Exercise 6.3

A. **Find I.F and solve**

(i) $\mathbf{3ydx + 2xdy = 0}$

Solution: Given that,

$$3ydx + 2xdy = 0 \qquad \ldots\ldots (i)$$

Comparing above equation with Mdx + Ndy = 0 then,

$M = 3y$, $N = 2x$

So, $\frac{\partial M}{\partial y} = 3$, $\frac{\partial N}{\partial x} = 2$.

Thus, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. So, (i) is not exact. Then,

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{3-2}{2x} = \frac{1}{2x}$$

Therefore, I.F. $= e^{\int f(x)\,dx} = e^{\int \frac{1}{2x}dx} = e^{\frac{1}{2}\log x} = e^{\log x^{\frac{1}{2}}} = x^{1/2}$

Multiplying (i) by I.F.

$3y\,x^{1/2}\,dx + 2x \cdot x^{1/2}\,dy = 0.$

Then, $M = 3x^{1/2}y$ & $N = 2x^{3/2}$

Now, it solution is,

$\int M\,dx + \int (\text{terms of N not containing x})dy = 2c^2$

$\Rightarrow \int 3x^{1/2}y\,dx + \int 0\,dy = 2c^2$

$\Rightarrow 3y \times \frac{2}{3}x^{3/2} = 2c^2 \Rightarrow 2c^2 = 2x^{3/2}y \Rightarrow c = x^3y^2.$

(ii) xdy – ydx = 0

Solution: Given that,

$xdy - ydx = 0 \Rightarrow -y\,dx + xdy = 0$

$\Rightarrow ydx + (-x)\,dy = 0$ ...... (i)

Comparing above equation with Mdx + Ndy = 0 then,

$M = y$ and $N = -x$

So, $\frac{\partial M}{\partial y} = 1$, $\frac{\partial N}{\partial x} = -1$

Thus, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. So, (i) is not exact. Then,

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1+1}{-x} = -\frac{2}{x} = f(x)$$

Therefore, I.f. $= e^{\int f(x)\,dx} = e^{\int -\frac{2}{x}dx} = e^{-2\log x} = e^{\log x^{-2}} = x^{-2} = \frac{1}{x^2}$

Multiplying (i) by I.F.,

$\frac{y}{x^2}dx + \left(-\frac{x}{x^2}\right)dy = 0$

$\Rightarrow \frac{y}{x^2}dx + \left(-\frac{1}{x}\right)dy$

Then, $M = \frac{y}{x^2}$ & $N = \frac{-1}{x}$

Now, it solution is,

$\int Mdx + \int (\text{terms of N not containing})\,dy = c_1$

$\Rightarrow \int \frac{y}{x^2}du + \int o\,dy = c_1$

$\Rightarrow y \times \frac{-1}{x} = c_1 \Rightarrow c_1 = -\frac{y}{x} \Rightarrow y = cx$ for $c = -c_1$.

(iii) $2dx - e^{y-x}\,dy = 0.$

Solution: Given that,

$2dx - e^{y-x}\,dy = 0$

$\Rightarrow 2dx - e^y \cdot e^{-x}\,dy = 0$ ...... (i)

Comparing above equation with Mdx + Ndy = 0 then,

$M = 2$ and $N = -e^y e^{-x}$

So. $\frac{\partial M}{\partial y} = 0$, and $\frac{\partial N}{\partial x} = e^y e^{-x}$

Thus. $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. So, (i) is not exact. Then,

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{0 - e^y e^{-x}}{-e^y e^{-x}} = \frac{-e^y e^{-x}}{-e^y e^{-x}} = 1$$

Therefore, I.f. $= e^{\int f(x)\,dx} = e^{\int dx} = e^x$

Multiplying (i) by I.F.

$2e^x\,dx - e^y\,dy = 0$

Then, $M = 2e^x$, $N = -e^y$

Now, it solution is,

$\int M\,dx + \int (\text{terms of N not containing x})\,dy = c$

$\Rightarrow \int 2e^x\,dx + \int -e^y\,dy = c$

$\Rightarrow 2e^x - e^y = c.$

(iv) y cos x dx + 2sin x dy = 0.

Solution: Given that,

$y\cos x\,dx + 2\sin x\,dy = 0$ ...(i)

Comparing above equation with Mdx + Ndy = 0 then,

$M = y \cos x$ and $N = 2 \sin x$

So, $\frac{\partial M}{\partial y} = \cos x$, $\frac{\partial N}{\partial x} = 2 \cos x$

Thus, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. So, (i) is not exact. Then,

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{\cos x - 2\cos x}{2 \sin x} = -\frac{1}{2} \cot x = f(x)$$

$$If = e^{\int f(x)\, dx} = e^{-\int \frac{1}{2} \cot x} = e^{-\frac{1}{2} \log \sin x} = (\sin)^{-1/2} = \frac{1}{\sqrt{\sin x}}$$

Multiplying (i) by I.F.

$$y \frac{\cos x}{\sqrt{\sin x}} dx + 2 \frac{\sin x}{\sqrt{\sin x}} dy = 0$$

$$M = y \frac{\cos x}{\sqrt{\sin x}}, N = 2\sqrt{\sin x}$$

Now, it solution is,

$\int M\, dx + \int$(terms of N not containing x) dy = c

$$\Rightarrow \int y \frac{\cos x}{\sqrt{\sin x}} dx + \int 0\, dy = c$$

$$\Rightarrow y \int \frac{\cos x}{\sqrt{\sin x}} dx = c$$

Put $v = \sin x$ $dv = \cos dx$

$$y \int \frac{dv}{\sqrt{v}} = c$$

$$\Rightarrow y \times \frac{v^{1/2}}{\frac{1}{2}} = c \Rightarrow 2y\, v^{1/2} = c \Rightarrow 2y\sqrt{\sin x} = c \Rightarrow y^2 \sin x = c.$$

(v) 2 cosh x cosy dx = sinh x sin y dy

Solution: Given that,

$2\cosh x \cos y\, dx - \sinh x \sin y\, dy = 0$ ...... (i)

Comparing above equation with Mdx + Ndy = 0 then,

$M = 2 \cosh x \cos y$ and $N = -\sinh x \sin y$

So, $\frac{\partial M}{\partial y} = -2 \cosh x \sin y$ and $\frac{\partial N}{\partial x} = -\cosh x \sin y$

Thus, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. So, (i) is not exact. Then,

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{-2 \cosh x \sin y + \cos hx \sin y}{-\sin hx \sin y} = \frac{-\cos hx \sin y}{-\sin hx \sin y} =$$

cothx

$$I.F. = e^{\int f(x)\, dx} = e^{\int \coth x\, dx} = e^{\log \sin hx} = \sin hx$$

Multiplying (i) by I.F.

$$2 \cos hx \sin hx \cos y\, dx - \sin h^2 x \sin y\, dy = 0$$

So, $M = 2\cosh x \sinh x \cos y$ and $N = -\sinh^2 x \sin y$

Now, it solution is,

$\int Mdu + \int$(terms of N not containing x) dy = c

$$\Rightarrow \int 2 \cos hx \sin hx \cos y + \int o\, dy = c$$

$$\Rightarrow 2 \cos y \int \cos hx \sin hx\, dx = c$$

Put, $v = \sinh x$. So, $dv = \cosh x\, dx$. So that,

$$2 \cos y \int v\, dv = c \Rightarrow 2 \cos y \frac{v^2}{2} = c$$

$$\Rightarrow \sinh^2 x \cos y = c.$$

(vi) $(2 \cos y + 4x^2)\, dx = x \sin y\, dy$

Solution: Given that,

$(2\cos y + 4x^2)dx - x \sin y\, dy = 0$ ...... (i)

Comparing above equation with Mdx + Ndy = 0 then,

$M = 2\cos y + 4x^2$ and $-x \sin y$

So, $\frac{\partial M}{\partial y} = -2 \sin y$ and $\frac{\partial N}{\partial x} = -\sin y$

Thus, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. So, (i) is not exact. Then,

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{-2 \sin y + \sin y}{-x \sin y} = \frac{-\sin y}{-x \sin y} = \frac{1}{x} = f(x)$$

$$I.F. = e^{\int f(x)\, dx} = e^{\int \frac{1}{x} dx} = ed^{\log x} = x$$

Multiplying (i) by I.F.

$$(2x\cos y + 4x^3)\, dx - x^2 \sin y\, dy = 0$$

Then, $M = 2x \cos y + 4x^3$, $N = -x^2 \sin y$

Now, it solution is,

$\int M\,dx + \int(\text{terms of N not containing x})\,dy = c$

$\Rightarrow \int(2x \cos y + 4x^3)\,dx + \int 0\,dy = c$

$\Rightarrow 2\cos y \int x\,dx + 4\int x^3\,dx = c$

$\Rightarrow 2\cos y \times \frac{x^2}{2} + 4 \times \frac{x^4}{4} = c$

$\Rightarrow c = x^2 \cos y + x^4$

(vii) $2x \tan y\,dx + \sec^2 y\,dy = 0$

Solution: Given that,

$2x \tan y\,dx + \sec^2 y\,dy = 0$ ...... (i)

Comparing above equation with $Mdx + Ndy = 0$ then,

$M = 2x \tan y$ and $N = \sec^2 y$

So, $\frac{\partial M}{\partial y} = 2x \sec^2 y, \frac{\partial N}{\partial x} = 0.$

Thus, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. So, (i) is not exact. Then,

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{2x \sec^2 y - 0}{\sec^2 y} = 2x.$$

Therefore, I.F. $= e^{\int f(x)\,dx} = e^{2x^2/2} = e^{x^2}$

Multiplying (i) by I.F.

$2xe^{x^2} - \tan y\,dx + e^{x^2} \sec^2 y\,dy = 0$

So, $M\ 2x\,e^{x^2} \tan y$ and $N = e^{x^2} \sec^2 y$

Now, it solution is,

$\int M\,dx + \int(\text{terms of N not containing x})\,dy = c$

$\Rightarrow \int 2xe^{x^2} \tan y\,dx + \int o\,dy = c$

$\Rightarrow \tan \int 2x\,e^{x^2}\,dx = c$

$\Rightarrow e^{x^2} \tan y = c$ $\left[\text{put } u = x^2,\ e^u \frac{dy}{dx} = 2x \Rightarrow du = 2x\,dx\right]$

(viii) $x^{-1} \cosh y\,dx + \sin hy\,dy = 0$ ...(i)

Solution: Given that,

$2x \tan y\,dx + \sec^2 y\,dy = 0$ ...(i)

Comparing above equation with $Mdx + Ndy = 0$ then,

$M = x^{-1} \cos hy$ and $N = \sin hy$

So, $\frac{\partial M}{\partial y} = \frac{1}{x} \sin hy\ \frac{\partial N}{\partial x} = 0$

Thus, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. So, (i) is not exact. Then,

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{\frac{1}{x}\sin hy}{\sin hy} = \frac{1}{x}$$

Therefore, I.F. $= e^{\int 1/x\,dx} = e^{\log x} = x.$

Multiplying (i) by I.F.,

$\frac{1}{x} \times x \cos hy\,dx + x \sin hy\,dy = 0$

$\Rightarrow M = \cosh hy$ and $N = x \sinh y$

Now, it solution is,

$\int M\,dx + \int(\text{terms of N not containing x})\,dy = c$

$\Rightarrow \int \cos hy\,dx + \int 0\,dy = c \Rightarrow \cos hy \int dx = c$

$\Rightarrow x \cos hy = c.$

B. Solve the following differential equation:

(i) $(1 + x^2)\,dy + 2xy\,dx = 0$

Solution: Given that,

$(1 + x^2)\,dy + 2xy\,dx = 0$ ...... (i)

Comparing above equation with $Mdx + Ndy = 0$ then,

$N = (1 + x^2)$ and $M = 2xy$

So, $\frac{\partial M}{\partial y} = 2x,$ and $\frac{\partial N}{\partial x} = 2x$

Thus, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. So, (i) is exact. Then, its solution is

$\int M\,dx + \int(\text{terms of N not containing dy}) = c$

$\Rightarrow \int 2xy\,dx + \int dy = c \Rightarrow 2y \times \frac{x^2}{2} + y = c. \Rightarrow y(1 + x^2) = c.$

ii) $y\,dx + x(1 + y)\,dy = 0$

Solution: Given that,

$y\,dx + x(1 + y)\,dy = 0$ ...... (i)

Comparing above equation with $Mdx + Ndy = 0$ then,

$M = y$ and $N = x(1 + x)$

So, $\frac{\partial M}{\partial y} = 1$, $\frac{\partial N}{\partial x} = (1 + y)$

Thus, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. So, (i) is not exact. Then,

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{1 + y - 1}{y} = \frac{y}{y} = 1 = f(y)$$

Therefore, I.F. $= e^{\int f(x)\, dx} = e^{\int dy} = e^y$

Multiplying equation (i) by I.F.,

$$ye^y\, dx + xe^y\,(1 + y)\, dy = 0.$$

So, $M = ye^y$, $N = xe^y\,(1 + y)$

Now, it solution is,

$$\int M\, dx + \int (\text{terms of N not containing x})\, dy = c$$

$$\Rightarrow \int ye^y\, dx + \int 0\, dy = c$$

$$\Rightarrow ye^y \int dx + 0 = c \Rightarrow xye^y = c.$$

(iii) $\frac{\mathbf{3y\cos 3x\, dx - \sin 3x\, dy}}{\mathbf{y^2}} = \mathbf{0}$

Solution: Given that,

$$\frac{3y\cos 3x\, dx - \sin 3x\, dy}{y^2} = 0$$

$$\Rightarrow 3y\cos 3x\, dx - \sin 3x\, dy = 0 \qquad \ldots\ldots (i)$$

Comparing above equation with Mdx + Ndy = 0 then,

$M = 3y\cos 3x$ and $N = -\sin 3x$

So, $\frac{\partial M}{\partial y} = 3\cos 3x$, $\frac{\partial N}{\partial x} = -3\cos 3x$

Thus, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. So, (i) is not exact. Then,

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{3\cos 3x + 3\cos 3x}{-\sin 3x} = -\frac{6\cos 3x}{\sin 3x} = -6\cot 3x$$

Therefore, I.F. $= e^{\int f(x)\, dx} = e^{-6\int \cot 3x\, dx} = e^{-\frac{6}{3}\log(\sin 3x)} = e^{\log(\sin 3x)^{-2}} = (\sin 3x)^{-2}$

Multiplying equation (i) by I.F.

$$3y\cos 3x \frac{1}{\sin^2 3x}\, dx - \frac{\sin 3x}{\sin^2 3x}\, dy = 0$$

$$\Rightarrow 3y\cot 3x \,.\, \text{cosec}\, 3x\, dx - \text{cosec}\, 3x\, dy = 0$$

Then, $M = 3y\cot 3x.\ \text{cosec}\, 3x$, $N = \text{cosec}\, 3x$

Now, it solution of (i) is,

$$\int M\, dx + \int (\text{terms of N not containing x})\, dy = c$$

$$\Rightarrow \int 3y\cot 3x.\text{cosec}\, 3x\, dx + \int o\, dy = c$$

$$\Rightarrow 3y\frac{\text{cosec}\, 3x}{3} + 0 = c \Rightarrow \frac{y}{\sin 3x} = c \Rightarrow y = c\sin 3x.$$

(iv) $\mathbf{xy' + y + 4 = 0}$

Solution: Given that,

$$xy' + y + 4 = 0 \Rightarrow x\frac{dy}{dx} + (y + 4) = 0$$

$$\Rightarrow xdy + (y + 4)\, dx = 0.$$

Comparing above equation with Mdx + Ndy = 0 then,

$M = (y + 4)$ and $N = x$

So, $\frac{\partial M}{\partial y} = 1$, $\frac{\partial N}{\partial x} = 1$

Thus, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. So, (i) is exact. Then, its solution is,

$$\int M\, dx + \int (\text{terms of N not containing dy}) = c$$

$$\Rightarrow \int (y + 4)\, dx + \int 0 = c$$

$$\Rightarrow (y + 4)\, x = c \Rightarrow xy + 4x = c \Rightarrow xy = c - 4x \Rightarrow y = \frac{c}{x} - 4.$$

C. Solve the following

(i) $\mathbf{(x + y)\, dx + (y - x)\, dy = 0}$

Solution: Given that,

$$(x + y)\, dx + (y - x)\, dy = 0 \qquad \ldots\ldots (i)$$

$$\Rightarrow x\, dx + y\, dx + y\, dy - x\, dy = 0$$

$$\Rightarrow x\, dx + y\, dy - (x\, dy - y\, dx)$$

Dividing by $x^2 + y^2$

$$\frac{x\, dx + y\, dy}{x^2 + y^2} - \frac{x\, dy - y\, dx}{x^2 + y^2} = 0$$

$$\Rightarrow \frac{1}{2}\frac{(2x\, dx + 2y\, dy)}{x^2 + y^2} - \left(\frac{x\, dy - y\, dx}{x^2 + y^2}\right) = c$$

$$\Rightarrow \frac{1}{2} d\{\log(x^2 + y^2)\} - d\left\{\tan^{-1}\left(\frac{y}{x}\right)\right\} = 0$$

Integrating both side

$$\frac{1}{2}\int d\{\log(x^2+y^2)\} - \int d\left\{\tan^{-1}\left(\frac{y}{x}\right)\right\} = 0$$

$$\Rightarrow \frac{1}{2}\log(x^2+y^2) - \tan^{-1}\left(\frac{y}{x}\right) = c.$$

D. Solve the following initial value problems.

(i) $(y-1)dx + (x-3)\,dy = 0,\ y(0) = \frac{2}{3}$

Solution: Given that,

$$(y-1)dx + (x-3)\,dy = 0 \quad \text{...... (i)}$$

$$y(0) = \frac{2}{3} \quad \text{....... (ii)}$$

Comparing (i) with Mdx + Ndy = 0 then,

$M = (y-1)$ and $N = (x-3)$

So, $\frac{\partial M}{\partial y} = 1,\quad \frac{\partial N}{\partial x} = 1$

This shows that, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ So, (i) is exact. Then, its solution is,

$\int M\,dx + \int$ (terms of N not containing dy) = c

$$\Rightarrow \int(y-1)\,dx + \int -3\,dy = c$$

$$\Rightarrow (y-1)x + -3y = c$$

$$\Rightarrow x(y-1) - 3y = c \quad \text{..... (iii)}$$

Since, $y(0) = \frac{2}{3}$, then (iii) gives, $0(y-1) - 3\times\frac{2}{3} = c \Rightarrow c = -2$

Therefore (iii) becomes,

$$x(y-1) - 3y = -2 \Rightarrow xy - x - 3y + 2 = 0$$

$$\Rightarrow xy - 3y - x + 3 - 1 = 0$$

$$\Rightarrow xy - 3y - x + 3 = 1$$

$$\Rightarrow y(x-3) - 1(x-3) = 1$$

$$\Rightarrow (x-3)(9-1) = 1.$$

(ii) $3x^2y^4dx + 4x^3y^3dy = 0,\ y(1) = 2$

Solution: Given that,

$$3x^2y^4dx + 4x^3y^3dy = 0 \quad \text{...... (i)}$$

$$y(1) = 2 \quad \text{...... (ii)}$$

Comparing (i) with Mdx + Ndy = 0 then,

$M = 3x^2y^4,$ and $N = 4x^3y^3$

So, $\frac{\partial M}{\partial y} = 12x^2y^3,\quad \frac{\partial N}{\partial x} = 12x^2y^3$

This shows that, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. So, (i) is exact. Then, its solution is,

$\int M\,dx + \int$ (terms of N not containing dy) = c

$$\Rightarrow \int 3x^2y^4\,dx + \int 0\,dy = c \quad \Rightarrow 3y^4 \times \int x^2\,dx = c$$

$$\Rightarrow 3y^4 \times \frac{x^2}{3}\,c$$

$$\Rightarrow c = x^3y^4 \quad \text{...... (iii)}$$

Since, $y(1) = 2$ then (iii) gives, $c = 1 \times 2^4 = 16$

Therefore (iii) becomes,

$$x^3y^4 = 16.$$

(iii) $y' = \frac{1-x}{1+y'}$ $y(1) = 0$.

Solution: Given that,

$$y' = \frac{1-x}{1+y'} \quad \text{...... (i)}$$

$$y(1) = 0 \quad \text{...... (ii)}$$

Then (i) becomes, $\frac{dy}{dx} = \frac{(1-x)}{(1+y)}$

$$\Rightarrow (1+y)\,dy = (1-x)dx$$

Integrating, $\int(1+y)\,dy = \int(1-x)\,dx$

$$\Rightarrow y + \frac{y^2}{2} = x - \frac{x^2}{2}$$

$$\Rightarrow \frac{2y+y^2}{2} = \frac{2x-x^2}{2}$$

$$\Rightarrow x^2 - 2x + y^2 + 2y = 0$$

$$\Rightarrow x^2 - 2x + 1 + y^2 + 2y + 1 = 2$$

$$\Rightarrow (x-1)^2 + (y+1)^2 = 2.$$

(iv) $2dx + \sec x \cos y\,dy = 0,\ y(0) = 0$.

Solution: Given that,

$$2dx + \sec x \cos y\,dy = 0 \quad \text{....... (i)}$$

$$y(0) = 0 \quad \text{........ (ii)}$$

Comparing (i) with Mdx + Ndy = 0 then,

$M = 2$, and $N = \sec \cos y$

$\frac{\partial M}{\partial y} = 0$, $\frac{\partial N}{\partial x} = \cos y . \sec x . \tan x$

Thus, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. So, (i) is not exact. Then,

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{0 - \cos y . \sec x . \tan x}{\sec x . \cos y} = \frac{-\cos y . \sec x . \tan x}{\sec x . \cos y} = -\tan x$$

Therefore,

$$\text{I.F.} = e^{\int f(x)\,dx} = e^{-\int \tan x\,dx} = e^{-\log \sec x} = e^{\log (\sec x)^{-1}} = (\sec x)^{-1} = \frac{1}{\sec x} = \cos x$$

Multiplying equation (i) by I.F.,

$$2 \cos x\, dx + \sec x . \cos x \cos y\, dy = 0$$

So, $M = 2 \cos x$, $N = \sec x . \cos x . \cos y = \cos y$

Now, it solution of (i) is,

$$\int M\,dx + \int (\text{terms of N not containing x})\,dy = c$$

$$\Rightarrow \int 2 \cos x\, dx + \int \cos y\, dy = c$$

$$\Rightarrow 2 \sin x + \sin y = c \qquad \text{...... (iii)}$$

Since, $y(0) = 0$, then (iii) gives, $2 \sin 0 + \sin 0 = c \Rightarrow c = 0$.

Now equation (ii) becomes

$$2 \sin x + \sin y = 0.$$

(v) $2 \sin y\, dx + \cos y\, dy = 0,\ y(0) = \frac{\pi}{2}$

Solution: Given that,

$$2 \sin y\, dx + \cos y\, dy = 0 \qquad \text{....... (i)}$$

$$y(0) = \frac{\pi}{2} \qquad \text{...... (ii)}$$

Comparing (i) with Mdx + Ndy = 0 then,

$M = 2 \sin y$, and $N = \cos y$

So, $\frac{\partial M}{\partial y} = 2 \cos y$, and $\frac{\partial N}{\partial x} = 0$

Thus, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. So, (i) is not exact. Then,

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2 \cos y - 0}{\cos y} = \frac{2 \cos y}{\cos y} = 2 = f(x)$$

Therefore, $\text{I.F.} = e^{\int f(x)\,dx} = e^{2\int dx} = e^{2x}$

Multiplying equation (i) by I.F.,

$$2 \sin y\, e^{2x}\, dx + e^{2x} \cos y\, dy = 0$$

So, $M = 2e^{2x} \sin y$, $N = e^{2x} \cos y$

Now, it solution of (i) is,

$$\int M\,dx + \int (\text{terms of N not containing x})\,dy = c$$

$$\Rightarrow \int 2e^{2x} \sin y\, dx + \int 0\, dy = c$$

$$\Rightarrow 2 \sin y \int e^{2x} = c$$

$$\Rightarrow 2\frac{e^{2x}}{2} \sin y = c \Rightarrow e^{2x} \sin y = c \qquad \text{...... (iii)}$$

Since, $y(0) = \frac{\pi}{2}$, then (iii) gives, $e^{0} \sin \frac{\pi}{2} = c \Rightarrow c = 1$.

Then (iii) becomes,

$$e^{2x} \sin y = 1$$

(vi) $2xy\, dy = (x^2 + y^2)\, dx,\ y(1) = 2$

Solution: Given that,

$$(x^2 + y^2)\, dx - 2xy\, dy = 0 \qquad \text{...... (i)}$$

$$y(1) = 2 \qquad \text{...... (ii)}$$

Comparing (i) with Mdx + Ndy = 0 then,

$M = (x^2 + y^2)$ and $N = -2xy$

So, $\frac{\partial M}{\partial y} = (0 + 2y) = 2y$ and $\frac{\partial N}{\partial x} = -2y$

Thus, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. So, (i) is not exact. Then,

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2y + 2y}{-2xy} = \frac{4y}{-2xy} = \frac{-2}{x}$$

Therefore, $\text{I.F.} = e^{\int f(x)\,dx} = e^{\int -2/x\,dx} = e^{-2\int 1/x\,dx} = e^{-2\log x} = e^{\log x^{-2}} = x^{-2} = \frac{1}{x^2}$

Multiplying equation (i) by I.F.,

$$\frac{(x^2 + y^2)}{x^2}\,dx - \frac{2xy}{x^2}\,dy = 0.$$

$\Rightarrow \left(1+\frac{y^2}{x^2}\right)dx - \frac{2y}{x}dy = 0$

So. $M = \left(1+\frac{y^2}{x^2}\right)$ and $N = \frac{-2y}{x}$

Now, it solution of (i) is,

$\int M\,dx + \int(\text{terms of N not containing x})\,dy = c$

$\Rightarrow \int\left(1+\frac{y^2}{x^2}\right)dx + \int 0\,dy = c$

$\Rightarrow \left(x - \frac{y^2}{x}\right) = c$ ...... (iii)

Since $y(1) = 2$ then (iii) gives, $c = \left(1 - \frac{2^2}{1}\right) = -3$

Now, (iii) becomes,

$x - \frac{y^2}{x} = -3 \Rightarrow x^2 - y^2 = -3x \Rightarrow y^2 = x^2 + 3x \Rightarrow y = \sqrt{x^2+3x}$.

(vii) $[(x+1)e^x - e^y]\,dx = xe^y dy,\ y(1) = 0$

Solution: Given that,

$[(x+1)e^x - e^y]\,dx - xe^y\,dy = 0$ ...... (i)

$y(1) = 0$ ...... (ii)

Comparing (i) with $Mdx + Ndy = 0$ then,

$M = (x+1)e^x - e^y$ and $N = -xe^y$

So, $\frac{\partial M}{\partial y} = 0 - e^y = -e^y$ $\quad \frac{\partial N}{\partial x} = -e^y$

This shows that, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. So, (i) is exact. Then, its solution is,

$\int M\,dx + \int(\text{terms of N not containing x})\,dy = c$

$\Rightarrow \int[(x+1)e^x - e^y\}\,dx + \int 0\,dy = c$

$\Rightarrow \int(xe^x + e^x - e^y)\,dx = c$

$\Rightarrow xe^x - e^x + e^x - e^y x = c$

$\Rightarrow xe^x - xe^y = c$ ...... (ii)

Since $y(1) = 0 \Rightarrow 1.e^1 = 1.e^0 = c \Rightarrow (e-1) = c$

Now (ii) becomes,

$xe^x - xe^y = e - 1.$

(viii) $2\sin 2x \sinh y\,dx - \cos 2x \cosh y\,dy = 0,\ y(0) = 1$

Solution: Given that,

$2\sin 2x \sin hy\,dx - \cos 2x \cos hy\,dy = 0$ ...... (i)

$y(0) = 1$ ...... (ii)

Comparing (i) with $Mdx + Ndy = 0$ then,

$M = 2\sin 2x \sinh y$ and $N = -\cos 2x \cos hy$

So, $\frac{\partial M}{\partial y} = 2\sin 2x \cos hy,$ $\quad \frac{\partial N}{\partial x} = 2\sin 2x.\cos hy$

This shows that, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. So, (i) is exact. Then, its solution is,

$\int M\,dx + \int(\text{terms of N not containing x})\,dy = c$

$\Rightarrow \int 2\sin 2x \cos hy\,dx + \int 0\,dy = c$

$\Rightarrow 2\sin hy \int \sin 2x = c$

$\Rightarrow 2\sin hy \times -\frac{\cos 2x}{2} = c$

$\Rightarrow -\sin hy \cos 2x = c$ ...... (iii)

Since, $y(0) = 1$ then (iii) gives,

$-\sinh 1 \cos 0 = c \Rightarrow c = -\sinh 1$

Now, equation (i) becomes,

$-\sin hy \cos 2x = -\sin h1$

$\Rightarrow \sinh y \cos 2x = \sinh 1$

E. Solve the following differential equations:

(i) $x\,dy - y\,dx = (x^2 + y^2)\,dx$

Solution: Given that

$x\,dy - y\,dx = (x^2+y^2)\,dx$ ....(1)

Put, $x = r\cos\theta$, $y = r\sin\theta$ then $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$. So,

$$\frac{d\theta}{dx} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{x\frac{dy}{dx} - y}{x^2}\right) = \frac{1}{x^2+y^2}\left(x\frac{dy}{dx} - y\right) = \frac{x\,dy - y\,dx}{r^2\,dx}$$

$\Rightarrow r^2\,d\theta = x\,dy - y\,dx$

Then (1) becomes,

$r^2 d\theta = r^2\,dx \Rightarrow d\theta = dx$

Integrating we get,

$$\theta = x + c \Rightarrow \tan^{-1}\left(\frac{y}{x}\right) = x + c$$

$$\Rightarrow \quad y = x \tan(x + c)$$

This is the solution of given equation.

(ii) $y(2xy + e^x)\, dx = e^x\, dy$

Solution: Given that,

$$y(2xy + e^x)\, dx = e^x\, dy$$

$$\Rightarrow \quad 2xy^2\, dx + ye^x\, dx = e^x\, dy$$

$$\Rightarrow \quad 2x\, dx = \frac{e^x\, dy - ye^x\, dx}{y^2}$$

$$\Rightarrow \quad d(x^2) = -d\left(\frac{e^x}{y}\right)$$

Integrating we get,

$$x^2 = -\frac{e^x}{y} + C \quad \Rightarrow \quad x^2 + \frac{e^x}{y} = C.$$

(iii) $xdy - ydx = xy^2\, dx$

Solution: Given that,

$$xdy - ydx = xy^2\, dx$$

$$\Rightarrow \quad \frac{xdy - ydx}{y^2} = x\, dx \quad \Rightarrow \quad -\left(\frac{y\, dx - x\, dy}{y^2}\right) = d\left(\frac{x^2}{2}\right)$$

$$\Rightarrow \quad -d\left(\frac{x}{y}\right) = d\left(\frac{x^2}{2}\right)$$

Integrating we get,

$$-\frac{x}{y} = \frac{x^2}{2} - c \quad \Rightarrow \quad \frac{x^2}{2} + \frac{x}{y} = c.$$

(iv) $x^2y\, dx - (x^3 + y^3)\, dy = 0$

Solution: Given that,

$$x^2y\, dx - (x^3 + y^3)\, dy = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x^2y}{x^3 + y^3}$$

This is a homogeneous equation. So, put $y = vx$. Then $\frac{dy}{dx} = v + x \cdot \frac{dv}{dx}$. Therefore,

$$v + x\frac{dv}{dx} = \frac{x^3v}{x^3 + v^3x^3} = \frac{v}{1 + v^3}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{v}{1 + v^3} - v = \frac{v - v - v^4}{1 + v^3} = -\frac{v^4}{1 + v^3}$$

$$\Rightarrow \quad \left(\frac{1 + v^3}{v^4}\right) dv + \frac{dx}{x} = 0$$

$$\Rightarrow \quad \left(\frac{1}{v^4} + \frac{1}{4} \cdot \frac{4v^3}{v^4}\right) dv + \frac{dx}{x} = 0$$

Integrating we get,

$$\frac{v^{-3}}{-3} + \frac{1}{4}\log(v^4) + \log(x) = c$$

$$\Rightarrow \quad -\frac{x^3}{3y^3} + \frac{1}{4}\log\left(\frac{y^4}{x^4}\right) + \log(x) = c.$$

(v) $(x^2 + y^2 + 1)\, dx - 2xy\, dy = 0$

Solution: Given that,

$$(x^2 + y^2 + 1)\, dx - 2xy\, dy = 0 \qquad \ldots\ldots(1)$$

Comparing it with $Mdx + Ndy = 0$ then we get,

$$M = x^2 + y^2 + 1 \quad \text{and} \quad N = -2xy$$

So, $\frac{\partial M}{\partial y} = 2y$ and $\frac{\partial N}{\partial x} = -2y$

Thus, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. So, the equation is not exact.

Therefore, for the integrating factor,

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{-2xy}(2y + 2y) = \frac{4y}{-2xy} = -\frac{2}{x}$$

So,

$$\text{I.F.} = e^{\int -2/x\, dx} = e^{-2\log x} = x^{-2} = \frac{1}{x^2}.$$

Then multiplying (1) by I. F. then,

$$\left(\frac{x^2 + y^2 + 1}{x^2}\right) dx - \left(\frac{2xy}{x^2}\right) dy = 0$$

This is exact. So, its solution is

$$\int M dx + \int (\text{terms of N that not included } x)\, dy = c$$

$$\Rightarrow \quad \int \left(\frac{x^2 + y^2 + 1}{x^2}\right) dx\, dx + \int 0\, dy = c$$

$$\Rightarrow \quad \int (1 + y^2 \cdot x^{-2} + x^{-2})\, dx = c$$

$$\Rightarrow \quad x + y^2 \cdot \frac{x^{-1}}{-1} + \frac{x^{-1}}{-1} = c$$

$$\Rightarrow \quad x - \frac{y^2}{x} - \frac{1}{x} = c$$

(vi) $(y^4 + 2y)\,dx + (xy^3 + 2y^4 - 4x)dy = 0$

Solution: Given that,

$$(y^4 + 2y)\,dx + (xy^3 + 2y^4 - 4x)dy = 0 \qquad \text{.........(1)}$$

Comparing it with Mdx + Ndy = 0 then we get,

$$M = y^4 + 2y \qquad \text{and} \quad N = xy^3 + 2y^4 - 4x$$

So, $$\frac{\partial M}{\partial y} = 4y^3 + 2 \qquad \text{and} \quad \frac{\partial N}{\partial x} = y^3 - 4$$

This shows that $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. So, the equation (1) is not exact. So for the integrating factor of (1)

$$\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{(y^4 + 2y)}(y^3 - 4 - 4y^3 - 2)$$

$$= \frac{-3\,(y^3 + 2)}{y(y^3 + 2)} = -\frac{3}{y}$$

So, I.F. $= e^{-\int(3/y)\,dy} = e^{-3\log y} = \frac{1}{y^3}$

Multiplying (1) by I. F. then,

$$\left(\frac{y^4 + 2y}{y^3}\right)dx + \left(\frac{xy^3 + 2y^4 - 4x}{y^3}\right)dy = 0$$

$$\Rightarrow \quad (y + 2y^{-2})\,dx + (x + 2y - 4xy^{-3})\,dy = 0$$

This is exact. So, its solution is,

$$\int M dx + \int(\text{terms of N which is free from x})\,dy = c$$

$$\Rightarrow \quad \int(y + 2y^{-2})\,dx + \int 2y\,dy = c$$

$$\Rightarrow \quad \left(y + \frac{2}{y^2}\right)x + y^2 = c.$$

(vii) $(3xy - 2ay^2)\,dx + (x^2 - 2axy)\,dy = 0$

Solution: Given that,

$$(3xy - 2ay^2)\,dx + (x^2 - 2axy)\,dy = 0 \qquad \text{.........(1)}$$

Comparing it with Mdx + Ndy = 0 then we get,

$$M = 3xy - 2ay^2 \qquad \text{and} \quad N = x^2 - 2axy$$

So, $$\frac{\partial M}{\partial y} = 3x - 4ay \qquad \text{and} \quad \frac{\partial N}{\partial x} = 2x - 2ay$$

This shows that $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. So, the equation (1) not exact. So, for the integrating factor of (1).

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{x(x - 2ay)}(3x - 4ay - 2x + 2ay)$$

$$= \frac{x - 2ay}{x(x - 2ay)} = \frac{1}{x}$$

So, I.F. $= e^{\int dx/x} = e^{\log x} = x.$

Now, multiplying (1) by I. F. Then,

$$x\,(3axy - 2ay^2)\,dx + x(x^2 - 2axy)\,dy = 0$$

This is exact. So, its solution is

$$\int Mdx + \int (\text{terms of N which is free from x})\,dy = c$$

$$\Rightarrow \int x\,(3ay - 2ay^2)\,dx + \int 0\,dy = c$$

$$\Rightarrow 3ay \int x^2\,dx - 2ay^2 \int x\,dx = c$$

$$\Rightarrow ayx^3 - ay^2x^2 = c$$

$$\Rightarrow ax^2\,(xy - y^2) = c.$$