EXERCISE 4.6

Show that the value under integral sign is exact in the plane and evaluate integral.

1. $\int_{(-1,5)}^{(4,3)} (3z^2\,dx + 6xz\,dz)$ **[2009 Spring – Short]**

Solution: Given Integral is,

$$I = \int_{(-1,5)}^{(4,3)} (3z^2\,dx + 6xz\,dz) \quad \text{... (i)}$$

Here the integrand value of (i) is,

$$3z^2\,dx + 6xz\,dz \quad \text{... (ii)}$$

Comparing (ii) with $F_1dx + F_2dz$ then,

$$F_1 = 3z^2 \quad \text{and} \quad F_2 = 6xz.$$

Here,

$$\frac{\delta F_1}{\delta z} = 6z \quad \text{and} \quad \frac{\delta F_2}{\delta x} = 6z$$

This shows that $\frac{\delta F_1}{\delta z} = \frac{\delta F_2}{\delta x}$. So, the value (ii) is exact. Therefore,

$$I = \int_a^b d[\int F_1 dx + \int(\text{terms free from x in } F_2)\,dz]$$

$$= \int_{(-1,5)}^{(4,3)} d(\int 3z^2\,dx) = [3xz^2]_{(-1,5)}^{(4,3)} = 108 + 75 = 183$$

Thus, $I = 183$.

2. $\int_{(4,3/2)}^{(4,1/2)} (2x\,\text{Sin}\pi y\,dx + \pi x^2\,\text{Cos}\pi y\,dy)$. **[2010 Fall; 2005 Fall – Short]**

Solution: Given integral is,

$$I = \int_{(4,3/2)}^{(4,1/2)} (2x\,\text{Sin}\pi y\,dx + \pi x^2\,\text{Cos}\pi y\,dy) \quad \text{....... (i)}$$

Here the integrand value of (i) is,

$$2x\,\text{Sin}\pi y\,dx + \pi x^2\,\text{Cos}\pi y\,dy \quad \text{....... (ii)}$$

Comparing (ii) with $F_1dx + F_2\,dy$ then,

$$F_1 = 2x\,\text{Sin}\pi y \quad \text{and} \quad F_2 = \pi x^2 - \text{Cos}\pi y.$$

Here,

$$\frac{\delta F_1}{\delta y} = 2\pi x\,\text{Cos}\pi y \quad \text{and} \quad \frac{\delta F_2}{\delta x} = 2\pi x\,\text{Cos}\pi y$$

This shows that $\frac{\delta F_1}{\delta y} = \frac{\delta F_2}{\delta x}$. So, the value (ii) is exact. Therefore,

$$I = \int_a^b d[\int F_1 dx + \int(\text{terms free from x in } F_2)\, dy]$$

i.e. $$I = \int_{(4,3/2)}^{(4,1/2)} d(\int 2x \sin\pi y\, dx)$$

$$= \int_{(3,3/2)}^{(4,1/2)} d(x^2\, 3\text{Sin}\pi y) = [x^2 \text{ Sin}\pi y]_{(3,3/2)}^{(4,1/2)}$$

$$= 16 \text{ Sin}\frac{\pi}{2} - 9 \text{ Sin}\frac{3\pi}{2} = 16 + 9 = 25$$

Thus, I = 25.

3. $$\int_{(0,0,0)}^{(4,1,2)} (3y dx + 3x dy + 2z\, dz)$$ **[2009 Fall – Short]**

[2011 Fall Q.No. 6(b) OR] [2010 Spring Q.No. 6(a)] [2003 Fall Q.No. 4(b) OR]

Solution: Given integral is,

$$I = \int_{(0,0,0)}^{(4,1,2)} (3y dx + 3x dy + 2z\, dz) \quad \text{.......... (i)}$$

Here, the integrand value of (i) is,

$$3y\, dx + 3x\, dx + 2z\, dz \quad \text{.......... (ii)}$$

Comparing (ii) with $F_1 = 3y$, $F_2 = 3x$, and $F_3 = 2z$. Then,

$$\frac{\delta F_1}{\delta y} = 3,\ \frac{\delta F_2}{\delta x} = 3,\ \frac{\delta F_1}{\delta z} = 0,\ \frac{\delta F_3}{\delta x} = 0,\ \frac{\delta F_2}{\delta z} = 0,\ \frac{\delta F_3}{\delta y} = 0.$$

This shows that,

$$\frac{\delta F_1}{\delta y} = \frac{\delta F_2}{\delta x},\ \frac{\delta F_1}{\delta z} = \frac{\delta F_3}{\delta x}, \text{ and } \frac{\delta F_2}{\delta z} = \frac{\delta F_3}{\delta y}$$

So, the value in (ii) is exact. Therefore,

$$I = \int_a^b d[\int F_1 dx + \int(\text{terms free from x in } F_2)\, dy + \int(\text{terms free from x and y in } F_3)\, dz]$$

i.e. $$I = \int_{(0,0,0)}^{(4,1,2)} d(\int 3y\, dx + \int 0\, dy + \int 2z\, dz)$$

$$= \int_{(0,0,0)}^{(4,1,2)} d(3xy + z^2) = [3xy + z^2]_{(0,0,0)}^{(4,1,2)} = (12 + 4) - 0 = 16.$$

Thus, I = 16.

4. $$\int_{(0,0,0)}^{(4,1,2)} e^{x-y+z^2} (dx - dy + 2zdx)$$

Solution: Given integrals,

$$I = \int_{(0,0,0)}^{(4,1,2)} e^{x-y+z^2} (dx - dy + 2zdx) \quad \text{.......... (i)}$$

Here the integrand value of (i) is,

$$e^{x-y+z^2}(dx - dy + 2z\, dz) \text{.......... (ii)}$$

Comparing (ii) with $F_1dx + F_2dy + F_3dz$ then we get,

$$F_1 = e^{x-y+z^2},\quad F_2 = -e^{x-y+z^2} \text{ and } F_3 = 2ze^{x-y+z^2}$$

Then,

$$\frac{\delta F_1}{\delta y} = -e^{x-y+z^2},\quad \frac{\delta F_2}{\delta x} = -e^{x-y+z^2},\quad \frac{\delta F_3}{\delta x} = -2ze^{x-y+z^2},$$

$$\frac{\delta F_1}{\delta z} = -2ze^{x-y+z^2},\quad \frac{\delta F_2}{\delta z} = -2ze^{x-y+z^2},\quad \frac{\delta F_3}{\delta y} = -2ze^{x-y+z^2}$$

This shows that,

$$\frac{\delta F_1}{\delta y} = \frac{\delta F_2}{\delta x},\quad \frac{\delta F_3}{\delta x} = \frac{\delta F_1}{\delta z},\quad \frac{\delta F_2}{\delta z} = \frac{\delta F_3}{\delta y}$$

So, the value in (ii) is exact. Therefore,

$$I = \int_a^b d[\int F_1 dx + \int(\text{terms free from x in } F_2)\, dy + \int(\text{terms free from x and y in } F_3)\, dz]$$

i.e. $$I = \int_{(0,0,0)}^{(4,1,2)} d(\int e^{x-y+z^2} dx + \int 0\, dy + \int 0\, dz)$$

$$I = \int_{(0,0,0)}^{(4,1,2)} d(e^{x-y+z^2}) = [e^{x-y+z^2}]_{(0,0,0)}^{(2,4,0)} = e^{2-4+0} - e^{0-0+0}$$

$$= e^{-2} - e^0 = e^{-2} - 1$$

Thus, $I = e^{-2} - 1$.

5. $$\int_{(0,2,3)}^{(1,1,1)} [yz \text{ Sinh}(xz)\, dx + \text{Cosh}(xz)\, dy + xy \text{ Sinh}(xz)\, dz]$$

Solution: Given integral is,

$$I = \int_{(0,2,3)}^{(1,1,1)} [yz \text{ Sinh}(xz)\, dx + \text{Cosh}(xz)\, dy + xy \text{ Sinh}(xz)\, dz] \text{ .... (i)}$$

Here, the integrand value is, ... (ii)

$$yz \text{ Sinh}(xz)\, dx + \text{Cosh}(xz)\, dy + xy \text{ Sinh}(xz)\, dz$$

$$I = \int_a^b d[\int F_1 dx + \int(\text{terms free from x in } F_2)\, dy]$$

i.e. $I = \int_{(4,3/2)}^{(4,1/2)} d(\int 2x \sin\pi y\, dx)$

$$= \int_{(3,3/2)}^{(4,1/2)} d(x^2\, 3\text{Sin}\pi y) = [x^2 \text{Sin}\pi y]_{(3,3/2)}^{(4,1/2)}$$

$$= 16\,\text{Sin}\frac{\pi}{2} - 9\,\text{Sin}\frac{3\pi}{2} = 16 + 9 = 25$$

Thus, $I = 25$.

3. $\int_{(0,0,0)}^{(4,1,2)} (3ydx + 3xdy + 2z\, dz)$ **[2009 Fall – Short]**

[2011 Fall Q.No. 6(b) OR] [2010 Spring Q.No. 6(a)] [2003 Fall Q.No. 4(b) OR]

Solution: Given integral is,

$$I = \int_{(0,0,0)}^{(4,1,2)} (3ydx + 3xdy + 2z\, dz) \quad \text{.......... (i)}$$

Here, the integrand value of (i) is,

$$3y\,dx + 3x\,dx + 2z\,dz \quad \text{........... (ii)}$$

Comparing (ii) with $F_1 = 3y$, $F_2 = 3x$, and $F_3 = 2z$. Then,

$$\frac{\delta F_1}{\delta y} = 3,\ \frac{\delta F_2}{\delta x} = 3,\ \frac{\delta F_1}{\delta z} = 0,\ \frac{\delta F_3}{\delta x} = 0,\ \frac{\delta F_2}{\delta z} = 0,\ \frac{\delta F_3}{\delta y} = 0.$$

This shows that,

$$\frac{\delta F_1}{\delta y} = \frac{\delta F_2}{\delta x},\ \frac{\delta F_1}{\delta z} = \frac{\delta F_3}{\delta x}, \text{ and } \frac{\delta F_2}{\delta z} = \frac{\delta F_3}{\delta y}$$

So, the value in (ii) is exact. Therefore,

$$I = \int_a^b d[\int F_1 dx + \int(\text{terms free from x in } F_2)\, dy + \int(\text{terms free from x and y in } F_3)\, dz]$$

i.e. $I = \int_{(0,0,0)}^{(4,1,2)} d(\int 3y\,dx + \int 0\,dy + \int 2z\,dz)$

$$= \int_{(0,0,0)}^{(4,1,2)} d(3xy + z^2) = [3xy + z^2]_{(0,0,0)}^{(4,1,2)} = (12 + 4) - 0 = 16.$$

Thus, $I = 16$.

4. $\int_{(0,0,0)}^{(4,1,2)} e^{x-y+z^2} (dx - dy + 2zdx)$

Solution: Given integrals,

$$I = \int_{(0,0,0)}^{(4,1,2)} e^{x-y+z^2} (dx - dy + 2zdx) \quad \text{.......... (i)}$$

Here the integrand value of (i) is,

$$e^{x-y+z^2}(dx - dy + 2z\, dz) \text{.......... (ii)}$$

Comparing (ii) with $F_1dx + F_2dy + F_3dz$ then we get,

$$F_1 = e^{x-y+z^2},\quad F_2 = -e^{x-y+z^2} \text{ and } \quad F_3 = 2ze^{x-y+z^2}$$

Then,

$$\frac{\delta F_1}{\delta y} = -e^{x-y+z^2},\quad \frac{\delta F_2}{\delta x} = -e^{x-y+z^2},\quad \frac{\delta F_3}{\delta x} = -2ze^{x-y+z^2},$$

$$\frac{\delta F_1}{\delta z} = -2ze^{x-y+z^2},\quad \frac{\delta F_2}{\delta z} = -2ze^{x-y+z^2},\quad \frac{\delta F_3}{\delta y} = -2ze^{x-y+z^2}$$

This shows that,

$$\frac{\delta F_1}{\delta y} = \frac{\delta F_2}{\delta x},\quad \frac{\delta F_3}{\delta x} = \frac{\delta F_1}{\delta z},\quad \frac{\delta F_2}{\delta z} = \frac{\delta F_3}{\delta y}$$

So, the value in (ii) is exact. Therefore,

$$I = \int_a^b d[\int F_1 dx + \int(\text{terms free from x in } F_2)\, dy + \int(\text{terms free from x and y in } F_3)\, dz]$$

i.e. $I = \int_{(0,0,0)}^{(4,1,2)} d(\int e^{x-y+z^2} dx + \int 0\, dy + \int 0\, dz)$

$$I = \int_{(0,0,0)}^{(4,1,2)} d(e^{x-y+z^2}) = [e^{x-y+z^2}]_{(0,0,0)}^{(2,4,0)} = e^{2-4+0} - e^{0-0+0}$$

$$= e^{-2} - e^0 = e^{-2} - 1$$

Thus, $I = e^{-2} - 1$.

5. $\int_{(0,2,3)}^{(1,1,1)} [yz\,\text{Sinh}\,(xz)\,dx + \text{Cosh}\,(xz)\,dy + xy\,\text{Sinh}\,(xz)\,dz]$

Solution: Given integral is,

$$I = \int_{(0,2,3)}^{(1,1,1)} [yz\,\text{Sinh}\,(xz)\,dx + \text{Cosh}\,(xz)\,dy + xy\,\text{Sinh}\,(xz)\,dz] \text{ .... (i)}$$

Here, the integrand value is,

$$yz\,\text{Sinh}\,(xz)\,dx + \text{Cosh}\,(xz)\,dy + xy\,\text{Sinh}\,(xz)\,dz \text{ .... (ii)}$$

Then,

$$\frac{\delta F_1}{\delta y} = z\ \text{Sinh}\ (xz),\quad \frac{\delta F_2}{\delta x} = z\ \text{Sinh}\ (xz),\quad \frac{\delta F_1}{\delta z} = y\ \text{Sinh}\ (xz) + xyz\ \text{Cosh}\ (xz),$$

$$\frac{\delta F_3}{\delta x} = y\ \text{Sinh}\ (xz) + xyz\ \text{Cosh}\ (xz),\quad \frac{\delta F_2}{\delta z} = x\ \text{Sinh}\ (xz), \frac{\delta F_3}{\delta y} = x\ \text{Sinh}\ (xz)$$

This shows that

$$\frac{\delta F_1}{\delta y} = \frac{\delta F_2}{\delta x},\quad \frac{\delta F_1}{\delta z} = \frac{\delta F_3}{\delta x},\quad \frac{\delta F_2}{\delta z} = \frac{\delta F_3}{\delta y}.$$

So, the value in (ii) is exact. Therefore,

$$I = \int_a^b d[\textstyle\int F_1 dx + \int(\text{terms free from x in } F_2)\ dy + \int(\text{terms free from x and y in } F_3)\ dz]$$

i.e. $$I = \int_{(0,2,3)}^{(1,1,1)} d[\textstyle\int yz\ \text{Sinh}\ (xz)\ dx + \int 0\ dy + \int 0\ dz]$$

$$= \int_{(0,2,3)}^{(1,1,1)} d(y\ \text{Cosh}(xz)) = [y\ \text{Cosh}(xz)]_{(0,2,3)}^{(1,1,1)}$$

$$= \text{Cosh}\ 1 - 2\ \text{Cosh}\ 0 = \text{Cosh}\ 1 - 2.$$

Thus, $I = \text{Cosh}\ 1 - 2$.

6. $$\int_{(0,0,1)}^{(1,\pi/4,2)} [2xyz^2 dx + (x^2z^2 + z\ \text{Cosyz})\ dy + (2x^2yz + y\ \text{Cosyz})\ dz]$$

Solution: Given integral is,

$$I = \int_{(0,0,1)}^{(1,\pi/4,2)} [2xyz^2 dx + (x^2z^2 + z\ \text{Cosyz})\ dy + (2x^2yz + y\ \text{Cosyz})\ dz] \quad \text{..... (i)}$$

Here, the integrand value of (i) is,

$$2xyz^2 dx + 6x^2z^2 + z\ \text{Cosyz})\ dy + (2x^2yz + y\ \text{Cosyz})\ dz \quad \text{..... (ii)}$$

Comparing (ii) with $F_1dx + F_2dy + F_3dz$ then we get,

$$F_1 = 2xyz^2,\quad F_2 = x^2z^2 + z\ \text{Cosyz},\quad F_3 = 2x^2yz + y\ \text{Cosyz}$$

Then,

$$\frac{\delta F_1}{\delta y} = 2xz^2,\quad \frac{\delta F_2}{\delta x} = 2xz^2,\quad \frac{\delta F_1}{\delta z} = 4xyz,\quad \frac{\delta F_3}{\delta x} = 4xyz,$$

$$\frac{\delta F_2}{\delta z} = 2x^2z + \text{Cosyz} + yz\ \text{Cosyz},\quad \frac{\delta F_3}{\delta y} = 2x^2z + \text{Cosyz} + yz\ \text{Cosyz}$$

This shows that,

$$\frac{\delta F_1}{\delta y} = \frac{\delta F_2}{\delta x},\quad \frac{\delta F_1}{\delta z} = \frac{\delta F_3}{\delta x},\quad \frac{\delta F_2}{\delta z} = \frac{\delta F_3}{\delta y}$$

So the integrand value (ii) is exact. Therefore,

$$I = \int_a^b d[\textstyle\int F_1 dx + \int(\text{terms free from x in } F_2)\ dy + \int(\text{terms free from x and y in } F_3)\ dz]$$

i.e. $$I = \int_{(0,0,0)}^{(1,\pi/4,2)} d[\textstyle\int 2xyz^2\ dx + \int z\ \text{cosyz}\ dy + \int 0\ dz]$$

$$I = \int_{(0,0,0)}^{(1,\pi/4,2)} ((2xyz^2\ dx + x^2 z^2 dy + 2x^2yz\ dz + z\ \text{Cosyz}\ dy + y\ \text{Cosyz}\ dz)$$

$$= \int_{(0,0,0)}^{(1,\pi/4,2)} d(x^2yz^2 + \text{Sinyz}) = [x^2yz + \text{Sinyz}]_{(0,0,1)}^{(1,\pi/4,2)}$$

$$= \frac{4\pi}{4} + 3\ \text{Sin}\frac{\pi}{2} - 0 - \text{Sin}0 = \pi + 1.$$

Thus, $I = \pi + 1$.

7. $$\int_{(0,1)}^{(2,3)} [(2x + y^3)\ dx + (3xy^2 + 4)\ dy]$$ **[2009 Spring – Short]**

Solution: Here,

$$I = \int_{(0,1)}^{(2,3)} [(2x + y^3)\ dx + (3xy^2 + 4)\ dy] \quad \text{.......... (i)}$$

The integrand value of (i) is,

$$(2x + y^3)\ dx + (3xy^2 + 4)\ dy \quad \text{.......... (ii)}$$

Comparing (ii) with $F_1\ dx + F_2\ dy$ then we get,

$$\frac{\delta F_1}{\delta y} = 3y^2 \quad \text{and} \quad \frac{\delta F_2}{\delta x} = 3y^2$$

This shows that $\frac{\delta F_1}{\delta y} = \frac{\delta F_2}{\delta x}$. So, the value (ii) is exact. Therefore,

$$I = \int_a^b d[\textstyle\int F_1 dx + \int(\text{terms free from x in } F_2)\ dy]$$

i.e. $$I = \int_{(0,1)}^{(2,3)} d[\textstyle\int(2x + y^3)\ dx + \int 4\ dy]$$

$$I = \int_{(0,1)}^{(2,3)} (2x\ dx + (y^3 dx + 3xy^2 dy) + 4dy)$$

$$= \int_{(0,1)}^{(2,3)} d(xy^3 + x^2 + 4y) = [xy^3 + x^3 + 4y]_{(0,1)}^{(2,3)}$$

$$= (54 + 4 + 12) - (0 + 0 + 4) = 70 - 4 = 66.$$

Thus, I = 66.

8. $\int_{(-1,2)}^{(3,1)} [(y^2 + 2xy)\, dx + (x+2 + 2xy)\, dy]$

Solution: Similar to 7.

9. $\int_{(1,0,2)}^{(-2,1,3)} [(6xy^3 + 2z^2)\, dx + 9x^2y^2\, dy + (4xz + 1)\, dz]$

Solution: Similar to 6.

10. $\int_{(0,1,1/2)}^{(\pi/2,3,2)} [y^2 \cos x\, dx + (2y \sin x + e^{2z})\, dy + 2ye^{2z}\, dz]$ **[2012 Fall Q.No. 4(a) OR]**

Solution: Here,

$$I = \int_{(0,1,1/2)}^{(\pi/2,3,2)} [y^2 \cos x\, dx + (2y \sin x + e^{2z})\, dy + 2y\, e^{2z}\, dz] \quad \text{......... (i)}$$

The integrand value of (i) is,

$$y^2 \cos x\, dx + (2y \sin x + e^{2z})\, dy + 2y\, e^{2z}\, dz$$

Comparing (ii) with $F_1dx + F_2dy + F_3dz$ then we get,

$$F_1 = y^2 \cos x, \qquad F_2 = 2y \sin x + e^{2z}, \qquad F_3 = 2ye^{2z}$$

Then,

$$\frac{\delta F_1}{\delta y} = 2y\cos x,\ \frac{\delta F_2}{\delta x} = 2y \cos x,\ \frac{\delta F_1}{\delta z} = 0,\ \frac{\delta F_3}{\delta x} = 0,\ \frac{\delta F_2}{\delta z} = 2e^{2z},\ \frac{\delta F_3}{\delta y} = 2e^{2z}$$

This shows that,

$$\frac{\delta F_1}{\delta y} = \frac{\delta F_2}{\delta x}, \quad \frac{\delta F_1}{\delta z} = \frac{\delta F_3}{\delta x}, \quad \frac{\delta F_2}{\delta z} = \frac{\delta F_3}{\delta y}$$

So, the value (ii) is exact. Therefore,

$$I = \int_a^b d[\textstyle\int F_1 dx + \int(\text{terms free from x in } F_2)\, dy + \int(\text{terms free from x and y in } F_3)\, dz]$$

i.e. $$I = \int_{(0,1,1/2)}^{(\pi/2,3,2)} d[\textstyle\int y^2 \cos x\, dx + \int e^{2z}\, dy + \int 0\, dz]$$

$$I = \int_{(0,1,1/2)}^{(\pi/2,3,2)} (y^2 \cos x\, dx + 2y \sin x\, dy + e^{2z}\, dz)$$

$$= \int_{(0,1,1/2)}^{(\pi/2,3,2)} d\,(y^2 \sin x + ye^{2z}) = [y^2 \sin x + ye^{2z}]_{(0,1,1/2)}^{(\pi/2,3,2)}$$

$$= (9 \sin\tfrac{\pi}{2} + 3\, e^4) - (\sin 0 + e^1) = 3e^4 + 9 - e$$

Thus, $I = 3e^4 - e + 9$.

EXERCISE 4.7

A. Using Greens theorem, evaluate the following integrals:

1. $\oint_c (y dx + 2x dy)$, **C: the boundary of the square** $0 < x < 1,\ 0 < y < 1$ **(counterclockwise).**

Solution: Given that, the integral is,

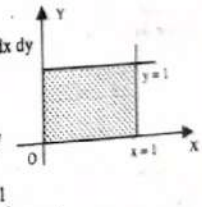
$$I = \oint_c (y\, dx + 2x\, dy) \quad \text{......... (i)}$$

where, c is the path $0 \le x \le 1,\ 0 \le y \le 1$ (in counter clockwise).

Comparing the given integral I with the integral $\oint_c [F_1\, dx + F_2\, dy]$ then we get,

$$F_1 = y \ \text{and} \ F_2 = 2x$$

By Green's theorem we have,



$$\oint_c [F_1\, dx + F_2\, dy] = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\, dy$$

$$= \int_0^1 \int_0^1 (2 - 1)\, dx\, dy$$

$$= \int_0^1 \int_0^1 1\, dx\, dy = \int_0^1 dy = 1.$$

Thus, $\oint_c y dx + 2x\, dy = 1$ for $0 \le x \le 1,\ 0 \le y \le 1$.

2. $\oint_c [2xy\ dx + (e^x + x^2)\ dy]$, **C: the boundary of the triangle with vertices (0, 0), (1, 0), (1, 1) (clockwise).**

Solution: Given that,

$$I = \oint_c [2xy\,dx + (e^x + x^2)\,dy] \quad \ldots\ldots\ldots\ldots (i)$$

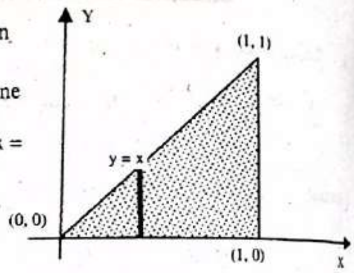
And the region is bounded by a triangle having vertices (0, 0), (1, 0) and (1, 1) in clockwise direction.

Comparing the given integral I with the integral $\oint_c [F_1\,dx + F_2\,dy]$ then we get,

$$F_1 = 2xy \text{ and } F_2 = e^x + x^2$$

By Green's theorem we have,

$$\oint_c [F_1\,dx + F_2\,dy] = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$$



Since the region of I is shown in figure in which has counterclockwise direction.

In the figure y varies from y = 0 to the line joining (0, 0) and (1, 1). That is y varies from y = 0, to y = x. and x moves from x = 0 to x = 1.

Then, (i) becomes,

$$\oint_c [2xy\,dx + (e^x + x^2)\,dy]$$

$$= \int_0^1\int_0^x e^x dy dx = \int_0^1 e^x [y]_0^x\,dx = [xe^x - e^x]_0^1 = (e - e) - (0 - 1) = 1$$

Thus, $\oint_c [2xy\,dx + (e^x + x^2)\,dy] = 1.$

Since the direction of the force is in clockwise. So,

$$\oint_c [2xy\,dx + (e^x + x^2)\,dy] = -1.$$

3. $\oint_c [(3x^2 + y)\,dx + 4y^2\,dy]$, **C: the boundary of the triangle with vertices (0, 0), (1, 0), (0, 2): counterclockwise.**

[2009 Spring Q.No. 4(a); 2006 Spring Q.No. 4(a) OR]

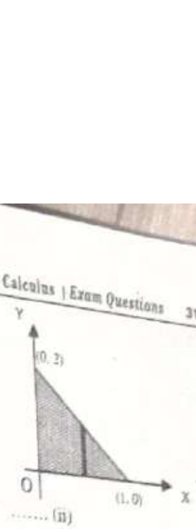
Solution: Given that,

$$I = \oint_c [(3x^2 + y)\,dx + 4y^2\,dy] \quad \ldots\ldots\ldots\ldots (i)$$

And the region is the triangle having vertices (0, 0), (1, 0) and (0, 2) in counter wise direction.

Comparing the given integral I with the integral

$\oint_c [F_1\,dx + F_2\,dy]$ then we get,



$$F_1 = 3x^2 + y \text{ and } F_2 = 4y^2$$

By Green's theorem we have,

$$\oint_c [F_1\,dx + F_2\,dy] = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy \quad \ldots\ldots (ii)$$

In the figure, y varies from y = 0 to the line joining points (1, 0) and (0, 2). That is, y varies from y = 0 to y = −2x + 2. And x moves from x = 0 to x = 1.

Then (i) becomes,

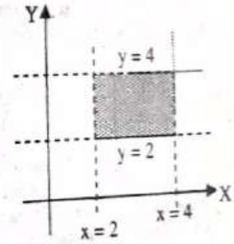
$$I = \iint_R [0 - 1]\,dA = \int_0^1\int_0^{2-2x} (-1)\,dy\,dx = -\int_0^1 (2 - 2x)\,dx$$

$$= -[2 - 2x]_0^1$$

$$= -(2 - 1) = -1$$

Thus, $\oint_c [(3x^2 + y)dx + 4y^2\,dy] = -1.$

4. $\oint_c (x^2 + y^2)\,dy$, **C: the boundary of the of the square $2 \le x \le 4, 2 \le y \le 4$.**



Solution: Given that,

$$I = \oint_c (x^2 + y^2)dy \quad \ldots\ldots (i)$$

And the boundary of c are $2 \le x \le 4, 2 \le y \le 4$.

Comparing the given integral I with the integral

$\oint_c [F_1\,dx + F_2\,dy]$ then we get,

$$F_1 = 3x^2 + y \text{ and } F_2 = 4y^2$$

By Green's theorem we have,

$$\oint_c [F_1\,dx + F_2\,dy] = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy \quad \ldots\ldots (ii)$$

Now (i) becomes,

$$I = \oint_c (x^2 + y^2)dy = \iint_R (2x)\,dA \quad [\because F_1 = 0]$$

$$= \int_2^4\int_2^4 2x\,dxdy \quad [\because \text{ using the boundaries}]$$

$$= \int_2^4 [x^2]_2^4 = 12\int_2^4 dy = 12\times(4-2) = 24$$

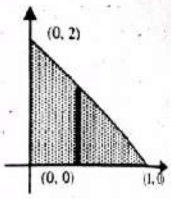
Thus, $\oint_c (x^2+y^2)\,dy = 24$.

5. $\oint_c [(x^3-3y)\,dx + (x+\text{Sin}y)\,dy]$, **C: the boundary of the triangle with vertices (0, 0), (1, 0), (0, 2).**

Solution: Given that,

$$I = \oint_c [(x^3-3y)\,dx + (x+\text{Siny})\,dy] \quad \ldots (i)$$

And the boundaries of has vertices (0, 0), (1, 0) and (0, 2).
Comparing the given integral I with the integral



$\oint_c [F_1\,dx + F_2\,dy]$ then we get,

$$F_1 = 3x^2 + y \quad \text{and} \quad F_2 = 4y^2$$

By Green's theorem we have,

$$\oint_c [F_1\,dx + F_2\,dy] = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dy\,dx \quad \ldots\ldots (ii)$$

From the figure, the region of integration (path) of $\vec{F}$ has boundaries with vertices at (0, 0), (1, 0) and (0, 2). On the region y varies from y = 0 to y = 2 – 2x (line joining the points (1, 0) and (0, 2). And x moves from x = 0 to x = 1.
Therefore, (iii) becomes,

$$I = 4\int_0^1\int_0^{2-2x} dydx = 4\int_0^1 (2-2x)\,dx = 4\,[2x-x^2]_0^1 = 4(2-1) = 4.$$

Thus, $\oint_c [(x^3-3y)\,dx + (x+\sin y)\,dy] = 4$

B. Using Green's theorem, evaluate the live integral $\oint_c \vec{F}(r).d\vec{r}$ counterclockwise around the boundary C of the region R, where

1. $\vec{F} = (x^2e^y, y^2e^x)$, **C the rectangle with vertices (0, 0), (2, 0), (2, 3), (0, 3).** **[2003 Spring Q.No. 4(a) OR]**

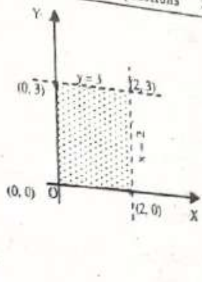
Solution: Given that.

$$\vec{F} = (x^2e^y, y^2e^x)$$

Then,

$$\text{Curl } \vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ x^2e^y & y^2e^x & 0 \end{vmatrix}$$

$$= (y^2e^x - x^2e^y)\,\vec{k}$$



So,

$$\text{Curl } \vec{F}.\vec{k} = (y^2e^x - x^2e^y)\,\vec{k}.\vec{k} = y^2e^x - x^2e^y$$

Now, by Green's theorem, we have,

$$\oint_c \vec{F}\,d\vec{r} = \iint_R (\text{curl } \vec{F}.\vec{k})\,dA = \iint_R (y^2e^x - x^2e^y)\,dA \quad \ldots\ldots\ldots (i)$$

Given that the path of $\vec{F}$ is C: the rectangle having vertices (0, 0), (2, 0), (2, 3) and (0, 3).
From the figure, y varies from y = 0 to y = 3 and x moves from x = 0 to x = 2
Therefore (i) becomes,

$$\oint_c \vec{F}.d\vec{r} = \int_0^2\int_0^3 (y^2e^x - x^2e^y)\,dydx$$

$$= \int_0^2 \left[\frac{y^3e^x}{3} - x^2e^y\right]_0^3 dx = \int_0^2 (9e^x - x^2e^3 + x^2)\,dx$$

$$= \left[9e^x - \frac{x^3}{3}e^3 + \frac{x^3}{3}\right]_0^2$$

$$= 9e^2 - \frac{8}{3}e^3 + \frac{8}{3} - 9$$

$$= 9(e^2-1) + \frac{8}{3}(1-e^3)$$

Thus, $\oint_c \vec{F}.d\vec{r} = 9\,(e^2-1) + \frac{8}{3}(1-e^3)$.

2. $\vec{F} = (y, -x)$, **C the circle** $x^2+y^2 = \frac{1}{4}$

Solution: Given that,

$$\vec{F} = (y, -x)$$

Then,

$$\text{Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ y & -x & 0 \end{vmatrix} = (-1-1)\,\vec{k} = -2\vec{k}$$

And,

$$\text{Curl } \vec{F}.\vec{k} = -2\vec{k}.\vec{k} = -2$$

By Green's theorem we have,

$$\oint_c \vec{F}.d\vec{r} = \iint_R (\text{curl}\,\vec{F}.\vec{k})\,dA = -2\iint_R dA \qquad \text{.......... (i)}$$

Given that the of $\vec{F}$ is $x^2 + y^2 = \frac{1}{4}$. That is the path is a circle having radius $\frac{1}{2}$. So, changing the Cartesian from to polar with $x = r\cos\theta$ and $y = r\sin\theta$. Then $dxdy = rdrd\theta$.

Also, radius of region is $r = \frac{1}{2}$. And the angle θ varies from $\theta = 0$ to $\theta = 2\pi$.

Therefore, (i) becomes,

$$\oint_c \vec{F}.d\vec{r} = -2\int_0^{1/2}\int_0^{2\pi} r\,drd\theta = -2\int_0^{1/2} r.\,2\pi dr = -4p\left[\frac{r^2}{2}\right]_0^{1/2} = -4\pi.\frac{1}{8} = \frac{-\pi}{2}$$

Thus, $\oint_c \vec{F}.d\vec{r} = \frac{-\pi}{2}$

3. $\vec{F}$ = grad (sinx.cosy), C is the ellipse $25x^2 + 9y^2 = 225$.

Solution: Given that,

$$\vec{F} = \text{grad (sinx cosy)}$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{i}\frac{\partial}{\partial y}\right)(\text{sinx cosy}) = \text{cosx cosy}\,\vec{i} - \text{sinx siny}\,\vec{j}$$

So,

$$\text{Curl }\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ \text{cosx cosy} & -\text{sinx siny} & 0 \end{vmatrix}$$

$$= (-\text{cosx siny} + \text{cosx siny})\,\vec{k} = 0\,\vec{k}.$$

Therefore, $\text{Curl }\vec{F}.\vec{k} = 0$

By Green's theorem we have,

$$\oint_c \vec{F}.d\vec{r} = \iint_R (\text{curl}\,\vec{F}.\vec{k})\,dA = \iint_R 0\,dA = 0.$$

Thus, $\oint_c \vec{F}.d\vec{r} = 0.$

4. $\vec{F} = (\tan 0.2x, x^5y)$, R: $x^2 + y^2 \le 25, y \ge 0$.

Solution: Given that,

$$\vec{F} = (\tan 0.2x,\ x^5y)$$

Then,

$$\text{Curl }\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ \tan 0.2x & x^5y & 0 \end{vmatrix} = 5x^4y\,\vec{k}$$

So,

$$\text{Curl }\vec{F}.\vec{k} = 5x^4y\,\vec{k}.\vec{k} = 5x^4y$$

By Green's theorem, we have,

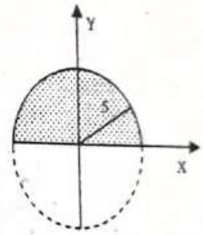
$$\oint_c \vec{F}.d\vec{r} = \iint_R (\text{curl}\,\vec{F}.\vec{k})\,dA = 5\iint_R (x^4y)\,dA \qquad \text{......... (i)}$$

Given that the path of $\vec{F}$ is in the region $x^2 + y^2 \le 25, y \ge 0$. Clearly the region is a half circle having radius $r = 5$.

Thus, $r = 5$ and θ varies from $\theta = 0$ to $\theta = \pi$.

Transforming the coordinate in to polar from then,

$x = r\cos\theta$, $y = r\sin\theta$ and $dydx = rdrd\theta$.



Then, (i) becomes,

$$\oint_c \vec{F}.d\vec{r} = \int_0^5\int_0^\pi 5r^4\cos^4\theta\,r\sin\theta.\,r\,d\theta\,dr$$

$$= \int_0^5\int_0^\pi 5r^6\cos^4\theta\sin\theta\,d\theta\,dr$$

Put $\cos\theta = u$ then $-\sin\theta\,d\theta = du$. Also, $\theta = 0 \Rightarrow u = 1$, $\theta = \pi \Rightarrow u = -1$

Then,

$$\oint_c \vec{F}.d\vec{r} = -\int_0^5 5r^6\int_1^{-1} u^4\,dudr$$

$$= -5\int_0^5 r^6\left[\frac{u^5}{5}\right]_1^{-1} dr = -5\int_0^5 r^6\left(\frac{-1-1}{5}\right) dr$$

$$= 5\times\frac{2}{5}\left[\frac{r^7}{7}\right]_0^5 = \frac{2\times 5^7}{7}.$$

Thus, $\oint_c \vec{F}.d\vec{r} = \frac{2\times 5^7}{7}.$

5. $\vec{F} = [\frac{e^y}{x}\ e^y\log x + 2x)$, R: $1 + x^4 \le y \le 2$.

Solution: Given that,

$$\vec{F} = \left(\frac{e^y}{x},\ e^y\log x + 2x\right)$$

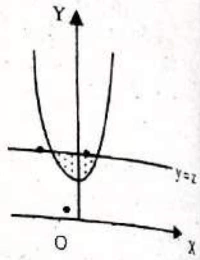
Then,

$$\text{Curl } \vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ \frac{e^y}{x} & e^y \log x + 2x & 0 \end{vmatrix} = \left(\frac{e^y}{x} + 2 - \frac{e^y}{x}\right)\vec{k} = 2\vec{k}$$

So, $\text{Curl } \vec{F}.\vec{k} = 2\vec{k}.\vec{k} = 2$

Now, by Green's theorem we have,

$$\oint_c \vec{F}\,d\vec{r} = \iint_R (\text{curl } \vec{F}.\vec{k})dA = 2\iint_R dA \quad \dots (i)$$



Also, given that the path of region of $\vec{F}$ is $1 + x^4 \leq y \leq 2$

For the curve $1 + x^4 = y$

x	0	± 1	± 2
y	1	2	17

And the curve $y = 2$ is a straight line.

From the figure, the region is bounded by $1 + x^2 \leq y \leq 2$ and solving the curves $y = 2$ and $y = x^2 + 1$ then we get $x = \pm 1$.

Now, (i) becomes,

$$\oint_c \vec{F}.d\vec{r} = 2\int_{-1}^{1}\int_{1+x^4}^{2} dy\,dx$$

$$= 2\int_{-1}^{1} [y]_{1+x^4}^{2} dx = 2\int_{-1}^{1} (2 - 1 - x^4)\,dx$$

$$= 2\int_{-1}^{1} (1 - x^4)\,dx$$

$$= 2\left[x - \frac{x^5}{5}\right]_{-1}^{1} = \left[\left(1 - \frac{1}{5}\right) - \left(-1 + \frac{1}{5}\right)\right]$$

$$= 2\left(2 - \frac{2}{5}\right)$$

$$= 4\left(5 - \frac{1}{5}\right) = \frac{16}{5}.$$

Thus, $\oint_c \vec{F}.d\vec{r} = \frac{16}{5}$.

C. Use Green's theorem to evaluate the line integrals:

1. $\oint_c [(x^2 + y^2)dx + xy^2\,dy]$; **where C is the closed curve determined by $y^2 = x$ and $y = -x$ with $0 \leq x \leq 1$.**

Solution: Given that,

$$I = \oint_c [(x^2 + y^2)\,dx + xy^2 dy] \quad \dots\dots\dots\dots (i)$$

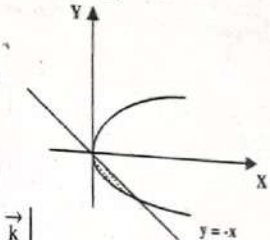
Where, the path c is determined by $y^2 = x$ and $y = -x$ for $0 < x < 1$.

Clearly, $y^2 = x$ is a parabola having vertex at (0, 0) and line of symmetry is $y = 0$.

And, the line $y = -x$ passes through (0, 0) and (1, −1). From the figure, y varies from $y = -\sqrt{x}$ to $y = -x$. And x moves $x = 0$ to $x = 1$.

By Green's theorem, we have,

$$\oint_c \vec{F}\,d\vec{r} = \iint_R (\text{curl } \vec{F}.\vec{k})dA \quad \dots\dots\dots\dots (ii)$$



Comparing (i) with $\oint_c \vec{F}.d\vec{r}$ then, we get,

$$\vec{F} = (x^2 + y^2)\,\vec{i} + xy^2\,\vec{j}$$

Then,

$$\text{Curl } \vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ x^2 + y^2 & xy^2 & 0 \end{vmatrix} = (y^2 - 2y)\,\vec{k}$$

So,

$$\text{Curl } \vec{F}.\vec{k} = (y^2 - 2y)\,\vec{k}.\vec{k} = y^2 - 2y$$

Then (ii) becomes,

$$\oint_c [(x^2 + y^2)dx + xy^2\,dy] = \int_0^1 \int_{-\sqrt{x}}^{-x} (y^2 - 2y)\,dy\,dx$$

$$= \int_0^1 \left[\frac{y^3}{3} - y^2\right]_{\sqrt{x}}^{-x} dx$$

$$= \int_0^1 \left(\frac{-x^3}{3} - x^2 + \frac{x\sqrt{x}}{3} + x\right) dx$$

$$= \left[\frac{-x^4}{12} - \frac{x^3}{3} + \frac{x^{5/2}}{15/2} + \frac{x^2}{2}\right]_0^1$$

$$= \frac{-1}{12} - \frac{1}{3} + \frac{2}{15} + \frac{1}{2}$$

$$= \frac{-5 - 20 + 8 + 30}{60} = \frac{13}{60}.$$

Thus, $I = \frac{13}{60}$.

2. $\oint_c [x^2y^2\,dx + (x^2 - y^2)\,dy]$; **where C is the square with vertices (0, 0), (1, 0), (1, 1), (0, 1).**

Solution: Given that,

$$I = \oint_c [x^2y^2 + (x^2 - y^2)\,dy] \quad \ldots\ldots\ldots\ldots \text{(i)}$$

With C is a square having vertices (0, 0), (1, 0), (1, 1) and (0, 1)

Comparing (i) with $\oint_c \vec{F}.d\vec{r}$ then, we get,

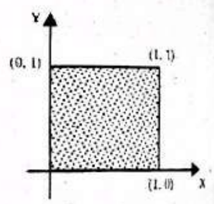
$$\vec{F} = x^2y^2\,\vec{i} + (x^2 - y^2)\,\vec{j}$$

Then,

$$\text{Curl}\,\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ x^2y^2 & x^2 - y^2 & 0 \end{vmatrix} = (2x - 2x^2y)\,\vec{k}$$

So,

$$\text{Curl}\,\vec{F}.\vec{k} = (2x - 2x^2y)\,\vec{k}.\vec{k} = (2x - 2x^2y)$$



Now, by Green's theorem,

$$\oint_c \vec{F}.d\vec{r} = \iint_R (\text{curl}\,\vec{F}.\vec{k})\,dA$$

So,

$$\oint_c [x^2y^2dx + (x^2 - y^2)\,dy] = \iint_R (2x - 2x^2y)\,dA \quad \ldots\ldots \text{(ii)}$$

Given that the region of the force is the square shown in figure. In which, y varies from y = 0 to the line joining the points (0, 1) and (1, 1). That is, from y = 0 to y = 1. And x moves from x = 0 to x = 1.

Therefore (ii) becomes,

$$\oint_c [x^2y^2dx + (x^2 - y^2)\,dy] = \int_0^1\int_0^1 (2x - 2x^2y)\,dy\,dx$$

$$= \int_0^1 [2x - 2x^2y]_0^1\,dx = \int_0^1 (2x - x^2)\,dx$$

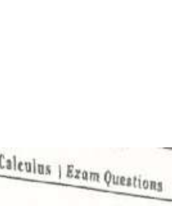
$$= \left[x^2 - \frac{x^3}{3}\right]_0^1 = 1 - \frac{1}{3} = \frac{2}{3}$$

Thus, $I = \frac{2}{3}$.

$\oint_c xy\,dx + (y + x)\,dy)]$, **where C is the circle $x^2 + y^2 = 1$.**

Solution: Given that,

$$I = \oint_c [xy.dx + (y + x)\,dy] \quad \ldots\ldots\ldots \text{(i)}$$



where c is a circle $x^2 + y^2 = 1$

Comparing (i) with $\oint_c \vec{F}.d\vec{r}$, then we get,

$$\vec{F} = xy\,\vec{i} + (y + x)\,\vec{j}$$

Then,

$$\text{Curl}\,\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ xy & y + x & 0 \end{vmatrix} = (1 - x)$$

Then,

$$\text{Curl}\,\vec{F}.\vec{k} = (1 - x)\,\vec{k}.\vec{k} = 1 - x$$

By Green's theorem,

$$\oint_c \vec{F}d\vec{r} = \iint_R (\text{curl}\,\vec{F}.\vec{k})\,dA$$

So,

$$\oint_c [xy\,dx + (y + x)\,dy] = \iint_R (1 - x)\,dA$$

$$= \int_0^1\int_0^{2\pi} (1 - r\cos\theta)\,r\,d\theta\,dr \text{ [Changing in polar form]}$$

$$= \int_0^1 [\theta - r\sin\theta]_0^{2\pi}\,r\,dr = \int_0^1 2\pi r\,dr \text{ [}\because \sin 2\pi = 0 = \sin 0\text{]}$$

$$= [\pi r^2]_0^1 = \pi.$$

Thus, $I = \pi$.

4. $\oint_c [xy\,dx + \sin y\,dy]$, **where C is the triangle with vertices (1, 1), (2, 2), (3, 0).**

Solution: Given that,

$$I = \oint_c (xy\,dx + \sin y\,dy) \quad \ldots\ldots\ldots\ldots \text{(i)}$$

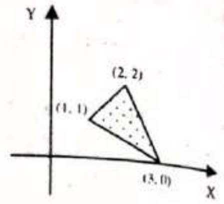
with c is a triangle having vertices at (1, 1), (2, 2) and (3, 0).

Comparing (i) with $\oint_c \vec{F}.d\vec{r}$ then we get,

$$\vec{F} = xy\,\vec{i} + \sin y\,\vec{j}$$

So,

$$\text{Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ xy & \sin y & 0 \end{vmatrix} = -x\vec{F}$$



Then $\quad \text{Curl } \vec{F}.\vec{k} = -x\,\vec{k}.\vec{k} = -x$

Now, by Green's theorem,

$$\oint_c \vec{F}.d\vec{r} = \iint_R (\text{curl } \vec{F}.\vec{k})\,dA$$

So, $\quad I = \iint_R (-x)\,dA \quad$ .... (ii)

Since the region C is shown in figure.

Here, the equation of line joining (1, 1) and (2, 20 is, y = x.

The equation of line joining (1, 1) and (3, 0) is, $y = \frac{-1}{2}(x - 3)$.

The equation of line joining (2, 2) and (3, 0) is, $y = 6 - 2x$.

From the figure, C is bounded from $y = \frac{3-x}{2}$ to $y = x$ in which x moves from $x = 1$ to $x = 2$. And the region is moves from $x = 2$ to $x = 3$ in which it is bounded by the lines $y = \frac{3-x}{2}$ to $y = 6 - 2x$.

Therefore, (ii) becomes,

$$I = -\int_1^2 \int_{[(3-x)/2]}^{x} dx - \int_2^3 \int_{[(3-x)/2]}^{6-x} x\,dy\,dx$$

$$= -\int_1^2 x\,[y]_{[(3-x)/2]}^{6-x}\,dx - \int_2^3 [y]_{[(3-x)/2]}^{6-x}\,dx$$

$$= -\int_1^2 x\left(x - \frac{3-x}{2}\right)dx - \int_2^3 x\left(6 - 2x - \frac{3-x}{2}\right)dx$$

$$= -\int_1^2 \left(\frac{2x^2 - 3x + x^2}{2}\right)dx - \int_2^3 \left(\frac{12x - 4x^2 - 3x + x^2}{2}\right)dx$$

$$= -\frac{1}{2}\int_1^2 (3x^2 - 3x)\,dx - \frac{1}{2}\int_2^3 (9x - 3x^2)\,dx$$

$$= -\frac{1}{2}\left[x^3 - \frac{3x^2}{2}\right]_1^2 - \frac{1}{2}\left[\frac{9x^2}{2} - x^3\right]_2^3$$

$$= -\frac{1}{2}\left[8 - 6 - 1 + \frac{3}{2} + \frac{81}{2} - 27 - 18 + 8\right] = -\frac{1}{2}\left[-36 + \frac{84}{2}\right]$$

$$= -\frac{1}{2}(-36 + 42) = \frac{-6}{2} = -3$$

Thus, I = –3.

3. $\oint_c \left[\frac{y^2}{(1+x^2)}\,dx + 2y\tan^{-1}x\,dy\right]$; **where C the hypocycloid** $x^{2/3} + y^{2/3} = 1$.

Solution: Given that,

$$I = \oint_c \left(\frac{y^2}{1+x^2}\,dx + 2y\tan^{-1}x\,dy\right) \quad \text{.......... (i)}$$

where C is $x^{2/3} + y^{2/3} = 1$

Comparing (i) with $\oint_c \vec{F}.d\vec{r}$ then we get,

$$\vec{F} = \frac{y^2}{1+x^2}\,\vec{i} + 2y\tan^{-1}x\,\vec{j}$$

Then,

$$\text{Curl } \vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ y^2/(1+x^2) & 2y\tan^{-1}x & 0 \end{vmatrix} = \frac{2y}{1+x^2} - \frac{2y}{1+x^2} = 0$$

So, $\quad \text{Curl } \vec{F}.\vec{k} = 0$

By Green's theorem we have,

$$\oint_c \vec{F}.d\vec{r} = \iint_R (\text{curl } \vec{F}.\vec{k})\,dA$$

So, $\quad I = \iint_R 0\,dA = 0.$

4. $\oint_c [(x + y)\,dx + (y + x^2)\,dy]$, **where C is the boundary of the region between the circles** $x^2 + y^2 = 1$ **and** $x^2 + y^2 = 4$.

Solution: Given that

$$I = \oint_c [(x+y)\,dx + (y+x^2)\,dy] \qquad \text{.......... (i)}$$

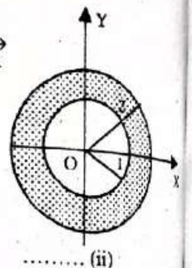
Where C is the region between $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$

Comparing (i) with $\oint_c \vec{F}\,d\vec{r}$ then we get,

$$\vec{F} = (x+y)\,\vec{i} + (y+x^2)\,\vec{j}$$

Then,

$$\text{Curl}\,\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\delta}{\delta x} & \dfrac{\delta}{\delta y} & \dfrac{\delta}{\delta z} \\ x+y & y+x^2 & 0 \end{vmatrix} = (2x-1)\,\vec{k}$$



So,

$$\text{Curl}\,\vec{F}.\vec{k} = (2x-1)\,\vec{k}.\vec{k} = 2x - 1$$

Since, by Green's theorem we have,

$$\oint_c \vec{F}\,d\vec{r} = \iint_R (\text{curl}\,\vec{F}.\vec{k})\,dA = \iint_R (2x-1)\,dA \qquad \text{......... (ii)}$$

Given that the force $\vec{F}$ works on the region in between $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Clearly the first circle has radius 1 and second has radius 2.

Therefore, the feasible region is in between $r = 1$ to $r = 2$.

Also, the region moves from $\theta = 0$ to $\theta = 2\pi$.

Therefore changing the integrand in (ii) in to polar from as $x = r\cos\theta$ and $dxdy = rdrd\theta$.

So that,

$$\oint_c \vec{F}\,d\vec{r} = \int_0^{2\pi}\int_1^2 (2r\cos\theta - 1)\,rdrd\theta$$

$$= \int_1^2\int_0^{2\pi} (2r^2\cos\theta - r)\,d\theta\,dr$$

$$= \int_1^2 [2r^2\sin\theta - r\theta]_0^{2\pi}\,dr = -2\pi\int_1^2 r\,dr \quad [\because \sin 2\pi = \sin 0]$$

$$= -\pi[r^2]_1^2 = -\pi(4-1) = -3\pi.$$

Thus, $\oint_c [(x+y)\,dx + (y+x^2)\,dy] = -3\pi$

7. **$\oint_c [15xy\,dx + x^3\,dy]$, where C is the closed curve consisting of the graphs of $y = x^2$ and $y = 2x$ between the points (0, 0) and (2, 4).**

Solution: Given that,

$$I = \oint_c (5xy\,dx + x^3\,dy) \qquad \text{.............. (i)}$$

Where c is the closed curve obtained by the graph of the curve $y = x^2$ and $y = 2x$ in between (0, 0) to (2, 4).

Comparing (i) with $\oint_c \vec{F}\,d\vec{r}$ then we get,

$$\vec{F} = 5xy\,\vec{i} + x^3\,\vec{j}$$

So,

$$\text{Curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\delta}{\delta x} & \dfrac{\delta}{\delta y} & \dfrac{\delta}{\delta z} \\ 5xy & x^3 & 0 \end{vmatrix} = (3x^2 - 5x)\,\vec{k}$$

Then, $\text{Curl}\,\vec{F}.\vec{k} = (3x^2 - 5x)\,\vec{k}\,\vec{k} = 3x^2 - 5x$

Since by Green's theorem we have,

$$\oint_c \vec{F}.d\vec{r} = \iint_R (\text{curl}\,\vec{F}.\vec{k})\,dA = \iint_R (3x^2 - 5x)\,dA \qquad \text{............ (ii)}$$

Given that $\vec{F}$ work on the region of common part of $y = x^2$ and $y = 2x$ in between (0, 0) to (2, 4).

Therefore, (ii) becomes,

$$I = \iint_R (3x^2 - 5x)\,dA$$

$$= \int_0^2\int_{2x}^{x^2} (3x^2 - 5x)\,dy\,dx = \int_0^2 [3x^2y - 5x^3]_{2x}^{x^2}\,dx$$

$$= \int_0^2 [(3x^4 - 5x^3) - (6x^3 - 10x^2)]\,dx$$

$$= \int_0^2 (3x^4 - 11x^3 + 10x^2)\,dx$$

$$= \left[\frac{3x^5}{5} - \frac{11x^4}{4} + \frac{10x^3}{3}\right]_0^2$$

$$= \frac{96}{5} - \frac{11\times 16}{4} + \frac{80}{3} = \frac{288 - 660 + 400}{15} = \frac{28}{15}$$

Thus, $I = \dfrac{28}{15}$.

8. $\oint_C [2xy\,dx + (x^2+y^2)\,dy]$, where C is the ellipse $4x^2 + 9y^2 = 36$.

Solution: Given that,

$$I = \oint_C [2xy\,dx + (x^2+y^2)\,dy] \quad \text{......... (i)}$$

where C is $4x^2 + 9y^2 = 36$.

Comparing (i) with $\oint_C \vec{F}\cdot d\vec{r}$ then we get,

$$\vec{F} = 2xy\,\vec{i} + (x^2+y^2)\,\vec{j}$$

Then,

$$\text{Curl } \vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ 2xy & x^2+y^2 & 0 \end{vmatrix} = (2x-2x)\vec{k} = 0\vec{k}$$

So, Curl $\vec{F}\cdot\vec{k} = 0\vec{k}\cdot\vec{k} = 0$.
By Green's theorem we have,

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R (\text{curl }\vec{F}\cdot\vec{k})\,dA$$

So, $\oint_C [2xy\,dx + (x^2+y^2)\,dy] = \iint_R 0\,dA = 0.$

EXERCISE 4.8

Evaluate $\iint_S \vec{F}\cdot\vec{n}\,dA$, where

1. $\vec{F} = (3x^2, y^2, 0)$, **S:** $\vec{r} = (u, v, 2u + 3v)$, $0 \le u \le 2, -1 \le v \le 1$.
Solution: Given that,

$$\vec{F} = (3x^2, y^2, 0) = 3x^2\vec{i} + y^2\vec{j} + 0\vec{k}$$

And $\vec{r} = (u, v, 2u+3v) = u\,\vec{i} + v\,\vec{j} + (2u+3v)\,\vec{k}$.

Then, $\vec{r}_u = (\vec{i} + 2\vec{k})$ and $\vec{r}_v = \vec{j} + 3\vec{k}$.
So that,

$$\vec{N} = \vec{r}_u\,\vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{vmatrix} = -2\vec{i} - 3\vec{j} + \vec{k}.$$

Since we have,

$$\iint_S \vec{F}\cdot\vec{n}\,dA = \iint_R \vec{F}(\vec{r})\cdot\vec{N}\,du\,dv \quad \text{......... (i)}$$

Since, $\vec{r} = x\,\vec{i} - y\,\vec{j} + z\,\vec{k}$. And given that, $\vec{r} = u\,\vec{i} + v\,\vec{j} + (2u+3v)\vec{k}$.
So that, $\vec{F} = 3x^2\,\vec{i} + y^2\,\vec{j}$
This implies that,

$$\vec{F}(\vec{r})\cdot\vec{N} = (3u^2\,\vec{i} + v^2\,\vec{j})\cdot(-2\vec{i} - 3\vec{j} + \vec{k}) = -6u^2 - 3v^2$$

Also, given that the region is $0 \le u \le 2$, $-1 \le v \le 1$.
Thus, (i) become,

$$\iint_S \vec{F}\cdot\vec{n}\,dA = \int_{-1}^{1}\int_{0}^{2}(-6u^2 - 3v^2)\,du\,dv$$

$$= \int_{-1}^{1}[-2u^3 - 3v^2u]_0^2\,dv = \int_{-1}^{1}(-16 - 6v^2)\,dv$$

$$= [-16v - 2v^3]_{-1}^{1}$$

$$= (-16-2) - (16+2) = -18 - 18 = -36$$

Thus, $\iint_S \vec{F}\cdot\vec{n}\,dA = -36.$

2. $\vec{F} = (e^{2y}, e^{-2z}, e^{2x})$, **S :** $\vec{r} = (3\cos u, 3\sin u, v)$, $0 \le u \le \frac{\pi}{2}$, $0 \le v \le 2$.

Solution: Given that, $\vec{F} = (e^{2y}, e^{-2z}, e^{2x})$ and $\vec{r} = (3\,\text{Cos}u, 3\,\text{Sin}u, v)$.

So, $\vec{r}_u = (-3\,\text{Sin}u, 3\,\text{Cos}u, 0)$ and $\vec{r}_v = (0, 0, 1)$
Then,

$$\vec{N} = \vec{r}_u\times\vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3\sin u & 3\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 3\,\text{Cos}u\,\vec{i} + 3\,\text{Sin}u\,\vec{j} = (3\,\text{Cosv}, \text{Sinu}, 0).$$

Since we know that $\vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k} = (x, y, z)$ and given that $\vec{r} = (3\,\text{Cosu}, 3\,\text{Sinu}, v)$ then we get

$$x = 3\,\text{Cos}u, \quad y = 3\,\text{Sin}u, \quad z = v$$

Then, $\vec{F}(\vec{r}) = (e^{6\,\text{Sin}u}, e^{-2v}, e^{6\,\text{Cos}u})$
So that,

$$\vec{F}(\vec{r})\cdot\vec{N} = (e^{6\,\text{Sin}u}, e^{-2v}, e^{6\,\text{Cos}u})\cdot(3\,\text{Cos}u, 3\,\text{Sin}u, 0)$$

$$= 3\,\text{Cos}u\,e^{6\,\text{Sin}u} + 3\,\text{Sin}u\,e^{-2v}$$

Since we have, $\iint_S \vec{F}\cdot\vec{n}\,dA = \iint_R \vec{F}(\vec{r})\cdot\vec{N}\,du\,dv$ .... (i)

Also given that the region is $0 \le u \le \frac{\pi}{2}, 0 \le v \le 2$.

Then (i) becomes,

$$\iint_S \vec{F}.\vec{n}\, dA = \int_0^2\int_0^{\frac{\pi}{2}} (3\cos u\, e^{6\sin u} + 3\sin u\, e^{-2v})\, du\, dv$$

$$= \int_0^2 \left[\frac{e^{6\sin u}}{2} + (-3)\cos u\, e^{-2v}\right]_0^{\pi/2} dv$$

$$= \int_0^2 \left(\frac{1}{2}e^6 + 3e^{-2v} - \frac{1}{2}\right) dv$$

$$[\because \operatorname{Sin}\frac{\pi}{2} = 1 = \operatorname{Cos}0, \operatorname{Sin}0 = 0 = \operatorname{Cos}\frac{\pi}{2}]$$

$$= \left[\frac{1}{2}e^6 v + \frac{3e^{-2v}}{-2} - \frac{v}{2}\right]_0^2$$

$$= e^6 - \frac{3}{2}(e^{-4} - 1) - 1 \quad = e^6 - \frac{3}{2}e^{-4} + \frac{1}{2}.$$

Thus, $\iint_S \vec{F}.\vec{n}\, dA = e^6 - \frac{3}{2}e^{-4} + \frac{1}{2}$.

3. $\vec{F} = (x - z, y - x, z - y)$, S: $\vec{r} = (u\cos v, u\sin v, u)$, $0 \le u \le 3, 0 \le v \le 2\pi$.
[2004 Spring Q.No. 4(a)]

Solution: Similar to Q. No. 1 and Q. No. 2.

4. $\vec{F} = (0, x, 0)$, S: $x^2 + y^2 + z^2 = 1, x \ge 0, y \ge 0, z \ge 0$.

Solution: Given that

$\vec{F} = (0, x, 0)$ and $x^2 + y^2 + z^2 = 1$ for $x \ge 0, y \ge 0, z \ge 0$.

Set, $x = u$, $y = v$ then $z = \sqrt{1 - u^2 - v^2}$.

Since we have,

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = u\vec{i} + v\vec{j} + \sqrt{1 - u^2 - v^2}\,\vec{k}.$$

Then, $\vec{r}_u = \vec{i} - \frac{u\vec{k}}{\sqrt{1 - u^2 - v^2}}$ and $\vec{r}_v = \vec{j} - \frac{v\vec{k}}{\sqrt{1 - u^2 - v^2}}$.

Since the sphere $x^2 + y^2 + z^2 = 1$ has radius $r = 1$. And given that the region is only the part $x \ge 0, y \ge 0, z \ge 0$ that implies the angle $\theta = \frac{\pi}{2}$.

So, set the Cartesian form u, v is polar form as,

$u = r\cos\theta, \quad v = r\sin\theta$

So, $\vec{r}_u = \vec{i} - \frac{r\cos\theta\,\vec{k}}{\sqrt{1 - r^2}}$, $\vec{r}_v = \vec{j} - \frac{r\sin\theta\,\vec{k}}{\sqrt{1 - r^2}}$, $du\,dv = r\, dr\, d\theta$.

Since we have, $\iint_S \vec{F}.\vec{n}\, dA = \iint_R \vec{F}(\vec{r}).\vec{N}\, du\, dv$ .... (i)

where $\vec{N} = \vec{r}_u \times \vec{r}_v$

Since, $\vec{F} = (0, x, 0)$ and $\vec{r} = (x, y, z) = (u, v, \sqrt{1 - u^2 - v^2})$.

Then, $\vec{F}(\vec{r}) = u\vec{j}$

And,

$$\vec{N} = \vec{r}_u \times \vec{r}_v$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -u/\sqrt{1 - u^2 - v} \\ 0 & 1 & -v/\sqrt{1 - u^2 - v^2} \end{vmatrix} = \frac{u\vec{i}}{\sqrt{1 - u^2 - v^2}} + \frac{v\vec{j}}{\sqrt{1 - u^2 - v^2}} + \vec{k}$$

Then,

$$\vec{F}(\vec{r}).\vec{N} = (u\vec{j}).\left(\frac{u\vec{i}}{\sqrt{1 - u^2 - v^2}} + \frac{v\vec{j}}{\sqrt{1 - u^2 - v^2}} + \vec{k}\right)$$

$$= \frac{uv}{\sqrt{1 - u^2 - v^2}} = \frac{r^2\sin\theta\cos\theta}{\sqrt{1 - r^2}} = \frac{r^2\sin 2\theta}{2\sqrt{1 - r^2}}.$$

Now (i) becomes,

$$\iint_S \vec{F}.\vec{n}\, dA = \int_0^{\pi/2}\int_0^1 \frac{r^2\sin 2\theta}{2\sqrt{1 - r^2}}\, r\, dr\, d\theta$$

$$= \int_0^1 \frac{r^3\, dr}{2\sqrt{1 - r^2}} \int_0^{\pi/2} \operatorname{Sin}2\theta\, d\theta.$$

$$= \int_0^1 \frac{r^3\, dr}{2\sqrt{1 - r^2}} \left[\frac{-\cos 2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{-1}{4}\int_0^1 \frac{r^3\, dr}{\sqrt{1 - r^2}}(\operatorname{Cos}\pi - \operatorname{Cos}0) = \frac{2}{4}\int_0^1 \frac{r^3\, dr}{\sqrt{1 - r^2}}$$

Put $r = \operatorname{Sin}\theta$ then $dr = \operatorname{Cos}\theta\, d\theta$. Also $r = 0 \Rightarrow \theta = 0$, $r = 1 \Rightarrow 1 = \frac{\pi}{2}$. Then.

$$\iint_S \vec{F}.\vec{n}\, dA = \frac{1}{2}\int_0^{\pi/2} \frac{\sin^3\theta.\cos\theta\, d\theta}{\cos\theta}$$

$$= \frac{1}{2}\int_0^{\pi/2} \operatorname{Sin}^3\theta\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \operatorname{Sin}^3\theta \cos^0\theta\, d\theta$$

$$= \frac{\Gamma\left(\frac{3+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{3+0+2}{2}\right)} \quad [\because \text{ Using beta and gamma function}]$$

$$= \frac{\Gamma(2)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{5}{2}\right)}$$

$$= \frac{1!\sqrt{\pi}}{2\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\sqrt{\pi}}$$

$$\left[\because \Gamma(m) = m!; \Gamma(m+1) = m\Gamma(m); \Gamma(1/2) = \sqrt{\pi}\right]$$

$$= \frac{1}{3}$$

Thus, $\iint_S \vec{F}.\vec{n}\, dA = \frac{1}{3}$.

5. $\vec{F} = (x, y, z)$, S: $\vec{r} = (u \cos v, u \sin v, u^2)$, $0 \le u \le 4, -\pi \le v \le \pi$.

Solution: Similar to Q. No. 2.

6. **$\vec{F} = (18z, -12, 3y)$ and S is the surface of the plane $2x + 3y + 6z = 12$ in the first octant.**

Solution: Given that, $\vec{F} = (18z, -12, 3y)$
and the surface is, $2x + 3y + 6z = 12$

in the first octant set, $x = u, y = v$ then $z = \frac{12 - 2u - 3v}{6}$.

Since we have,

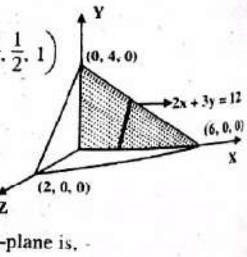
$$\vec{r} = (x, y, z) = \left(u, v, \frac{12 - 2u - 3v}{6}\right)$$

So, $\vec{r_u} = \left(1, 0, \frac{-2}{6}\right) = \left(1, 0, \frac{-1}{3}\right)$ and $\vec{r_v} = \left(0, 1, \frac{-3}{6}\right) = \left(0, 1, \frac{-1}{2}\right)$

Then,

$$\vec{N} = \vec{r_u} \times \vec{r_v}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1/3 \\ 0 & 1 & -1/2 \end{vmatrix} = \frac{\vec{i}}{3} + \frac{\vec{j}}{3} + \vec{k} = \left(\frac{1}{3}, \frac{1}{2}, 1\right)$$



So that,

$$\vec{F}(\vec{r}).\vec{N} = (36 - 64 - 9v, -12, 3v).\left(\frac{1}{3}, \frac{1}{2}, 1\right)$$

$$= 12 - 2u - 3v - 6 + 3v = 6 - 2u$$

The projection of the plane $2x + 3y + 6z = 12$ is xy–plane is,

$$2x + 3y = 12, z = 0.$$

in which y varies from $y = 0$ to $y = \frac{12 - 2x}{3}$ and on the region, x moves fro. $x = 0$ to $x = 6$.

Since $x = u$, $y = v$. therefore (i) becomes,

$$\iint_S \vec{F}.\vec{n}\, dA = \int_0^6 \int_0^{(12-2u)/3} (6 - 2u)\, dv du$$

$$= \int_0^6 [6v - 2uv]_0^{(12-2u)/3} du$$

$$= \int_0^6 (24 - 4u - 8u + \frac{4u^2}{3})\, du$$

$$= \int_0^6 \left(24 - 12u + \frac{4u^2}{3}\right) du$$

$$= [24u - 6u^2 + \frac{4u^2}{3}]_0^6$$

$$= 144 - 216 + 96 = 24$$

Thus

$$\iint_S \vec{F}.\vec{n}\, dA = 24.$$

7. **$\vec{F} = (12x^2y, -3yz, 2z)$ and S is the portion of the plane $x + y + z = 1$ included in the first octant.** **[2010 Fall Q.No. 4(b)]**

Solution: Given that $\vec{F} = (12x^2y, -3yz, 2z)$.

And surface is $x + y + z = 1$ in first octant.

Set $x = u$ and $y = v$ then $z = 1 - u - v$

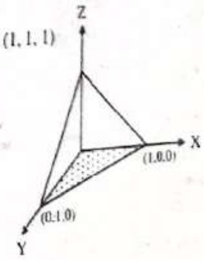
Here,

$$\vec{r} = (x, y, z) = (u, v, 1 - u - v)$$

So, $\vec{r}_u = (1, 0, -1)$ and $\vec{r}_v = (0, 1, -1)$

Then,

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \vec{i} + \vec{j} + \vec{k} = (1, 1, 1)$$



By surface integral we have,

$$\iint_S \vec{F}.\vec{n}\, dA = \iint_R \vec{F}.\vec{N}\, dx\, dy \quad \text{.......(i)}$$

where, $\vec{N} = \vec{r}_x \times \vec{r}_y = \vec{r}_u \times \vec{r}_v = (1, 1, 1)$

Here,

Solution: Given that $\vec{F} = (x, y, z)$.

And the surface is the upper half of the sphere $x^2 + y^2 + z^2 = a^2$.

The projection of the surface in xy–plane is the circle $x^2 + y^2 = a^2$ in which y varies $= -\sqrt{a^2 - x^2}$ to $y = \sqrt{a^2 - x^2}$ and x moves from $x = -a$ to $x = a$.

Set x = u and y = v then $z = \sqrt{a^2 - x^2 - y^2} = \sqrt{a^2 - u^2 - v^2}$

Here,

$$\vec{F} = (x, y, z) = (u, v, \sqrt{a^2 - u^2 - v^2})$$

Now, by surface integral we have.

$$\iint_S \vec{F}.\vec{n}\, dA = \iint_R \vec{F}.\vec{N}\, dx\, dy \quad \text{......(i)}$$

where. $\vec{N} = \vec{r}_x \times \vec{r}_y = \vec{r}_u \times \vec{r}_v$ and dx dy = du dv

Since we have, $\vec{r} = (x, y, z) = (u, v, \sqrt{a^2 - u^2 - v^2}))$

Then,

$$\vec{r}_u = \left(1, 0, \frac{-u}{\sqrt{a^2 - u^2 - v^2}}\right) \text{ and } \vec{r}_v = \left(0, 1, \frac{-v}{\sqrt{a^2 - u^2 - v^2}}\right)$$

So,

$$\vec{N} = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -u/\sqrt{a^2 - u^2 - v^2} \\ 0 & 1 & -v/\sqrt{a^2 - u^2 - v^2} \end{vmatrix}$$

$$= \frac{u}{\sqrt{a^2 - u^2 - v^2}}\vec{i} + \frac{v}{\sqrt{a^2 - u^2 - v^2}}\vec{j} + \vec{k}$$

Then,

$$\vec{F}.\vec{N} = \frac{u^2}{\sqrt{a^2 - u^2 - v^2}} + \frac{v^2}{\sqrt{a^2 - u^2 - v^2}} + \sqrt{a^2 - u^2 - v^2}$$

$$= \frac{u^2 + v^2 + a^2 - u^2 - v^2}{\sqrt{a^2 - u^2 - v^2}} = \frac{a^2}{\sqrt{a^2 - u^2 - v^2}}$$

Then (i) becomes,

$$\iint_S \vec{F}.\vec{n}\, dA = \int_{-a}^{a} \int_{-\sqrt{a^2 - u^2}}^{\sqrt{a^2 - u^2}} \frac{a^2}{\sqrt{a^2 - u^2 - v^2}}\, dv\, du \quad \text{......(ii)}$$

Put $u = r\cos\theta$, $v = r\sin\theta$ then $r^2 = u^2 + v^2$. Also, dv du = r dr dθ.

Moreover, the radius of the circle $u^2 + v^2 = a^2$ is r = a and θ varies from θ = 0 to θ = 2π. Then (ii) becomes,

$$\iint_S \vec{F}.\vec{n}\, dA = \int_0^{2\pi} \int_0^{a} \frac{a^2}{\sqrt{a^2 - r^2}}\, r\, dr\, d\theta$$

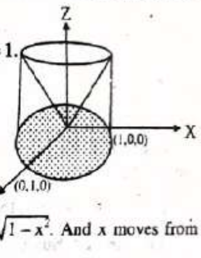
Put $a^2 - r^2 = p$ then $-2r\, dr = dp$. Also, $r = 0 \Rightarrow p = a^2$, $r = a \Rightarrow p = 0$. Then

$$\iint_S \vec{F}.\vec{n}\, dA = \int_0^{2\pi} \int_{a^2}^{0} a^2 p^{-1/2}\left(-\frac{dp}{2}\right) d\theta$$

$$= -\frac{a^2}{2}\left[\frac{p^{1/2}}{1/2}\right]_{a^2}^{0} [\theta]_0^{2\pi} = -a^2(0 - a)(2\pi - 0) = 2\pi a^3.$$

Thus, $\iint_S \vec{F}.\vec{n}\, dA = 2\pi a^3$.

10. **Find $\int\int_S (\vec{F}, \vec{n})\, ds$, where $\vec{F} = 2\vec{i} + 5\vec{j} + 3\vec{k}$ and S is the portion of the cone $Z = \sqrt{x^2 + y^2}$ that is inside the cylinder $x^2 + y^2 = 1$.**



Solution: Given that, $\vec{F} = 2\vec{i} + 5\vec{j} + 3\vec{k}$

And S is the portion of the cone $z = \sqrt{x^2 + y^2}$ inside in the cylinder $x^2 + y^2 = 1$.

That means, the projection of the portion in xy–plane is $x^2 + y^2 = 1$.

On the projection y varies from $y = -\sqrt{1 - x^2}$ to $y = \sqrt{1 - x^2}$. And x moves from x = – 1 to x = 1.

Now, by surface integral

$$\iint_S \vec{F}.\vec{n}\, ds = \iint_R (\vec{F}.\vec{N})\, dy\, dx \quad \text{........(i)}$$

where, $\vec{N} = \vec{r}_u \times \vec{r}_v$

Here, $\vec{r} = (x, y, z)$.

Put x = u and y = v then $z = \sqrt{u^2 + v^2}$

So, $\vec{r} = (u, v, \sqrt{u^2 + v^2})$

Then, $\vec{r}_u = \left(1, 0, \frac{u}{\sqrt{u^2 + v^2}}\right)$ and $\vec{r}_v = \left(0, 1, \frac{v}{\sqrt{u^2 + v^2}}\right)$

Therefore,

$$\vec{N} = \vec{r}_u \times \vec{r}_v$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & u/\sqrt{u^2 + v^2} \\ 0 & 1 & v/\sqrt{u^2 + v^2} \end{vmatrix} = \left(\frac{-u}{\sqrt{u^2 + v^2}}, \frac{-v}{\sqrt{u^2 + v^2}}, 1\right)$$

Then,

$$\vec{F}.\vec{N} = \frac{-2u - 5v + 3\sqrt{u^2 + v^2}}{\sqrt{u^2 + v^2}}$$

Now, (i) becomes,

$$\iint_S \vec{F}.\vec{n}\, ds = \int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} \left(\frac{-2u-5v+3\sqrt{u^2+v^2}}{\sqrt{u^2+v^2}}\right) dv\, du$$

Set $u = r\cos\theta$, $v = r\sin\theta$. Then on the circle, $r = 0$ to $r = 1$ and θ varies from $\theta = 0$ to $\theta = 2\pi$.

Therefore,

$$\iint_S \vec{F}.\vec{n}\, ds = \int_0^1\int_0^{2\pi} \frac{-2r\cos\theta - 5r\sin\theta + 3r}{r}\, r\, d\theta\, dr$$

$$= \int_0^1 r\,[-2\sin\theta + 5\cos\theta + 3\theta]_0^{2\pi}\, dr = \int_0^1 r\, dr.\, 6\pi = 6\pi.\frac{1}{2} = 3\pi.$$

11. Find the flux of $\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$ through the surface S is the first octant portion of the plane $2x + 3y + z = 6$.

Similar to Q. 10.

12. Let S be the part of the graph of $z = 9 - x^2 - y^2$ with $z \geq 0$. If $\vec{F} = 3x\vec{i} + 3y\vec{j} + z\vec{k}$. Find the flux of $\vec{F}$ through S. **[2009 Fall Q.No. 4(a)]**

Solution: Given that $\vec{F} = (3x, 3y, z)$.

And S is part of $z = 9 - x^2 - y^2$ with $z \geq 0$.

Clearly, the projection of the parabolid in xy–plane is a circle $x^2 + y^2 = 9$.

By surface integral,

$$\iint_S \vec{F}.\vec{n}\, dA = \iint_R (\vec{F}.\vec{N})\, dx\, dy \quad \text{......(i)}$$

where, $\vec{N} = \vec{r}_x \times \vec{r}_y$

Since $\vec{r} = (x, y, z) \Rightarrow \vec{r} = (x, y, 9 - x^2 - y^2)$

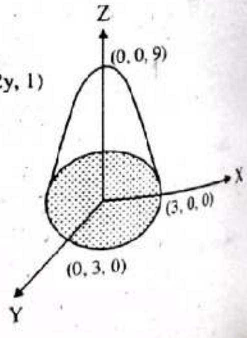
Then $\vec{r}_x = (1, 0, -2x)$, $\vec{r}_y = (0, 1, -2y)$.

For the circle, set $x = r\cos\theta$, $y = r\sin\theta$ then $z = 9 - r^2$.

On the circle, radius $r = 3$ and angular variation $\theta = 2\pi$.

Also,

$$\vec{N} = \vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix} = (2x, 2y, 1)$$



Then

$$\begin{aligned} \vec{F}.\vec{N} &= 6x^2 + 6y^2 + z \\ &= 6(x^2 + y^2) + 9 - (x^2 + y^2) \\ &= 6r^2 + 9 - r^2 \\ &= 9 + 5r^2 \end{aligned}$$

Now, (i) becomes,

$$\begin{aligned} \iint_S \vec{F}.\vec{n}\, dS &= \int_0^{2\pi}\int_0^3 (9 + 5r^2)\, r\, dr\, d\theta \\ &= \int_0^{2\pi}\left[\frac{9r^2}{2} + \frac{5}{4}r^4\right]_0^3 d\theta \\ &= \int_0^{2\pi}\left(\frac{81}{2} + \frac{405}{4}\right) d\theta \\ &= \left(81 + \frac{405}{2}\right).\frac{1}{2}\times 2\pi = \frac{162 + 405}{2}\pi = \frac{567\pi}{2} \end{aligned}$$

Thus, $\iint_S \vec{F}.\vec{n}\, dA = \frac{567\pi}{2}$.

13. $\vec{F} = (x, z, y)$, S is the hemisphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$.

Solution: Given that $\vec{F} = (x, z, y)$.

And the surface S is the hemisphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$.

Here,

$$\nabla.\vec{F} = 1 + 0 + 0 = 1.$$

Now,

$$\begin{aligned} \iint_S \vec{F}.\vec{n}\, dA &= \text{volume of the hemisphere.} \\ &= \frac{1}{2}\times\frac{4}{3}\times\pi\times(2)^3 \\ &= \frac{16\pi}{3} \end{aligned}$$

14. $\vec{F} = 3x\vec{i} + xz\vec{j} + z^2\vec{k}$, S is the surface of the region bounded by the paraboloid $z = 4 - x^2 - y^2$ and the xy-plane.

Solution: Similar to Q. 12.

EXERCISE – 4.9

Evaluate $\iint_S \vec{F}.\vec{n}\, dA$, by using Gauss divergence theorem of the **following data:**

1. **$\vec{F} = (x^2, 0, z^2)$, S is the box $|x| \leq 1$, $|y| \leq 3$, $|z| \leq 2$.**

Solution: Given that $\vec{F} = (x^2, 0, \geq 2)$ and the surface is the box $|x| \leq 1$, $|y| \leq 3$, $|z| \leq 2$.

By Gauss divergence theorem, we have,