Exercise 9.3

Change the Cartesian integral into an equivalent polar integral and evaluate the polar integral.

(1) $$\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\,dx$$

Solution: Given integral is,

$$I = \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\,dx$$

Here, the region of integration is bounded by $y = \sqrt{a^2 - x^2}$ to $y = -\sqrt{a^2 - x^2}$, $x = -a$ to $x = a$. That is the region of integration is the circle $x^2 + y^2 = a^2$ as shown in figure. For the radical strip in the region, r varies from $r = 0$ to $r = a$. In order to cover the region such type of radical strip varies from $\theta = 0$ to $\theta = 2\pi$.

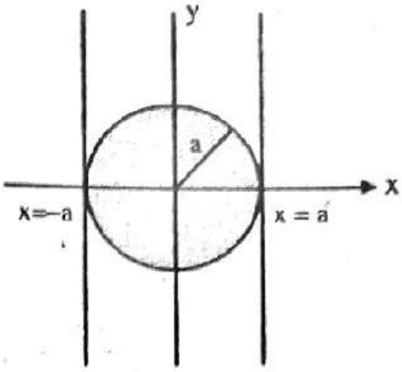
Therefore, the region of integration is, $0 \le r \le a$ and $0 \le \theta \le 2\pi$.

Then the above integral change to

$$I = \int_{0}^{2\pi}\int_{0}^{a} r\,dr\,d\theta = \int_{0}^{2\pi}\left[\frac{r^2}{2}\right]_0^a d\theta$$

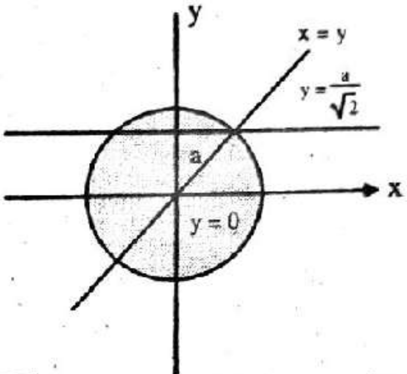
$$= \int_0^{2\pi} \frac{a^2}{2} d\theta$$

$$= \frac{a^2}{2} [\theta]_0^{2\pi} = \frac{a^2}{2} \times 2\pi = a^2\pi$$



Thus $I = a^2\pi$.

$$\int_0^{a/\sqrt{2}} \int_y^{\sqrt{a^2 - y^2}} \mathbf{x\, dx\, dy}$$



Solution: Given integral is,

$$I = \int_0^{a/\sqrt{2}} \int_y^{\sqrt{a^2 - y^2}} x\, dx\, dy$$

Here, the variables x varies from $x = y$ to $x = \sqrt{a^2 - y^2}$ and the variable y varies from $y = 0$ to $y = \frac{a}{\sqrt{2}}$.

Thus, the region of integration is from the line $x = y$ to the circle $x^2 + y^2 = a^2$. Also, the region moves from origin to the length $\frac{a}{\sqrt{2}}$ toward y-axis.

Now, changing the region to polar we substitute $x = r\cos\theta$ and $y = r\sin\theta$. Clearly the circle has radius $r = 0$ to $r = a$. So, the region of integration is, $0 \leq r \leq a$.

And, to find θ, the region is bounded by the line $x = y$. So, $\theta = \frac{\pi}{4}$.

Thus the angular form moves from $\theta = 0$ to $\theta = \frac{\pi}{4}$.

Also, $dx\, dy = r\, dr\, d\theta$.

Then the above integral changes to

$$I = \int_0^{\pi/4} \int_0^{a} r\cos\theta .\, r dr.\, d\theta$$

$$= \int_0^{\pi/4} \int_0^{a} r^2 \cos\theta .\, dr.\, d\theta$$

$$= \int_0^{\pi/4} \cos\theta \left[\frac{r^3}{3}\right]_0^a d\theta = \int_0^{\pi/4} \frac{a^3}{3} \cos\theta .\, d\theta = \frac{a^3}{3} [\sin\theta]_0^{\pi/4} = \frac{a^3}{3\sqrt{2}}$$

Thus, $I = \frac{a^3}{3\sqrt{2}} = \frac{a^3\sqrt{2}}{6}$.

(3) $\int_0^2 \int_0^x y\, dy\, dx$

Solution: Given integral is,

$$I = \int_0^2 \int_0^x y\, dy\, dx.$$

Here, the variables x varies from $x = 0$ to $x = 2$ and the variable y varies from $y = 0$ to $y = x$.

Thus, in the region of integrating limits for y are $y = 0$ to the line $y = x$. Also, the region moves from origin to the length 2 units toward x-axis.

Now, changing the region to polar we substitute $x = r\cos\theta$ and $y = r\sin\theta$.

Then, to find r, $x = 0$, $r\cos\theta = 0$, $r = 0$, $\qquad x = 2$, $r\cos\theta = 2$, $r = 2\sec\theta$

Therefore, $0 \le r \le 2\sec\theta$.

And, to find θ, $y = 0$, $r\sin\theta = 0$, $\theta = 0$ $\qquad y = x$, $r\sin\theta = r\cos\theta$, $\tan\theta = 1 = \tan\frac{\pi}{4} \Rightarrow \theta = \frac{\pi}{4}$

So, the above integral change to,

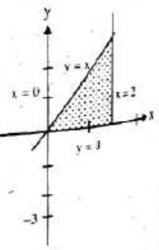
$$I = \int_0^{\pi/4}\int_0^{\pi/4} r\sin\theta\, r dr\, d\theta = \int_0^{\pi/4}\int_0^{2\sec\theta} r^2 \sin\theta\, dr d\theta$$

$$= \int_0^{\pi/4} \sin\theta \left[\frac{r^3}{3}\right]_0^{2\sec\theta} d\theta$$

$$= \int_0^{\pi/4} \sin\theta \frac{(2\sec\theta)^3}{3} d\theta$$

$$= \frac{8}{3}\int_0^{\pi/4} \sin\theta.\sec^3\theta\, d\theta$$

$$= \frac{8}{3}\int_0^{\pi/4} \tan\theta.\sec^2\theta\, d\theta$$



Put, $\tan\theta = t$ then $\sec^2\theta.\, d\theta = dt$. Also, $\theta = 0 \Rightarrow t = 0$ and $\theta = \frac{\pi}{4} \Rightarrow t = 1$. Then,

$$I = \frac{8}{3}\int_0^1 t\, dt = \frac{8}{3}\left[\frac{t^2}{2}\right]_0^1 = \frac{4}{3}.$$

(4) $\int_0^3 \int_0^{x\sqrt{3}} \frac{dy\, dx}{\sqrt{x^2 + y^2}}$

Solution: Given integral is,

$$I = \int_0^3 \int_0^{x\sqrt{3}} \frac{dy\, dx}{\sqrt{x^2 + y^2}}$$

Here, the variables x varies from $x = 0$ to $x = 3$ and the variable y varies from $y = 0$ to $y = x\sqrt{3}$.

Now, changing the region to polar we substitute $x = r\cos\theta$ and $y = r\sin\theta$.

And, to find r, $x = 0$, $r\cos\theta = 0$, $r = 0$ $\qquad x = 3$, $r\cos\theta = 3$, $r = 3\sec\theta$

Also, to find θ, $y = 0$, $r\sin\theta = 0$, $\theta = 0$ $\qquad y = x\sqrt{3}$, $r\sin\theta = \sqrt{3}\, r\cos\theta$, $\tan\theta = \sqrt{3} = \tan\frac{\pi}{3} \Rightarrow \theta = \frac{\pi}{3}$.

Now, the above integration change to,

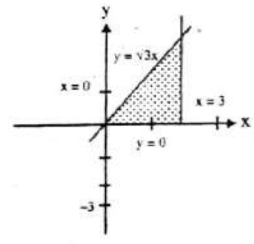
$$I = \int_0^{\pi/3}\int_0^{3\sec\theta} \frac{r\, dr\, d\theta}{\sqrt{(r\cos\theta)^2 + (r\sin\theta)^2}}$$

$$= \int_0^{\pi/3}\int_0^{3\sec\theta} \frac{r\, dr\, d\theta}{r}$$

$$= \int_0^{\pi/3} [r]_0^{3\sec\theta} d\theta$$

$$= \int_0^{\pi/3} 3\sec\theta.\, d\theta = 3\int_0^{\pi/3} \sec\theta.\, d\theta$$

$$= 3\left[\log(\sec\theta + \tan\theta)\right]_0^{\pi/3}$$



$$= 3\left[\log\left(\sec\frac{\pi}{3} + \tan\frac{\pi}{3}\right) - \log(\sec 0 + \tan 0)\right]$$
$$= 3\,[\log(2 + \sqrt{3})].$$

Thus, $I = 3\,[\log(2 + \sqrt{3})]$.

(5) $\displaystyle\int_{-a}^{a}\int_{0}^{\sqrt{a^2 - x^2}} e^{-(x^2+y^2)}\, dy\, dx$ **[2003 Fall Q. No. 3(a)]**

Solution: Given integral is,

$$I = \int_{-a}^{a}\int_{0}^{\sqrt{a^2 - x^2}} e^{-(x^2+y^2)}\, dy\, dx$$

Here, the variables y varies from $y = 0$ to $y = \sqrt{a^2 - x^2}$ and the variable x varies from $x = -a$ to $x = a$.

Thus, the region of integration is the half circle that is in only the positive region of y.

Now, changing the region to polar we substitute $x = r\cos\theta$ and $y = r\sin\theta$.

Clearly the circle has radius $r = 0$ to $r = a$. So, the region of integration is, $0 \le r \le a$.

And to find θ,

$$y = 0 \qquad\qquad y = \sqrt{a^2 - x^2}$$
$$\Rightarrow r\sin\theta = 0 \qquad\qquad \Rightarrow r\sin\theta = \sqrt{a^2 - r^2\cos^2\theta}$$
$$\Rightarrow \sin\theta = 0 \qquad\qquad \Rightarrow \sin\theta = \sin\theta \quad [\text{being } r = a]$$
$$\Rightarrow \theta = 0, \pi$$

Also, $dx\,dy = r\,d\theta\,dr$.

So that the above integral changes to,

$$I = \int_{0}^{\pi}\int_{0}^{a} e^{-(r^2\cos 2\theta + r^2\sin 2\theta)}\, r\, dr.\, d\theta$$
$$= \int_{0}^{\pi}\int_{0}^{a} e^{-r^2}\, r\, dr\, d\theta$$

Put, $t = r^2$ then $\frac{dt}{dr} = 2r \Rightarrow \frac{dt}{2} = r\,dr$. Also, $r = 0 \Rightarrow t = 0$ and $r = a \Rightarrow t = a^2$.

Then,

$$I = \int_{0}^{\pi}\int_{0}^{a^2} e^{-t}\frac{dt}{2}\, d\theta$$

$$= \frac{1}{2}\int_{0}^{\pi}\left[\frac{e^{-t}}{-1}\right]_0^{a^2} d\theta$$

$$= \frac{1}{-2}\int_{0}^{\pi}(e^{-a^2} - 1)\, d\theta = \frac{1}{2}(1 - e^{-a^2})\int_{0}^{\pi} d\theta = \frac{1}{2}(1 - e^{-a^2})\,[\theta]_0^{\pi}$$
$$= \frac{\pi}{2}(1 - e^{-a^2}).$$

Thus, $I = \frac{\pi}{2}(1 - e^{-a^2})$.

(6) $\displaystyle\int_{1}^{2}\int_{0}^{x} \frac{dy\, dx}{\sqrt{x^2 + y^2}}$

Solution: Given integral is,

$$I = \int_{1}^{2}\int_{0}^{x} \frac{dy\, dx}{\sqrt{x^2 + y^2}}$$

Here, the variables x varies from $x = 1$ to $x = 2$ and the variable y varies from $y = 0$ to $y = x$.

Now, changing the region to polar we substitute $x = r\cos\theta$ and $y = r\sin\theta$.

Then, to find r,

$$x = 1, \qquad\qquad x = 2$$
$$\Rightarrow r\cos\theta = 1 \qquad\qquad \Rightarrow r\cos\theta = 2$$
$$\Rightarrow r = \sec\theta \qquad\qquad \Rightarrow r = 2\sec\theta$$

And to find θ,

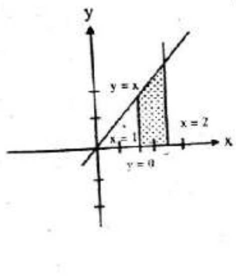
$$y = 0 \qquad\qquad y = x$$
$$\Rightarrow r\sin\theta = 0 \qquad\qquad \Rightarrow r\sin\theta = r\cos\theta$$
$$\Rightarrow \sin\theta = 0 \qquad\qquad \Rightarrow \tan\theta = 1 = \tan\frac{\pi}{4}$$
$$\Rightarrow \theta = 0 \qquad\qquad \Rightarrow \theta = \frac{\pi}{4}$$

Thus, $0 \le \theta \le \frac{\pi}{4}$.

So that, the above integral changes to

$$I = \int_{0}^{\pi/4}\int_{\sec\theta}^{2\sec\theta} \frac{r\, dr.\, d\theta}{\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}}$$

$$= \int_{0}^{\pi/4} [r]_{\sec\theta}^{2\sec\theta}\, d\theta$$



$$= \int_0^{\pi/4} \sec\theta.\, d\theta$$

$$= [\log(\sec\theta + \tan\theta)]_0^{\pi/4}$$

$$= \log\left(\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right) - \log(\sec\theta + \tan\theta) \;\; = \log(\sqrt{2} + 1).$$

Thus, $I = \log(\sqrt{2} + 1)$.

(7) $\displaystyle\int_0^2 \int_0^{\sqrt{4-y^2}} \cos(x^2 + y^2)\, dx\, dy$

Solution: Given integral is,

$$I = \int_0^2 \int_0^{\sqrt{4-y^2}} \cos(x^2 + y^2)\, dx\, dy$$

Here, the variables x varies from $x = 0$ to $x = \sqrt{4 - y^2}$ and the variable y varies from $y = 0$ to $y = 2$.

Thus, the region of integration is from the line $x = 0$ to the circle $x^2 + y^2 = 2^2$.

Also, the region moves from origin to the length 2 toward y-axis.

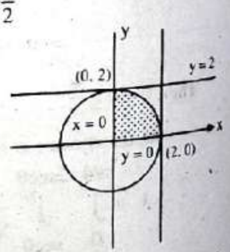
Now, changing the region to polar we substitute $x = r\cos\theta$ and $y = r\sin\theta$.

Clearly the circle has radius $r = 0$ to $r = 2$. So, the region of integration is, $0 \le r \le 2$.

And, to find θ,

$y = 0$ | $y = 2$
$\Rightarrow r\sin\theta = 0$ | $\Rightarrow r\sin\theta = 2$
$\Rightarrow \sin\theta = 0$ | $\Rightarrow 2\sin\theta = 2$ [Being r = 2]
$\Rightarrow \theta = 0$ | $\Rightarrow \theta = \frac{\pi}{2}$

The above integral change to



$$I = \int_0^{\pi/2} \int_0^2 \cos(r^2\cos^2\theta + r^2\sin^2\theta)\, r\, dr.\, d\theta$$

$$= \int_0^{\pi/2} \int_0^2 \cos r^2 r\, dr.\, d\theta$$

Put $r^2 = t$ then $\frac{dt}{dr} = 2r \Rightarrow \frac{dt}{2} = r\, dr$. Also, $r = 0 \Rightarrow t = 0$ and $r = 2 \Rightarrow t = 4$.

Then,

$$I = \int_0^{\pi/2} \int_0^4 \cos t\, \frac{dt}{2}\, d\theta = \frac{1}{2}\int_0^{\pi/2} [\sin t]_0^4\, d\theta = \frac{\sin 4}{2}\int_0^{\pi/2} d\theta = \frac{\pi}{4}\sin 4.$$

Thus, $I = \frac{\pi}{4}\sin 4$.

(8) $\displaystyle\int_0^a \int_y^a \frac{x\, dx\, dy}{x^2 + y^2}$

Solution: Given integral is,

$$I = \int_0^a \int_y^a \frac{x\, dx\, dy}{x^2 + y^2} \quad \dots (i)$$

Here, the region be $y \le x \le a$, $0 \le y \le a$.

Now, reversing the order of integration, for which x varies from $x = 0$ to $x = a$. Also, the strip moves from $y = 0$ to $y = x$. Therefore, after changing the order of integration of (i), it becomes,

$$I = \int_0^a \int_0^x \frac{x\, dy\, dx}{x^2 + y^2} \quad \dots (ii)$$

Here, the variables y varies from $y = 0$ to $y = x$ and the variable x varies from $x = 0$ to $x = a$.

Now, changing the region to polar we substitute $x = r\cos\theta$ and $y = r\sin\theta$.

Then to find r,

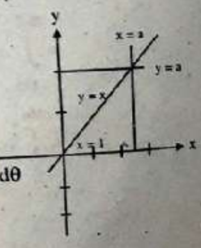
$x = 0$, | $x = a$
$\Rightarrow r\cos\theta = 0$, | $\Rightarrow r\cos\theta = a$
$\Rightarrow r = 0$, | $\Rightarrow r = a\sec\theta$

So, $0 \le r \le a\sec\theta$.

And, to find θ,

$y = 0$, | $y = x$
$\Rightarrow r\sin\theta = 0$, | $\Rightarrow r\sin\theta = r\cos\theta$
$\Rightarrow \theta = 0$. | $\Rightarrow \tan\theta = 1 = \tan\frac{\pi}{4} \Rightarrow \theta = \frac{\pi}{4}$.

So that, the above integral changes to,



$$I = \int_0^{\pi/4} \int_0^{a\sec\theta} \frac{r\cos\theta.\, r\, dr.\, d\theta}{r^2}$$

$$= \int_0^{\pi/4} \cos\theta\, [r]_0^{a\sec\theta}\, d\theta = \int_0^{\pi/4} \cos\theta \times a\sec\theta.\, d\theta$$

$$= a[\theta]_0^{\pi/4} = \frac{\pi a}{4}.$$

Thus, $I = \frac{\pi a}{4}$.

(9) $\displaystyle\int_0^{4a}\int_{y^2/4a}^{y} \frac{x^2-y^2}{x^2+y^2}\,dx\,dy$

Solution: Given integral is,

$$I = \int_0^{4a}\int_{y^2/4a}^{y} \frac{x^2-y^2}{x^2+y^2}\,dx\,dy$$

Here, the variables x varies from $x = \frac{y^2}{4a}$ to $x = y$ and the variable y varies from $y = 0$ to $y = 4a$.

Now, changing the region to polar we substitute $x = r\cos\theta$ and $y = r\sin\theta$.

Then, to find r,

$x = 0,$	$x = 4a$
$\Rightarrow r\cos\theta = 0$	$\Rightarrow r\cos\theta = 4a$
$\Rightarrow r = 0$	$\Rightarrow r = 4a\sec\theta$

And, to find θ,

$y = 0$	$y = x$
$\Rightarrow r\sin\theta = 0$	$\Rightarrow r\sin\theta = r\cos\theta$
$\Rightarrow \theta = 0$	$\Rightarrow \tan\theta = 1 = \tan\frac{\pi}{4} \Rightarrow \theta = \frac{\pi}{4}$

The above integration is changes to

$$I = \int_0^{\pi/4}\int_0^{4a\sec\theta} \frac{r^2(\cos^2\theta - \sin^2\theta)}{r^2}\, r\,dr\,d\theta$$

$$= \int_0^{\pi/4}\int_0^{4a\sec\theta} \cos 2\theta .\, r\,dr.\,d\theta$$

$$= \int_0^{\pi/4} \cos 2\theta \left[\frac{r^2}{2}\right]_0^{4a\sec\theta} d\theta$$

$$= \int_0^{\pi/4} \cos 2\theta .\, \frac{16a^2\sec^2\theta}{2}\, d\theta$$

$$= 8a^2 \int_0^{\pi/4} \left(\frac{\cos^2\theta - \sin^2\theta}{\cos^2\theta}\right) d\theta$$

$$= 8a^2 \int_0^{\pi/4} (1 - \tan^2\theta)\, d\theta$$

$$= 8a^2 \int_0^{\pi/4} (1 - \sec^2\theta + 1)\, d\theta$$

$$= 8a^2 \int_0^{\pi/4} (2 - \sec^2\theta)\, d\theta = 8a^2\,[2\theta - \tan\theta]_0^{\pi/4} = 8a^2\left[\frac{\pi}{2} - 1\right].$$

Thus, $I = 8a^2\left[\frac{\pi}{2} - 1\right]$.

(10) $\displaystyle\int_0^{a}\int_{y}^{a} \frac{x^2\,dx\,dy}{\sqrt{x^2+y^2}}$

Solution: Given integral is,

$$I = \int_0^{a}\int_{y}^{a} \frac{x^2\,dx\,dy}{\sqrt{x^2+y^2}}$$

Here, the variables y varies from $y = 0$ to $y = a$ and the variable x varies from $x = y$ to $x = a$.

Now, changing the region to polar we substitute $x = r\cos\theta$ and $y = r\sin\theta$.

Then, to find r,

$x = 0,$	$x = a$
$\Rightarrow r\cos\theta = 0$	$\Rightarrow r\cos\theta = a$
$\Rightarrow r = 0$	$\Rightarrow r = a\sec\theta.$

So, $0 \le r \le a\sec\theta$.

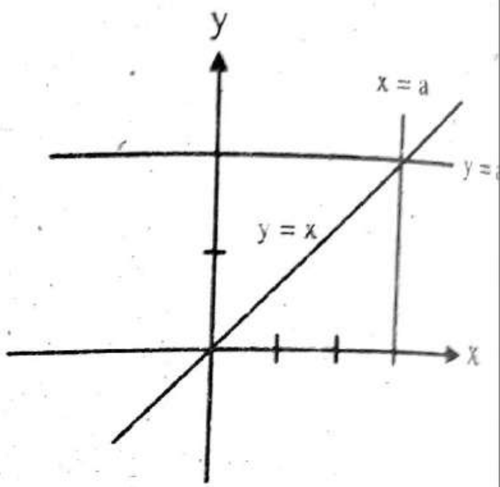
And, to find θ,

$y = 0$	$y = x$
$\Rightarrow r\sin\theta = 0$	$r\sin\theta = r\cos\theta$
$\Rightarrow \theta = 0$	$\tan\theta = 1 = \tan 45° \Rightarrow \theta = \frac{\pi}{4}$

So, $0 \le \theta \le \frac{\pi}{4}$

So that the above integral changes to

$$I = \int_0^{\pi/4} \int_0^{a\sec\theta} \frac{(r\cos\theta)^2\, r dr.\, d\theta}{\sqrt{(r\cos\theta)^2 + (r\sin\theta)^2}}$$



$$= \int_0^{\pi/4} \int_0^{a\sec\theta} \frac{r^2\cos^2\theta \; dr\, d\theta}{r}$$

$$= \int_0^{\pi/4} \cos^2\theta \left[\frac{r^3}{3}\right]_0^{a\sec\theta} d\theta$$

$$= \int_0^{\pi/4} \cos^2\theta \times \frac{a^3\sec^3\theta}{3}\, d\theta$$

$$= \frac{a^3}{3} \int_0^{\pi/4} \sec\theta\, d\theta \quad = \frac{a^3}{3} [\log(\sec\theta + \tan\theta)]_0^{\pi/4}$$

$$= \frac{a^3}{3}\left[\log\left(\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right) - \log(\sec 0 + \tan 0)\right]$$

$$= \frac{a^3}{3} [\log(\sqrt{2} + 1)].$$

Thus, $I = \frac{a^3}{3} [\log(\sqrt{2} + 1)]$.