Mathematics for Machine Learning

Course Code: LMC312-15BF

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Course includes :

- Logic
- Sets and Types
- Probability
- Linear Algebra and Vector Space
- Matrix and Linear Operators
- Differential Calculus
- Statistics and Inference

Unit 1 : LOGIC

- 1.1 Logic
- 1.2 Propositional Equivalence
- 1.3Predicate and Quantifiers
- 1.4 Methods of Proofs

1.1 Logic : Propositions, Proposition variables, truth table, Boolean operators, tautology, Contradiction, Converse, Inverse, Contrapositive, translating English sentences

Discrete Objects/Concepts and Structures

PART I

- Propositions
- Predicates
- Proofs
- Sets
- Functions
- Orders of Growth
- Algorithms
- Integers
- Summations
- Sequences
- Strings
- Permutations
- Combinations
- Probability

PART II

- Relations
- Graphs
- Logic Circuits
- Automata

Uses for Discrete Math in Computer Science

- Advanced algorithms & data structures
- Programming language compilers & interpreters
- Computer networks
- Operating systems
- Computer architecture
- Database management systems
- Cryptography
- Error correction codes
- Graphics & animation algorithms, game engines, etc.…

Propositional Logic

Logic

- Specifically concerned with whether reasoning is correct.
- Study of reasoning.
- Focuses on the relationship among statements, not on the content of any particular statement.
- Gives precise meaning to mathematical statements.

Propositional Logic

- the logic that deals with statements (propositions) and compound statements built from simpler statements using so-called Boolean connectives.

Definition of a Proposition

A proposition (denoted p, q, r, …) is simply:

- a statement (i.e., a declarative sentence)
- with some definite meaning, (not vague or ambiguous)
- having a truth value that's either true (T) or false (F) , it is never both, neither, or somewhere "in between!"
- However, you might not know the actual truth value, and, the truth value might depend on the situation or context.
- Later, we will study probability theory, in which we assign degrees of certainty ("between" T and F) to propositions.
- But for now: think True/False only! (or in terms of 1 and 0)

Examples of Propositions

- It is raining.(In given situation)
- Beijing is the capital of China. (T)
- 2 + 2 = 5. (F)
- 1 + 2 = 3. (T)

NOTE : A fact-based declaration is a proposition, even if no one knows whether it is true

- 11213 is prime.
- There exists an odd perfect number.

Examples of Non -Propositions

The following are NOT propositions:

- Who’s there? (interrogative, question)
- Just do it! (imperative, command)
- La la la la la. (meaningless interjection)

- Yeah, I sorta dunno, whatever... (vague)
- 1 + 2 (expression with a non-true/false value)
- x + 2 = 5 (declaration about semantic tokens of non-constant value)

Some Popular Boolean Operators

Formal Name	Nickname	Symbol
Negation operator	NOT	$\neg$
Conjunction operator	AND	$\wedge$
Disjunction operator	OR	v
Implication operator	IMPLIES	$\rightarrow$
Bi-conditional operator	IFF	$\leftrightarrow$

The Negation Operator

- Negation operator A unary operator inverts the truth value of its operand
- The unary negation operator "¬" (NOT) transforms a proposition into its logical negation.
- E.g. If p = "I have brown hair." then ¬p = "It is not the case that I have brown hair" or "I do not have brown hair."
- The truth table for NOT:

p	$\neg p$
T	F
F	T

The Conjunction Operator (^)

- The binary conjunction operator "^" (AND) combines two propositions to form their logical conjunction.
- E.g. If p = "I will have salad for lunch." and q = "I will have steak for dinner."

then, p^q = "I will have salad for lunch and I will have steak for dinner."

Conjunction Truth Table

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

The Disjunction Operator

- The binary disjunction operator "v" (OR) combines two propositions to form their logical disjunction.
- E.g. If p = "My car has a bad engine." and q = "My car has a bad carburetor."

then, pvq = "My car has a bad engine, or my car has a bad carburetor."

- Meaning is like "and/or" in informal English
- Note that pvq means that p is true, or q is true, or both are true!

So, this operation is also called inclusive or, because it includes the possibility that both p and q are true.

Disjunction Truth Table

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

The Implication Operator (→)

- The conditional statement (aka implication) p →q states that p implies q.

i.e., If p is true, then q is true

but if p is not true, then q could be either true or false.

- E.g., let p = “You study hard.” q = “You will get a good grade.”

p → q = “If you study hard, then you will get a good grade.” (else, it could go either way)

Here,

p: hypothesis or antecedent or premise

q: conclusion or consequence

Implication Truth Table

- $p \rightarrow q$ is false only when p is true but q is not true.
- $p \rightarrow q$ does not require that p or q are ever true!
- E.g. "$(1=0)\rightarrow$ pigs can fly" is TRUE!

P	Q	P→Q
T	T	T
T	F	F
F	T	T
F	F	T

Examples of Implications

- “If this lecture ever ends, then the sun will rise tomorrow.” True or False?($T \rightarrow T$)
- “If 1+1=6, then Trump is president.” True or False?($F \rightarrow T$)
- “If the moon is made of green cheese, then I am richer than Bill Gates.” True or False?($F \rightarrow F$)
- “If Tuesday is a day of the week, then I am a penguin.” True or False?($T \rightarrow F$)

English Phrases Meaning $p \rightarrow q$

- "p implies q"
- "if p, then q"
- "if p, q"
- "when p, q"
- "whenever p, q"
- "q if p"
- "q when p"
- "q whenever p"
- "p only if q"
- "p is sufficient for q"
- "q is necessary for p"
- "q follows from p"
- "q is implied by p"

Converse, Inverse, Contrapositive

Some terminology, for an implication $p \to q$:

Then,

- Its converse is: $q \to p$.
- Its inverse is: $\neg p \to \neg q$.
- Its contrapositive: $\neg q \to \neg p$

- One of these three has the same meaning (same truth table)

as $p \to q$. Can you figure out which?

Truth table of converse,Inverse,contrapositive

p	q	p → q	q → p	¬p → ¬q	¬q → ¬p
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

Some Examples

p: Today is Easter

q: Tomorrow is Monday

Now, $p \rightarrow q$

If today is Easter then tomorrow is Monday.

Converse:

$\mathbf{q \rightarrow p}$

If tomorrow is Monday then today is Easter.

Inverse:

$\mathbf{\neg p \rightarrow \neg q}$

If today is not Easter then tomorrow is not Monday

Contrapositive:

$\mathbf{\neg q \rightarrow \neg p}$

If tomorrow is not Monday then today is not Easter.

The Bi-conditional Operator ($\leftrightarrow$)

The bi-conditional statement $p \leftrightarrow q$ states that p if and only if (iff) q.

p = "It is below freezing."

q = "It is snowing."

Then,

$p \leftrightarrow q$

"It is below freezing if and only if it is snowing."

OR "That it is below freezing is necessary and sufficient for it to be snowing"

Biconditional Truth Table

- p is necessary and sufficient for q
- If p then q, and conversely
- p iff q

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Need to know

- $p \leftrightarrow q$ is equivalent to $(p \rightarrow q) \wedge (q \rightarrow p)$.
- $p \leftrightarrow q$ means that p and q have the same truth value.
- $p \leftrightarrow q$ does not imply that p and q are true.

Boolean Operations Summary

We have seen 1 unary operator and 5 binary operators. What are they? Their truth tables are below.

P	Q	¬p	p^q	pvq	p→q	p↔q
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

For an implication $p \rightarrow q$

Its converse is: $q \rightarrow p$

Its inverse is: $\neg p \rightarrow \neg q$

Its contrapositive: $\neg q \rightarrow \neg p$

Translating English Sentences

Let,

p = “It rained last night”,

q = “The sprinklers came on last night,”

r = “The lawn was wet this morning.”

Translate each of the following into English:

$\neg p$ = “It did not rain last night

$\neg r \wedge p$ = “The lawn was not wet this morning, and it rained last night.”

$\neg r \vee p \vee q$=“The lawn wasn’t wet this morning, or it rained last night, or the sprinklers came on last night.”

Another Example

Find the converse of the following statement.

“Raining tomorrow is a sufficient condition for my not going to town.”

Now,

Step 1:

Assign propositional variables to component propositions.

p: It will rain tomorrow

q: I will not go to town

Step 2:

Symbolize the assertion: $p \rightarrow q$

Step 3:

Symbolize the converse: $q \rightarrow p$

Step 4:

Convert the symbols back into words.

“If I don’t go to town then it will rain tomorrow” or

“Raining tomorrow is a necessary condition for my not going to town.”

CLASS-WORK

1. What are the contrapositive, the converse, and the inverse of the conditional statement

"The home team wins whenever it is raining?"

2. Construct a truth table for each of the compound propositions.

a) $p \wedge \neg p$

b) $p \vee \neg p$

c) $(p \vee \neg q) \rightarrow q$

d) $(p \vee q) \rightarrow (p \wedge q)$

e) $(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$

Tautology and contradiction

- A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a tautology.
- A compound proposition that is always false is called a contradiction.
- A compound proposition that is neither a tautology nor a contradiction is called a contingency.
- Consider the truth tables of $p\vee\neg p$ and $p\wedge\neg p$, . Because $p\vee\neg p$ is always true, it is a tautology. Because $p\wedge\neg p$ is always false, it is a contradiction.

p	**¬p**	**pv¬p**	**P ^¬p**
T	F	T	F
F	T	T	F

Logically equivalent

- The compound propositions p and q are called logically equivalent if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent.
- De Morgan's Laws.

$$\neg(p \wedge q) \equiv \neg p \vee \neg q \quad \neg(p \vee q) \equiv \neg p \wedge \neg q$$

these two laws are logically equivalent.

Two compound proposition has same truth values, then they are said to be logically equivalent.

Example

$\neg(p \vee q)$and $\neg p \wedge \neg q$ are logically equivalent. $\neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q)$is a tautology and that these compound propositions are logically equivalent.

P	**q**	**pvq**	**¬(p^q)**	**¬p**	**¬q**	**¬p^¬q**
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

SOLVE NOW :

1.Show that $p \rightarrow q$ and$\neg p \vee q$ are logically equivalent.

2.show that $p \vee (q \wedge r)$and $(p \vee q) \wedge (p \vee r)$Are Logically Equivalent.

3. show that$\neg(p \rightarrow q)$and $p \wedge \neg q$ are logically equivalent.

4. Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent

Class-work

1. Show that$(p \rightarrow q) \wedge (p \rightarrow r)$and $p \rightarrow (q \wedge r)$are logically equivalent.
2. Show that $(p \rightarrow r) \wedge (q \rightarrow r)$and $(p \vee q) \rightarrow r$ are logically equivalent.
3. Show that$(p \rightarrow q) \vee (p \rightarrow r)$and $p \rightarrow (q \vee r)$are logically equivalent.
4. Show that $(p \rightarrow q) \wedge (q \rightarrow r) \rightarrow (p \rightarrow r)$ is a tautology.
5. Show that $(p \vee q) \wedge (\neg p \vee r) \rightarrow (q \vee r)$is a tautology

Logically equivalent

Equivalence	Name
$P \wedge T \equiv p$, $p \vee F \equiv p$	Identity laws
$P \vee T \equiv T$,$p \wedge F \equiv F$	Domination laws
$P \vee p \equiv p$,$p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$P \vee q \equiv q \vee p$,$p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$, $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$,$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws

Logically equivalent

Equivalence	Name
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan’s laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv T$ $p \wedge \neg p \equiv F$	Negation laws

Home-work

1)Verify all the laws of logical equivalences.

2)$(p \rightarrow r) \vee (q \rightarrow r) \leftrightarrow (p \wedge q) \rightarrow r$ and $(p \vee q) \wedge (\neg p \wedge \neg q)$ is contradiction.

Predicates and Quantifers

<u>Definition:</u> Let p(x) be a statement involving the variable x and let D be a set . we call p a propositional function(w.r.t. D) if for each x in D, p(x) is a proposition with certain truth value i.e. when the variable in the propositional function are assigned values, the resulting statement become a proposition with certain truth value.

<u>example:</u>

Let p(n)”n is an odd integer” is a proposition function(predicates) where D be the set of positive integers. then for all n in D,p(n) is a proposition with certain truth value.

Examples

1) n^2+2n is an odd integer; where domain of discourse is set of positive integers.

2)x^2-x-6=0;where domain of discourse is set of real numbers

3)Let Q(x,y) denote he statement ”x=y+3”what is the truth value of the propositions Q(1,2) and Q(3,0)?

Quantification

The process of binding propositional variable over the given domain is called quantification.

In predicate logic ,there are two type of quantification.

1)universal quantification

2)existential quantification

Universal quantification

Let p(x) be the propositional function with domain of discourse D , the statement

for every x , p(x)

For all x , p(x)

$\forall$ x ,p(x) is called the universal quantified statement. And the symbol $\forall$ is called universal quantifier.

Examples

1) Let $P(x)$be the statement “$x+1>x$.” What is the truth value of the quantification $\forall x P(x)$, where the domain consists of all real numbers?

Solution:

Because $P(x)$is true for all real numbers x, the quantification $\forall x, P(x)$ is true.

2) Let $Q(x)$ be the statement “$x<2$.”What is the truth value of the quantification $\forall x, Q(x)$, where the domain consists of all real numbers?

Solution:

$Q(x)$ is not true for every real number x, because, for instance, $Q(3)$ is false. That is, $x = 3$ is a counterexample for the statement $\forall x, Q(x)$. Thus $\forall x, Q(x)$ is false.

Examples contd…

- 3) What is the truth value of $\forall x$, $P(x)$, where $P(x)$ is the statement “x2 < 10” and the domain consists of the positive integers not exceeding 4?

  Solution:

  The statement $\forall x$, $P(x)$ is the same as the conjunction $P(1) \wedge P(2) \wedge P(3) \wedge P(4)$, because the domain consists of the integers 1, 2, 3, and 4. Because $P(4)$, which is the statement “42 < 10,” is false, it follows that $\forall x, P(x)$ is false.

Existential quantifer

The existential quantification of P(x)is the proposition

"There exists an element x in the domain such that P(x)." We use the notation $\exists x, P(x)$ for the existential quantification of P(x).

Here $\exists$ is called the existential quantifier.

Besides the phrase "there exists,"we can also express existential quantification in many other ways, such as by using the words "for some," "for at least one," or "there is." The existential quantification $\exists x, P(x)$ is read as "There is an x such that P(x),"

"There is at least one x such that P(x),"

or

"For some x,P(x)."

Quantifer

statement	When true	When false
$\forall x, P(x)$	P(x)is true for every x.	There is an x for which P(x)is false
$\exists x, P(x)$	There is an x for which P(x)is true	There is an x for which P(x)is false

Some Questions:

1. **Let P(x)denote the statement “x>3.” What is the truth value of the quantification ∃x,P(x), where the domain consists of all real numbers?**

Solution:

Because “x>3” is sometimes true—for instance, when x = 4—the existential quantification of P(x), which is∃ x,P(x), is true.

2. **Let Q(x)denote the statement “x = x +1.”What is the truth value of the quantification ∃x, Q(x), where the domain consists of all real numbers?**

Solution:

Because Q(x) is false for every real number x, the existential quantification of Q(x), which is ∃x, Q(x), is false.

Class-work

1) Let P(x)be the statement "x = x^2." If the domain consists of the integers, what are these truth values? a) P(0) b) P(1) c) P(2) d) P(−1) e) ∃ x, P(x) f) ∀x ,P(x)

2) Determine the truth value of each of these statements if the domain consists of all integers. a) ∀n(n+1 > n) b) ∃n(2n = 3n) c) ∃n(n =−n) d) ∀n(3n ≤ 4n) .

3)Verify de-morgan's law for logic.

A) ¬∃x, P(x) : ∀x, ¬P(x)

B) ¬∀x,P(x): ∃x,¬P(x)

Direct-Proof

A direct proof shows that a conditional statement

$p \rightarrow q$ is true by showing that if p is true,

then q must also be true,

so that the combination p true and q false never occurs.

In a direct proof, we assume that p is true and use axioms, definitions, and previously proven theorems, together with rules of inference,

to show that q must also be true.

Examples

1) Direct proof of the theorem “If n is an odd integer, then n2 is odd.”
2) Direct proof that if m and n are both perfect squares, then nm is also a perfect square.
3) Prove that if n is an integer and 3n+2 is odd, then n is odd.
4) Prove that if n = ab, where a and b are positive integers, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.
5) Prove that the sum of two rational numbers is rational.
6) Prove that if n is an integer and n2 is odd, then n is odd

Indirect-proof

Indirect proof has two way

A) contraposition method:

use of the fact that the conditional statement

$p \rightarrow q$ is equivalent to its contrapositive, $\neg q \rightarrow \neg p$.

This means that the conditional statement $p \rightarrow q$ can be proved by showing that its contrapositive ,$\neg q \rightarrow \neg p$, is true.

In a proof by contraposition of $p \rightarrow q$,

we take $\neg q$ as a premise, and using axioms, definitions, and previously proven theorems, together with rules of inference,

we show that $\neg p$ must follow.

Indirect-proof

B. Contradiction method:

Suppose we want to prove that a statement p is true.

Furthermore, suppose that we can find a contradiction q such that $\neg p \rightarrow q$ is true.

Because q is false, but $\neg p \rightarrow q$ is true, we can conclude that $\neg p$ is false, which means that p is true.

How can we find a contradiction q that might help us prove that p is true in this way?

Examples

1) Prove that $\sqrt{2}$ is irrational by giving a proof by contradiction.
2) Give a proof by contradiction of the theorem "If 3n+2 is odd, then n is odd."
3) Show that if n is an integer and n^3+5 is odd, then n is even using a) a proof by contraposition. b) a proof by contradiction
4) Prove that if n is an integer and 3n+2 is even, then n is even using a) a proof by contraposition. b) a proof by contradiction