Exercise 6.12

Find a general solution of the following equating by method of variation of parameter.

(1) $y'' + y = \sec x$

[2006 Spring Q. No. 4(b) OR]

Solution: Given equation is, $y'' + y = \sec x$. ... (i)

This is second order non-homogeneous equation. Then its solution is,

$$y(x) = y_h(x) + y_p \quad \text{... (ii)}$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

Here, the homogeneous equation of (i) is,

$$y'' + y = 0$$

Its auxiliary equation is

$$m^2 + 1 = 0 \Rightarrow m = \pm i$$

So, its solution is,

$$y_h(x) = (A \cos x + B \sin x)$$

And, for particular solution,

We have,

$$y_1 = \cos x \quad \text{and} \quad y_2 = \sin x,$$

then, $y'_1 = -\sin x$ and $y_2' = \cos x$

$R = \sec x$

So, the Wronskian is,

$$W(y_1, y_2) = y_1y'_2 - y_2y'$$
$$= \cos x.\cos x - \sin x(-\sin x) = \cos^2 x + \sin^2 x = 1.$$

Thus,

$$y_p = -y_1\int \frac{y_2R}{w}dx + y_2\int \frac{y_1R}{w}dx$$
$$= -\cos x\int \frac{\sin x\ \sec x}{1}dx + \sin x\int \frac{\cos x.\ \sec x}{1}$$
$$= -\cos x\int \tan x\, dx + \sin\int dx$$
$$= -\cos x \log(\sec x) + x \sin x$$

Now (ii) becomes,

$$y(x) = y_h(x) + y_p$$
$$= A\cos x + B\sin x - \cos x\log(\sec x) + x\sin x.$$
$$= \cos x\,(A - \log(\sec x)) + \sin x\,(B + x).$$

(2) $\mathbf{y'' - 2y' + y = \frac{12e^x}{x^3}}$

Solution: Given equation is, $y'' - 2y' + y = \frac{12e^x}{x^3}$ ... (i)

This is second order non-homogeneous equation. Then its solution is,

$$y(x) = y_h(x) + y_p \quad \text{... (ii)}$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

Here, the homogeneous equation of (i) is,

$$y'' - 2y' + y = 0$$

Its auxiliary equation is

$$m^2 - 2m + 1 = 0 \Rightarrow (m-1)^2 = 0 \Rightarrow m = 1, 1.$$

Its solution is, $y_h(x) = (c_1 + c_2x)\,e^x$

And, for particular solution,

We have,

$y_1 = e^x$ and $y_2 = xe^x$,

So, $y'_1 = e^x$ and $y'_2 = xe^x + e^x$

And, $R = \frac{12e^x}{x^3}$

So, the Wronskian is,

$$W(y_1, y_2) = y_1y_2' - y_2y_1'$$
$$= e^x(xe^x + e^x) - xe^x.e^x = xe^{2x} + e^{2x} - xe^{2x} = e^{2x}$$

Thus,

$$y_p = -y_1\int \frac{y_2R}{w}dx + y_2\int \frac{y_1R}{w}dx$$

$$= -e^x\int \frac{xe^x}{e^{2x}} \times \frac{12e^x}{x^3} + xe^x\int \frac{e^x}{e^{2x}}.\frac{12e^x}{x^3}$$
$$= -e^x\,12\int x^{-2}dx + 12xe^x\int x^{-3}dx$$
$$= -12e^x \times \frac{-1}{x} + 12xe^x \times \frac{-1}{2x^2} = \frac{12e^x}{x} - \frac{6e^x}{x} = \frac{6e^x}{x}$$

Now (ii) becomes,

$$y(x) = y_h(x) + y_p$$
$$= (c_1 + c_2x)\,e^x + \frac{6e^x}{x}.$$

(3) $\mathbf{y'' - 4y' + 4y = 6 + \frac{e^{2x}}{x}}$ **[2009 Fall Q. No. 5(b)]**

Solution: Given equation is, $y'' - 4y' + 4y = 6 + \frac{e^{2x}}{x}$ ... (i)

This is second order non-homogeneous equation. Then its solution is,

$$y(x) = y_h(x) + y_p \quad \text{... (ii)}$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

Here, the homogeneous equation of (i) is,

$$y'' - 4y' + 4y = 0$$

Its auxiliary equation is

$$m^2 - 4m + 4 = 0 \Rightarrow (m-2)^2 = 0$$
$$\Rightarrow m = 2, 2.$$

Its solution is, $y_h(x) = (c_1 + c_2x)\,e^{2x}$

And, for particular solution,

We have,

$y_1 = e^{2x}$ and $y_2 = xe^{2x}$

So, $y'_1 = 2e^{2x}$ and $y'_2 = e^{2x} + 2xe^{2x}$

Also, $R = 6 + \frac{e^{2x}}{x}$

So, the Wronskian is,

$$W(y_1, y_2) = y_1y_2' - y_2y_1'$$
$$= e^{2x}(e^{2x} + 2xe^{2x}) - xe^{2x}.2e^{2x}$$
$$= e^{4x} + 2xe^{4x} - 2xe^{4x} = e^{4x}$$

Then,

$$y_p = -y_1\int \frac{y_2R}{w}dx + y_2\int \frac{y_1R}{w}dx$$
$$= -e^{2x}\int \frac{xe^{2x} \times \left(6 + \frac{e^{2x}}{x}\right)}{e^{4x}} + xe^{2x}\int \frac{e^{2x}\left(6 + \frac{e^{2x}}{x}\right)}{e^{4x}}dx$$
$$= -e^{2x}\int\left(\frac{6xe^{2x}}{e^{4x}} + \frac{xe^{4x}}{xe^{4x}}\right)dx + xe^{2x}\int\left(\frac{e^{2x}}{e^{4x}} + \frac{e^{4x}}{xe^{4x}}\right)dx$$

$$= -e^{2x}\int(6xe^{-2x}+1)\,dx + xe^{2x}\int(6e^{-2x}+x^{-1})\,x$$

$$= -e^{2x}\left\{6\left(\frac{xe^{-2x}}{-2}-\frac{e^{-2x}}{4}\right)+x\right\} + xe^{2x}\left(\frac{6e^{-2x}}{-2}+\log x\right)$$

$$= 3x + \frac{3}{2} - xe^{2x} - 3x + x\log xe^{2x}$$

$$\Rightarrow \quad y_p = 1.5 - xe^{2x} + x\log xe^{2x}$$

Now (ii) becomes,

$$y(x) = y_h(x) + y_p$$
$$= (c_1 + c_2x)\,e^{2x} + x\log x\, e^{2x} - xe^{2x} + 1.5$$
$$= (c_1 + c_2x + x\log x - x)\,e^{2x} + 1.5$$

(4) $y'' + 2y' + y = 4e^{-x}\log x$

Solution: Given equation is, $y'' + 2y' + y = 4e^{-x}\log x$ ... (i)

This is second order non-homogeneous equation. Then its solution is,

$$y(x) = y_h(x) + y_p \qquad \text{... (ii)}$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

Here, the homogeneous equation of (i) is,

$$y'' + 2y' + y = 0$$

Its auxiliary equation is

$$m^2 + 2m + 1 = 0 \Rightarrow (m+1)^2 = 0 \Rightarrow m = -1, -1$$

Its solution is, $\quad y_h(x) = (c_1 + c_2x)\,e^{-x}$

And, for particular solution,

We have,

$y_1 = e^{-x}$ and $y_2 = xe^{-x}$

So, $y_1' = -e^{-x}$ and $y_2' = -xe^{-x} + e^{-x}$

Also, $R = 4e^{-x}\log x$

So, the Wronskian is,

$$W(y_1, y_2) = y_1y_2' - y_2y_1'$$
$$= e^{-x}(-xe^{-x} + e^{-x}) - xe^{-x}(-e^{-x})$$
$$= -xe^{-2x} + e^{-2x} + xe^{-2x}$$
$$= e^{-2x}$$

Then,

$$y_p = -y_1\int\frac{y_2R}{w}\,dx + y_2\int\frac{y_1R}{w}\,dx$$

$$= -e^{-x}\int\frac{xe^{-x}\,4e^{-x}\log x}{e^{-2x}}\,dx + xe^{-x}\int\frac{e^{-x}\,4e^{-x}\log x}{e^{-2x}}\,dx$$

$$= -e^{-x}\int\frac{4xe^{-2x}\log x}{e^{-2x}}\,dx + xe^{-x}\int\frac{4e^{-2x}\log x}{e^{-2x}}\,dx$$

$$= -4e^{-x}\left[\log x\int x\,dx - \left\{\int\frac{d\log x}{dx}\int x\,dx\right\}dx\right] + 4xe^{-x}\left\{\log x\int dx - \int\frac{d\log x}{dx}\int dx\right\}$$

$$= -4e^{-x}\left(\frac{x^2}{2}\log x - \int\frac{1}{x}\frac{x^2}{2}\,dx\right) + 4xe^{-x}\left(x\log x - \int\frac{1}{x}\cdot x\,dx\right)$$

$$= -4e^{-x}\left(\frac{x^2}{2}\log x - \frac{x^2}{4}\right) + 4xe^{-x}(x\log x - x)$$

$$= -2x^2e^{-x}\log x + x^2e^{-x} + 4x^2e^{-x}\log x - 4x^2e^{-x}$$

$$\Rightarrow \quad y_p = 2x^2e^{-x}\log x - 3x^2e^{-x}$$

Now (ii) becomes,

$$y(x) = y_h(x) + y_p$$
$$= (c_1 + c_2x)\,e^{-x} + x^2e^{-x}(2\log x - 3).$$

(5) $y'' + 2y' + 2y = 2e^{-x}\sec^3 x$

Solution: Given equation is, $y'' + 2y' + 2y = 2e^{-x}\sec^3 x$ ... (i)

This is second order non-homogeneous equation. Then its solution is,

$$y(x) = y_h(x) + y_p \qquad \text{... (ii)}$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

Here, the homogeneous equation of (i) is,

$$y'' + 2y' + 2y = 0$$

Its auxiliary equation is

$$m^2 + 2m + 2 = 0 \Rightarrow m = \frac{-2\pm\sqrt{4-8}}{2} = \frac{-2\pm\sqrt{4i^2}}{2} = \frac{-2\pm 2i}{2} = (-1 \pm i).$$

Thus, its solution is $y_h(x) = e^{-x}(A\cos x + B\sin x)$

And, for particular solution,

We have,

$y_1 = e^{-x}\cos x$ and $y_2 = -e^{-x}\sin x$

So, $y_1' = -e^{-x}\cos x - e^{-x}\sin x$ and $y_2' = -e^{-x}\sin x + e^{-x}\cos x$

Also, $R = 2e^{-x}\sec^3 x$

So, the Wronskian is,

$$W(y_1, y_2) = y_1y_2' - y_2y_1'$$
$$= e^{-x}\cos x(-e^{-x}\sin x + e^{-x}\cos x) - e^{-x}\sin x(-e^{-x}\cos x - e^{-x}\sin x)$$
$$= -e^{-2x}\cos x.\sin x + e^{-2x}\cos^2 x + e^{-2x}\sin x.\cos x + e^{-2x}\sin^2 x$$
$$= e^{-2x}$$

Then,

$$y_p = -y_1\int\frac{y_2R}{w}\,dx + y_2\int\frac{y_1R}{w}\,dx$$

$$= -e^{-x}\cos x\int\frac{e^{-x}\sin x \times 2e^{-x}\sec^3 x}{e^{-2x}}\,dx + e^{-x}\sin x\int\frac{e^{-x}\cos x\times 2e^{-x}\sec^3 x}{e^{-2x}}\,dx$$

$$= -e^{-x}\cos x\int 2\tan x\sec^2 x\,dx + e^{-x}\sin x\int\sec^2 x\,dx$$

Put, $\tan x = v$ then, $dv = \sec^2 x\,dx$. So,

$$= -e^{-x}\cos x\int 2v\,dv + 2e^{-x}\sin x\int dv = -e^{-x}\cos x\,v^2 + 2e^{-x}\sin x\,.\,v$$
$$= -e^{-x}\cos x\tan^2 x + 2e^{-x}\sin x.\tan x$$

$$= e^{-x}(-\cos x \tan^2 x + 2\sin x.\tan x)$$

$$= e^{-x}\left(-\cos x.\frac{\sin^2 x}{\cos^2 x} + 2\sin x.\tan x\right)$$

$$= e^{-x}(-\tan x.\sin x + 2\sin x.\tan x)$$

$\Rightarrow$ $y_p = e^{-x}(\sin x.\tan x)$.

Now (ii) becomes,

$$y(x) = y_h(x) + y_p = e^{-x}(A\cos x + B\sin x) + e^{-x}\sin x.\tan x$$

$$= e^{-x}(A\cos x + B\sin x + \sin x.\tan x)$$

(6) $\mathbf{y'' - 2y' + y = 35e^x x^{3/2}}$

Solution: Given equation is, $y'' - 2y' + y = 35e^x x^{3/2}$ ... (i)

This is second order non-homogeneous equation. Then its solution is,

$$y(x) = y_h(x) + y_p \quad \text{... (ii)}$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

Here, the homogeneous equation of (i) is,

$$y'' - 2y' + y = 0$$

Its auxiliary equation is

$$m^2 - 2m + 1 = 0 \quad \Rightarrow (m-1)^2 = 0 \Rightarrow m = 1, 1.$$

Its solution is, $y_h(x) = (c_1 + c_2 x)e^x$

And, for particular solution,

We have, $y_1 = e^x$ and $y_2 = xe^x$

So, $y_1' = e^x$ and $y_2' = xe^x$

Also, $R = 35e^x x^{3/2}$

So, the Wronskian is,

$$W(y_1, y_2) = y_1y_2' - y_2y_1'$$

$$= e^x(xe^x + e^x) - xe^x.e^x = xe^{2x} + e^{2x} - xe^{2x} = e^{2x}$$

Then,

$$y_p = -y_1\int\frac{y_2R}{w}dx + y_2\int\frac{y_1R}{w}dx$$

$$= -e^x\int\frac{xe^x.35e^x x^{3/2}}{e^{2x}} + xe^x\int\frac{e^x.35e^x.x^{3/2}}{e^{2x}}dx$$

$$= -35e^x\int x^{5/2}dx + 35xe^x\int x^{3/2}dx = -35e^x \times \frac{2}{7}x^{7/2} + 35xe^x \times \frac{2}{5}x^{5/2}$$

$$= -10x^{7/2}e^x + 14x^{5/2}e^x$$

$$= 4e^x x^{7/2}$$

Now (ii) becomes,

$$y(x) = y_h(x) + y_p = (c_1 + c_2x)e^x + 4e^x x^{7/2}$$

$$= e^x(c_1 + c_2x + 4x^{7/2}).$$

(7) $\mathbf{y'' + y' = x}$

Solution: Given equation is, $y'' + y' = x$. ... (i)

This is second order non-homogeneous equation. Then its solution is,

$$y(x) = y_h(x) + y_p \quad \text{.... (ii)}$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

Here, the homogeneous equation of (i) is,

$$y'' + y' = 0$$

Its auxiliary equation is,

$$m^2 + m = 0 \Rightarrow m(m+1) = 0 \Rightarrow m = 0, -1.$$

Its solution is, $y_h(x) = c_1 + c_2e^{-x}$

And, for particular solution,

We have, $y_1 = 1$, and $y_2 = e^{-x}$

So, $y_1' = 0$ $y_2' = -e^{-x}$

Also, $R = x$.

So, the Wronskian is,

$$W(y_1, y_2) = y_1y_2' - y_2y_1'$$

$$= 1 \times -e^{-x} - e^{-x}.0 = -e^{-x}$$

Then,

$$y_p = -y_1\int\frac{y_2R}{w}dx + y_2\int\frac{y_1R}{w}dx$$

$$= -1\int\frac{e^{-x}x}{-e^{-x}} + e^{-x}\int\frac{1.x}{-e^{-x}}dx = \int x\,dx + e^{-x}\int -x e^x.dx = \frac{x^2}{2} - e^{-x}(xe^x - e^x)$$

$$\Rightarrow \quad y_p = \frac{x^2}{2} - x + 1.$$

Now (ii) becomes,

$$y(x) = y_h(x) + y_p$$

$$= c_1 + c_2e^{-x} + \frac{x^2}{2} - x + 1.$$

(8) $\mathbf{y'' + y = \sin x}$

Solution: Given equation is, $y'' + y = \sin x$. ... (i)

This is second order non-homogeneous equation. Then its solution is,

$$y(x) = y_h(x) + y_p \quad \text{... (ii)}$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

Here, the homogeneous equation of (i) is,

$$y'' + y = 0$$

So, its auxiliary equation

$$m^2 + 1 = 0 \Rightarrow m = \pm i.$$

Its solution is,

$$y_h(x) = A\cos x + B\sin x$$

And, for particular solution,

We have, $y_1 = \cos x$ and $y_2 = \sin x$

So, $\quad y_1' = -\sin x \qquad y_2' = \cos x$

So, the Wronskian is,

$$W(y_1, y_2) = \cos x \cos x + \sin x . \sin x = \cos^2 x + \sin x^2 x = 1$$

Then,

$$y_p = -y_1 \int \frac{y_2 R}{w} dx + y_2 \int \frac{y_1 R}{w} dx$$

$$= -\cos x \int \frac{\sin x . \sin x}{1} dx + \sin x \int \sin x . \cos x \, dx$$

$$= -\cos x \int \left(\frac{1 - \cos 2x}{2}\right) dx + \frac{\sin x}{2} \int \sin 2x \, dx$$

$$= -\cos x \left(\frac{1}{2} x - \frac{\sin 2x}{4}\right) + \frac{\sin x}{2}\left(\frac{-\cos 2x}{2}\right)$$

$$= \frac{-x \cos x}{2} + \frac{\sin 2x . \cos x}{4} - \frac{\cos 2x . \sin x}{4} = \frac{-x \cos x}{2} + \frac{1}{4} \sin(2x - 4)$$

$$= \frac{-x \cos x}{2} + \frac{\sin x}{4}$$

Now (ii) becomes,

$$y(x) = y_h(x) + y_p$$

$$= A \cos x + B \sin x - \frac{x \cos x}{2} + \frac{\sin x}{4}$$

(9) $y'' + 2y' + y = e^{-x}$

Solution: Given equation is, $y'' + 2y' + y = e^{-x}$ ... (i)

This is second order non-homogeneous equation. Then its solution is,

$$y(x) = y_h(x) + y_p \qquad \text{... (ii)}$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

Here, the homogeneous equation of (i) is,

$$y'' + 2y' + y = 0$$

So, its auxiliary equation is,

$$m^2 + 2m + 1 = 0 \Rightarrow (m + 1)^2 = 0 \Rightarrow m = -1, -1.$$

Its solution is, $\quad y_h(x) = (c_1 + c_2 x) e^{-x}$

And, for particular solution,

We have, $\quad y_1 = e^{-x}$ and $\quad y_2 = xe^{-x}$

So, $\quad y_1' = -e^{-x}$ and $\quad y_2' = -xe^{-x} + e^{-x}$

Also, $\quad R = e^{-x}$

So, the Wronskian is,

$$W(y_1, y_2) = y_1 y_2' - y_2 y_1'$$

$$= e^{-x}(e^{-x} - xe^{-x}) - xe^{-x} \times -e^{-x}$$

$$= e^{-2x} - xe^{-2x} + xe^{-2x}$$

$$= e^{-2x}$$

Then,

$$y_p = -y_1 \int \frac{y_2 R}{w} dx + y_2 \int \frac{y_1 R}{w} dx$$

$$= -e^{-x} \int \frac{xe^{-x} . e^{-x}}{e^{-2x}} dx + xe^{-x} \int \frac{e^{-x} . e^{-x}}{e^{-2x}} dx = -e^{-x} \frac{x^2}{2} + xe^{-x} \times x$$

$$= x^2 e^{-x} - \frac{x^2}{2} e^{-x} = \frac{x^2 e^{-x}}{2}$$

Now (ii) becomes,

$$y(x) = y_h(x) + y_p$$

$$= (c_1 + c_2 x) e^{-x} + \frac{x^2}{2} e^{-x}$$

(10) $y'' - y = e^x$

Solution: Given equation is, $y'' - y = e^x$ ... (i)

This is second order non-homogeneous equation. Then its solution is,

$$y(x) = y_h(x) + y_p \qquad \text{... (ii)}$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

Here, the homogeneous equation of (i) is,

$$y'' - y = 0$$

So, its auxiliary equation is,

$$m^2 - 1 = 0 \Rightarrow m = \pm 1.$$

Its solution is, $y_h(x) = c_1 e^x + c_2 e^{-x}$

And, for particular solution,

We have, $\quad y_1 = e^x$ and $\quad y_2 = e^{-x}$

So, $\quad y_1' = e^x$ and $\quad y_2' = -e^{-x}$

Also, $\quad R = e^x$

So, the Wronskian is,

$$W(y_1, y_2) = y_1 y_2' - y_2 y_1'$$

$$= e^x \times e^{-x} - e^{-x} . e^x = -1 - 1 = -2$$

Then,

$$y_p = -y_1 \int \frac{y_2 R}{w} dx + y_2 \int \frac{y_1 R}{w} dx$$

$$= -e^x \int \frac{e^{-x} . e^x}{-2} dx + e^{-x} \int \frac{e^x . e^x}{-2} = \frac{xe^x}{2} - \frac{e^{-x} . e^{2x}}{4} = \frac{xe^x}{2} - \frac{e^x}{4}$$

Now (ii) becomes,

$$y(x) = y_h(x) + y_p$$

$$= c_1 e^x + c_2 e^{-x} + \frac{x}{2} e^x - \frac{e^x}{4}$$

(11) $y'' + 4y' + 5y = 10$ **[2003 Fall Q. No. 4(b)]**

Solution: Given equation is, $y'' + 4y' + 5y = 10$ ... (i)

This is second order non-homogeneous equation. Then its solution is,

$$y(x) = y_h(x) + y_p \quad \dots \text{(ii)}$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

Here, the homogeneous equation of (i) is,

$$y'' + 4y' + 5y = 0$$

So, its auxiliary equation is $m^2 + 4m + 5 = 0$

$$m = \frac{-4 \pm \sqrt{16 - 20}}{2} = \frac{-4 \pm 2i}{2} = -2 \pm i$$

Therefore, its solution is

$$y_h(x) = e^{-2x}(A\cos x + B\sin x)$$

And, for particular solution,

We have,

$$y_1 = e^{-2x}\cos x \quad \text{and} \quad y_2 = e^{-2x}\sin x$$

So, $y_1' = -2e^{-2x}\cos x - \sin x\, e^{-2x}$, $y_2' = -2e^{-2x}\sin x + e^{-2x}\cos x$

Also, $R = 10$.

So, the Wronskian is,

$$\begin{aligned} W(y_1, y_2) &= y_1y_2' - y_2y_1' \\ &= e^{-2x}\cos x\,(e^{-2x}\cos x - 2e^{-2x}\sin x) - e^{-2x}\sin x\,(-2e^{-2x}\cos x - \sin x e^{-2x}) \\ &= e^{-4x} \end{aligned}$$

Then,

$$\begin{aligned} y_p &= -y_1\int \frac{y_2R}{w}dx + y_2\int \frac{y_1R}{w}dx \\ &= -e^{-2x}\cos x\int \frac{e^{-2x}\sin x 10}{e^{-4x}}dx + e^{-2x}\sin x\int e^{-2x}\cos x.\frac{10}{e^{-4x}}dx \\ &= -10e^{-2x}\cos x\int e^{2x}\sin x\,dx + 10e^{-2x}\sin x\int e^{2x}\cos x\,dx \\ &= -10e^{-2x}\cos x\times\frac{4}{5}\left(\frac{\sin x e^{2x}}{2} - \frac{\cos x e^{2x}}{4}\right) + 10e^{-2x}\sin x\times\frac{4}{5}\left(\frac{\cos x e^{2x}}{2} + \frac{e^{2x}\sin x}{4}\right) \\ &= -8e^{-2x}\cos x\times\frac{\sin x e^{2x}}{2} + 8e^{-2x}\cos x\frac{\cos e^{2x}}{4} + 8e^{-2x}\sin x.\frac{\cos x e^{2x}}{2} + 8x^{-2x}\sin x.\frac{e^{2x}\sin x}{4} \\ &= 2\cos^2 x + 2\sin^2 x \\ &= 2 \end{aligned}$$

Now (ii) becomes,

$$\begin{aligned} y(x) &= y_h(x) + y_p \\ &= e^{-2x}(A\cos x + B\sin x) + 2 \end{aligned}$$

(12) $y'' - 4y' + 4y = \frac{e^{2x}}{x}$

Solution: Given equation is, $y'' - 4y' + 4y = \frac{e^{2x}}{x}$ ... (i)

This is second order non-homogeneous equation. Then its solution is,

$$y(x) = y_h(x) + y_p \quad \dots \text{(ii)}$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

Here, the homogeneous equation of (i) is,

$$y'' - 4y' + 4y = 0$$

So, its auxiliary equation is,

$$m^2 - 4m + 4 = 0 \Rightarrow (m-2)^2 = 0 \Rightarrow m = 2, 2.$$

Its solution is,

$$y_h(x) = (c_1 + c_2x)e^{2x}$$

And, for particular solution,

We have, $y_1 = e^{2x}$ and $y_2 = xe^{2x}$

So, $y_1' = 2e^{2x}$ and $y_2' = 2xe^{2x} + e^{2x}$

Also, $R = \frac{e^{2x}}{x}$

So, the Wronskian is,

$$\begin{aligned} W(y_1, y_2) &= y_1y_2' - y_2y_1' \\ &= e^{2x}(2xe^{2x} + e^{2x}) - xe^{2x}(2e^{2x}) = 2xe^{4x} + e^{4x} - 2xe^{4x} \\ &= e^{4x} \end{aligned}$$

Then,

$$\begin{aligned} y_p &= -y_1\int \frac{y_2R}{w}dx + y_2\int \frac{y_1R}{w}dx = -e^{2x}\int \frac{xe^{2x}}{e^{4x}}.\frac{e^{2x}}{x}dx + xe^{2x}\int \frac{e^{2x}}{e^{4x}}.\frac{e^{2x}}{x}dx \\ &= -xe^{2x} + x\log x\, e^{2x} \end{aligned}$$

Now (ii) becomes,

$$\begin{aligned} y(x) = y_h(x) + y_p &= (c_1 + c_2x)e^{2x} + x\log x e^{2x} - xe^{2x} \\ &= (c_1 + c_2x + x\log x - x)e^{2x} \end{aligned}$$

(13) $y'' + 2y' + y = e^{-x}\cos x$

Solution: Given equation is, $y'' + 2y' + y = e^{-x}\cos x$ ... (i)

This is second order non-homogeneous equation. Then its solution is,

$$y(x) = y_h(x) + y_p \quad \dots \text{(ii)}$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

Here, the homogeneous equation of (i) is,

$$y'' + 2y' + y = 0$$

So, its auxiliary equation is

$$m^2 + 2m + 1 = 0 \Rightarrow (m+1)^2 = 0 \Rightarrow m = -1, -1$$

Therefore, its solution is,

$$y_h(x) = (c_1 + c_2x)e^{-x}$$

And, for particular solution,

We have, $y_1 = e^{-x}$ and $y_2 = xe^{-x}$

So, $y_1' = -e^{-x}$, $y_2' = e^{-x} - xe^{-x}$

Also, $R = e^{-x}\cos x$

So, the Wronskian is,

$$
\begin{aligned}
W(y_1, y_2) &= y_1y_2' - y_2y_1' \\
&= e^{-x}(e^{-x} - xe^{-x}) + xe^{-x}e^{-x} \\
&= e^{-2x} - xe^{2x} + xe^{-2x} \\
&= e^{-2x}
\end{aligned}
$$

Then, $y_p = -y_1\int \frac{y_2R}{w}dx + y_2\int \frac{y_1R}{w}$

$$
\begin{aligned}
&= -e^{-x}\int \frac{xe^{-x}.\,e^{-x}\cos x}{e^{-2x}} + xe^{-x}\int \frac{e^{-x}.\,e^{-x}\cos x}{e^{-2x}}dx \\
&= -e^{-x}\int x\cos x\,dx + xe^{-x}\int \cos x\,dx \\
&= -e^{-x}(x\sin x + \cos x) + xe^{-x}\sin x \\
&= -xe^{-x}\sin x - e^{-x}\cos x + xe^{-x}\sin x
\end{aligned}
$$

$\Rightarrow \quad y_p = -e^{-x}\cos x$

Now (ii) becomes,

$$
\begin{aligned}
y(x) = y_h(x) + y_p &= (c_1 + c_2x)e^{-x} - e^{-x}\cos x \\
&= (c_1 + c_2x - \cos x)e^{-x}
\end{aligned}
$$

(14) $y'' - 2y' + y = \frac{e^x}{x^3}$

Solution: Given equation is, $y'' - 2y' + y = \frac{e^x}{x^3}$ ... (i)

This is second order non-homogeneous equation. Then its solution is,

$y(x) = y_h(x) + y_p$ ... (ii)

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

Here, the homogeneous equation of (i) is,

$y'' - 2y' + y = 0$

So, its auxiliary equation is,

$m^2 - 2m + 1 = 0 \Rightarrow (m-1)^2 = 0 \Rightarrow m = 1, 1.$

Its solution is,

$y_h(x) = (c_1 + c_2x)e^x$

And, for particular solution,

We have, $y_1 = e^x$ and $y_2 = xe^x$

So, $y_1' = e^x$ $\quad y_2' = e^x + xe^x$

Also, $R = \frac{e^x}{x^3}$

So, the Wronskian is,

$$
\begin{aligned}
W(y_1, y_2) = y_1y_2' - y_2y_1' &= e^x(e^x + xe^x) - xe^x.\,e^x \\
&= e^{2x} + xe^{2x} + xe^{-2x} \\
&= e^{2x}
\end{aligned}
$$

Then,

$$y_p = -y_1\int \frac{y_2R}{w}dx + y_2\int \frac{y_1R}{w}dx = -e^x\int \frac{xe^x}{e^{2x}}.\frac{e^x}{x^3}dx + xe^x\int \frac{e^x}{e^{2x}}.\frac{e^x}{x^3}dx$$

$$
\begin{aligned}
&= -e^x\int x^{-2}dx + xe^x\int x^{-3}dx \\
&= -\frac{e^x}{-x} + \frac{xe^x}{-2x^2} \\
&= \frac{e^x}{2x}
\end{aligned}
$$

Now (ii) becomes,

$$
\begin{aligned}
y(x) &= y_h(x) + y_p \\
&= (c_1 + c_2x)e^x + \frac{e^x}{2x}
\end{aligned}
$$

(15) $(D^2 - 2D + 1)y = 3x^{1/2}e^x$

Solution: Given equation is, $(D^2 - 2D + 1)y = 3x^{1/2}e^x$

$\Rightarrow y'' - 2y' + y = 3x^{1/2}e^x$ ... (i)

This is second order non-homogeneous equation. Then its solution is,

$y(x) = y_h(x) + y_p$ ... (ii)

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

Here, the homogeneous equation of (i) is,

$y'' - 2y' + y = 0$

So, its auxiliary equation is,

$m^2 - 2m + 1 = 0 \Rightarrow m = 1, 1.$

Its solution is,

$y_h(x) = (c_1 + c_2x)e^x$

And, for particular solution,

We have, $y_1 = e^x$ and $y_2 = xe^x$

So, $y_1' = e^x$ $\quad y_2' = e^x + xe^x$

Also, $R = 3x^{1/2}e^x$

So, the Wronskian is,

$$
\begin{aligned}
W(y_1, y_2) = y_1y_2' - y_2y_1' &= e^x(e^x + xe^x) - e^x xe^x \\
&= e^{2x} + x\,e^{2x} - x\,e^{2x} \\
&= e^{2x}
\end{aligned}
$$

Then,

$$
\begin{aligned}
y_p &= -y_1\int \frac{y_2R}{w}dx + y_2\int \frac{y_1R}{w}dx \\
&= -e^x\int \frac{xe^x.\,3x^{1/2}.\,e^x}{e^{2x}} + xe^x\int \frac{e^x.\,3x^{1/2}.\,e^x}{e^{2x}}dx \\
&= -e^x\int 3x^{3/2}dx + xe^x\int 3x^{1/2}dx \\
&= -3e^x\times\frac{2}{7}x^{7/2} + 3xe^x.\frac{2}{5}x^{5/2} \\
&= -\frac{6}{7}e^x x^{7/2} + \frac{6}{5}e^x.\,x^{7/2} = \frac{-30e^x x^{7/2} + 42e^x x^{7/2}}{35}
\end{aligned}
$$

$\Rightarrow \quad y_p = \frac{12e^x}{35}x^{7/2}$

Now (ii) becomes,

$$y(x) = y_h(x) + y_p = (c_1 + c_2x)\,e^x + \frac{12}{35}e^x\,x^{7/2}$$

$$= \left(c_1 + c_2x + \frac{12}{35}x^{7/2}\right)e^x$$

(16) $(D^2 + 4D + 4)\,y = \frac{2e^{-2x}}{x^2}$

Solution: Given equation is, $(D^2 + 4D + 4)\,y = \frac{2e^{-2x}}{x^2}$

$$\Rightarrow y'' + 4y' + 4y = \frac{2e^{-2x}}{x^2} \qquad \ldots \text{(i)}$$

This is second order non-homogeneous equation. Then its solution is,

$$y(x) = y_h(x) + y_p \qquad \ldots \text{(ii)}$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

Here, the homogeneous equation of (i) is,

$$y'' + 4y' + 4y = 0$$

So, its auxiliary equation is

$$m^2 + 4m + 4 = 0 \Rightarrow (m+2)^2 = 0 \Rightarrow m = -2, -2.$$

Its solution is,

$$y_h(x) = (c_1 + c_2x)\,e^{-2x}$$

And, for particular solution,

We have, $y_1 = e^{-2x}$ and $y_2 = xe^{-2x}$

So, $y_1' = -2e^{-2x}$, $y_2' = (e^{-2x} - 2xe^{-2x})$

Also, $R = \frac{2e^{-2x}}{x^2}$

So, the Wronskian is,

$$W(y_1, y_2) = y_1y_2' - y_2y_1' = e^{-2x}(e^{-2x} - 2xe^{-2x}) + xe^{-2x}.\,2e^{-2x}$$
$$= e^{-4x} - 2xe^{-4x} + 2xe^{-4x}$$
$$= e^{-4x}$$

Then,

$$y_p = -y_1\int\frac{y_2R}{w}dx + y_2\int\frac{y_1R}{w}dx$$

$$= -e^{-2x}\int\frac{xe^{-2x}}{e^{-4x}}.\frac{2e^{-2x}}{x^2} + xe^{-2x}\int\frac{e^{-2x}}{e^{-4x}}.\frac{2e^{-2x}}{x^2}dx$$

$$= -e^{-2x}\int\frac{2}{x}dx + xe^{-2x}\int\frac{2}{x^2}dx$$

$$= -2e^{-2x}\log x - 2e^{-2x}$$

Now (ii) becomes,

$$y(x) = y_h(x) + y_p = (c_1 + c_2x)\,e^{-2x} - 2e^{-2x}\log x - 2e^{-2x})$$
$$= (c_1 + c_2x - 2\log x - 2)\,e^{-2x}$$

(17) $y'' + 4y = 4\tan 2x$

Solution: Given equation is, $y'' + 4y = 4\tan 2x. \qquad \ldots$ (i)

This is second order non-homogeneous equation. Then its solution is,

$$y(x) = y_h(x) + y_p \qquad \dots (ii)$$

where y_h be the solution of homogeneous part of (i) and y_p be the particular solution of (i).

Here, the homogeneous equation of (i) is,

$$y'' + 4y = 0$$

So, its auxiliary equation is,

$$m^2 + 4 = 0 \Rightarrow m^2 = 4i^2 \Rightarrow m = \pm 2i$$

Its solution is,

$$y_h(x) = (A\cos 2x + B\sin 2x)$$

And, for particular solution,

We have, $y_1 = \cos 2x$ and $y_2 = \sin 2x$

So, $y_1' = -2\sin 2x$ $\quad y_2' = 2\cos 2x$

Also, $R = 4\tan x$.

So, the Wronskian is,

$$W(y_1, y_2) = y_1y_2' - y_2y_1' = \cos 2x \times 2\cos 2x + \sin 2x.\, 2\sin 2x$$
$$= 2\cos^2 2x + 2\sin^2 2x = 2$$

Then,

$$y_p = -y_1\int \frac{y_2R}{w}dx + y_2\int \frac{y_1R}{w}dx$$

$$= -\cos 2x\int \frac{\sin 2x.\, 4\tan 2x}{2} + 4\sin 2x\int \frac{\cos 2x.\, \tan 2x}{2}dx$$

$$= -\cos 2x\int \frac{2\sin^2 2x}{\cos 2x}dx + 2\sin 2x\int \sin 2x dx$$

$$= -2\cos 2x\int \frac{(1-\cos^2 2x)}{\cos 2x}dx + 2\sin 2x\int \sin 2x\, dx$$

$$= -2\cos 2x\int (\sec 2x - \cos 2x)\, dx + 2\sin 2x\int \sin 2x\, dx$$

$$= -2\cos 2x\left\{\frac{\log(\sec 2x + \tan 2x)}{2} - \frac{\sin 2x}{2}\right\} + 2\sin 2x\frac{(-\cos 2x)}{2}$$

$$= -2\cos 2x \log(\sec 2x + \tan 2x) + \sin 2x.\, \cos 2x - \sin 2x\cos 2x$$

$$= -\cos 2x\log(\sec 2x + \tan 2x)$$

Now (ii) becomes,

$$y(x) = y_h(x) + y_p$$
$$= A\cos 2x + B\sin 2x - \cos 2x\log(\sec 2x + \tan 2x)$$