Exercise 6.5

Solve the following differential equation:

1. $\frac{dy}{dx} - y \tan x = -y^2 \sec x.$ **[2003 Fall Q. No. 4(**

Solution: Given that, $\frac{dy}{dx} - y \tan x = -y^2 \sec x$

$$\Rightarrow -\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{y}\tan x = \sec x \quad \ldots\ldots (i)$$

Put $u = \frac{1}{y}$ then $\frac{du}{dx} = -\frac{1}{y^2}\frac{dy}{dx}$. So, the equation (i) becomes,

$$\frac{du}{dx} + u \tan x = \sec x \quad \ldots\ldots (ii)$$

This is linear differential equation of first order in u.

Comparing (ii) with the equation $\frac{dy}{dx} + py = Q$ then we get,

$$P = \tan x, \qquad Q = \sec x$$

Then, I.F. $e^{\int pdx} = e^{\int \tan x} = e^{\log \sec x} = \sec x.$

Now, multiplying (ii) by I.F. and then taking integration both sides, so that (ii) becomes,

$$\boxed{u \times \text{I.F.} = \int Q \times \text{I.F.}\ dx + c}$$

$$u \times \sec x = \int \sec x \times \sec x\ dx + c$$

$$\Rightarrow \quad u \times \sec x = \int \sec^2 x\ dx + c \quad \Rightarrow \quad u \times \sec x = \tan x + c$$

$$\Rightarrow \quad \frac{1}{y} \sec x = \tan x + c$$

$$\Rightarrow \quad \sec x = y (\tan x + c).$$

2. $\dfrac{dy}{dx} + xy = xy^{-1}$.

Solution: Given that, $\dfrac{dy}{dx} + xy = xy^{-1} \Rightarrow y\dfrac{dy}{dx} + xy^2 = x$ ...... (i)

Put $u = y^2$ then $\dfrac{du}{dx} = 2y\dfrac{dy}{dx} \Rightarrow \dfrac{1}{2}\dfrac{du}{dx} = y\dfrac{dy}{dx}$. So, the equation (i) becomes,

$$\frac{1}{2}\frac{du}{dx} + xu = x$$

$$\Rightarrow \frac{du}{dx} + 2x\,u = 2x \qquad \text{...... (ii)}$$

This is linear differential equation of first order in u.

Comparing (ii) with the equation $\dfrac{dy}{dx} + py = Q$ then we get,

$$P = 2x, \qquad Q = 2x$$

Then, $\text{I.F.} = e^{\int p dx} = e^{\int 2x\, dx} = e^{x^2}$

Now, multiplying (ii) by I.F. and then taking integration both sides, so that (ii) becomes,

$$\boxed{u \times \text{I.F.} = \int Q \times \text{I.F.}\ dx + c}$$

Let: y+t = x²

$$\Rightarrow u \times e^{x^2} = \int 2x\, e^{x^2}\, dx + c$$

$$\Rightarrow u \times e^{x^2} = e^{x^2} + c \Rightarrow y^2 \times e^{x^2} = e^{x^2} + c \Rightarrow y^2 = 1 + ce^{-x^2}$$

(3) $x\dfrac{dy}{dx} + y \log y = xye^x$ **[2009 Spring Q. No. 4(a)]**

Solution: Given that, $x\dfrac{dy}{dx} + y \log y = xye^x \Rightarrow \dfrac{1}{y}\dfrac{dy}{dx} + \dfrac{1}{x}\log y = e^x$ ...... (i)

Put $u = \log(y)$ then $\dfrac{du}{dx} = \dfrac{1}{y}\dfrac{dy}{dx}$. So, the equation (i) becomes,

$$\frac{du}{dx} + \frac{1}{x}u = e^x \qquad \text{...... (ii)}$$

This is linear differential equation of first order in u.

Comparing (ii) with the equation $\frac{dy}{dx} + py = Q$ then we get,

$$P = \frac{1}{x} \quad \text{and} \quad Q = e^x$$

Then, I.F. $= e^{\int pdx} = e^{\int 1/x\, dx} = e^{\log x} = x.$

Now, multiplying (ii) by I.F. and then taking integration both sides, so that (ii) becomes,

$$\boxed{u \times \text{I.F.} = \int Q \times \text{I.F.}\ dx + c}$$

$$u \times x = \int e^x \times x\, dx + c \;\Rightarrow\; \underline{u \times x = xe^x - e^x + c}$$

$$\Rightarrow (\log y)\, x = xe^x - e^x + c.$$

4. $\dfrac{dy}{dx} + \dfrac{1}{x} = \dfrac{e^y}{x^2}$

Solution: Given that, $\dfrac{dy}{dx} + \dfrac{1}{x} = \dfrac{e^y}{x^2} \;\Rightarrow\; \dfrac{1}{e^y}\dfrac{dy}{dx} + \dfrac{1}{xe^y} = \dfrac{1}{x^2}$ ...... (i)

Put, $u = \frac{1}{e^y} = e^{-y}$ then $\frac{du}{dx} = -e^{-y}\frac{dy}{dx} \Rightarrow -\frac{du}{dx} = \frac{1}{e^y}\frac{dy}{dx}$.

So, the equation (i) becomes,

$$-\frac{du}{dx} + \frac{1}{x}u = \frac{1}{x^2}$$

$$\Rightarrow \frac{du}{dx} + \left(-\frac{1}{x}\right)u = \left(-\frac{1}{x^2}\right) \qquad \text{...... (ii)}$$

This is linear differential equation of first order in u.

Comparing (ii) with the equation $\frac{dy}{dx} + py = Q$ then we get,

$$P = -\frac{1}{x}, \qquad Q = -\frac{1}{x^2}$$

Then, I.F. $= e^{\int pdx} = e^{-\int 1/x dx} = e^{-\log x} = e^{\log (x)^{-1}} = \frac{1}{x}$

Now, multiplying (ii) by I.F. and then taking integration both sides, so that (ii) becomes,

$$\boxed{u \times \text{I.F.} = \int Q \times \text{I.F.}\ dx + c}$$

$$u \times \frac{1}{x} = -\int \frac{1}{x^2} \times \frac{1}{x}\, dx + c$$

$$\Rightarrow u \times \frac{1}{x} = -\int x^{-3}\, dx + c \;\Rightarrow\; \frac{u}{x} = \frac{1}{2x^2} + c$$

$$\Rightarrow \frac{u \times 2x^2}{x} = 1 + 2c\, x^2$$

$$\Rightarrow \frac{1}{e^y}\, 2x = 1 + 2c\, x^2 \;\Rightarrow\; 2x = (1 + 2cx^2)\, e^y$$

5. $2\dfrac{dy}{dx} = \dfrac{y}{x} + \dfrac{y^2}{x^2}$

Solution: Given that, $2\dfrac{dy}{dx} - \dfrac{y}{x} = \dfrac{y^2}{x^2} \;\Rightarrow\; \dfrac{2}{y^2}\dfrac{dy}{dx} - \dfrac{1}{xy} = \dfrac{1}{x^2}$ ...... (i)

Put $u = -\frac{1}{y}$ then $\frac{du}{dx} = \frac{1}{y^2}\frac{dy}{dx}$. So, the equation (i) becomes,

$$2\frac{du}{dx} + \frac{u}{x} = \frac{1}{x^2}$$

$$\Rightarrow \frac{du}{dx} + \frac{u}{2x} = \frac{1}{2x^2} \qquad \text{...... (ii)}$$

This is linear differential equation of first order in u.

Comparing (ii) with the equation $\frac{dy}{dx} + py = Q$ then we get,

$$P = \frac{1}{2x}, \qquad Q = \frac{1}{2x^2}$$

Then, I.F. $= e^{\int pdx} = e^{\int 1/2x\, dx} = e^{1/2 \int 1/x\, dx} = e^{1/2 \log x} = e^{\log x^{1/2}} = x^{1/2}$

Now, multiplying (ii) by I.F. and then taking integration both sides, so that (ii) becomes,

$$\boxed{u \times \text{I.F.} = \int Q \times \text{I.F.}\ dx + c}$$

$$u \times x^{1/2} = \int x^{1/2} \times \frac{1}{2x^2}\, dx + c$$

$$\Rightarrow u \times x^{1/2} = \frac{1}{2}\int x^{-3/2}\, dx + c = \frac{1}{2}\frac{x^{-1/2}}{-\frac{1}{2}} + c = c - \frac{1}{x^{1/2}}$$

$$\Rightarrow -\frac{1}{y}x^{1/2} = c - \frac{1}{x^{1/2}} \;\Rightarrow\; \frac{x}{y} = (1 - cx^{1/2}).$$

6. **$xy' + y = y^2 \log x$**

Solution: Given that, $x\dfrac{dy}{dx} + y = y^2 \log x \;\Rightarrow\; \dfrac{x}{xy^2}\dfrac{dy}{dx} + \dfrac{y}{xy^2} = \dfrac{y^2}{xy^2}\log x$

$$\Rightarrow \frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = \frac{1}{x}\log x \qquad \text{...... (i)}$$

Put $u = \frac{1}{y} = y^{-1}$. Then, $\frac{du}{dx} = -\frac{1}{y^2}\frac{dy}{dx} \Rightarrow -\frac{du}{dx} = \frac{1}{y^2}\frac{dy}{dx}$.

So, the equation (i) becomes,

$$-\frac{du}{dx} + \frac{u}{x} = \frac{1}{x}\log x$$

$$\Rightarrow \frac{du}{dx} + \left(-\frac{1}{x}\right)u = \left(-\frac{1}{x}\log x\right). \qquad \text{...... (ii)}$$

This is linear differential equation of first order in u.

Comparing (ii) with the equation $\frac{dy}{dx} + py = Q$ then we get,

$$P = -\frac{1}{x}, \qquad Q = -\frac{1}{x}\log x$$

Then, I.F. $= e^{\int pdx} = e^{-\int 1/x\, dx} = e^{-\log x} = e^{\log x^{-1}} = \frac{1}{x}$

Now, multiplying (ii) by I.F. and then taking integration both sides, so that (ii) becomes,

$$\boxed{u \times \text{I.F.} = \int Q \times \text{I.F. } dx + c}$$

$$u \times \frac{1}{x} = -\int \frac{1}{x}\log x \times \frac{1}{x}$$

$$\Rightarrow \frac{u}{x} = -\left[\log \int x^{-2}dx - \int\left\{\frac{d(\log x)}{dx}\int x^{-2}dx\right\} dx\right]$$

$$\Rightarrow \frac{u}{x} = -\left[-\frac{\log x}{x} - \int \frac{1}{x} \times -\frac{1}{x}dx\right] = -\left[-\frac{\log x}{x} + \int x^{-2}\,dx\right]$$

$$= -\left[-\frac{\log x}{x} + \frac{1}{x}\right] + c = \frac{\log x}{x} + \frac{1}{x} + c.$$

$$\Rightarrow \frac{1}{xy} = \frac{\log x + 1}{x} + c = (\log x + 1) + cx$$

$$\Rightarrow cxy + y(\log x + 1) = 1.$$

(7) $y' + \frac{1}{x}\text{Tan}y = \frac{1}{x^2}\text{Tan}y\ \text{Sin}y$

Solution: Given that, $\quad y' + \frac{1}{x}\text{Tan}y = \frac{1}{x^2}\text{Tan}y\ \text{Sin}y$

$$\Rightarrow \frac{1}{\text{Tan}y\ \text{Sin}y}\frac{dy}{dx} + \frac{1}{x}\frac{\text{Tan}y}{\text{Tan}y\ \text{Sin}y} = \frac{1}{x^2}$$

$$\Rightarrow \cot y\ \text{cosec}y\frac{dy}{dx} + \frac{1}{x}\text{cosec}y = \frac{1}{x^2} \qquad \ldots\ldots (i)$$

Put u = cosec y. Then, $\frac{du}{dx} - \text{cosec}y\ \cot y + \frac{dy}{dx} \Rightarrow -\frac{du}{dx} = \text{cosec}y\ \cot y \cdot \frac{dy}{dx}$

So, the equation (i) becomes,

$$-\frac{du}{dx} + \frac{1}{x^u} = \frac{1}{x^2}$$

$$\Rightarrow -\frac{du}{dx} + \left(-\frac{1}{x}u\right) = \left(-\frac{1}{x^2}\right) \qquad \ldots\ldots (ii)$$

This is linear differential equation of first order in u.

Comparing (ii) with the equation $\frac{dy}{dx} + py = Q$ then we get,

$$P = -\frac{1}{x}, Q = -\frac{1}{x^2}$$

Then, I.F. $= e^{\int pdx} = e^{-\int 1/x\, dx} = e^{-\log x} = e^{\log x^{-1}} = \frac{1}{x}$

Now, multiplying (ii) by I.F. and then taking integration both sides, so that (ii) becomes,

$$\boxed{u \times \text{I.F.} = \int Q \times \text{I.F. } dx + c}$$

$$u \times \frac{1}{x} = \int -\frac{1}{x^2} \times \frac{1}{x} + c$$

$$\Rightarrow \frac{u}{x} = -\frac{-x^2}{-2} + c$$

$$\Rightarrow \frac{\text{cosec}y}{x} = \frac{1}{2x^2} + c \;\Rightarrow\; \text{cosec}y = \frac{1 + 2cx^2}{2x^2} \;\Rightarrow\; 2x = 1\,(1 = 2cx^2)\ \text{sin}y.$$

(8) $2xy' = 10x^3y^5 + y$

Solution: Given that, $\quad 2xy' = 10x^3y^5 + y \;\Rightarrow\; \frac{1}{y^5}\frac{dy}{dx} - \frac{1}{2xy^4} = 5x^2 \qquad \ldots\ldots (i)$

Put $u = y^{-4}$. Then, $\frac{du}{dx} = -4y^{-5}\frac{dy}{dx}$. So, the equation (i) becomes,

$$-\frac{1}{4}\frac{du}{dx} - \frac{u}{2x} = 5x^2 \qquad \Rightarrow \frac{du}{dx} + \frac{4u}{2x} = -20x^2$$

$$\Rightarrow \frac{du}{dx} + \left(\frac{2}{x}\right)u = (-20x^2). \qquad \ldots\ldots (ii)$$

This is linear differential equation of first order in u.

Comparing (ii) with the equation $\frac{dy}{dx} + py = Q$ then we get,

$$P = \frac{2}{x}, \qquad Q = -20x^2$$

Then, I.F. $= e^{\int pdx} = e^{-\int 2/x\, dx} = e^{2\log x} = e^{\log x^2} = x^2$

Now, multiplying (ii) by I.F. and then taking integration both sides, so that (ii) becomes,

$$\boxed{u \times \text{I.F.} = \int Q \times \text{I.F. } dx + c}$$

$$u \times x^2 = -\int 20x^2 \times x^2\, dx + c$$

$$\Rightarrow u \times x^2 = -20\frac{x^5}{5} + c \;\Rightarrow\; -20\frac{1}{y^4}x^2 = -4x^5 + c \;\Rightarrow x^2 = 4x^5y^4 + xy$$

(9) $y' + 2y = y^2$

Solution: Given that, $\; y' + 2y = y^2 \;\Rightarrow\; \frac{1}{y^2}\frac{dy}{dx} + 2\frac{1}{y} = 1 \qquad \ldots\ldots (i)$

Put $u = \frac{1}{y} = y^{-1}$ then $\frac{du}{dx} = -\frac{1}{y^2}\frac{dy}{dx} \;\Rightarrow\; -\frac{du}{dx} = -\frac{1}{y^2}\frac{dy}{dx}$.

So, the equation (i) becomes,

$$-\frac{du}{dx} + 2u = 1$$

$$\Rightarrow \frac{du}{dx} + (-2)u = -1.$$

This is linear differential equation of first order in u.

Comparing (ii) with the equation $\frac{dy}{dx} + py = Q$ then we get,

$$P = -2, \qquad Q = -1$$

Then, I.F. $= e^{\int pdx} = e^{\int -2dx} = e^{-2x}$

Now, multiplying (ii) by I.F. and then taking integration both sides, so that (ii) becomes,

$$\boxed{u \times I.F. = \int Q \times I.F.\ dx + c}$$

$$u \times e^{-2x} = \int -1 \times e^{-2x}\, dx + c$$

$$\Rightarrow u \times e^{-2x} = -\frac{e^{-2x}}{-2} + c$$

$$\Rightarrow 2u = 1 + 2ce^{2x} \Rightarrow 2 \times \frac{1}{y} = 1 + 2ce^{2x} \Rightarrow 2 = y(1 + 2ce^{2x}).$$

(10) $y' + \frac{y}{3} = \left(\frac{1-2x}{3}\right) y^4$

Solution: Given that, $y' + \frac{y}{3} = \left(\frac{1-2x}{3}\right) y^4 \Rightarrow \frac{1}{y^4}\frac{dy}{dx} + \frac{1}{3y^3} = \left(\frac{1-2x}{3}\right)$ ...... (i)

Put, $u = \frac{1}{y^3} = y^{-3}$ then $\frac{du}{dx} = -3y^{-4}\frac{dy}{dx} \Rightarrow -\frac{1}{3}\frac{du}{dx} = \frac{1}{y^4}\frac{dy}{dx}$.

So, the equation (i) becomes,

$$-\frac{1}{3}\frac{du}{dx} + \frac{1}{3}u = \left(\frac{1-2x}{3}\right)$$

$$\Rightarrow \frac{du}{dx} + (-1)u = (2x-1) \qquad \text{...... (ii)}$$

This is linear differential equation of first order in u.

Comparing (ii) with the equation $\frac{dy}{dx} + py = Q$ then we get,

$$P = -1, \qquad Q = (2x-1)$$

Then, I.F. $= e^{\int pdx} = e^{-\int dx} = e^{-x}$

Now, multiplying (ii) by I.F. and then taking integration both sides, so that (ii) becomes,

$$\boxed{u \times I.F. = \int Q \times I.F.\ dx + c}$$

$$\Rightarrow u \times e^{-x} = \int (2x-1)e^{-x} + c$$

$$\Rightarrow u \times e^{-x} = -(2x-1)e^{-x} - 2e^{-x} + c$$

$$\Rightarrow u = -(2x-1) - 2 + ce^{x} \Rightarrow \frac{1}{y^3} = -(2x+1) + ce^{x}$$

$$\Rightarrow y^{-3} = ce^{x} - 2x - 1.$$

(11) $y' = \frac{1}{6e^y - 2x}$

Solution: Given that,

$$y' = \frac{1}{6e^y - 2x} \Rightarrow \frac{dy}{dx} = \frac{1}{6e^y - 2x} \Rightarrow \frac{dx}{dy} = 6e^y - 2x$$

$$\Rightarrow \frac{dx}{dy} + 2x = 6e^y \qquad \text{.........(1)}$$

This is linear differential equation in x whose integrating factor is

$$I.F. = e^{\int 2dy} = e^{2y}$$

Now, multiplying (1) by I.F. and then taking integration on both sides then,

$$x \cdot e^{2y} = \int e^{2y} \cdot 6e^y\, dy + c$$

$$= 2\int 3^{3y}(3dy) + c = 2 \cdot e^{3y} + c$$

$$\Rightarrow x = 2e^y + ce^{-2y}.$$

(12) $y' - x^2y = y^2e^{-x^3/2}$

Solution: Given that, $y' - x^2y = y^2e^{-x^3/2} \Rightarrow \frac{1}{y^2}\frac{dy}{dx} - \frac{x^2}{y} = e^{-x^3/2}$

Put $u = -\frac{1}{y} = -y^{-1}$. Then, $\frac{du}{dx} = y^{-2}\frac{dy}{dx} \Rightarrow \frac{du}{dx} = \frac{1}{y^2} = \frac{dy}{dx}$.

So, the equation (i) becomes,

$$\frac{du}{dx} + u\,x^2 = e^{-x^3/2} \qquad \text{...... (ii)}$$

This is linear differential equation of first order in u.

Comparing (ii) with the equation $\frac{dy}{dx} + py = Q$ then we get,

$$P = x^2, \qquad Q = e^{-x^3/2}$$

Then, I.F. $= e^{\int pdx} = e^{\int x^2dx} = e^{x^3/3}$

Now, multiplying (ii) by I.F. and then taking integration both sides, so that (ii) becomes,

$$\boxed{u \times I.F. = \int Q \times I.F.\ dx + c}$$

$$u \times e^{x^3/3} = \int e^{-x^3/2 + x^3/3}\, dx + c$$

$$\Rightarrow u \times e^{x^3/3} = \int e^{-x^3/3} + \int e^{-x^3/6} + c$$

$$\Rightarrow -\frac{1}{y} \times e^{x^3/3} = \frac{e^{-x^3/6}}{-\frac{1}{2}x^2} + c \Rightarrow -e^{x^3/3} = -\frac{e^{-x^3/6}}{x^2}y + cy$$

$$\Rightarrow e^{x^3/3} = \frac{e^{-x^3/6}}{x^2} y - cy.$$

13. $e^y(y' + 1) = e^x$

Solution: Given that, $e^y(y' + 1) = e^x \Rightarrow e^y \frac{dy}{dx} + e^y = e^x$ ...... (i)

Put $u = e^y$. Then, $\frac{du}{dx} = e^y \frac{dy}{dx}$. So, the equation (i) becomes,

$$\frac{du}{dx} + u = e^x \quad \text{...... (ii)}$$

This is linear differential equation of first order in u.

Comparing (ii) with the equation $\frac{dy}{dx} + py = Q$ then we get,

$$P = 1 \quad \text{and} \quad Q = e^x$$

Then, I.F. $= e^{\int pdx} = e^{\int dx} = e^x$

Now, multiplying (ii) by I.F. and then taking integration both sides, so that (ii) becomes,

$$\boxed{u \times \text{I.F.} = \int Q \times \text{I.F.}\, dx + c}$$

$$u \times e^x = \int e^x \times e^x\, dx + c = \int e^{2x}\, dx + c$$

$$\Rightarrow e^y e^x = \frac{e^{2x}}{2} + c \Rightarrow e^{x+y} = \frac{e^{2x}}{2} + c.$$

14. **$y(2xy + e^x)\, dx = e^x\, dy$**

Solution: Given that, $y(2xy + e^x)\, dx = e^x\, dy$

$$\Rightarrow \frac{2xy^2 + ye^x}{e^x} = \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = 2xy^2e^{-x} + y$$

$$\Rightarrow \frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y} = 2xe^{-x} \quad \text{...... (i)}$$

Put, $u = -\frac{1}{y} = -y^{-1}$. Then, $\frac{du}{dx} = \frac{1}{y^2}\frac{dy}{dx}$. So, the equation (i) becomes,

$$\frac{du}{dx} + u = 2xe^{-x} \quad \text{...... (ii)}$$

This is linear differential equation of first order in u.

Comparing (ii) with the equation $\frac{dy}{dx} + py = Q$ then we get,

$$P = 1 \quad \text{and} \quad Q = 2xe^{-x}$$

Then, I.F. $= e^{\int pdx} = e^{\int dx} = e^x$

Now, multiplying (ii) by I.F. and then taking integration both sides, so that (ii) becomes,

$$\boxed{u \times \text{I.F.} = \int Q \times \text{I.F.}\, dx + c}$$

$$u \times e^x = \int 2xe^{-x} \times e^x\, dx + c$$

$$\Rightarrow u \times e^x = 2\int x\, dx + c$$

$$\Rightarrow \frac{-1}{y} e^x = x^2 + c \Rightarrow e^x = -y(x^2 + c).$$

15. **$\tan y\, y' + \tan x = \cos y \cdot \cos^2 x$**

Solution: Given that, $\tan y\, y' + \tan x = \cos y \cdot \cos^2 x$

$$\Rightarrow \frac{\tan y}{\cos y}\frac{dy}{dx} + \frac{\tan x}{\cos y} = \cos^2 x$$

$$\Rightarrow \sec y \cdot \tan y \frac{dy}{dx} + \sec y \tan x = \cos^2 x \quad \text{...... (i)}$$

Put, $u = \sec y$. Then, $\frac{du}{dx} = \sec y \cdot \tan y \frac{dy}{dx}$. So, the equation (i) becomes,

$$\frac{du}{dx} + u \tan x = \cos^2 x \quad \text{...... (ii)}$$

This is linear differential equation of first order in u.

Comparing (ii) with the equation $\frac{dy}{dx} + py = Q$ then we get,

$$P = \tan x, \quad Q = \cos^2 x$$

Then, I.F. $= e^{\int pdx} = e^{\int \tan x\, dx} = e^{\log \sec x} = \sec x$

Now, multiplying (ii) by I.F. and then taking integration both sides, so that (ii) becomes,

$$\boxed{u \times \text{I.F.} = \int Q \times \text{I.F.}\, dx + c}$$

$$u \times \sec x = \int \cos^2 x \sec x\, dx + c$$

$$\Rightarrow u \sec x = \int \cos x\, dx + c \Rightarrow \sec y \sec x = \sin x + c$$

$$\Rightarrow \sec y = (\sin x + c)\cos x.$$

16. **$y' + y \tan x = y^3 \cos x$**

Solution: Given that, $y' + y \tan x = y^3 \cos x$

$$\Rightarrow \frac{1}{y^3}\frac{dy}{dx} + \frac{1}{y^2}\tan x = \cos x \quad \text{...... (i)}$$

Put, $u = \frac{1}{y^2} = y^{-2}$. Then, $\frac{du}{dx} = -2y^{-3}\frac{dy}{dx} \Rightarrow -\frac{1}{2}\frac{du}{dx} = \frac{1}{y^3}\frac{dy}{dx}$.

So, the equation (i) becomes,

$$-\frac{1}{2}\frac{du}{dx} + u \tan x = \cos x$$

$$\Rightarrow \frac{du}{dx} - 2u \tan x = -2\cos x \quad \text{....... (ii)}$$

This is linear differential equation of first order in u.

Comparing (ii) with the equation $\frac{dy}{dx} + py = Q$ then we get,

$$P = -2 \tan x, \qquad Q = -2 \cos x$$

Then, I.F. $= e^{\int p dx} = e^{-2\int \tan x} = e^{-2 \log \sec x} = e^{\log(\sec x)^{-2}} = (\sec x)^{-2} = \frac{1}{\sec^2 x}$.

Now, multiplying (ii) by I.F. and then taking integration both sides, so that (ii) becomes,

$$\boxed{u \times \text{I.F.} = \int Q \times \text{I.F. } dx + c}$$ note

$$u \times \frac{1}{\sec^2 x} = \int -2 \cos x \times \frac{1}{\sec^2 x} dx + c$$

$$\Rightarrow \quad u \cos^2 x = -2 \int \cos^3 x \, dx + c$$

$$\Rightarrow \quad u \cos^2 x = -2 \int \left(\frac{\cos 3x + 3 \cos x}{4}\right) dx + c$$

$$\Rightarrow \quad u \cos^2 x = -\frac{1}{2}\left(\frac{\sin 3x}{3} + 3 \sin x\right) + c$$

$$\Rightarrow \quad \frac{1}{y^2} \cos^2 x = -\frac{1}{2} \frac{(\sin 3x + 9 \sin x)}{3} + c$$

$$\Rightarrow \quad \cos^2 x = -y^2 \left\{\frac{1}{2} \frac{(\sin 3x + 9 \sin x)}{2} - c\right\}.$$

17. $(x^3y^2 + xy)\, dx = dy$

Solution: Given that, $(x^3y^2 + xy)\, dx = dy \Rightarrow \frac{dy}{dx} - xy = x^3y^2$

$$\Rightarrow \frac{1}{y^2}\frac{dy}{dx} - \frac{x}{y} = x^3$$

Put, $u = \frac{1}{-y} = -y^{-1}$. Then, $\frac{du}{dx} = \frac{1}{y^2}\frac{dy}{dx}$. So, the equation (i) becomes,

$$\frac{du}{dx} + ux = x^3 \qquad \ldots\ldots \text{(ii)}$$

This is linear differential equation of first order in u.

Comparing (ii) with the equation $\frac{dy}{dx} + py = Q$ then we get,

$$P = x, \qquad Q = x^3$$

Then, I.F. $= e^{\int p dx} = e^{\int x dx} = e^{x^2/2}$

Now, multiplying (ii) by I.F. and then taking integration both sides, so that (ii) becomes,

$$\boxed{u \times \text{I.F.} = \int Q \times \text{I.F.}\ dx + c}$$

$$u \times e^{x^2/2} = \int x^2 \times e^{x^2/2}\ dx + c$$

$$\Rightarrow \quad u \times e^{x^2/2} = \int x^2 \times x \times e^{x^2/2}\ dx + c$$

Put, $t = \frac{x^2}{2}$ then $dt = xdx$. So,

$$u \times e^{x^2/2} = \int 2\ t\ e^t\ dt + c$$

$$= \int 2\ t\ e^t\ dt + c \ = 2[te^t - e^t] + c \ = 2\left[\frac{x^2}{2} e^{x^2/2} - e\ e^{x^2/2}\right] + c$$

$$\Rightarrow \quad u \times e^{x^2/2} = x^2\ e^{x^2/2} - 2e^{x^2/2} + c \ \Rightarrow \quad u = x^2 - 2 + c\ e^{x^2/2}$$

$$\Rightarrow \quad -\frac{1}{y} = x^2 - 2 + ce^{x^2/2}.$$

$e^{-x^2/2}$ //