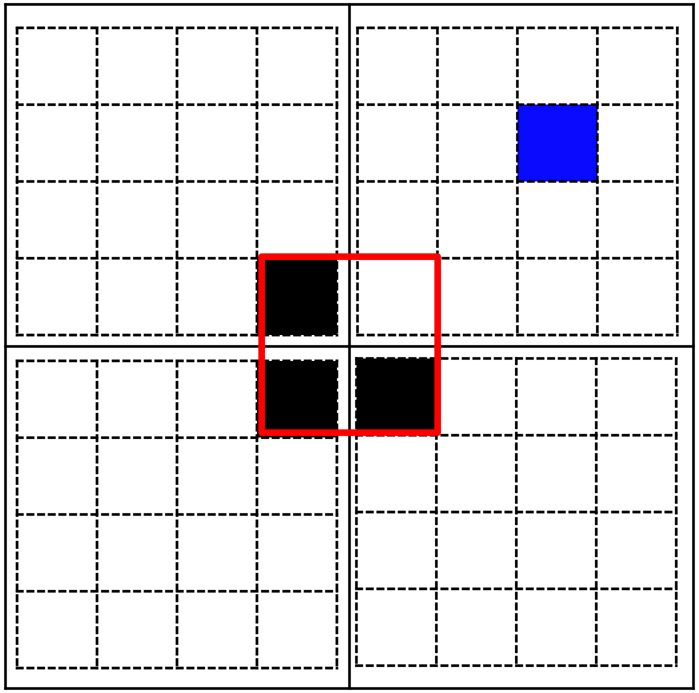
The Tiling problem



Consider the four central squares, and remove one tile per quadrant.

This looks like placing a tromino at the the center.

Then each quadrant can be tiled by the IH.



Funny Example

Any group of students are all of the same gender.

Base step: one person can certainly only be one gender.

IH: assume all students are of the same gender, for all sets of size k.

IS: prove it for a set of k+1 students.

Given a set of k+1 students. Consider the subset formed by removing a student. Let us remove the $(k+1)^{st}$ person.

The remaining 1, …, k students are of the same gender (let say, only girls), by the IH.

$\underline{S_1, S_2, \ldots, S_k}, S_{k+1}$

In particular, #2 is a girl. Therefore, all students from 2 to (k+1) are all girls, again by the IH.

$S_1, \underline{S_2, \ldots, S_{k+1}}$

$P(1) \not\Rightarrow P(2)$

Therefore, all students are girls.

Example 3

Prove that 8^n - 2^n is divisible by 6 for $n > 0$.

Let P(k) be the predicate "*$8^k - 2^k$ is divisible by 6*"

Base case: k =1, $8^k - 2^k = 8 - 2 = 6$. True.

IH: assume P(k) is true for some $k > 0$.

IS: we need to prove that P(k+1) is also true.

Proof of IS:

$$P(k+1) = 8^{k+1} - 2^{k+1} = 8 \cdot 8^k - 2 \cdot 2^k$$

Let us examine P(k+1).

$$= (6+2) \cdot 8^k - 2 \cdot 2^k$$

$$= 6 \cdot 8^k + 2 \cdot (8^k - 2^k)$$

Now, by the IH P(k+1) is divisible by 6.

Hence, by mathematical induction *P(n)* is true for all integers $n > 0$.

Exercise 1

Prove that $3^n + 7^n - 2$ is divisible by 8 for $n > 0$.

Let P(k) be the predicate "*$3^k + 7^k - 2$ is divisible by 8*"

Base case: k =1, $3^k + 7^k - 2 = 3 + 7 - 2 = 8$. True.

IH: assume P(k) is true for some k > 0.

IS: we need to prove that P(k+1) is also true.

Proof of IS:

Let us examine P(k+1).

$$
\begin{aligned}
3^{k+1} + 7^{k+1} - 2 &= 3\cdot 3^k + 7\cdot 7^k - 2 \\
&= 3\cdot 3^k + (3 + 4)\cdot 7^k - 2 \\
&= 3\cdot(3^k + 7^k - 2) + 4\cdot 7^k + 4
\end{aligned}
$$

Note, $4\cdot 7^k + 4$ is divisible by 8, if $7^k + 1$ is even, or if 7^k is odd.

That is so, since the product of odd integers is odd.

Soundness of Induction

How do we know that INDUCTION really works?

$$[P(0) \wedge \forall k, (P(k) \rightarrow P(k+1)] \Longrightarrow \forall n,\ P(n)$$

Proof by contradiction.

Assume that for statement P(n), we can establish the base step P(0), and the induction step, but nonetheless it is not true that P(n) holds for all n.

So, for some values of n, P(n) is false.

Soundness of Induction

Let n_0 be the least such n that $P(n_0)$ is false.

Certainly, n_0 cannot be 0, due to the base case

Thus, $\exists n_1$ such that $n_1 < n_0$ and $n_0 = n_1 + 1$.

Now, by our choice of n_0, this means that $P(n_1)$ holds, since $n_1 < n_0$.

Then by the Induction Step, $P(n_1 + 1)$ also holds.

But $P(n_1 + 1)$ is the same as $P(n_0)$, so $P(n_0)$ is true

and we have a contradiction.

Least Element Principle

In the proof we pick n_0 to be the least n that $P(n_0)$ is false.

Does the set of integers have the least element?

Least Element Principle.

Every non-empty subset of the natural numbers must contain a least element.

We have used this principal implicitly in the irrationality proofs.

Lecture 8, slide 9. Assume $\sqrt{2}$ is rational, so $\sqrt{2} = \frac{p}{q}$, for some integers p and q, where q is the least denominator among the fractions.

Example 4

Let S be a finite set of $n \geq 0$ elements. Prove that $|\mathcal{P}(S)| = 2^n$.

All subsets of a set S is called a power set $\mathcal{P}(A)$. Lecture 6.

$\mathcal{P}(\{1, 2\}) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$.

Let P(k) be the proposition that a set with k elements has 2^k subsets.

Base case: $\mathcal{P}(\emptyset) = \{\emptyset\}$. True

IH: Assume P(k) is true for some $k \geq 0$.

IS: Prove P(k + 1) is also true.

Let T be a set with k + 1 elements. We can write $T = S \cup \{x\}$.

As an example, consider $\mathcal{P}(\{1, 2, x\})$

Example 4 (cont.)

As an example, consider $\mathcal{P}(\{1, 2, x\}) = \{\emptyset, \{1\}, \{2\}, \{1, 2\},$
$\{x\}, \{1, x\}, \{2, x\}, \{1, 2, x\}\}$.

In this example we see that $\mathcal{P}(\{1, 2, x\})$ consists of two groups: one (in red) does not contain x, while the second group does contain x.

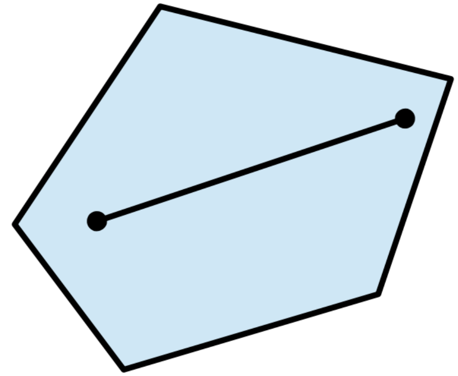
It follows, the size of $|\mathcal{P}(\{1, 2, x\})| = 2 \cdot |\mathcal{P}(\{1, 2\})|$

Therefore, $|\mathcal{P}(T)| = |\mathcal{P}(S \cup \{x\})| = 2 \cdot |\mathcal{P}(S)| =_{(IH)} = 2 \cdot 2^k = 2^{k+1}$.

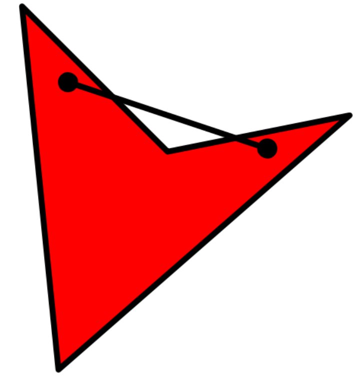
By mathematical induction, it follows that P(n) is true for all nonnegative integers n.

Example 5

A convex polygon is a polygon where, for any two points in or on the polygon, the line between those points is contained within the polygon.



convex



non-convex

Theorem. The sum of the internal angles of a convex n-gon is $(n - 2) \cdot 180°$ for $n \geq 3$.

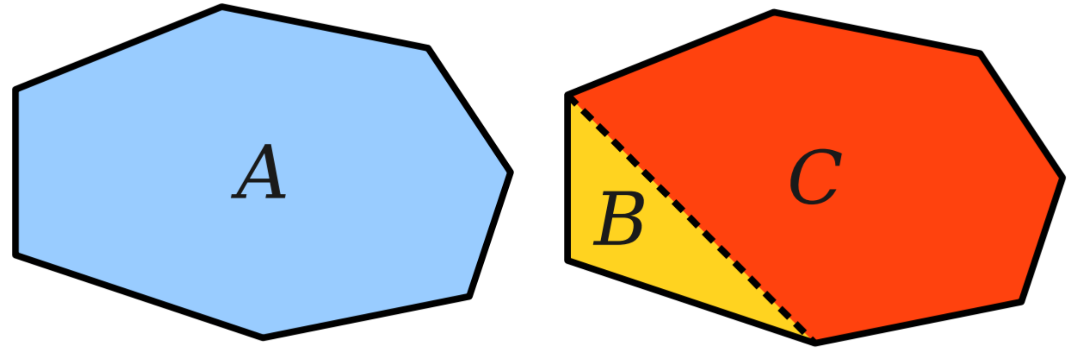
Let P(k) be the statement of the theorem.

Base case: k = 3.True. The result was shown in high school. Angles in a triangle add up to 180°

IH: Assume P(k) is true for some $k \geq 3$.

IS: Prove P(k + 1) is also true; the sum of angles is $(k - 1) \cdot 180°$.

Continue



Let A be an arbitrary convex polygon with $k+1$ vertices.

Take any three consecutive vertices in A and draw a line as shown.

The sum of the angles in A is equal to the sum of the angles in triangle B and the sum of the angles in convex polygon C.

The sum of the angles in $A = 180° + (k - 2) \cdot 180° = (k - 1) \cdot 180°$

Thus P(k + 1) holds, completing the induction.

Induction misproof

Theorem. *For $n > 0$* $$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \ldots + \frac{1}{(n-1) \times n} = \frac{3}{2} - \frac{1}{n}$$

Let P(k) be the statement of the theorem.

Base case: k = 1. $$\frac{1}{1 \times 2} = \frac{3}{2} - \frac{1}{k}$$

The problem is with the base case!

IH: Assume P(k) is true for some k > 0.

IS: Prove P(k + 1) is also true.

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \cdots + \frac{1}{(k-1) \times k} + \frac{1}{k \times (k+1)} = \frac{3}{2} - \frac{1}{k} + \frac{1}{k \times (k+1)}$$

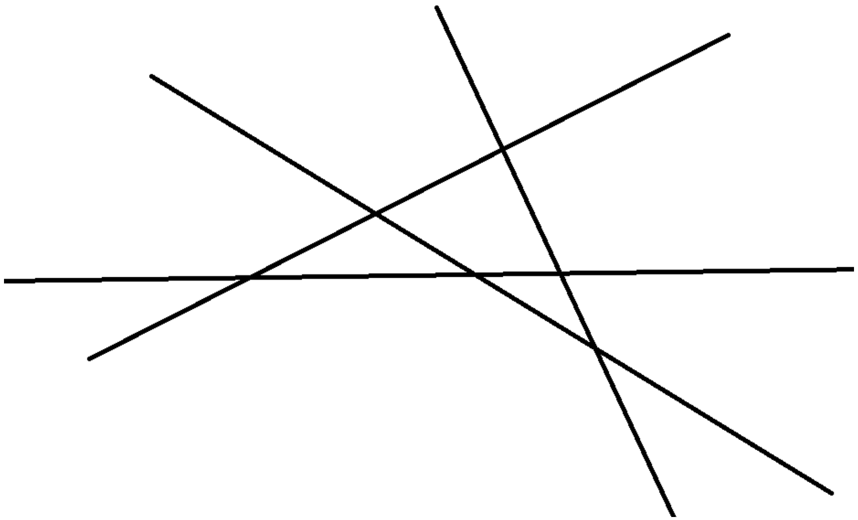
$$= \frac{3}{2} - \frac{1}{k} + \left(\frac{1}{k} - \frac{1}{k+1}\right)$$

$$= \frac{3}{2} - \frac{1}{k+1}$$

This is what we needed to show.

Example 6

Imaging cutting the plane with $n \geq 0$ lines in such a way that no pair of lines is parallel, and no three lines intersect in a point. Into how many regions is the plane cut?



$n = 0$: 1 region

$n = 1$: 2 regions

$n = 2$: 4 regions

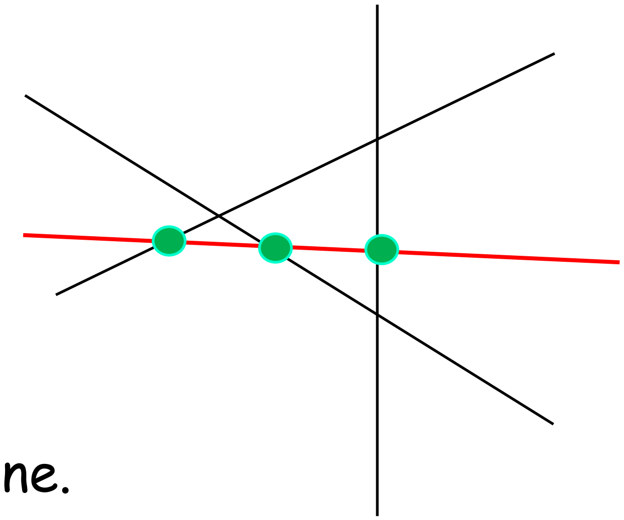
$n = 3$: 7 regions

<u>Theorem</u>.

n lines (as above) separate the plane in $(n^2 + n + 2)/2$ regions.

Continue

Let P(k) be the statement of the theorem.



Base case: we have already proved.

IH: Assume P(k) is true for some $k \geq 0$.

IS: Prove $P(k + 1)$ is also true.

Imagine a plane with k lines. Draw another line.

The new $(k + 1)$-st line will cross all other k lines.

So, it will make k intersections. None of these intersections are at the same point on the new line.

The points of intersection divide the new line into $k + 1$ segments.

Continue

The points of intersection thus divide the new line into k + 1 segments.

Each of these k + 1 segments divides its region into two.

Thus, this adds k + 1 regions.

Altogether, $(k^2 + k + 2)/2 + (k + 1)$ regions.

A bit of algebra,

$$
\begin{aligned}
(k^2 + k + 2)/2 + (k + 1) &= (k^2 + k + 2 + 2k + 2)/2 \\
&= (k^2 + 2k + 1 + k + 1 + 2)/2 \\
&= ((k + 1)^2 + (k + 1) + 2)/2
\end{aligned}
$$

This is what we needed to show.

Strong Induction

$$[P(0) \wedge (P(0) \wedge P(1) \wedge \ldots \wedge P(k)) \rightarrow P(k+1)] \Longrightarrow \forall n,\ P(n)$$

Let $P(n)$ be a statement (predicate) for each integer $n \geq 0$.

To prove that $P(n)$ is true for all $n \geq 0$, we do:

Base case(s): Show that $P(0)$ holds.

Induction Hypothesis (IH): Assume that $P(j)$ holds for $j = 0, 1, \ldots, k$.

Induction Step (IS): Show that $P(0) \wedge P(1) \wedge \ldots \wedge P(k)$ implies $P(k+1)$.

Intuition

<u>Theorem</u>. *Every natural number > 1 is prime or can be factored into primes.*

Let P(k) ="integer k > 1 be prime or can be factored into primes".

Base case: k = 2. True, since 2 is prime

IH: Assume it for k.

IS: Prove it for k+1.

How can we go from k to k+1. For example, knowing that 9 can be factored into primes, how do we prove this for 10?

A different approach. Assume 2, 3,…, k all can be factored into primes.

Then any composite number k+1 can be represented as a product

$$k + 1 = x \cdot y$$
$$x, y > 1 \text{ and } x, y \leq n$$

Example 7

<u>Theorem</u>. *Every natural number > 1 is prime or can be factored into primes.*

Let P(k) ="integer k > 1 be prime or can be factored into primes".

Base case: k = 2. True, since 2 is prime

IH: Assume that numbers 2, 3, 4, …, k are prime or can be factored into primes.

IS: Prove it for k+1.

Case 1. Let k+1 be prime.

Case 2. Let k+1 be composite.

Any composite number k+1 can be represented as a product

$$k + 1 = x \cdot y$$

$$x, y > 1 \text{ and } x, y \leq k$$

By the IH, P(x) and P(y) are true, therefore, P(k+1) is true.

The Principle of Strong Induction

Let P(n) be a statement (predicate) for each integer $n \geq a \geq 0$.

The principle of induction is a way of proving that P(n) is true for $\forall n \geq a$.

$$[P(a) \land (P(a) \land P(a+1) \land \ldots \land P(k)) \rightarrow P(k+1)] \Longrightarrow \forall n \geq a,\ P(n)$$

Base case(s): Show that P(a) holds.

Induction Hypothesis (IH): Assume that P(j) holds for $a \leq j \leq k$.

Induction Step (IS): Show that P(k+1) is true.

Then we conclude that P(n) is true for all integers $n \geq a$.

Example 8

<u>Theorem</u>. *Every natural number > 1 is prime or can be factored into primes.*

Let P(k) ="integer k > 1 be prime or can be factored into primes".

Base case: k = 2. True, since 2 is prime

IH: Assume that numbers 2, 3, 4, …, k are prime or can be factored into primes.

IS: Prove it for k+1.

Case 1. Let k+1 be prime.

Case 2. Let k+1 be composite.

Any composite number k+1 can be represented as a product

$$k + 1 = x \cdot y$$
$$x, y > 1 \text{ and } x, y \leq k$$

By the IH, P(x) and P(y) are true, therefore, P(k+1) is true.

Exercise 2

Suppose that you begin with a chocolate bar made up of n squares by m squares. At each step, you choose a piece of chocolate that has more than two squares and snap it in two along any line, vertical or horizontal. Eventually, it will be reduced to single squares. Show by induction that the number of snaps required to reduce it to single squares is $n \cdot m - 1$.

Proof. Let P(k) ="a bar of k squares requires k - 1 snaps".

Base case: k = 1. It requires no snaps. 1 – 1 = 0. True.

IH: Assume that P(j) is true for $1 \leq j \leq k$.

k+1 squares

IS: Given a chocolate bar of k+1 squares.

Break a bar into two pieces of k_1 and k_2 squares, s.t. $k_1 + k_2 = k+1$.

By the IH, $P(k_1) = k_1-1$ and $P(k_2) = k_2-1$ breaks.

It follows, $P(k+1) = 1 + P(k_1+k_2) = 1+k_1-1+k_2-1 = (k_1+k_2) -1 = k+1-1 = k$