Exercise 6.2

A. Find the general solution of the following equations:

(i) $(x + y + 1)y' = 1$.

Solution: Given equation is

$$(x + y + 1)y' = 1 \quad \text{......(1)}$$

Put $x + y + 1 = u$ then $1 + y' = u'$ where $u' = \frac{du}{dx}$. Then (1) becomes,

$$u(u' - 1) = 1 \Rightarrow uu' - u = 1 \Rightarrow \frac{uu'}{1+u} = 1$$

$$\Rightarrow \left(1 - \frac{1}{1+u}\right) u' = 1$$

Integrating we get,

$$u - \log(1 + u) = x + c_1$$

$$\Rightarrow (x + y + 1) - \log(1 + x + y + 1) = x + c_1$$

$$\Rightarrow y - \log(x + y + 1) = x + c_1 - x - 1$$

$$= c \qquad \text{for } c_1 - 1 = c$$

$$\Rightarrow y - \log(x + y + z) = c$$

This is required general solution of (1).

(ii) $(x + y)^2 y' = a^2$

Solution: Given equation is

$$(x + y)^2 y' = a^2 \quad .....(1)$$

Put $x + y = u$ then $1 + y' = u'$ where $u' = \frac{du}{dx}$. Then (1) becomes,

$$u^2(u' - 1) = a^2 \Rightarrow u^2u' = a^2 + u^2$$

$$\Rightarrow \frac{u^2}{a^2 + u^2} u' = 1$$

$$\Rightarrow \left(1 - \frac{a^2}{a^2 + u^2}\right) u = 1$$

Integrating we get,

$$u - a^2 . \left(\frac{1}{a}\right) \tan^{-1}\left(\frac{u}{a}\right) = x + c$$

$$\Rightarrow u - a\tan^{-1}\left(\frac{u}{a}\right) = x + c \Rightarrow x + y - a\tan^{-1}\left(\frac{x+y}{a}\right) = x + c$$

$$\Rightarrow y - a\tan^{-1}\left(\frac{x+y}{a}\right) = c$$

$$\Rightarrow \tan^{-1}\left(\frac{x+y}{a}\right) = \frac{y-c}{a}$$

$$\Rightarrow x + y = a\tan\left(\frac{y-c}{a}\right)$$

(iii) $xy' = x + y$

Solution: Given equation is

$$xy' = x + y \quad .....(1)$$

$$\Rightarrow y' = 1 + \frac{y}{x} \quad .....(2)$$

Put $\frac{y}{x} = u$ then $y = xu$. So, $y' = u + xu'$. Then (2) becomes,

$$u + xu' = 1 + u$$

$$\Rightarrow xu' = 1 \Rightarrow u' = \frac{1}{x}$$

$$\Rightarrow du = \frac{dx}{x}$$

Integrating we get,

$$u = \log(x) + c \Rightarrow \frac{y}{x} = \log(x) + c$$

$$\Rightarrow y = x\log(x) + cx$$

This is the required general solution.

(iv) $x^2y' = y^2 + xy + x^2$

Solution: Given equation is

$$x^2y' = y^2 + xy + x^2 \quad .....(1)$$

$$\Rightarrow y' = \frac{y^2}{x^2} + \frac{y}{x} + 1 \quad ....(2)$$

Put $\frac{y}{x} = u \Rightarrow y = xu$. Then $y' = u + xu'$. So, (2) becomes,

$$u + xu' = u^2 + u + 1 \Rightarrow xu' = u^2 + 1$$

$$\Rightarrow \frac{du}{u^2 + 1} = \frac{dx}{x}$$

Integrating we get,

$$\tan^{-1}(u) = \log(x) + c$$

$$\Rightarrow u = \tan(\log(x) + c) \Rightarrow \frac{y}{x} = \tan(\log(x) + c)$$

$$\Rightarrow y = x\tan(\log(x) + c)$$

(v) $y' = \frac{y - x}{y + x} = \frac{y/x - 1}{y/x + 1}$

Solution: Given equation is

$$y' = \frac{y - x}{y + x} = \frac{y/x - 1}{y/x + 1} \quad .....(1)$$

Put, $\frac{y}{x} = u \Rightarrow y = xu$ then $y' = xu' + u$. So, (1) becomes,

$$xu' + u = \frac{u - 1}{u + 1} \Rightarrow xu' = \frac{u - 1}{u + 1} - u = \frac{u - 1 - u^2 - u}{u + 1} = -\frac{u^2 + 1}{u + 1}$$

$$\Rightarrow \left(\frac{u+1}{u^2+1}\right) du = \frac{dx}{x}$$

$$\Rightarrow \left[\frac{1}{2}\cdot\frac{2u}{u^2+1} + \frac{1}{u^2+1}\right] du = \frac{dx}{x}$$

Integrating we get,

$$\frac{1}{2}\log(u^2+1) + \tan^{-1}(u) = \log(x) + c_1$$

$$\Rightarrow \frac{1}{2}\log\left(\frac{x^2+y^2}{x^2}\right) + \tan^{-1}\left(\frac{y}{x}\right) = \log(x) + c_1$$

$$\Rightarrow \log(x^2+y^2) - 2\log(x^2) + 2\tan^{-1}\left(\frac{y}{x}\right) = c$$

$$\Rightarrow \log(x^2+y^2) - 2\log(x^2) + 2\tan^{-1}\left(\frac{y}{x}\right) = c$$

$$\Rightarrow \log\left(\frac{x^2+y^2}{x^4}\right) + 2\tan^{-1}\left(\frac{y}{x}\right) = c$$

This is required general solution.

(vi) $y' = \sin(x+y) + \cos(x+y)$

Solution: Given equation is

$$y' = \sin(x+y) + \cos(x+y) \quad \text{......(1)}$$

Put $x + y = u$ then $1 + y' = u'$. Then (1) becomes,

$$u' - 1 = \sin u + \cos u$$

$$\Rightarrow \frac{du}{1 + \sin x + \cos x} = dx$$

Integrating we get,

$$\int \frac{du}{1 + \sin x + \cos x} = \int dx \quad \text{......(2)}$$

Set, $\tan\frac{u}{2} = t \tan \sec^2\left(\frac{u}{2}\right)\frac{du}{2} = dt \Rightarrow du = \frac{2dt}{1+t^2}$

Also, $\sin u = \frac{2t}{1+t^2}$ and $\cos u = \frac{1-t^2}{1+t^2}$

Therefore,

$$\int \frac{du}{1+\sin x+\cos x} = \int \frac{2dt}{(1+t^2)+2t+(1-t^2)}$$

$$= \frac{2}{2}\int \frac{dt}{1+t} = \log(1+t) + c = \log\left(1+\tan\frac{u}{2}\right) + c$$

Thus (2) gives

$$\log\left(1+\tan\frac{u}{2}\right) = x + c$$

$$\Rightarrow \log\left(1+\tan\left(\frac{x+y}{2}\right)\right) = x + c$$

(vii) $xy' - y = x\sqrt{x^2+y^2}$

Solution: Given equation is

$$xy' - y = x\sqrt{x^2+y^2} \Rightarrow x\frac{dy}{dx} - y = x\sqrt{x^2+y^2}$$

$$\Rightarrow \frac{dy}{dx} = \sqrt{x^2+y^2} = \frac{y}{x}$$

Put $y = ux$ then $\frac{dy}{dx} = u + x\frac{du}{dx}$. So that,

$$u + x\frac{du}{dx} = \sqrt{x^2+x^2u^2} + u \Rightarrow x\frac{du}{dx} = x\sqrt{1+u^2}$$

$$\Rightarrow \frac{du}{\sqrt{1+u^2}} = dx$$

Integrating we get,

$$\log(u+\sqrt{1+u^2}) = x + \log c \Rightarrow u + \sqrt{1+u^2} = ce^x$$

$$\Rightarrow \frac{y}{x} + \sqrt{1+\frac{y^2}{x^2}} = ce^x$$

$$\Rightarrow y + \sqrt{x^2+y^2} = cxe^x$$

(viii) $\cos(x+y)\,dy = dx$

Solution: Given equation is

$$\cos(x+y)\,dy = dx \quad \text{........(1)}$$

Put $x + y = u$ then $1 + y' = u'$. Then (1) becomes,

$$\cos u\,(u' - 1) = 1$$

$$\Rightarrow \cos u.\,u' = 1 + \cos u$$

$$\Rightarrow \frac{\cos u.\,u'}{1+\cos u} = 1$$

$$\Rightarrow \left(\frac{\cos\frac{u}{2} - \sin\frac{2u}{2}}{2\cos^2\frac{u}{2}}\right)\cdot\frac{du}{dx} = 1 \Rightarrow \left(1 - \tan^2\frac{u}{2}\right) du = 2dx$$

$$\Rightarrow \left(1 - \tan^2\frac{u}{2} + 1\right) du = 2dx$$

$$\Rightarrow \left(2 - \sec^2\frac{u}{2}\right) du = 2dx$$

Integrating we get,

$$2u - \frac{\tan\frac{u}{2}}{\frac{1}{2}} = 2x + 2c \Rightarrow 2u - 2\tan\frac{u}{2} = 2x + 2c$$

$$\Rightarrow u - \tan\frac{u}{2} = x + c$$

$$\Rightarrow x + y - \tan\left(\frac{x+y}{2}\right) = x + c$$

$$\Rightarrow y - \tan\left(\frac{x+y}{2}\right) = c$$

$$\Rightarrow \tan\left(\frac{x+y}{2}\right) = y - c$$

(ix) $y' = (y - x)^2$

Solution: Given equation is

$y' = (y - x)^2$ ....(1)

Put. $y - x = u$ then $y' - 1 = u'$. Then (1) becomes,

$1 + u' = u^2$

$$\Rightarrow \frac{du}{u^2 - 1} = dx$$

Integrating we get,

$$\int \frac{du}{u^2-1} = \int dx \Rightarrow \frac{1}{2}\log\left(\frac{u-1}{u+1}\right) = x + \log(c_1)$$

$$\Rightarrow \log\left(\frac{u-1}{u+1}\right) = 2x + \log(c) \qquad \text{for } c = c_1^2$$

$$\Rightarrow \frac{u-1}{u+1} = ce^{2x}$$

$$\Rightarrow u - 1 = (u+1)\,ce^{2x}$$

$$\Rightarrow u(1 - ce^{2x}) = 1 + ce^{2x}$$

$$\Rightarrow u = \frac{1 + ce^{2x}}{1 - ce^{2x}}$$

$$\Rightarrow y = x + \left(\frac{1 + ce^{2x}}{1 - ce^{2x}}\right)$$

This is required general solution.

(x) $xy' = e^{-xy} - y$

Solution: Given equation is

$xy' = e^{-xy} - y$ ....(1)

Put $xy = u$ then $xy' + y = u'$ $\Rightarrow xy' = u' - \frac{u}{x}$. Then (1) becomes,

$$u' - \frac{u}{x} = e^{-u} - \frac{u}{x} \Rightarrow u' = e^{-u}$$

$$\Rightarrow e^{u}\,du = dx$$

Integrating we get,

$$e^{u} + x + c \Rightarrow e^{xy} = x + c \Rightarrow xy = \log(x + c)$$

$$\Rightarrow y = \frac{1}{x}\log(x + c)$$

This is required general solution.

(xi) $y' = \frac{y - x + 1}{y - x + 5}$

Solution: Given equation is

$$y' = \frac{y - x + 1}{y - x + 5} \qquad ....(1)$$

Put $y - x = u$ then $y' - 1 = u'$. Then (1) becomes

$$1 + u' = \frac{u+1}{u+5} \Rightarrow u' = \frac{u+1}{u+5} - 1 = \frac{-4}{u+5}$$

$$\Rightarrow (4 + 5)\,du = -4dx$$

Integrating we get,

$$\frac{u^2}{2} + 5u = -4x + c_1$$

$$\Rightarrow u^2 + 10u = -x + c \qquad \text{for } c = 2c_1$$

$$\Rightarrow (y - x)^2 + 10(y - x) = -x + c$$

$$\Rightarrow (y - x)^2 + 10y - 2x = c$$

This is required general solution.

(xii) $y' = \frac{1 - 2y - 4x}{1 + y + 2x}$

Solution: Given equation is

$$y' = \frac{1 - 2y - 4x}{1 + y + 2x} = \frac{1 - 2(y + 2x)}{1 + (y + 2x)} \qquad ......(1)$$

Put, $y = 2x + u$ then $y' + 2 = u'$. Then (1) becomes,

$$u' - 2 = \frac{1 - 2u}{1 + u} \Rightarrow u' = \frac{1 - 2u}{1 + u} + 2 = \frac{1 - 2u + 2 + 2u}{1 + u} = \frac{3}{1 + u}$$

$$\Rightarrow (1 + u)\,du = 3dx$$

Integrating we get,

$$u + \frac{u^2}{2} = 3x + c \Rightarrow 2u + u^2 = 6x + c$$

$$\Rightarrow 2y + 4x + (y + 2x)^2 = 6x + c$$

$$\Rightarrow (y + 2x)^2 + 2y - 2x = c$$

This is required general solution.

(xiii) $\frac{dy}{dx} + 1 = e^{x+y}$

Solution: Given equation is

$$\frac{dy}{dx} + 1 = e^{x+y} \quad ....(1)$$

Put $x + y = u$ then $1 + \frac{dy}{dx} = \frac{du}{dx}$. Then (1) becomes,

$$1 + \frac{du}{dx} + 1 = e^u \Rightarrow \frac{du}{dx} = e^u - 2$$

$$\Rightarrow (e^u - 2)\, du = dx$$

Integrating we get,

$$e^u - 2u = x + c \Rightarrow e^{x+y} - 2x - 2y = x + c$$

$$\Rightarrow e^{x+y} - 2x - 2y = c$$

(xiv) $\frac{dy}{dx} + 1 = e^{x-y}$

Solution: Given equation is

$$\frac{dy}{dx} + 1 = e^{x-y} \quad ....(1)$$

Put $x - y = u$ then $1 - \frac{dy}{dx} = \frac{du}{dx}$. Then (1) becomes,

$$1 - \frac{du}{dx} + 1 = e^u \Rightarrow \frac{du}{dx} = 2 + e^u$$

$$\Rightarrow (2 + e^u)\, du = dx$$

Integrating we get,

$$2u + e^u = x + c$$

$$\Rightarrow 2x - 2y + e^{x-y} = x + c$$

$$\Rightarrow 3x - 2y + e^{x-y} = c$$

(xv) $(x^2 + 2xy + y^2 + 1)\frac{dy}{dx} = x + y$

Solution: Given equation is

$$(x^2 + 2xy + y^2 + 1)\frac{dy}{dx} = x + y \quad ......(1)$$

$$\Rightarrow (x + y)^2 + 1 \cdot \frac{dy}{dx} = (x + y) \quad .....(2)$$

Put $x + y = u$ then $1 + \frac{dy}{dx} = \frac{du}{dx}$. Then (2) becomes,

$$(u^2 + 1)\left(\frac{du}{dx} - 1\right) = u \quad \Rightarrow (u^2 + 1)\frac{du}{dx} = (u + u^2 + 1)$$

$$\Rightarrow \left(\frac{u^2 + 1}{u^2 + u + 1}\right) du = dx$$

$$\Rightarrow \left(1 - \frac{1}{u^2 + u + 1}\right) du = dx$$

Taking integration on both sides,

$$\int du - \int \frac{du}{u^2 + u + 1} = \int dx \quad ....(3)$$

Here,

$$\int \frac{du}{u^2 + u + 1} = \int \frac{du}{\left(u + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{u + \frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + c_1$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2u + 1}{\sqrt{3}}\right) + c_1$$

Then (3) becomes,

$$u - \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2u + 1}{\sqrt{3}}\right) = x + c \Rightarrow x + y - \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2x + 2y + 1}{\sqrt{3}}\right) = x + c$$

$$\Rightarrow y - \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2x + 2y + 1}{\sqrt{3}}\right) = c$$

(xvi) $\frac{\mathbf{x\,dx + y\,dy}}{\mathbf{x\,dy - y\,dx}} = \sqrt{\frac{\mathbf{1 - (x^2 + y^2)}}{\mathbf{x^2 + y^2}}}$

Solution: Here,

$$\frac{x\,dx + y\,dy}{x\,dy - y\,dx} = \sqrt{\frac{1 - (x^2 + y^2)}{x^2 + y^2}} \quad ....(i)$$

Put $x = r\cos\theta$, $y = r\sin\theta$. Then $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$. Then,

$$2r\frac{dr}{dx} = 2x + 2y\frac{dy}{dx} \qquad \text{and} \qquad \frac{d\theta}{dx} = \frac{1}{1 + \left(\frac{y}{x}\right)^2}\left(\frac{x\frac{dy}{dx} - y}{x^2}\right)$$

$$\Rightarrow r\,dr = x\,dx + y\,dy \qquad \Rightarrow \frac{d\theta}{dx} = \frac{1}{x^2 + y^2}\left(\frac{x\,dy - y\,dx}{dx}\right)$$

$$\Rightarrow r^2 d\theta = x\,dy - y\,dx$$

Then (i) becomes,

$$\frac{r\,dr}{r^2\,d\theta} = \sqrt{\frac{1-r^2}{r^2}} \Rightarrow \frac{dr}{d\theta} = \sqrt{1-r^2}$$

$$\Rightarrow \frac{dr}{\sqrt{1-r^2}} = d\theta$$

$$\Rightarrow \sin^{-1}(r) = \theta + c$$

$$\Rightarrow r = \sin(\theta + c)$$

$$\Rightarrow r^2 = \sin^2(\theta + c)$$

$$\Rightarrow x^2 + y^2 = \sin^2\left(\tan^{-1}\left(\frac{y}{x}\right) + c\right).$$

B. Solve the following initial value problem.

(i) $2x^2yy' = \tan(x^2y^2) - 2xy^2,\ y(1) = \sqrt{\frac{\pi}{2}}$

Solution: Given equation is

$$2x^2yy' = \tan(x^2y^2) - 2xy^2 \quad \text{with } y(1) = \sqrt{\frac{\pi}{2}}$$

Here,

$$2x^2yy' = \tan(x^2y^2) - 2xy^2 \quad \text{.......(1)}$$

Put, $x^2y^2 = u$ then $2x^2y.\ y' + 2xy^2 = u'$. Then (1) becomes,

$$u' = \tan u \Rightarrow \frac{du}{\tan u} = dx$$

$$\Rightarrow \frac{\cos u}{\sin u}du = dx$$

Integrating we get,

$$\log(\sin u) = x + c \Rightarrow \sin u = e^{(x+c)}$$

$$\Rightarrow \sin(x^2y^2) = e^{x+c} \quad \text{......(2)}$$

Since, we have $y(1) = \sqrt{\frac{\pi}{2}}$. So, (1) gives

$$\sin\left(\frac{\pi}{2}\right) = e^{1+c} \Rightarrow 1 = e^{1+c} \Rightarrow 1 + c = \log(1) = 0 \Rightarrow c = -1.$$

Thus, (2) becomes,

$$\sin(x^2y^2) = e^{x-1}$$

(ii) $y' = \frac{y-x}{y-x-1},\ y(-5) = 5$

Solution: Given equation is

$$y' = \frac{y-x}{y-x-1} \quad \text{with } y(-5) = 5$$

Here,

$$y' = \frac{y-x}{y-x-1} \quad \text{......(i)}$$

Put, $y - x = u$ then $y' - 1 = u'$. So, (1) becomes,

$$1 + u' = \frac{u}{u-1} \Rightarrow u' = \frac{u}{u-1} - 1 = \frac{1}{u-1}$$

$$\Rightarrow (u+1)\,du = dx$$

Integrating we get,

$$\frac{u^2}{2} + u = x + c_1 \Rightarrow u^2 + 2u = 2x + c \quad \text{for } 2c_1 = c$$

$$\Rightarrow (y-x)^2 + 2(y-x) = 2x + c$$

$$\Rightarrow (y-x)^2 + 2y = 4x + c \quad \text{......(2)}$$

Since we have $y(-5) = 5$. So, (2) gives

$$(5+5)^2 + 10 = 20 + c$$

$$\Rightarrow c = 100 + 10 - 20 = 90.$$

Therefore, (2) becomes,

$$(y-x)^2 + 2y - 4x = 90$$

(iii) $(2x - 4y + 5)y' + (x - 2y + 3) = 0,\ y(2) = 25$

Solution: Given equation is

$$(2x - 4y + 5)y' + (x - 2y + 3) = 0 \quad \text{with } y(2) = 25$$

Here,

$$(2x - 4y + 5)y' + (x - 2y + 3) = 0$$

$$\Rightarrow (2(x-2y) + 5)y' + (x - 2y + 3) = 0 \quad \text{.....(1)}$$

Put $x - 2y = u$ then $1 - 2y' = u' \Rightarrow y' = \frac{1-u'}{2}$. Then (1) becomes,

$$(2u+5)\left(\frac{1-u'}{2}\right) + (u+3) = 0 \Rightarrow 2u + 5 - (2u+5)u' - 2u + 6 = 0$$

$$\Rightarrow (2u+5)du = 11dx$$

Integrating we get,

$$u^2 + 5u = 11x + c \Rightarrow (x-2y)^2 + 5(x-2y) = 11x + c$$

$$\Rightarrow (x-2y)^2 - 10y - 6x = c \quad \text{......(2)}$$

Since we have, $y(0) = \frac{\pi}{2}$. So, (2) gives

$$\sin\left(\frac{\pi}{2}\right) = e^c \Rightarrow 1 = e^c \Rightarrow c = \log(1) = 0.$$

Therefore, (2) becomes,

$$\sin(y-x) = e^{x^2/2}$$

(iv) $y' - x \tan (y - x) = 1,\ y(0) = \frac{\pi}{2}$

Solution: Given that,

$$y' - x \tan (y - x) = 1 \quad ....(1)$$

$$y(0) = \frac{\pi}{2} \quad ....(2)$$

Here, $y' - x \tan (y - x) = -1$

Put $y - x = u$ then $y' - 1 = u'$.

So, $u' + 1 - x \tan u = 1 \Rightarrow u' - x \tan u = 0$

$$\Rightarrow \frac{du}{-\tan u} + x\,dx = 0$$

Integrating we get,

$$\log (\sin u) + \frac{x^2}{2} = c$$

$$\Rightarrow \log (\sin (y - x)) + \frac{x^2}{2} = c \quad ....(3)$$

Since $y(0) = \frac{\pi}{2}$ then (3) gives us,

$$\log\left(\sin\frac{\pi}{2}\right) + 0 = c \Rightarrow c = 0 \qquad [\because \log(1) = 0]$$

Therefore (3) becomes,

$$\log (\sin (y - x)) + \frac{x^2}{2} = 0$$

$$\Rightarrow \sin(y - x) = e^{-x^2/2}$$

This is the solution of given equation.

(v) $xy' = y + 3x^4 \cos^2\left(\frac{y}{x}\right),\ y(1) = 0$

Solution: Given equation is

$$xy' = y + 3x^4 \cos^2\left(\frac{y}{x}\right) \text{ with } y(1) = 0$$

Here, $$xy' = y + 3x^4 \cos^2\left(\frac{y}{x}\right) \quad .....(1)$$

Put, $\frac{y}{x} = u$, $y = xu$ then $y' = u + xu'$. Then (1) becomes,

$$x(u + xu') = xu + 3x^4 \cos^2 u \Rightarrow x^2u' = 3x^4 \cos^2 u$$

$$\Rightarrow u' = 3x^2 \cos^2 u$$

$$\Rightarrow \sec^2 u\, du = 3x^2 dx$$

Integrating we get,

$$\tan u = x^3 + c$$

$$\Rightarrow \tan\left(\frac{y}{x}\right) = x^3 + c \quad ....(2)$$

Since we have, $y(1) = 0$. So, (2) gives,

$$\tan 0 = 1 + c \Rightarrow c = -1.$$

Therefore (2) becomes,

$$\tan\left(\frac{y}{x}\right) = x^3 - 1 \Rightarrow \frac{y}{x} = \tan^{-1}(x^3 - 1)$$

$$\Rightarrow y = x \tan^{-1}(x^3 - 1).$$

(vi) $xyy' = 2y^2 + 4x^2,\quad y(2) = 4$

Solution: Given equation is

$$xyy' = 2y^2 + 4x^2, \quad y(2) = 4$$

Here,

$$xy.y' = 2y^2 = 2y^2 + 4x^2$$

$$\Rightarrow \frac{y}{x}.y' = 2\frac{y^2}{x^2} + 4 \quad ....(1) \quad [\because \text{dividing by } x^2]$$

This is a homogeneous equation. So, put $\frac{y}{x} = u$ then,

$y' = u + xu'$. Then (1) becomes,

$$\Rightarrow u(u + xu') = 2u^2 + 4 \Rightarrow u^2 + xuu' = 2u^2 + 4$$

$$\Rightarrow xuu' = u^2 + 4$$

$$\Rightarrow \left(\frac{u}{u^2 + 4}\right) du = \frac{dx}{x}$$

$$\Rightarrow \frac{1}{2}\left(\frac{2u}{u^2 + 4}\right) du = \frac{dx}{x}$$

Integrating we get,

$$\frac{1}{2}\log(u^2 + 4) = \log(x) + \log(c)$$

$$\Rightarrow \sqrt{u^2 + 4} = cx \Rightarrow \sqrt{\left(\frac{y}{x}\right)^2 + 4} = cx$$

$$\Rightarrow \sqrt{y^2 + 4x^2} = cx^2 \quad ......(2)$$

Since we have $y(2) = 4$. So, (2) gives,

$$\sqrt{16 + 16} = c.4 \Rightarrow 4\sqrt{2} = c.4 \Rightarrow c = \sqrt{2}$$

Therefore (2) becomes,

$$\sqrt{y^2 + 4x^2} = \sqrt{2}.\, x^2 \Rightarrow y^2 + 4x^2 = 2x^4 \qquad [\because \text{squaring on both sides}]$$

$$\Rightarrow y = \sqrt{2x^4 - 4x^2}$$

C. **Find the solution of the following homogeneous differential equations.**

(i) $\frac{dy}{dx} + \frac{y}{x} = \left(\frac{y}{x}\right)^2$

Solution: Given equation is

$$\frac{dy}{dx} + \frac{y}{x} = \left(\frac{y}{x}\right)^2 \quad \text{......(1)}$$

This is homogeneous equation of first order.

Put, $\frac{y}{x} = u \Rightarrow y = xu$. Then $\frac{dy}{dx} = u + x.\frac{du}{dx}$. Then (1) becomes,

$$u + x.\frac{du}{dx} + u = u^2 \Rightarrow x.\frac{du}{dx} = u^2 - 2u \Rightarrow \frac{du}{u^2 - 2u} = \frac{dx}{x}$$

$$\Rightarrow \frac{du}{u(u-2)} = \frac{dx}{x}$$

$$\Rightarrow \frac{1}{2}\left(\frac{1}{u-2} - \frac{1}{u}\right) du = \frac{dx}{x}$$

Integrating we get,

$$\frac{1}{2}[\log(u-2) - \log(u)] = \log(x) + \log(c_1)$$

$$\Rightarrow \log\left(\frac{u-2}{u}\right)^{1/2} = \log(c_1x) \Rightarrow \frac{u-2}{u} = cx^2 \qquad \text{for } c_1^2 = c$$

$$\Rightarrow \frac{y-2x}{y} = cx^2$$

$$\Rightarrow y - 2x = cx^2y$$

(ii) $x(x-y)\,dy = y(x+y)dx$

Solution: Given equation is,

$$x(x-y)\,dy = y(x+y)\,dx$$

$$\Rightarrow x(x-y)\frac{dy}{dx} = y(x+y) \quad \text{...........(i)}$$

This is a homogeneous equation. So, put $y = vx$ then $\frac{dy}{dx} = v + x\frac{dv}{dx}$. So that,

$$x(x - vx)\left(v + x\frac{dv}{dx}\right) = vx(x + vx)$$

$$\Rightarrow (1-v)\left(v + x\frac{dv}{dx}\right) = v(1 = v)$$

$$\Rightarrow (1-v)\,x\frac{dv}{dx} = v(1+v) - v(1-v) = v(1+v-1+v) = 2v^2$$

$$\Rightarrow \frac{1-v}{v^2}dv = \frac{2dx}{x}$$

$$\Rightarrow \left[v^{-2} - \frac{1}{2}\left(\frac{2v}{v^2}\right)\right]dv = 2\frac{dx}{x}$$

Integrating we get,

$$\frac{v^{-1}}{-1} - \frac{1}{2}\log(v^2) = 2\log(x) + \log c$$

$$\Rightarrow -\frac{1}{v} - \log(v) = \log(x^2) + \log c$$

$$\Rightarrow -\frac{x}{y} - \log\left(\frac{y}{x}\right) = \log(x^2) + \log(c)$$

$$\Rightarrow -\frac{x}{y} = \log\left(x^2 \times \frac{y}{x}\right) + \log(c)$$

$$\Rightarrow -\frac{x}{y} = \log(xy) + \log(c) = \log(cxy)$$

$$\Rightarrow cxy = e^{-x/y}$$

This is the solution of given equation.

(iii) $\frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right)$

Solution: Given equation is

$$\frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right) \quad \text{......... (1)}$$

This is homogeneous equation.

So, put $y = ux$ then $\frac{dy}{dx} = u + x\frac{du}{dx}$. Then (1) becomes,

$$u + x\frac{du}{dx} = u + \tan u \Rightarrow x\frac{du}{dx} = \tan u$$

$$\Rightarrow \left(\frac{\cos u}{\sin u}\right)du = \frac{dx}{x}$$

Integrating we get,

$$\log(\sin u) = \log(x) + \log(c)$$

$$\Rightarrow \sin u = cx$$

$$\Rightarrow \sin\left(\frac{y}{x}\right) = cx$$

(iv) $x\sin\left(\frac{y}{x}\right)dy = \left(y\sin\frac{y}{x} - x\right)dx$

Solution: Given equation is

$$x\sin\left(\frac{y}{x}\right)dy = \left(y\sin\frac{y}{x} - x\right)dx$$

$$\Rightarrow \sin\left(\frac{y}{x}\right).\frac{dy}{dx} = \frac{y}{x}\sin\left(\frac{y}{x}\right) - 1 \quad \text{.......(1)}$$

Put $y = ux$ then $\frac{dy}{dx} = u + x.\frac{du}{dx}$. Then (1) becomes,

$$\sin u.\left(u + x\frac{du}{dx}\right) = u \sin u - 1 \Rightarrow \quad u \sin u + x \sin u \frac{du}{dx} = u \sin u - 1$$

$$\Rightarrow \quad \sin u \, du = -\frac{du}{x}$$

Integrating we get,

$$-\cos u = -\log(x) + c \quad \Rightarrow \quad \cos\left(\frac{y}{x}\right) = \log(x) - c$$

$$\Rightarrow \quad \log(x) = \cos\left(\frac{y}{x}\right) + c$$

(v) $\mathbf{(1 + e^{x/y})\,dx + e^{x/y}\left(1 - \frac{x}{y}\right) dy = 0}$

Solution: Given equation is

$$(1 + e^{x/y})\,dx + e^{x/y}\left(1 - \frac{x}{y}\right) dy = 0$$

$$\Rightarrow \quad (1 + e^{x/y})\frac{dx}{dy} + e^{x/y}\left(1 - \frac{x}{y}\right) = 0 \quad ....(1)$$

Put $\frac{x}{y} = u \Rightarrow x = yu$. Then $\frac{dx}{dy} = u + y.\frac{du}{dy}$. So (1) becomes,

$$(1 + e^{u})\left(u + y\frac{du}{dy}\right) + e^{u}(1 - u) = 0$$

$$\Rightarrow \quad u + ue^{u} + (1 + e^{u})y\frac{du}{dy} + e^{u} - ue^{u} = 0$$

$$\Rightarrow \quad \left(\frac{1 + e^{u}}{u + e^{u}}\right) du = -\frac{dy}{y} \quad ......(2)$$

Set $u + e^{u} = t$ then $(1 + e^{u})\,du = dt$. So, (2) becomes,

$$\frac{dt}{t} = -\frac{dy}{y}$$

Integrating we get,

$$\log(t) = -\log(y) + \log(c)$$

$$\Rightarrow \quad t = \frac{c}{y} \quad \Rightarrow \quad u + e^{u} = \frac{c}{y} \Rightarrow \frac{x}{y} + e^{x/y} = \frac{c}{y}$$

$$\Rightarrow \quad x + y\,e^{x/y} = c.$$

(vi) $\mathbf{x\frac{dy}{dx} = y - \sqrt{x^2 + y^2}}$

Solution: Given equation is

$$x\frac{dy}{dx} = y - \sqrt{x^2 + y^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} - \sqrt{1 + \frac{y^2}{x^2}} \qquad \ldots\ldots(1)$$

Put $y = xu$ then $\frac{dy}{dx} = x \cdot \frac{du}{dx} + u$. Then (1) becomes

$$x \cdot \frac{du}{dx} + u = u - \sqrt{1 + u^2}$$

$$\Rightarrow \frac{du}{\sqrt{1 + u^2}} = -\,dx$$

Integrating we get

$$\log\left(u + \sqrt{u^2 + 1}\right) = -\log(x) + \log c \qquad \left[\because \int \frac{dx}{\sqrt{x^2 + a^2}} = \log\left(x + \sqrt{x^2 + a^2}\right) + c\right]$$

$$\Rightarrow u + \sqrt{u^2 + 1} = \frac{c}{x} \qquad \Rightarrow \frac{y}{x} + \sqrt{\frac{y^2}{x^2} + 1} = \frac{c}{x}$$

$$\Rightarrow y + \sqrt{x^2 + y^2} = c$$

(vii) $x\,dy - y\,dx = \sqrt{x^2 + y^2}\,dx$

Solution: Given equation is

$$x\,dy - y\,dx = \sqrt{x^2 + y^2}\,dx$$

$$\Rightarrow \frac{dy}{dx} - \frac{y}{x} = \sqrt{1 + \frac{y^2}{x^2}} \qquad \ldots\ldots(1) \quad [\because \text{dividing by } x\,dx]$$

Put $\frac{y}{x} = u$ then $y = xu$. So, $\frac{dy}{dx} = u + x\frac{dy}{dx}$. Then (1) becomes,

$$u + x\frac{du}{dx} - u = \sqrt{1 + u^2} \qquad \Rightarrow x\frac{du}{dx} = \sqrt{1 + u^2}$$

$$\Rightarrow \frac{du}{\sqrt{1 + u^2}} = \frac{dx}{x}$$

Integrating we get,

$$\log\left(u + \sqrt{1 + u^2}\right) = \log(x) + \log(c)$$

$$\Rightarrow u + \sqrt{1 + u^2} = cx \qquad \Rightarrow \frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}} = cx$$

$$\Rightarrow y + \sqrt{x^2 + y^2} = cx^2$$