Exercise 9.1

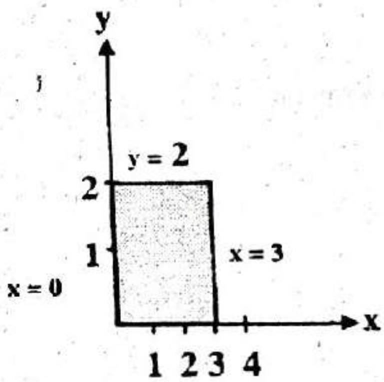
1. **Evaluate the following integrates and sketch the region over which each integration takes place.**

(i) $\int_0^3 \int_0^2 (4-y^2)\, dy\, dx$

Solution: Given integral is

$$I = \int_0^3 \int_0^2 (4-y^2)\, dy\, dx$$

Here the region of integration is bounded below by y = 0, above by y = 2, on the left by x = 0 and on the right by x = 3. On these bases the region of integration is as shown in figure.



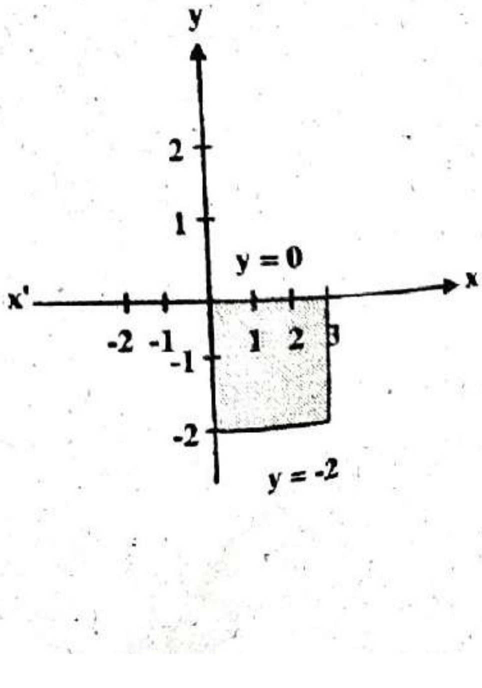
Now,

$$I = \int_0^3 \int_0^2 (4-y^2)\, dy\, dx = \int_0^3 \left[4y - \frac{y^3}{3}\right]_0^2 dx$$

$$= \int_0^3 \left(8 - \frac{8}{3}\right) dx$$

$$= \frac{16}{3}\int_0^3 dx = \frac{16}{3}[x]_0^3 = \frac{16}{3} \times 3 = 16.$$

Thus, $I = \int_0^3 \int_0^2 (4-y^2)\, dy\, dx = 16.$



(ii) $\int_0^3 \int_{-2}^0 (x^2y - 2xy)\, dy\, dx$

Solution: Given integral is

$$I = \int_0^3 \int_{-2}^0 (x^2y - 2xy)\, dy\, dx$$

Here the region of integration is bounded below by $y = -2$, above by $y = 0$, on the left by $x = 0$ and on the right by $x = 3$. On these bases the region of integration is as shown in figure.

Now,

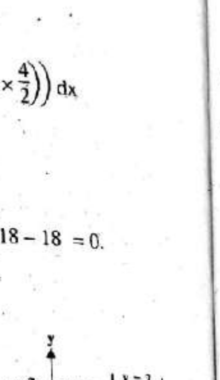
$$I = \int_0^3 \int_{-2}^0 (x^2y - 2xy)\, dy\, dx$$

$$= \int_0^3 \left[x^2\frac{y^2}{2} - 2x\frac{y^2}{2}\right]_{-2}^0 dx = \int_0^3 \left(0 - \left(x^2 \times \frac{4}{2} - 2x \times \frac{4}{2}\right)\right) dx$$

$$= \int_0^3 (4x - 2x^2)\, dx$$

$$= \left[4\frac{x^2}{2} - 2\frac{x^3}{3}\right]_0^3 = 18 - 18 = 0.$$

Thus, $I = \int_0^3 \int_{-2}^0 (x^2y - 2xy)\, dy\, dx = 0.$



(iii) $\int_1^2 \int_{-1}^2 (12xy^2 - 8x^3)\, dy\, dx$

Solution: Given integral is

$$I = \int_1^2 \int_{-1}^2 (12x\, y^2 - 8x^3)\, dy\, dx$$

Here the region of integration is bounded below by $y = -1$, above by $y = 2$, on the left by $x = 1$ and on the right by $x = 2$. On these bases the region of integration is as shown in figure.

Now,

$$I = \int_1^2 \int_{-1}^2 (12x\, y^2 - 8x^3)\, dy\, dx$$

$$= \int_1^2 \left[12x\frac{y^3}{3} - 8x^3y\right]_{-1}^2 dx$$

$$= \int_1^2 \left[\left(12x \times \frac{8}{3} - 16x^3\right) - \left(12x \times \frac{-1}{3} + 8x^3\right)\right] dx$$

$$= \int_1^2 (32x - 16x^3 + 4x - 8x^3)\, dx = \int_1^2 (36x - 24x^3)\, dx$$

$$= \left[36 \times \frac{x^2}{2} - 24 \times \frac{x^4}{4}\right]_1^2$$

$$= \left(36 \times \frac{4}{2} - 24 \times \frac{16}{4} - \frac{36}{2} + \frac{24}{4}\right) = 72 - 96 - 18 + 6 = -36$$

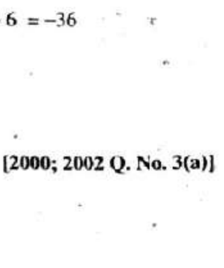
Thus, $I = \int_1^2 \int_{-1}^2 (12x\, y^2 - 8x^3)\, dy\, dx = -36.$

(iv) $\int_0^\pi \int_0^x$ **x siny dy dx** **[2000; 2002 Q. No. 3(a)]**

Solution: Given integral is

$$I = \int_0^\pi \int_0^x x \sin y\, dy\, dx$$

Here the region of integration is bounded below by $y = 0$, above by $y = x$, on the left by $x = 0$ and on the right by $x = \pi$. On these bases the region of integration is as shown in figure.



Now,

$$I = \int_0^\pi \int_0^x x \sin y\, dy\, dx$$

$$= \int_0^\pi x\,[-\cos y]_0^x\, dx = -\int_0^\pi x\,(\cos x - \cos 0)\, dx$$

$$= -\int_0^\pi (x\cos x - x)\, dx$$

$$= -\left[x\sin x + \cos x - \frac{x^2}{2}\right]_0^\pi$$

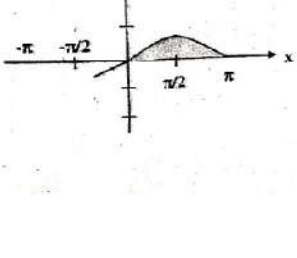
$$= -\left[\left(\pi \sin\pi + \cos\pi - \frac{\pi^2}{2}\right) - (0 + \cos 0 - 0)\right]$$

$$= -\left(-1 - \frac{\pi^2}{2} - 1\right)$$

$$= -\left(-2 - \frac{\pi^2}{2}\right) = 2 + \frac{\pi^2}{2} = \left(\frac{4 + \pi^2}{2}\right).$$

Thus, $I = \int_0^\pi \int_0^x x \sin y\, dy\, dx = \left(\frac{4 + \pi^2}{2}\right).$

(v) $\int_0^\pi \int_0^{\sin x}$ **y dy dx**



Solution: Given integral is

$$I = \int_0^\pi \int_0^{\sin x} y\, dy\, dx$$

Here the region of integration is bounded below by $y = 0$, above by $y = \sin x$, on the left by $x = 0$ and on the right by $x = \pi$. On these bases the region of integration is as shown in figure.

Now,

$$I = \int_0^{\pi}\int_0^{\sin x} y\,dy\,dx = \int_0^{\pi}\left[\frac{y^2}{2}\right]_0^{\sin x} dx$$

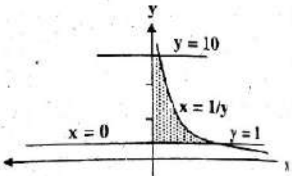
$$= \int_0^{\pi}\frac{\sin^2 x}{2}\,dx$$

$$= \frac{1}{2}\int_0^{\pi}\sin^2 x\,dx = \frac{1}{2}\int_0^{\pi}\left(\frac{1-\cos 2x}{2}\right)dx$$

$$= \frac{1}{4}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi} = \frac{1}{4}[\pi - 0] = \frac{\pi}{4}$$

Thus, $I = \int_0^{\pi}\int_0^{\sin x} y\,dy\,dx = \frac{\pi}{4}$.

(vi) $\int_{10}^{1}\int_{0}^{1/y} y\,e^{xy}\,dx\,dy$



Solution: Given integral is

$$I = \int_{10}^{1}\int_{0}^{1/y} y\,e^{xy}\,dx\,dy$$

Here the region of integration is bounded left by $x = 0$, right by $x = \frac{1}{y}$, on above by $y = 10$ and on below by $y = 1$. On these bases the region of integration is as shown in figure.

Now,

$$I = \int_{10}^{1}\int_{0}^{1/y} y\,e^{xy}\,dx\,dy$$

$$= \int_{10}^{1}\left[\frac{e^{xy}}{y}\right]_0^{1/y} dy = \int_{10}^{1}\left(\frac{y}{y}\left[e^{\frac{1}{y}\times y} - e^0\right]\right)dy$$

$$= \int_{10}^{1}(e-1)\,dy \quad = e\int_{10}^{1}dy - \int_{10}^{1}dy$$

$$= e\,[y]_{10}^{1} - [y]_{10}^{1}$$

$$= e\,[1-10] - [1-10]$$

$$= -9e + 9$$

$$= 9 - 9e.$$

Thus, $I = \int_{10}^{1}\int_{0}^{1/y} y\,e^{xy}\,dx\,dy = 9 - 9e.$

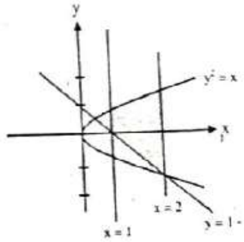
(vii) $\int_{1}^{2}\int_{1-x}^{\sqrt{x}} x^2 y\,dy\,dx$

Solution: Given integral is

$$I = \int_1^2\int_{1-x}^{\sqrt{x}} x^2 y\,dy\,dx.$$

Here the region of integration is bounded by $y = 1 - x$, and by $y = \sqrt{x}$. Since the line $y = 1 - x$ passes through the points (0, 1) and (1, 0). And curve $y = \sqrt{x}$ is a parabola that has vertex at (0, 0) and has line of symmetry $y = 0$. So, the parabola is right openward.

Also, the region is bounded on the left by $x = 1$ and on the right by $x = 2$. On these bases the region of integration is as shown in figure.



Now,

$$I = \int_1^2\int_{1-x}^{\sqrt{x}} x^2 y\,dy\,dx$$

$$= \int_1^2 x^2\left[\frac{y^2}{2}\right]_{1-x}^{\sqrt{x}}$$

$$= \int_1^2 x^2\left[\frac{x}{2} - \frac{(1-x)^2}{2}\right]dx$$

$$= \int_1^2 x^2\left(\frac{x - 1 + 2x - x^2}{2}\right)dx$$

$$= \frac{1}{2}\int_1^2 x^2(3x - 1 - x^2)\,dx$$

$$= \frac{1}{2}\int_1^2 (3x^3 - x^2 - x^4)\,dx$$

$$= \frac{1}{2}\left[3\times\frac{x^4}{4} - \frac{x^3}{3} - \frac{x^5}{5}\right]_1^2 = \frac{1}{2}\left[\left(3\times\frac{16}{4} - \frac{8}{3} - \frac{32}{5}\right) - \left(\frac{3}{4} - \frac{1}{3} - \frac{1}{5}\right)\right]$$

$$= \frac{1}{2}\left[12 - \frac{8}{3} - \frac{32}{5} - \frac{3}{4} + \frac{1}{3} + \frac{1}{5}\right]$$

$$= \frac{1}{2}\left(\frac{720 - 160 - 384 - 45 + 20 + 12}{60}\right) = \frac{163}{120}$$

Thus, $I = \int_1^2\int_{1-x}^{\sqrt{x}} x^2 y\,dy\,dx = \frac{163}{120}$.

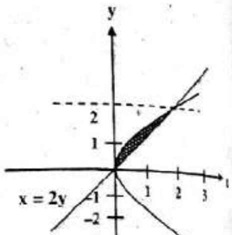
(viii) $\displaystyle\int_0^2\int_{y^2}^{2y} (4x - y)\,dx\,dy$

Solution: Given integral is

$$I = \int_0^2\int_{y^2}^{2y} (4x - y)\,dx\,dy$$

Here the region of integration is bounded by $x = y^2$, and by $x = 2y$.
Since the curve $y^2 = x$ is a parabola that has vertex at (0, 0) and has line of symmetry $y = 0$. So, the parabola is right openward. And, line $x = 2y$ passes through the points (0, 0) and (2, 1).
Also, the region is bounded on below by $y = 0$ and on above by $y = 2$.
On these bases the region of integration is as shown in figure.

Now,



$$I = \int_0^2\int_{y^2}^{2y} (4x - y)\,dx\,dy$$

$$= \int_0^2 \left[\frac{4x^2}{2} - xy\right]_{y^2}^{2y} dy$$

$$= \int_0^2 \left\{\left(4\times\frac{4y^2}{2} - 2y^2\right) - \left(\frac{4y^4}{2} - y^3\right)\right\} dy$$

$$= \int_0^2 (8y^2 - 2y^2 - 2y^4 + y^3)\,dy$$

$$= \int_0^2 (6y^2 - 2y^4 + y^3)\,dy$$

$$= \left[6\times\frac{y^3}{3} - 2\times\frac{y^5}{5} + \frac{y^4}{4}\right]_0^2 = \left(6^2\times\frac{8}{3} - 2\times\frac{32}{5} + \frac{16}{4}\right)$$

$$= \left(16 - \frac{64}{5} + 4\right) = \frac{80 - 64 + 20}{5} = \frac{36}{5}.$$

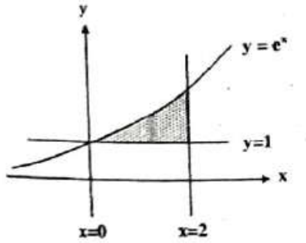
Thus, $I = \displaystyle\int_0^2\int_{y^2}^{2y} (4x - y)\,dx\,dy = \frac{36}{5}$.

(2) Evaluate, the integrate by interchanging the equivalent integral obtained by reversing the order of integration if necessary.

(i) $\displaystyle\int_0^2\int_1^{e^x} dy\,dx$ [2010 Spring-Short]

Solution: Given integral be

$$I = \int_0^2\int_1^{e^x} dy\,dx \qquad \ldots (1)$$



Here, the region of integration is bounded below by $y = 1$, above by $y = e^x$, on the left by $x = 0$ and the right by $x = 2$. On these bases the region of integration is as shown in figure.

Now,

$$I = \int_0^2\int_1^{e^x} dy\,dx$$

$$= \int_0^2 [y]_1^{e^x} dx = \int_0^2 (e^x - 1)\,dx = [e^x - x]_0^2 = (e^2 - 2 - e^0 + 0) = e^2 - 3.$$

Thus, $I = \displaystyle\int_0^2\int_1^{e^x} dy\,dx = e^2 - 3$.

(ii) $\displaystyle\int_0^{\sqrt{2}}\int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} y\,dx\,dy$

Solution: Given integral be

$$I = \int_0^{\sqrt{2}}\int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} y\,dx\,dy \qquad \ldots (1)$$

Here, the region is given by $-\sqrt{4-2y^2} \le x \le \sqrt{4-2y^2}$; $0 \le y \le \sqrt{2}$.

Since,

$$x = \sqrt{4-2y^2} \quad\Rightarrow\quad x^2 = 4 - 2y^2$$

$$\Rightarrow\quad \frac{x^2}{4} + \frac{y^2}{2} = 1,$$ which is an ellipse having centre at (0, 0) and has vertex at $(\pm 2, \pm\sqrt{2})$.

Thus, the integral (1) has the region of shaded portion as shown in the figure-1, that has horizontal strip.

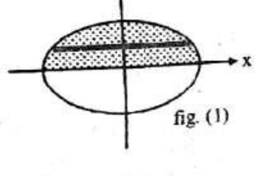


fig. (1)

Now, reversing the order of integration we take the vertical strip as in figure–2 for which y varies from $y = 0$ to the ellipse $y = \sqrt{\frac{1}{2}(4 - x^2)}$. Also, the strip moves from $x = -2$ to $x = 2$ (these are x-coordinates of vertices of the ellipse). Then,

$$I = \int_{x=-2}^{2} \int_{y=0}^{\sqrt{\frac{1}{2}(4-x^2)}} y\,dy\,dx$$

$$= \int_{x=-2}^{2} \left[\frac{y^2}{2}\right]_0^{\sqrt{\frac{1}{2}(4-x^2)}} dx$$

$$= \frac{1}{2}\int_{x=-2}^{2} \left[\frac{1}{2}(4-x^2) - 0\right] dx$$

$$= \frac{1}{4}\int_{x=-2}^{2} (4-x^2)\,dx$$

$$= \frac{1}{4}\left[4 - \frac{x^3}{3}\right]_{-2}^{2} = \frac{1}{4}\left[\left(8 - \frac{8}{3} - \left(-8 + \frac{8}{3}\right)\right)\right] = \frac{1}{4}\cdot\frac{32}{3} = \frac{8}{3}$$

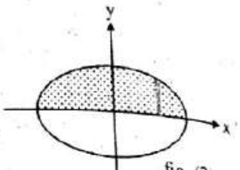


fig. (2)

Thus, $I = \frac{8}{3}$.

(iii) $\int_0^{\pi}\int_x^{\pi} \left(\frac{\sin y}{y}\right) dy\,dx$ **[2004 Spring Q. No. 3(a)]**

Solution: Given integral is

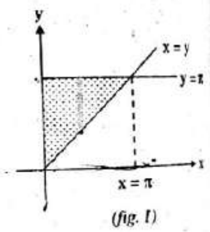
$$I = \int_0^{\pi}\int_x^{\pi} \left(\frac{\sin y}{y}\right) dy\,dx \quad \text{......... (1)}$$

Here the region of integration is bounded by $y = x$, and by $y = \pi$.

Since the line $y = x$ passes through the points (0, 0) and (1, 1). And the line $y = \pi$ is a straight line that is parallel to x-axis.

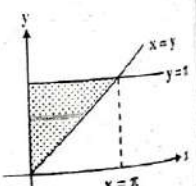
Next, the line $x = 0$ is y-axis. And the line $x = \pi$ is a straight line that is parallel to y-axis.

Then, the region generated by (1) is the shaded portion that has vertical strip, as shown in figure-1.



(fig. 1)

Now, reversing the order of integration, we take the horizontal strip as in figure 2 for which x varies from $x = 0$ to $x = y$. Also, the strip moves from $y = 0$ to $y = \pi$. Therefore, after changing the order of integration of (1), it becomes,



(fig. 2)

$$I = \int_{y=0}^{\pi} \int_{x=0}^{y} \left(\frac{\sin y}{y}\right) dx\,dy$$

$$= \int_{y=0}^{\pi} \left(\frac{\sin y}{y}\right) \int_{x=0}^{y} dx\,dy$$

$$= \int_{y=0}^{\pi} \left(\frac{\sin y}{y}\right)\cdot [x]_0^y\,dy = \int_{y=0}^{\pi} \left(\frac{\sin y}{y}\cdot y\right) dy = \int_{y=0}^{\pi} \sin y\,dy$$

$$= [-\cos y]_0^{\pi}$$

$$= -[\cos\pi - \cos 0]$$

$$= -[-1 - 1] = 2$$

Thus, $I = 2$.

(iv) $\int_0^1\int_y^1 x^2 e^{xy}\,dx\,dy$ **[2002 Q. No. 3(a)]**

Solution: Given integral is,

$$I = \int_0^1\int_y^1 x^2 e^{xy}\,dx\,dy \quad \text{.......... (1)}$$

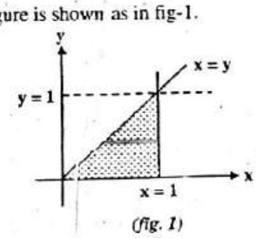
Here the region of integration is bounded by $x = y$, and by $x = 1$.

Since the line $y = x$ passes through the points (0, 0) and (1, 1). And the line $x = 1$ is a straight line that is parallel to y-axis.

Next, the line $y = 0$ is x-axis. And the line $y = 1$ is a straight line that is parallel to x-axis.

On the basis of these boundaries the sketch of figure is shown as in fig-1.

Clearly, the region generated by (1) is the shaded portion that has horizontal strip, as in figure 1.



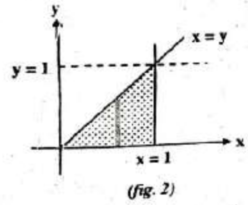
(fig. 1)

Now, interchanging the order of integration, we take the vertical strip as in figure – 2 for which y varies from $y = 0$ to $y = x$. Also, the strip moves from $x = 0$ to $x = 1$.

Therefore, after changing the order of integration, the integral (1) becomes,

$$I = \int_0^1\int_0^x x^2 e^{x}\,dy\,dx$$

$$= \int_0^1 x^2 \int_0^x e^{xy}\,dy\,dx$$



(fig. 2)

$$= \int_0^1 x^2\left[\frac{e^{xy}}{x}\right]_0^x dx = \int_0^1 x^2\left(\frac{e^{x^2}-1}{x}\right)dx$$

$$= \int_0^1 x\,e^{x^2}\,dx - \int_0^1 x\,dx$$

$$= I_1 - I_2$$

Here,

$$I_1 = \int_0^1 x\,e^{x^2}\,dx$$

Put, $x^2 = t$ then $2x\,dx = dt$. Also, $x = 0 \Rightarrow t = 0$, $x = 1 \Rightarrow t = 1$. Then,

$$I_1 = \frac{1}{2}\int_0^1 e^t\,dt = \frac{1}{2}[e^t]_0^1 = \frac{e^1 - e^0}{2} = \frac{e-1}{2}$$

and

$$I_2 = \int_0^1 x\,dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}$$

Then (2) becomes,

$$I = \frac{e-1}{2} - \frac{1}{2} = \frac{e-2}{2}$$

Thus, $I = \frac{e-2}{2}$

(v) $\displaystyle\int_0^{\infty}\int_{\sqrt[3]{x}}^{2}\left(\frac{1}{1+y^4}\right)dy\,dx$

Solution: Given integral is

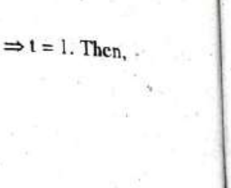
$$I = \int_0^{\infty}\int_{\sqrt[3]{x}}^{2}\left(\frac{1}{1+y^4}\right)dy\,dx \quad \text{........ (1)}$$

(fig. 1)

Here the region is $\sqrt[3]{x} \le y \le 2,\ 0 \le x \le 8$.

Clearly, the region generated by (1) the shaded portion that has vertical strip, as shown in figure 1.

Now, interchanging the order of integration, we get the region has horizontal strip as in figure 2 which x varies from $x = 0$ to $x = y^3$. Also, the strip moves from $y = 0$ to $y = 2$. Thus, after interchanging the order of integration (1) deduces to



(fig. 2)

$$I = \int_0^2\int_0^{y^3}\left(\frac{1}{1+y^4}\right)dx\,dy$$

$$= \int_0^2\left(\frac{1}{1+y^4}\right)\int_0^{y^3}dx\,dy$$

$$= \int_0^2 \frac{1}{1+y^4}\cdot y^3\,dy$$

Put $y^4 = t$ then $4y^3\,dy = dt$. Also, $y = 0 \Rightarrow t = 0$, $y = 2 \Rightarrow y = 16$. Then,

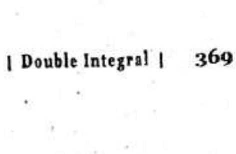
$$I = \int_0^{16}\left(\frac{1}{1+t}\right)\cdot\frac{dt}{4} = \frac{1}{4}[\log(1+t)]_0^{16} = \frac{1}{4}\log(17) \quad [\because \log(1) = 0]$$

Thus, $I = \frac{1}{4}\log(17)$.

(vi) $\displaystyle\int_0^1\int_{2x}^{2} e^{y^2}\,dy\,dx$

Solution: Given integral is

$$I = \int_0^1\int_{2x}^{2} e^{y^2}\,dy\,dx \quad \text{.......(1)}$$



(fig. 1)

Here the region of integration is bounded by $y = 2x$, and by $y = 2$.

Since the line $y = 2x$ passes through the points (0, 0) and (2, 1). And the line $y = 2$ is a straight line that is parallel to x-axis.

Next, the line $x = 0$ is y-axis. And the line $x = 1$ is a straight line that is parallel to y-axis.

On the basis of these boundaries the sketch of figure is shown as in fig-1.

(fig. 2)

Clearly, the required region generated by the integral (1) is the shaded portion that has vertical strip as shown in figure 1.

Now, reversing the order of integration, region has horizontal strip as in figure 2 in which x varies from $x = 0$ to $x = \frac{y}{2}$. Also, the strip moves from $y = 0$ to $y = 2$. Then (1) becomes,

$$I = \int_0^2 \int_0^{y/2} e^{y^2}\, dx\, dy = \int_0^2 e^{y^2} \cdot \frac{y}{2}\, dy$$

Put, $y^2 = t$ then $2y\, dy = dt$. Also, $y = 0 \Rightarrow t = 0$, $y = 2 \Rightarrow t = 4$. Then,

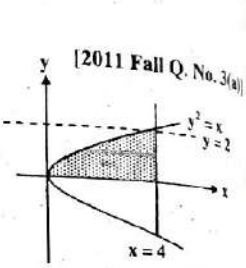
$$I = \frac{1}{2}\int_0^4 e^t \frac{dt}{2} = \frac{1}{4}\int_0^4 e^t\, dt = \frac{1}{4}\left[e^t\right]_0^4 = \frac{1}{4}(e^4 - 1).$$

Thus, $I = \dfrac{(e^4 - 1)}{4}$

(vii) $\displaystyle\int_0^2 \int_{y^2}^4 y\cos(x^2)\, dx\, dy$ **[2011 Fall Q. No. 3(a)]**

Solution: Given integral is

$$I = \int_0^2 \int_{y^2}^4 y\cos(x^2)\, dx\, dy \quad \text{........ (1)}$$



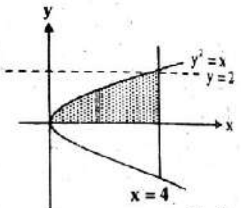
(fig. 1)

Here the region of integration is bounded by $x = y^2$, and by $x = 4$.

Since the curve $y^2 = x$ is a parabola that has vertex at (0, 0) and has line of symmetry $y = 0$. So, the parabola is right openward. And, line $x = 4$ is a straight line that is parallel to y-axis.

Also, the region is bounded on below by $y = 0$ and on above by $y = 2$.

On these bases the region of integration is as shown in figure.



(fig. 2)

Clearly, the region generated by (1) is the shaded portion that has horizontal strip, as shown in figure 1.

Now interchanging the order of integration of (1), the integral takes the vertical strip as in figure 2 in which y varies from $y = 0$ to $y = \sqrt{x}$. Also, the strip moves from $x = 0$ to $x = 4$.

Therefore (1) reduces to

$$I = \int_0^4 \int_0^{\sqrt{x}} y\cos(x^2)\, dy\, dx$$

$$= \int_0^4 \cos(x^2)\left[\frac{y^2}{2}\right]_0^{\sqrt{x}} dx$$

$$= \int_0^4 \cos(x^2) \cdot \frac{x}{2}\, dx$$

Put $x^2 = t$ then $2x\, dx = dt$. Also, $x = 0 \Rightarrow t = 0$, $x = 4 \Rightarrow t = 6$. Then,

$$I = \int_0^{16} \cos t \frac{dt}{4} = \frac{1}{4}[\sin t]_0^6 = \frac{\sin(6)}{4}$$

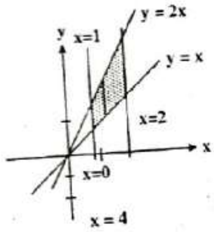
Thus, $I = \dfrac{\sin(16)}{4}$

3. Integrate f(x, y) over the region R:

(i) $f(x, y) = \dfrac{x}{y}$ **over the region in the first quadrant bounded by the lines $y = x$, $y = 2x$, $x = 1$, $x = 2$.**

Solution: Given that he region is in the first quadrant bounded by the lines $y = x$, $y = 2x$, $x = 1$, $x = 2$.

Here the region of integration is bounded below by $y = 2x$, above by $y = x$, on the left by $x = 1$ and on the right by $x = 2$. Since the line $y = 2x$ passes through (0, 0) and (2, 1). Also, the line $y = x$ passes through (0, 0) and (1, 1). On these bases the region of integration is as shown in figure.



Now, taking vertical strip then,

Now,

$$I = \int_1^2 \int_x^{2x} \frac{x}{y}\, dy\, dx$$

$$= \int_1^2 x \int_x^{2x} \frac{1}{y}\, dy\, dx$$

$$= \int_1^2 x\, (\log y)_x^{2x}\, dx$$

$$= \int_1^2 x\, (\log 2x - \log x)\, dx$$

$$= \int_1^2 x \log\frac{2x}{x}\, dx = \log 2 \times \left[\frac{x^2}{2}\right]_1^2 = \log 2 \times \left(\frac{4}{2} - \frac{1}{2}\right) = \frac{3}{2}\log 2.$$

Thus, $I = \dfrac{3}{2}\log 2$.

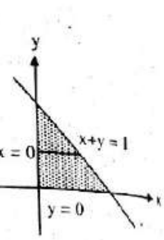
(ii) If $f(x, y) = y - \sqrt{x}$ over the triangular region cut from the first quadrant by the lines $x + y = 1$.

Solution: Given that he region is in the first quadrant bounded by the lines $x + y = 1$.

Here the region of integration is bounded by the line $x + y = 1$ hat passes through (1, 0) and (0, 1). Also, the region is bounded by the axes. On these bases the region of integration is as shown in figure.

Now, taking horizontal strip,

$$\int_0^1\int_0^{1-y}(y - \sqrt{x})\,dx\,dy = \int_0^1\left\{y\,[x]_0^{1-y} - \left[\frac{x^{3/2}}{\frac{3}{2}}\right]_0^{1-y}\right\}dy$$



$$= \int_0^1\left(y(1-y) - \frac{2}{3}(1-y)\frac{3}{2}\right)dy$$

$$= \int_0^1\left[y - y^2 - \frac{2}{3}(1-y)\frac{3}{2}\right]dy$$

$$= \left[\frac{y^2}{2} - \frac{y^3}{3} - \frac{2}{3}\frac{(1-y)^{5/2}}{5/2\,(-1)}\right]_0^1 = \left[\frac{y^2}{2} - \frac{y^3}{3} + \frac{2}{5}(1-y)^{5/2}\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{3} + \frac{4}{15}\{(1-1)^{5/2} - (1-0)^{5/2}\}$$

$$= \frac{1}{2} - \frac{1}{3} - \frac{4}{15} = \frac{15 - 10 - 8}{30} = \frac{-3}{30} = \frac{-1}{10}$$

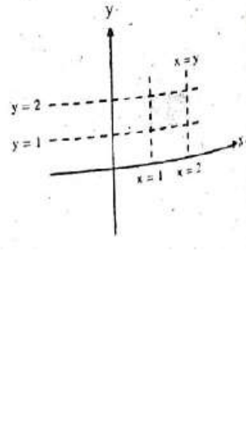
Thus, $\displaystyle\int_0^1\int_0^{1-y}(y - \sqrt{x})\,dx\,dy = \frac{-1}{10}$.

(iii) $f(x, y) = \dfrac{1}{xy}$ over the rectangle R: $1 \le x \le 2, 1 \le y \le 2$.

Solution: Given that, $f(x, y) = \dfrac{1}{xy}$ over the region R: $1 \le x \le 2$, $1 \le y \le 2$.

Here the region of integration is bounded below by $y = 1$, above by $y = 2$, on the left by $x = 1$ and on the right by $x = 2$. On these bases the region of integration is as shown in figure.

Now,



$$I = \int_{y=1}^{2}\int_{x=1}^{2} f(x, y)\,dx\,dy$$

$$= \int_{y=1}^{2}\int_{x=1}^{2}\left(\frac{1}{xy}\right)dx\,dy$$

$$= \int_{y=1}^{2}\frac{1}{y}\int_{x=1}^{2}\left(\frac{1}{xy}\right)dx\,dy$$

$$= \int_{y=1}^{2}\frac{1}{y}\,[\log x]_1^2\,dy = \log(2)\int_{y=1}^{2}\frac{dy}{y} \qquad [\because \log(1) = 0]$$

$$= \log(2)\,[\log(y)]_1^2$$

$$= \log(2)\log(2) \qquad [\because \log(1) = 0]$$

$$= [\log(2)]^2$$

Thus, $I = [\log(2)]^2$

4. Evaluate, the following integrals by changing the order of integration.

(i) $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}} y^2\,dy\,dx$

Solution: Given integral is

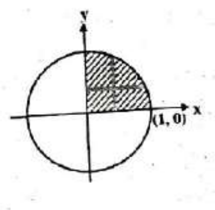
$$I = \int_0^1\int_0^{\sqrt{1-x^2}} y^2\,dy\,dx \qquad \text{.........} (1)$$

Here, the region of integral is R: $0 \le x \le 1$, $0 \le y \le \sqrt{1 - x^2}$.

Since the region $y = \sqrt{1 - x^2}$ is a circle having radius $r = 1$. And, $y = 0$ and $x = 0$ are the axes.

On the bases the region of integration is he shaded portion in the figure.

Put $x = r\cos\theta$, $y = r\sin\theta$. Then $dx\,dy = r\,dr\,d\theta$. Then by the figure $r = 0$, $r = 1$ and $\theta = 0$, $\theta = \frac{\pi}{2}$. Then (i) becomes,



$$I = \int_{r=0}^{1}\int_{\theta=0}^{\pi/2} r^2\sin^2\theta\, r\,d\theta\,dr$$

$$= \int_{r=0}^{1} r^3\int_{\theta=0}^{\pi/2}\frac{1 - \cos 2\theta}{2}\,d\theta\,dr$$

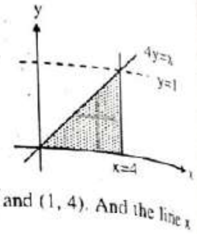
$$= \int_0^1\frac{r^3}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2} dr = \int_0^1\frac{r^3}{2}\frac{\pi}{2}\,dr = \frac{\pi}{4}\left[\frac{r^4}{4}\right]_0^1 = \frac{\pi}{16}$$

[2010 Spring Q. No. 3(a)]

(ii) $\displaystyle\int_0^1\int_{4y}^{4} e^{x^2}\,dx\,dy$

Solution: Given integral is

$$I = \int_0^1 \int_{4y}^4 e^{x^2}\, dx\, dy \qquad \text{......... (1)}$$



Here the region of integration is bounded by $x = 4y$, and by $x = 4$.

Since the line $x = 4y$ passes through the points (0, 0) and (1, 4). And the line $x = 4$ is a straight line that is parallel to y-axis.

Next, the line $y = 0$ is x-axis. And the line $y = 1$ is a straight line that is parallel to x-axis.

On the basis of these boundaries the sketch of figure is shown as in fig.

Clearly the region generated by (1) is the shaded portion in the corresponding figure that has horizontal strip.

Now, changing the order of integration of (1), the region takes vertical strip in which y varies from $y = 0$ to $y = \frac{x}{4}$. Also the strip moves from $x = 0$ to $x = 4$.

Then,

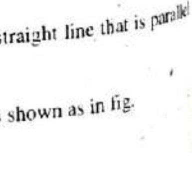
$$I = \int_0^4 \int_0^{x/4} e^{x^2}\, dy\, dx = \int_0^4 e^{x^2}\, [y]_0^{x/4}\, dx = \frac{1}{4}\int_0^4 x\, e^{x^2}\, dx$$

Put $x^2 = t$ then $2x\, dx = dt$. Also, $x = 0 \Rightarrow t = 0$, $x = 4 \Rightarrow t = 16$. So that,

$$I = \frac{1}{4}\int_0^{16} e^t \frac{dt}{2} = \frac{1}{8}\,[e^t]_0^{16} = \frac{e^{16} - 1}{8} \qquad [\because e^0 = 1]$$

Thus, $I = \dfrac{e^{16} - 1}{8}$.

(iii) $\displaystyle\int_0^4 \int_y^4 \frac{x\, dx\, dy}{x^2 + y^2}$



Solution: Given integral is

$$I = \int_0^4 \int_y^4 \frac{x\, dx\, dy}{x^2 + y^2} \qquad \text{........ (1)}$$

Here the region of integration is bounded by $x = y$, and by $x = 4$.

Since the line $x = y$ passes through the points (0, 0) and (1, 1). And the line $x = 4$ is a straight line that is parallel to y-axis.

Next, the line $y = 0$ is x-axis. And the line $y = 4$ is a straight line that is parallel to x-axis.

On the basis of these boundaries the sketch of figure is shown as in fig.

So, the region of integration generated by integral (1) is the shaded portion that has horizontal strip.

Now, changing the order of integration, the region takes the vertical strip in which y varies from $y = 0$ to $y = x$. And, the strip moves from $x = 0$ to $x = 4$.

Therefore,

$$I = \int_0^4 \int_y^x \frac{x}{x^2 + y^2}\, dy\, dx$$

$$= \int_0^4 x \cdot \left[\frac{1}{x}\tan^{-1}\left(\frac{y}{x}\right)\right]_0^x dx \qquad \left[\because \int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right]$$

$$= \int_0^4 \tan^{-1}\left(\frac{x}{x}\right) dx \qquad [\because \tan^{-1} 0 = 0]$$

$$= \int_0^4 \tan^{-1}(1)\, dx = \int_0^4 \frac{\pi}{4}\, dx \qquad \left[\because \tan\frac{\pi}{4} = 1 \Rightarrow \tan^{-1}(1) = \frac{\pi}{4}\right]$$

$$= \frac{\pi}{4}\,[x]_0^4 = \frac{\pi}{4} \cdot 4 = \pi.$$

Thus, $I = \pi$.

(iv) $\displaystyle\int_0^{a/\sqrt{2}} \int_y^{\sqrt{a^2 - y^2}} x\, dx\, dy$ **[2009 Spring Q. No. 3(a)]**

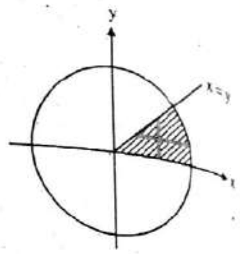
Solution: Given integral is

$$I = \int_0^{a/\sqrt{2}} \int_y^{\sqrt{a^2 - y^2}} x\, dx\, dy$$

Here, the region of integration is, $y \le x \le \sqrt{a^2 - y^2}$, $0 \le y \le \frac{a}{\sqrt{2}}$.

Clearly, the region of integration is the shaded part over the given region that has horizontal strip.

Now, changing order of integration, the integral takes vertical strip in which the strip bounded for $x = 0$ to $x = \frac{a}{\sqrt{2}}$ by the curves $y = 0$ and $y = x$ and for $x = \frac{a}{\sqrt{2}}$ to $x = a$ by $y = 0$ and $y = \sqrt{a^2 - x^2}$.



$$I = \int_{y}^{a/\sqrt{2}} \int_{0}^{x} x\,dy\,dx + \int_{a/\sqrt{2}}^{a} \int_{0}^{\sqrt{a^2-x^2}} x\,dy\,dx$$

$$= \int_{0}^{a/\sqrt{2}} x\,[y]_0^x\,dx + \int_{a/\sqrt{2}}^{a} [y]_0^{\sqrt{a^2-x^2}}\,dx$$

$$= \int_{0}^{a/\sqrt{2}} x^2\,dx + \int_{a/\sqrt{2}}^{a} x\sqrt{a^2-x^2}\,dx$$

$$= \frac{1}{3}\left(\frac{a}{\sqrt{2}}\right)^3 + \int_{a/\sqrt{2}}^{a} x\sqrt{a^2-x^2}\,dx$$

$$= \frac{a^3}{6\sqrt{2}} + \int_{a/\sqrt{2}}^{a} x\sqrt{a^2-x^2}\,dx$$

Put $a^2 - x^2 = t^2$ then $-2x\,dx = 2t\,dt$. Also, $x = \frac{a}{\sqrt{2}} \Rightarrow t = \frac{a}{\sqrt{2}}$ and $x = a \Rightarrow t = 0$.

Then, $$I = \frac{a^3}{6\sqrt{2}} + \int_{0}^{a/\sqrt{2}} t^2\,dt = \frac{a^3}{6\sqrt{2}} + \frac{1}{3}\left(\frac{a}{\sqrt{2}}\right)^3 = \frac{2a^3}{6\sqrt{2}} = \frac{a^3}{3\sqrt{2}}$$

Thus, $I = \frac{a^3}{3\sqrt{2}}$.

(v) $\displaystyle\int_{0}^{1} \int_{0}^{\sqrt{2-x^2}} \frac{x\,dydx}{\sqrt{x^2+y^2}}$ **[2006 Spring Q. No. 3(a)]**

Solution: Given integral is

$$I = \int_{0}^{1} \int_{0}^{\sqrt{2-x^2}} \frac{x\,dydx}{\sqrt{x^2+y^2}}.$$

Here, region of integration is $x \le y \le \sqrt{2-x^2}$, $0 \le x \le 1$.

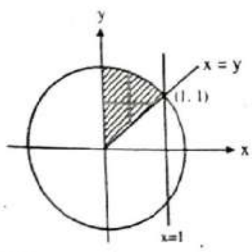
Clearly, the required region has vertical strip.

Now, changing the order of integration, the region takes horizontal strip in which x varies from $x = 0$ to $x = y$ for $y = 0$ to $y = 1$. And, x varies from $x = 0$ to $x = \sqrt{2-y^2}$ for $y = 1$ to $y = \sqrt{2}$. Therefore,

$$I = \int_{0}^{1}\int_{0}^{y} \frac{x\,dydx}{\sqrt{x^2+y^2}} + \int_{1}^{\sqrt{2}} \int_{0}^{\sqrt{2-y^2}} \frac{x\,dxdy}{\sqrt{x^2+y^2}}$$

Put $x^2 + y^2 = t$ then $2x\,dx = dt$. Also, $x = 0 \Rightarrow t = y^2$, and $x = y \Rightarrow t = 2y^2$. Also $x = \sqrt{2-y^2} \Rightarrow t = 2$.

Then,



$$I = \frac{1}{2}\int_{0}^{1}\int_{y^2}^{2y^2} \frac{dt\,dy}{\sqrt{t}} + \frac{1}{2}\int_{1}^{\sqrt{2}}\int_{y^2}^{2} \frac{dt\,dy}{\sqrt{t}}$$

$$= \frac{1}{2}\int_{0}^{1} \left[\frac{t^{1/2}}{1/2}\right]_{y^2}^{2y^2} dy + \frac{1}{2}\int_{1}^{\sqrt{2}} \left[\frac{t^{1/2}}{1/2}\right]_{y^2}^{2} dy$$

$$= \int_{0}^{1} [(2y^2)^{1/2} - (y^2)^{1/2}]\,dy + \int_{1}^{\sqrt{2}} [(2)^{1/2} - (y^2)^{1/2}]\,dy$$

$$= \int_{0}^{1} y(\sqrt{2}-1)\,dy + \int_{1}^{\sqrt{2}} (\sqrt{2}-y)\,dy$$

$$= (\sqrt{2}-1)\left[\frac{y^2}{2}\right]_0^1 + \left[y\sqrt{2} - \frac{y^2}{2}\right]_1^{\sqrt{2}}$$

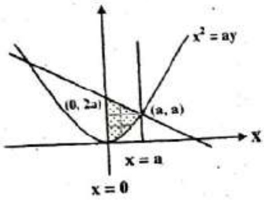
$$= (\sqrt{2}-1)\frac{1}{2} + \left[\left(2-\frac{2}{2}\right) - \left(\sqrt{2}-\frac{1}{2}\right)\right]$$

$$= \frac{\sqrt{2}-1}{2} + 1 - \sqrt{2} + \frac{1}{2}$$

$$= \frac{\sqrt{2}-1+2-2\sqrt{2}+1}{2} = \frac{2-\sqrt{2}}{2} = \left(1-\frac{1}{\sqrt{2}}\right)$$

Thus, $I = \left(1-\frac{1}{\sqrt{2}}\right)$.

(vi) $\displaystyle\int_{0}^{a} \int_{\frac{x^2}{a}}^{2a-x} xy\,dy\,dx$



Solution: Given integral is

$$I = \int_{0}^{a} \int_{\frac{x^2}{a}}^{2a-x} xy\,dy\,dx$$

Here, the region of integration is $\frac{x^2}{a} \le y \le 2a - x$; $0 \le x \le a$.

Clearly, the required region (shaded part in the figure) has vertical strip.

Now, changing the order of integration, the region takes horizontal strip that is bounded by three different curves.

Since, the strip is bounded by $x = 0$ and $x = \sqrt{ay}$ for $y = 0$ to $y = a$ and by $x = 0$, $x = 2a - y$ for $y = a$ to $y = 2a$.

Then,

$$I = \int_0^a \int_0^{\sqrt{ay}} xy\,dx\,dy + \int_0^{2a} \int_0^{(2a-y)} xy\,dx\,dy$$

$$= \int_0^a a\left[\frac{x^2}{2}\right]_0^{\sqrt{ay}} dy + \int_0^{2a} y\left[\frac{x^2}{2}\right]_0^{(2a-y)} dy$$

$$= \int_0^a \frac{a}{2} y^2\,dy + \frac{1}{2}\int_a^{2a} y\,(4a^2 + y^2 - 4ay)\,dy$$

$$= \frac{a}{2}\left[\frac{y^3}{3}\right]_0^a + \frac{4a^2}{2}\left[\frac{y^2}{2}\right]_a^{2a} + \frac{1}{2}\left[\frac{y^4}{4}\right]_a^{2a} - \frac{4a}{2}\left[\frac{y^3}{3}\right]_a^{2a}$$

$$= \frac{a^4}{6} + a^2(4a^2 - a^2) + \frac{1}{8}(16a^4 - a^4) - \frac{4a}{6}(8a^3 - a^3)$$

$$= \frac{a^4}{6} + 3a^4 + \frac{15a^4}{8} - \frac{28a^4}{6}$$

$$= \frac{8a^4 + 144a^4 + 90a^4 - 224a^4}{48} = \frac{18a^4}{48} = \frac{3a^4}{8}$$

Thus, $I = \frac{3a^4}{8}$.

(vii) $$\int_0^b \int_0^{\left(\frac{a\sqrt{b^2-y^2}}{b}\right)} xy\,dx\,dy$$

Solution: Given integral is

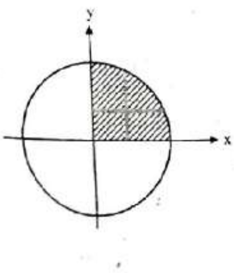
$$I = \int_0^b \int_0^{\left(\frac{a\sqrt{b^2-y^2}}{b}\right)} xy\,dx\,dy$$

Here, the region of integration be $0 \le y \le b$, $0 \le x \le \frac{a\sqrt{b^2-y^2}}{b}$.

Clearly the required region is the shaded portion that has horizontal strip.

Now, by changing the order of integration, the region takes vertical strip that is bounded by $y = 0$ to $y = \frac{b\sqrt{a^2-x^2}}{a}$. And, $0 \le x \le a$.

Then,



$$I = \int_0^a \int_0^{\frac{b\sqrt{a^2-x^2}}{a}} xy\,dy\,dx$$

$$= \int_0^a x\left[\frac{y^2}{2}\right]_0^{\frac{b\sqrt{a^2-x^2}}{a}} dx$$

$$= \int_0^a \frac{x}{2}\left[\frac{b^2(a^2-x^2)}{a^2}\right] dx$$

$$= \frac{1}{2a^2}\int_0^a (a^2b^2x - b^2x^3)\,dx$$

$$= \frac{a^2b^2}{2a^2}\left[\frac{x^2}{2}\right]_0^a - \frac{b^2}{2a^2}\left[\frac{x^4}{4}\right]_0^a = \frac{a^4b^2}{4a^2} - \frac{a^4b^2}{8a^2} = \frac{a^2b^2}{4} - \frac{a^2b^2}{8} = \frac{a^2b^2}{8}$$

Thus, $I = \frac{a^2b^2}{8}$

(viii) $$\int_0^a \int_{\sqrt{ax}}^0 \frac{y^2\,dy\,dx}{y^4 - a^2y^2}$$

Solution: Given integral is

$$I = \int_0^a \int_{\sqrt{ax}}^0 \frac{y^2\,dy\,dx}{y^4 - a^2y^2} = \int_0^a \int_{\sqrt{ax}}^0 \frac{dy\,dx}{\sqrt{y^2 - a^2}}$$

Here, the region of integration is, $0 \le x \le a$, $\sqrt{ax} \le y \le 0$.

Clearly, the region generated by I, is the shaded portion in the figure that has vertical strip.

Now, changing the order of integration of I, the region takes horizontal strip in which x varies from $x = \frac{y^2}{a}$ to $x = a$. For this, the strip moves from $y = 0$ to $y = -a$. Then,

dx dy

$$I = \int_{-a}^0 \int_{y^2/a}^a \frac{dy\,dx}{y^2 - a^2}$$

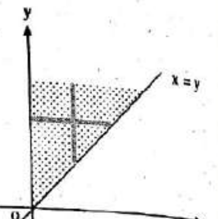
$$= \int_{-a}^0 \left(\frac{1}{y^2 - a^2}\right)[x]_{y^2/a}^a\,dy$$

$$= \int_{-a}^{0} \left(\frac{1}{y^2 - a^2}\right)\left(a - \frac{y^2}{a}\right) dy$$

$$= \frac{1}{a}\int_{-a}^{0} \frac{a^2 - y^2}{y^2 - a^2}\, dy = -\frac{1}{a}\int_{-a}^{0} dy = -\frac{1}{a}[y]_{-a}^{0} = -\frac{1}{a}(0 + a) = -1$$

Thus, $I = -1$.

(ix) $\displaystyle\int_0^{\infty}\int_x^{\infty} \left(\frac{e^{-y}}{y}\right) dy\, dx$ **[2009 Fall Q. No. 3(a)]**



Solution: Given integral is

$$I = \int_0^{\infty}\int_x^{\infty} \left(\frac{e^{-y}}{y}\right) dy\, dx$$

Here, region of integration is $0 \leq x < \infty$, $x \leq y < \infty$.

Clearly, the region generated by I is the shaded portion in the figure that has vertical strip that moves from $x = 0$ to $x \to \infty$.

Now, changing the order of integration of I, the region takes horizontal strip in which x varies from $x = 0$ to $x = y$ that moves from $y = 0$ to $y \to \infty$. Then,

$$I = \int_0^{\infty}\int_0^{y} \left(\frac{e^{-y}}{y}\right) dx\, dy$$

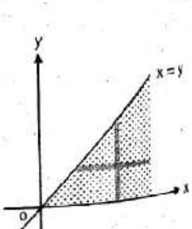
$$= \int_0^{\infty} \left(\frac{e^{-y}}{y}\right) \int_0^{y} dx\, dy$$

$$= \int_0^{\infty} \left(\frac{e^{-y}}{y}\right) [x]_0^{y}\, dy$$

$$= \int_0^{\infty} \frac{e^{-y}}{y} \cdot y\, dy = \int_0^{\infty} e^{-y}\, dy = \left[\frac{e^{-y}}{-1}\right]_0^{\infty} = \left(\frac{e^{-\infty} - e^{0}}{-1}\right) = \left(\frac{0-1}{-1}\right) = 1$$

Thus, $I = 1$.

(x) $\displaystyle\int_0^{\infty}\int_0^{x} x\, e^{-x^2/y}\, dy\, dx$



Solution: Given integral is

$$I = \int_0^{\infty}\int_0^{x} x\, e^{-x^2/y}\, dy\, dx$$

Here, the region of integration is $0 \leq y \leq x$, $0 \leq x < \infty$.

Clearly, the region generated by I is the shaded portion in the figure that has vertical strip in which y varies from $y = 0$ to $y = x$ that moves from $x = 0$ to $x \to \infty$.

Now, changing the order of integration, the region has horizontal strip in which x varies from $x = y$ to $x \to \infty$ that moves from $y = 0$ to $y \to \infty$. Then,

$$I = \int_0^{\infty}\int_y^{\infty} x\, e^{-x^2/y}\, dy\, dx$$

Put $\frac{x^2}{y} = t$ then $\frac{2x}{y} dx = dt$. And $x = y \Rightarrow t = y$, $x \to \infty \Rightarrow t \to \infty$. So that,

$$I = \int_0^{\infty}\int_y^{\infty} e^{-t}\, y \frac{dt}{2}\, dy$$

$$= \int_0^{\infty} \frac{y}{2}\int_y^{\infty} e^{-t}\, dt\, dy = \int_0^{\infty} \frac{y}{2}\left[\frac{e^{-t}}{-1}\right]_y^{\infty} dy = \int_0^{\infty} \frac{y}{2}\left(\frac{0 - e^{-y}}{-1}\right) dy = \frac{1}{2}\int_0^{\infty} y\, e^{-y}\, dy$$

$$\Rightarrow I = \frac{1}{2}\left[y\left(\frac{e^{-y}}{-1}\right) - (1)\frac{e^{-y}}{(-1)^2}\right]_0^{\infty} \quad [\because \text{Applying integrating by parts}]$$

$$= \frac{1}{2} \cdot 1 = \frac{1}{2}$$

Thus, $I = \frac{1}{2}$

(xi) $\displaystyle\int_0^{3}\int_1^{\sqrt{4-y}} (x + y)\, dx\, dy$

Solution: Given integral is

$$I = \int_0^{3}\int_1^{\sqrt{4-y}} (x + y)\, dx\, dy$$

Here, the region of integrtion is $0 \leq y \leq 3$, $1 \leq x \leq \sqrt{4 - y}$.

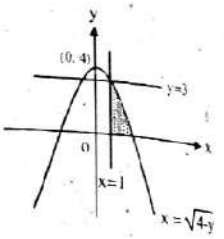
Since, $x = \sqrt{4 - y} \Rightarrow x^2 = -(y - 4)$ which is a parabola having vertex at $(0, 4)$ and down-open ward.

Clearly, the region determined by I is the shaded portion has horizontal strip in which the strip is bounded by $x = 1$ and $x = \sqrt{4-y}$ and it moves from $y = 0$ to $y = 3$.

Now, changing the order of integration of I, the region takes vertical strip that is bounded by $y = 0$ and $y = 4 - x^2$ and it moves from $x = 1$ to $x = 2$.

Then,

$$I = \int_1^2 \int_0^{\sqrt{4-x^2}} (x+y)\, dy\, dx$$



$$= \int_1^2 \left[xy + \frac{y^2}{2}\right]_{y=0}^{4-x^2} dx$$

$$= \int_1^2 \left[x(4-x^2) + \frac{1}{2}(4-x^2)^2\right] dx$$

$$= \int_1^2 \left(4x - x^3 + 8 + \frac{x^4}{2} - 4x^2\right) dx$$

$$= \left[\frac{4x^2}{2} - \frac{x^4}{4} + 8x + \frac{x^5}{10} - \frac{4x^3}{3}\right]_1^2$$

$$= \left(\frac{16}{2} - \frac{16}{4} + 16 - \frac{32}{10} - \frac{32}{3}\right) - \left(\frac{4}{2} - \frac{1}{4} + 8 + \frac{1}{10} - \frac{4}{3}\right)$$

$$= 16\left(\frac{1}{2} - \frac{1}{4} + 1 + \frac{2}{10} - \frac{2}{3}\right) - \left(2 - \frac{1}{4} + 8 + \frac{1}{10} - \frac{4}{3}\right)$$

$$= 16\left(\frac{30 - 15 + 60 + 12 - 40}{60}\right) - \left(\frac{120 - 15 + 480 + 6 - 80}{6}\right)$$

$$= \frac{1}{60}[16(102 - 55) - (606 - 95)] = \frac{1}{60}(752 - 511) = \frac{241}{60}$$

Thus, $I = \frac{241}{60}$

(xii) $$\int_0^a \int_{x/a}^{\sqrt{x/a}} (x^2 + y^2)\, dy\, dx$$

Solution: Given integral is

$$I = \int_0^a \int_{x/a}^{\sqrt{x/a}} (x^2 + y^2)\, dy\, dx$$

Here, the region of integration of I is, $0 \le x \le a$, $\frac{x}{a} \le y \le \sqrt{\frac{x}{a}}$.

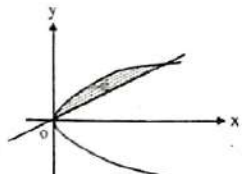
Clearly, the region generated by I, is the shaded portion in the figure that has vertical strip in which the strip bounded by $y = \frac{x}{a}$ and $y = \sqrt{\frac{x}{a}}$ and it moves from $x = 0$ to $x = a$.

Now, changing the order of integration, the region has horizontal strip that is bounded by $x = ay$ and $x = ay^2$ and the strip moves from $y = 0$ to $y = 1$.

$\therefore$ at $x = a$, $y = \frac{x}{a} = \frac{a}{a} = 1$.

Then,

$$I = \int_0^1 \int_{ay^2}^{ay} (x^2 + y^2)\, dx\, dy$$



$$= \int_0^1 \left[\frac{x^3}{3} + xy^2\right]_{x=ay^2}^{ay} dy$$

$$= \int_0^1 \left(\frac{a^3y^3}{3} + ay^3 - \frac{a^3y^6}{3} - ay^4\right) dy$$

$$= \left[\left(\frac{a^3}{3} + a\right)\frac{y^4}{4} - \frac{a^3}{3}\cdot\frac{y^7}{7} - a\cdot\frac{y^5}{5}\right]_0^1$$

$$= \left(\frac{a^3}{3} + a\right)\cdot\frac{1}{4} - \frac{a^3}{21} - \frac{a}{5}$$

$$= a^3\left(\frac{1}{12} - \frac{1}{21}\right) + a\left(\frac{1}{4} - \frac{1}{5}\right)$$

$$= a^3\left(\frac{21 - 12}{252}\right) + a\left(\frac{5-4}{20}\right) = a^3\left(\frac{9}{252}\right) + \frac{a}{20} = \frac{a^3}{28} + \frac{a}{20}$$

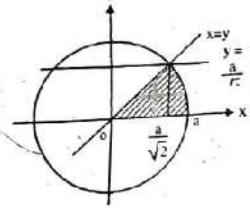
Thus, $I = \frac{a^3}{28} + \frac{a}{20}$

(xiii) $$\int_0^{a/\sqrt{2}} \int_y^{\sqrt{a^2 - y^2}} \log(x^2 + y^2)\, dx\, dy \text{ for } a > 0.$$

Solution

Given integral is

$$I = \int_0^{a/\sqrt{2}} \int_y^{\sqrt{a^2 - y^2}} \log(x^2 + y^2)\, dx\, dy \text{ for } a > 0.$$



Here, the region of integration is

$$0 \le y \le \frac{a}{\sqrt{2}},\ y \le x \le \sqrt{a^2 - y^2}$$

Clearly, the region generated by I is the shaded part in the figure that has horizontal strip in which the strip is bounded by $x = y$ and $x = \sqrt{a^2 - y^2}$ and it moves from $y = 0$ to $y = \frac{a}{\sqrt{2}}$.

Now, changing the order of integration, the region has vertical strip. From figure it is clearly that the strip is bounded by $y = 0$ and $y = x$ to till when the strip moves form $x = 0$ to $x = \frac{a}{\sqrt{2}}$ and the strip is bounded by $y = 0$ and $y = \sqrt{a^2 - x^2}$ when the strip moves from $x = \frac{a}{\sqrt{2}}$ to $x = a$.

Then,

$$I = \int_{0}^{a/\sqrt{2}} \int_{0}^{x} \log(x^2 + y^2)\, dy\, dx + \int_{a/\sqrt{2}}^{a} \int_{0}^{\sqrt{a^2 - x^2}} \log(x^2 + y^2)\, dy\, dx$$

$$= \int_{0}^{a/\sqrt{2}} \int_{0}^{x} \log(x^2 + y^2)\, dy\, dx + \int_{a/\sqrt{2}}^{a} \int_{0}^{\sqrt{a^2 - x^2}} \log(x^2 + y^2)\, dy\, dx$$

$$= \int_{0}^{a/\sqrt{2}} \int [\log(x^2 + y^2) \int dy - \int \left\{ \frac{d \log(x^2 + y^2)}{dy} \int dy \right\} dy]_0^x\, dx +$$

$$\int_{a/\sqrt{2}}^{a} [\log(x^2 + y^2) \int dy - \int \left\{ \frac{d \log(x^2 + y^2)}{dy} \int dy \right\} dy]_0^{\sqrt{a^2 - x^2}}\, dx$$

$$= \int_{0}^{a/\sqrt{2}} \int [\log(x^2 + y^2) \times y - \frac{1}{x^2 + y^2} \times 2y \times y dy]_0^x\, dx +$$

$$\int_{a/\sqrt{2}}^{a} \int [\log(x^2 + y^2) \times y - \int \left\{ \frac{1}{x^2 + y^2} \times 2y.\ y\, dy \right\}_0^{\sqrt{a^2 - x^2}} dx$$

$$= \int_{0}^{a/\sqrt{2}} \int \left[y \log(x^2 + y^2) - 2 \int \left(\frac{x^2 + y^2}{x^2 + y^2} - \frac{x^2}{x^2 + y^2} \right) dy \right]_0^x dx +$$

$$\int_{a/\sqrt{2}}^{a} \left[y \log(x^2 + y^2) - 2 \int \left(\frac{x^2 + y^2}{x^2 + y^2} - \frac{x^2}{x^2 + y^2} \right) dy \right]_0^{\sqrt{a^2 - x^2}} dx$$

$$= \int_{0}^{a/\sqrt{2}} \int \left[y \log(x^2 + y^2) - 2y + \frac{2x^2}{x} \tan^{-1} \frac{y}{x} \right]_0^x dx +$$

$$\int_{a/\sqrt{2}}^{a} \int \left[y \log(x^2 + y^2) - 2y + \frac{2x^2}{x} \tan^{-1} \frac{y}{x} \right]_0^{\sqrt{a^2 - x^2}}$$

$$= \int_{0}^{a/\sqrt{2}} \int \left(x \log 2x^2 - 2x + 2x \times \frac{\pi}{2} \right) dx +$$

$$\int_{a/\sqrt{2}}^{a} \sqrt{a^2 - x^2} \log(x^2 + \sqrt{a^2 - x^2}) - 2\sqrt{a^2 - x^2} + 2x \tan^{-1} \frac{\sqrt{a^2 - x^2}}{x}$$

$$= \left[\log 2x^2 \int x - \int \left(\frac{d \log 2x^2}{dx} \int x\, dx \right) dx - \frac{2x^2}{2} + \pi \frac{x^2}{2} \right]_0^{a/\sqrt{2}} +$$

$$[\log(x^2 + \sqrt{a^2 - x^2}) \int \sqrt{a^2 - x^2} dx - \int \frac{d}{dx} \log(x^2 + \sqrt{a^2 - x^2} \int \sqrt{a^2 - x^2} -$$

$$2 \int \sqrt{a^2 - x^2}\, dx + \tan^{-1} \frac{\sqrt{a^2 - x^2}}{x} \int 2x\, dx - \int \frac{d}{dx} \tan^{-1} \sqrt{a^2 - x^2} \int 2x.\, dx]_{a/\sqrt{2}}^{a}$$

$$= \left[\frac{x^2}{2} \log 2x^2 - \int \frac{1}{2x^2} \times 4x \times \frac{x^2}{2} dx - x^2 + \frac{\pi x^2}{2} \right]_0^{a/\sqrt{2}} + [\log(x^2 + \sqrt{a^2 - x^2}) (\frac{x}{2}$$

$$\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}) - \int \frac{1}{x^2 + \sqrt{a^2 - x^2}} (2x - \frac{2x}{\sqrt{a^2 - x^2}}) (\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}$$

$$\frac{x}{a}) dx - 2\frac{x}{2} \sqrt{a^2 - x^2} - \frac{2a^2}{2} \sin^{-1} \frac{x}{a} + \tan^{-1} \frac{\sqrt{a^2 - x^2}}{x} \times 2\frac{x^2}{2} - \int \frac{1}{1 + \left(\frac{a^2 - x^2}{x^2} \right)}$$

$$\frac{2x^2}{2} dx]_{a/\sqrt{2}}^{a}$$

$$= \left[\frac{x^2}{2} \log 2x^2 - \frac{x^2}{2} - x^2 + \frac{\pi x^2}{2} \right]_0^{a/\sqrt{2}} +$$

$$\left[\int \left(\frac{2x}{x^2 + \sqrt{a^2 - x^2}} - \frac{2x}{x^2 \sqrt{a^2 - x^2} + (a^2 - x^2)} \right) \right] + x^2 \tan^{-1} \frac{\sqrt{a^2 - x^2}}{x} - \int \frac{x^2}{a^2} \times x^2$$

$$dx \Big]_{a/\sqrt{2}}^{a}$$

$$= \frac{\pi a^2}{4} \left(\log a - \frac{1}{2} \right)$$