Exercise 10.1

1. **Find the first partial derivatives of F.**

(i) $\mathbf{f(x, y) = 2x^4y^3 - xy^2 + 3y + 1}$

Solution: Given that, $f(x, y) = 2x^4y^3 - xy^2 + 3y + 1$.

Then, $\dfrac{\partial f}{\partial x} = 8x^3y^3 - y^2$ and $\dfrac{\partial f}{\partial y} = 6x^4y^2 - 2xy + 3$.

(ii) $\mathbf{f(x, y) = xe^y + y\sin x}$

Solution: Given that, $f(x, y) = x\,e^y + y.\sin x$

Then, $\dfrac{\partial f}{\partial x} = e^y + y\cos x$ and $\dfrac{\partial f}{\partial y} = xe^y + \sin x$.

(iii) $\mathbf{f(x, y) = x\cos\left(\frac{x}{y}\right)}$

Solution: Given that, $f(x, y) = x\cos\left(\dfrac{x}{y}\right)$

Then, $$\frac{\partial f}{\partial x} = x\frac{\partial\left\{\cos\left(\frac{x}{y}\right)\right\}}{\partial\left(\frac{x}{y}\right)} \times \frac{\partial\left(\frac{x}{y}\right)}{\partial x} + \cos\left(\frac{x}{y}\right).1$$

$$= -\frac{x}{y}\sin\left(\frac{x}{y}\right) + \cos\left(\frac{x}{y}\right) = \cos\left(\frac{x}{y}\right) - \frac{x}{y}\sin\left(\frac{x}{y}\right).$$

and $$\frac{\partial f}{\partial y} = x\frac{\partial\left\{\cos\left(\frac{x}{y}\right)\right\}}{\partial\left(\frac{x}{y}\right)} \times \frac{\partial\left(\frac{x}{y}\right)}{\partial y} = \frac{x^2}{y^2}\left(\frac{x}{y}\right).$$

(iv) $f(x, y, z) = 3x^2z + xy^2$

Solution: Given that, $f(x, y, z) = 3x^2z + xy^2$

Then, $\frac{\partial f}{\partial x} = 6xz + y^2$ $\quad \frac{\partial f}{\partial y} = 2xy$ $\quad$ and $\quad \frac{\partial f}{\partial z} = 3x^2$

(v) $f(r, s, t) = r^2 e^{2s} \cos t$

Solution: Given that, $f(r, s, t) = r^2 e^{2s} \cos t$

Then, $\frac{\partial f}{\partial r} = 2re^{2s} \cos t$ $\quad \frac{\partial f}{\partial s} = 2r^2e^{2s} \cos t$ $\quad$ and $\quad \frac{\partial f}{\partial t} = -r^2e^{2s} \sin t.$

(vi) $f(x, y, z) = xe^z - ye^x + ze^{-y}$

Solution: Given that, $f(x, y, z) = xe^z - ye^x + ze^{-y}$

Then, $\frac{\partial f}{\partial x} = e^z - ye^x$ $\quad \frac{\partial f}{\partial y} = -e^x - ze^{-y}$ $\quad$ and $\quad \frac{\partial f}{\partial z} = xe^z + e^{-y}$

2. Verify that $u_{xy} = u_{yx}$

(i) $u = xy^4 - 2x^2y^3 - 4x^2 + 3x$

Solution: Given that, $u = xy^4 - 2x^2y^3 - 4x^2 + 3x.$

Differentiating,

$u_x = y^4 - 4xy^3 - 8x + 3$ $\quad$ and $\quad u_y = 4xy^3 - 6x^2y^2$

$(u_x)_y = 4y^3 - 12xy^2$ $\quad (u_y)_x = 4y^3 - 12xy^2$

This shows that, $u_{xy} = u_{yx}$.

(ii) $u = x^3e^{-2y} + y^{-2} \cos x$

Solution: Given that, $u = x^3e^{-2y} + y^{-2} \cos x$

Differentiating,

$u_x = 3x^2e^{-2y} - y^{-2} \sin x$ $\quad$ and $\quad u_y = -2x^3e^{-2y} - 2y^{-3}\cos x$

$u_{xy} = -6x^2e^{-2y} + 2y^{-3}\sin x$ $\quad u_{yx} = -6x^2e^{-2y} + 2y^{-3}\sin x$

This shows that, $u_{xy} = u_{yx}$

(iii) $u = \frac{x^2}{x + y}$

Solution: Given that, $u = \frac{x^2}{x + y}$

Differentiating,

$$u_x = \frac{(x + y)\, 2x - x^2}{(x + y)^2}$$

$$= \frac{2x^2 + 2xy - x^2}{(x + y)^2} = \frac{x^2 + 2xy}{(x + y)^2}$$

$$u_{xy} = \frac{(x + y)^2\, 2x - (x^2 + 2xy)\, 2\,(x + y).1}{(x + y)^4}$$

$$= \frac{(x + y)^2\, 2x - (x^2 + 2xy)\,(2x + 2y)}{(x + y)^4}$$

$$= \frac{2x^3 + 4x^2y - 2x^3 - 6x^2y - 2xy^2 - 2xy^2}{(x + y)^4} = -\frac{(2x^2y + 2xy^2)}{(x + 4)^4} = -\frac{2xy}{(x + y)^3}$$

And,

$$u_y = \frac{(x + y) \times 0 - x^2.1}{(x + y)^2} = \frac{-x^2}{(x + y)^2}$$

$$u_{yx} = \frac{-\{(x + y)^2 .\, 2x - x^2\, 2(x + y).1}{\{(x + y)^2\}^2}$$

$$= -\frac{(2x^3 + 4x^2y + 2xy^2 - 2x^3 - 2x^2y)}{(x + y)^4} = -\frac{(2x^2y + 2xy^2)}{(x + y)^4} = -\frac{2xy}{(x + 4)^4}$$

This shows that, $u_{xy} = u_{yx}$.

(iv) $u = y^2 e^{x^2} + \frac{1}{x^2y^3}$

Solution: Here, $u = y^2 e^{x^2} + \frac{1}{x^2y^3}$

So, $u_x = 2xe^{x^2}y^2 + \frac{-2x^{-3}}{y^3}$ $\quad$ and $\quad u_y = 2y\, e^{x^2} + \frac{-3y^{-4}}{x^2}$

Also,

$u_{xy} = 4xy\, e^{x^2} + 6x^{-3}y^{-4}$ $\quad$ and $\quad u_{yx} = 4xye^{x^2} + 6x^{-3}\, y^{-4}$

Thus, $u_{xy} = u_{yx}$.

(v) $u = \sqrt{x^2 + y^2 + z^2}$

Solution: Given that, $u = \sqrt{x^2 + y^2 + z^2}$

Differentiating,

$$u_x = \frac{1}{2\sqrt{x^2 + y^2 + z^2}} \times 2x = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$$

$$u_{xy} = \frac{\sqrt{x^2 + y^2 + z^2} \times 0 - x\frac{1}{2\sqrt{x^2 + y^2 + z^2}} \times 2y}{(x^2 + y^2 + z^2)} = -\frac{xy}{(x^2 + y^2 + z^2)^{3/2}}$$

And,

$$u_y = \frac{1}{2\sqrt{x^2 + y^2 + z^2}} \times 2y = \frac{y}{\sqrt{x^2 + y^2 + z^2}}$$

$$u_{yx} = \frac{\sqrt{x^2 + y^2 + z^2} \times 0 - y\frac{1}{2\sqrt{x^2 + y^2 + z^2}} \times 2x}{(x^2 + y^2 + z^2)} = -\frac{xy}{(x^2 + y^2 + z^2)^{3/2}}$$

This shows that, $u_{xy} = u_{yx}$.

(vi) $u = 3x^2y^3z + 2xy^4z^2 - yz$

Solution: Given that, $u = 3x^2y^3z + 2xy^4z^2 - yz$

Differentiating,

$u_x = 6xy^3z + 2y^4z^2$ $\quad$ and $\quad u_y = 9x^2y^2z + 8xy^3z^2 - z$

$$u_{xy} = 18xy^2z + 8y^3z^2 \qquad u_{yx} = 18xy^2z + 8y^3z^2$$

This shows that, $u_{xy} = u_{yx}$.

(vii) $\mathbf{u = \sin^{-1}\left(\frac{y}{x}\right)}$

Solution: Given that, $u = \sin^{-1}\left(\frac{y}{x}\right)$

Differentiating,

$$u_x = \frac{1}{\sqrt{1-\left(\frac{y}{x}\right)^2}} \times -\frac{y}{x^2}$$

$$= -\frac{y}{x^2\frac{\sqrt{x^2-y^2}}{x}} = -\frac{y}{x\sqrt{x^2-y^2}} = -\frac{y}{\sqrt{x^4-x^2y^2}}$$

$$u_{xy} = -\frac{\left\{x\sqrt{x^2-y^2}\,.\,1 - y\frac{1}{2\sqrt{x^4-x^2y^2}} \times -2x^2y\right\}}{(x^4-x^2y^2)}$$

$$= -\frac{x^2(x^2-y^2)+x^2y^2}{(x^4-x^2y^2)^{3/2}} = -\frac{x^4}{(x^4-x^2y^2)^{3/2}} = \frac{x}{(x^2-y^2)^{3/2}}$$

and,

$$u_y = \frac{1}{\sqrt{1-\left(\frac{y}{x}\right)^2}} \times \frac{1}{x} = \frac{x}{\sqrt{x^2-y^2}} \times \frac{1}{x} = \frac{1}{\sqrt{x^2-y^2}}$$

$$u_{yx} = \frac{\sqrt{x^2-y^2}\,.\,d(1) - \frac{1}{2\sqrt{x^2-y^2}} \times 2x}{(x^2-y^2)} = -\frac{x}{(x^2-y^2)^{3/2}}$$

This shows that, $u_{xy} = u_{yx}$.

(viii) $\mathbf{u = \log(x)\tan^{-1}(x^2+y^2)}$

Solution: Given that, $u = \log(x)\tan^{-1}(x^2+y^2)$

Differentiating,

$$u_x = \tan^{-1}(x^2+y^2)\frac{1}{x} + \log x\frac{1}{1+(x^2+y^2)^2} \times 2x$$

$$= \frac{\tan^{-1}(x^2+y^2)}{x} + \frac{2x\log x}{1+(x^2+y^2)^2}$$

$$u_{xy} = \frac{x\frac{1}{1+(x^2+y^2)^2} \times 2y - \tan^{-1}(x^2+y^2) \times 0}{x^2} + \frac{\{1+(x^2+y^2)^2\}\{(2\log x+2)\} - 2x\log x \times 2(x^2+y^2) \times 2y}{\{1+(x^2+y^2)^2\}^2}$$

$$= \frac{2xy}{x^2\{1+(x^2+y^2)^2\}} + \frac{2(1+\log x)\{1+(x^2+y^2)^2\} - 8xy\log x\,(x^2+y^2)}{\{1+(x^2+y^2)^2\}^2}$$

and,

$$u_y = \log x\frac{1}{1+(x^2+y^2)^2} \times 2y = \frac{2y\log x}{1+(x^2+y^2)^2}$$

$$u_{yx} = \frac{\{1+(x^2+y^2)^2\}\,2\frac{y}{x} - 2y\log x\,2(x^2+y^2) \times 2x}{\{1+(x^2+y^2)^2\}^2}$$

$$= \frac{\frac{2y}{x}\{1+(x^2+y^2)^2\} - 8xy\log x\,(x^2+y^2)}{\{1+(x^2+y^2)^2\}^2}$$

This shows that, $u_{xy} = u_{yx}$.

(ix) $\mathbf{u = ax^2 + 2hxy + by^2}$

Solution: Given that, $u = ax^2 + 2hxy + by^2$

Differentiating,

$$u_x = 2ax + 2hy \quad \text{and} \quad u_y = 2hx + 2by$$

$$u_{xy} = 2h \qquad u_{yx} = 2h$$

This shows that, $u_{xy} = u_{yx}$.

(x) $\mathbf{u = \tan^{-1}\left(\frac{y}{x}\right)}$

Solution: Given that, $u = \tan^{-1}\left(\frac{y}{x}\right)$

Differentiating,

$$u_x = \frac{1}{1+\left(\frac{y}{x}\right)^2} \times -\frac{y}{x^2} = -\frac{y}{x^2} \times \frac{x^2}{(x^2+y^2)} = -\frac{y}{(x^2+y^2)}$$

$$u_{xy} = -\frac{[(x^2+y^2)\,.\,1 - y\,.\,2y]}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

And,

$$u_y = \frac{1}{1+\frac{y^2}{x^2}} \times \frac{1}{x} = \frac{x}{x^2+y^2}$$

$$u_{yx} = \frac{(x^2+y^2)\,.\,1 - x\,.\,2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

This shows that, $u_{xy} = u_{yx}$.

(xi) $\mathbf{u = \log\left(\frac{x^2+y^2}{xy}\right)}$

Solution: Given that, $u = \log\left(\frac{x^2+y^2}{xy}\right)$

Differentiating,

$$u_x = \frac{\frac{1}{x^2+y^2}}{xy} \times \frac{xy \,.\, 2x - (x^2+y^2) \,.\, y}{x^2y^2}$$

$$= \frac{2x^2y - x^2y - y^3}{xy(x^2+y^2)} = \frac{x^2y - y^3}{xy(x^2+y^2)} = \frac{y(x^2-y^2)}{xy(x^2+y^2)} = \frac{(x^2-y^2)}{(x^3+xy^2)}$$

$$u_{xy} = \frac{(x^3+xy^2) \times -2y - (x^2-y^2) \times 2xy}{(x^3+xy^2)^2}$$

$$= \frac{-2x^3y - 2xy^3 - 2x^3y + 2xy^3}{(x^3+xy^2)^2}$$

$$= -\frac{4x^3y}{x^2(x^2+y^2)^2} = -\frac{4xy}{(x^2+y^2)^2} = -\frac{4x^3y}{(x^3+xy^2)^2}$$

And,

$$u_y = \frac{\frac{1}{x^2+y^2}}{xy} \times \frac{xy \,.\, 2y - (x^2+y^2).x}{x^2y^2}$$

$$= \frac{2xy^2 - xy^2}{xy(x^2+y^2)} = \frac{xy^2 - x^3}{xy(x^2+y^2)} = \frac{x(y^2-x^2)}{xy(x^2+y^2)} = \frac{y^2-x^2}{y(x^2+y^2)}$$

$$u_{yx} = \frac{y(x^2+y^2) \times -2x - (y^2-x^2)\,2xy}{\{y(x^2+y^2)\}^2}$$

$$= \frac{-2x^3y - 2xy^3 - 2xy^3 + 2x^3y}{y^2(x^2+y^2)^2} = \frac{-4xy^3}{y^2(x^2+y^2)^2} = \frac{-4xy}{(x^2+y^2)^2}$$

This shows that, $u_{xy} = u_{yx}$.

(xii) $u = e^{ax}$. sin by

Solution: Given that, $u = e^{ax}$. sin by

Differentiating,

$u_x = ae^{ax}$ sin by and $u_y = be^{ax}$ cos by

$u_{xy} = abe^{ax}$ cosby $u_{yx} = abe^{ax}$. cosby

This shows that, $u_{xy} = u_{yx}$.

(xiii) $u = \log(x \sin y + y \sin x)$

Solution: Given that, $u = \log(x \sin y + y \sin x)$

Differentiating,

$$u_x = \frac{1}{(x \sin y + y\sin x)}(\sin y + y\cos x)$$

$$u_{xy} = \frac{(x \sin y + y \sin x)(\cos y + \cos x) - (\sin y + y\cos x)(\cos y + \sin x)}{(x\sin y + y\sin x)^2}$$

$$= \frac{(x\sin y \cos y + x\sin y \cos x + y\sin x \cos y + y\sin x.\cos x) - (x\sin y.\cos y + \sin x.\sin y + x y\cos x.\cos y + y\sin x.\cos x)}{(x\sin y + y\sin x)^2}$$

$$= \frac{x\sin y \cos y + x\sin y \cos x + y\sin x \cos y + y\sin x \cos x - x\sin y \cos y - \sin x \sin y - xy\cos x \cos y - y\sin x.\cos x}{(x\sin y + y\sin x)^2}$$

$$= \frac{x\sin y \cos x + y\sin x \cos y - \sin x \sin y - xy\cos x \cos y}{(x\sin y + y\sin x)^2}$$

And,

$$u_y = \frac{1}{(x\sin y + y\sin x)}(x\cos y + \sin x)$$

$$u_{yx} = \frac{(x\sin y + y\sin x)(\cos y + \cos x) - (x\cos y + \sin x)(\sin y + y\cos x)}{(x\sin y + y\sin x)^2}$$

This shows that, $u_{xy} = u_{yx}$.

(xiv) $u = e^{x^2} + xy + y^2$

Solution: Given that, $u = e^{x^2} + xy + y^2$

Differentiating,

$u_x = 2xe^{x^2} + y$ and $u_y = x + 2y$

$u_{xy} = 1$ $u_{yx} = 1$

This shows that, $u_{xy} = u_{yx}$.

3. If $u = x^2 + y^2 + z^2$ show that $ux_x + yu_y + zu_z = 2u$.

Solution: Given that, $u = x^2 + y^2 + z^2$

Differentiating,

$u_x = 2x$ $u_y = 2y$ and $u_z = 2z$

Now,

$$xu_x + yu_y + zu_z = x \times 2x + y \times 2y + z \times 2z$$
$$= 2x^2 + 2y^2 + 2z^2$$
$$= 2(x^2 + y^2 + y^2) = 2u$$

Thus, $ux_x + yu_y + zu_z = 2u$.

4. If $u = x^2y + y^2z + z^2x$ show that $u_x + u_y + u_z = (x + y + z)^2$.

Solution: Given that, $u = x^2y + y^2z + z^2x$

Differentiating,

$u_x = 2xy + z^2$ $u_y = x^2 + 2yz$, $u_z = y^2 + 2zx$

Now,

$$u_x + u_y + u_z = 2xy + z^2 + x^2 + 2yz + y^2 + 2zx$$
$$= (x^2 + y^2 + z^2 + 2xy + 2yz + 2zx)$$
$$= (x + y + z)^2$$

Thus, $u_x + u_y + u_z = (x + y + z)^2$.

5. (i) If $f(x, y, z) = e^{x/y} + e^{y/z} + e^{z/x}$ show that $x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} + z\frac{\partial F}{\partial z} = 0$.

[2004 Fall Q. No. 2(b)]

Solution: Given that, $f(x, y, z) = e^{x/y} + e^{y/z} + e^{z/x}$

Differentiating,

$$\frac{\partial F}{\partial x} = \frac{1}{y} e^{x/y} - \frac{z}{x^2} e^{z/x} \quad \Rightarrow x.\frac{\partial F}{\partial x} = \frac{x}{y} e^{x/y} - \frac{z}{x} e^{z/x}$$

$$\frac{\partial F}{\partial y} = -\frac{x}{y^2} e^{x/y} + \frac{1}{z} e^{y/z} \quad \Rightarrow y.\frac{\partial F}{\partial y} = \frac{x}{y} e^{y/z} - \frac{x}{y} e^{x/y}$$

And $\frac{\partial F}{\partial z} = -\frac{y}{z^2} e^{y/z} + \frac{1}{x} e^{z/x} \quad \Rightarrow z.\frac{\partial F}{\partial z} = \frac{z}{x} e^{z/x} - \frac{y}{z} e^{y/x}$

Now,

$$x.\frac{\partial F}{\partial x} + y.\frac{\partial F}{\partial y} + z.\frac{\partial F}{\partial z} = \frac{x}{y} e^{x/y} - \frac{z}{x} e^{z/x} + \frac{x}{y} e^{y/z} - \frac{x}{y} e^{x/y} + \frac{z}{x} e^{z/x} - \frac{y}{z} e^{y/x}$$

$$= 0.$$

Thus, $x.\frac{\partial F}{\partial x} + y.\frac{\partial F}{\partial y} + z.\frac{\partial F}{\partial z} = 0.$

(ii) If $V = (\sqrt{x^2 + y^2 + z^2})$ show that $V_{xx} + V_{yy} + V_{zz} = \frac{2}{V}$.

Solution: Given that, $V = (\sqrt{x^2 + y^2 + z^2})$.

Differentiating,

$$V_x = \frac{1}{2\sqrt{x^2 + y^2 + z^2}} \times 2x = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$$

$$V_{xx} = \frac{(x^2 + y^2 + z^2)^{1/2} . 1 - x \frac{1}{2\sqrt{x^2 + y^2 + z^2}} \times 2x}{(x^2 + y^2 + z^2)}$$

$$= \frac{x^2 + y^2 + z^2 - x^2}{(x^2 + y^2 + z^2)^{3/2}} = \frac{y^2 + z^2}{\sqrt{x^2 + y^2 + z^2}^{3/2}}$$

And, $V_y = \frac{1}{2\sqrt{x^2 + y^2 + z^2}} \times 2y = \frac{y}{\sqrt{x^2 + y^2 + z^2}}$

$$V_{yy} = \frac{(x^2 + y^2 + z^2)^{1/2} . 1 - y \frac{1}{2\sqrt{x^2 + y^2 + z^2}} \times 2y}{(x^2 + y^2 + z^2)} = \frac{x^2 + z^2}{\sqrt{(x^2 + y^2 + z^2)^{3/2}}}$$

Also,

$$V_z = \frac{1}{2\sqrt{x^2 + y^2 + z^2}} \times 2z = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$V_{zz} = \frac{(x^2 + y^2 + z^2)^{1/2} . 1 - z \frac{1}{2\sqrt{x^2 + y^2 + z^2}} \times 2z}{(x^2 + y^2 + z^2)} = \frac{x^2 + y^2}{\sqrt{(x^2 + y^2 + z^2)^{3/2}}}$$

Now,

$$V_{xx} + V_{yy} + V_{zz} = \frac{y^2 + z^2 + x^2 + z^2 + x^2 + y^2}{(x^2 + y^2 + z^2)^{3/2}}$$

$$= \frac{2(x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)\sqrt{x^2 + y^2 + z^2}} = \frac{2}{V}$$

Thus, $V_{xx} + V_{yy} + V_{zz} = \frac{2}{V}$.

(6) If $u = \log(\sqrt{x^2 + y^2 + z^2})$ show that: $(x^2 + y^2 + z^2)\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y2} + \frac{\partial^2 u}{\partial z^2}\right] = 1.$

[2007 Fall Q. No. 2(b)]

Solution: Given that, $u = \log(\sqrt{x^2 + y^2 + z^2})$

Differentiating,

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{x^2 + y^2 + z^2}} \times \frac{2x}{2\sqrt{x^2 + y^2 + z^2}} = \frac{x}{(x^2 + y^2 + z^2)}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2 + y^2 + z^2). 1 - x(2x)}{(x^2 + y^2 + z^2)^2} = \frac{y^2 + z^2 - x^2}{(x^2 + y^2 + z^2)^2}$$

Similarly,

$$\frac{\partial^2 u}{\partial y^2} = \frac{x^2 + z^2 - y^2}{(x^2 + y^2 + z^2)^2} \quad \text{And} \quad \frac{\partial^2 u}{\partial z^2} = \frac{x^2 + y^2 - z^2}{(x^2 + y^2 + z^2)^2}$$

Now,

$$(x^2 + y^2 + z^2)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$= (x^2 + y^2 + z^2)\left(\frac{y^2 + z^2 - x^2 + x^2 + z^2 - y^2 + x^2 + y^2 - z^2}{(x^2 + y^2 + z^2)^2}\right)$$

$$= \frac{(x^2 + y^2 + z^2)^2}{(x^2 + y^2 + z^2)^2} = 1.$$

Thus, $(x^2 + y^2 + z^2)\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y2} + \frac{\partial^2 u}{\partial z^2}\right] = 1.$

(7) If $V = f(x, y, z)$, show that $x^2 V_{xx} = y^2 V_{yy} = z^2 V_{zz}$

Solution: Given that, $V = F(x\ y\ z)$

Differentiating V w. r. t. x,

$V_x = F'(x\ y\ z)\ yz$ And $V_{xx} = F''(x\ y\ z)\ y^2 . z^2$

Multiplying by x^2 both sides,

$x^2 V_{xx} = F''(xyz)\ x^2y^2z^2$ ....(i)

Next, differentiating V w. r. t. y,

$V_y = F'(xyz). Xz$ and $V_{yy} = F''(xyz). x^2 z^2$

Multiplying by y^2

$y^2 V_{yy} = F''(xyz)\ x^2y^2z^2$ ....(ii)

Also, differentiating V w. r. t. z,

$V_z = F'(xyz). xy$ and $V_{zz} = F''(xyz). x^2y^2$

Multiplying by z^2

$z^2 V_{zz} = F''(xyz)\ x^2y^2z^2$ ....(iii)

Now, from (i), (ii) & (iii), we get,

$x^2 V_{xx} = y^2 V_{yy} = z^2 V_{zz}$

(8) If $x = r\cos\theta$, $y = r\sin\theta$, showthat

(i) $\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}=\frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2\right]$ (ii) $\frac{\partial^2 r}{\partial x^2}\cdot\frac{\partial^2 r}{\partial y^2}=\left(\frac{\partial^2 r}{\partial x.\partial y}\right)^2$.

Solution: Let, $x = r\cos\theta$ and $y = r\sin\theta$.

Then, $x^2+y^2=r^2$

Differentiating partially w. r. t. 'x' then

$$2x = 2r.\frac{\partial r}{\partial x} \Rightarrow \frac{\partial r}{\partial x}=\frac{x}{r}.$$

Again differentiating partially w. r. t. x, then,

$$\frac{\partial^2 r}{\partial x^2}=\frac{r.1-x.\frac{\partial r}{\partial x}}{r^2}=\frac{r-x.\frac{x}{r}}{r^2}\quad\left[\text{using }\frac{\partial r}{\partial x}=\frac{x}{r}\right]$$

$$=\frac{r^2-x^2}{r^3}=\frac{r^2(1-\cos^2\theta)}{r^3}\quad[\text{Being } x = r\cos\theta]$$

$$=\frac{r^2(\sin^2\theta)}{r^3}=\frac{y^2}{r^3}\quad[\text{Being } y = r\sin\theta]$$

Thus, $\frac{\partial^2 r}{\partial x^2}=\frac{y^2}{r^3}$

Similarly,

$$\frac{\partial r}{\partial y}=\frac{y}{r}\quad\text{and}\quad\frac{\partial^2 r}{\partial y^2}=\frac{x^2}{r^3}$$

Also, differentiating $\frac{\partial r}{\partial x}$ partially w. r. t. y, then,

$$\frac{\partial^2 r}{\partial x.\partial y}=\frac{\partial}{\partial y}\left(\frac{x}{r}\right)=\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial y}=-x.\frac{1}{r^2}\times\frac{y}{r}=\frac{-xy}{r^3}$$

Now,

(i) $\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}=\frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2\right]$

$\Rightarrow \frac{y^2}{r^3}+\frac{x^2}{r^3}=\frac{1}{r}\left[\left(\frac{x}{r}\right)^2+\left(\frac{y}{r}\right)^2\right]$

$\Rightarrow \frac{x^2+y^2}{r^3}=\frac{x^2+y^2}{r^3}$

This proves (i).

(ii) $\frac{\partial^2 r}{\partial x^2}\cdot\frac{\partial^2 r}{\partial y^2}=\left(\frac{\partial^2 r}{\partial x.\partial y}\right)^2$

$\Rightarrow \frac{y^2}{r^3}\cdot\frac{x^2}{r^3}=\left(-\frac{xy}{r^3}\right)^2$

$\Rightarrow \frac{x^2y^2}{r^6}=\frac{x^2y^2}{r^6}$

This proves (ii).

(9) If $u = x^2y + y^2z + z^2x$, show that: $u_x + u_y + u_z = (x + y + z)^2$.

Solution: Given that, $u = x^2y + y^2z + z^2x$

Differentiating we get,

$$\frac{\partial u}{\partial x}=2xy+z^2\qquad\frac{\partial u}{\partial y}=x^2+2yz\qquad\frac{\partial u}{\partial z}=y^2+2xy$$

Now,

$$u_x+u_y+u_z=2xy+z^2+x^2+2yz+y^2+2xz$$

$$=x^2+y^2+z^2+2xy+2yz+2xz$$
$$=(x+y+z)^2$$

Thus, $u_x + u_y + u_z = (x + y + z)^2$.

(10) If $u=\frac{x}{y}+\frac{y}{z}+\frac{z}{x}$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0$.

Solution: Let, $u=\frac{x}{y}+\frac{y}{z}+\frac{z}{x}$

Differentiating we get,

$$\frac{\partial u}{\partial x}=\frac{1}{y}-\frac{z}{x^2}\qquad\frac{\partial u}{\partial y}=\frac{1}{z}-\frac{x}{y^2}\qquad\frac{\partial u}{\partial z}=-\frac{y}{z^2}+\frac{1}{x}$$

Now,

$$x.\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=x\left(\frac{1}{y}-\frac{z}{x^2}\right)+y\left(\frac{1}{z}-\frac{x}{y^2}\right)+z\left(-\frac{y}{z^2}+\frac{1}{x}\right)$$

$$=\frac{x}{y}-\frac{z}{x}+\frac{y}{z}-\frac{x}{y}-\frac{y}{z}+\frac{z}{x}$$

$$=0$$

Thus, $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0$.

(11) If $u=\begin{vmatrix}x^2 & y^2 & z^2\\ x & y & z\\ 1 & 1 & 1\end{vmatrix}$. Show that: $u_x + u_y + u_z = 0$.

Solution: Let, $u=\begin{vmatrix}x^2 & y^2 & z^2\\ x & y & z\\ 1 & 1 & 1\end{vmatrix}=x^2(y-z)-y^2(x-z)+z^2(x-y)$

$$=x^2y-x^2z-xy^2+y^2z+xz^2-yz^2$$

Differentiating we get,

$$u_x=2xy-2xz-y^2+z^2$$
$$u_y=x^2-2xy+2yz-z^2$$

and, $u_z=-x^2+y^2+2xz-2yz$

Now,

$$u_x+u_y+u_z=2xy-2xz-y^2+z^2+x^2-2xy+2yz-z^2-x^2+y^2+2xz-2yz$$
$$=0.$$

Thus, $u_x + u_y + u_z = 0$.

(12) If $u = \log(e^x + e^y)$, show that $rt - s^2 = 0$ where $r=\frac{\partial^2 u}{\partial x^2}$, $s=\frac{\partial^2 u}{\partial x\partial y}$, $t=\frac{\partial^2 u}{\partial y^2}$.

Solution: Let, $u = \log(e^x + e^y)$

Differentiating we get,

$$\frac{\partial u}{\partial x}=\frac{1}{e^x+e^y}e^x=\frac{e^x}{e^x+e^y}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(e^x + e^y)\, e^x - e^x.\, e^x}{(e^x + e^y)^2} = \frac{e^{2x} + e^{xy} - e^{2x}}{(e^x + e^y)^2} = \frac{e^{xy}}{(e^x + e^y)^2}$$

And,

$$\frac{\partial u}{\partial y} = \frac{1}{e^x + e^y} e^y = \frac{e^y}{e^x + e^y}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{(e^x + e^y)\, e^y - e^y.\, e^y}{(e^x + e^y)^2} = \frac{e^{xy}}{(e^x + e^y)^2}$$

Also, $$\frac{\partial^2 u}{\partial x.\, \partial y} = \frac{(e^x + e^y)\, 0 - e^y.\, e^x}{(e^x + e^y)^2} = -\frac{e^{xy}}{(e^x + e^y)^2}$$

Now,

$$rt - s^2 = \frac{\partial^2 u}{\partial x^2}.\frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x.\, \partial y}\right)^2 = \frac{e^{xy}}{(e^x + e^y)^2} \cdot \frac{e^{xy}}{(e^x + e^y)^2} - \left(\frac{e^{xy}}{(e^x + e^y)^2}\right)^2$$

$$= \frac{e^{x^2 y^2}}{\{(e^x + e^y)\}^2} - \frac{e^{x^2 y^2}}{\{(e^x + e^y)\}^2}$$

$$= 0$$

Thus, $rt - s^2 = 0$.

(13) If $u = \tan^{-1} \dfrac{(xy)}{\sqrt{1 + x^2 + y^2}}$. Show that $\dfrac{\partial^2 u}{\partial x.\, \partial y} = (1 + x^2 + y^2)^{-3/2}$.

Solution: Let, $u = \tan^{-1} \dfrac{(xy)}{\sqrt{1 + x^2 + y^2}}$

Differentiating we get,

$$\frac{\partial u}{\partial x} = \frac{1}{1 + \left(\dfrac{(xy)}{\sqrt{1 + x^2 + y^2}}\right)^2} \times \frac{(1 + x^2 + y^2)^{1/2}.\, y - xy \dfrac{1}{2\sqrt{1 + x^2 + y^2}} \times 2x}{(1 + x^2 + y^2)}$$

$$= \frac{(1 + x^2 + y^2)}{1 + x^2 + y^2 + x^2 y^2} \times \frac{(1 + x^2 + y^2) y - x^2 y}{(1 + x^2 + y^2)^{3/2}}$$

$$= \frac{(1 + x^2 + y^2)\,(y + x^2 y + y^3 - x^2 y)}{(1 + x^2 + y^2 + x^2 y^2)\,(1 + x^2 + y^2)^{3/2}}$$

$$= \frac{(y + y^3)}{(1 + x^2 + y^2 + x^2 y^2)\,(1 + x^2 + y^2)^{1/2}}$$

$$= \frac{y(1 + y^2)}{(1 + y^2)\,(1 + x^2)\,(1 + x^2 + y^2)^{1/2}}$$

$$= \frac{y}{(1 + x^2)\,(1 + x^2 + y^2)^{1/2}}$$

And, $$\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{(1 + x^2)} \left\{\frac{(1 + x^2 + y^2)^{1/2}.\, 1 - y \dfrac{1}{2\sqrt{1 + x^2 + y^2}} \times 2y}{(1 + x^2 + y^2)}\right\}$$

$$= \frac{1}{(1 + x^2)} \left\{\frac{1 + x^2 + y^2 - y^2}{(1 + x^2 + y^2)^{3/2}}\right\}$$

$$= \frac{(1 + x^2)}{(1 + x^2)\,(1 + x^2 + y^2)^{3/2}}$$

$$= (1 + x^2 + y^2)^{-3/2}$$

Thus, $\dfrac{\partial^2 u}{\partial x.\, \partial y} = (1 + x^2 + y^2)^{-3/2}$.

(14) If $u = e^{xyz}$, show that $\dfrac{\partial^3 u}{\partial x\, \partial y\, \partial z} = (1 + 3xyz + x^2 y^2 z^2)\, e^{xyz}$.

Solution: Let, $u = e^{xyz}$

Differentiating we get,

$$\frac{\partial u}{\partial x} = e^{xyz}.\, yz$$

And, $$\frac{\partial^2 u}{\partial x \partial y} = yz.\, (e^{xyz}.\, xz) + e^{xyz}.\, z = xyz^2.\, e^{xyz} + e^{xyz}.\, z$$

Also, $$\frac{\partial^2 u}{\partial x \partial y \partial z} = xy\,(2z.\, e^{xyz} + e^z.e^{xyz}.xy) + e^{xyz}.xyz + e^{xyz}.1$$

$$= 2xyz\, e^{xyz} + x^2 y^2 z^2\, e^{xyz} + xyze^{xyz} + e^{xyz}$$

$$= e^{xyz}\,(2xyz + x^2 y^2 z^2 + xyz + 1)$$

$$= (1 + 3xyz + x^2 y^2 z^2)\, e^{xyz}$$

Thus, $\dfrac{\partial^3 u}{\partial x\, \partial y\, \partial z} = (1 + 3xyz + x^2 y^2 z^2)\, e^{xyz}$.

(15) If $u = \log(x^2 + y^2) + \tan^{-1}\left(\dfrac{y}{x}\right)$, show that: $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

Solution: Let, $u = \log(x^2 + y^2) + \tan^{-1}\left(\dfrac{y}{x}\right)$

Differentiating we get,

$$\frac{\partial u}{\partial x} = \frac{1}{x^2 + y^2} 2x + \frac{1}{1 + \dfrac{y^2}{x^2}} \times \left(-\frac{y}{x^2}\right) = \frac{2x}{x^2 + y^2} + \frac{-y}{x^2 + y^2} = \frac{2x - y}{x^2 + y^2}$$

And, $$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2 + y^2).\, 2 - (2x - y).\, 2x}{(x^2 + y^2)^2}$$

$$= \frac{2x^2 + 2y^2 - 4x^2 + 2xy}{(x^2 + y^2)^2} = \frac{2y^2 + 2xy - 2x^2}{(x^2 + y^2)^2}$$

Also, $$\frac{\partial u}{\partial y} = \frac{1}{x^2 + y^2} 2y + \frac{1}{1 + \dfrac{y^2}{x^2}} \times \frac{1}{x} = \frac{2y}{x^2 + y^2} + \frac{x}{x^2 + y^2} = \frac{x + 2y}{x^2 + y^2}$$

And, $$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2 + y^2).\, 2 - (x + 2y).\, 2y}{(x^2 + y^2)^2}$$

$$= \frac{2x^2 + 2y^2 - 2xy - 4y^2}{(x^2+y^2)^2} = \frac{2x^2 - 2xy - 2y^2}{(x^2+y^2)^2}$$

Now,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{2y^2 + 2xy - 2x^2 + 2x^2 - 2xy - 2y^2}{(x^2+y^2)^2} = 0.$$

Thus, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$

(16) If $u = \tan^{-1}\left(\frac{2xy}{x^2 - y^2}\right)$, show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

Solution: Let, $u = \tan^{-1}\left(\frac{2xy}{x^2 - y^2}\right)$

Differentiating we get,

$$\frac{\partial u}{\partial x} = \frac{1}{1 + \frac{4x^2y^2}{(x^2-y^2)^2}} \times \frac{(x^2-y^2)\,2y - 2xy\,.\,2x}{(x^2-y^2)^2}$$

$$= \frac{(x^2-y^2)^2}{(x^2-y^2)^2 + 4x^2y^2} \times \frac{2x^2y - 2y^3 - 4x^2y}{(x^2-y^2)^2}$$

$$= \frac{-2y^3 - 2x^2y}{(x^2+y^2)^2}$$

$$= \frac{-2y\,(x^2+y^2)}{(x^2+y^2)^2} = \frac{-2y}{x^2+y^2}$$

And,

$$\frac{\partial^2 u}{\partial x^2} = -\left(\frac{(x^2+y^2)\,.\,0 - 2y\,.\,2x}{(x^2+y^2)^2}\right) = \frac{4xy}{(x^2+y^2)^2}$$

Again,

$$\frac{\partial u}{\partial y} = \frac{1}{1 + \frac{4x^2y^2}{(x^2-y^2)^2}} \times \frac{(x^2-y^2)\,2y - 2x - 2xy.\,x - 2y}{(x^2-y^2)^2}$$

$$= \frac{(x^2-y^2)^2}{(x^2+y^2)^2} \times \frac{2x^3 - 2xy^2 + 4xy^2}{(x^2-y^2)^2}$$

$$= \frac{2x^3 + 2xy^2}{(x^2+y^2)^2}$$

$$= \frac{2x\,(x^2+y^2)}{(x^2+y^2)^2} = \frac{2x}{x^2+y^2}$$

Also,

$$\frac{\partial^2 u}{\partial y^2} = -\left(\frac{(x^2+y^2)\,.\,0 - 2x\,.\,2y}{(x^2+y^2)^2}\right) = -\frac{4xy}{(x^2+y^2)^2}$$

Now,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{4xy}{(x^2+y^2)^2} - \frac{4xy}{(x^2+y^2)^2} = 0$$

Thus, this proves the statement.

(17) If $u = \tan(y + ax) - (y - ax)^{3/2}$ show that $\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial^2 u}{\partial y^2}$.

Solution: Let, $u = \tan(y + ax) - (y - ax)^{3/2}$

Differentiating we get,

$$\frac{\partial u}{\partial x} = \sec^2(y + ax) \times a - \frac{3}{2}(y - ax)^{1/2} \times -a$$

$$= \frac{2a\sec^2(y + ax) + 3a(y - ax)^2}{2}$$

And,

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2}\left[2a.\sec(y + ax).\,\sec(y + ax)\tan(y + ax).a + 3a.\frac{1}{2\sqrt{y - ax}} \times -a\right]$$

Also,

$$\frac{\partial f}{\partial y} = \sec^2(y + ax).\,1 - \frac{3}{2}(y - ax)^{1/2}.\,1$$

And,

$$\frac{\partial^2 u}{\partial y^2} = 2\sec(y + ax).\sec(y + ax).\,\tan(y + ax).\,1 - \frac{3}{2} \times \frac{1}{2}(y - ax)^{-1/2}$$

$$= 2\sec^2(y + ax)\,.\,\tan(y + ax) - \frac{3}{4}(y - ax)^{-1/2}$$

Now,

$$\frac{\partial^2 u}{\partial x^2} = a^2\left[2\sec^2(y + ax).\,\tan(y + ax) - \frac{3}{4}(y - ax)^{-1/2}\right]$$

$$= a^2 \frac{\partial^2 u}{\partial y^2}$$

This proves the requirement.

(18) If $u = \log(x^2 + y^2 + z^2)$ show that $x\frac{\partial^2 u}{\partial x.\,\partial z} = y\frac{\partial^2 u}{\partial z.\,\partial x} = z\frac{\partial^2 u}{\partial x.\,\partial y}$.

Solution: Let, $u = \log(x^2 + y^2 + z^2)$

Differentiating we get,

$$\frac{\partial u}{\partial x} = \frac{1}{(x^2 + y^2 + z^2)} \times 2x = \frac{2x}{(x^2 + y^2 + z^2)}$$

And,

$$\frac{\partial^2 u}{\partial x.\,\partial y} = \frac{(x^2 + y^2 + z^2)\,.\,0 - 2x\,.\,2y}{(x^2 + y^2 + z^2)^2} = \frac{-4xy}{(x^2 + y^2 + z^2)^2}$$

Then,

$$z\frac{\partial^2 u}{\partial x.\,\partial y} = \frac{-4xyz}{(x^2 + y^2 + z^2)^2} \qquad \text{.......(i)}$$

Also,

$$\frac{\partial^2 u}{\partial x.\,\partial z} = \frac{(x^2 + y^2 + z^2) - 2x\,.\,2z}{(x^2 + y^2 + z^2)^2} = \frac{-4xz}{(x^2 + y^2 + z^2)^2}$$

Then,

$$y\frac{\partial^2 u}{\partial x.\,\partial z}=\frac{-4xyz}{(x^2+y^2+z^2)^2} \quad \text{.......(ii)}$$

Next,

$$\frac{\partial u}{\partial y}=\frac{1}{(x^2+y^2+z^2)}\times 2y-\frac{2y}{(x^2+y^2+z^2)}$$

And,

$$\frac{\partial^2 u}{\partial y.\,\partial z}=\frac{(x^2+y^2+z^2).\,0-2y.\,2z}{(x^2+y^2+z^2)^2}=\frac{-4yz}{(x^2+y^2+z^2)^2}$$

Then,

$$x\frac{\partial^2 u}{\partial y.\,\partial z}=\frac{-4xyz}{(x^2+y^2+z^2)^2} \quad \text{.......(iii)}$$

Now, from (i), (ii) and (iii), we observe,

$$x\frac{\partial^2 u}{\partial y.\,\partial z}=y\frac{\partial^2 u}{\partial x.\,\partial z}=z\frac{\partial^2 u}{\partial x.\,\partial y}.$$

(19) If $u=x^2\tan^{-1}\left(\frac{y}{x}\right)-y^2\left(\tan^{-1}\frac{x}{y}\right)$, show that $\frac{\partial^2 u}{\partial x\,\partial y}=\frac{x^2-y^2}{x^2+y^2}$.

Solution: Let, $u=x^2\tan^{-1}\left(\frac{y}{x}\right)-y^2\left(\tan^{-1}\frac{x}{y}\right)$

Differentiating we get,

$$\frac{\partial u}{\partial x}=2x.\tan^{-1}\left(\frac{y}{x}\right)+x^2\frac{1}{1+\frac{y^2}{x^2}}\times\left(-\frac{y}{x^2}\right)-y^2\frac{1}{1+\frac{x^2}{y^2}}\times\frac{1}{y}$$

$$=2x.\tan^{-1}\left(\frac{y}{x}\right)-\frac{x^2y}{x^2+y^2}-\frac{y^3}{x^2+y^2}$$

And,

$$\frac{\partial^2 u}{\partial x\,\partial y}=\left(2x.\frac{1}{1+\frac{y^2}{x^2}}\times\frac{1}{x}\right)-\left(\frac{(x^2+y^2)\,x^2-x^2y.\,2y}{(x^2+y^2)^2}\right)-\left(\frac{(x^2+y^2)\,3.y^2-y^3(2y)}{(x^2+y^2)^2}\right)$$

$$=\frac{2x^2}{x^2+y^2}-\frac{x^4+x^2y^2-2x^2y^2}{(x^2+y^2)^2}-\frac{3x^2y^2+3y^4-2y^4}{(x^2+y^2)^2}$$

$$=\frac{2x^2(x^2+y^2)-x^4+x^2y^2-3x^2y^2-y^4}{(x^2+y^2)^2}$$

$$=\frac{2x^4+2x^2y^2-x^4-2x^2y^2-y^4}{(x^2+y^2)^2}$$

$$=\frac{x^4-y^4}{(x^2+y^2)^2}=\frac{(x^2-y^2)\,(x^2+y^2)}{(x^2+y^2)^2}=\frac{x^2-y^2}{x^2+y^2}$$

Thus, $\frac{\partial^2 u}{\partial x\,\partial y}=\frac{x^2-y^2}{x^2+y^2}$.

(20) If $u=\sqrt{x^2+y^2+z^2}$, show that

(i) $\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=1$ (ii) $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{2}{u}$

Solution: Let, $u=\sqrt{x^2+y^2+z^2}$

Differentiating w. r. t. x we get,

$$\frac{\partial u}{\partial x}=\frac{1}{2\sqrt{x^2+y^2+z^2}}\times 2x=\frac{x}{\sqrt{x^2+y^2+z^2}}$$

And,

$$\frac{\partial^2 u}{\partial x^2}=\frac{(x^2+y^2+z^2)^{1/2}.\,1-x.\,\frac{1}{2\sqrt{x^2+y^2+z^2}}\times 2x}{(x^2+y^2+z^2)}$$

$$=\frac{x^2+y^2+z^2-x^2}{(x^2+y^2+z^2)^{3/2}}$$

$$=\frac{y^2+z^2-x^2}{(x^2+y^2+z^2)^{3/2}}$$

Similarly, differentiating w. r. t. y and z then we get,

$$\frac{\partial f}{\partial y}=\frac{y}{\sqrt{x^2+y^2+z^2}} \qquad \frac{\partial u}{\partial z}=\frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$\frac{\partial^2 u}{\partial y^2}=\frac{x^2+z^2}{(x^2+y^2+z^2)^{3/2}} \qquad \frac{\partial^2 u}{\partial z^2}=\frac{x^2+y^2}{(x^2+y^2+z^2)^{3/2}}$$

(i)

Now,

$$\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=\frac{x^2}{x^2+y^2+z^2}+\frac{y^2}{x^2+y^2+z^2}+\frac{z^2}{x^2+y^2+z^2}$$

$$=\frac{x^2+y^2+z^2}{x^2+y^2+z^2}=1.$$

Thus, $\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=1$.

(ii)

Now,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{y^2+z^2+x^2+z^2+x^2+y^2}{(x^2+y^2+z^2)^{3/2}}$$

$$=\frac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{3/2}}=\frac{2}{\sqrt{x^2+y^2+z^2}}=\frac{2}{u}$$

Thus, $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{2}{u}$.

(21) If $u=\frac{1}{\sqrt{t}}e^{\left(-\frac{x^2}{4a^2t}\right)}$, show that $\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}$.

Solution: Let,

$$u=\frac{1}{\sqrt{t}}e^{\left(-\frac{x^2}{4a^2t}\right)}$$

Then,

$$\frac{\partial u}{\partial t} = -\frac{1}{2}t^{-3/2}\, e^{\left(-\frac{x^2}{4a^2t}\right)} + \frac{1}{\sqrt{t}}\, e^{\left(-\frac{x^2}{4a^2t}\right)}\left(\frac{x^2}{4a^2t^2}\right)$$

And,

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{t}}\, e^{\left(-\frac{x^2}{4a^2t}\right)}\left(\frac{-2x}{4a^2t}\right)$$

Also,

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\sqrt{t}}\, e^{\left(-\frac{x^2}{4a^2t}\right)}\left(\frac{-2x}{4a^2t}\right)^2 + \frac{1}{\sqrt{t}}\, e^{\left(-\frac{x^2}{4a^2t}\right)}\left(\frac{-2x}{4a^2t}\right)$$

So,

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\sqrt{t}}\, e^{\left(-\frac{x^2}{4a^2t}\right)}\left(\frac{x^2}{4a^2t^2}\right) - \frac{1}{2}t^{-3/2}\, e^{\left(-\frac{x^2}{4a^2t}\right)} = \frac{\partial u}{\partial t}$$

$$\Rightarrow \quad \frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2}$$

(22) If $u = (1 - 2xy + y^2)^{-1/2}$, show that $x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = y^2u^3$

Solution: Let, $\quad u = (1 - 2xy + y^2)^{-1/2}$

Differentiating we get,

$$\frac{\partial u}{\partial x} = -\frac{1}{2}(1 - 2xy + y^2)^{-3/2} \times (-2y) \;= y(1 - 2xy + y^2)^{-3/2}$$

Then,

$$x.\frac{\partial u}{\partial x} = xy\,(1 - 2xy + y^2)^{-3/2} \quad \text{.......(i)}$$

Next,

$$\frac{\partial u}{\partial y} = -\frac{1}{2}(1 - 2xy + y^2)^{-3/2} \times (-2x + 2y).$$

Then,

$$y\frac{\partial u}{\partial y} = (xy - y^2)\,(1 - 2xy + y^2)^{-3/2} \quad \text{.......(ii)}$$

Now,

$$x.\frac{\partial u}{\partial x} - y.\frac{\partial u}{\partial y} = \frac{xy}{(1 - 2xy + y^2)^{3/2}} - \frac{xy - y^2}{(1 - 2xy + y^2)^{3/2}}$$

$$= \frac{xy - xy + y^2}{(1 - 2xy + y^2)^{3/2}}$$

$$= y^2\,(1 - 2xy + y^2)^{-1/2})^3 = y^2 \times u^3$$

Thus, $x.\frac{\partial u}{\partial x} - y.\frac{\partial u}{\partial y} = y^2 \times u^3$.

(23) If $u = e^x\,(x\cos y - y\sin y)$, show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

Solution: Let, $\quad u = e^x\,(x\cos y - y\sin y)$.

Differentiating we get,

$$\frac{\partial u}{\partial x} = \cos y\,(e^x + xe^x) - e^x y\sin y$$

And,

$$\frac{\partial^2 u}{\partial x^2} = \cos y(e^x + e^x + xe^x) - e^x y\sin y$$

$$= 2e^x\cos y + xe^x\cos y - e^x y\sin y \quad \text{....... (i)}$$

Also,

$$\frac{\partial u}{\partial y} = -xe^x\sin y - e_x\,(\sin y + y\cos y)$$

And, $$\frac{\partial^2 u}{\partial y^2} = -xe^x\cos y - e^x(\cos y + \cos y - y\sin y)$$

$$= -xe^x\cos y - 2e^x\cos y + e^x y\sin y \quad \text{.......(ii)}$$

From (i) and (ii)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2e^x\cos y + xe + x\cos y - e^x y\sin y - xe^x\cos y - 2e^x\cos y +$$
$e^x y\sin y$

$$= 0$$

Thus, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

(24) If $u = \log(x^3 + y^3 + z^3 - 3xyz)$. Show that: $\frac{\partial u}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x + y + z}$.

Solution: Let, $\quad u = \log(x^3 + y^3 + z^3 - 3xyz)$.

Differentiating we get,

$$\frac{\partial u}{\partial x} = \frac{1}{x^3 + y^3 + z^3 - 3xyz}(3x^2 - 3yz) = \frac{3x^2 - 3yz}{(x^3 + y^3 + z^3 - 3xyz)}$$

And,

$$\frac{\partial u}{\partial y} = \frac{1}{x^3 + y^3 + z^3 - 3xyz}(3y^2 - 3xz) = \frac{3y^2 - 3xz}{(x^3 + y^3 + z^3 - 3xyz)}$$

Also,

$$\frac{\partial u}{\partial z} = \frac{1}{x^3 + y^3 + z^3 - 3xyz}(3z^2 - 3xy) = \frac{3z^2 - 3xy}{(x^3 + y^3 + z^3 - 3xyz)}$$

Now,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3x^2 - 3yz - 3xz + 3z^2 - 3xy}{(x^3 + y^3 + z^3 - 3xyz)}$$

$$= \frac{3(x^2 + y^2 - z^2 - xy - xz - yz)}{(x^3 + y^3 + z^3 - 3xyz)}$$

$$= \frac{3(x^2 + y^2 + z^2 - xy - xz - yz)}{(x + y + z)\,(x^2 + y^2 + z^2 - xy - xz - yz)} = \frac{3}{(x + y + z)}.$$

Then, $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{3}{(x+y+z)}$.

(25) If $u=\log(x^3+y^3+z^3-3xyz)$ show that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{-3}{(x+y+z)^2}$.

[2002 Q. No. 2(b)]

Solution: Let, $u=\log(x^3+y^3+z^3-3xyz)$

Differentiating we get,

$$\frac{\partial u}{\partial x}=\frac{1}{(x^3+y^3+z^3-3xyz)}\times(3x^2-3yz)$$

$$=\frac{3x^2-3yz}{(x^3+y^3+y^3-3xyz)}$$

Also,

$$\frac{\partial^2 u}{\partial x^2}=\frac{(x^3+y^3+z^3-3xyz)(6x)-(3x^2-3yz)(3x^2-3yz)}{(x^3+y^3+z^3-3xyz)^2}$$

$$=\frac{6x^4+6xy^3+6xz^3-18x^2yz-(9x^4-9x^2yz-9x^2yz+9y^2z^2)}{(x^3+y^3+z^3-3xyz)^2}$$

$$=\frac{6x^4+xy^3+6xz^3-18x^2yz-9x^4+18x^2yz-9y^2z^2}{(x^3+y^3+z^3-3xyz)^2}$$

$$=\frac{-3x^4+6xy^3+6xz^3-9y^2z^2}{(x^3+y^3+z^3-3xyz)^2}$$

Similarly,

$$\frac{\partial u}{\partial y}=\frac{3y^2-3xz}{(x^3+y^3+z^3-3xyz)}\quad\text{and}\quad\frac{\partial^2 u}{\partial y^2}=\frac{6x^3y-3y^4+6yz^3-9x^2z^2}{(x^3+y^3+z^3-3xyz)^2}$$

Also, $$\frac{\partial u}{\partial z}=\frac{3z^2-3xy}{(x^3+y^3+z^3-3xyz)}\quad\text{and}\quad\frac{\partial^2 u}{\partial z^2}=\frac{6x^3z+6y^3z-3z^4-9x^2y^2}{(x^3+y^3+z^3-3xyz)^2}$$

Now,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}$$

$$=\frac{-3x^4+6xy^3+6xz^3-9y^2z^2+6x^3y-3y^4+6yz^3-9x^2z^2+6x^3z+6y^3z-3z^4-9x^2y^2}{(x^3+y^3+z^3-3xyz)^2}$$

$$=\frac{-3(x^4+y^4+z^4)-9(x^2y^2+y^2z^2+x^2z^2)+6(xy^3+xz^3+x^3y+yz^3+x^3z+y^3z)}{(x^3+y^3+z^3-3xyz)^2}$$

$$=\frac{-3}{(x+y+z)^2}$$

Thus, $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\frac{-3}{(x+y+z)^2}$

(26) If $x=r\cos\theta$, $y=r\sin\theta$ so that $r^2=x^2+y^2$ and $\theta=\tan^{-1}\left(\frac{y}{x}\right)$ prove that:

(i) $\frac{\partial r}{\partial x}=\frac{\partial x}{\partial r}$ **(ii)** $\frac{1}{r}\frac{\partial r}{\partial\theta}=r\cdot\frac{\partial\theta}{\partial r}$ **(iii)** $\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}=0$.

Solution: Let, $x=r\cos\theta$, $y=r\sin\theta$. Then $x^2+y^2=r^2$ and $\tan\theta=\frac{y}{x}$.

(i) Since $r^2=x^2+y^2$. So, $2r\frac{\partial r}{\partial x}=2x\Rightarrow\frac{\partial r}{\partial x}=\frac{x}{r}$

Also, $x=r\cos\theta$. So, $\frac{\partial x}{\partial r}=\cos\theta=\frac{x}{r}$

Thus, $\frac{\partial r}{\partial x}=\frac{\partial x}{\partial r}$.

(ii) Since we know that,

$$\frac{\partial r}{\partial\theta}=\frac{\partial r}{\partial x}\cdot\frac{\partial x}{\partial\theta}+\frac{\partial r}{\partial y}\cdot\frac{\partial y}{\partial\theta}$$

Also, $$\frac{\partial r}{\partial\theta}=\frac{\partial\theta}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial\theta}{\partial y}\cdot\frac{\partial y}{\partial r}$$

Since, $r^2=x^2+y^2$.

So, $2r\cdot\frac{\partial r}{\partial x}=2x\Rightarrow\frac{\partial r}{\partial x}=\frac{x}{r}$ and $2r\cdot\frac{\partial r}{\partial y}=2y\Rightarrow\frac{\partial r}{\partial y}=\frac{y}{r}$

Also, $x=r\cos\theta$ and $y=r\sin\theta$

So, $\frac{\partial x}{\partial\theta}=r(-\sin\theta)=-y$ So, $\frac{\partial y}{\partial\theta}=r\cos\theta=x$

Moreover, we have, $\theta=\tan^{-1}\left(\frac{y}{x}\right)$. So,

$$\frac{\partial\theta}{\partial x}=\frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right)=\frac{x^2}{x^2+y^2}\cdot\left(-\frac{y}{x^2}\right)=\frac{-y}{x^2+y^2}=-\frac{y}{r^2}$$

and

$$\frac{\partial\theta}{\partial y}=\frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{1}{x}\right)=\frac{x^2}{x^2+y^2}\cdot\left(\frac{1}{x}\right)=\frac{x}{r^2}$$

Now,

$$\frac{\partial r}{\partial\theta}=\frac{\partial r}{\partial x}\cdot\frac{\partial x}{\partial\theta}+\frac{\partial r}{\partial y}\cdot\frac{\partial y}{\partial\theta}=\frac{x}{r}(-y)+\frac{y}{r}\cdot x=0$$

$$\frac{\partial\theta}{\partial r}=\frac{\partial\theta}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial\theta}{\partial y}\cdot\frac{\partial y}{\partial r}=\left(-\frac{y}{r^2}\right)\left(\frac{x}{r}\right)+\frac{x}{r^2}\frac{y}{r}\qquad[\because\text{ using (i)}]$$

$$=0$$

This shows that the proof part is trivial.

(iii) Since we have,

$$\theta=\tan^{-1}\left(\frac{y}{x}\right)$$

From (ii) $\frac{\partial\theta}{\partial x} = -\frac{y}{x^2+y^2}$ and $\frac{\partial\theta}{\partial y} = \frac{x}{x^2+y^2}$

Then, $\frac{\partial^2\theta}{\partial x^2} = \frac{2xy}{(x^2+y^2)^2}$ and $\frac{\partial^2\theta}{\partial y^2} = \frac{-2xy}{(x^2+y^2)^2}$

Thus, $\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = \frac{2xy}{(x^2+y^2)^2} - \frac{2xy}{(x^2+y^2)^2} = 0$

(27) If $z = f(x + ay) + (x - ay)$, prove that: $\frac{\partial^2 z}{\partial y^2} = a^2\frac{\partial^2 z}{\partial x^2}$

Solution: Let, $z = f(x + ay) + (x - ay)$

Differentiating we get,

$$\frac{\partial z}{\partial y} = f'(x + ay) . a + (-a) \quad \text{and,} \quad \frac{\partial^2 z}{\partial y^2} = f''(x + ay) . a^2$$

Also, $\frac{\partial z}{\partial x} = f'(x + ay) . 1 + 1$ and $\frac{\partial^2 z}{\partial x^2} = f''(x + ay)$

Now, $\frac{\partial^2 z}{\partial y^2} = f''(x + ay).a^2 = a^2\frac{\partial^2 z}{\partial x^2}$

(28) If $z(x + y) = x^2 + y^2$, show that $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$.

Solution: Let, $z = \frac{x^2+y^2}{x+y}$

Differentiating we get,

$$\frac{\partial z}{\partial x} = \frac{(x+y) . 2x - (x^2+y^2) . 1}{(x+y)^2}$$

$$= \frac{2x^2 + 2xy - x^2 - y^2}{(x+y)^2} = \frac{x^2 + 2xy - y^2}{(x+y)^2}$$

And, $\frac{\partial z}{\partial y} = \frac{(x+y) . 2y - (x^2+y^2) . 1}{(x+y)^2}$

$$= \frac{2xy + 2y^2 - x^2 - y^2}{(x+y)^2} = \frac{y^2 + 2xy - x^2}{(x+y)^2}$$

Now,

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$$

$$\Rightarrow \left\{\frac{x^2 + 2xy - y^2}{(x+y)^2} - \frac{y^2 + 2xy + x^2}{(x+y)^2}\right\}^2 = \left\{1 - \frac{x^2 + 2xy - y^2}{(x+y)^2} - \frac{y^2 + 2xy - x^2}{(x+y)^2}\right\}$$

$$\Rightarrow \left\{\frac{x^2 + 2xy - y^2 - y^2 - 2xy + x^2}{(x+y)^2}\right\}^2 = 4\left(\frac{x^2 + 2xy + y^2 - x^2 - 2xy + y^2 - y^2 - 2xy + x^2}{(x+y)^2}\right)$$

$$\Rightarrow \left\{\frac{2(x^2 - y^2)}{(x + y)^2}\right\}^2 = 4\left(\frac{x^2 + 2xy + y^2 - x^2 - 2xy + y^2 - y^2 - 2xy + x^2}{(x + y)^2}\right)$$

$$\Rightarrow \left\{\frac{2(x + y)(x - y)}{(x + y)^2}\right\}^2 = 4\frac{(x - y)^2}{(x + y)^2}$$

$$\Rightarrow \frac{4(x - y)^2}{(x + y)^2} = 4\frac{(x - y)^2}{(x + y)^2}$$

This proves that $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$.