EXERCISE – 4.9

Evaluate $\iint_S \vec{F}.\vec{n}\, dA$, by using Gauss divergence theorem of the following data:

1. **$\vec{F} = (x^2, 0, z^2)$, S is the box $|x| \le 1, |y| \le 3, |z| \le 2$.**

Solution: Given that $\vec{F} = (x^2, 0, \ge 2)$ and the surface is the box $|x| \le 1$, $|y| \le 3$, $|z| \le 2$.

By Gauss divergence theorem, we have,

$$\iint_S \vec{F}\cdot\vec{n}\,dA = \iiint_T \operatorname{div}\vec{F}\,dv \quad \text{......... (i)}$$

Here, $\operatorname{div}\vec{F} = \nabla.\vec{F} = \left(\vec{i}\frac{\delta}{\delta x} + \vec{j}\frac{\delta}{\delta y} + \vec{k}\frac{\delta}{\delta z}\right).(x^2, 0, z^2) = 2x + 2z$

Now (i) becomes,

$$\iint_S \vec{F}\cdot\vec{n}\,dA = \int_{-1}^{1}\int_{-3}^{3}\int_{-2}^{2}(2x+2z)\,dz\,dy\,dx$$

$$= \int_{-1}^{1}\int_{-3}^{3}[2xz + z^2]_{2}^{-2}\,dydx$$

$$= \int_{-1}^{1}\int_{-3}^{3}(4x+4+4x-4)\,dydx$$

$$= \int_{-1}^{1}\int_{-3}^{3}8x\,dydx$$

$$= \int_{-1}^{1}[8xy]_{3}^{-3}\,dx = \int_{-1}^{1}[24x+24x]dx$$

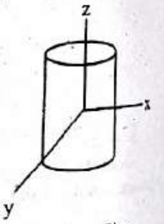
$$= \int_{-1}^{1}48x\,dx = [24x^2]_{1}^{-1} = 24 - 24 = 0.$$

2. $\vec{F}$ = (cosy, sinx, cosz), S is the surface $x^2 + y^2 \le 4$, $|z| \le 2$.

Solution: Given that, $\vec{F}$ = (Cosy, Sinx, Cosz)

And the surface is $x^2 + y^2 \le 4$, $|z| \le 2$.

By Gauss divergence theorem, we have,



$$\iint_S \vec{F}\cdot\vec{n}\,dA = \iiint_T \operatorname{div}\vec{F}\,dv \quad \text{... (i)}$$

Here, $\operatorname{Div}\vec{F} = \nabla.\vec{F} = \left(\vec{i}\frac{\delta}{\delta x} + \vec{j}\frac{\delta}{\delta y} + \vec{k}\frac{\delta}{\delta z}\right).(\text{Cosy, Sinx, Cosz}) = -\text{Sinz}.$

Since $x^2 + y^2 = 4$ is a circle in xy–plane in which y varies from $y = 0$ to $y = \pm\sqrt{4-x^2}$ and on the surface x moves from $x = -2$ to 2.

Therefore (i) become

$$\iint_S \vec{F}\cdot\vec{n}\,dA = \int_{-2}^{2}\int_{-2}^{2}\int_{-2\sqrt{4-x^2}}^{2\sqrt{4-x^2}}(-\text{Sinz})\,dy\,dx\,dz$$

$$= \int_{-2}^{2}\text{Sinz}\,dz\int\int_{-2\sqrt{4-x^2}}^{2\sqrt{4-x^2}}dy\,dx$$

$$= (0)\int\int_{-2\sqrt{4-x^2}}^{2\sqrt{4-x^2}}dy\,dx.$$

$\left[\because \int_{-a}^{a} f(x)\,dx = 0 \text{ if } f(x) \text{ is odd and sinz is odd function.}\right]$

$$= 0.$$

3. $\vec{F} = (4x, x^2y, -x^2z)$, S is the surface of the tetrahedron with vertices (0, 0, 0), (1, 0, 0), (0, 1, 0) and (0, 0, 1).

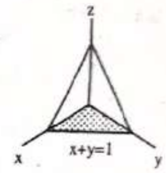
Solution: Given that, $\vec{F} = (4x, x^2y, -x^2z)$.

And the surface is the tetrahedron with vertices (0, 0, 0), (1, 0, 0), (0, 1, 0) and (0, 0, 1).

Thus, the region of integration is

$$0 \le x \le 1, 0 \le y \le 1 - x, 0 \le z \le 1 - y - x$$

By Gauss divergence theorem, we have,



$$\iint_S \vec{F}\cdot\vec{n}\,dA = \iiint_T \operatorname{div}\vec{F}\,dv \quad \text{... (i)}$$

Here,

$$\operatorname{Div}\vec{F} = \nabla.\vec{F} = \left(\vec{i}\frac{\delta}{\delta x} + \vec{j}\frac{\delta}{\delta y} + \vec{k}\frac{\delta}{\delta z}\right).(4x, x^2y, -x^2z) = 4 + x^2 - x^2$$
$$= 4.$$

Then (i) becomes,

$$\iint_S \vec{F}\cdot\vec{n}\,dA = \int_0^1\int_0^{1-x}\int_0^{1-x-y}4\,dz\,dy\,dx$$

$$= 4\int_0^1\int_0^{1-x}[z]_0^{1-x-y}\,dydx$$

$$= 4\int_0^1\int_0^{1-x}(1-x-y)dydx$$

$$= 4\int_0^1\left[y - xy - \frac{y^2}{2}\right]_0^{1-x}dx.$$

$$= \frac{4}{2}\int_0^1\left[1 - x - x(1-x) - \frac{(1-x)^2}{2}\right]dx.$$

$$= 2\int_0^1(1 - 2x + x^2)\,dx = 2\left[x - x^2 + \frac{x^3}{3}\right]_0^1$$

$$= 2\left(1 - 1 + \frac{1}{3}\right) = \frac{2}{3}.$$

4. $\vec{F} = (x^3, y^3, z^3)$, **S is the sphere $x^2 + y^2 + z^2 = 9$.**

Solution: Given that, $\vec{F} = (x^3, y^3, z^3)$.

And the surface is a sphere $x^2 + y^2 + z^2 = 9$, that has radius 3.

By Gauss divergence theorem, we have,

$$\iint_S \vec{F}.\vec{n}\,dA = \iiint_T \operatorname{div}\vec{F}\,dv \qquad \text{......... (i)}$$

Here,

$$\operatorname{div}\vec{F} = \nabla.\vec{F} = \left(\vec{i}\frac{\delta}{\delta x} + \vec{j}\frac{\delta}{\delta y} + \vec{k}\frac{\delta}{\delta z}\right).(x^3, y^3, z^3)$$

$$= 3x^2 + 3y^2 + 3z^2 = 3(x^2 + y^2 + z^2) = 3 \times 9 = 27.$$

Clearly, the sphere has limits $x = \pm 3$, $y = \pm\sqrt{9 - x^2}$ and $z = \pm\sqrt{1 - x^2 - y^2}$.

Then (i) becomes,

$$\iint_S \vec{F}.\vec{n}\,dA = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} 27\,dz\,dy\,dx$$

$$= 27\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dz\,dy\,dx$$

Since $\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dz\,dy\,dx$, is a sphere of radius 3. So the volume of the sphere is $\frac{4}{3}\pi r^2$. That is,

$$\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dz\,dy\,dx = \frac{4}{3}\pi\,9 = 12\pi.$$

Thus,

$$\iint_S \vec{F}.\vec{n}\,dA = 27\,(12\pi) = 372\pi.$$

5. $\vec{F} = (4xz, -y^2, yz)$, **S is the surface of the cube bounded by the plane $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.**

Solution: Given that, $\vec{F} = (4xz, -y^2, yz)$.

And the surface is bounced by $x = 0$, $y = 0$, $y = 1$, $z = 0$, $z = 1$.

By Gauss divergence we have,

$$\iint_S \vec{F}.\vec{n}\,dA = \iiint_T \operatorname{div}\vec{F}\,dv \qquad \text{......... (i)}$$

Here,

$$\operatorname{div}\vec{F} = \nabla.\vec{F} = \left(\vec{i}\frac{\delta}{\delta x} + \vec{j}\frac{\delta}{\delta y} + \vec{k}\frac{\delta}{\delta z}\right).(4xz, -y^2, yz)$$

$$= 4z - 2y + y = 4z - y.$$

Then (i) becomes,

$$\iint_S \vec{F}.\vec{n}\,dA = \int_0^1\int_0^1\int_0^1 (4z - y)\,dz\,dy\,dx$$

$$= \int_0^1\int_0^1 [2z^2 - yz]_0^1\,dy\,dx$$

$$= \int_0^1\int_0^1 (2 - y)\,dy\,dx$$

$$= \int_0^1\left[2y - \frac{y^2}{2}\right]_0^1 dx = \int_0^1\left(2 - \frac{1}{2}\right)dx = \int_0^1 \frac{3}{2}\,dx = \frac{3}{2}[x]_0^1 = \frac{3}{2}$$

6. $\vec{F} = (4x, -2y^2, z^2)$ **and s is the surface bounding the region $x^2 + y^2 = 4$, $z = 0, z = 3$.**

Solution: Given that, $\vec{F} = (4x, -2y^2, z^2)$.

And the surface is bounded by $x^2 + y^2 = 4$, $z = 0$, $z = 3$.

Clearly the circle $x^2 + y^2 = 4$ is bounded by $y = \pm\sqrt{4 - x^2}$ in which x moves from $x = -2$ to $x = 2$.

By Gauss divergence theorem we have,

$$\iint_S \vec{F}.\vec{n}\,dA = \iiint_T \operatorname{div}\vec{F}\,dv \qquad \text{......... (i)}$$

Here,

$$\operatorname{div}\vec{F} = \nabla.\vec{F} = \left(\vec{i}\frac{\delta}{\delta x} + \vec{j}\frac{\delta}{\delta y} + \vec{k}\frac{\delta}{\delta z}\right).(4x, -2y^2, z^2)$$

$$= 4 - 4y + 2z.$$

Then (i) becomes,

$$\iint_S \vec{F}.\vec{n}\,dA = \int_0^3\int_{-2}^{2}\int_{\sqrt{4-x}}^{\sqrt{4-x}} (4 - 4y + 2z)\,dy\,dx\,dz$$

$$= \int_0^3\int_{-2}^{2} [4y - 2y^2 + 2yz]_{\sqrt{4-x^2}}^{-\sqrt{4-x^2}}$$

$$= \int_0^3\int_{-2}^{2} [8\sqrt{4-x^2} - 0 + 4z\sqrt{4-x^2}]\,dx\,dz$$

$$= \int_{-2}^{2}\int_{0}^{3}\left[8\sqrt{4-x^2} + 4z\sqrt{4-x^2}\right] dz\,dx$$

$$= \int_{-2}^{2}\left[8\sqrt{4-x^2}\, z + 2z^2\sqrt{4-x^2}\right]_0^3 dz$$

$$= \int_{-2}^{2}(24\sqrt{4-x^2} + 18\sqrt{4-x^2})dx$$

$$= 42\int_{-2}^{2}\sqrt{4-x^2}\,dx$$

$$= 42\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]$$

$$= 42\left[(0 + 2\text{Sin}^{-1}(1)) - (0 + 2\,\text{Sin}^{-1}(-2))\right]$$

$$= 42\left[2\text{Sin}^{-1}(1) + 2\,\text{Sin}^{-1}(1)\right] \qquad [\because \text{Sin}(-\theta) = -\text{Sin}\theta]$$

$$= 168\,\text{Sin}^{-1}(1) = 168\frac{\pi}{2} = 84\pi.$$

7. $\vec{F} = (9x, y\cosh^2 x, -z\sinh^2 x)$**, S: the ellipsoid** $4x^2 + y^2 + 9z^2 = 36$.

Solution: Similar to Q.4.

[Hints: Use $\cosh^2 x - \sinh^2 x = 1$. And obtain limits as in Q.4.]

8. $\vec{F} = (\sin x, y, z)$**, S is the surface of** $0 \le x \le \pi/2,\ x \le y \le z,\ 0 \le z \le 1$.

Solution: Given that, $\vec{F} = (\text{Sin}x, y, z)$ and surface is, $0 \le x \le \frac{\pi}{2},\ x \le y \le z,\ 0 \le z \le 1$.

By Gauss divergence theorem, we have,

$$\iint_S \vec{F}.\vec{n}\,dA = \iiint_T \text{div}\,\vec{F}\,dv \qquad \text{......... (i)}$$

Here,

$$\text{div}\,\vec{F} = \left(\vec{i}\frac{\delta}{\delta x} + \vec{j}\frac{\delta}{\delta y} + \vec{k}\frac{\delta}{\delta z}\right).(\text{Sin}x, y, z) = \text{Cos}x + 1 + 1 = 2 + \text{Cos}x$$

Then (i) becomes,

$$\iint_S \vec{F}.\vec{n}\,dA = \int_0^1\int_0^{\frac{\pi}{2}}\int_x^z (2 + \text{Cos}x)\,dy\,dx\,dz$$

$$= \int_0^1\int_0^{\frac{\pi}{2}}[2y + y\,\text{Cos}x]_x^z\,dx\,dz$$

$$= \int_0^1 [2xz + z\,\text{Sin}x - x^2 - x\,\text{Sin}x - \text{Cos}x]_0^{\pi/2}\,dx$$

$$= \int_0^1\left[xz + z - \frac{\pi^2}{4} - \frac{\pi}{2} + 1\right]dx \quad [\because \text{Sin}\frac{\pi}{2} = 1 = \text{Cos}0]$$

$$[\because \text{Sin}0 = 0 = \text{Cos}\frac{\pi}{2}]$$

$$= \left[\frac{xz^2}{2} + \frac{z^2}{2} - \frac{\pi^2 z}{4} - \frac{\pi z}{2} + z\right]_0^1$$

$$= \frac{\pi}{2} + \frac{1}{2} - \frac{\pi^2}{4} - \frac{\pi}{2} + 1$$

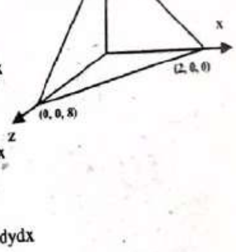
$$= \frac{3}{2} - \frac{\pi^2}{4}.$$

B. $\iint_S \phi\,dv$**, where** $\phi = 45x^2y$ **and v is the closed region bounded by the planes** $4x + 2y + z = 8,\ x = 0,\ y = 0,\ z = 0$.

Solution: Given that, $\varnothing = 45x^2y$

And the surface is bounded by $4x + 2y + z = 8,\ x = 0,\ y = 0,\ z = 0$.

Clearly the region is bounded $x = 0$ and $x = 2$, $y = 0$ and $y = 4 - 2x$, $z = 0$ and $z = 8 - 4x - 2y$.



Now,

$$\iiint \varnothing\,dv = \int_0^2\int_0^{4-2x}\int_0^{(8-4x-2y)} 45x^2y\,dz\,dy\,dx$$

$$= 45\int_0^2\int_0^{4-2x} x^2y\,[z]_0^{8-4x-2y}\,dy\,dx$$

$$= 45\int_0^2\int_0^{4-2x} x^2y\,(8 - 4x - 2y)\,dy\,dx$$

$$= 45\int_0^2\int_0^{4-2x}(8x^2y - 4x^3y - 2x^2y^2)\,dy\,dx$$

$$= 45\int_0^2\left[4x^2y^2 - 2x^3y^2 - \frac{2x^2y^3}{3}\right]_0^{4-2x} dx$$

$$= 45\int_0^2\left[(4x^2 - 2x^3)(4 - 2x)^2 - \frac{2x^2}{3}(4 - 2x)^3\right]dx$$

$$= 45\int_0^2 [(4x^2 - 2x^3)(16 - 16x + 4x^2) - \frac{2x^2}{3}(64 - 8x^3 - 96x + 48x^2)]\,dx$$

$$= \frac{45}{3}\int_0^2 [192x^2 - 192x^3 + 48x^4 - 96x^3 + 96x^4 - 24x^5 - 123x^2 + 16x^3 + 192x^3 - 96x^4]\, dx$$

$$= 15\int_0^2 (64x^2 - 96x^3 + 48x^4 - 8x^5)\, dx$$

$$= 120\int_0^2 (8x^2 - 12x^3 + 6x^4 - x^5)\, dx$$

$$= 120\left[\frac{8x^3}{3} - 3x^4 + \frac{6x^5}{5} - \frac{x^6}{6}\right]_0^2$$

$$= 120\left[\frac{64}{3} - 48 + \frac{192}{5} - \frac{64}{6}\right]$$

$$= 120\left(\frac{32}{3} - 48 + \frac{192}{5}\right) = 120\left(\frac{160 - 720 + 576}{15}\right) = 8\times 16 = 128.$$

C. If $\vec{F} = (2x^2 - 3z, -2xy, -4x)$, then evaluate $\iiint (\nabla\times\vec{F})\, dv$, where V is the closed region bounded by the planes $x = 0, y = 0, z = 0$ and $2x + 2y + z = 4$.

Solution: Given that, $\vec{F} = (2x^2 - 3z, -2xy, -4x)$.

And the region is bounded by $x = 0, y = 0, z = 0$ and $2x + 2y + z = 4$.

Then the region is bounded by $x = 0$ and $x = 2$, $y = 0$ and $y = 2 - x$, $z = 0$ and $z = 4 - 2x - 2y$.

Here,

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta y} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ 2x^2 - 3z & -2xy & -4x \end{vmatrix} = 0\,\vec{i} + (4 - 3)\,\vec{j} + (-2y)\,\vec{k} = (0, 1, -2y)$$

Now,

$$\iiint(\nabla\times\vec{F})\, dv = \int_0^2\int_0^{(2-x)}\int_0^{(4-2x-2y)} (\vec{j} - 2y\,\vec{k})\, dzdydx$$

$$= \int_0^2\int_0^{(2-x)} [z\,\vec{j} - 2yz\,\vec{k}]_0^{4-2x-2y}\, dydx$$

$$= \int_0^2\int_0^{(2-x)} [(4 - 2x - 2y)\,\vec{j} - 2y\,(4 - 2x - 2y)\,\vec{k}]dydx$$

$$= \int_0^2\int_0^{(2-x)} [(4 - 2x - 2y)\,\vec{j} - (8y - 4xy - 4y^2)\,\vec{k}\,]dydx$$

$$= \int_0^2 [(4 - 2xy - y^2)\,\vec{j} - \left(4y^2 - 2xy^2 - \frac{4y^3}{3}\right)\vec{k}\,]_0^{2-x}\, dx$$

$$= \int_0^2 [[4(2 - x) - 2x\,(2 - x) - (2 - x)^2]\,\vec{j} - [4\,(2 - x)^2 - 2x\,(2 - x)^2 - \frac{4}{3}(2 - x)^3\,\vec{k}\,]]_0^{2-x}\, dx$$

$$= \int_0^2 [(8 - 4x - 4x + 2x^2 - 4 - x^2 + 4x)\,\vec{j} - (16 + 4x^2 - 16x - 8x - 2x^3 + 8x^2 - \frac{4}{3}(8 - x3 - 12x + 6x^2)\,\vec{k}\,]dx$$

$$= \int_0^2 [(4 - 4x + x^2)\,\vec{j} - \frac{1}{3}(16 - 24x + 12x^2 - 2x^3)\,\vec{k}\,]\, dx$$

$$= [\left(4x - 2x^2 + \frac{x^3}{3}\right)\vec{j} - \frac{1}{3}\left(16x - 12x^2 + 4x^3 - \frac{2x^4}{4}\right)\vec{k}\,]_0^2$$

$$= \left(8 - 8 + \frac{8}{3}\right)\vec{j} - \frac{1}{3}(32 - 48 + 32 - 8)\,\vec{k}$$

$$= \frac{8}{3}\,\vec{j} - \frac{8}{3}\,\vec{k} = \frac{8}{3}(\vec{j} - \vec{k})$$

D. Using the Gauss divergence theorem, find $\iint_S (\vec{F}.\vec{n})\, ds$, where

1. **$\vec{F} = y\sin x\,\vec{i} + y^2 z\,\vec{j} + (x + 3z)\,\vec{k}$ and S is the surface of the region bounded by the planes $x = \pm 1, y = \pm 1, z = \pm 1$.**

Solution: Given that, $\vec{F} = y\,\mathrm{Sin}x\,\vec{i} + y^2 z\,\vec{j} + (x + 3z)\,\vec{k}$.

And the surface of region is bounded by $x = \pm 1$, $y = \pm 1$, $z = \pm 1$.

By Gauss divergence theorem, we have,

$$\iint_S (\vec{F}.\vec{n})ds = \iiint_T \mathrm{div}\,\vec{F}\, dv \qquad \text{.........(10)}$$

Here,

$$\mathrm{div}\,\vec{F} = \nabla.\vec{F} = \left(\vec{i}\frac{\delta}{\delta x} + \vec{j}\frac{\delta}{\delta y} + \vec{k}\frac{\delta}{\delta z}\right)(y\,\mathrm{Sin}x\,\vec{i} + y^2 z\,\vec{j} + (x + 3z)\,\vec{k})$$

$$= y\,\mathrm{Cos}x + 2yz + 3$$

Now (i) becomes,

$$\iint_S (\vec{F}.\vec{n})ds = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}(y\,\mathrm{Cos}x + 2yz + 3)dxdydz$$

$$= \int_{-1}^{1}\int_{-1}^{1} [y\,\text{Sin}x + 2xyz + 3x]_{-1}^{1}\, dydz$$

$$= \int_{-1}^{1}\int_{-1}^{1} [y\,\text{Sin}1 - y\,\text{Sin}(-1) + 2yz - 2(-1)\,yz + 3 - 3(-1)]\,dydz$$

$$= \int_{-1}^{1}\int_{-1}^{1} (2y\,\text{Sin}1 + 4yz + 6)\, dydz$$

$$= \int_{-1}^{1} [y^2\,\text{so}1 + 2y^2z + 6y)]_{-1}^{1}\, dz$$

$$= \int_{-1}^{1} (\text{Sin}1 - \text{Sin}1 + 2z - 2z + 6 + 6)\, dz$$

$$= \int_{-1}^{1} 12dz = [12z]_{-1}^{1} = 12\,(1+1) = 24.$$

2. $\vec{F} = yz\,\vec{i} + xz\,\vec{j} + xy\,\vec{k}$; **S is the graph of** $x^{2/3} + y^{2/3} + z^{2/3} = 1$.

Solution: Given that, $\vec{F} = yz\,\vec{i} + xz\,\vec{j} + xy\,\vec{k} = (yz, xz, xy)$.

And the surface of region is the graph bounced by $x^{2/3} + y^{2/3} + z^{2/3} = 1$.

By Gauss divergence theorem, we have

$$\iint_S \vec{F}.\vec{n}\, ds = \iiint_T \text{div}\,\vec{F}\, dv \qquad \text{.........(i)}$$

Hare,

$$\text{div}\,\vec{F} = \nabla.\vec{F} = \left(\vec{i}\frac{\delta}{\delta x} + \vec{j}\frac{\delta}{\delta y} + \vec{k}\frac{\delta}{\delta z}\right).(yz\,\vec{i} + xz\,\vec{j} + xy\,\vec{k})$$
$$= 0 + 0 + 0 = 0.$$

Therefore (i) becomes,

$$\iint_S (\vec{F}.\vec{n})\, ds = \iiint_T 0\, dv = 0.$$

EXERCISE 4.10

Evaluate $\oint_c \vec{F}.d\vec{r}$ **by using stocks theorem:**

1. $\vec{F} = (z^2, 5x, 0)$, **S is the square** $0 \le x \le 1, 0 \le y \le 1, z = 1$.

[2004 Spring Q.No. 4(b)]

Solution: Given that, $\vec{F} = (z^2, 5x, 0)$ and the surface is $0 \le x \le 1, 0 \le y \le 1, z = 1$.

$$\oint_c \vec{F}.d\vec{r} = \iint_S (\nabla.\vec{F}).\vec{N}\,dxdy \qquad \text{.........(i)}$$

where, $\vec{N} = \vec{r_x} \times \vec{r_y}$.

Here, $\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ z^2 & 5x & 0 \end{vmatrix} = 2z\,\vec{j} + 5\,\vec{k} = (0, 2z, 5)$.

Since we have, $\vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$.

$$\Rightarrow \vec{r} = x\,\vec{i} + y\,\vec{j} + \vec{k}.$$

So,

$$\vec{r_x} = \vec{i} \quad \text{and} \quad \vec{r_y} = \vec{j}.$$

Then,

$$\vec{N} = \vec{r_x} \times \vec{r_y} = \vec{i} \times \vec{j} = \vec{k} = (0, 0, 1).$$

So that, $(\nabla\times\vec{F}).\vec{N} = (0, 2z, 5).(0, 0, 1) = 5$.

Now (i) becomes

$$\oint_c \vec{F}.d\vec{r} = \int_0^1\int_0^1 5dxdy = 5\int_0^1 [x]_0^1\, dy = 5\int_0^1 dy = 5[y]_0^1 = 5$$

Thus $\oint_c \vec{F}.d\vec{r} = 5$.

2. $\vec{F} = (e^x, e^x\,\text{siny}, e^x\,\text{cosy})$, **S:** $z = y^2, 0 \le x \le 4, 0 \le y \le 2$.

Solution: Given that $\vec{F} = (e^x, e^x\,\text{Siny}, e^x\,\text{Cosy})$

And the surface is, $0 \le x \le 4, 0 \le y \le 2, z = y^2$.

By Stoke's theorem we have,

$$\oint_c \vec{F}.d\vec{r} = \iint_S (\nabla\times\vec{F}).\vec{N}\, dxdy \qquad \text{.........(i)}$$

where, $\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ e^x & e^x\,\text{siny} & e^x\,\text{cosy} \end{vmatrix}$

$$= (-e^x\,\text{Siny} - e^x\,\text{Siny})\,\vec{i} + e^x\,\vec{j} + 0\,\vec{k}$$

We have,

$$\vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k} = x\,\vec{i} + y\,\vec{j} + y^2\,\vec{k} \qquad [\because z = y^2]$$

Then, $\vec{r_1} = \vec{i}$ and $\vec{r_y} = \vec{j} + 2y\,\vec{k}$.

So that,

$$\vec{N} = \vec{r_x} \times \vec{r_y} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & 2y \end{vmatrix} = -2y\vec{j} + \vec{k}$$

Therefore,

$$(\nabla\times\vec{F}).\vec{N} = (-2e^z \operatorname{Sin}y\,\vec{i} + e^z\vec{j} + 0\vec{k}).(0\vec{i} - 2y\vec{j} + \vec{k}) = -2ye^z$$

Now, (i) becomes,

$$\oint_c \vec{F}.d\vec{r} = -\int_0^4\int_0^2 2ye^z\,dydx = -\int_0^4\int_0^2 2ye^{y^2}\,dydx$$

Set $y^2 = t$ then $2ydy = dt$. Also $y = 0 \Rightarrow t = 0$, $y = 2 \Rightarrow t = 4$. Then,

$$\oint_c \vec{F}.d\vec{r} = -\int_0^4\int_0^2 e^t\,dtdx = -\int_0^4 [e^t]_0^4\,dx = -\int_0^4 (e^4 - e^0)\,dx = -[xe^4 - x]_0^4$$

$$\Rightarrow \oint_c \vec{F}.d\vec{r} = -(4e^4 - 4) = 4(1 - e^4).$$

3. $\vec{F} = (y^2, z^2, x^2)$, S the portion of the parabolid $x^2 + y^2 = z$, $y \geq 0$, $z \leq 1$.

Solution: Given that, $\vec{F} = (y^2, z^2, x^2)$ and the surface is, $x^2 + y^2 = z$, $y \geq 0$, $z \leq 1$.

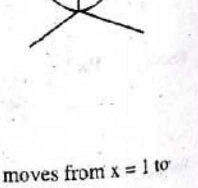
By Stokes theorem, we have,

$$\iint_S (\nabla\times\vec{F}).\vec{N}\,dxdy = \oint_c \vec{F}.d\vec{r} \qquad \text{......... (i)}$$

where, $\vec{N} = \vec{r_x} \times \vec{r_y}$.

Here, $$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ y^2 & z^2 & x^2 \end{vmatrix} = -2z\vec{i} - 2x\vec{j} - 2y\vec{k}.$$

Since S is a paraboloid with z is linear and z may have maximum value 1. Therefore $z = 1$.



We have, $\vec{r} = x\vec{i} + y\vec{j} + \vec{k}$. So,

$\vec{r_x} = \vec{i}$ and $\vec{r_y} = \vec{j}$.

Then,

$$\vec{N} = \vec{r_x}\times\vec{r_y} = \vec{i}\times\vec{j} = \vec{k}.$$

Therefore,

$$(\nabla\times\vec{F}).\vec{N} = (-2z\vec{i} - 2x\vec{j} - 2y\vec{k}).\vec{k} = -2y$$

Given that the surface is $x^2 + y^2 = z$, $z \leq 1$, $y \geq 0$.

This gives that y varies from $y = 0$ to $y = \sqrt{1 - x^2}$ and x moves from $x = 1$ to $x = 1$.

Therefore,

$$\iint_S (\nabla\times\vec{F}).\vec{N}\,dx\,dy = \int_{-1}^{1}\int_0^{\sqrt{1-x^2}} (-2y)\,dy\,dx$$

$$= -\int_{-1}^{1} [y^2]_0^{\sqrt{1-x^2}}\,dx$$

$$= -\int_{-1}^{1} (1 - x^2)\,dx$$

$$= -\left[x - \frac{x^3}{3}\right]_{-1}^{1} = -\left[\left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right)\right]$$

$$= -\left(1 - \frac{1}{3} + 1 - \frac{1}{3}\right)$$

$$= -\left(2 - \frac{2}{3}\right) = -\frac{4}{3}$$

Thus, $\oint_c \vec{F}.d\vec{r} = -\frac{4}{3}$.

4. $\vec{F} = (-5y, 4x, z)$, C is the circle $x^2 + y^2 = 4$, $z = 1$.

Solution: Given that, $\vec{F} = (-5y, 4x, z)$ and the surface is $x^2 + y^2 = 4$, $z = 1$.

By Stoke's theorem, we have,

$$\iint_S (\nabla\times\vec{F}).\vec{N}\,dxdy = \oint_c \vec{F}.d\vec{r} \qquad \text{......... (i)}$$

where, $\vec{N} = \vec{r_x}\times\vec{r_y}$.

Here, $$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ -5y & 4x & z \end{vmatrix} = 0\vec{i} + 0\vec{j} + (4 + 5)\vec{k} = (0, 0, 9).$$

We have, $\vec{r} = x\vec{i} + y\vec{j} + \vec{k}$. So,

$\vec{r_x} = \vec{i}$ and $\vec{r_y} = \vec{j}$.

So that, $\vec{N} = \vec{r_x}\times\vec{r_y} = \vec{i}\times\vec{j} = \vec{k} = (0, 0, 1)$.

Therefore, $(\nabla\times\vec{F}).\vec{N} = (0, 0, 9).(0, 0, 1) = 9$.

Given that the surface is $x^2 + y^2 = 4$, $z = 1$.

Clearly, the surface is a circle in which y varies from $y = -\sqrt{4 - x^2}$ to $y = \sqrt{4 - x^2}$ and on the region x moves from $x = -2$ to $x = 2$.

Therefore,

$$\iint_S (\nabla\times\vec{F}).\vec{N}\,dxdy = 9\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dydx$$

$$= 9\int_{-2}^{2}(2\sqrt{4-x^2})\,dx$$

$$= 18\left[\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_{-2}^{2}$$

$$= 18\,[0 + 2\,\text{Sin}^{-1}(1) - 0 - 2\,\text{Sin}^{-1}(-1)]$$

$$= 18\times 4\times\frac{\pi}{2}\quad [\because \text{Sin}^{-1}(-\theta) = -\text{Sin}^{-1}(\theta)]$$

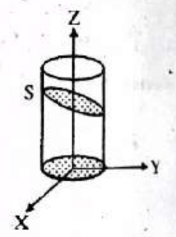
$$= 36\pi$$

Thus, by (i), $\oint_c \vec{F}.d\vec{r} = = 36\pi$.

5. $\vec{F} = (4z, -2x, 2x)$, C is the circle $x^2 + y^2 = 1$, $z = y + 1$.

Solution: Given that, $\vec{F} = (4z, -2x, 2x)$ and the surface is,
$x^2 + y^2 = 1$, $z = y + 1$.

By Stoke's theorem we have



$$\iint_S (\nabla\times\vec{F}).\vec{N}\,dxdy = \oint_c \vec{F}.d\vec{r} \quad \text{.... (i)}$$

where, $\vec{N} = \vec{r_x}\times\vec{r_y}$.

Here, $\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ 4z & -2x & 2x \end{vmatrix} = 0\,\vec{i} + (4-2)\,\vec{j} + (-2)\,\vec{k} = (0, 2, -2)$

Since we have, $\vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k} = x\,\vec{i} + y\,\vec{j} + (y+1)\,\vec{k}$. Then,

$\vec{r_x} = \vec{i} = (1, 0, 0)$ and $\vec{r_y} = \vec{j} + \vec{k} = (0, 1, 1)$.

So that,

$$\vec{N} = \vec{r_x}\times\vec{r_y} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 0\,\vec{i} - \vec{j} + \vec{k} = (0, -1, 1).$$

Then, $(\nabla.\vec{F}).\vec{N} = (0, 2, -2).(0, -1, 1) = 0 - 2 - 2 = -4$

Given surface on xy plane is $x^2 + y^2 = 1$ which is a circle in which y varies from $y = -\sqrt{1-x^2}$ to $\sqrt{1-x^2}$ and x moves on the region from $x = -1$ to $x = 1$.

Therefore,

$$\iint_S (\nabla\times\vec{F}).\vec{N}\,dx\,dy = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (-4)\,dydx$$

$$= -4\int_{-1}^{1}[y]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\,dx$$

$$= -4\int_{-1}^{1}2\sqrt{1-x^2}\,dx$$

$$= -8\left[\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}\left(\frac{x}{1}\right)\right]_{-1}^{1}$$

$$= -8\left[0+\frac{1}{2}\sin^{-1}(1) - 0 - \frac{1}{2}\sin^{-1}(-1)\right]$$

$$= -8.\text{Sin}^{-1}(1) \qquad [\because \text{Sin}^{-1}(-\theta) = -\text{Sin}^{-1}\theta]$$

$$= -8.\frac{\pi}{2} \qquad [\because \text{Sin}^{-1}(1) = \frac{\pi}{2}]$$

$$= -4\pi$$

Thus, by (i), $\oint_c \vec{F}.d\vec{r} = -4\pi$.

6. $\vec{F} = (0, xyz, 0)$, C is the boundary of the triangle with vertices (1, 0, 0), (0, 1, 0), (0, 0, 1).

Solution: Given that, $\vec{F} = (0, xyz, 0)$.

And the surface is a triangle having vertices (1, 0, 0), (0, 1, 0) and (0, 0, 1).

By Stoke's theorem we have,

$$\iint_S (\nabla\times\vec{F}).\vec{N}\,dxdy = \oint_c \vec{F}.d\vec{r} \quad \text{.........(i)}$$

where, $\vec{N} = \vec{r_x}\times\vec{r_y}$

Here, $\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ 0 & xyz & 0 \end{vmatrix} = -xy\,\vec{i} + yz\,\vec{k} = (-xy, 0, yz).$

Since the equation of plane that passes through (1, 0, 0), (0, 1, 0) and (0, 0, 1) be

$$x + y + z = 1 \quad \text{........ (ii)}$$

Since we have, $\vec{r} = x\,\vec{i} + y\,\vec{j} + (1 - x - y)\,\vec{k}$ [∵ uSing (ii)]

Then,

$\vec{r_x} = \vec{i} - \vec{k} = (1, 0, -1)$ and $\vec{r_y} = \vec{j} - \vec{k} = (0, 1, -1)$.

So that,

$$\vec{N} = \vec{r_x} \times \vec{r_y} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \vec{i} + \vec{j} + \vec{k} = (1, 1, 1).$$

Therefore, $(\nabla\times\vec{F}).\vec{N} = (-xy, 0, yz)(1, 1, 1)$

$= -xy + 0 + yz$

$= y(-x + z)$

$= y(-x + 1 - x - y) = y(1 - 2x - y) = y - 2xy - y^2$

Since the surface is the plane $x + y + z = 1$. On the xy–plane, the projection of the plane is $x + y = 1$ in which x varies from $x = 0$ to $x = 1 - y$ and y moves from $y = 0$ to $y = 1$.

Therefore,

$$\iint_s (\nabla\times\vec{F}).\vec{N}\,dydx = \int_0^1\int_0^{1-y}(y - 2xy - y^2)\,dxdy$$

$$= \int_0^1 [xy - x^2y - xy^2]_0^{1-y}\,dy$$

$$= \int_0^1 [(1-y)\,y - (1-y)^2\,y - (1-y)\,y^2]dy$$

$$= \int_0^1 (y - y^2 - y + 2y^2 - y^3 - y^2 + y^3)\,dy$$

$$= \int_0^1 0\,dy = 0\int_0^1 dy = 0.$$

Thus, by (i), $\oint_c \vec{F}.d\vec{r} = 0.$

7. **$\vec{F} = (y^3, 0, x^3)$, C is the boundary of the triangle with vertices (1, 0, 0), (0, 1, 0), (0, 0, 1).**

Solution: Similar to Q. No. 6.

8. **$\vec{F} = (x^2 + y^2, -2xy, 0)$, C is the rectangle hounded by the lines $x = \pm a$, $y = 0$, $y = b$.**

Solution: Given that $\vec{F} = (x^2 + y^2, -2xy, 0)$.

And the surface is a rectangle bounded by $x = \pm a$, $y = 0$, $y = b$.

By Stoke's theorem we have;

$$\iint_s (\nabla\times\vec{F}).\vec{N}\,dxdy = \oint_c \vec{F}.d\vec{r} \quad \text{......... (i)}$$

where, $\vec{N} = \vec{r_x} \times \vec{r_y}$

Here, $$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ x^2 + y^2 & -2xy & 0 \end{vmatrix} = 0\,\vec{i} + 0\,\vec{j} + (-2y - 2y)\,\vec{k} = (0, 0, -4y)$$

Since z is independent to x and y. Therefore.

$\vec{r_x} = \vec{i} = (1, 0, 0)$ and $\vec{r_y} = \vec{j} = (0, 1, 0)$.

Then, $\vec{N} = \vec{r_x} \times \vec{r_y} = \vec{i} \times \vec{j} = \vec{k} = (0, 0, 1)$.

Therefore,

$$\iint_s (\nabla\times\vec{F}).\vec{N}\,dxdy = \int_0^b\int_{-a}^a (-4y)\,dxdy$$

$$= \int_0^b [-4xy]_{-a}^a\,dy = \int_0^b -4y(a + a)\,dy$$

$$= -4a\,[y^2]_0^b = -4a\,(b^2 - 0) = -4ab^2$$

Thus (i) gives, $\oint_c \vec{F}.d\vec{r} = -4ab^2$.

$\vec{F} = (2x - y, -yz^2 - y^2z)$, S is the upper half surface of $x^2 + y^2 + z^2 = 1$, bounded by its projection on xy plane.

OR **Verify Stoke's theorem for the vector function, $\vec{F} = (2x - y)\,\vec{i} - yz^2\,\vec{j} - y^2z\,\vec{k}$ where S is the surface of the sphere $x^2 + y^2 + z^2 = 1$ above the xy plane and C its boundary.** **[2001 Q.No. 4(b) OR]**

Solution: Given that $\vec{F} = (2x - y, -yz^2, -y^2z)$.

And the region is the upper half of the sphere $x^2 + y^2 + z^2 = 1$ that is bounded by its projection on xy–plane.

By Stoke's theorem, we have,

$$\iint_s (\nabla\times\vec{F}).\vec{N}\,dxdy = \oint_c \vec{F}.d\vec{r} \quad \text{......... (i)}$$

where, $\vec{N} = \vec{r_x} \times \vec{r_y}$

Here, $$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ 2x - y & -yz^2 & -y^2z \end{vmatrix}$$

$$= (-2yz + 2yz)\,\vec{i} + (0 - 0)\,\vec{j} + (0 + 1))\,\vec{k} = (0, 0, 1).$$

Since we have,

Then, $\vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k} = x\,\vec{i} + y\,\vec{j} + \sqrt{1 - x^2 - y^2}\,\vec{k}$

$$\vec{r_x} = \vec{i} - \frac{x}{\sqrt{1-x^2-y^2}}\vec{k} \qquad \text{and } \vec{r_y} = \vec{j} - \frac{y}{\sqrt{1-x^2-y^2}}\vec{k}$$

So that, $\vec{N} = \vec{r_x} \times \vec{r_y}$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -x/\sqrt{1=x^2-y^2} \\ 0 & 1 & -y/\sqrt{1-x^2-y^2} \end{vmatrix} = \frac{x}{\sqrt{1-x^2-y^2}}\vec{i} + \frac{y}{\sqrt{1-x^2-y^2}}\vec{j} + \vec{k}.$$

Then,

$$(\nabla \times \vec{F}).\vec{N} = 1.$$

Since the region is the projection of $x^2 + y^2 + z^2 = 1$ on xy–plane. So the region of integration is $x^2 + y^2 = 1$, $z = 0$.

This is a circle with radius $r = 1$.

Setting, $x = \text{Cos}\theta$ and $y = \text{Sin}\theta$ then $dx\,dy = r\,dr\,d\theta$. Also, θ varies from $\theta = 0$ to $\theta = 2\pi$. Then,

$$\iint_s (\nabla \times \vec{F}).\vec{N}\,dx\,dy = \int_0^{2\pi}\int_0^1 r\,dr\,d\theta = \int_0^{2\pi}\left[\frac{r^2}{2}\right]_0^1 d\theta = \frac{1}{2}[\theta]_0^{2\pi} = \frac{1}{2}2\pi = \pi.$$

Then (i) gives, $\oint_c \vec{F}.d\vec{r} = \pi.$

10. $\vec{F} = (y^2, x^2, (x + z))$, C is the boundary of the triangle with vertices at (0, 0, 0), (1, 0, 0) and (1, 1, 0).

Solution: Similar to Q. No. 7.

11. $\vec{F} = y^2\vec{i} + z^2\vec{j} + x^2\vec{k}$, S is the first octant portion of the plane $x + y + z = 1$.

Solution: Given that $\vec{F} = y^2\vec{i} + z^2\vec{j} + x^2\vec{k}$.

And the surface is the portion of the plane $x + y + z = 1$ in the first octant.

By Stoke's theorem we have,

$$\iint_s (\nabla \times \vec{F}).\vec{N}\,dxdy = \oint_c \vec{F}.d\vec{r} \quad \text{.........(i)}$$

where $\vec{N} = \vec{r_x} \times \vec{r_y}$.

Here,
$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ y^2 & z^2 & x^2 \end{vmatrix} = = -2z\vec{i} - 2x\vec{j} - 2y\vec{k}.$$

Since we have,

Then, $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = x\vec{i} + y\vec{j} + (1 - x - y)\vec{k}$ $\quad$ [$\because x + y + z = 1$]

So that, $\vec{r_x} = \vec{i} - \vec{k}$ and $\vec{r_j} = \vec{j} - \vec{k}$.

$$\vec{N} = \vec{r_x} \times \vec{r_y} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \vec{i} + \vec{j} + \vec{k}.$$

Therefore,

$$\begin{aligned}(\nabla \times \vec{F}).\vec{N} &= (-2z\vec{i} - 2x\vec{j} - 2y\vec{k}).(\vec{i} + \vec{j} + \vec{k}) \\ &= -2z - 2x - 2y \\ &= -2(x + y + z) = -2(1) = -2.\end{aligned}$$

The projection of the surface plane $x + y + z = 1$ on xy–plane is $x + y = 1$, $z = 0$. In which y varies from $y = 0$ to $y = 1 - x$ and x moves from $x = 0$ t $x = 1$.

Therefore,

$$\begin{aligned}\iint_s (\nabla \times \vec{F}).\vec{N}\,dxdy &= \int_0^1\int_0^{1-x} (-2)\,dydx \\ &= -2\int_0^1 [y]_{-1}^{1-x}\,dx \\ &= -2\int_0^1 (1-x)dx = -2\left[x - \frac{x^2}{2}\right]_0^1 = -2\left(1 - \frac{1}{2}\right) = -1.\end{aligned}$$

Thus, by (i), $\oint_c \vec{F}.d\vec{r} = -1.$

12. $\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$, S is the hemisphere $z = (a^2 - x^2 - y^2)^{1/2}$.

Solution: Given that $\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$.

And the surface is a hemisphere, $z = (a^2 - x^2 - y^2)^{1/2}$.

By Stoke's theorem we have,
$$\iint_s (\nabla \times \vec{F}).\vec{N}\,dxdy = \oint_c \vec{F}.d\vec{r} \quad \text{......... (i)}$$

where, $\vec{N} = \vec{r_x} \times \vec{r_y}$.

Here,
$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ z & x & y \end{vmatrix} = \vec{i} + \vec{j} + \vec{k}$$

Since we have,

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = x\vec{i} + y\vec{j} + \sqrt{a^2 - x^2 - y^2}\,\vec{k}$$

Then,

$$\vec{r_x} = \vec{i} - \frac{x}{\sqrt{a^2 - x^2 - y^2}}, \qquad \vec{r_y} = \vec{j} - \frac{y}{\sqrt{a^2 - x^2 - y^2}}$$

So that,

$$\vec{N} = \vec{r_x} \times \vec{r_y}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -x/\sqrt{a^2-x^2-y^2} \\ 0 & 1 & -y/\sqrt{a^2-x^2-y^2} \end{vmatrix} = \frac{x}{\sqrt{a^2-x^2-y^2}}\vec{i} + \frac{y}{\sqrt{a^2-x^2-y^2}}\vec{j} + \vec{k}$$

Then, $(\nabla\times\vec{F}).\vec{N} = \dfrac{x}{\sqrt{a^2-x^2-y^2}} + \dfrac{y}{\sqrt{a^2-x^2-y^2}} + 1 = \dfrac{x+y}{\sqrt{a^2-x^2-y^2}} + 1.$

Given surface is a hemisphere $z = \sqrt{a^2-x^2-y^2}$ that has radius $r = a$.

Set $x = r\text{Cos}\theta$, $y = r\,\text{Sin}\theta$ then $dx\,dy = r\,dr\,d\theta$. And the angular region moves from $\theta = 0$ to $\theta = 2\pi$.

Then,

$$\iint_S (\nabla\times\vec{F}).\vec{N}\,dxdy = \int_0^{2\pi}\int_0^a \left[\frac{[r(\cos\theta+\sin\theta)]}{\sqrt{a^2-r^2(\cos^2\theta+\sin^2\theta)}} + 1\right] r\,dr\,d\theta$$

$$= \int_0^a\int_0^{2\pi}\left(\frac{[r^2(\cos\theta+\sin\theta)]}{\sqrt{a^2-r^2}} + r\right) d\theta\,dr$$

$$= \int_0^a \left[\frac{r^2(\sin\theta-\cos\theta)}{\sqrt{a^2-r^2}} + r\,\theta\right]_0^{2\pi}$$

$$= \int_0^a \left[\frac{r^2\times 0}{\sqrt{a^2-r^2}} + 2r\pi\right] dr = \int_0^a (2r\pi)\,dr = \pi[r^2]_0^a = \pi a^2$$

Thus, by (i), $\oint_C \vec{F}.d\vec{r} = \pi a^2.$

13. $\vec{F} = 2y\vec{i} + e^z\vec{j} - \tan^{-1}x\vec{k}$ **and S is the portion of the paraboloid** $z = 4 - x^2 - y^2$ **cut off by the xy–plane.**

Solution: Given that $\vec{F} = 2y\vec{i} + e^z\vec{j} - \tan^{-1}x\vec{k}$

and the surface is $z = 4 - x^2 - y^2$ that cut off by xy–plane.

By stake's theorem we have,

$$\oint_C \vec{F}.d\vec{r} = \iint_S (\text{curl }\vec{F}.\vec{N})\,ds \qquad \ldots\ldots\ldots \text{(i)}$$

where, $\vec{N} = \vec{r}_x\times\vec{r}_y = \vec{k} = (0, 0, 1).$

Here,

$$\text{curl}.\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2y & e^z & -\tan^{-1}x \end{vmatrix} = \left(e^z, \frac{1}{1+x^2}, -2\right).$$

Then, curl $\vec{F}.\vec{N} = -2.$

Given that the surface $z = 4 - x^2 - y^2$ is cut off by xy–plane. So, on the projection of the surface in xy–plane is $x^2 + y^2 = 4$. This is a circle with radius 2 and angular variation is 2π. Therefore, (i) becomes

$$\oint_C \vec{F}.d\vec{r} = -2\int_0^2\int_0^{2\pi} r\, d\theta\, dr \quad \text{being the parabolid is downward}$$

$$= -2\left[\frac{r^2}{2}\right]_0^2 [\theta]_0^{2\pi} = -2\left(\frac{4-0}{2}\right)(2\pi - 0) = -8\pi.$$

14. $\vec{F} = y^2\vec{i} + 2x\vec{j} + 5y\vec{k}$, **S is the hemisphere** $z = (4 - x^2 - y^2)^{1/2}$.

Solution: Similar to Q. No. 12