Exercise 6.8

(1) Are the following functions linearly dependent or independent on the gien **interval?**

(i) Cosx, Sinx (any interval)

Solution: Let, $y_1 = \text{Cosx}$ and $y_2 = \text{Sinx}$

Now, $\dfrac{y_1(x)}{y_2(x)} = \dfrac{\cos x}{\sin x} = \cot x$,which is not a constant

So, cosx & sinx are linearly independent.

(ii) $x^2, x^3\ (o < x < 1)$

Solution: Let, $y_1 = x^2$ and $y_2 = x^3$

Now, $\frac{y_1(x)}{y_2(x)} = \frac{x^2}{x^3} = \frac{1}{x}$,which is not a constant.

So, $x^2 - 4, -3x^2 + 12$ are linearly dependent.

(iv) $1, e^{4x}\ (x < 0)$

Solution: Let, $y_1 = 1$ and $y_2 = e^{4x}$

Now, $\frac{y_1(x)}{y_2(x)} = \frac{1}{e^{4x}} = e^{-4x}$,which is not a constant.

So, $1, e^{4x}$ are linearly dependent.

(v) $\log x, \log x^2\ (x \geq 0)$

Solution: Let, $y_1 = \log x$ and $y_2 = \log x^2$

Now, $\frac{y_1(x)}{y_2(x)} = \frac{\log x}{\log x^2} = \frac{\log x}{2\log x} = \frac{1}{2}$,which is a constant

So, $\log x, \log x^2$ are linearly dependent.

(2) **Find a general solution of the following**

(i) $y'' - a^2y = 0$

Solution: Given equation is, $y'' - a^2y = 0$

So, its auxiliary equation is,

$$m^2 - a^2 = 0 \Rightarrow (m)^2 = (\pm a)^2 \Rightarrow m = a, -a \text{ (distinct real root)}$$

So, the general solution of given equation is,

$$y(x) = c_1e^{ax} + c_2e^{-ax}$$

(ii) $y'' - 4y' + 3y = 0$

Solution: Given equation is, $y'' - 4y' + 3y = 0$

So, its auxiliary equation is,

$$m^2 - 4m + 3 = 0 \Rightarrow m^2 - 3m - m + 3 = 0$$

$$\Rightarrow m(m-3) - 1(m-3) = 0$$

$$\Rightarrow (m-3)(m-1) = 0$$

$$\Rightarrow m = 3, 1 \text{(distinct real root)}$$

So, the general solution of given equation is,

$$y(x) = c_1e^{3x} + c_2e^{x}$$

(iii) $y'' + y' = 0$

Solution: Given equation is, $y'' + y' = 0$

So, its auxiliary equation is,

$$m^2 + m = 0 \Rightarrow m(m+1) = 0 \Rightarrow m = 0, -1 \text{ (distinct root)}$$

So, the general solution of given equation is,

$$y(x) = c_1e^0 + c_2e^{-x}$$
$$\Rightarrow y(x) = c_1 + c_2e^{-x}$$

(iv) $16y'' + 24y' + 9y = 0$ **[2002 Q. No. 4(b)]**

Solution: Given equation is, $16y'' + 24y' + 9y = 0.$

So, its auxiliary equation is,

$$16m^2 + 24m + 9 = 0 \Rightarrow (4m)^2 + 2.4m.3 + (3)^{3=0}$$
$$\Rightarrow (4m+3)^2 = 0 \Rightarrow m = -\frac{3}{4}, -\frac{3}{4} \text{ (equal root)}$$

So, the general solution of given equation is,

$$y(x) = (c_1 + c_2x)e^{\frac{-3}{4x}}$$

(v) $2y'' + 5y' - 12 = 0$

Solution: Given equation is, $2y'' + 5y' - 12 = 0$

So, its auxiliary equation is,

$$2m^2 + 5m - 12 = 0 \Rightarrow 2m^2 + 8m - 3m - 12 = 0$$
$$\Rightarrow 2m(m+4) - 3(m+4) = 0$$
$$\Rightarrow (m+4)(2m-3) = 0$$
$$\Rightarrow m = -4, \frac{3}{2} \text{ (distinct real root)}$$

So, the general solution of given equation is,

$$y(x) = c_1e^{-4x} + c_2e^{\frac{3}{2x}}$$

(3) Solve the following initial value problems.

(i) $y'' - y = 0, y(0) = 6, y'(0) = 4$

Solution: Given equation is, $y'' - y = 0$ ...... (i)

$y(0) = 6, y'(0) = 4$ ...... (ii)

So, its auxiliary equation is,

$$m^2 - 1 = 0 \Rightarrow m = \pm 1$$

So, its general solution is

$$y(x) = c_1e^x + c^2e^{-x} \quad \text{...... (iii)}$$

Since, by (ii) $y(0) = 6$ then (iii) gives,

$$6 = c_1e^0 + c_2e^0 \Rightarrow c_1 + c_2 = 6 \quad \text{.... (A)}$$

Now, differentiating (iii) we get,

$$y'(x) = c_1e^x - c_2e^{-x}$$

Since, by (ii) $y'(0) = 4$ then

$$-4 = c_1e^0 = c_2e^0 \Rightarrow c_1 - c_2 = -4 \quad \text{.... (B)}$$

Solving (A) and (B) we get,

$$c_1 = 1, c_2 = 5$$

Now, equations (iii) becomes

$$y(x) = e^x + 5e^{-x}$$

(ii) $y'' - 3y' + 2y = 0, y(0) = 0, y'(0) = 0$

Solution: Given equation is, $y'' - 3y' + 2y = 0$ ...... (i)

$y(0) = 0, y'(0) = 0$ ...... (ii)

So, its auxiliary equation is,

$$m^2 - 3m + 2 = 0 \Rightarrow m^2 - 2m - m + 2 = 0$$
$$\Rightarrow m(m-2) - 1(m-2) = 0$$
$$\Rightarrow (m-2)(m-1) = 0$$
$$\Rightarrow m = 2, 1$$

So, its general solution is,

$$y(x) = c_1e^{2x} + c_2e^x \quad \text{...... (iii)}$$

Since, by (ii), $y(0) = 0$ then (iii) gives,

$$0 = c_1e^0 + c_2e^0 \Rightarrow c_1 + c_2 = 0 \quad \text{... (A)}$$

Now, differentiating (iii) we get,

$$y'(x) = 2c_1e^{2x} + c_2e^x$$

Since, by (ii), $y'(0) = 0$ then

$$0 = 2c_1e^0 + c_2e^0 \Rightarrow 2c_1 + c_2 = 0 \quad \text{... (B)}$$

Solving (A) and (B) we get,

$$c_1 = 0 \text{ and } c_2 = 0$$

Now, equation (iii) becomes,

$$y(x) = 0$$

(iii) $y'' - 4y' + 3y = 0,\ y(0) = -1,\ y'(0) = -5$

Solution: Given equation is, $y'' - 4y' + 3y = 0$ ...... (i)

$y(0) = -1,\ y'(0) = -5$ ...... (ii)

So, its auxiliary equation is,

$$m^2 - 4m + 3 = 0 \Rightarrow m^2 - 3m - m + 3 = 0$$

$$\Rightarrow m(m-3) - 1(m-3) = 0$$

$$\Rightarrow (m-3)(m-1) = 0$$

$$\Rightarrow m = 3, 1$$

So, its general solution is,

$$y(x) = c_1e^{3x} + c_2e^{x} \qquad \text{... (A)}$$

Since, by (ii), $y(0) = -1$ then (iii) gives,

$$-1 = c_1e^0 + c_2e^0 \Rightarrow c_1 + c_2 = -1 \qquad \text{... (B)}$$

Now, differentiating (iii) we get,

$$y'(x) = 3c_1e^{3x} + c_2e^{x}$$

Since, by (ii), $y'(0) = -5$ then

$$-5 = 3c_1e^{3x} + c_2e^{x} \Rightarrow 3c_1 + c_2 = -5 \qquad \text{... (B)}$$

Solving (A) and (B) we get,

$$c_1 = -2 \text{ and } c_2 = 1$$

Now, equation (i) becomes

$$y(x) = -2e^{3x} + e^{x}$$

$$\Rightarrow y(x) = e^{x} - 2e^{3x}$$