EXERCISE 4.11

A. Find the line integral $\int_C \vec{F}.d\vec{r}$

1. $\vec{F}$ = (ycosxy, xcosxy, e^z), C is the straight line segment from $(\pi, 1, 0)$ to $\left(\frac{1}{2}, \pi, 1\right)$

Solution: Given that $\vec{F}$ = (yCosxy, xCosxy, e^z).

And the line is from $(\pi, 1, 0)$ to $\left(\frac{1}{2}, \pi, 1\right)$

Since we have, $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$. Then $d\vec{r} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$.
So that,

$$\begin{aligned}\vec{F}.d\vec{r} &= (y\text{Cos}xy,\ x\ \text{Cos}xy,\ e^z).(dx, dy, dz)\\ &= y\ \text{Cos}xy\ dx + x\ \text{Cos}xy\ dy + e^z\ dz = d(\text{Sin}xy) + d(e^z).\\ &= d(\text{Sin}xy + e^z).\end{aligned}$$

Now,

$$\int_C \vec{F}.d\vec{r} = \int_{(\pi,1,0)}^{(1/2,\pi,1)} d(\text{Sin}xy + e^z) = [\text{Sin}xy + e^z]_{(\pi,1,0)}^{(1/2,\pi,1)}$$

$$= \left(\sin\frac{\pi}{2} + e^1\right) - (\text{Sin}\pi + e^0)$$

$$= 1 + e - 0 - 1 = e.$$

Thus, $\int_C \vec{F}.d\vec{r} = e.$

2. $\vec{F} = (y^2, 2xy + \sin x, 0)$, C the boundary of $0 \le x \le \pi/2, 0 \le y \le 2, z = 0$.

Solution: Given that, $\vec{F} = (y^2, 2xy + \text{Sin}x, 0)$

And the surface is bounded by the boundaries $0 \le x \le \frac{\pi}{2}, 0 \le y \le 2, z = 0.$

Since the surface is a closed surface, by Stoke's theorem we have,

$$\int_c \vec{F}.d\vec{r} = \iint_s (\nabla\times\vec{F}).\vec{N}\,dx\,dy \;\ldots\ldots (i)$$

We have $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \Rightarrow \vec{r} = x\vec{i} + y\vec{j} + 0\vec{k}$ **[Being z = 0]**

Then, $\vec{N} = \vec{r_x}\times\vec{r_y} = \vec{k} = (0, 0, 1)$.

Here,

$$(\nabla\times\vec{F}).\vec{N} = (0, 0, \text{Cos}x).(0, 0, 1) = \text{Cos}x.$$

Now,

$$\iint_s (\nabla\times\vec{F}).\vec{N}\,dx\,dy = \int_0^2\int_0^{\pi/2} \text{Cos}x\,dx\,dy$$

$$= \int_0^2 [\text{Sin}x]_0^{\pi/2}\,dy = \int_0^2 dy \quad [\because \text{Sin}\tfrac{\pi}{2} = 1, \text{Sin}0 = 0]$$

$$= [y]_0^2 = 2$$

Thus, by (i), $\oint_c \vec{F}.d\vec{r} = 2$.

3. **$\vec{F} = (\cos\pi y, \sin\pi x, \cos\pi x)$, C the boundary of $0 \le x \le \frac{1}{2}$, $0 \le y \le 4$, $z = x$.**

Solution: Given that, $\vec{F} = (\text{Cos}\pi x, \text{Sin}\pi x, \text{Cos}\pi x)$.

And the region is bounded by $0 \le x \le \frac{1}{2}$, $0 \le y \le 4$, $z = x$

Since the region is a closed surface, by Stoke's theorem we have,

$$\oint_c \vec{F}.d\vec{r} = \iint_s (\nabla\times\vec{F}).\vec{N}\,dxdy \;\ldots\ldots (i)$$

We have $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \Rightarrow \vec{r} = x\vec{i} + y\vec{j} + x\vec{k}$ [Being z = x]

Then, $\vec{N} = \vec{r_x}\times\vec{r_y} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -\vec{i} + \vec{j} = (-1, 0, 1)$.

Therefore,

$$(\nabla\times\vec{F}).\vec{N} = (0, \pi\,\text{Sin}\pi x, \pi\,(\text{Cos}\pi x + \text{Sin}\pi y).(-1, 0, 1).$$
$$= \pi(\text{Cos}\pi x + \text{Sin}\pi y).$$

Now,

$$\iint_s (\nabla\times\vec{F}).\vec{N}\,dxdy = \pi\int_0^4\int_0^{1/2} (\text{Cos}\pi x + \text{Sin}\pi y)\,dxdy$$

$$= \pi\int_0^4 \left[\frac{\sin\pi x}{\pi} + x\sin\pi y\right]_0^{1/2} dy$$

$$= \frac{\pi}{\pi}\int_0^4 \left[\sin\frac{\pi}{2} + \pi.\frac{1}{2}\sin\pi y\right] dy \qquad = 0]$$

$$= \int_0^4 (1 + \frac{\pi}{2}\,\text{Sin}\pi y)\,dy \qquad [\because \text{Sin}\tfrac{\pi}{2} = 1]$$

$$= \left[y - \frac{\pi}{2}\frac{\cos\pi y}{\pi}\right]_0^4$$

$$= \left(4 - \frac{1}{2}\cos4\pi\right) - \left(0 - \frac{1}{2}\cos0\right)$$

$$= 4 - \frac{1}{2} - 0 + \frac{1}{2} \qquad [\because \text{Cos}0 = \text{Cos}4\pi = 1]$$

Thus, by (i) $\oint_c \vec{F}.d\vec{r} = 4$.

4. **$\vec{F} = (8xy, 4x^2, 2\cos2z)$, C the helix $\vec{r} = (\cos t, \sin t, t)$, $0 \le t \le \pi/4$.**

Solution: Given that, $\vec{F} = (8xy, 4x^2, 2\,\text{Cos}2z)$.

And the surface is a helix, $\vec{r} = (\text{Cos}t, \text{Sin}t, t)$ for $0 < t < \frac{\pi}{4}$.

Since we have $\vec{r} = (x, y, z)$. So, comparing with given term then, we see that.

$x = \text{Cos}t$, $y = \text{Sin}t$, $z = t$.

Therefore, $\vec{F} = (8\,\text{Cost Sint}, 4\,\text{Cos}^2t, 2\,\text{Cos}2t)$.

Since $\vec{r} = (\text{Cost}, \text{Sint}, t)$. So, $d\vec{r} = (-\text{Sint}, \text{Cost}, 1)\,dt$. So that,

$$\begin{aligned}
\vec{F}.d\vec{r} &= [-8\,\text{Cost Sin}^2t + \text{Cos}^3t + 2\,\text{Cos}2t]\,dt \\
&= [-8\,\text{Cost}\,(1 - \text{Cos}^2t) + 4\,\text{Cos}^3t + 2\,\text{Cos}2t]\,dt \\
&= [-8\,\text{Cost} + 8\,\text{Cos}^2t + 4\,\text{Cos}^3t + 2\,\text{Cos}2t]\,dt \\
&= [2\,\text{Cos}2t - 8\,\text{Cost} + 12\,\text{Cos}^3t]\,dt \\
&= [2\,\text{Cos}2t - 8\,\text{Cost} + 12\,\text{Cost}\,(1 - \sin^2t)]\,dt \\
&= [2\,\text{Cos}2t - 8\,\text{Cost} + 12\,\text{Cost} - 12\,\text{Sin}^2t\,\text{Cost}]\,dt \\
&= [2\,\text{Cos}2t + 4\,\text{Cost} - 12\,\text{Sin}^2t\,\text{Cost}]\,dt
\end{aligned}$$

Now,

$$\oint_c \vec{F}.d\vec{r} = \int_0^{\pi/4} (2\,\text{Cos}2t + 4\,\text{Cost} - 12\,\text{Sin}^2t\,\text{Cost})\,dt$$

$$= [\text{Sin}2t + 4\,\text{Sint}]_0^{\pi/4} - 12\int_0^{\pi/4} \text{Sin}^2t\cos t\,dt$$

$$= \text{Sin}\frac{\pi}{2} + 4\,\text{Sin}\frac{\pi}{4} - 12\int_0^{\pi/4} \text{Sin}^2t\cos t\,dt$$

$$= 1 + 4\left(\frac{1}{\sqrt{2}}\right) - 12\int_0^{\pi/4} \sin^2 t \cos t \, dt$$

Set $\sin t = u$ then $\cos t \, dt = du$. Also, $t = 0 \Rightarrow u = 0$, $t = \frac{\pi}{4} \Rightarrow u = \frac{1}{\sqrt{2}}$. Then,

$$\oint_c \vec{F}.d\vec{r} = 1 + 2\sqrt{2} - 12\int_0^{1/\sqrt{2}} u^2 \, du$$

$$= 1 + 2\sqrt{2} - 12\left[\frac{u^3}{3}\right]_0^{1/\sqrt{2}}$$

$$= 1 + 2\sqrt{2} - 12\left(\frac{1}{\sqrt{2}}\right)^3$$

$$= 1 + 2\sqrt{2} - 4\frac{1}{2\sqrt{2}}$$

$$= 1 + 2\sqrt{2} - \sqrt{2} = 1 + \sqrt{2}(2-1) = 1 + \sqrt{2}.$$

5. $\vec{F} = (e^x, e^y, e^z)$, C: $x = \log y$, $z = \log y$, $1 \le y \le 2$.

Solution: Given that, $\vec{F} = (e^x, e^y, e^z)$ and region is $x = \log y = z$, $1 \le y \le 2$.

Then, $\vec{F} = (e^{\log y}, e^y, e^{\log y}) = (y, e^y, y)$.

Since we have,

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = \log y\,\vec{i} + y\,\vec{j} + \log y\,\vec{k}$$

So, $d\vec{r} = \left(\frac{1}{y}\vec{i} + \vec{j} + \frac{1}{y}\vec{k}\right) dy$

Then,

$$\vec{F}.d\vec{r} = \left[(y, e^y, y).\left(\frac{1}{y}, 1, \frac{1}{y}\right)\right] dy = (1 + e^y + 1)dy$$

$$= (2 + e^y)\,dy.$$

Now,

$$\oint_c \vec{F}.d\vec{r} = \int_1^2 (2 + e^y)\,dy = \int_1^2 (1 + e^y + 1)\,dy$$

$$= \int_1^2 (2 + e^y)\,dy$$

$$= [2y + e^y]_1^2 = 4 + e^2 - 2 - e^1 = 2 + e^2 - e^1$$

Thus, $\oint_c \vec{F}.d\vec{r} = 2 + e^2 - e^1$.

6. $\vec{F} = (x^3, e^{3y}, e^{-3z})$, C : $x^2 + 9y^2 = 9$, $z = x^2$.

Solution: Given that, $\vec{F} = (x^3, e^{3y}, e^{-3z})$ and the region is $x^2 + 9y^2 = 9$, $z = x^2$.

Since we have,

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = x\vec{i} + y\vec{j} + x^2\vec{k}$$

So,

$$\vec{r_x} = \vec{i} + 2x\vec{k} = (1, 0, 2x), \quad \vec{r_y} = \vec{j} = (0, 1, 0)$$

By Stoke's theorem we have,

$$\iint_s (\nabla\times\vec{F}).\vec{N}\,dxdy = \oint_c \vec{F}.d\vec{r} \quad \text{......... (i)}$$

where, $\vec{N} = \vec{r_x}\times\vec{r_y}$

Here, $\vec{N} = \vec{r_x}\times\vec{r_y} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2x \\ 0 & 1 & 0 \end{vmatrix} = (-2x, 0, 1).$

And,

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ x^3 & e^{3y} & e^{-3x^2} \end{vmatrix} = (0, 6x\,e^{-3x^2}, 0)$$

Then,

$$(\nabla\times\vec{F}).\vec{N} = (0, 6x\,e^{-3x^2}, 0).(-2x, 0, 1) \qquad = 0 + 0 + 0 = 0.$$

Now,

$$\iint_s (\nabla\times\vec{F}).\vec{N}\,dxdy = \iint_s 0\,dxdy = 0$$

Thus, by (i), $\oint_c \vec{F}.d\vec{r} = 0.$

7. $\vec{F} = (\sin\pi x, z, 0)$, C the boundary of the triangle with vertices (0, 0, 0), (1, 0, 0), (1, 1, 0).

Solution: Given that, $\vec{F} = (\sin\pi x, z, 0)$.

And the surface is a triangle that has vertices (0, 0, 0), (1, 0, 0) and (1, 1, 0). Therefore, the surface is the plane $z = 0$ that passes all through points.

We have, $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = x\vec{i} + y\vec{j}$ $[\because z = 0]$

So $\vec{r_x} = \vec{i} = (1, 0, 0)$ and $\vec{r_y} = \vec{j} = (0, 1, 0)$

By Stoke's theorem we have,

$$\iint_s (\nabla\times\vec{F}).\vec{N}\,dxdy = \oint_c \vec{F}.d\vec{r} \quad \text{......... (i)}$$

where, $\vec{N} = \vec{r_x}\times\vec{r_y}$

Here, $\vec{N} = \vec{r_x}\times\vec{r_y} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 0\,\vec{i} + 0\,\vec{j} + \vec{k} = (0, 0, 1).$

And,

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ \sin\pi x & z & 0 \end{vmatrix} = -\vec{i} + 0\,\vec{j} + \vec{k} = (-1, 0, 0).$$

Therefore,

$$(\nabla\times\vec{F}).\vec{N} = (-1, 0, 0).(0, 0, 1) = 0 + 0 + 0 = 0.$$

Now, $\displaystyle\iint_S (\nabla\times\vec{F}).\vec{N}\,dxdy = \iint_S 0\,dx\,dy = 0.$

Then (i) gives, $\displaystyle\oint_C \vec{F}.d\vec{r} = 0.$

B. Find $\displaystyle\iint_S \vec{F}.\vec{n}\,dA$, where

1. $\vec{F} = (x, y)$, S: $z = 2x + 5y$, $0 \le x \le 2$, $-1 \le y \le 1$.

Solution: Given that, $\vec{F} = (x, y) = (x, y, 0)$

and S is $z = 2x + 5y$ for $0 \le x \le 2, -1 \le y \le 1$.

Since we have, $\vec{r} = (x, y, z) = (x, y, 2x + 5y)$

So, $\vec{r}_x = (1, 0, 2)$ and $\vec{r}_y = (0, 1, 5)$.

Now, $I = \displaystyle\iint_S \vec{F}.\vec{n}\,dA$ ....(i)

with $\vec{n} = \vec{r}_x\times\vec{r}_y$

Here,

$$\vec{n} = \vec{r}_x\times\vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2 \\ 0 & 1 & 5 \end{vmatrix} = -2\vec{i} - 5\vec{j} + \vec{k}$$

Then, $\vec{F}.\vec{n} = -2x - 5y$.

Therefore (i) becomes,

$$I = \int_0^2\int_{-1}^1 (-2x - 5y)\,dy\,dx$$

$$= -\int_0^2 \left[2xy + \frac{5y^2}{2}\right]_{-1}^1 dx$$

$$= -\int_0^2 \left(2x + \frac{5}{2} + 2x - \frac{5}{2}\right) dx \quad = -4\int_0^2 x\,dx$$

$$= -4\left[\frac{x^2}{2}\right]_0^2 = -4\left(\frac{4}{2} - 0\right) = -8.$$

2. $\vec{F} = (0, 20y, 2z^3)$, S: the surface of $0 \le x \le 6$, $0 \le y \le 1$, $0 \le z \le y$.

Solution: Given that, $\vec{F} = (0, 20y, 2z^3)$.

And the surface is bounded by $0 \le x \le 6$, $0 \le y \le 1$, $0 \le z \le y$.

By Gauss divergence theorem we have,

$$\iiint_T \text{div}\,\vec{F}\,dv = \iint_S \vec{F}.\vec{n}\,dA \quad \text{....... (i)}$$

Here,

$$\text{div}\,\vec{F} = \nabla.\vec{F} = \left(\vec{i}\frac{\delta}{\delta x} + \vec{j}\frac{\delta}{\delta y} + \vec{k}\frac{\delta}{\delta z}\right).(0, 20y, 2z^3)$$

$$= 0 + 20 + 6z^2 = 20 + 6z^2$$

Then,

$$\iiint_T \text{div}\,\vec{F}\,dv = \int_0^6\int_0^1\int_0^{6y} (20 + 6z^2)\,dz\,dy\,dx$$

$$= \int_0^6\int_0^1 [20z + 2z^3]_0^y\,dydx$$

$$= \int_0^6\int_0^1 (20y + 2y^3)\,dy\,dx$$

$$= \int_0^6 \left[10y^2 + \frac{2y^4}{4}\right]_0^1 dx$$

$$= \int_0^6 \left[10 + \frac{2}{4} - 0\right] dx$$

$$= \frac{42}{4}\int_0^6 dx = \frac{21}{2}[x]_0^6 = \frac{21}{2}\times 6 = 63.$$

Thus by (i), $\displaystyle\iint_S \vec{F}.\vec{n}\,dA = 63.$

3. $\vec{F} = (0, x^2, -xz)$, S: $\vec{r} = (u, u^2, v)$, $0 \le u \le 1$, $-2 \le v \le 2$.

Solution: Given that, $\vec{F} = (0, x^2, -xz)$. And $\vec{r} = (u, u^2, v)$.

So $\vec{r}_u = (1, 24, 0)$ and $\vec{r}_v = (0, 0, 1)$

Then,

$$\vec{N} = \vec{r}_u\times\vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 24 & 0 \\ 0 & 0 & 1 \end{vmatrix} = (24, -1, 0).$$

Comparing $\vec{r} = (u, u^2, v)$ with $\vec{r} = (x, y, z)$ then we get,

$x = u, \quad y = u^2, \quad z = v.$

Then, $\vec{F}(\vec{r}) = (0, u^2, -uv)$.

So,

$$\vec{F}(\vec{r}).\vec{N} = (0, u^2, -uv).(2u, -1, 0) = 0 - u^2 + 0 = -u^2.$$

Now,

$$\iint_R \vec{F}(\vec{r}).\vec{N}\, du\, dv = \int_0^1 \int_{-2}^{2} (-u^2)\, dv\, du$$

$$= \int_0^1 [-u^2 v]_{-2}^{2}\, du = \int_0^1 [-u^2(2+2)]\, du$$

$$= -4\int_0^1 u^2\, du = -4\left[\frac{u^3}{3}\right]_0^1 = \frac{-4}{3}.$$

Then by (i), $\iint_S \vec{F}.\vec{N} dA = \frac{-4}{3}$.

4. $\vec{F} = (1, 1, 1)$, S: $x^2 + y^2 + 4z^2 = 4, z \geq 0$.

Solution: Given that $\vec{F} = (1, 1, 1) = \vec{i} + \vec{j} + \vec{k}$.

And the surface is $x^2 + y^2 + 4z^2 = 4, z \geq 0$.

By Gauss divergence theorem, we have,

$$\iiint_T \text{div}\vec{F} dv = \iint_S \vec{F}.\vec{n}\, dA \quad \text{......... (i)}$$

Here,

$$\text{div}\vec{F} = \nabla.\vec{F} = \left(\vec{i}\frac{\delta}{\delta x} + \vec{j}\frac{\delta}{\delta y} + \vec{k}\frac{\delta}{\delta z}\right).(\vec{i} + \vec{j} + \vec{k})$$
$$= 0 + 0 + 0 = 0$$

Then,

$$\iiint_T \text{div}\vec{F} dv = \iiint_T 0\, dv = 0.$$

Thus by (i), $\iint_S \vec{F}.\vec{n}\, dA = 0$.

5. $\vec{F} = (x + z, y + z, x + y)$, S is the sphere $x^2 + y^2 + z^2 = 9$.

Solution: Given that, $\vec{F} = (x + z, y + z, x + y)$.

And the surface is a sphere, $x^2 + y^2 + z^2 = 9$.

Then, div. $\vec{F} = \nabla.\vec{F} = 1 + 1 + 0 = 2$

Clearly on the sphere, $z = \pm\sqrt{9 - x^2 - y^2}$ and on the projection in xy–plane, $y = \pm\sqrt{9 - x^2}$. And, x moves from x = – 3 to 3.

Clearly, the sphere has symmetrical time hemisphere.

So, $z = 0$ to $\sqrt{9 - x^2 - y^2}$, $y = 0$ to $\sqrt{1 - x^2}$ and $x = 0$ to 3.

Now, by Gauss divergence theorem,

$$\iint_S \vec{F}.\vec{n}\, dA = \iiint_V \text{div}.\vec{F}\, dv$$

$$= 2^3 \int_0^3 \int_0^{\sqrt{3 - x^2}} \int_0^{\sqrt{9 - x^2 - y^2}} 2\, dz\, dy\, dx$$

$$= 16 \int_0^3 \int_0^{\sqrt{9 - x^2}} \sqrt{9 - x^2 - y^2}\, dy\, dx$$

$$= 16 \int_0^3 \left[\frac{y}{2}\sqrt{9 - x^2 - y^2} + \left(\frac{3 - x^2}{2}\right)\sin^{-1}\left(\frac{y}{\sqrt{9 - x^2}}\right)\right]_0^{\sqrt{9 - x^2}} dx$$

$$= 16 \int_0^3 \left(\frac{9 - x^2}{2}\right)\sin^{-1}(1)\, dx$$

$$= \frac{16\pi}{4}\int_0^3 (9 - x^2)\, dx \quad [\because \sin^{-1}(1) = \pi/2]$$

$$= 4\pi\left[9x - \frac{x^3}{3}\right]_0^3 = 4\pi(27 - 9) = 72\pi.$$

Thus, $\iint_S \vec{F}.\vec{n}\, dA = 72\pi$.

OTHER IMPORTANT QUESTION FROM FINAL EXAM

VELOCITY, ACCELARATION, GRADIENT, DIVERGENCE, CURL, DIRECTIONA; DERIVATIVES, SOLENOIDAL, IRROTATIONAL, CONSERVATIVE

2014 Fall Q.No. 4(b)

For curve $x = 3t, y = 3t^2, z = 2t^3$ show that $[\vec{r}, \vec{r}', \vec{r}''] = 180$ at $t = 1$.

Solution: Given that $x = 3t, y = 3t^2, z = 2t^3$. Then $\vec{r} = (3t, 3t^2, 2t^3)$.

Then,

$\vec{r}' = (3, 6t, 6t^2)$ and $\vec{r}'' = (0, 6, 12t)$.

Now,

$$[\vec{r}, \vec{r}', \vec{r}''] = \begin{vmatrix} 3t & 3t^2 & 2t^3 \\ 3 & 6t & 6t^2 \\ 0 & 6 & 12t \end{vmatrix}$$

$$= 3t(72t^2 - 36t^2) - 3(36t^3 - 12t^3) = 108t^3 + 72t^3 = 180t^3.$$

Thus, at t = 1,

$$[\vec{r}, \vec{r}', \vec{r}''] = 180.$$

2014 Spring Q. No. 2(a)

Define directional derivative of f in the direction of $\vec{a}$, find the directional derivative of $f = 4xz^3 - 3x^2yz^2$ in the direction of z–axis at P(2, –1, 2).

Solution: First Part: See the definition of directional derivative.

Second Part: See Exercise 4.2 Q. No. 3(vii).

2014 Fall Q. No. 4(a)

Define directional derivative of a function f in the direction of $\vec{a}$. Find the directional derivative of a function $f = x^2 - y^2 + 2z^2$ at the point A(1, 2, 3) in the direction of $\vec{a} = \vec{i} + \vec{j} + \vec{k}$.

Solution: First Part: See the definition of directional derivative.

Second Part: Given surface is, $f = x^2 - y^2 + 2z^2$

Then,

$$\text{grad}(f) = \nabla f = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)f = 2x\vec{i} - 2y\vec{j} + 4z\vec{k}$$

At point A(1, 2, 3). $\quad \text{grad}(f) = 2\vec{i} - 4\vec{j} + 12\vec{k}$.

Also given that $\vec{a} = \vec{i} + \vec{j} + \vec{k}$

Then the unit vector of $\vec{a}$ is,

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{\vec{i} + \vec{j} + \vec{k}}{\sqrt{1+1+1}} = \frac{1}{3}(\vec{i} + \vec{j} + \vec{k}).$$

Now the directional derivative of f along $\vec{a}$ at p is,

$$\nabla f.\hat{a} = \left(2\vec{i} - 4\vec{j} + 12\vec{k}\right).\left(\frac{1}{\sqrt{3}}(\vec{i} + \vec{j} + \vec{k})\right)$$

$$= \frac{1}{\sqrt{3}}(2 - 4 + 12) = \frac{1}{\sqrt{3}}(10).$$

2012 Fall Q.No. 3(b)

Show that the vector $F = (x^2 - yz)i + (x^2y + xz + 2yz^2)j + (2y^2z + xy)k$ is conservative and find $\varnothing$ such that $F = \nabla\varnothing$.

Solution: Given that,

$$\vec{F} = (x^2 - yz)\vec{i} + (x^2y + xz + 2yz^2)\vec{j} + (2y^2z + xy)\vec{k}$$

The function $\vec{F}$ is conservative only if curl $\vec{F} = 0$.

Here,

$$\text{curl}\,\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 - yz & x^2y + xz + 2yz^2 & 2y^2z + xy \end{vmatrix}$$

$$= (4yz + x - x - 4yz)\vec{i} + (-y - y)\vec{j} + (2xy + z - z)\vec{k}$$

$$= -2y\vec{j} + 2xy\vec{k}.$$

This shows that the function is not a conservative.

*Note: **This shows question should be corrected as***

$$\vec{F} = (2xy^2 + yz)\vec{i} + (2x^2y + xz + 2yz^2)\vec{i} + (2y^2z + xy)\vec{k}$$

and see Ex. 4.5, Q. O(iv).

11 Fall Q.No. 4(a): 2010 Spring Q.No. 6(b)

If $\vec{v} = x^2yz\vec{i} + xy^2z\vec{j} + xyz^2\vec{k}$, find (i) div (curl $\vec{v}$) and (ii) curl (curl $\vec{v}$).

Solution: Given that $\vec{v} = x^2yz\vec{i} + xy^2z\vec{j} + xyz^2\vec{k}$

Then,

$$\text{Curl}\,\vec{v} = \nabla\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2yz & xy^2z & xyz^2 \end{vmatrix}$$

$$= (xz^2 - xy^2)\vec{i} + (x^2y - yz^2)\vec{j} + (y^2z - x^2z)\vec{k}$$

Now,

(i) $\text{Div.}(\text{curl}\,\vec{v}) = \nabla.(\text{curl}\,\vec{v}) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(\text{curl}\,\vec{v})$

$$= (z^2 - y^2) + (x^2 - z^2) + (y^2 - x^2)$$

$$= 0$$

(ii) $$\text{curl}(\text{curl}\,\vec{v}) = \nabla\times(\text{curl}\,\vec{v}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xz^2 - xy^2 & x^2y - yz^2 & y^2z - x^2z \end{vmatrix}$$

$$= 4(yz\vec{i} + zx\vec{j} + xy\vec{k}).$$

11 Spring Q.No. 3(a)

Define gradient of a scalar function. If $\varnothing = x^3 + y^3 + z^3 - 3xyz$. Find div (grad $\varnothing$) ; and curl (grad $\varnothing$).

Solution: Given that $\phi = x^3 + y^3 + z^3 - 3xyz$

Then,

$$\text{grad}\,\phi = \nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial y}\right)(x^3 + y^3 + z^3 - 3xyz)$$

$$= (3x^2 - 3yz)\vec{i} + (3y^2 - 3zx)\vec{j} + (3z^2 - 3xy)\vec{k}$$

Now,

$$\text{div}(\text{grad}\ \phi) = \nabla.(\text{grad}\ \phi)$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial y}\right).(\text{grad}\ \phi)$$

$$= 6x + 6y + 6z = 6(x + y + z)$$

And,

$$\text{curl}(\text{grad}\ \phi) = \nabla\times(\text{grad}\ \phi)$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3x^2 - 3yz & 3y^2 - 3zx & 3z^2 - 3xy \end{vmatrix}$$

$$= (3x - 3x)\vec{i} - (3y + 3y)\vec{j} + (-3z + 3z)\vec{k} = 0.$$

Thus, $\text{div}(\text{grad}\ \phi) = 6(x + y + z)$ and $\text{curl}(\text{grad}\ \phi) = 0$.

2010 Fall Q.No. 3(b)

Define directional derivative of the function f in the direction $\vec{a}$. Derive the expression of directional derivative of f in the direction $\vec{a}$. Find directional derivate of $f = xy^2 + yz^3$ at (2, –1, 1) along the direction of the normal to the surface S: $x\log z - y^2 + 4 = 0$ at (–1, 2, 1).

Solution: **First Part:** See the definition of directional derivative.

Second Part: See the derivation.

Third Part: See the solution of Exercise 4.2, Q. 3(viii).

2010 Spring Q. No. 4(c)

Find the directional derivative of the function $f = x^2 + 3y^2 + 4z^2$ at (1, 0, 1) in the direction of $\vec{a} = -\vec{i} - \vec{j} + \vec{k}$.

Solution: Similar to 2010 Fall.

2009 Fall Q.No. 3(b)

Define Divergence and Curl of a vector. If $\varnothing = \log(x^2 + y^2 + z^2)$ find div (grad$\varnothing$) and curl (grad$\varnothing$).

Solution: **First Part:** See the definition of divergence and curl of a vector.

Second Part: See the solution of Exercise 4.3, Q. 11.

2005 Fall Q.No. 3(b)

If $\vec{v}$ = grad $(x^3 + y^3 + z^3 - 3xyz)$ find div $\vec{v}$ and curl $\vec{v}$.

Solution: See the problem part of 2011 Spring.

2006 Spring Q.No. 3(b)

If $\vec{u} = y\vec{i} + z\vec{j} + x\vec{k}$, and $\vec{v} = yz\vec{i} + zx\vec{j} + xy\vec{k}$ find curl $(\vec{u}\times\vec{v})$ and grad$(\vec{u}.\vec{v})$.

Solution: Given that,

Then,

$$\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ y & z & x \\ yz & zx & xy \end{vmatrix}$$

$$= (xyz - zx^2)\vec{i} - (xy^2 - xyz)\vec{j} + (xyz - yz^2)\vec{k}$$

Now,

$$\text{curl}(\vec{u}\times\vec{v}) = \nabla\times(u\times\vec{v})$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xyz - zx^2 & xyz - xy^2 & xyz - yz^2 \end{vmatrix}$$

$$= (xz - z^2 - xy)\vec{i} + (xy - x^2 - yz)\vec{j} + (yz - y^2 - zx)\vec{k}$$

and,

$$\vec{u}.\vec{v} = (y, z, x).(yz, zx, xy) = y^2z + z^2x + x^2y$$

So,

$$\text{grad}\ (\vec{u}.\vec{v}) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(\vec{u}.\vec{v})$$

$$= (z^2 + 2xy)\vec{i} + (x^2 + 2yz)\vec{j} + (y^2 + 2zx)\vec{k}.$$

2013 Spring Q. No. 2(a) OR; 2008 Spring Q.No. 3(a)

Define directional derivative of function f(x) in the direction $\vec{a}$. Find directional derivative of $f = xy^2 + yz^3$ at (2, –1, 1) along the direction of the normal to the surface S: $x\log z - y^2 + 4 = 0$ at (–1, 2, 1).

Solution: See the first and third part of 2010 Fall.

2008 Spring Q.No. 3(a) OR

Define Divergence and Curl of a vector function. If f be a continuous and differential scalar values function then prove that curl (grad f) = 0.

Solution: **First Part:** See the definitions.

Second Part: See the relative theorem.

2007 Fall Q.No. 3(a)

Define divergence and curl of a vector. Define directional derivative of f in the direction of $\vec{a}$. Find the directional derivative of $f = x^2 + 3y^2 + 4z^2$ in the direction $\vec{a} = -\vec{i} - \vec{j} + \vec{k}$ at P(1, 0, 0).

Solution: **First Part:** See the definition of divergence, curl of a vector directional derivative.

Second Part: Given surface is, $f = x^2 + 3y^2 + 4z^2$

Then,

$$\text{grad}(f) = \nabla f = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) f = 2x\,\vec{i} + 6y\,\vec{j} + 8z\,\vec{k}$$

At point P(1, 0, 0), $\quad \text{grad}(f) = 2\vec{i}$.

Also given that $\vec{a} = -\vec{i} - \vec{j} + \vec{k}$

Then the unit vector of $\vec{a}$ is,

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{-\vec{i} - \vec{j} + \vec{k}}{\sqrt{1+1+1}} = \frac{1}{3}(-\vec{i} - \vec{j} + \vec{k})$$

Now the directional derivative of f along $\vec{a}$ at p is,

$$\nabla f.\hat{a} = 2\vec{i}.\left(\frac{1}{\sqrt{3}}(-\vec{i} - \vec{j} + \vec{k})\right) = -\frac{1}{\sqrt{3}}$$

2004 Spring Q.No. 3(a) OR

If $\vec{v} = x^2y\,\vec{i} + xz\,\vec{j} + 2yz\,\vec{k}$, find: i. div $\vec{v}$ $\qquad$ ii. curl $\vec{v}$.

Solution: Given that $\vec{v} = x^2y\,\vec{i} + xz\,\vec{j} + 2yz\,\vec{k}$

Then,

$$\text{div.}(\vec{v}) = \nabla.\vec{v} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).\vec{v} = 2xy + 0 + 2y$$
$$= 2y(x+1).$$

And,

$$\text{curl }\vec{v} = \nabla\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2y & xz & 2yz \end{vmatrix}$$
$$= (2z - x)\vec{i} + (0-0)\vec{j} + (z - x^2)\vec{k}$$
$$= (2z - x)\vec{i} + (z - x^2)\vec{k}.$$

2004 Fall Q.No. 3(a)

If $\vec{r_1} = x^2yz\,\vec{i} - 2xz^2\vec{f} + xz^2\vec{k}$ and $\vec{r_2} = 2z\,\vec{i} - y\,\vec{j} + x^2\vec{k}$ find the value of $\frac{\delta^2}{\delta y\delta x}(\vec{r_1}\times\vec{r_2})$.

Solution: Given that, $\quad \vec{r}_1 = x^2yz\,\vec{i} - 2xz^2\,\vec{j} + xz^2\,\vec{k}$

Then,

$$\vec{r}_1\times\vec{r}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x^2yz & -2xz^2 & xz^2 \\ 2z & -y & x^2 \end{vmatrix}$$
$$= (-2x^3z^2 + xyz^2)\vec{i} - (x^4yz - 2xz^3)\vec{j} + (-x^2y^2z + 4xz^3)\vec{k}$$

Now,

$$\frac{\partial^2}{\partial y\partial x}(\vec{r}_1\times\vec{r}_2) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}(\vec{r}_1\times\vec{r}_2)\right)$$
$$= \frac{\partial}{\partial y}\{(-6x^2z^2 + xz^2)\vec{i} + (2z^3 - 4x^3yz)\vec{j} + (4z^3 - 2xy^2z)\vec{k}\}$$
$$= 0\vec{i} - 4x^3z\vec{j} - 4xyz\vec{k}$$
$$= -4xz(x^2\vec{j} + y\vec{k}).$$

2004 Fall Q.No. 3(b)

If $\vec{u} = x^2y\,\vec{i} - 2xz\,\vec{j} + 2yz\,\vec{k}$ find curl (curl $\vec{u}$).

Solution: Similar to 2011 Fall Q. No. 4(a–ii).

2003 Fall Q.No. 3(a) OR

Find the directional derivative of f(xyz) $= 2x^2 + 3y^2 + z^2$ at the point (2, 1, 3) in the direction of vector $\vec{a} = \vec{i} - 2\vec{k}$.

Solution: Similar to problem part of 2007 Fall Q. No. 3(a).

2003 Fall Q.No. 3(b)

Define divergence and curl of a vector $\vec{v}$. If $\vec{v}$ is the vector function, then prove that div (curl $\vec{v}$) = 0.

Solution: First Part: See the definition of divergence and curl of a vector.

Second Part: Similar to 2007 Fall 3(a).

2003 Spring Q.No. 3(a)

If $\vec{r_1} = (2t+1)\vec{i} - t^2\vec{j} + 3t^3\vec{k}$ and $\vec{r_2} = t^2\vec{i} - t\vec{j} + (t-1)\vec{k}$. Find $\frac{d}{dt}(\vec{r_1}\times\vec{r_2})$.

Solution: Given that

$$\vec{r_1} = (2t+1)\vec{i} - t^2\vec{j} + 3t^2\vec{k} \qquad \text{and} \quad \vec{r_2} = t^2\vec{i} - t\vec{j} + (t-1)\vec{k}$$

Then,

$$\vec{r_1}\times\vec{r_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2t+1 & -t^2 & 3t^3 \\ t^2 & -t & t-1 \end{vmatrix}$$
$$= (t^2 - t^3 + 3t^4) - (2t^2 - 2t + t - 1 - 3t^5)\vec{j} + (t^4 - 2t^2 - t)\vec{k}$$

Now,

$$\frac{d}{dt}(\vec{r_1}\times\vec{r_2}) = (2t - 3t^2 + 12t^3) - (4t - 1 - 15t^4)j + (4t3 - 4t - 1)k.$$

2003 Spring Q.No. 3(b)

If $\varnothing = \log(x^2 + y^2 + z^2)$, find div (grad$\varnothing$).

Solution: Given that $\phi = \log(x^2 + y^2 + z^2)$

Then,

$$\text{grad}\,\phi = \nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\phi$$

$$= \left(\frac{1}{x^2+y^2+z^2}\right)(2x\vec{i} + 2y\vec{j} + 2z\vec{k})$$

Now,

$$\text{div.}(\text{grad}\,\phi) = \nabla\cdot(\text{grad}\,\phi)$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left\{\left(\frac{1}{x^2+y^2+z^2}\right)(x\vec{i} + y\vec{j} + z\vec{k})\right\}$$

$$= \left(\frac{2}{x^2+y^2+z^2}\right)(1+1+1)$$

$$= \frac{6}{x^2+y^2+z^2}$$

2002 Q.No. 3(a) OR

Find the derivative of $\left[\vec{r}\ \ \frac{d\vec{r}}{dt}\ \ \frac{d^2\vec{r}}{dt^2}\right]$.

Solution: Let $R = [\vec{r}, \vec{r}', \vec{r}'']$

Then

$$\frac{dR}{dt} = \left[\frac{d\vec{r}}{dt}\cdot\frac{d\vec{r}}{dt}\cdot\frac{d^2\vec{r}}{dt^2}\right] + \left[\vec{r}\cdot\frac{d^2\vec{r}}{dt^2}\cdot\frac{d^2\vec{r}}{dt^2}\right] + \left[\vec{r}\cdot\frac{d\vec{r}}{dt}\cdot\frac{d^3\vec{r}}{dt^3}\right]$$

Since in a scalar triple product if two component has same value then the product value is zero. So,

$$\left[\frac{d\vec{r}}{dt}\cdot\frac{d\vec{r}}{dt}\cdot\frac{d^2\vec{r}}{dt^2}\right] = 0 = \left[\vec{r}\cdot\frac{d^2\vec{r}}{dt^2}\cdot\frac{d^2\vec{r}}{dt^2}\right]$$

Therefore, the derivative of $\left[\vec{r}\ \ \frac{d\vec{r}}{dt}\ \ \frac{d^2\vec{r}}{dt^2}\right]$ is $\left[\vec{r}\cdot\frac{d\vec{r}}{dt}\cdot\frac{d^3\vec{r}}{dt^3}\right]$.

2002 Q.No. 3(b)

If $\vec{f} = x^2y\vec{i} - xz\vec{j} + 4yz\vec{k}$ **find div (curl** $\vec{f}$**).**

Solution: Given that $\vec{f} = x^2y\vec{i} - xz\vec{j} + 4yz\vec{k}$

Then,

$$\text{curl}\,\vec{f} = \nabla\times\vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2y & -xz & 4yz \end{vmatrix}$$

$$= (4z + x)\vec{i} + (0-0)\vec{j} + (-z - x^2)\vec{k}$$

$$= (4z + x)\vec{i} - (x^2 + z)\vec{k}$$

Now,

$$\text{div.}(\text{curl}\,\vec{f}) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(\text{curl}\,\vec{f}) = 1 - 1 = 0.$$

2002 Q.No. 3(a)

Find the directional derivatives of the function $f = xy + yz + zx$ in the direction of the vector $\vec{a} = 2\vec{i} + 3\vec{j} + 6\vec{k}$ at the point (3, 1, 2).

Solution: Similar to 2007 Fall 3(a).

2002 Q.No. 3(a) OR

Define divergence and curl of vector $\vec{v}$. If $\vec{v} = x^2yz\vec{i} + xy^2z\vec{j} + xyz^2\vec{k}$, find (i) div $\vec{v}$ and (ii) curl $\vec{v}$.

Solution: See the definition of Divergence.

For problem part, see exam question solution of 2011 Fall.

SIMPLE INTEGRATION, LINE INTEGRAL, WORK DONE, FLUX, EXACTNESS

2014 Spring Q.No. 3(a)

Show that the value under the integral sign is exact and evaluate the integral

$$\int_{(4,0,3)}^{(-1,1,2)} [(yz+1)dx + (xz+1)dy + (xy+1)dz]$$

Solution: Given integral is, $I = \int_{(4,0,3)}^{(-1,1,2)} [(yz+1)dx + (xz+1)dy + (xy+1)dz]$ ... (i)

Comparing the value under the integral sign on (i) with $F_1dx + F_2dy + F_3dz$ then we get,

$$F_1 = yz + 1, \quad F_2 = xz + 1, \quad F_3 = xy + 1$$

Then,

$$\frac{\partial F_1}{\partial y} = z, \quad \frac{\partial F_1}{\partial z} = y, \quad \frac{\partial F_2}{\partial z} = x, \quad \frac{\partial F_2}{\partial x} = z, \quad \frac{\partial F_3}{\partial x} = y, \quad \frac{\partial F_3}{\partial y} = x$$

Here,

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}$$

So, the value in (i) is exact.

Now,

$$I = \int_{(4,0,3)}^{(-1,1,2)} [(yz+1)dx + (xz+1)dy + (xy+1)dz]$$

$$= \int_{(4,0,3)}^{(-1,1,2)} d\left[\int (yz+1)dx + \int dy + \int dz\right]$$

$$= \int_{(4,0,3)}^{(-1,1,2)} d\,[xyz + x + y + z]$$

$$= [xyz + x + y + z]_{(4,0,3)}^{(-1,1,2)}$$

$$= (-2 - 1 + 1 + 2) - (0 + 4 + 0 + 3) = -7..$$

2006 Spring Q.No. 4(a)

What do you mean by exact integral? Show that the expression within the integral sign is exact and evaluate it,

$$\int_{(0,2,3)}^{(0,2,3)} (yz \sinh zx\, dx + \cosh zx\, dy + xy \sinh zx\, dz).$$

Solution: First Part – See the definition of exact definition.

Second Part – See solution of Exercise 4.6 Q. No. 5.

2005 Fall Q.No. 3(a)

Let f be a continuous and differentiable scalar valued function, then show that curl (grad f) = 0. And find unit normal on the surface $\vec{r} = e^{z}\vec{i} + e^{x}\vec{j} + e^{y}\vec{k}$ at (2, 3, 4). If $\vec{r} = 5t^2\vec{i} + t\vec{j} - t^3\vec{k}$, find $\int_1^2 \left(\vec{r} \times \frac{d\vec{r}}{dt}\right) dt$.

Solution: Given that, $\vec{r} = 5t^2\vec{i} + t\vec{j} - t^3\vec{k}$

Then, $\frac{d\vec{r}}{dt} = 10t\,\vec{i} + \vec{j} - 3t^2\vec{k}$

So that,

$$\vec{r} \times \frac{d\vec{r}}{dt} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 5t^2 & t & -t^3 \\ 10t & 1 & -3t^2 \end{vmatrix}$$

$$= (-3t^3 + t^3)\vec{i} + (-10t^4 + 10t^4)\vec{j} + (5t^2 - 10t^2)\vec{k}$$

$$= -2t^3\vec{i} + 5t^4\vec{j} - 5t^2\vec{k}$$

Now,

$$\int_1^2 \left(\vec{r} \times \frac{d\vec{r}}{dt}\right) dt = \int_1^2 (-2t^3\vec{i} + 5t^4\vec{j} - 5t^2\vec{k})\, dt$$

$$= \left[-\frac{2t^4}{4}\vec{i} + t^5\vec{j} - \frac{5t^3}{3}\vec{k}\right]_1^2$$

$$= \frac{-2(16-1)}{4}\vec{i} + (32-1)\vec{j} - \frac{5(8-1)\vec{k}}{3}$$

$$= -\frac{15}{2}\vec{i} + 31\vec{j} - \frac{35}{3}\vec{k}$$

2004 Spring Q.No. 4(b) OR

Show that the form under the integral sign is exact and evaluate

$$\int_{(0,\pi)}^{(3,\pi/2)} [e^x \cos y\, dx - e^x \sin y\, dy].$$

Solution: Given integral is

$$I = \int_{(0,\pi)}^{(3,\pi/2)} [e^x \cos y dx - e^x \sin y\, dy] \quad ....(i)$$

Comparing the value under the integral sign in (i) with $F_1dx + F_2dy$ then we get,

$F_1 = e^x \cos y$ and $F_2 = -e^x \sin y$

Then $\frac{\partial F_1}{\partial y} = -e^x \sin y$ and $\frac{\partial F_2}{\partial x} = -e^x \sin y$

Thus, $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$. So, the value is exact.

Now, (i) becomes,

$$I = \int_{(0,\pi)}^{(3,\pi/2)} d(e^x \cos y) = [e^x \cos y]_{(0,\pi)}^{(3,\pi/2)} = e^3 \cos\frac{\pi}{2} - e^0 \cos\pi = 0 - 1(-1) = 1$$

Thus,

$$I = \int_{(0,\pi)}^{(3,\pi/2)} (e^x \cos y\, dx - e^x \sin y\, dy) = 1.$$

2002 Fall Q.No. 4(b) OR

Show that the form under the integral sign is exact and then evaluate

$$\int_{(0,0,0)}^{(a,b,c)} (2xy^2\, dx + 2x^2y\, dy + dz).$$

Solution: Given integral is, $I = \int_{(0,0,0)}^{(a,b,c)} (2xy^2dx + 2x^2y\, dy + dz)$ ———(i)

Comparing the value under the integral sign on (i) with $F_1dx + F_2dy + F_3dz$ then we get,

$F_1 = 2xy^2$, $F_2 = 2x^2y$, $F_3 = 1$

Then,

$\frac{\partial F_1}{\partial y} = 4xy$, $\frac{\partial F_1}{\partial z} = 0$, $\frac{\partial F_2}{\partial z} = 4xy$, $\frac{\partial F_3}{\partial x} = 0 = \frac{\partial F_3}{\partial y}$

Here,

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}.$$

So, the value in (i) is exact.

Now,

$$I = \int_{(0,0,0)}^{(a,b,c)} d(x^2y^2 + z) = [x^2y^2 + z]_{(0,0,0)}^{(a,b,c)} = a^2b^2 + c.$$

2011 Fall Q.No. 6(b); 2010 Spring Q.No. 5(b)

Calculate $\int_c \vec{f}.d\vec{r}$, where $\vec{f}$ = [cosh x, sinh y, e^z], C : $\vec{r}$ = [t, t^2, t^3] from (0, 0, 0) to (2, 4, 8).

Solution: Given that $\vec{f}$ = (cohx, sinhy, e^z) and $\vec{r}$ = (t, t^2, t^3).

Then $d\vec{r} = (1, 2t, 3t^2)\,dt$.

Since we know $\vec{r}$ = (x, y, z). So, comparing it with $\vec{r}$ = (t, t^2, t^3) we get,

$x = t,\ y = t^2,\ z = t^3$

Then $\vec{f} = (\cosh t, \sinh t^2, e^{t^3})$ and t moves from 0 to 2.

So, $\vec{f}\,d\vec{r} = (\cosh t + 2t\sinh t^2 + 3t^2e^{t^3})\,dt$

Now,

$\int_c \vec{f}.d\vec{r}$ from (0, 0, 0) to (2, 4, 8) is

$$\int_{(0,0,0)}^{(2,4,8)} \vec{r}.d\vec{r} = \int_0^2 (\cosh t + 2t\sinh t^2 + 3t^2e^{t^3})\,dt$$

Put $t^2 = u$ and $t^3 = v$ then,

$$= \int_0^2 \cosh t\,dt + \int_0^4 \sinh u\,du + \int_0^8 e^v\,dv$$

$$= [\sinh t]_0^2 + [\cosh u]_0^4 + [e^v]_0^8$$

$$= \sinh 2 + \cosh 4 - 1 + e^8 - 1$$

$$= \sinh 2 + \cosh 4 + e^8 - 2.$$

2011 Spring Q.No. 4(a)

Prove that $\int_c \vec{F}.d\vec{r} = 2\pi^2 - 8\pi$, where $\vec{F}$ = (x − y, y − z, z − x); C: (2cost, t, 2sint) from (2, 0, 0) to (2, 2π, 0).

Solution: Similar to 2011 Fall 6(b).

2011 Spring Q.No. 4(b)

Evaluate $\oint_c (-xy^2\,dx + x^2y\,dy)$, where c is the boundary of the region in the first quadrant bounded by $y = 1 - x^2$ counter clockwise.

Solution: By Greens theorem in plane, we get

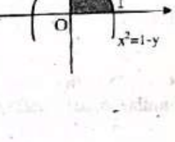
$$\oint_c (F_1dx + F_2dy) = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy \quad \text{......... (1)}$$

Then,

$$\oint_c (-xy^2dx + x^2ydy) = \iint_R (2xy + 2xy)dxdy$$

$$= 4\iint_R dy\,dxdy \quad \text{......... (2)}$$

We have C is the boundary of the region in the first quadrant bounded by $y = 1 - x^2$, show in figure below.



From (2),

$$\oint_c (-xy^2dx + x^2\,ydy) = 4\int_0^1 \int_0^{\sqrt{1-y}} xydxdy$$

$$= 4\int_0^1 y\left[\frac{x^2}{2}\right]_0^{\sqrt{1-y}} dy$$

$$= \frac{4}{2}\int_0^1 y(1-y)\,dy$$

$$= 2\int_0^1 (y - y^2)\,dy = 2\left[\frac{y^2}{2} \frac{y^3}{3}\right]_0^1 = 2\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3}$$

Thus, $\oint_c (-xy^2dx + x^2\,ydy) = \frac{1}{3}$.

2011 Fall Q.No. 5(b)

Find the flux integral of $\vec{F}$ = [x, y, z] through the surface S, where S is the first octant portion of the plane 2x + 3y + z = 6.

Solution: Similar to the solution of 2011 Spring Q. 3(b).

<u>2005 Fall Q.No. 4(b)</u>

If $\vec{F} = 4xy\,\vec{i} + 8y\,\vec{j} + 3z\,\vec{k}$, find the line integral of $\vec{F}$ along the curve $y = 3x$, $z = 2x$ from (0, 0, 0) to (1, 3, 2).

Solution: Given that $\vec{F} = 4xy\,\vec{i} + 8y\,\vec{j} + 3z\,\vec{k}$

And along the curve $y = 3x$, $z = 2x$.

Since we have, $\vec{r} = (x, y, z) = (x, 3x, 2x)$. So, $d\vec{r} = (1, 3, 2)\,dx$.

Now, line integral of $\vec{F}$ along $y = 3x$, $z = 2x$ from (0, 0, 0) to (1, 3, 2) is

$$\int_c \vec{F}.d\vec{r} = \int_0^1 ((12x^2, 24x, 6x).(1, 3, 2))\,dx$$

$$= \int_0^1 (12x^2 + 72x + 12x)\,dx = [4x^3 + 36x^2 + 6x^2]_0^1$$

$$= 4 + 36 + 6 = 46.$$

Thus, $\int_c \vec{F}.d\vec{r} = 46.$

<u>2004 Fall Q.No. 4(a)</u>

Calculate $\int_c \vec{F}(\vec{r}).d\vec{r}$ if $\vec{F} = [xy, x^2y^2]$, where "c" is the quarter circle from (2, 0) to (0, 2) with centre at (0, 0).

Solution: Similar to 2012 Fall Q.4(a).

<u>2004 Spring Q.No. 3(b)</u>

Find the work done by the force $\vec{F} = 4xy\,\vec{i} + 8y\,\vec{j} + 2\,\vec{k}$ along the curve $y = 2x$, $z = 2x$ from (0, 0, 0) to (3, 6, 6).

Solution: Given that $\vec{F} = 4xy\,\vec{i} + 8y\,\vec{j} + 2\,\vec{k}$

And along the curve $y = 2x$, $z = 2x$

Since we have, $\vec{r} = (x, y, z) = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$

i.e. $\vec{r} = x\,\vec{i} + 2x\,\vec{j} + 2x\,\vec{k}$

Then, $d\vec{r} = (\vec{i} + 2\vec{j} + 2\vec{k})\,dx$.

So that,

$$\vec{F}.d\vec{r} = (4xy\,\vec{i} + 8y\,\vec{j} + 2\vec{k}).(\vec{i} + 2\vec{j} + 2\vec{k})\,dx$$
$$= (8x^2\,\vec{i} + 16x\,\vec{j} + 2\vec{k}).(\vec{i} + 2\vec{j} + 2\vec{k})\,dx$$
$$= (8x^2 + 32x + 4)\,dx$$

Now, work done by $\vec{F}$ along $y = 2x$, $z = 2x$ from (0, 0, 0) to (3, 6, 6) is,

$$\int_c \vec{F}.d\vec{r} = \int_0^3 (8x^2 + 32x + 4)\,dx = \left[\frac{8x^3}{3} + 16x^2 + 4x\right]_0^3$$
$$= 8 \times 9 + 16 \times 9 + 12$$
$$= 9[8 + 16] + 12 = 216 + 12 = 228.$$

Thus, the work done by the force $\vec{F}$ along the given curve is 228.

<u>02 Q.No. 4(a)</u>

Evaluate the line integral $\oint_c \vec{F}.d\vec{r}$ counter clockwise around the boundary C of the region R where $\vec{F} = [\text{Sin}y\ \cos x]$, R be the triangle with vertices (0, 0), (π, 0), (π, 1).

Solution: See the problem part of 2007 Fall.

<u>02 Q.No. 4(a) OR</u>

Evaluate the line integral of $\vec{F} = [3y^2, x - y^4]$ over C the square with vertices (1, 1), (–1, 1), (–1, –1), (1, –1) counter clockwise.

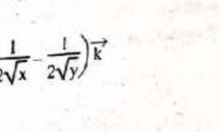
Solution: Similar to 2007 Fall.

SURFACE INTEGRAL BY USING GREEN'S THEOREM

<u>12 Fall Q.No. 4(a)</u>

State Green theorem. Evaluate $\int_c [\sqrt{y}\,dx + \sqrt{x}\,dy]$ where C is the triangle with vertices (1,1), (3,1) and (2,2).

Solution: We have to evaluate $\int_c (\sqrt{y}dx + \sqrt{x}dy)$ around the triangle having verticals (1, 1), (3, 1) and (2, 2).



By Green's theorem we have,

$$\int_c \vec{F}.d\vec{r} = \iint_R (\text{curl}\,\vec{F}.\vec{k})\,dA \quad \text{..........(i)}$$

Comparing $\int_c (\sqrt{y}\,dx + \sqrt{x}\,dy)$ with $\int_c \vec{F}\,d\vec{r}$ then we get,

Then, $\vec{F} = (\sqrt{y}, \sqrt{x})$ and $\vec{r} = (x, y)$.

$$\text{curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \sqrt{y} & \sqrt{x} & 0 \end{vmatrix} = \left(\frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{y}}\right)\vec{k}$$

Then. $\text{curl}\,\vec{F}.\vec{k} = \frac{1}{2}\left(\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{y}}\right)$

Since the region of I is as shown in figure with shaded portion in which y varies from x = y [equation of line passes through (1, 1) and (2, 2)] to y = 4 – x [equation of line passes through (3, 1) and (2, 2)] and on the region x moves from x = 1 to x = 2.

Then, (i) gives,

$$\oint_c \vec{F}\,d\vec{r} = \oint_{c'} (\sqrt{y}\,dx + \sqrt{x}dy)$$

$$= \int_1^2 \int_x^{4-x} \frac{1}{2}\left(\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{y}}\right) dy\,dx$$

$$= \frac{1}{2}\int_1^2 \left[\frac{y}{\sqrt{x}} - \frac{y^{1/2}}{1/2}\right]_x^{4-x} dx$$

$$= \frac{1}{2}\int_1^2 \left(\frac{4-x-x}{\sqrt{x}} - 2\left(\sqrt{4-x} - \sqrt{x}\right)\right) dx$$

$$= \frac{1}{2}\int_1^2 \left(\frac{4}{\sqrt{x}} - 2\sqrt{x} - 2\sqrt{4-x} + 2\sqrt{x}\right) dx$$

$$= \frac{1}{2}\int_1^2 \left(\frac{4}{\sqrt{x}} - 2\sqrt{4-x}\right) dx$$

$$= \frac{1}{2}\left[\frac{4\,x^{1/2}}{1/2} - 2\left(\frac{\sqrt{x}\sqrt{4-x}}{2} + \frac{4}{2}\sin^{-1}\frac{\sqrt{x}}{2}\right)\right]_1^2$$

$$= \frac{1}{2}\left[8\,(\sqrt{2}-1) - (\sqrt{2}\,\sqrt{2} - \sqrt{3}) - 4\left(\sin^{-1}\frac{\sqrt{2}}{2} - \sin^{-1}\frac{1}{2}\right)\right]$$

$$= \frac{1}{2}\left[8\sqrt{2} - 2 + \sqrt{3} - 4\,\frac{\pi}{4} + \frac{4\pi}{6}\right] \quad \left[\because \sin^{-1}\frac{1}{\sqrt{2}} = \frac{\pi}{4},\ \sin^{-1}\frac{1}{2} = \frac{\pi}{6}\right]$$

$$= 4\sqrt{2} + \sqrt{3} - 5 - \frac{\pi}{6}$$

Thus, $\oint_c (\sqrt{y}\,dx + \sqrt{x}\,dy) = \frac{\pi}{6}$, around the triangle.

2011 Fall Q.No. 4(b)

State Green theorem. Use it to evaluate the integral $\vec{F}.\,d\vec{r}$, where $\vec{F} = [x^2, y^2]$, c: the square whose vertices are (0, 0), (1, 0), (1, 1), and (0, 1).

Solution: First Part: See the statement of Green's theorem.

Second Part: Similar problem for rectangle to 2012 Fall Q.No. 4(a).

2010 Fall Q.No. 4(a)

State Greens theorem in plane. Evaluate $\oint_c [5xydx + x^3dy]$, where C is the closed curve consisting of the graph of $y = x^2$ and $y = 2x$ between the points (0,0) and (2,4).

Solution: First Part: See the statement of Green's theorem.

Second Part: See Exercise 4.7 Q. No. 7.

2008 Spring Q.No. 4(a)

State Green's theorem and use it to evaluate the integral $\oint_c (2xy^3dx + 3x^2y^2dy)$, c: $x^2 + y^2 = 1$ counter clockwise.

Solution: First Part: See the statement of Green's theorem.

Second Part: See Exercise 4.7 Q. No. 4.

2007 Fall Q.No. 3(b); 2003 Fall Q.No. 4(a)

State Green's theorem. Use it to evaluate the line integral $\oint_c \vec{F}(r).d\vec{r}$ counter clock wise where $\vec{F}$ = [siny, cosx] and C is the triangle with vertices (0, 0), (π, 0) and (π, 1).

Solution: First Part: See the statement of Green's theorem.

Second Part: By Greens theorem

$$\oint_c \vec{F}.d\vec{r} = \iint_R (\text{curl}\,\vec{F}.\vec{k})\,dA \qquad \text{...... (1)}$$

where, $\vec{F} = \sin y\,\vec{i} + \cos x\,\vec{j}$.

Then,

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \sin y & \cos x & 0 \end{vmatrix} = \vec{k}\,(-\sin x - \cos y)$$

Thus from (1),

$$\oint_c \vec{F}.d\vec{r} = -\iint_R (\sin x + \cos y)\,dA \qquad \text{....... (2)}$$

We have R is the triangle with vertices (0, 0), (π, 0), (π, 1).

So, (2) becomes,

$$\oint_c \vec{F}.d\vec{r} = -\int_0^1 \int_{\pi y}^{\pi} (\cos y + \sin x)\,dxdy$$

$$= -\int_0^1 [x\cos y - \cos x]_{\pi y}^{\pi}\, dy$$

$$= -\int_0^1 (\pi\cos y - \cos\pi - \pi y\cos y + \cos\pi y)\, dy$$

$$= -\int_0^1 (\pi\cos y + 1 - \pi y\cos y + \cos\pi y)\, dy$$

$$= -\left[\pi\sin y + y - \pi(y\sin y + \cos y) + \frac{\sin\pi y}{\pi}\right]_0^1$$

$$= -(\pi\sin 1 + 1 - \pi\sin 1 - \pi\cos 1 + \pi)$$

Thus, $\oint_C \vec{F}.d\vec{r} = (\pi\cos 1 - \pi - 1)$.

<u>2002 Q.No. 3(b)</u>

State Green's theorem and then use it to evaluate $\int_C \vec{F}.d\vec{r}$, where $\vec{F} = e^{x+y}\vec{i} + e^{x-y}\vec{j}$ where C is the boundary of the triangle region $x \le y \le 2x$, $0 \le x \le 1$.

Solution: Given that, $\vec{F} = e^{x+y}\vec{i} + e^{x-y}\vec{j} = e^x e^y\vec{i} + e^x.e^{-y}\vec{j}$

And boundaries of the region be $x \le y \le 2x$ $0 \le x \le 1$.

By Green's theorem we have,

$$\oint_C \vec{F}.d\vec{r} = \iint_R (\text{curl } \vec{F}.\vec{k})\, dA \qquad \text{———(i)}$$

Here,

$$\text{curl } \vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^x.e^y & e^x.e^{-y} & 0 \end{vmatrix} = (e^x e^{-y} - e^x e^y)\,\vec{k}$$

Then $\text{curl } \vec{F}.\vec{k} = e^x e^{-y} - e^x e^y$

Now, (i) becomes with boundaries $x \le y \le 2x$, $0 \le x \le 1$,

$$\oint_C \vec{F}.d\vec{r} = \int_0^1\int_x^{2x} e^x[e^{-y} - e^y]\, dy\, dx$$

$$= \int_0^1 e^x\left[\frac{e^{-y}}{-1} - e^y\right]_x^{2x} dx$$

$$= \int_0^1 e^x[(-e^{-2x} - e^{2x}) - (-e^{-x} - e^x)]dx$$

$$= \int_0^1 (-e^{-x} - e^{3x} + e^0 + e^{2x})\, dx$$

$$= \int_0^1 (e^{2x} - e^{3x} - e^{-x} + 1)\, dx$$

$$= \left[\frac{e^{2x}}{2} - \frac{e^{3x}}{3} - \frac{e^{-x}}{-1} + x\right]_0^1 = \left(\frac{e^2}{2} - \frac{e^3}{3} + e^{-1} + 1\right) - \left(\frac{1}{2} - \frac{1}{3} + 1 + 0\right)$$

$$= \frac{e^2}{2} - \frac{e^3}{3} - \frac{1}{e} - \frac{1}{6}$$

<u>001 Q.No. 4(a)</u>

Evaluate the following integral by using Green's theorem. $\int[(3x^2 - 8y^2)\, dx + (4y - 6xy)\, dy]$ where C is the boundary of the region defined by $y^2 = x$, $y = x^2$.

Solution: First Part: See the statement of Green's theorem.

Second Part: Similar to 2002 Q.3(b).

Similar Questions

<u>013 Spring Q. No. 2(b)</u>

State Greens theorem in plane. Evaluate $\oint_C [(3x^2 + y)dx + 4y^2\, dy]$, where C is the boundary of the triangle with vertices (0, 0), (1, 0), (0, 2) counterclockwise.

<u>014 Spring Q. No. 2(b)</u>

State Green's theorem in a plane, and find $\oint_C \vec{F}.d\vec{r}$ where $\vec{F} = (2x - y - z)i + (x + y - z^2)\vec{j} + (3x - 2y + 4z)k$ around the circle $x^2 + y^2 = a^2$, $z = 0$.

<u>013 Spring Q. No. 3(a)</u>

Evaluate $\iint_S \vec{F}.\vec{n}\, dA$, where $\vec{F} = (18z, -12, 3y)$, s is the surface of the plane $2x + 3y + 6z = 12$ in the first octant.

<u>014 Fall Q. NO. 5(a)</u>

Evaluate: $\iint_S \vec{F}.n dA$ where $\vec{F} = (x^2, e^y, 1)$, S: $x + y + z = 1$ $x \ge 0, y \ge 0, z \ge 0$.

<u>014 Spring Q. No. 3(b)</u>

Evaluate the surface integral $\iint_S (\vec{F}.\vec{n})dA$, where $\vec{F} = (x^2; 0, 3y^2)$ and S is the portion of the plane $x + y + z = 1$ in the first octant.

CLOSED CURVE INTEGRAL BY USING STOKE'S THEOREM

2012 Fall Q.No. 4(b)

State Stoke's theorem. Evaluate $\oint_c \vec{F} dr$ where $F = (y^2 + z^2 + x^2)$ and C the portion of the sphere $x^2 + y^2 + (z-1)^2 = 1, y \geq 0, z \leq 1$.

Solution: First Part: See the statement of Stoke's theorem.

Second Part: Similar to 2002, 4(b).

2009 Fall Q.No. 4(b)

State Stokes theorem. Evaluate $\oint_c \vec{F}.d\vec{r}$, where $\vec{F} = (z, x, y)$, S: the hemisphere $z = (a^2 - x^2 - y^2)^{1/2}$.

Solution: First Part: See the statement of Stoke's theorem.

Second Part: See Exercise 4.10 Q. No. 12.

2009 Spring Q.No. 4(b) OR

Evaluate $\oint_c \vec{F}.d\vec{r}$ where $\vec{F} = \left(y, \frac{z}{2}, \frac{3y}{2}\right)$, C is the circle of $x^2 + y^2 + z^2 = 6z, z = x + 3$.

Solution: We know by Stoke's theorem,

$$\oint_C \vec{F}.d\vec{r} = \iint_R (\text{curl}\vec{F}.\vec{r})dA \quad \text{........(1)}$$

Here, $\vec{F} = y\vec{i} + \frac{z}{2}\vec{j} + \frac{3y}{2}\vec{k}$.

Then,

$$\text{Curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & z/2 & 3/2\cdot y \end{vmatrix} = \vec{i}\left(\frac{3}{2} - \frac{1}{2}\right) - \vec{j}(0) + \vec{k}(-1)$$

$$= \vec{i} - \vec{k} = \vec{G}(\text{say}).$$

Then equation (1) can be written as, by surface integral,

$$\oint_C \vec{F}.d\vec{r} = \iint_S (\vec{G}.\vec{n})\,dA = \iint_R \vec{G}.\vec{N}\,dx\,dy \quad \text{.......(2)}$$

where $\vec{N} = \vec{r}_x \times \vec{r}_y$

and $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \Rightarrow \vec{r} = x\vec{i} + y\vec{j} + (x+3)\vec{k}$.

Differentiate partially, w. r. t. x and y, we get

$$\vec{r}_x = \vec{i} + \vec{k} \text{ and } \vec{r}_y = \vec{j}.$$

Then $\vec{N} = \vec{r}_x \times \vec{r}_y = (\vec{i} + \vec{k}) \times (\vec{j}) = \vec{k} - \vec{i}$.

So that (2) reduces as,

$$\oint_C \vec{F}.d\vec{r} = \iint_R (-2)\,dxdy = -2\iint_R dx\,dy \quad \text{.......... (3)}$$

The given surface $x^2 + y^2 + z^2 = 6z, z = x + 3$ can be written as,

$$x^2 + y^2 + (x+3)^2 = 6(x+3)$$

$$\Rightarrow x^2 + y^2 + x^2 = 9$$

$$\Rightarrow 2x^2 + y^2 = 9$$

$$\Rightarrow \frac{x^2}{9/2} + \frac{y^2}{9} = 1$$

Thus (3) becomes,

$$\oint_C \vec{F}.d\vec{r} = -2\times[\text{Area of the ellipse } 2x^2 + y^2 = 9]$$

$$= -2 \times \pi \frac{3}{\sqrt{2}}.3$$

Hence, $\oint_c \vec{F}.d\vec{r} = -9\sqrt{2}\,\pi$.

[20]08 Spring Q.No. 4(b) OR

State Stokes theorem. Evaluate $\oint_c \vec{F}.d\vec{r}$, where $\vec{F} = (y^3, 0, x^3)$ and C is the boundary of the triangle with vertices (1, 0, 0), (0, 1, 0) and (0, 0, 1).

Solution: First Part: See the statement of Stoke's theorem.

Second Part: See Exercise 4.10 Q. No. 7.

[20]06 Spring Q.No. 4(b)

State Stokes theorem and evaluate: $\int_c \vec{F}.d\vec{r}$, where $\vec{F} = y\vec{i} + \frac{z}{2}\vec{j} + \frac{3y}{2}\vec{k}$, C is the boundary of the circle $x^2 + y^2 + z^2 = 6z, z = x + 3$.

Solution: First Part: See the statement of Stoke's theorem.

Second Part: See 2009 Spring Q. No. 4(b) OR.

[20]03 Fall Q.No. 4(b)

State Stock's theorem. Evaluate $\int_c F.r'(s)\,ds$ where $F = [2y^2, x, -z^3]$, C the circle $x^2 + y^2 = a^2, z = b\ (> 0)$.

Solution: First Part: See the statement of Stoke's theorem.

Second Part: Similar to 2007 Fall 3(a).

2002 Q.No. 4(b)

State Stoke's theorem and hence evaluate the surface integral is $\iint (\nabla \times \vec{F}).n\, dA$, where $\vec{F} = [y^2, -x^2, 0]$. S the semi-circular disc $x^2 + y^2 \le 4$, $y \ge 0$ and $z = 0$.

Solution: First Part: See the statement of Stoke's theorem.

Second Part: We know by stokes theorem

$$\oint_C \vec{F}.d\vec{r} = \iint_S (\text{curl } \vec{F}.\vec{n})\, dA \qquad \text{.........(1)}$$

We have

$$\vec{F} = y^2\vec{i} - x^2\vec{j}$$

Then $\text{curl } \vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 & -x^2 & 0 \end{vmatrix} = \vec{i}(0) - \vec{j}(0) + \vec{k}(-2x - 2y)$

$$= -2(x + y)\vec{k} = \vec{G} \text{ (say)}.$$

Then from equation (1) we get

$$\oint_C \vec{F}.d\vec{r} = \iint_S (\vec{G}.\vec{n})\, dA, \qquad \text{where } \vec{G} = -2(x + y)\vec{k}$$

$$= \iint_R (\vec{G}.\vec{N})\, dx\, dy \qquad \text{.......(2)}$$

(by definition of surface integral.)

where $\vec{N} = \vec{r}_x \times \vec{r}_y$

Since, $\vec{r} = x\vec{i} + y\vec{j} + 0\vec{k}$. Then, $\vec{r}_x = \vec{i}$ and $\vec{r}_y = \vec{j}$.

So, $\vec{N} = \vec{i} \times \vec{j} = \vec{k}$ and $\vec{G} = -2(x + y)\vec{k}$.

Now (2) becomes,

$$\oint_C \vec{F}.d\vec{r} = -2\iint_R (x + y)\, dxdy = -2\int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (x+y)\, dxdy$$

Since the surface is $x^2 + y^2 \le 4$ and $y \ge 0$. Therefore,

$$\oint_C \vec{F}.d\vec{r} = -2\int_0^2 \left[\frac{x^2}{2} + yx\right]_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} dy$$

$$= -2\int_0^2 y^2\sqrt{4 - y^2}\, dy$$

$$= -4\int_0^2 y\sqrt{4 - y^2}\, dy$$

$$= \left[\frac{4}{3}(4 - y^2)^{3/2}\right]_0^2 = \frac{4}{3}[0 - 4^{3/2}] = -\frac{4}{3} \times 2^3 = -32/3$$

Hence, $\oint_C \vec{F}.d\vec{r} = -32/3$.

2002 Q.No. 4(b)

Evaluate the line integral using Stoke's theorem $\int_C \vec{F}.\vec{r}'(s)\, ds$, where $\vec{F} = [y, xz^3, -zy^3]$; C, the circle $x^2 + y^2 = 4$, $z = -3$.

Solution: We know, by stokes theorem

$$\oint_C (\vec{F}.\vec{r}'(s))\, ds = \iint_R (\text{curl } \vec{F}.\vec{n})\, ds \qquad \text{........ (1)}$$

Here, $\vec{F} = y\vec{i} + xz^3\vec{j} - zy^3\vec{k}$

Then,

$$\text{curl } \vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & xz^3 & -zy^3 \end{vmatrix}$$

$$= \vec{i}(-3zy^2 - 3xz^2) - \vec{j}(0) + \vec{k}(z^3 - 1) = \vec{G} \text{ (say)}.$$

Then (1) becomes,

$$\oint_C (\vec{F}.\vec{r})ds = \iint_S (\vec{F}.\vec{n})ds = \iint_R (\vec{F}.\vec{N})\, dx\, dy \qquad \text{....... (2)}$$

Given that C is the circle $x^2 + y^2 = 4$, $z = -3$.

Then, $\vec{N} = \vec{k}$ and $\vec{G}.\vec{N} = (z^3 - 1)$.

Therefore, (2) reduces as,

$$\oint_C (\vec{F}.\vec{r})ds = \iint (z^3 - 1)\, dxdy$$

$$= -28 \iint dx\, dy$$

$$= -28 \times (\text{Area of the circle } x^2 + y^2 = 4)$$

$$= -28 \times \pi(2)^2$$

$$= -112\pi$$

Thus we get, $\oint_C (\vec{F}.\vec{r})ds = -112\pi$.

2001 Q.No. 4(b) OR

Verify Stoke's theorem for the vector function. Evaluate $\oint_C \vec{F}\, dr$ where $F = (2x - y)\vec{i} - yz^2\vec{j} - y^2z\vec{k}$ where S is the surface of the sphere $x^2 + y^2 + z^2 = 1$ above the xy plane and C its boundary.

Solution: First Part: See the statement of Stoke's theorem.

Second Part: See Exercise 4.10 Q. No. 9.

Similar Questions

2014 Spring Q. No. 3(b) OR

State Stokes theorem. Find $\oint_c \vec{F}.d\vec{r}$ if $\vec{F} = (y^2, z^2, x^2)$, S is the first portion of the plane $x + y + z = 1$.

2013 Fall Q. No. 6(b)

Find $\oint_c \vec{F}.d\vec{r}$ if $\vec{F} = [y^2, 2xy + \sin x, 0]$, where c is the boundary of $0 \le x \le \frac{\pi}{2}$, $0 \le y \le 2$ by using Stoke's Theorem.

2013 Fall Q. No. 5(a)

State Stoke's theorem. Evaluate $\oint_c \vec{F}.d\vec{r}$ where $F = [y, xz^3, -zy^3]$, C the circle $x^2 + y^2 = 4, z = -3$.

Solution: See Statement of Stoke's theorem and see 2002 Q. No. 4(b).

VOLUME INTEGRAL BY USING GUASS DIVERGENCE THEOREM

2013 Fall Q.No. 4(b); 2012 Fall Q.No. 4(b) OR; 2004 Fall Q.No. 4(b); 2003 Spring Q.No. 4(b)

Evaluate $\iint_S \vec{F}.\vec{n}\,dA$ if $\vec{F} = [x^2, e^y, 1]$; S: $x + y + z = 1, x \ge 0, y \ge 0, z \ge 0$.

Solution: Given that

$$\vec{F} = (x^2, e^y, 1)$$

and the surface is. $x + y + z = 1$, for $x \ge 0, y \ge 0, z \ge 0$

By Gauss divergence theorem we have,

$$\iint_R \vec{F}.\vec{n}\,dA = \iiint_V \text{div}\,\vec{F}\,dV \quad \text{.........(i)}$$

Here,

$$\text{div}.\vec{F} = \nabla.\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(x^2\,\vec{i} + e^y\,\vec{j} + \vec{k})$$

$$= 2x + e^y$$

The plane $x + y + z = 1$ with $x \ge 0, y \ge 0, z \ge 0$ is as shown in figure.

Clearly z varies from $z = 0$ to the plane $x + y + z = 1 \Rightarrow 1 - x - y$. And the variable y varies in xy-plane from $y = 0$ to $x + y = 1 \Rightarrow y = 1 - x$.

Also, on the region x moves from $x = 0$ to $x = 1$.

Then, (i) becomes,

$$\iint_S \vec{F}.\vec{n}\,dA = \int_0^1\int_0^{1-x}\int_0^{1-x-y}(2x + e^y)\,dz\,dy\,dx$$

$$= \int_0^1\int_0^{1-x}[2xz + ze^y]_0^{1-x-y}$$

$$= \int_0^1\int_0^{1-x}[2x(1 - x - y) + (1 - x - y)e^y]\,dy\,dx$$

$$= \int_0^1\int_0^{1-x}[2x - 2x^2 - 2xy + (1 - x)e^y - ye^y]\,dy\,dx$$

$$= \int_0^1[2xy - 2x^2y - xy^2 + (1 - x)e^y - ye^y + e^y]_0^{1-x}\,dx$$

$$= \int_0^1[(2x - x^2)(1 - x) - x(1 - x)^2 + (2 - x)e^{1-x} - (2 - x)e^0 - (1 - x)e^{1-x}]\,dx$$

$$= \int_0^1[2x - 2x^2 - x^2 + x^3 - x - x^3 + 2x^2 + (2 - x - 1 + x)e^{1-x} - 2 + x]\,dx$$

$$= \int_0^1[2x - x^2 - 2 + e\,e^{-x}]\,dx$$

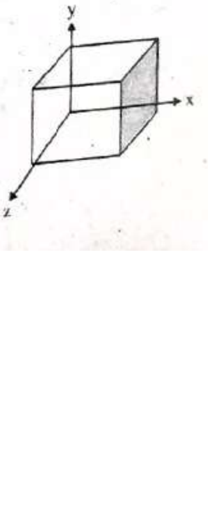
$$= \left[x^2 - \frac{x^3}{3} - 2x + e.\frac{e^{-x}}{-1}\right]_0^1 = \left(1 - \frac{1}{3} - 2 - e.e^{-1}\right) + e$$

$$= -\frac{7}{3} + e \qquad [\because e.e^{-1} = 1]$$

Thus, $\iint_S \vec{F}.\vec{n}\,dA = e - \frac{7}{3}$.

2011 Fall Q.No. 5(a)

State Gauss Divergence Theorem. Using Gauss divergence theorem, evaluate the integral $\iint_S \vec{F}.\vec{n}\,ds$, where $\vec{F} = [4x, 2y^2, z^2]$ and s is the surface of the cube $|x| \le 1, |y| \le 1, |z| \le 1$.



Solution: Given that $\vec{F} = (4x, 2y^2, z^2)$

and the surface is the cube $|x| \le 1, |y| \le 1, |z| \le 1$.

By Gauss divergence theorem we have,

$$\iint_S \vec{F}.\vec{n}\,ds = \iiint_V \text{div}\,\vec{F}\,dV \quad \text{.......(i)}$$

Here,

$$\operatorname{div}\vec{F} = \nabla.\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).(4x\,\vec{i} + 2y^2\vec{j} + z^2\vec{k})$$

$$= 4 + 4y + 2z$$

Since the surface is a cube with $|x| \le 1$, $|y| \le 1$, $|z| \le 1$.

So, $-1 \le x \le 1$, $-1 \le y \le 1$, $-1 \le z \le 1$.

Now, by (i)

$$\iint_S (\vec{F}.\vec{n})\,ds = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} (4 + 4y + 2z)\,dxdydz$$

$$= \int_{-1}^{1}\int_{-1}^{1} [4x + 4xy + 2xz]_{-1}^{1}\,dy\,dz$$

$$= \int_{-1}^{1}\int_{-1}^{1} [4(1+1) + 4y(1+1) + 2z(1+1)]\,dydz$$

$$= 4\int_{-1}^{1}\int_{-1}^{1} (2 + 2y + z)\,dydz$$

$$= 4\int_{-1}^{1} [2y + y^2 + zy]_{-1}^{1}\,dz$$

$$= 4\int_{-1}^{1} [2(1+1) + (1-1) + z(1+1)]\,dz$$

$$= 4\int_{-1}^{1} (4 + 2z)\,dz = 4[4z + z^2]_{-1}^{1}$$

$$= 4[4(1+1) + (1-1)] = 4 \times 8 = 32.$$

Thus, $\iint_S (\vec{F}.\vec{n})\,ds = 32.$

<u>2011 Spring Q.No. 3(b); 2007 Fall Q.No. 4(a)</u>

Evaluate the surface integral $\iint_S (\vec{F}.\vec{n})ds$, where $\vec{F} = (x^2, 0, 3y^2)$ and S is the portion of the plane $x + y + z = 1$ in the first octant.

Solution: Similar to 2012 Fall Q. No. 4(b).

<u>2010 Spring Q.No. 5(a)</u>

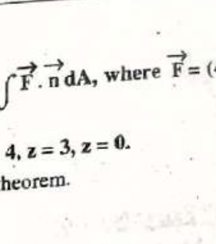
State Gauss Divergence theorem. Use it to evaluate $\iint_S \vec{F}.\vec{n}\,dA$, where $\vec{F} = (4x, -2y^2, z^2)$, S is the surface bounding the region $x^2 + y^2 = 4$, $z = 3$, $z = 0$.

Solution: First Part: See the statement of Gauss Divergence theorem.

Second Part: See Exercise 4.9 Q. No. A–6.

<u>2008 Spring Q.No. 4(b)</u>

Evaluate $\iint_S \vec{F}.\vec{n}\,dA$ where $\vec{F} = (y^3, x^3, z^3)$, s: $x^2\ 4y^2 = 1$, $x \ge 0$, $y \ge 0$, $0 \le z \le h$.



Solution: Given that $\vec{F} = (y^3, x^3, z^3)$

and surface is $x^2 + 4y^2 = 1$ for $x \ge 0$, $y \ge 0$, $0 \le z \le h$.

By Gauss divergence theorem,

$$\iint_S \vec{F}.\vec{n}\,dA = \iiint_V \operatorname{div}\vec{F}\,dV \quad \text{.........(i)}$$

Here,

$$\operatorname{div}\vec{F} = \nabla.\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(y^3\vec{i} + x^3\vec{j} + z^3\vec{k}) = 3z^2$$

Given surface is a ellipsoid having height h on first quadrant.

So, x varies on the region from $x = 0$ to $x = \sqrt{1-4y^2}$ and y moves from $y = 0$ to $y = \frac{1}{2}$. Also, the region moves from $z = 0$ to $z = h$.

Now, (i) becomes,

$$\iint_S \vec{F}.\vec{n}\,dA = \int_0^{1/2}\int_0^{\sqrt{1-4y^2}}\int_0^{h} 3z^2\,dzdxdy$$

$$= \int_0^{1/2}\int_0^{\sqrt{1-4y^2}} [z^3]_0^h\,dx\,dy$$

$$= h^3\int_0^{1/2} [x]_0^{\sqrt{1-4y^2}}\,dy$$

$$= h^3\int_0^{1/2} \sqrt{1-4y^2}\,dy$$

$$= 2h^3\int_0^{1/2} \sqrt{\frac{1}{4} - y^2}\,dy$$

$$= 2h^3\left[\frac{y\sqrt{\frac{1}{4}-y^2}}{2} + \frac{1/4}{2}\sin^{-1}\frac{y}{1/2}\right]_0^{1/2}$$

$$= 2h^3\left[0 + \frac{1}{8}\sin^{-1}(1)\right]$$

$$= \frac{h^3}{4}\cdot\frac{\pi}{2} = \frac{\pi h^3}{8} \qquad [\because \sin^{-1}(1) = \frac{\pi}{2}].$$

2007 Fall Q.No. 4(b)

State Gauss divergence theorem for the surface integral. Evaluate $\iint_S (\vec{F}.\vec{n})\, ds$, where $\vec{F} = (e^x, e^y, e^z)$ and S is the surface of the cube $|x| \le 1$, $|y| \le 1$, and $|z| \le 1$.

Solution: First Part: See the statement of Gauss Divergence theorem.

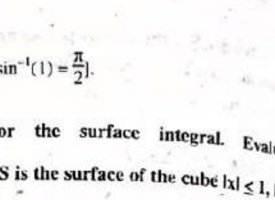
Second Part: Similar to 2011 Fall Q. No. 5(a).

2005 Fall Q.No. 4(a)

Define surface integral of $\vec{F}$ on the surface S. Evaluate $\iint_S \vec{F}.\vec{n}\, dA$ where $\vec{F}$ = (x, 3y, 6z) and S is the surface of the cone $\sqrt{x^2+y^2} \le Z$, $0 \le z \le 3$.

Solution: First Part: See the definition of surface integral.

Second Part: By Gauss divergence theorem



$$\iint_S \vec{F}.\vec{n}\, dA = \iiint \operatorname{div}\vec{F}\, dv$$

We have,

$$\vec{F} = x\vec{i} + 3y\vec{j} + 6z\vec{k}$$

Then, $\operatorname{div}.\vec{F} = 1 + 3 + 6 = 10.$

and we have given surface is $\sqrt{x^2+y^2} \le z$, $0 \le z \le 3$ is shown in figure.

Then from equation (1), we get

$$\iint_S \vec{F}.\vec{n}\, dA = \iiint 10dv = 10\iiint dv$$

= 10×volume of cone represented by the surface S

$$= 10 \times \frac{1}{3}\pi(3)^2.3$$

Thus, $\iint_S \vec{F}.\vec{n}\, dA = 90\pi.$

Note: If div $\vec{F}$ is not a constant value. Then we put the limits of x, y and z are as follows.

$$0 \le z \le 3; -\sqrt{z} \le y \le \sqrt{z}; -\sqrt{z^2-y^2} \le x \le \sqrt{z^2-y^2}.$$

2002 Q.No. 4(a)

Evaluate $\iint_S \vec{F}.\vec{n}\, dA$, where $\vec{F}$ = [$3x^2$, y^2, 0]; S: $\vec{r}$ = [u, v, 2u + 3v], for $0 \le u \le 2; -1 \le v \le 1$.

Solution: Similar to 2004 Spring 4(a).

2001 Q.No. 4(b)

State Gauss's divergence theorem and use this to evaluate $\iint_S [(x^3 - yz)\,\vec{i} - 2x^2y\,\vec{j}].\vec{n}\, dA$, where S is the surface of the cube bounded by the planes x = 0, x = 1, y = 1, z = 1.

Solution: First Part: See the statement of Gauss Divergence theorem.

Second Part: Similar to 2011 Fall 5(a).

Similar Questions

2014 Fall Q. No. 5(b) OR

State Gauss Divergence Theorem. Evaluate $\int_C \vec{F}.\vec{n}\, dA$ by using Green's Theroem if $\vec{F} = \left[\frac{e^y}{x}, e^y \ln x + 2x\right]$, R: $1 + x^4 \le y \le 2$.

SHORT QUESTIONS FROM FINAL EXAMINATION

2012 Fall: If $\vec{r} \times \frac{d^2r^4}{dt^2}$, show that $\vec{r} \times \frac{d^2\vec{r}}{dt} = 0$.

Question is incomplete.

2011 Spring: Find the curl of $\vec{F} = 2y\,\vec{j} + 5x\,\vec{k}$.

Solution: Given that $\vec{F} = 2y\,\vec{j} + 5x\,\vec{k}$.

Then,

$$\operatorname{curl}\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 2y & 5x \end{vmatrix} = 0\vec{i} - 5\vec{j} + 0\vec{k}.$$

Thus, $\operatorname{curl}\vec{F} = -5\vec{j}$.

2010 Spring: Find the unit tangent vector to the curve $\vec{r} = [t, t^2, t^3]$.

Solution: See the solution part of Q. 11, Exercise 4.3.

2009 Spring: Find the directional derivative of the scalar valued function $f(x) = x^2 + y^2$, at (1, 2) in the direction $\vec{a} = 2\vec{i} - \vec{j}$.

Solution: Given that $f = x^2 + y^2$ and $\vec{a} = 2\vec{i} - \vec{j}$.

Then grad (f) $= \nabla f = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y}\right) f = 2x\,\vec{i} + 2y\,\vec{j}$

and $\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{2\vec{i} - \vec{j}}{\sqrt{4+1}} = \frac{1}{\sqrt{5}}(2\vec{i} - \vec{j})$

Now, directional derivative of f at (1, 2) along $\vec{a}$ is;

$$D_{\vec{a}} f = \text{grad (f)} \cdot \hat{a} \text{ at } (1, 2)$$

$$= \frac{1}{\sqrt{5}}(4x - 2y) \text{ at } (1, 2)$$

$$= \frac{1}{\sqrt{5}}(4 - 4) = 0.$$

2008 Spring: Find the divergence of the vector $\vec{v} = (x^2 + yz)\vec{i} + (y^2 + zx)\vec{j} + (z^2 + xy)\vec{k}$.

Solution: See the solution part of Q. 4(ii), Exercise 4.3.

2007 Fall: If $\vec{r} = \vec{a}$ coswt + b sinwt, show that $\vec{r} \times \frac{d\vec{r}}{dt} = w\,\vec{a} \times \vec{b}$ where $\vec{a}$ and $\vec{b}$ are constant vectors.

Solution: Let $\vec{r} = \vec{a}\cos wt + \vec{b}\sin wt$. Then, $\frac{d\vec{r}}{dt} = -w\,\vec{a}\sin wt + w\,\vec{b}\cos wt$.

Now,

$$\vec{r} \times \frac{d\vec{r}}{dt} = w(\vec{a} \times \vec{b})\cos^2 wt - w(\vec{b} \times \vec{a})\sin^2 wt \quad [\because \vec{a} \times \vec{a} = 0]$$

$$= w(\vec{a} \times \vec{b})\cos^2 wt + w(\vec{a} \times \vec{b})\sin^2 wt$$

$$= w(\vec{a} \times \vec{b})(\cos^2 wt + \sin^2 wt) = w(\vec{a} \times \vec{b}).$$

Thus, $\vec{r} \times \frac{d\vec{r}}{dt} = w(\vec{a} \times \vec{b})$.

2006 Spring: If f(x, y, z) = xyz, show that $\nabla.(\nabla f) = 0$.

Solution: Let f = xyz.

Then, $\nabla f = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(xyz) = yz\,\vec{i} + zx\,\vec{j} + xy\,\vec{k}$

and,

$$\nabla.\nabla f = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).(yz\,\vec{i} + zx\,\vec{j} + xy\,\vec{k})$$

$$= \frac{\partial}{\partial x}yz + \frac{\partial}{\partial y}zx + \frac{\partial}{\partial z}xy = 0 + 0 + 0 = 0.$$

Thus, $\nabla.\nabla f = 0$.

Alternative solution

Let f = xyz. Since we have div(grad f) = 0.

That is $\nabla.(\nabla f) = 0$

2006 Spring: Find $\frac{d}{dt}(\vec{r}.\vec{r})$ where $\vec{r} = t\,\vec{i} + 3t^2\,\vec{j} + 4t^3\,\vec{k}$.

Solution: Let $\vec{r} = (t, 3t^2, 4t^3)$. Then $\vec{r}.\vec{r} = (t, 3t^2, 4t^3).(t, 3t^2, 4t^3)$

$$= t^2 + 9t^4 + 16t^6$$

So, $\frac{d}{dt}(\vec{r}.\vec{r}) = 2t + 36t^3 + 96t^5$.

2005 Fall: If $\vec{v} = 3t^2\,\vec{i} + 3t\,\vec{j} - (3t + 2)\,\vec{k}$, evaluate $\int_2^3 \vec{v}\,dt$.

Solution: Let $\vec{v} = (3t^2, 3, -3t - 2)$. Then,

$$\int_2^3 \vec{v}\,dt = 3\vec{i}\int_2^3 t^2 dt + 3\vec{j}\int_2^3 t\,dt - \vec{k}\left[3\int_2^3 t\,dt - 2\int_2^3 dt\right]$$

$$= 3\vec{i}\left[\frac{t^3}{3}\right]_2^3 + 3\vec{j}\left[\frac{t^2}{2}\right]_2^3 - \vec{k}\left[3\frac{t^2}{2} - 2t\right]_2^3$$

$$= \vec{i}(27 - 8) + \frac{3\vec{j}}{2}(9 - 4) - \vec{k}\left(\frac{27}{2} - 6 - 6 + 4\right)$$

$$= 19\,\vec{i} + \frac{15}{2}\vec{j} - \frac{11}{2}\vec{k}.$$

2004 Fall: Find the directional derivative of $f(x, y, z) = 2x^{-2} + 3y^2 + z^2$ at P (1, 2, 3) in the direction of $\vec{a} = \vec{i} - 2\vec{k}$.

Solution: Similar to 2009 Spring.

2004 Fall: If the divergence of $\vec{F} = 2x\,\vec{i} + y\,\vec{j} + pz\,\vec{k}$ is zero find the value of p.

Solution: Let $\vec{F} = (2x, y, pz)$.

Then div. $\vec{F} = \nabla.\vec{F} = \left(\vec{i}\frac{\partial}{\partial z} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).\vec{F} = 2 + 1 + P = 3 + P$

Given that div. $\vec{F} = 0$. Then $3 + P = 0 \Rightarrow P = -3$.

Thus, value of P is -3.

2004 Spring: Find the gradient of $f = xy + yz + zx$.

Solution: Let $f = xy + yz + zx$. Then,

$$\text{grad } f = \nabla f = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(xy + yz + zx)$$

$$= (y + z)\vec{i} + (x + z)\vec{j} + (x + y)\vec{k}.$$

Thus, gradient of f is $(y + z, x + z, x + y)$.

2004 Spring: If $\vec{a} = 3t^2\vec{i} + 4t^3\vec{j}$ and $\vec{b} = 5t^2\vec{i} + 4t\vec{j}$ find $d/dt(\vec{a}.\vec{b})$.

Solution: Let $\vec{a} = (3t^2, 4t^3)$, $\vec{b} = (5t^2, 4t)$

Then

$$\vec{a}.\vec{b} = (3t^2, 4t^3).(5t^2, 4t) = 15t^4 + 16t^4 = 31t^4.$$

Now,

$$\therefore \frac{d}{dt}(\vec{a}.\vec{b}) = 124\,t^3$$

2004 Spring: If $\vec{r} = t^2\vec{i} + (2t + 1)\vec{j} + 3t\vec{k}$. Find $|d^2\vec{r}/dt^2|$.

Solution: Let $\vec{r} = (t^2, 2t + 1, 3t)$.

Then,

$$\frac{d\vec{r}}{dt} = (2t, 2, 3) \quad \text{and} \quad \frac{d^2\vec{r}}{dt^2} = (2, 0, 0).$$

Thus, $\frac{d^2\vec{r}}{dt^2} = 2\vec{i}$.

2003 Fall: Find the gradient of $f = x^3 + y^3 + z^3 - 3xyz$.

Solution: Similar to 2004 Spring.

2003 Fall: If $\frac{d\vec{a}}{dt} = \vec{c}\times\vec{a}, \frac{d\vec{b}}{dt} = \vec{c}\times\vec{b}$. Show that $\frac{d}{dt}(\vec{a}\times\vec{b}) = \vec{c}\times(\vec{a}\times\vec{b})$.

Solution: Let $\frac{d\vec{a}}{dt} = \vec{c}\times\vec{a}$ and $\frac{d\vec{b}}{dt} = \vec{c}\times\vec{b}$.

Now,

$$\frac{d}{dt}(\vec{a}\times\vec{b}) = \frac{d\vec{a}}{dt}\times\vec{b} + \vec{a}\times\frac{d\vec{b}}{dt}$$

$$= (\vec{c}\times\vec{a})\times\vec{b} + \vec{a}\times(\vec{c}\times\vec{b})$$

$$= (\vec{b}.\vec{c})\vec{a} - (\vec{b}.\vec{a})\vec{c} + (\vec{a}.\vec{b})\vec{c} - (\vec{a}.\vec{c})\vec{b}$$

[$\because$ Using cross product of three vectors]

$$= (\vec{b}.\vec{c})\vec{a} - (\vec{a}.\vec{c})\vec{b} = \vec{c}\times(\vec{a}\times\vec{b}).$$

Thus, $\frac{d}{dt}(\vec{a}\times\vec{b}) = \vec{c}\times(\vec{a}\times\vec{b})$.

2003 Spring: Find the directional derivative of $f(x, y, z) = x^2 + y^2 + z^2$ at (1, 2, 1) in the direction $\vec{a} = 2\vec{i} - 2\vec{j} + \vec{k}$.

Solution: Similar to 2009 Spring.

2003 Spring: If the divergence of $\vec{F} = 2px\vec{i} + y\vec{j} + z\vec{k}$ is zero, find the value of p.

Solution: Similar to 2004 Fall.

2002: If $\vec{V} = x^2y\vec{i} + y^2z\vec{j} + z^2x\vec{k}$ find div $\vec{V}$.

Solution: Let $\vec{v} = (x^2y, y^2z, z^2x)$.

Then,

$$\text{div.}\vec{r} = \nabla.\vec{v} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right).\vec{v} = (2xy + 2yz + 2zx).$$

2002: Find the gradient of $f = x^2 + y^2 + z^2$.

Solution: Similar to 2004 Spring.

2001: If $\vec{v} = x^2yz\,\vec{i} - xy^2z\vec{j} - xyz^2\vec{k}$ find div. $\vec{v}$.

Solution: Similar to 2004 Spring.

2001: Evaluate $\int_c (y^2dx - x^2\,dy)$ counter the clockwise along the circle $x^2 + y^2 + = 1$ from (1, 0) to (0, 1).

Solution: Given integral is, $\int_c (y^2dx - x^2dy)$.

Comparing it with $\int_c (F_1dx + F_2dy)$ then we get, $F_1 = y^2$, $F_2 = -x^2$.

Also, given that the integral moves along $x^2 + y^2 = 1$ in counterclockwise direction. In which y varies from $y = -\sqrt{1 - x^2}$ to $y = \sqrt{1 - x^2}$ and moves from $x = -1$ to $x = 1$.

Now, by Green's theorem,

$$\oint_c (F_1dx + F_2dy) = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dy\,dx$$

Then,

$$\oint_C (y^2dx - x^2dy) = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(-\frac{\partial x^2}{\partial x} - \frac{\partial y^2}{\partial y}\right) dy\, dx$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (-2x - 2y)\, dy\, dx$$

$$= \int_{-1}^{1} [-2xy - y^2]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\, dx$$

$$= -\int_{-1}^{1} (4x\sqrt{1-x^2} + 1 - x^2 - 1 + x^2)\, dx$$

$$= -\int_{-1}^{1} 4x\sqrt{1-x^2}\, dx$$

Put $1 - x^2 = t^2$ then $-2x\, dx = 2t\, dt \Rightarrow x\, dx = -t\, dt$. Also, $x = 0 \Rightarrow t = 1$, $x = 1 \Rightarrow t = 0$.

Now,

$$\oint_C (y^2dx - x^2dy) = -2\int_{1}^{0} 4t\,(-t\,dt)$$

$$= -8\int_{1}^{0} t^2\, dt = -8\left[\frac{t^3}{3}\right]_0^1 = -8\left(0 - \frac{1}{3}\right) = \frac{8}{3}.$$

Similar Questions

2013 Fall Q. No. 7(a): Find unit tangent vector to the curve $\vec{r} = t^2\vec{i} + 2t\vec{j} - t^3\vec{k}$ at t = 1.

2013 Spring Q. No. 7(d): Check the exactness condition for value under the integral sign $\int_{(0,\pi)}^{(3,\pi/2)} (e^x \cos y\,dx - e^x \sin y\,dy)$ and evaluate the integral if it is exact.

2014 Fall Q. No. 7(b): If $\phi = e^{xy}$, find grad ϕ.

2014 Spring Q. No. 7(b): If $\vec{r} = \vec{a}e^{nt} + \vec{b}e^{-nt}$, where $\bar{a}$ & $\bar{b}$ are constant vectors, show $\frac{d^2\bar{r}}{dt^2} - n^2\bar{r} = 0$.

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