## Exercise 11.1

1. Test the following surface for maxima, minima and saddle points. Find the functional values at these points.

(a) $z = x^2 + xy + y^2 + 3x - 3y + 4$

**Solution:** Given that

$$z = x^2 + xy + y^2 + 3x - 3y + 4$$

Then,

$z_x = 2x + y + 3$ and $z_y = x + 2y - 3$

$z_{xx} = 2,$ $z_{yy} = 2$

Also, $z_{xy} = 1$

For extreme point, set,

$z_x = 0$ and $z_y = 0$

$\Rightarrow 2x + y + 3 = 0$ $\Rightarrow x + 2y - 3 = 0$

Solving these equations we get,

$x = -3$ and $y = 3$.

Now,

$$z_{xx} = 2 > 0.$$

and, $\begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 = 3 > 0.$

Therefore, z is minimum at (-3, 3) and minimum value is,

$$z = 9 - 9 + 9 - 9 - 9 + 4 = -5.$$

**(b) $z = 5xy - 7x^2 + 3x - 6y + 2$**

**Solution:** Given function is,

$$z = 5xy - 7x^2 + 3x - 6y + 2$$

Then,

$z_x = 5y - 14x + 3$ and $z_y = 5x - 6$

$z_{xx} = -14$ $z_{yy} = 0$

Also, $z_{xy} = 5$

For extreme point, set,

$z_x = 0$ and $z_y = 0$

$\Rightarrow 5x - 14x + 3 = 0$ $\Rightarrow 5x - 6 = 0$

Solving these equations we get

$x = \frac{6}{5}$ and $y = \frac{69}{5}$

Now,

$$z_{xx} = -14 < 0.$$

and $\begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} -14 & 5 \\ 5 & 0 \end{vmatrix} = 0 - 25 = -25 < 0.$



This shows that the function is saddle at $\left(\frac{6}{5}, \frac{69}{5}\right)$.

And value of z at the point is

$$z = \frac{2070}{25} - \frac{252}{25} + \frac{18}{5} - \frac{414}{5} + 2$$

$$= \frac{2070 - 252 + 90 - 2070 + 50}{25} = -\frac{112}{25}.$$

**(c) $z = x^2 + xy + 3x + 2y + 5$**

**Solution:** Given function is,

$$z = x^2 + xy + 3x + 2y + 5$$

Then,

$z_x = 2x + y + 3$ and $z_y = x + 2$

$z_{xx} = 2$ $z_{yy} = 0$

Also, $z_{xy} = 1$

For extreme point, set,

$z_x = 0$ and $z_y = 0$

$\Rightarrow 2x + y + 3 = 0$ $\Rightarrow x + 2 = 0$

Solving these equations we get,

$x = -2, y = 1.$

Now,

$$z_{xx} = 2 > 0$$

and $\begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} = -1 < 0.$

This shows that the function is saddle at (-2, 1).

And, the value of z at the point is,

$$z = 4 - 2 - 6 + 2 + 5 = 3.$$

**(d) $z = 2xy - 5x^2 - 2y^2 + 4x - 4$**

**Solution:** Given function is

$$z = 2xy - 5x^2 - 2y^2 + 4x - 4$$

Then,

$z_x = 2y - 10x + 4$ and $z_y = 2x - 4y$

$z_{xx} = -10$ $z_{yy} = -4$

Also, $z_{xy} = 2$

For extreme point, set,

$z_x = 0$ and $z_y = 0$

$\Rightarrow 2y - 10x + 4 = 0$ $\Rightarrow 2x - 4y = 0$

Solving these equations we get,

$x = \frac{4}{9}$ and $y = \frac{2}{9}$.

Now, at $(x, y) = \left(\frac{4}{9}, \frac{2}{9}\right)$,

$$z_{xx} = -10 < 0.$$

and $\begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} -10 & 2 \\ 2 & -4 \end{vmatrix} = 40 - 4 = 36 > 0.$

This shows that z is maximum at $\left(\frac{4}{9}, \frac{2}{9}\right)$. And maximum value is,

$$z = \frac{16}{81} - \frac{80}{81} - \frac{8}{81} + \frac{16}{9} - 4$$
$$= \frac{16 - 80 - 8 + 144 - 324}{81} = -\frac{252}{81} = -\frac{28}{81}.$$

**(e) $z = x^2 + xy + y^2 + 3y + 3$**

**Solution:** Given function is

$$z = x^2 + xy + y^2 + 3y + 3$$

Then,

$z_x = 2x + y$ and $z_y = x + 2y + 3$

$z_{xx} = 2$ $\quad z_{yy} = 2$

Also, $z_{xy} = 1$

For extreme point, set,

$z_x = 0$ and $z_y = 0$

$\Rightarrow \quad 2x + y = 0 \qquad \Rightarrow \quad x + 2y + 3 = 0.$

Solving these equations we get,

$x = 1$ and $y = -2$.

Now, at $(x, y) = (1, -2)$,

$$z_{xx} = 2 > 0$$

and $\begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 = 3 > 0$

This shows that z is minimum at (1, –2). And minimum value at the point is,

$$z = 1 - 2 + 4 - 6 + 3 = 0.$$

**(f) $z = 2x^2 + 3xy + 4y^2 - 5x + 2y$**

**Solution:** Given function is

$$z = 2x^2 + 3xy + 4y^2 - 5x + 2y$$

Then,

$z_x = 4x + 3y - 5$ and $z_y = 3x + 8y + 2$

$z_{xx} = 4$ $\quad z_{yy} = 8$

Also $z_{xy} = 3$

For extreme point, set,

$z_x = 0$ and $z_y = 0$

$\Rightarrow \quad 4x + 3y - 5 = 0 \qquad \Rightarrow \quad 3x + 8y + 2 = 0$



Solving these equations we get,

$x = 2$ and $y = -1$.

Now, at $(x, y) = (2, -1)$,

$$z_{xx} = 4 > 0$$

and $\begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} 4 & 3 \\ 3 & 8 \end{vmatrix} = 32 - 9 = 23 > 0.$

This shows that z is minimum at (x, y) = (2, –1). And minimum value at the point is,

$$z = 8 - 6 + 4 - 10 - 2 = -6.$$

**(g) $z = x^2 - 4xy + 4y^2 - 5x + 2y$**

**Solution:** Given function is

$$z = x^2 - 4xy + 4y^2 - 5x + 2y$$

Then,

$z_x = 2x - 4y - 5$ and $z_y = -4x + 8y + 2$

$z_{xx} = 2$ $\quad z_{yy} = 8$

Also, $z_{xy} = -4$

For extreme point, set,

$z_x = 0$ and $z_y = 0$

$\Rightarrow \quad 2x - 4y - 5 = 0 \qquad \Rightarrow \quad -4x + 8y + 2 = 0$

Solving these equations we get,

$x = \frac{1}{6}$ and $y = \frac{4}{3}$

Now, at the point $(x, y) = \left(\frac{1}{6}, \frac{4}{3}\right)$,

$$z_{xx} = 2 > 0$$

and $\begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} 2 & -4 \\ -4 & 8 \end{vmatrix} = 16 - 16 = 0.$

This shows that the z gives no information at $(x, y) = \left(\frac{1}{6}, \frac{4}{3}\right)$.

**(h) $z = x^2 - y^2 - 2x + 4y + 6$**

**Solution**

Given function is

$$z = x^2 - y^2 - 2x + 4y + 6$$

Then,

$z_x = 2x - 2$ and $z_y = -2y + 4$

$z_{xx} = 2$ $\quad z_{yy} = -2$

Also $z_{xy} = 0$

For extreme point, set,

$z_x = 0$ and $z_y = 0$

$\Rightarrow \quad 2x - 2 = 0 \qquad \Rightarrow \quad -2y + 4 = 0$

Solving these equations we get,

$x = 1$ and $y = 2$.

Now, at the point $(x, y) = (1, 2)$,

$z_{xx} = 2 > 0$

and $\begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix} = -4 - 0 = -4 < 0$

This shows that z is saddle at $(x, ) = (1, 2)$. And the value of z at the point is

$z = 1 - 4 - 2 + 8 + 6 = 9.$

**(i) $z = x^2 + 2xy$**

**Solution:** Given function is

$z = x^2 + 2xy$

Then,

$z_x = 2x + 2y \quad$ and $\quad z_y = 2x$

$z_{xx} = 2 \qquad z_{yy} = 0$

Also, $z_{xy} = 2$

For extreme point, set,

$z_x = 0 \quad$ and $\quad z_y = 0$

$\Rightarrow \quad 2x + 2y = 0 \qquad \Rightarrow \quad 2x = 0$

Solving these equations we get,

$x = 0$ and $y = 0$.

Now, at the point $(x, y) = (0, 0)$,

$z_{xx} = 2 > 0$

and $\begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} 2 & 2 \\ 2 & 0 \end{vmatrix} = 0 - 4 = -4$

This shows that z is saddle at $(x, y) = (0, 0)$. And the value of z at the point is

$z = 0 + 0 = 0$

**(j) $z = x^2 + xy + y^2 + x - 4y + 5$**

**Solution:** Given function is

$z = x^2 + xy + y^2 + x - 4y + 5$

Then,

$z_x = 2x + y + 1 \quad$ and $\quad z_y = x + 2y - 4$

$z_{xx} = 2 \qquad z_{yy} = 2$

Also, $z_{xy} = 1$

For extreme point, set,

$z_x = 0 \quad$ and $\quad z_y = 0$

$\Rightarrow 2x + y + 1 = 0 \qquad \Rightarrow x + 2y - 4 = 0$

Solving these equations we get,

$x = -2$ and $y = 3$.



Now, at the point $(x, y) = (-2, 3)$,

$z_{xx} = 2 > 0$

and $\begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 = 3 > 0.$

This shows that z is minimum at $(x, y) = (-2, 3)$. And minimum value is

$z = 4 - 6 + 9 - 2 - 12 + 5$

$= -2$

**(k) $z = 3x^2 - xy + 2y^2 - 8x + 9y + 10$**

**Solution:** Given function is

$z = 3x^2 - xy + 2y^2 - 8x + 9y + 10$

Then,

$z_x = 6x - y - 8 \quad$ and $\quad z_y - x + 4y + 9$

$z_{xx} = 6 \qquad z_{yy} = 4$

Also, $z_{xy} = -1$

For extreme point, set,

$z_x = 0 \quad$ and $\quad z_y = 0$

$\Rightarrow 6x - y - 8 = 0 \qquad \Rightarrow -x + 4y + 9 = 0$

Solving these equations we get,

$x = 1$ and $y = -2$

Now, at the point $(x, y) = (1, -2)$,

$z_{xx} = 6 > 0$

and, $\begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} 6 & -1 \\ -1 & 4 \end{vmatrix} = 24 - 1 = 23 > 0$

This shows that z is minimum at $(1, -2)$. And minimum value at the point is

$z = 3 + 2 + 8 - 8 - 18 + 10 = -3.$

**(l) $z = x^3 - y^3 - 2xy + 6$**

**Solution:** Given function is

$z = x^3 - y^3 - 2xy + 6$

Then $z_x = 3x^2 - 2y \quad$ and $\quad z_y = -3y^2 - 2x$

$z_{xx} = 6x \qquad z_{yy} = -6y$

Also, $z_{xy} = -2$

For extreme point, set,

$z_x = 0 \quad$ and $\quad z_y = 0$

$\Rightarrow 3x^2 - 2y = 0 \qquad \Rightarrow -3y^2 - 2x = 0$

Solving these equations, we get,

$x = 0, \frac{2}{3}$ and $y = 0, \frac{2}{3}$

Now, at the point $(x, y) = (0, 0)$,

$z_{xx} = 6.0 = 0$

and, $\begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} 0 & -2 \\ -2 & 0 \end{vmatrix} = -4 < 0.$

This shows that z is saddle at (x, y) = (0, 0).

And value of z at the point is

$z = 0 - 0 - 0 + 6 = 6.$

Next, at the point $(x, y) = \left(\frac{2}{3}, \frac{2}{3}\right)$.

$z_{xx} = 6 \times \frac{2}{3} = 4 > 0$

and, $\begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} 4 & -2 \\ -2 & -4 \end{vmatrix} = -16 - 4 = -20 < 0.$

This shows that z is saddle at $(x, y) = \left(\frac{2}{3}, \frac{2}{3}\right)$.

And value of z at the point is

$$z = \frac{8}{27} - \frac{8}{27} - \frac{8}{9} + 6 = \frac{-8 + 54}{9} = \frac{46}{9}.$$

**(m) $z = 6x^2 - 2x^3 + 3y^2 + 6xy$**

**Solution:** Given function is

$z = 6x^2 - 2x^3 + 3y^2 + 6xy$

Then,

$z_x = 12x - 6x^2 + 6y$ and $z_y = 6y + 6x$

$z_{xx} = 12 - 12x$ $z_{yy} = 6$

Also, $z_{xy} = 6$

For extreme point, set,

$z_x = 0$ and $z_y = 0$

$\Rightarrow 12x - 6x^2 + 6y = 0 \quad \Rightarrow \quad 6y + 6x = 0$

Solving these equations we get,

$x = 0, 1$ and $y = 0, -1$.

Now, at the point (x, y) = (0, 0),

$z_{xx} = 12 - 0 = 12 > 0$

and, $\begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} 12 & 6 \\ 6 & 6 \end{vmatrix} = 72 - 36 = 36 > 0$

This shows that z is minimum at (0, 0). And, minimum value at the point is

$z = 0.$

Next, at the point (1, -1),

$z_{xx} = 12 - 12 = 0$

and, $\begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} 0 & 6 \\ 6 & 6 \end{vmatrix} = 0 - 36 = -36 < 0.$

This shows that z is saddle at (1, -1).



$z = 6 - 2 + 3 - 6 = 1.$

**(n) $z = x^3 + y^3 - 3xy + 15$**

**Solution:** Given function is

$z = x^3 + y^3 - 3xy + 15$

Then, $z_x = 3x^2 - 3y$ and $z_y = 3y^2 - 3x$

$z_{xx} = 6x$ $z_{yy} = 6y$

Also, $z_{xy} = -3$

For extreme point, set,

$z_x = 0$ and $z_y = 0$

$\Rightarrow 3x^2 - 3y = 0 \quad \Rightarrow 3y^2 - 3x = 0$

Solving these equations we get,

$x = 0, 1$ and $y = 0, 1$.

Now, at point (x, y) = (0, 0),

$z_{xx} = 6.0 = 0$

and, $\begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} 0 & -3 \\ -3 & 0 \end{vmatrix} = -9 < 0.$

This shows that z is saddle at (0, 0). And value of z at the point is

$z = 0 + 0 - 0 + 15 = 15.$

Next, at point (x, y) = (1, 1).

$z_{xx} = 6.1 = 6 > 0$

and, $\begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} 6 & -3 \\ -3 & 6 \end{vmatrix} = 36 - 9 = 27 > 0.$

This shows that z is minimum at (x, y) = (1, 1). And minimum value of z at the point is,

$z = 1 + 1 - 3 + 15 = 14.$

**(o) $z = 4xy - x^4 - y^4$**

**Solution:** Given function is

$z = 4xy - x^4 - y^4$

Then

$z_x = 4y - 4x^3$ and $z_y = 4x - 4y^3$

$z_{xx} = -12x^2$ $z_{yy} = -12y^2$

Also, $z_{xy} = 4$

For extreme point, set,

$z_x = 0$ and $z_y = 0$

$\Rightarrow 4y - 4x^3 = 0 \quad \Rightarrow 4x - 4y^3 = 0$

Solving these equations we get

$x = 1, -1, 0$ and $y = 1, -1, 0$

Now, at point (x, y) = (0, 0),

$$z_{xx} = -12 \,.\, 0 = 0 \quad \text{and} \quad \begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} 0 & 4 \\ 4 & 0 \end{vmatrix} = -16 < 0.$$

This shows that z is saddle at (0, 0).

And, value of z at (0, 0) is

$$z = 0.$$

Next, at point (x, y) = (1, 1)

$$z_{xx} = -12 < 0$$

and, $\begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} -12 & 4 \\ 4 & -12 \end{vmatrix} = 144 - 16 = 128 > 0$

This shows that z is maximum at (1, 1). And, the maximum value of z at (1, 1) is

$$z = 4 - 1 - 1 = 2$$

Next, at point (x, y) = (-1, -1)

$$z_{xx} = -12 < 0$$

and, $\begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} -14 & 4 \\ 4 & -12 \end{vmatrix} = 144 - 16 = 128 > 0$

This shows that z is maximum at (-1, -1). And maximum value of z at (-1, -1) is,

$$z = 4 - 1 - 1 = 2.$$

**(p) $u = 16 - (x + 2)^2 - (y - 2)^2$**

**Solution:** Given function is

$$u = 16 - (x + 2)^2 - (y - 2)^2$$

Then,

$$u_x = -2(x + 2) \quad \text{and} \quad u_y = -2(y - 2)$$
$$u_{xx} = -2 \qquad\qquad u_{yy} = -2$$

Also, $u_{xy} = 0$

For extreme point, set,

$$u_x = 0 \quad \text{and} \quad u_y = 0$$
$$\Rightarrow -2(x + 2) = 0 \qquad \Rightarrow -2(y - 2) = 0$$

Solving these equations we get,

$$x = -2 \text{ and } y = 2.$$

Now, at point (x, y) = (-2, 2),

$$u_{xx} = -2 < 0$$

and, $\begin{vmatrix} u_{xx} & u_{xy} \\ u_{yx} & u_{yy} \end{vmatrix} = \begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} = 4 > 0.$

This shows that u is maximum at (x, y) = (-2, 2). And, maximum value of u at (-2, 2) is

$$u = 16 - (-2 + 2)^2 - (2 - 2)^2 = 16.$$



**(q) $z = x^3 - x^2 - y^2 + xy$**

**Solution:** Given function is

$$z = x^3 - x^2 - y^2 + xy$$

Then,

$$z_x = 3x^2 - 2x + y \quad \text{and} \quad z_y = -2y + x$$
$$z_{xx} = 6x - 2 \qquad\qquad z_{yy} = -2$$

Also, $z_{xy} = 1$

For extreme point, set,

$$z_x = 0 \quad \text{and} \quad z_y = 0$$
$$\Rightarrow 3x^2 - 2x + y = 0 \qquad \Rightarrow -2y + x = 0$$

Solving these equations we get,

$$x = 0, \frac{1}{2} \text{ and } y = 0, \frac{1}{4}$$

Now, at point (x, y) = (0, 0),

$$z_{xx} = 0 - 2 = -2 < 0$$

and, $\begin{vmatrix} z_{xx} & z_{xy} \\ z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} = 4 - 1 = 3 > 0.$

This shows that z is maximum at (x, y) = (0, 0). And, maximum value of z at (0, 0) is,

$$z = 0.$$

**2. Find the extreme and stationary points of f.**

**(i) $f(x, y) = -x^2 - 4x - y^2 + 2y - 1$**

**Solution:** Given function is

$$f(x, y) = -x^2 - 4x - y^2 + 2y - 1$$

Then,

$$f_x = -2x - 4 \quad \text{and} \quad f_y = -2y + 2$$
$$f_{xx} = -2 \qquad\qquad f_{yy} = -2$$

Also, $f_{xy} = 0$

For extreme point, set,

$$f_x = 0 \quad \text{and} \quad f_y = 0$$
$$\Rightarrow -2x - 4 = 0 \qquad \Rightarrow -2y + 2 = 0$$

Solving these equations we get,

$$x = -2 \text{ and } y = 1.$$

Now, at point (x, y) = (-2, 1),

$$f_{xx} = -2 < 0$$

and, $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} = 4 > 0.$

This shows that f is maximum at (-2, 1) and maximum value at (-2, 1) is,

$$f(-2, 1) = -4 + 8 - 1 + 2 - 1 = 4.$$

**(ii)** $f(x, y) = x^2 + 4y^2 - x + 2y$

**Solution:** Given function is

$$f(x, y) = x^2 + 4y^2 - x + 2y$$

Then

$f_x = 2x - 1$ and $f_y = 8y + 2$

$f_{xx} = 2$ $\quad f_{yy} = 8$

Also, $f_{xy} = 0$

For extreme point, set,

$f_x = 0$ and $f_y = 0$

$\Rightarrow 2x - 1 = 0$ $\quad \Rightarrow 8y + 2 = 0$

Solving these equations we get,

$$x = \frac{1}{2} \text{ and } y = -\frac{1}{4}$$

Now, at point $(x, y) = \left(\frac{1}{2}, -\frac{1}{4}\right)$,

$$f_{xx} = 2 > 0$$

and, $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 8 \end{vmatrix} = 16 > 0$

This shows that $f(x, y)$ is minimum at $(x, y) = \left(\frac{1}{2}, -\frac{1}{4}\right)$ and minimum value of f is,

$$f\left(\frac{1}{2}, -\frac{1}{4}\right) = \frac{1}{4} + \frac{4}{16} - \frac{1}{2} - \frac{2}{4}$$

$$= \frac{1 + 1 - 2 - 2}{4} = \frac{-2}{4} = -\frac{1}{2}$$

**(iii)** $\mathbf{f(x, y) = x^2 + 2xy + 3y^2}$

**Solution:** Given function is

$$f(x, y) = x^2 + 2xy + 3y^2$$

Then,

$f_x = 2x + 2y$ and $f_y = 2x + 6y$

$f_{xx} = 2$ $\quad f_{yy} = 6$

Also, $f_{xy} = 2$

For extreme point, set,

$f_x = 0$ and $f_y = 0$

$\Rightarrow 2x + 2y = 0$ $\quad \Rightarrow 2x + 6y = 0$

Solving these equations we get,

$x = 0$ and $y = 0$.

Now, at point $(x, y) = (0, 0)$

$$f_{xx} = 2 > 0$$



and, $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2 & 2 \\ 2 & 6 \end{vmatrix} = 12 - 4 = 8 > 0$

This shows that f is minimum at (0, 0). And its minimum value at the point is,

$f(0, 0) = 0$.

**(iv)** $\mathbf{f(x, y) = x^3 + 3xy^2 - y^3}$

**Solution:** Given function is

$$f(x, y) = x^3 + 3xy^2 - y^3$$

Then,

$f_x = 3x^2 + 3y^2$ and $f_y = 6xy - 3y^2$

$f_{xx} = 6x$ $\quad f_{yy} = 6x - 6y$

Also, $f_{xy} = 6y$

For extreme point, set,

$f_x = 0$ and $f_y = 0$

$\Rightarrow 3x^2 + 3y^2 = 0$ $\quad \Rightarrow 6xy - 3y^2 = 0$

Solving these equations we get,

$x = 0, 1$ and $y = 0, -1$.

Now, at point $(x, y) = (0, 0)$,

$$f_{xx} = 6.0 = 0$$

and, $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0.$

This shows that f gives no information at (0, 0).

**(v)** $\mathbf{f(x, y, z) = 2x^2 + xy + 4y^2 + xy + z^2 + 2}$

**Solution:** Given function is

$$f(x, y, z) = 2x^2 + xy + 4y^2 + xy + z^2 + 2$$

Then,

$f_x = 4x + y + z$, $\quad f_y = x + 8y$ and $f_z = x + 2z$

$f_{xx} = 4$ $\quad f_{yy} = 8$ $\quad f_{zz} = 2$

$f_{xy} = 1$ $\quad f_{yz} = 0$ $\quad f_{zx} = 1$

For extreme point, set,

$f_x = 0$, $\quad f_y = 0$ and $f_z = 0$

$\Rightarrow 4x + y + z = 0$ $\quad \Rightarrow x + 8y = 0$ $\quad x + 2z = 0$

Solving these equations we get,

$x = 0$, $y = 0$ and $z = 0$.

Now, at point $(x, y, z) = (0, 0, 0)$,

(a) $f_{xx} = 4 > 0$

(b) $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 4 & 1 \\ 1 & 8 \end{vmatrix} = 32 - 1 = 31 > 0$

(c) $\begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix} = \begin{vmatrix} 4 & 1 & 1 \\ 1 & 8 & 0 \\ 1 & 0 & 2 \end{vmatrix} = 4(16-0) - 1(2-0) + 1(0-8)$

$= 64 - 2 - 8 = 54 > 0.$

This shows that f is minimum at (0, 0, 0) and minimum value is

$f(0, 0, 0) = 0 + 0 + 0 + 0 + 0 + 2 = 2.$

**(vi) $f(x, y, z) = 35 - (2x + 3)^2 - (y - 4)^2 - (z + 1)^2$**

**Solution:** Given function is

$$f(x, y, z) = 35 - (2x + 3)^2 - (y - 4)^2 - (z + 1)^2$$

Then,

$f_x = -2(2x + 3)$, $f_y = -2(y - 4)$ and $f_z = -2(z + 1)$

$f_{xx} = -4$ $\quad f_{yy} = -2$ $\quad f_z = -2$

$f_{xy} = 0$ $\quad f_{yz} = 0$ $\quad f_{zx} = 0$

For extreme point, set,

$f_x = 0$, $\quad f_y = 0$ and $\quad f_z = 0$

$\Rightarrow -2(2x + 3) = 0$ $\quad \Rightarrow -2(y - 4) = 0$ $\quad \Rightarrow -2(z + 1) = 0$

Solving these equations we get,

$$x = -\frac{3}{2}, y = 4, z = -1.$$

Now, at the point,

(a) $f_{xx} = -4 < 0$

(b) $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} -4 & 0 \\ 0 & -2 \end{vmatrix} = 8 > 0$

(c) $\begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix} = \begin{vmatrix} -4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{vmatrix} = -16 < 0$

This shows that f is maximum at the point. And maximum value is,

$$f\left(-\frac{3}{2}, 4, -1\right) = 35 - (-3 + 3)^2 - (4 - 4)^2 - (-1 + 1)^2$$

$$= 35.$$

**(vii) – (ix) Similar to (v) and (vi)**

**3. Similar to Q. No. 2.**

**4. Find the extreme value of $f = 48 - (x - 5)^2 - 3(y - 4)^2$ such that $x + 3y = 9$.**

**Solution:** Given that,

$$f = 48 - (x - 5)^2 - 3(y - 4)^2 \quad \text{......... (i)}$$

Such that, $x + 3y = 9 \Rightarrow x = 9 - 3y$.

Then (i) can be written as,

$$f(y) = 48 - (9 - 3y - 5)^2 - 3(y - 4)^2$$



$= 48 - (4 - 3y)^2 - 3(y - 4)^2$

$= 48 - 16 - 9y^2 + 24y - 3y^2 - 48 + 24y$

$= -12y^2 + 48y - 16$

So,

$f_y = -24y + 48$ and $f_{yy} = -24 < 0.$

For extreme point, set,

$f_y = 0 \Rightarrow -24y + 48 = 0$

$\Rightarrow y = 2.$

And, $f_{yy} = -24$ at $y = 2$.

Moreover,

$x = 9 - 6 = 3$ at $y = 2$

Thus, f(x, y) is maximum at (x, y) = (3, 2). And maximum value is

$f(3, 2) = 48 - (3 - 5)^2 - 3(2 - 4)^2$

$= 48 - 4 - 12$

$= 32.$

**5. Find the extreme value of $f = x^2 + y^2 + z^2$ such that $ax + by + cz = p$.**
**[1999 Q. No. 2(a); 2001 Q. No. 2(a)]**

**Solution:** Given that

$$f = x^2 + y^2 + z^2 \quad \text{...... (i)}$$

Such that, $ax + by + cz = p \Rightarrow z = \frac{p - ax - by}{c}$

Then (i) can be written as,

$$f = x^2 + y^2 + \left(\frac{p - ax - by}{c}\right)^2$$

So,

$f_x = 2x + \frac{2(p - ax - by)(-a)}{c}$ and $f_y = 2y + 2\left(\frac{p - ax - by}{c}\right)(-b)$

$f_{xx} = 2 + \frac{2a^2}{c}$ $\quad f_{yy} = 2 + \frac{2b^2}{c}$

Also, $f_{xy} = \frac{2ab}{c}$

For extreme point, set,

$f_x = 0$ and $f_y = 0$

$\Rightarrow 2x - \frac{2a}{c}(p - ax - by) = 0$ $\quad \Rightarrow 2y - \frac{2b}{c}(p - ax - by) = 0$

Solving we get,

$$x = \frac{pa}{a^2 + b^2 + c^2}, \quad y = \frac{pb}{a^2 + b^2 + c^2}$$

Now, at point (x, y),

$$f_{xx} = \left(2 + \frac{2a^2}{c}\right) > 0$$

$$\text{and, } \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2 + 2a^2/c & 2ab/c \\ 2ab/c & 2 + 2b^2/c \end{vmatrix}$$

$$= \frac{4}{c^2} \begin{vmatrix} c + a^2 & ab \\ ab & c + b^2 \end{vmatrix}$$

$$= \frac{4}{c^2} [(c + a^2)(c + b^2) - a^2b^2]$$

$$= \frac{4}{c^2} [c^2 + c(a^2 + b^2) + a^2b^2 - a^2b^2]$$

$$= \frac{4}{c^2} [c^2 + c(a^2 + b^2)] > 0$$

This shows that f is minimum at the point.

Here,

$$z = \frac{1}{c}(p - ax - by)$$

$$= \frac{1}{c}\left[p - a\left(\frac{pa}{a^2 + b^2 + c^2}\right) - b\left(\frac{pb}{a^2 + b^2 + c^2}\right)\right]$$

$$= \frac{1}{c\,(a^2 + b^2 + c^2)}(pa^2 + pb^2 + pc^2 - pa^2 - pb^2)$$

$$= \frac{pc}{a^2 + b^2 + c^2}$$

So, the maximum value of f is

$$f = \left(\frac{pa}{a^2 + b^2 + c^2}\right)^2 + \left(\frac{pb}{a^2 + b^2 + c^2}\right)^2 + \left(\frac{pc}{a^2 + b^2 + c^2}\right)^2$$

$$= p^2\left(\frac{a^2 + b^2 + c^2}{a^2 + b^2 + c^2}\right)$$

$$= \frac{p^2}{a^2 + b^2 + c^2}$$

**6. Find the maximum value of f = xyz, given x + y + z = 24.**

**Solution:** Given that,

$$f = xyz \qquad \text{........ (i)}$$

Such that, $x + y + z = 24 \Rightarrow z = 24 - x - y$

Then (i) can be written as

$$f = xy\,(24 - x - y)$$

$$= 24xy - x^2y - xy^2$$

So,

$$f_x = 24y - 2xy - y^2 \qquad \text{and} \qquad f_y = 24x - x^2 - 2xy$$

$$f_{xx} = -2y \qquad\qquad f_{yy} = -2x$$

$$f_{xy} = 24 - 2x$$

For extreme point, set,



$$f_x = 0 \qquad \text{and} \qquad f_y = 0$$

$$\Rightarrow 24y - 2xy - y^2 = 0 \qquad \Rightarrow 24x - x^2 - 2xy = 0$$

Solving these equations we get,

$x = 8$ and $y = 8$.

Now, at point $(x, y) = (8, 8)$,

$$f_{xx} = -16 < 0$$

$$\text{and, } \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} -16 & 8 \\ 8 & -16 \end{vmatrix} = 256 - 64 = 192 > 0$$

This shows that f is maximum. And maximum value of f is

$$f = (8)(8)(24 - 8 - 8) = 64\,(8) = 512.$$

**7. Find extreme value of $f = x^2 + y^2 + z^2$ such that $x + z = 1$ and $2y + z = 2$.**

**[2004 Spring Q. No. 2(a)]**

**Solution:** Given function is

$$f = x^2 + y^2 + z^2 \qquad \text{.......... (i)}$$

Such that, $x + z = 1$ and $2y + z = 2$

$$\Rightarrow x = 1 - z \qquad \Rightarrow y = \frac{1}{2}(2 - z)$$

Then (i) can be written as

$$f(z) = (1 - z)^2 + \left(1 - \frac{z}{2}\right)^2 + z^2 = 1 + z^2 - 2z + 1 + \frac{z^2}{4} - z + z^2$$

$$= 2 - 3z + \frac{9z^2}{4}$$

$$= \frac{1}{4}(8 - 12z + 9z^2) \qquad \text{........... (ii)}$$

So,

$$f_z = \frac{1}{4}(-12 + 18z) \qquad \text{and} \qquad f_{zz} = \frac{18}{4} = \frac{9}{2} > 0$$

For extreme point, set,

$$f_z = 0 \quad \Rightarrow \frac{1}{4}(-12 + 18z) = 0$$

$$\Rightarrow z = \frac{12}{18} = \frac{2}{3}$$

This shows that f is minimum at $z = \frac{2}{3}$. And minimum value of f at $z = \frac{2}{3}$ is

$$f\left(\frac{2}{3}\right) = \frac{1}{4}\left[8 - 12\left(\frac{2}{3}\right) + 9\left(\frac{2}{3}\right)^2\right]$$

$$= \frac{1}{4}(8 - 8 + 4) = 4.$$

8. Find the minimum value of $f = x^2 + xy + y^2 + 3z^2$ such that $x + 2y + 4z = 60$.

**Solution:** Given that,

$f = x^2 + xy + y^2 + 3z^2$ ........(i)

Such that, $x + 2y + 4z = 60$

$\Rightarrow x = 60 - 2y - 4z$

Then (i) becomes,

$f(y, z) = (60 - 2y - 4z)^2 + (60 - 2y - 4z)y + y^2 + 3z^2$

$= 3600 + 4y^2 + 16z^2 - 240y - 480z + 16yz + 60y - 2y^2 - 4yz + y^2 + 3z^2$

$= 3600 + 3y^2 - 180y + 12yz - 480z + 19z^2$

So,

$f_y = 6y - 180 + 12z$ and $f_z = 12 - 480 + 38z$

$f_{yy} = 6$ $f_{zz} = 38$

Also, $f_{yz} = 12$

For extreme point, set,

$f_y = 0$ and $f_z = 0$

$\Rightarrow 6y - 180 + 12 = 0$ $\Rightarrow 12y - 480 + 38z = 0$

Solving these equations we get,

$y = \frac{90}{7}$ and $z = \frac{60}{7}$

Then,

$$x = 60 - 2\left(\frac{90}{7}\right) - 4\left(\frac{60}{7}\right) = \frac{420 - 180 - 240}{7} = \frac{0}{7} = 0$$

Now, at point $\left(\frac{90}{7}, \frac{60}{7}\right)$

$f_{yy} = 6 > 0$

and $\begin{vmatrix} f_{yy} & f_{yz} \\ f_{yz} & f_{zz} \end{vmatrix} = \begin{vmatrix} 6 & 12 \\ 12 & 38 \end{vmatrix} = 228 - 144 = 84 > 0$

This shows that f is minimum at $\left(0, \frac{90}{7}, \frac{60}{7}\right)$. And the minimum value is

$$f = 0 + 0 + \left(\frac{90}{7}\right)^2 + 3\left(\frac{60}{7}\right)^2$$

$$= \frac{8100}{49} + \frac{10800}{49} = \frac{18900}{49} = \frac{2700}{7}$$

**9. Find the minimum value of $f = x^2 + y^2 + z^2$ such that $x + y + z = 1$ and $xyz = 1$.**

**Solution:** Given that,

$f = x^2 + y^2 + z^2$ ....... (i)

Such that, $x + y + z = 1 \Rightarrow y + z = 1 - x$



and $xyz = -1 \Rightarrow yz = -\frac{1}{x}$

Then (i) can be written as,

$f = x^2 + (y + z)^2 - 2yz$

$= x^2 + (1 - x)^2 + \frac{2}{x}$

$= 2x^2 - 2x + 1 + \frac{2}{x}$

So, $f_x = 4x - 2 - \frac{2}{x^2}$ and $f_{xx} = 4 + \frac{4}{x^3}$

For extreme point, set,

$f_x = 0 \Rightarrow 4x - 2 - \frac{2}{x^2} = 0$

$\Rightarrow 4x^3 - 2x^2 - 2 = 0$

$\Rightarrow 4x^3 - 4x^2 + 2x^2 - 2x + 2x - 2 = 0$

$\Rightarrow (x - 1)(4x^2 + 2x + 2) = 0$

Solving we get, $x = 1$, other result is invalid (imaginary result).

Now, at $x = 1$,

$f_{xx} = 4 + 4 = 8 > 0$

This shows that f is minimum at $x = 1$ and minimum value of f is,

$f = 1 + (1 - 1)^2 + \frac{2}{1} = 1 + 2 = 3.$

**10. Find the extreme value for the function $f(x, y) = x^2 + y^2$ under the condition $x + 4y = 2$.**

**Solution:** Given that,

$f(x, y) = x^2 + y^2$ ........(i)

Such that, $x + 4y = 2 \Rightarrow x = 2 - 4y$

Then (i) becomes,

$f(y) = (2 - 4y)^2 + y^2$

$= 4 + 16y^2 - 16y + y^2$

$= 4 - 16y + 17y^2$

So,

$f_y = -16 + 34y$ and $f_{yy} = 34 > 0$

For extreme point, set,

$f_y = 0 \Rightarrow -16 + 34y = 0$

$\Rightarrow y = \frac{8}{17}$

This shows that f is minimum and minimum value is

$$f = 4 - 16\left(\frac{8}{17}\right) + 17\left(\frac{8}{17}\right)^2$$

$$= \frac{68 - 128 + 64}{17} = \frac{4}{17}$$

**11. Find the minimum value of $f(x, y, z) = x^2 + y^2 + z^2$ such the $x + y + z = 3a^2$.**
**[2004 Fall, 2007 Fall, 2008 Fall, 2008 Spring Q. No. 2(a)]**

**Solution:** Given that,

$f(x, y, z) = x^2 + y^2 + z^2$ .........(i)

Such that, $x + y + z = 3a^2$

$\Rightarrow z = 3a^2 - x - y$

Then (i) becomes,

$$f(x, y) = x^2 + y^2 + (3a^2 - x - y)^2$$
$$= x^2 + y^2 + 9a^4 - 6a^2x - 6a^2y + 2xy + x^2 + y^2$$
$$= 2x^2 + 2y^2 + 9a^4 - 6a^2x - 6a^2y + 2xy$$

So,

$f_x = 4x - 6a^2 + 2y$ and $f_y = 4y - 6a^2 + 2x$

$f_{xx} = 4$ $\qquad$ $f_{yy} = 4$

Also, $f_{xy} = 2$

For extreme point, set,

$f_x = 0$ and $f_y = 0$

$\Rightarrow 2x - 6a^2 + 2y = 0$ $\qquad$ $\Rightarrow 2y - 6a^2 + 2x = 0$

Solving these equations we get,

$x = a^2$ and $y = a^2$

Then $z = 3a^2 - a^2 - a^2 = a^2$

Now, at point $(x, y) = (a^2, a^2)$,

$f_{xx} = 2 > 0$

and, $\begin{vmatrix} f_{yy} & f_{yz} \\ f_{yz} & f_{zz} \end{vmatrix} = \begin{vmatrix} 4 & 2 \\ 2 & 4 \end{vmatrix} = 16 - 4 = 12 > 0.$

This shows that f is minimum at $(a^2, a^2, a^2)$ and minimum value is

$f(a^2, a^2, a^2) = a^2 + a^2 + a^2 = 3a^2.$

**12. If the sum of three positive number is 8, what is the maximum value of their product.**

**Solution:** Let the numbers are x, y, z.

Given that the sum of these positive numbers is s.

So, $x + y + z = 8 \Rightarrow z = 8 - x - y$ .........(i)

And we have observe the maximum value of product of x, y, z.

So, let, $f = xyz$

Then,

$$f = xy(8 - x - y) \qquad [\because \text{using (i)}]$$
$$= 8xy - x^2y - xy^2$$

So that,

$f_x = 8y - 2xy - y^2$ and $f_y = 8x - x^2 - 2xy$



$f_{xx} = -2y$ $\qquad$ $f_{yy} = -2x$

Also, $f_{xy} = 8 - 2x - 2y$

For extreme point, set,

$f_x = 0$ and $f_y = 0$

$\Rightarrow 8y - 2xy - y^2 = 0$ $\qquad$ $\Rightarrow 8x - x^2 - 2xy = 0$

Solving these equations we get,

$x = \frac{8}{3}$ and $y = \frac{8}{3}$

Then,

$z = 8 - x - y \Rightarrow z = 8 - \frac{8}{3} - \frac{8}{3} = \frac{8}{3}$

Now, at point $(x, y) = \left(\frac{8}{3}, \frac{8}{3}\right)$.

$$f_{xx} = -2\left(\frac{8}{3}\right) = -\frac{16}{3} < 0$$

and, $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} -16/3 & -8/3 \\ -8/3 & -16/3 \end{vmatrix} = \frac{256}{9} - \frac{64}{9} = \frac{192}{9} > 0$

This shows that if maximum and the maximum value is,

$$f = \frac{8}{3} \times \frac{8}{3} \times \frac{8}{3} = \frac{512}{27}.$$

**13. Find the minimum values of $x^2 + y^2 + z^2$ where**
**(i) $x + y + z = 3a$ (ii) $xyz = a^3$**

**Solution: (i)** Given function is

$f(x, y, z) = x^2 + y^2 + z^2$ ........ (1)

Such that $x + y + z = 3a$

$\Rightarrow z = 3a - x - y$

Then (1) becomes,

$$f = x^2 + y^2 + (3a - x - y)^2$$
$$= x^2 + y^2 + 9a^2 + x^2 + y^2 - 6ax - 6ay + 2xy$$
$$= 2x^2 + 2y^2 + 2xy - 6ax - 6ay + 9a^2 \quad ....... (2)$$

So,

$f_x = 4a + 2y - 6a$ and $f_y = 4y + 2x - 6a$

$f_{xx} = 4$ $\qquad$ $f_{yy} = 4$

Also, $f_{xy} = 2$

For extreme point, set,

$f_x = 0$ and $f_y = 0$

$\Rightarrow 4x + 2y - 6a = 0$ $\qquad$ $\Rightarrow 4y + 2x - 6a = 0$

Solving these equations we get,

$x = a$ and $y = a$.

Then, $z = 3a - a = a$.

Now, at $(x, y) = (a, a)$.

$$f_{xx} = 4 > 0$$

and, $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 4 & 2 \\ 2 & 4 \end{vmatrix} = 16 - 4 = 12 > 0.$

This shows that f is minimum at (a, a, a). And minimum value of f at the point is,

$$f = a^2 + a^2 + a^2 = 3a^2$$

**Solution: (ii)** Given function is

$$f(x, y, z) = x^2 + y^2 + z^2 \qquad \text{.........(1)}$$

Such that $xyz = a^3 \Rightarrow z = \dfrac{a^3}{xy}$

Then (1) becomes,

$$f = x^2 + y^2 + \left(\frac{a^3}{xy}\right)^2$$

$$= x^2 + y^2 + \frac{a^6}{x^2y^2} \qquad \text{.......(2)}$$

So,

$$f_x = 2x - \frac{2a^6}{x^3y^2} \qquad \text{and} \qquad f_y = 2y - \frac{2a^6}{x^2y^3}$$

$$f_{xx} = 2 + \frac{6a^6}{x^4y^2} \qquad\qquad f_{yy} = 2 + \frac{6a^6}{x^2y^4}$$

Also, $f_{xy} = -\dfrac{4a^6}{x^3y^3}$

For extreme point, set,

$$f_x = 0 \qquad \text{and} \qquad f_y = 0$$

$$\Rightarrow 2x - \frac{2a^6}{x^3y^2} = 0 \qquad \Rightarrow 2y - \frac{2a^6}{x^2y^3} = 0$$

$$\Rightarrow x^4y^2 - a^6 = 0 \qquad \Rightarrow x^2y^4 - a^6 = 0$$

Solving these equations, we get,

$x = a$ and $y = a$

Then, $z = \dfrac{a^3}{a^2} = a$

Now, at point $(x, y) = (a, a)$,

$$f_{xx} = 2 + \frac{6a^6}{a^4a^2} = 2 + 6 = 8 > 0.$$

and, $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 8 & 4 \\ 4 & 8 \end{vmatrix} = 64 - 16 = 48 > 0$

This shows that f is minimum at $(x, y, z) = (a, a, a)$. And the minimum value of f at the point is,

$$f = a^2 + a^2 + a^2 = 3a^2.$$



**14. Find the extreme value for the function $x^2 + y^2$ under the condition $x + 4y = 2$.**

**Solution:** Given function is

$$f(x, y) = x^2 + y^2$$

Such that, $x + 4y = 2$

Repeated Question, **See Q. No. 10.**

**15. Show that the function $f(x, y) = y^2 + x^2y + x^4$ has a minimum value of (0, 0).**

**Solution:** Given function is

$$f(x, y) = y^2 + x^2y + x^4$$

So,

$$f_x = 2xy + 4x^3 \qquad \text{and} \qquad f_y = 2y + x^2$$

$$f_{xx} = 2y + 12x^2 \qquad\qquad f_{yy} = 2$$

Also, $f_{xy} = 2x$

For extreme point, set,

$$f_x = 0 \qquad \text{and} \qquad f_y = 0$$

$$\Rightarrow 2xy + 4x^3 = 0 \qquad \Rightarrow 2y + x^2 = 0$$

Solving these equations we get,

$x = 0$ and $y = 0$.

Now, at $(x, y) = (0, 0)$,

$f_{xx} = 0$ and $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 0 & 2 \end{vmatrix} = 0$

This shows that f gives no information at (0, 0).

**16. Find the minimum value of the function $x^2 + y^2 + z^2$ subject to $ax + by + cz = a + b + c$**

**Solution:** Given function is

$$f = x^2 + y^2 + z^2 \qquad \text{.....(1)}$$

Such that, $ax + by + cz = a + b + c$

$$\Rightarrow z = \frac{1}{c}(a + b + c - ax - by)$$

Then (1) becomes,

$$f = x^2 + y^2 + \frac{1}{c^2}(a + b + c - ax - by)^2 \qquad \text{.......(2)}$$

So, $f_x = 2x - \dfrac{2a}{c^2}(a + b + c - ax - by)$ and $f_y = 2y - \dfrac{2b}{c^2}(a + b + c - ax - by)$

$$f_{xx} = 2 + \frac{2a^2}{c^2} \qquad\qquad f_{yy} = 2 + \frac{2b^2}{c^2}$$

Also, $f_{xy} = \dfrac{2ab}{c^2}$

For extreme point, set,

$f_x = 0$ and $f_y = 0$

$\Rightarrow 2x - \frac{2a}{c^2}(a + b + c - ax - by) = 0 \qquad \Rightarrow 2y - \frac{2b}{c^2}(a + b + c - ax - by) = 0$

Solving we get,

$$x = \frac{a(a + b + c)}{a^2 + b^2 + c^2} \quad \text{and} \quad y = \frac{b(a + b + c)}{a^2 + b^2 + c^2}$$

Then, $z = \frac{c(a + b + c)}{a^2 + b^2 + c^2}$

Now, at point $(x, y) = \left(\frac{a + b + c}{a^2 + b^2 + c^2}\right)(a, b)$

$$f_{xx} = 2 + \frac{2a^2}{c^2} > 0$$

$$\text{and,} \quad \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2 + 2a^2/c^2 & 2ab/c^2 \\ 2ab/c^2 & 2 + 2b^2/c^2 \end{vmatrix}$$

$$= \frac{4}{c^4}\begin{vmatrix} a^2 + c^2 & ab \\ ab & b^2 + c^2 \end{vmatrix}$$

$$= \frac{4}{c^4}(a^2b^2 + a^2c^2 + b^2c^2 + c^4 - a^2b^2)$$

$$= \frac{4}{c^4}(a^2c^2 + b^2c^2 + c^4) > 0$$

This shows that f is minimum at $\left(\frac{a + b + c}{a^2 + b^2 + c^2}\right)(a, b, c)$. And minimum value is,

$$f = \left(\frac{a + b + c}{a^2 + b^2 + c^2}\right)^2 (a^2 + b^2 + c^2) = \frac{(a + b + c)^2}{a^2 + b^2 + c^2}$$

**17. A rectangular box, open at the top, is to have a volume of 32 c.c. Find the dimension of the box requiring least material for its construction.**

**[2009 Fall, 2009 Spring Q. No. 2(b)]**

**Solution:** Let length of box = x, breadth of box = y and height of box = z.

Given that volume of the box = 32 cc.

Since we have, volume of a box, v = xyz.

So, $xyz = 32 \Rightarrow z = \frac{32}{xy}$ ........(1)

Since the material to construct a box, is used in its surface.

And, we have the surface area of a open (at top) box is,

$$S = xy + 2yz + 2zx$$

$$= xy + (2y + 2z)\frac{32}{xy} \quad [\because \text{using (i)}]$$

$$= xy + \frac{64}{x} + \frac{64}{y} \quad .........(2)$$



So,

$$s_x = y - \frac{64}{x^2} \quad \text{and} \quad s_y = x - \frac{64}{y^2}$$

$$s_{xx} = +\frac{128}{x^3} \qquad s_y = \frac{128}{y^3}$$

Also, $s_{xy} = 1$

For extreme point, set,

$s_x = 0$ and $s_y = 0$

$\Rightarrow y - \frac{64}{x^2} = 0 \qquad \Rightarrow x - \frac{64}{y^2} = 0.$

Solving these equations we get,

x = 4 and y = 4.

Then, $z = \frac{32}{16} = 2$.

Now, at point (x, y) = (4, 4),

$$s_{xx} = \frac{128}{(4)^3} = \frac{128}{64} = 2 > 0$$

$$\text{and,} \begin{vmatrix} s_{xx} & s_{xy} \\ s_{yx} & s_{yy} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 = 3 > 0.$$

This shows that s is minimum at (4, 4, 2).

And minimum value of s is,

$s = 16 + 16 + 66 = 48$

Thus, the dimensions of the box are 4 cm, 4 cm and 2 cm.

**18. Find the dimension of the rectangular box, open at the top of maximum capacity whose surface is 432 sq.**

**Solution:** Consider, length of box = x, breadth box = z and height box = y.

Given that, surface area (s) = 432 sq. cm.

$$\Rightarrow xz + 2(xy + yz) = 432$$

$$\Rightarrow x(z + 2y) = 2yz = 432$$

$$\Rightarrow x = \frac{432 - 2yz}{z + 2y}$$

And, the capacity (v) = xyz

$$\Rightarrow v = \left(\frac{432 - 2z}{z + 2y}\right) yz = \frac{432yz - 2y^2z^2}{z + 2y}$$

$$\text{Then, } v_y = \frac{(z + 2y)(432z - 4yz^2) - 2(432yz - 2y^2z^2)}{(z + 2y)^2}$$

$$= \frac{432z^2 - 4yz^3 + 864yz - 8y^2z^2 - 864yz + 4y^2z^2}{(z + 2y)^2}$$

$$= \frac{432z^2 - 4yz^3 - 4y^2z^2}{(z + 2y)^2}$$

$$\text{And, } v_{yy} = \frac{(-4z^3 - 8yz^2)(z+9y)^2 - (432z^2 - 4yz^3 - 4y^2z^2)\,2(z+2y)\times 2}{(z+2y)^4}$$

$$= \frac{(-2z^3 - 8yz^2)(z+2y)^2 - (432z^2 - 4yz^3 - 4y^2z^2)(4z+8y)}{(z+2y)^4}$$

$$= \frac{(z+2y)(432y - 4y^2z) - (432yz - 2y^2z^2)}{(z+2y)^2}$$

$$= \frac{864y^2 - 2y^2z^2 - 8y^3z}{(z+2y)^2}$$

$$\text{Also, } v_{zz} = \frac{(z+2y)^2(-4y^2z - 8y^3) - (864y^2 - 2y^2z^2 - 8y^3z)\,2(z+2y)}{(z+2y)^4}$$

$$v_{yz} = \frac{(z+2y)^2(864z - 12yz^2 - 8y^2z) - (432z^2 - 4yz^3 - 4y^2z^2)\,2(z+2y)}{(z+2y)^4}$$

For maximum & minimum, set,

$v_y = 0$ and $v_z = 0$

$$\Rightarrow \frac{432z^2 - 4yz^3 - 4y^2z^2}{(z+2y)^2} = 0 \qquad \Rightarrow \frac{864y^2 - 2y^2z^2 - 8y^3z}{(z+2y)^2} = 0$$

$$\Rightarrow 432z^2 - 4yz^3 - 4y^2z^2 = 0 \qquad \Rightarrow 2y^2(432 - z^2 - 4yz) = 0$$

$$\Rightarrow 4z^2(108 - yz - y^2) = 0 \qquad \Rightarrow 432 - z^2 - 4yz = 0$$

$$\Rightarrow y(y+z) = 108 \quad \text{......(i)} \qquad \Rightarrow z(z+4y) = 432 \quad \text{........(ii)}$$

Dividing (i) by (ii) then,

$$\frac{y(y+z)}{z(z+4y)} = \frac{108}{432}$$

$$\Rightarrow 4y^2 + 4yz = z^2 + 4yz$$

$$\Rightarrow z = 2y$$

Putting the value of z in (i) so that,

$$y(y+2x) = 108$$

$$\Rightarrow 3y^2 = 108 \Rightarrow y^2 = 36 \Rightarrow y = 6$$

Then (i) and (ii) gives, $z = 12$, $x = 12$

$$v_{yy} = \frac{(-4\times12^3 - 8\times6\times12^2)(12+26)^2 - (432\times12^2 - 4\times6\times12^3 - 4\times6^2\times12^2)(4\times12+8\times6)}{(12+2\times6)^4}$$

$$= \frac{-7962624}{331776} = -24$$

And, $v_{zz} = -6$ and $v_{yz} = -6$

Now, $(v_{yy}v_{zz} - v_{yz}^2) = -24 \times -6 - (-6)^2 = 108 > 0$ and $v_{yy} = -24 < 0$

Therefore, volume is maximum when dimension is (12, 6, 12).

And, maximum value is, $v_{max.} = 12 \times 6 \times 12 = 864\ cm^3$

**(19) Prove that of all the rectangle parallelepiped of the same volume, the cube has the least surface.** **[2010 Spring Q. No. 2(b)].**

**Solution:** Let, length = x, breadth = y height = z

Then the volume of the parallelepiped is, (v) = xyz



$$\Rightarrow z = \frac{v}{xy}$$

Since the all part of the parallelepiped is closed.

So, the surface area of parallelepiped (s) = 2(xy + yz + zx)

$$\Rightarrow s = 2\left[xy + y.\frac{v}{xy} + \frac{v}{xy}.x\right]$$

$$\Rightarrow s = 2\left(xy + \frac{v}{x} + \frac{v}{y}\right)$$

Then different the above equation w. r. t. 'x'

$$s_x = 2\left(y - \frac{v}{x^2}\right) \qquad \text{And} \qquad s_{xx} = \frac{4v}{x^3}.$$

And, different the above equation w. r. t. 'y'

$$s_y = 2\left(x - \frac{v}{y^2}\right) \qquad \text{And} \qquad s_{yy} = \frac{4v}{y^3}$$

Also, $s_{xy} = 2$

For maxima and minima,

$s_x = 0$ $\qquad$ $s_y = 0$

$$\Rightarrow 2\left(y - \frac{v}{x^2}\right) = 0 \qquad \Rightarrow 2\left(x - \frac{v}{y^2}\right) = 0$$

$$\Rightarrow y = \frac{v}{x^2} \quad \text{.........(i)} \qquad \Rightarrow x = \frac{v}{y^2} \quad \text{........(ii)}$$

From (i) and (ii) we get,

$$y = \frac{v}{\frac{v^2}{y^4}} = \frac{v}{v^2} \times y^4 \Rightarrow v = y^3$$

So, (ii) gives, x = y. And therefore, z = x = y.

Here, $s_{xx} = \frac{4v}{x^3} = \frac{4x^3}{x^3} = 4 > 0$

And $(s_{xx}.s_{yy} - s_{xy}^2) = \frac{4v}{x^3}.\frac{4v}{y^3} - 4 = 4(4) - 4 = 16 - 4 = 12 > 0.$

So, s is minimum at x = y = z.

Therefore, the given parallelepiped is cube become x = y = z and surface area is least.

**(20) Prove that of all the rectangular parallelepiped of given surface, cube has the maximum volume.**

**Solution:** Let, length of parallelepiped (*l*) = x

Breadth of parallelepiped (b) = z

Height of parallelepiped (h) = y

Since the all part of the parallelepiped is closed.

Surface area of parallelepiped (s) = 2(xy + yz + zx)

$$y = \frac{s - 2zx}{2(x+z)}$$

And, the volume of the parallelepiped is, $(v) = xyz$

$$\Rightarrow v = xz\frac{(s - 2zx)}{2(x+z)} = \frac{sxz - 2z^2x^2}{2(x+z)}$$

Then, $v_x = \frac{1}{2}\frac{(x+z)(sz - 4z^2x) - (sxz - 2z^2x^2)}{(x+z)^2}$

$$= \frac{1}{2}\frac{sz^2 - 2z^2x^2 - 4z^3x}{(x+z)^2}$$

And, $v_{xx} = \frac{1}{2}\frac{(x+z)^2(-4xz^2 - 4z^3) - 2(x+z)(sz^2 - 2x^2z^2 - 4xz^3)}{(x+z)^4}$

$$= \frac{1}{2}\frac{(-4z^2x^2 - 4xz^3 - 4z^3x - 4z^4 - 2sz^2 + 4z^2x^2 + 8xz^3)}{(x+z)^3}$$

$$= \frac{-(2z^4 + sz^2)}{(x+z)^3}$$

Also, $v_{xz} = \frac{1}{2}\frac{(x+z)^2(2sz - 4zx^2 - 12xz^2) - 2(x+z)(sz^2 - 2z^2x^2 - 4xz^3)}{(x+z)^4}$

$$= \frac{1}{2}\frac{(2sxz - 4zx^3 - 12x^2z^2 + 2sz^2 - 4z^2x^2 - 12xz^3 - 2sz^2 + 4x^2z^2 + 8xz^3)}{(x+z)^3}$$

$$v_{xz} = \frac{sxz - 2x^3z - 6x^2z^2 - 4xz^3}{(x+z)^3}$$

Similarly,

$$v_z = \frac{1}{2}\frac{(sx^2 - 2x^2z^2 - 4x^3z)}{(x+z)^2} \quad \text{And} \quad v_{zz} = \frac{-(2x^4 + sx^2)}{(x+z)^3}$$

For extreme point, set,

$v_x = 0$

$\Rightarrow \frac{1}{2}\frac{sz^2 - 2x^2z^2 - 4z^3x}{(x+z)^2}$

$\Rightarrow sz^2 - 2x^2z^2 - 4xz^3 = 0$

$\Rightarrow s - 2x^2 - 4xz = 0$

$\Rightarrow z = \frac{s - 2x^2}{4x}$ ....(i)

$v_z = 0$

$\Rightarrow \frac{1}{2}\frac{(sx^2 - 2x^2z^2 - 4x^3z)}{(x+z)^2}$

$\Rightarrow s - 2z^2 - 4xz = 0$

$\Rightarrow x = \frac{s - 2z^2}{4z}$ .....(ii)

Then using (ii) to (i) we get,

$$z = \frac{s - 2(s - 2z^2)^2/16z^2}{4\frac{(s - 2z^2)}{4z}}$$

$$= \frac{(16sz^2 - 2s^2 + 8sz^2 - 8z^4)z}{(16z^2(s - 2z^2)}$$

$\Rightarrow 16sz^2 - 32z^4 = 16sz^2 - 2s^2 + 8sz^2 - 8z^4$

$\Rightarrow 2s^2 - 8sz^2 - 24z^4 = 0$

$\Rightarrow s^2 - 4sz^2 - 12z^4 = 0$



$\Rightarrow (s - 2z^2)^2 = (4z^2)^2$

$\Rightarrow s - 2z^2 = 4z^2 \quad \Rightarrow s = 6z^2 \quad \Rightarrow z = \sqrt{\frac{s}{6}}$

So that, at $z = \sqrt{\frac{s}{6}}$ we get, $x = \sqrt{\frac{s}{6}}$, $y = \sqrt{\frac{s}{6}}$

Thus, the given rectangular parallelepiped is cube because sides are $x = y = z$.

And at the point $x = y = z = \sqrt{\frac{s}{6}}$, the value of $v_{xx}$, $v_{zz}$ and $v_{xz}$ is,

$$v_{xx} = \frac{-(2z^4 + 2sz^2)}{(z+x)^3} = -\frac{\left\{2\left(\frac{\sqrt{s}}{\sqrt{6}}\right)^4 + 2s\left(\frac{\sqrt{s}}{\sqrt{6}}\right)^2\right\}}{\left(\frac{\sqrt{s}}{\sqrt{6}} + \frac{\sqrt{s}}{\sqrt{6}}\right)^3} = \frac{-\left(\frac{2s^2}{36} + 2s\frac{s}{6}\right)}{\left(\frac{2\sqrt{s}}{\sqrt{6}}\right)^3} = \frac{-7\sqrt{s}}{4\sqrt{6}}$$

Similarly, $v_{zz} = \frac{-7\sqrt{s}}{4\sqrt{6}}$

And, $v_{xz} = \frac{sxz - 2x^3z - 6x^2z^2 - 2xz^3}{(z+x)^3}$

$$= \frac{2 \times \frac{s}{6} - 2\frac{s^{3/2}}{6\sqrt{6}} \times \frac{s^{1/2}}{\sqrt{6}} - 6 \times \frac{s}{6} \times \frac{s}{6} - 2\frac{s^{1/2}}{\sqrt{6}} \cdot \frac{s^{3/2}}{6\sqrt{6}}}{\left(\frac{2\sqrt{s}}{\sqrt{6}}\right)^3}$$

$$= \frac{\frac{s^2}{6} - \frac{s^2}{18} - \frac{s^2}{18} - \frac{s^2}{6}}{\frac{8s^{3/2}}{6\sqrt{6}}} = -\frac{2s^2}{18} \times \frac{6\sqrt{6}}{8s^{3/2}} = -\frac{s^{1/2}\sqrt{6}}{12} = -\frac{s^{1/2}}{6\sqrt{6}}$$

Now,

$$v_{xx}.v_{zz} - (v_{xz})2 = \left(\frac{-7s^{1/2}}{4\sqrt{6}}\right) \times \left(\frac{-7s^{1/2}}{4\sqrt{6}}\right) - \left(\frac{-s^{1/2}}{6\sqrt{6}}\right)^2 = \frac{49s}{96} = \frac{s}{216} > 0$$

$$v_{xx} = \frac{-7\sqrt{s}}{4\sqrt{6}} < 0 \quad \text{(max.)}$$

The given rectangular parallelepiped is cube and having maximum volume.

**(21) (a) Find the points on the ellipse $x^2 + 2y^2 = 1$, where $f(x, y) = xy$ has its extreme values.**

**Solution:** Given that, $f(x, y) = xy$

Such that, $x^2 + 2y^2 = 1 \Rightarrow x^2 = 1 - 2y^2$

Then, $(f(x, y))^2 = x^2y^2 = (1 - 2y^2)\,y^2$

$= y^2 - 2y^4$

So, $f_y = 2y - 8y^3$ and $f_{yy} = 2 - 24y^2$.

For extreme point, set,

$f_y = 0 \Rightarrow 2y - 8y^3 = 0$

$\Rightarrow 1 - 4y^2 = 0$[Being $2 \neq 0$ and $y = 0$ gives $f = 0$ which is impossible]

$\Rightarrow y = \pm\frac{1}{2}$

And, at $y = \pm\frac{1}{2}$, we get $x = \pm\sqrt{1 - \frac{1}{2}} = \pm\sqrt{\frac{1}{2}}$

Now, at $y = \pm\frac{1}{2}$,

$f_{yy} = 2 - 24\frac{1}{4} = 2 - 6 = -4 < 0.$

So, the function is minimum at (x, y).

**(b) Find the maximum value of $f(x, y) = 9 - x^2 - y^2$ on the line $x + 3y = 12$.**

**Solution:** Given that, $f = 9 - x^2 - y^2$

Such that. $x = 12 - 3y$

Then,

$$f = 9 - (12 - 3y)^2 - y^2 = 9 - (144 - 72y + 9y^2) - y^2$$
$$= 9 - 144 + 72y - 9y^2 - y^2$$
$$= -10y^2 + 72y - 135$$

So, $f_y = -20y + 72$

And, $f_{yy} = -20 < 0$. This shows that f is maximum.

For extreme point, set,

$f_y = 0 \Rightarrow -20y = -72$

$\Rightarrow y = \frac{72}{20} = \frac{18}{5}$

And $x = 12 - 3 \times \frac{18}{5} = \frac{60 - 54}{5} = \frac{6}{5}$

Thus, the function has maxima at point $(x, y) = \left(\frac{1}{5}, \frac{18}{5}\right)$.

**(c) Find the extreme values of $f(x, y) = x^2y$ on the line $x + y = 3$.**

**Solution:** Given that, $f(x, y) = x^2y$

Such that, $x + y = 3 \Rightarrow y = 3 - x$.

Then, $f = x^2(3 - x)$

$= 3x^2 - x^3$

So, $f_x = 6x - 3x^2$ and $f_{xx} = 6 - 6x$

For extreme point, set,

$f_x = 0 \Rightarrow 6x = 3x^2 = 0$

$\Rightarrow 3x^2 = 6x$

$\Rightarrow x = 2.$



And, at $x = 2$, we get, $y = 1$.

Now, at $(x, y) = (2, 1)$, $f_{xx} = 6 - 6x = 6 - 6 \times 2 = -6 < 0$.

This shows that f is maximum at the point.

Thus, the given value is extreme at (2, 1).

**(d) Find the minimum value of $x + y$ subject of $xy = 16$.**

**Solution:** Given that, $f = x + y$

Such that, $xy = 16 \Rightarrow y = \frac{16}{x}$

Then, $f = x + \frac{16}{x} = \frac{x^2 + 16}{x}$

So, $f_x = \frac{x.2x - (x^2 + 16)}{x^2} = \frac{2x^2 - x^2 - 16}{x^2} = \frac{x^2 - 16}{x^2}$

And, $f_{xx} = \frac{x^2(2x) - (x^2 - 16)2x}{x^4} = \frac{2x^3 - 2x^3 + 32x}{x^4} = \frac{32}{x^3}$

For extreme point, set,

$fx = 0 \Rightarrow \frac{x^2 - 16}{x^2} = 0$

$\Rightarrow x^2 = 16$

$\Rightarrow x = \pm 4$

At point $x = 4$,

$f_{xx} = \frac{32}{64} = \frac{1}{2} > 0$

This shows that the function f is minimum at the point.

And at point $x = 4$, $y = \frac{16}{4} = 4$.

Thus, the function is minimum at (4, 4).

And at point $x = -4$,

$f_{xx} = \frac{32}{-64} = \frac{-1}{2} < 0$

This shows that the function f is maximum at the point.

And the minimum value (x, y) = (4, 4) be,

$f = 4 + 4 = 8.$

**(e) Find the maximum value of xy subject to $x + y = 16$.**

**Solution:** Given that, $f = xy$

Such that, $x + y = 16 \Rightarrow x = 16 - y$.

Then, $f = (16 - y).y = 16y - y^2$

So, $f_y = 16 - 2y$ and $f_{yy} = -2 < 0$

This shows that the function f is maximum.

For extreme point, set,

$f_y = 0 \Rightarrow 16 - 2y = 0.$

$\Rightarrow \quad y = 8.$

Therefore, $f_{max} = 16(8) - (8)^2 = 64$

Thus the maximum value of the function is 64.

**(g) The temperature T at any point (x, y, z) in space is $T = 400xyz^2$. Find the highest temperature on the unit sphere $x^2 + y^2 + z^2 = 1$.**

**Solution:** Given that, $T = 400\,xyz^2$

Such that, $x^2 + y^2 + z^2 = 1 \Rightarrow z^2 = 1 - x^2 - y^2$

So, $T = 400xy(1 - x^2 - y^2)$

$= 400xy - 400x^3y - 400xy^3$

Then, $T_x = 400y - 1200x^2y - 400y^3$ and $T_y = 400x - 400x^3 - 1200xy^2$

And, $T_{xx} = -2400\,xy$ $\quad T_{yy} = -2400xy$

Also, $T_{xy} = 400 - 1200x^2 - 1200y^2$

For extreme point, set,

| | |
|---|---|
| $T_x = 0$ | $T_y = 0$ |
| $\Rightarrow 400y - 1200x^2y - 400y^3 = 0$ | $\Rightarrow 400x - 400x^3 - 1200xy^2 = 0$ |
| $\Rightarrow 400y(1 - 3x^2 - y^2) = 0$ | $\Rightarrow 400x(1 - x^2 - 3y^2) = 0$ |
| $\Rightarrow 1 - 3x^2 = y^2$ | $\Rightarrow 1 - x^2 - 3y^2 = 0$ |
| $\Rightarrow y = \pm\sqrt{1 - 3x^2}$ ...... (i) | $\Rightarrow y = \pm\sqrt{\dfrac{1 - x^2}{3}}$ ....... (ii) |

From (i) and (ii)

$$\pm\sqrt{1 - 3x^2} = \pm\sqrt{\frac{1 - x^2}{3}} \Rightarrow 1 - 3x^2 = \frac{1 - x^2}{3}$$

$$\Rightarrow 3 - 9x^2 = 1 - x^2$$

$$\Rightarrow 8x^2 = 2$$

$$\Rightarrow x = \pm\frac{1}{2}$$

When $x = \frac{1}{2}$, $y = \sqrt{1 - \frac{3}{4}} = \frac{1}{2}$ and $z^2 = 1 - \frac{1}{4} - \frac{1}{4} = \frac{4 - 2}{4} \Rightarrow z = \frac{1}{2}$

Then at the point,

$$T_{xx} = -2400 \times \frac{1}{2} \times \frac{1}{2} = -600 = T_{yy}$$

And $\quad T_{xy} = 400 - 1200 \times \frac{1}{4} - 1200 \times \frac{1}{4} = -200.$

Now at the points are $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$

$T_{xx} = -600 < 0$

& $\quad (T_{xx}\,T_{yy} - T_{xy}^2) = -600 \times -600 - (-200)^2$

$= 360{,}000 - 40{,}000 > 0$

Thus, the highest temperature at point $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ is



$$T_{highest} = 400 \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = 50.$$

**(h) Show that $f(x, y) = y^2 + 2x^2y + 2x^4$ has a minimum value at (0, 0).**

**Solution:** Given that, $f = y^2 + 2x^2y + 2x^4$

Then, $f_x = 4xy + 8x^3$ and $f_y = 2y + 2x^2$

And $f_{xx} = 4y + 24x^2$ $\quad f_{yy} = 2$

Also, $f_{xy} = 4x$

For extreme point, set,

| | |
|---|---|
| $f_x = 0$ | $f_y = 0$ |
| $\Rightarrow 4xy + 8x^3 = 0$ | $\Rightarrow 2y + 2x^2 = 0$ |
| $\Rightarrow 4y + 8x^2 = 0$ | $\Rightarrow y = -x^2$ ......(ii) |
| $\Rightarrow y = -2x^2$ .......(i) | |

From (i) and (ii),

$-2x^2 = -x^2 \Rightarrow x^2 = 0 \Rightarrow x = 0.$

And, at $x = 0$, we get, $y = 0$.

Now, at $(x, y) = (0, 0)$

$f_{xx} = 0.$

This shows that f gives no information at $(x, y) = (0, 0)$.

**(i) Determine the maximum or minimum value of the function $20 - x^2 - y^2 = z^2$.**

**Solution:** Given that, $z^2 = 20 - x^2 - y^2 \quad \Rightarrow z = \sqrt{20 - x^2 - y^2}$

Then, $z_x = \dfrac{-x}{\sqrt{20 - x^2 - y^2}}$.

$$\text{And, } z_{xx} = \frac{\sqrt{20 - x^2 - y^2}\,(-1) - (-x)\dfrac{(-2x)}{2\sqrt{20 - x^2 - y^2}}}{20 - x^2 - y^2}$$

$$= \frac{x^2 + y^2 - 20 - x^2}{(20 - x^2 - y^2)\sqrt{20 - x^2 - y^2}}$$

$$= \frac{y^2 - 20}{(20 - x^2 - y^2)\sqrt{20 - x^2 - y^2}}$$

Also,

$$z_y = \frac{-y}{\sqrt{20 - x^2 - y^2}}$$

$$z_{yy} = \frac{x^2 - 20}{(20 - x^2 - y^2)\sqrt{20 - x^2 - y^2}}$$

And,

$$z_{xx} = \frac{0-(-x)\dfrac{(-2y)}{2\sqrt{20-x^2-y^2}}}{20-x^2-y^2}$$

$$= \frac{-xy}{(20-x^2-y^2)\sqrt{20-x^2-y^2}}$$

For extreme point, set,

$z_x = 0$ $\qquad$ $z_y = 0$

$\Rightarrow \dfrac{-x}{\sqrt{20-x^2-y^2}} = 0$ $\qquad$ $\Rightarrow \dfrac{-y}{\sqrt{20-x^2-y^2}} = 0$

$\Rightarrow x = 0$ $\qquad$ $\Rightarrow y = 0$

Then, at $(x, y) = (0, 0)$,

$$z_{xx} = \frac{-20}{20\sqrt{20}} = \frac{-1}{\sqrt{20}}, \qquad z_{yy} = \frac{-1}{\sqrt{20}} \quad \text{and} \quad z_{xy} = 0.$$

Now, at $(x, y) = (0, 0)$,

$$z_{xx} = \frac{-1}{\sqrt{20}} < 0.$$

$$z_{xx} \cdot z_{yy} - z_{xy}^2 = \left(\frac{-1}{\sqrt{20}}\right)\left(\frac{-1}{\sqrt{20}}\right) - 0 = \frac{1}{20} > 0.$$

This shows that z is maximum at $x = 0$, $y = 0$.

And the maximum value is, $z = 20$.

## OTHER QUESTIONS FROM SEMESTER END EXAMINATION

**2000, 2002 (II) Q. No. 2(a)**

**Find the minimum value of $u = x^2 + y^2 + z^2$ when $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$.**

**Solution:** We have, $f(x, y, z) = x^2 + y^2 + z^2$ and $\phi(x, y, z) = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$

Let $F = f + \lambda\phi$

Here, $f_x = 2x$, $\quad f_{xx} = 2$, $\quad f_{yx} = 0 = f_{xy}$ $\quad f_{zx} = 0 = f_{zx}$

$f_y = 2y$, $\quad f_{yy} = 2$, $\quad f_{yz} = f_{zy} = 0$

$f_z = 2z$, $\quad f_{zz} = 2$, $\quad f_{xz} = 0 = f_{zx}$

$\phi_x = -\frac{1}{x^2}$, $\quad \phi_y = -\frac{1}{y^2}$, $\quad \phi_z = -\frac{1}{z^2}$

We know, for extreme values:

$$F_x = 0 \Rightarrow f_x + \lambda\phi_x = 0 \quad \Rightarrow 2x - \frac{\lambda}{x^2} = 0 \quad \Rightarrow x = \left(\frac{\lambda}{2}\right)^{1/3}$$



$$F_y = 0 \Rightarrow f_y + \lambda\phi_y = 0 \quad \Rightarrow 2x - \frac{\lambda}{y^2} = 0 \quad \Rightarrow y = \left(\frac{\lambda}{2}\right)^{1/3}$$

$$F_z = 0 \Rightarrow f_z + \lambda\phi_z = 0 \quad \Rightarrow 2z - \frac{\lambda}{z^2} = 0 \quad \Rightarrow z = \left(\frac{\lambda}{2}\right)^{1/3}$$

$$F_\lambda = 0 \Rightarrow \phi = 0 \quad \Rightarrow \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1 \quad \Rightarrow \lambda = 54$$

Therefore, $x = 3$, $y = 3$, and $z = 3$.

Thus we get extreme value at (3, 3, 3).

Here,

$$|H_1| = \begin{vmatrix} 0 & \phi_x & \phi_y \\ \phi_x & f_{xx} & y_{yx} \\ \phi_y & f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 0 & -1/x^2 & -1/y^2 \\ -1/y^2 & 2 & 0 \\ -1/x^2 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 0 & -1/9 & -1/9 \\ -1/9 & 2 & 0 \\ -1/9 & 0 & 2 \end{vmatrix}$$

$$= -\frac{4}{81} < 0$$

$$|H_2| = \begin{vmatrix} 0 & \phi_x & \phi_y & \phi_z \\ \phi_x & f_{xx} & y_{yx} & f_{zx} \\ \phi_y & f_{xy} & f_{yy} & f_{zy} \\ \phi_z & f_{xz} & f_{yz} & f_{zz} \end{vmatrix} = \begin{vmatrix} 0 & -1/9 & -1/9 & -1/9 \\ -1/9 & 2 & 0 & 0 \\ -1/9 & 0 & 2 & 0 \\ -1/9 & 0 & 0 & 2 \end{vmatrix}$$

$$= \frac{1}{9}\begin{vmatrix} -1/9 & 0 & 0 \\ -1/9 & 2 & 0 \\ -1/9 & 0 & 2 \end{vmatrix} - \frac{1}{9}\begin{vmatrix} -1/9 & 2 & 0 \\ -1/9 & 0 & 0 \\ -1/9 & 0 & 2 \end{vmatrix} + \frac{1}{9}\begin{vmatrix} -1/9 & 2 & 0 \\ -1/9 & 0 & 2 \\ -1/9 & 0 & 0 \end{vmatrix}$$

$$= -\frac{4}{27} < 0$$

Therefore $f(x, y, z)$ is minimum at (3, 3, 3) and the minimum value is

$f(3, 3, 3) = 3^2 + 3^2 + 3^2 = 27.$

**2002 Q. No. 2(a)**

**Find the minimum value of $x^2 + y^2 + z^2$ having given $lx + my + nz = k$.**

**Solution:** See Exercise 11.1 Q. No. 5 with replacing a by *l*, b by m, c by n and p by k.

**2003 Fall Q. No. 2(a)**

**Write down the necessary condition that f(x, y, z) to have maximum or minimum value. Find the minimum value of $u = x^2 + xy + y^2 + 3z^2$ subject is the condition $x + 2y + 4z = 60$.**

**Solution:** First Part: See the condition.

Second Part: See Exercise 11.1 Q. No. 8.

**2006 Fall; 2011 Fall Q. No. 2(a)**

**If the sum of the dimension of a rectangular swimming pool is given. Prove that the amount of water in the pool is maximum when it is a cube.**

**Solution:** Let x, y and z be length, breadth and height of rectangular swimming pool.

Then we have $x + y + z = p$ (given) ........(1)

The amount of water $V = xyz$ ............(2)

We have to prove that the amount of water in the pool is maximum when it is a cube.

Here we have,

$$V = xyz \Rightarrow V = xy(p - x - y) \Rightarrow V = pxy - x^2y - xy^2$$

Then, $V_x = py - 2xy - y^2$, $\quad V_y = px - x^2 - 2xy$

$V_{xx} = -2y$, $\quad V_{xy} = p - 2x - 2y = V_y$, $\quad V_{yy} = -2x$

For extreme value, $V_x = 0 \Rightarrow py - 2xy - y^2 = 0 \Rightarrow p - 2x - y = 0$

and $\quad V_y = 0 \Rightarrow px - x^2 - 2xy = 0 \Rightarrow p - x - 2y = 0$

Since $x \neq 0$ and $y \neq 0$

We have,

$p - 2x - y = 0 \quad \Rightarrow \quad y = p - 2x$

and $\quad p - x - 2y = 0 \quad \Rightarrow \quad p - x - 2p + 4x = 0$

Solving we get,

$x = p/3,\ y = p/3$ a and $z = p/3$

At (p/3, p/3, p/3)

$$V_{xx} = -2p/3 = \frac{-2}{3}p < 0$$

and $\quad \begin{vmatrix} V_{xx} & V_{yx} \\ V_{xy} & V_{yy} \end{vmatrix} = \begin{vmatrix} -2p/3 & -p/3 \\ -p/3 & -2p/3 \end{vmatrix} = \frac{4p^2}{9} - \frac{p^2}{9} = \frac{3p^2}{9} = p^2/3 > 0.$

Thus V is maximum at (p/3, p/3, p/3). Thus the amount of water in the pool is maximum when it is a cube.

<u>2006 Spring Q. No. 2(a)</u>

**What are the criteria of a function of two independent variables to have extreme values? Find the extreme value of $x^2 + y^2 + z^2$ when $1/x + 1/y + 1/z = 1$.**

**Solution:** For criteria see the theoretical part of this chapter.

For the problem, see the solution of 2000.

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