OTHER QUESTIONS FROM SEMESTER END EXAMINATION

Second Order Differential Equation

1999 Q. No. 4(b); 2001 Q. No. 4(b)

Find the general solution of the differential equation. $y'' - 2y' + 2y = 2e^x \cos x$.

Solution: Given that, $y'' - 2y' + 2y = 2e^x \cos x$ ...... (i)

The auxiliary equation of homogeneous part of (i) is,

$$m^2 - 2m + 2 = 0 \Rightarrow m = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm 2i$$

So, its general equation is,

$$y_h(x) = e^{-x}(A \cos 2x + B \sin 2x) \quad \text{...... (ii)}$$

And for the particular solution of (i), let,

$$y_p = 2e^x(c_1 \sin x + c_2 \cos x)$$

$$\Rightarrow \quad y_p = 2c_1e^x \sin x + 2c_2e^x \cos x \quad \text{...... (iii)}$$

Then. $y'_p = 2c_1(e^x\cos x + e^x \sin x) + 2c_2(e^x \cos x - e^x \sin x)$

And. $y''_p = 2c_1(e^x\cos x - e^x \sin x + e^x \sin x + e^x \cos x) + 2c_2(e^x \cos x - e^x \sin x - e^x \cos x)$

$$\Rightarrow \quad y''_p = 4c_1e^x \cos x - 4c_2e^x \sin x$$

Then, the equation (ii) becomes

$4(c_1e^x\cos x - c_2e^x \sin x - c_1e^x \cos x - c_1e^x \sin x - c_2e^x \cos x + c_2e^x \sin x) = 2e^x \sin x$

$$\Rightarrow \quad -2c_1e^x \sin x - 2c_2e^x \cos x = e^x \sin x$$

Comparing coefficient on both side then,

$$-2c_1e^x = e^x \quad \text{and} \quad -2c_2e^x = 0$$

$$\Rightarrow c_1 = -\frac{1}{2} \qquad \Rightarrow c_2 = 0.$$

So the equation (iii) becomes,

$$y_p = e^x \sin x$$

Now, general equation of (i) is,

$$y(x) = y_h(x) + y_p$$
$$= e^{-x}(A \cos 2x + B \sin 2x) + e^x \sin x$$

2000 Q. No. 4(b)

Find the general solution of the differential equation, $y'' + 4y' + 3y = \sin x + 2\cos x$.

Solution: Given that, $y'' + 4y' + 3y = \sin x + 2\cos x$. ...... (i)

The auxiliary equation of homogeneous part of (i) is,

$$m^2 + 4m + 3 = 0 \Rightarrow m^2 + 3m + m + 3 = 0$$

$$\Rightarrow (m+3)(m-1) = 0 \Rightarrow m = 1, -3$$

So, its general equation is,

$$y_h(x) = c_1e^x + c_2e^{-3x} \quad \text{...... (ii)}$$

And for the particular solution of (i), let,

$$y_p = c_3 \cos x + c_4 \sin x \quad \text{...... (iii)}$$

Then. $y'_p = -c_3 \sin x + c_4 \cos x$ and $y''_p = -c_3 \cos x - c_4 \sin x$

So, equation is (i) becomes

$-c_3\cos x - c_4 \sin x - (-c_3 \sin x + c_4 \cos x) - 2(c_3 \cos x + c_4 \sin x) = \sin x + 2\cos x.$

$$\Rightarrow \quad \cos x(-c_3 - c_4 - 2c_3) + \sin x(-c_4 + c_3 - 2c_4) = \sin x + 2\cos x.$$

$$\Rightarrow \quad \cos x(-3c_3 - c_4) + \sin x(-c_3 + 3c_4) = \sin x + 2\cos x.$$

Comparing the coefficient on both side, then,

$$-3c_3 - c_4 = 2, \qquad c_3 - 3c_4 = 1.$$

Solving we get, $c_3 = \frac{-1}{2}$ and $c_4 = \frac{-1}{2}$.

Thus, equation (iii) becomes

$$y_p = \frac{-1}{2}(\cos x + \sin x)$$

Now, general equation of (i) is,

$$y(x) = y_h(x) + y_p$$
$$= c_1e^x + c_2e^{-3x} - \frac{1}{2}(\cos x + \sin x)$$

2002 Q. No. 4(b)

Solve $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 4x^2$.

Solution: Given that, $y'' + 3y' + 2y = 4x^2$ ..... (i)

The auxiliary equation of homogeneous part of (i) is,

$$m^2 + 3m + 2 = 0$$

$$\Rightarrow m^2 + 2m + m + 2 = 0 \Rightarrow (m+2)(m+1) = 0 \Rightarrow m = -1, -2.$$

So, its solution is,

$$y_h(x) = Ae^{-x} + Be^{-2x}$$

And for the particular solution of (i), let,

$$y_p = c_1x^2 + c_2x + c_3 \quad \text{....... (ii)}$$

Then, $y'_p = 2c_1x + c_2$ and $y''_p = 2c_1$

So that (i) becomes,

$$2c_1 + 3(2c_1x + c_2) + 2(c_1x^2 + c_2x + c_3) = 4x^2$$

$$\Rightarrow \quad 2c_1 + 6c_1x + 3c_2 + 2c_1x^2 + 2c_2x + 2c_3 = 4x^2$$

$$\Rightarrow \quad 2c_1x^2 + x(6c_1 + 2c_2) + (2c_1 + 3c_2 + 2c_3) = 4x^2$$

Comparing coefficient on both side then,

$$2c_1 = 4, \qquad 6c_1 + 2c_2 = 0, \qquad 2c_1 + 3c_2 + 2c_3 = 0.$$

Solving we get, $c_1 = 2$, $c_2 = -6$, $c_3 = 7$.

Therefore, (ii) becomes, $y_p = 2x^2 - 6x + 7$.

Now, general equation of (i) is,

$$y(x) = y_h(x) + y_p$$
$$y(x) = Ae^{-x} + Be^{-2x} + 2x^2 - 6x + 7.$$

2004 Fall; 2006 Fall Q. No. 4(b) OR

Solve by the method of variation of parameters: $y'' + y = \tan x$.

Solution: Given equation is, $y'' + y = \tan x$. ... (i)

This is second order non-homogeneous equation. Then its solution is,

$$y(x) = y_h(x) + y_p \quad \text{... (ii)}$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

Here, the homogeneous equation of (i) is,

$$y'' + y = 0$$

Its auxiliary equation is

$$m^2 + 1 = 0 \Rightarrow m = \pm i$$

So, its solution is,

$$y_h(x) = (A\cos x + B\sin x)$$

And, for particular solution,

We have,

$y_1 = \cos x$ and $y_2 = \sin x$,

then, $y'_1 = -\sin x$ and $y_2' = \cos x$

$R = \sec x$

So, the Wronskian is,

$$W(y_1, y_2) = y_1y'_2 - y_2y'$$
$$= \cos x.\cos x - \sin x(-\sin x) = \cos^2 x + \sin^2 x = 1.$$

Thus,

$$y_p = -y_1\int \frac{y_2R}{w}dx + y_2\int \frac{y_1R}{w}dx$$
$$= -\cos x\int \frac{\sin x \tan x}{1}dx + \sin x\int \frac{\cos x.\tan x}{1}$$
$$= -\cos x\int \frac{\sin^2 x}{\cos x}dx + \sin x\int \sin x\,dx$$
$$= -\cos x\int \frac{1-\cos^2 x}{\cos x}dx + \sin x\,(-\cos x)$$
$$= -\cos x\int(\sec x - \cos x)\,dx - \sin x.\cos x$$
$$= -\cos x\{\log(\sec x + \tan x) - \sin x\} - \sin x.\cos x$$
$$= -\cos x\{\log(\sec x + \tan x)\} + \cos x\sin x - \sin x.\cos x$$
$$= -\cos x\{\log(\sec x + \tan x)\}.$$

Now (ii) becomes,

$$y(x) = y_h(x) + y_p$$
$$= A\cos x + B\sin x - \cos x\{\log(\sec x + \tan x)\}.$$

Similar Question for Practice from Final Exam:

2006 Spring Q. No. 4(b)

Solve: $y'' - 4y' + 4y = x^2 + e^{2x}$

2007 Fall Q. No. 4(b)

Solve the equation $y'' - 5y' + 6y = e^{2x}$

2008 Spring Q. No. 4(b)

Find the general solution of the differential equation: $y'' - 4y' + 4y = 6 + \frac{e^{2x}}{x}$.

2010 Spring Q. No. 4(b)

Solve: $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 3e^{2x}$.

2011 Fall Q. No. 4(b)

Solve $y'' + 2y' + 2y = 4e^{-x}\sec^3 x$.

Extra Questions (Long Questions)

1999; 2001 Q. No. 5(b)

Solve the following initial value, problem $y''+5y'+6y = 0$, $y(0) = 2$, $y^2(0) = -3$

Solution: Given equation is, $y'' + 5y' + 6y = 0$ ...... (i)

$$y(0) = 2,\ y'(0) = -3 \quad \text{...... (ii)}$$

So, its auxiliary equation is,

$$m^2 + 5m + 6 = 0 \Rightarrow m^2 + 2m + 3m + 6 = 0$$
$$\Rightarrow (m+2)(m+3) = 0$$
$$\Rightarrow m = -2, -3$$

Therefore, the general solution of given equation (i) is,

$$y(x) = c_1e^{-2x} + c_2e^{-3x} \quad \text{... (iii)}$$

By (ii), we have, $2 = c_1 + c_2$ ... **(A)**

Differential equation (iii) w. r. t. x, then,

$$y'(x) = -2c_1e^{-2x} - 3c_2e^{-3x}$$

By (ii), we have, $-3 = -2c_1 - 3c_2$ ... **(B)**

Solving the equations (A) and (B) we get,

$$c_1 = 3,\ c_2 = -1.$$

Now, equation (iii) becomes,

$$y(x) = 3e^{-2x} - e^{-3x}$$

Similar Question for Practice from Final Exam:

2000; 2004 Fall Q. No. 5(b)

Solve the following initial value problem, $y'' + 2y' + 2y = 0$, $y(0) = 1$, $y'(0) = -1$

2007 Fall Q. No. 3(b)

Solve the following initial value problem, $y'' + 4y' + 5y = 0$, $y(0) = 2$, $y'(0) = -3$.

2002 Q. No. 5(b)

Solve the following initial value problem. $y'' + 2y' - 3y = 6e^{-2t}$, $y(0) = 2$, $y'(0) = -14$.

2003 Fall Q. No. 5(b)

Solve $y'' + 4y' + 4y = \sin t$; $y(0) = 1$, $y'(0) = 3$.

2008 Spring Q. No. 5(b)

Solve the initial value problem: $y'' + y' - y = 14 + 2x - 2x^2$, $y(0) = 0$, $y'(0) = 0$.

2010 Spring Q. No. 5(b)

Solve the initial value problem: $y'' + y = 2\cos x$, where $y(0) = 3$ and $y'(0) = 4$.

2006 Fall Q. No. 5(b)

Solve the initial value problem $y'' - y' - 2y = 3e^{2x}$, $y(0) = 0$, $y'(0) = -2$

SHORT QUESTIONS

1999; 2001: Find the roots of the characteristic equation of the differential equation $y'' + \pi^2 y = 0$.

Solution: Given differential equation is

$$y'' = \pi^2 y = 0 \qquad \text{.........(1)}$$

The characteristic equation of (i) is

$$m^2 + \pi^2 = 0 \quad \Rightarrow \quad m^2 = -\pi^2 = (\pi i)^2$$

$$\Rightarrow \quad m = \pm \pi i$$

These are required root of characteristic equation of (i).

2000: Find the roots of the characteristic equation of the differential equation: $y'' - 2y' + 10y = 0$.

Solution: Given differential equation is

$$y'' - 2y' + 10y = 0 \qquad \text{...... (i)}$$

The characteristic equation of (i) is

$$m^2 - 2m + 10 = 0$$

$$\Rightarrow m = \frac{2 \pm \sqrt{4 - 40}}{2} = \frac{2 \pm \sqrt{-36}}{2} = \frac{2 \pm \sqrt{(6i)^2}}{2} = 1 \pm 3i$$

These are required root of the characteristic equation of (i).