EXERCISE 4.4

1. **Evaluate:** (i) $\int_0^1 \{t\vec{i} + (t^2-2t)\vec{j} + 3t^2\vec{k}\}\,dt$ (ii) $\int_0^1 \{t\vec{i} + e^t\vec{j} + e^{-2t}\vec{k}\}\,dt$

Solution: (i) Here,

$$\int_0^1 \{t\vec{i} + (t^2-2t)\vec{j} + 3t^2\vec{k}\}\,dt = \left[\frac{t^2}{2}\vec{i} + \left(\frac{t^3}{3} - t^2\right)\vec{j} + t^3\vec{k}\right]_0^1$$

$$= \frac{\vec{i}}{2} - \frac{2\vec{j}}{3} + \vec{k}$$

(ii) Here,

$$\int_0^1 \{t\vec{i} + e^t\vec{j} + e^{-2t}\vec{k}\}\,dt = \left[\frac{t^2}{2}\vec{i} + e^t\vec{j} + \frac{e^{-2t}}{-2}\vec{k}\right]_0^1$$

$$= \left(\frac{1}{2}\vec{i} + e\vec{j} - \frac{e^{-2}}{2}\vec{k}\right) - \left(\vec{j} - \frac{\vec{k}}{2}\right)$$

$$= \frac{\vec{i}}{2} + (e-1)\vec{j} + \left(\frac{1-e^{-2}}{2}\right)\vec{k}.$$

2. **Evaluate:** (i) $\int_0^2 (\vec{r}.\vec{s})\,dt$ (ii) $\int_0^2 (\vec{r}\times\vec{s})\,dt$

where, $\vec{r} = t\vec{i} - t^2\vec{j} + (t-1)\vec{k}$, $\vec{s} = 2t^2\vec{i} + 6t\vec{k}$

Solution: Let $\vec{r} = t\vec{i} - t^2\vec{j} + (t-1)\vec{k}$ and $\vec{s} = 2t^2\vec{i} + 6t\vec{k}$

Then,

$$\vec{r}.\vec{s} = (t\vec{i} - t^2\vec{j} + (t-1)\vec{k}).(2t^2\vec{i} + 0\vec{j} + 6t\vec{k})$$
$$= 2t^3 + 0 + 6t^2 - 6t$$
$$= 2t^3 + 6t^2 - 6t$$

And,

$$\vec{r}\times\vec{s} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ t & -t^2 & (t-1) \\ 2t^2 & 0 & 6t \end{vmatrix} = -6t^3\vec{i} - (2t^3 - 2t^2 - 6t^2)\vec{j} + 2t^4\vec{k}$$

$$= -t^3\vec{i} + (8t^2 - 2t^3)\vec{j} + 2t^4\vec{k}$$

Now,

(i) $$\int_0^2 \vec{r}.\vec{s}\,dt = \int_0^2 (2t^3 + 6t^2 - 6t)\,dt = \left[\frac{2t^4}{4} + \frac{6t^3}{3} - \frac{6t^2}{2}\right]_0^2$$

$$= \frac{2(2)^4}{4} + \frac{6(2)^3}{3} - \frac{6(2)^2}{2}$$

$$= 2(2)^2 + 2(2)^3 - 3(2)^2 = 8 + 16 - 12 = 12.$$

(ii) $\int_0^2 (\vec{r}\times\vec{s})\,dt = \int_0^2 \{-6t^3\,\vec{i} + (8t^2 - 2t^3)\,\vec{j} + 2t^4\,\vec{k}\}\,dt$

$$= \left[-\frac{6t^4}{4}\vec{i} + \left(\frac{8t^3}{3} - \frac{2t^4}{4}\right)\vec{j} + \frac{2t^5}{5}\vec{k}\right]_0^2$$

$$= -\frac{6(2)^4}{4}\vec{i} + \left(\frac{8(2)^3}{3} - \frac{2(2)^4}{4}\right)\vec{j} + \frac{2(2)^5}{5}\vec{k}$$

$$= -24\vec{i} + \left(\frac{64}{3} - 8\right)\vec{j} + \frac{64}{5}\vec{k} \qquad = -24\vec{i} + \frac{40}{3}\vec{j} + \frac{64}{5}\vec{k}$$

Thus, (i) $\int_0^2 \vec{r}\cdot\vec{s}\,dt = 12$ (ii) $\int_0^2 (\vec{r}\times\vec{s})\,dt = -24\vec{i} + \frac{40}{3}\vec{j} + \frac{64}{5}\vec{k}$

3. Find the value of $\vec{r}$ satisfying the equation $\frac{d^2\vec{r}}{dt^2} = 6t\,\vec{i} - 24t^2\,\vec{j} + 4\sin t\,\vec{k}$ given that $\vec{r} = 2\vec{i} + \vec{j}$ and $\frac{d\vec{r}}{dt} = -\vec{i} - 3\vec{k}$ at t = 0.

Solution: Here, $\frac{d^2\vec{r}}{dt^2} = 6t\,\vec{i} - 24t^2\,\vec{j} + 4\sin t\,\vec{k}$ ....(i)

Integrating (i) with respect to t then

$$\frac{d\vec{r}}{dt} = 3t^2\vec{i} - 8t^3\vec{j} - 4\cos t\,\vec{k} + c \qquad ....(ii)$$

At t = 0, above equation (ii) gives,

$$\left.\frac{d\vec{r}}{dt^2}\right|_{at\ t=0} = -4\vec{k} + c \qquad ....(iii)$$

Given that, at t = 0, $\frac{d\vec{r}}{dt} = -\vec{i} - 3\vec{k}$. Then (iii) gives,

$$-\vec{i} - 3\vec{k} = -4\vec{k} + c$$

$$\Rightarrow\ c = -\vec{i} + \vec{k}$$

Therefore (ii) becomes,

$$\frac{d\vec{r}}{dt} = 3t^2\,\vec{i} - 8t^3\,\vec{j} - 4\cos t\,\vec{k} - \vec{i} + \vec{k} \qquad .....(iv)$$

Integrating (iv) w. r. t. t then

$$\vec{r} = t^3\vec{i} - 2t^4\vec{j} - 4\sin t\,\vec{k} - t\,\vec{i} + t\,\vec{k} + c \qquad ....(v)$$

At, t = 0, above equation (v) gives,

$$\vec{r}\,|_{at\ t=0} = 0 \qquad ....(vi)$$

Given that $\vec{r} = 2\vec{i} + \vec{j}$ at t = 0. Then (vi) gives,

$$2\vec{i} + \vec{j} = c$$

Therefore, (v) becomes,

$$\vec{r} = t^3\vec{i} - 2t^4\vec{j} - 4\sin t\,\vec{k} - t\,\vec{i} + t\,\vec{k} + 2\,\vec{i} + \vec{j}$$

$$\Rightarrow\ \vec{r} = (t^3 - t + 2)\,\vec{i} + (1 - 2t^4)\,\vec{j} + (t - 4\sin t)\,\vec{k}.$$

4. Let $\vec{a} = 12\cos 2t\,\vec{i} - 8\sin 2t\,\vec{j} + 16\,t\,\vec{k}$ be the acceleration of a particle at any time t. find the velocity $\vec{v}$ and displacement $\vec{r}$ at any time t and given $\vec{r} = \vec{0} = \vec{v}$ at t = 0.

Solution: Let $\vec{a} = 12\cos 2t\,\vec{i} - 8\sin 2t\,\vec{j} + 16t\,\vec{k}$ be acceleration at any time t. So,

$$\frac{d^2\vec{r}}{dt^2} = \vec{a} = 12\cos 2t\,\vec{i} - 8\sin 2t\,\vec{j} + 16t\,\vec{k} \qquad ....(i)$$

Also, given that, $\vec{r} = 0 = \vec{v}$ at t = 0.

Since $\vec{v}$ be the velocity at any time t. Therefore,

$$\vec{r} = 0 = \frac{d\vec{r}}{dt} \qquad \text{at } t = 0 \qquad ....(ii)$$

Integrating (i) w.r.t. t then

$$\frac{d\vec{r}}{dt} = 6\sin 2t\,\vec{i} + 4\cos 2t\,\vec{j} + 8t^2\,\vec{k} + c \qquad ......(iii)$$

at t = 0, (iii) gives,

$$0 = 0 + 4j + c\ \Rightarrow\ c = 4j \qquad [\because \text{using (ii)}]$$

Then (iii) becomes,

$$\frac{d\vec{r}}{dt} = 6\sin 2t\,\vec{i} + 4\cos 2t\,\vec{j} + 8t^2\,\vec{k} - 4j \qquad ....(iv)$$

Again integrating (iv) w.r.t. t then,

$$\vec{r} = -3\cos 2t\,\vec{i} - 2\sin 2t\,\vec{j} + \frac{8t^3}{3}\vec{k} - 4t\,\vec{j} + c \qquad ....(v)$$

At t = 0, (v) gives,

$$0 = -3\,\vec{i} - 0 + 0 - 0 + c\ \Rightarrow\ c = 3\,\vec{i} \qquad [\because \text{using (ii)}]$$

Then (v) becomes,

$$\vec{r} = -3\cos 2t\,\vec{i} - 2\sin 2t\,\vec{j} + \frac{8t^3}{3}\vec{k} - 4t\,\vec{j} + 3\,\vec{i}$$

$$\Rightarrow\ \vec{r} = (3 - 3\cos 2t)\,\vec{i} - (4t + 2\sin 2t)\,\vec{j} + \frac{8t^3}{3}\vec{k} \qquad ....(vi)$$

Thus, (iv) be the velocity and (vi) be the displacement of the particle at time t.

5. If $\vec{r}\times\frac{d^2\vec{r}}{dt^2} = \vec{0}$, show that $\vec{r}\times\frac{d\vec{r}}{dt} = \vec{a}$, where $\vec{a}$ is a constant vector.

Solution: Let, $\vec{r}\times\frac{d^2\vec{r}}{dt^2} = \vec{0}$ ....(i)

Then we wish to show $\vec{r}\times\frac{d\vec{r}}{dt} = \vec{a}$, for $\vec{a}$ is a constant vector.

Let $\vec{r}\times\frac{d\vec{r}}{dt} = \vec{a}$ exists. Differentiating w. r. t. t then,

$$\frac{d}{dt}\left(\vec{r}\times\frac{d\vec{r}}{dt}\right) = \frac{d}{dt}(\vec{a})\ \Rightarrow\ \frac{d\vec{r}}{dt}\times\frac{d\vec{r}}{dt} + \vec{r}\times\frac{d^2\vec{r}}{dt^2} = \vec{0}$$

$$\Rightarrow \vec{r} \times \frac{d^2\vec{r}}{dt^2} = 0 \qquad \left[\because \frac{d\vec{r}}{dt} \times \frac{d\vec{r}}{dt} = 0\right]$$

This holds by (i). Therefore $\vec{r} \times \frac{d\vec{r}}{dt} = \vec{a}$ exists.

6. Solve $\frac{d^2\vec{r}}{dt^2} = t\vec{a} + \vec{b}$, where $\vec{a}$ and $\vec{b}$ are constant vectors, given that $t = 0$, $\vec{r} = 0$ and $\frac{d\vec{r}}{dt} = \vec{u}$

Solution: Let, $\frac{d^2\vec{r}}{dt^2} = t\vec{a} + \vec{b}$ ....(i)

for $\vec{a}$ and $\vec{b}$ are constant vectors.

Also given that, $\vec{r} = 0$ and $\frac{d\vec{r}}{dt} = \vec{u}$; at $t = 0$ ....(ii)

Here integrating (i) w.r.t. t then

$$\frac{d\vec{r}}{dt} = \vec{a}\frac{t^2}{2} + \vec{b}t + \vec{c} \qquad \text{..........(iii)}$$

At t = 0, using (ii) the equation (iii) gives,

$$\vec{u} = \vec{a}.0 + \vec{b}.0 + \vec{c} \Rightarrow \vec{c} = \vec{u}$$

Therefore (ii) becomes,

$$\frac{d\vec{r}}{dt} = \frac{\vec{a}}{2}t^2 + \vec{b}t + \vec{u} \qquad \text{..........(iv)}$$

Again integrating (iv) w.r.t. t then,

$$\vec{r} = \frac{\vec{a}}{2}\cdot\frac{t^3}{3} + \vec{b}\frac{t^2}{2} + \vec{u}t + \vec{c} \qquad \text{.........(v)}$$

At t = 0, using (ii), the equation (v) gives

$$0 = \frac{\vec{a}}{2}.0 + \vec{b}.0 + \vec{u}.0 + \vec{c} \Rightarrow \vec{c} = 0$$

Therefore (v) becomes,

$$\vec{r} = \frac{\vec{a}}{6}t^3 + \frac{\vec{b}}{2}t^2 + \vec{u}t)$$

This is the solution of given equation.

EXERCISE 4.5

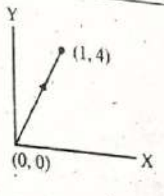
A. Calculate $\int_c \vec{F}.d\vec{r}$ for the following data. (If $\vec{F}$ is a force, this gives the work of the displacement along C).

1. $\vec{F} = (y^2, -x^2)$, C is the straight line from (0, 0) to (1, 4)

Solution: Given that, $\vec{F} = (y^2, -x^2)$ .... (i)

and the path is a straight line from (0, 0) to (1, 4).

Here, the equation of path C is



$$\boxed{y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)}$$

$$y - 0 = \frac{4 - 0}{1 - 0}(x - 0) \Rightarrow y = 4x$$

Put x = t then y = 4t. Also, t moves from t = 0 to t = 1.

Then, $\vec{r} = x\vec{i} + y\vec{j} \Rightarrow \vec{r} = t\vec{i} + 4t\vec{j}$

So, differentiating we get, $d\vec{r} = dt\vec{i} + 4dt\vec{j}$

So that,

$$\vec{F}.d\vec{r} = ((4t)^2\vec{i} - t^2\vec{j}).(dt\vec{i} + 4dt\vec{j}) = 16t^2dt - 4t^2dt = 12t^2dt.$$

Now,

$$\int_c \vec{F}.d\vec{r} = \int_0^1 12t^2 dt = \{4t^3\}_0^1 = 4.$$

2. $\vec{F} = (xy, x^2y^2)$, C is the quarter circle from (2, 0) to (0, 2) with center (0, 0).

Solution: Given that, $\vec{F} = (xy, x^2y^2) = xy\vec{i} + x^2y^2\vec{j}$

and the path of force $\vec{F}$ is the quarter circle from (2, 0) to (0, 2) with centre at (0, 0). Clearly, the length of (0, 0) to (2, 0) is 2. So, radius of the circle is 2. Therefore, the equation of path of curve is, $x^2 + y^2 = 4$.

Put x = t then $y = \sqrt{4 - t^2}$. Also, t moves from t = 2 to t = 0.

Then,

$$\vec{r} = x\vec{i} + y\vec{j} \Rightarrow \vec{r} = t\vec{i} + \vec{j}\sqrt{4 - t^2}$$

So, $d\vec{r} = dt\vec{i} - \frac{tdt}{\sqrt{4 - t^2}}\vec{j}$

So that,

$$\vec{F}.d\vec{r} = t\sqrt{4 - t^2}\vec{i} + t^2(4 - t^2)\vec{j} = t\sqrt{4 - t^2}\vec{i} + (4t^2 - t^4)\vec{j}$$

Now,

$$\int_c \vec{F}.d\vec{r} = \int_2^0 \left\{\left(t\sqrt{4 - t^2}\vec{i} + (4t^2 - t^4)\vec{j}\right).\left(dt\vec{i} - \frac{tdt}{\sqrt{4 - t^2}}\vec{j}\right)\right\}$$

$$= \int_2^0 \left(t\sqrt{4 - t^2} - \frac{t(4t^2 - t^4)}{\sqrt{4 - t^2}}\right)dt = \int_2^0 \{t\sqrt{4 - t^2} - t^3\sqrt{4 - t^2}\}\, dt$$

Put $4 - t^2 = u^2$ then $-2tdt = 2u\,du \Rightarrow tdt = -u\,du$. Also, $t = 2 \Rightarrow u = 0$ and $t = 0 \Rightarrow u = 2$. So,

$$\int_c \vec{F}.d\vec{r} = \int_2^0 \{-u^2 + u^2(4 - u^2)\}\, du$$

$$= \int_2^0 (3u^2 - u^4)du = \left[u^3 - \frac{u^5}{5}\right]_0^2 = 8 - \frac{32}{5} = \frac{40-32}{5} = \frac{8}{5}$$

3. $\vec{F} = [(x-y)^2, (y-x)^2]$, **C; $xy = 1$, $1 \le x \le 4$.**

Solution: Let, $\vec{F} = ((x-y)^2, (y-x)^2)$ and C: $xy = 1$ for $1 \le x \le 4$

Then, $\vec{r} = x\vec{i} + 4\vec{j} = x\vec{i} + \frac{1}{x}\vec{j}$. So, $d\vec{r} = dx\,\vec{i} - \frac{1}{x^2}dx\,\vec{j}$

So that,

$$\vec{F}.d\vec{r} = \{(x-y)^2\vec{i} + (y-x)^2\vec{j}\}.\left\{dx\,\vec{i} - \frac{1}{x^2}dx\,\vec{j}\right\}$$
$$= \left\{\left(x - \frac{1}{x}\right)^2\vec{i} + \left(\frac{1}{x} - x\right)^2\vec{j}\right\}.\left\{i - \frac{1}{x^2}\vec{j}\right\}dx$$
$$= \left\{\left(x - \frac{1}{x}\right)^2 - \frac{1}{x^2}\left(\frac{1}{x} - x\right)^2\right\}dx = \left\{x^2 - 2 + \frac{1}{x^2} - \frac{1}{x^4} + \frac{2}{x^2} - 1\right\}dx$$
$$= (x^2 - 3 + 3x^{-2} - x^{-4})\,dx.$$

Now,

$$\int_c \vec{F}.d\vec{r} = \int_1^4 (x^2 - 3 + 3x^{-2} - x^{-4})\,dx$$
$$= \left[\frac{x^3}{3} - 3x + \frac{3x^{-1}}{-1} - \frac{x^{-3}}{-3}\right]_1^4 = \left(\frac{64}{3} - 12 - \frac{3}{4} + \frac{1}{192}\right) - \left(\frac{1}{3} - 3 - 3 + \frac{1}{3}\right)$$
$$= \frac{63}{3} - 12 - \frac{3}{4} + \frac{1}{192} + 6 - \frac{1}{3}$$
$$= 21 - 6 - \frac{144 - 1 + 64}{192}$$
$$= 15 - \frac{207}{192} = \frac{2673}{192} = \frac{891}{64}.$$

4. $\vec{F} = (2z, x, -y)$, **C; $\vec{r} = (\cos t, \sin t, 2t)$ from $(0, 0, 0)$ to $(1, 0, 4\pi)$.**

Solution: Let, $\vec{F} = (2z, x, -y) = 2z\,\vec{i} + x\,\vec{j} - y\vec{k}$

and c: $\vec{r} = (\cos t, \sin t, 2t)$ from $(0, 0, 0)$ to $(1, 0, 4\pi)$.

Here,

$$\vec{r} = \cos t\,\vec{i} + \sin t\,\vec{j} + 2t\,\vec{k}$$

So, $d\vec{r} = (-\sin t\,\vec{i} + \cos t\,\vec{j} + 2\vec{k})\,dt$

and,

$$\vec{F} = 2(2t)\,\vec{i} + \cos t\,\vec{j} - \sin t\,\vec{k}$$
$$= 4t\,\vec{i} + \cos t\,\vec{j} - \sin t\,\vec{k}$$

Then

$$\vec{F}.d\vec{r} = (4t\,\vec{i} + \cos t\,\vec{j} - \sin t\,\vec{k}).(-\sin t\,\vec{i} + \cos t\,\vec{j} + 2\vec{k})\,dt$$
$$= (-4t\sin t + \cos^2 t - 2\sin t)\,dt$$
$$= \left(-4\sin t + \frac{1 + \cos 2t}{2} - 2\sin t\right)dt$$

Since the particle moves from $(0, 0, 0)$ to $(1, 0, 4\pi)$ along the curve.

So, $z = 0$ and $z = 4\pi$. i.e. $2t = 0$ and $2t = 4\pi$ $\Rightarrow t = 0$ ad $t = 2\pi$

Now,

$$\int_c F.d\vec{r} = \int_c \left(-4t\sin t + \frac{1 + \cos 2t}{2} - 2\sin t\right)dt$$
$$= \left[4t\cos t - 4\sin t + \frac{t}{2} + \frac{\sin 2t}{4} + 2\cos t\right]_0^{2\pi}$$
$$= 8\pi + \pi + 2 - 2 \qquad [\because \cos 2\pi = 1, \sin 2\pi = 0]$$
$$= 9\pi.$$

5. $\vec{F} = (e^x, e^{-y}, e^z)$, **C; $\vec{r} = (t, t^2, t)$ from $(0, 0, 0)$ to $(1, 1, 1)$.**

[2003 Spring Q.No. 4(a)]

Solution: Similar to Q. 4.

B. **Calculate $\int_c f\,ds$,**

6. **$f = x^2 + y^2$, C: $y = 3x$ from $(0, 0)$ to $(2, 6)$.**

Solution: Let, $f = x^2 + y^2$

and given that the path of integration is C: $y = 3x$ from $(0, 0)$ to $(2, 6)$.

Put $x = t$ then $y = 3t$. Also $x = 0 \Rightarrow t = 0$ and $x = 2 \Rightarrow t = 2$

And, f = t2 + (3t)2 = 10t2

Since, $\vec{r} = x\,\vec{i} + y\,\vec{j} = t\,\vec{i} + 3t\,\vec{j} = (\vec{i} + 3\vec{j})\,t$. So, $d\vec{r} = (\vec{i} + 3\vec{j})\,dt$

Since we have,

$$\frac{ds}{dt} = \sqrt{\frac{d\vec{r}}{dt}.\frac{d\vec{r}}{dt}} = \sqrt{(\vec{i} + 3\vec{j}).(\vec{i} + 3\vec{j})} = \sqrt{1 + 9} = \sqrt{10}$$

Now,

$$\int_c f\,ds = \int_0^2 f(t)\frac{ds}{dt}dt = \int_0^2 10t^2\sqrt{10}\,dt = \left[\frac{10t^3\sqrt{10}}{3}\right]_0^2 = \frac{80\sqrt{10}}{3}.$$

7. **$f = x^2 + y^2 + z^2$, C: $(\cos t, \sin t, 2t)$, $0 \le t \le 4\pi$.**

Solution: Given that, $f = x^2 + y^2 + z^2$

and the path of integration is C: $(\cos t, \sin t, 2t)$ for $0 \le t \le 4\pi$.

This shows that, $z = 2t$, $x = \cos t$ and $y = \sin t$.

So, $f = \cos^2 t + \sin^2 t + 4t^2 = 1 + 4t^2$

Since, $\vec{r} = x\,\vec{i} + y\,\vec{j} + z\vec{k}$ $\Rightarrow$ $\vec{r} = \cos t\,\vec{i} + \sin t\,\vec{j} + 2t\,\vec{k}$

So, $\frac{d\vec{r}}{dt} = (-\sin t\,\vec{i} + \cos t\,\vec{j} + 2\vec{k})$

Since, $\frac{ds}{dt} = \sqrt{\frac{d\vec{r}}{dt}.\frac{d\vec{r}}{dt}}$

$$= \sqrt{(-\sin t\ \vec{i} + \cos t\ \vec{j} + 2\vec{k}).(-\sin t\ \vec{i} + \cos t\ \vec{j} + 2\vec{k})}$$

$$= \sqrt{\sin^2 t + \cos^2 t + 4} \quad = \sqrt{1+4} = \sqrt{5}.$$

Now,

$$\int_c f.\,ds = \int_c f.\frac{ds}{dt}.\,dt = \int_0^{4\pi} (1+4t^2).\sqrt{5}\,dt$$

$$= \sqrt{5}\left[t + \frac{4t^3}{3}\right]_0^{4\pi} = \sqrt{5}\left(4\pi + \frac{256\pi^3}{3}\right).$$

8. $f = 1 + y^2 + z^2$, C: $\vec{r} = (t, \cos t, \sin t)$, $0 \le t \le \pi$.

Solution: Similar to 7.

9. $f = x^2 + (xy)^{1/3}$, C is the hypocycloid $\vec{r} = (\cos^3 t, \sin^3 t)$, $0 \le t \le \pi$.

Solution: Let, $f = x^2 + (xy)^{1/3}$

and the path of integrand is c : $\vec{r} = (\cos^3 t, \sin^3 t)$ for $0 \le t \le \pi$.

Here,

$$\vec{r} + x\vec{i} + y\vec{j} = \cos^3 t\,\vec{i} + \sin^3 t\,\vec{j}$$

So, $$\frac{d\vec{r}}{dt} = -3\cos^2 t \sin t\ \vec{i} + 3\sin^2 t \cos t\ \vec{j}$$

And, $f = x^2 + (xy)^{1/3} = \cos^6 t + \sin t \cos t$

Since, we have

$$\frac{ds}{dt} = \sqrt{\frac{d\vec{r}}{dt}\cdot\frac{d\vec{r}}{dt}} = \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t}$$

$$= \sqrt{9\cos^2 t \sin^2 t(\cos^2 t + \sin^2 t)} = 3\cos t \sin t$$

So that,

$$f.\frac{ds}{dt} = (\cos^6 t + \sin t \cos t)\ 3\cos t \sin t$$

$$= 3\cos^7 t \sin t + 3\sin^2 t \cos^2 t$$

$$= 3\cos^7 t \sin t + \frac{3}{4}\sin^2 2t \qquad [\because \sin 2A = 2\sin A \cos A]$$

$$= 3\cos^7 t \sin t + \frac{3}{4}\left(\frac{1-\cos 4t}{2}\right) = 3\cos^7 t \sin t + \frac{3}{8}(1-\cos 4t).$$

Now,

$$\int_c f\,ds = \int_c f\frac{ds}{dt}.\,dt = 3\int_0^{\pi}\cos^7 t \sin t\,dt + \frac{3}{8}\int_0^{\pi}(1-\cos 4t)\,dt$$

Put $\cos t = u$ then $-\sin t\,dt = du$. Also, $t = 0 \Rightarrow u = 1$ and $t = \pi \Rightarrow u = -1$. Therefore,

$$\int_c f\,ds = -3\int_1^{-1} u^7\,du + \frac{3}{8}\left[t - \frac{\sin 4t}{4}\right]_0^{\pi} = -3\left[\frac{u^8}{8}\right]_1^{-1} + \frac{3}{8}\pi \quad [\because \sin 4\pi = 0 = \sin 0]$$

$$= -\frac{3}{8}(1-1) + \frac{3\pi}{8} = \frac{3\pi}{8}$$

C. Show that $\int_c \vec{F}\,d\vec{r} = 3\pi$, given that $\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$ and C being the arc of the curve $\vec{r} = \cos t\ \vec{i} + \sin t\ \vec{j} + t\vec{k}$ from $t = 0$ to $t = 2\pi$.

Solution: Given that, $\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$

and c be the arc of $\vec{r} = \cos t\ \vec{i} + \sin t\ \vec{j} + t\vec{k}$ for $t = 0$ to $t = 2\pi$.

Since we have,

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \quad \Rightarrow \quad \vec{r} = \cos t\ \vec{i} + \sin t\ \vec{j} + t\vec{k}$$

So, $x = \cos t$, $y = \sin t$ and $z = t$.

Therefore, $\vec{F} = t\vec{i} + \cos t\ \vec{j} + \sin t\ \vec{k}$.

Then,

$$\vec{F}.d\vec{r} = (t\vec{i} + \cos t\ \vec{j} + \sin t\ \vec{k}).\{d(\cos t\ \vec{i} + \sin t\ \vec{j} + t\vec{k})\}$$

$$= (t\vec{i} + \cos t\ \vec{j} + \sin t\ \vec{k}).(-\sin t\ \vec{i} + \cos t\ \vec{j} + \vec{k})$$

$$= -t\sin t + \cos^2 t + \sin t$$

$$= -t\sin t + \frac{1+\cos 2t}{2} + \sin t$$

Now,

$$\int_c \vec{F}.d\vec{r} = \int_0^{2\pi}\left(-t\sin t + \frac{1+\cos 2t}{2} + \sin t\right)dt$$

$$= \left[(-t)(-\cos t) - (-1)(-\sin t) + \frac{t}{2} + \frac{\sin 2t}{4} - \cos t\right]_0^{2\pi}$$

$$= (2\pi + 0 + \pi + 0 - 1) - (0 - 0 + 0 + 0 - 1) \quad [\because \cos 2\pi = 1, \sin 2\pi = 0]$$

$$= 3\pi - 1 + 1 = 3\pi$$

This shows that $\int_c \vec{F}.d\vec{r} = 3\pi$.

D. Find the work done by the force $\vec{F} = (2y+3)\ \vec{i} + xz\ \vec{j} + (yz - x)\ \vec{k}$ when it moves a particle from the point (0, 0, 0) to the point (2, 1, 1) along the curve $x = 2t^2$, $y = t$, $z = t^3$. **[2008 Spring Q.No. 3(b)]**

Solution: Given that, $\vec{F} = (2y+3)\vec{i} + xz\vec{j} + (yz - x)\vec{k}$ ....(i)

that moves from (0, 0, 0) to (2, 1, 1) along $x = 2t^2$, $y = t$, $z = t^3$.

Then,

$$\vec{F} = (2t+3)\vec{i} + 2t^5\vec{j} + (t^4 - 2t^2)\vec{k}$$

Since, $y = t$ and $y = 0$, $y = 1$. So, $t = 0$ and $t = 1$.

We know that,

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \quad \Rightarrow \quad \vec{r} = 2t^2\vec{i} + t\vec{j} + t^3\vec{k}$$

Then,

$$d\vec{r} = (4t\vec{i} + \vec{j} + 3t^2\vec{k})\,dt$$

So that,

$$\vec{F}.d\vec{r} = ((2t+3)\vec{i} + 2t^5\vec{j} + (t^4 - 2t^2)\vec{k}).(2t\vec{i} + \vec{j} + 3t^2\vec{k})\,dt$$
$$= (8t^2 + 12t + 2t^5 + 3t^6 - 6t^4)\,dt$$

We have, the work done by the force $\vec{F}$ along the curve C: $\vec{r}$ is $\int_C \vec{F}.d\vec{r}$.

Now,

$$\int_C \vec{F}.d\vec{r} = \int_0^1 (3t^6 + 2t^5 - 6t^4 + 8t^2 + 12t)\,dt$$
$$= \left[\frac{3t^7}{7} + \frac{2t^6}{6} - \frac{6t^5}{5} + \frac{8t^3}{3} + \frac{12t^2}{2}\right]_0^1$$
$$= \frac{3}{7} + \frac{2}{6} - \frac{6}{5} + \frac{8}{3} + \frac{12}{2}$$
$$= \frac{3}{7} + \frac{1}{3} - \frac{6}{5} + \frac{8}{3} + 6$$
$$= \frac{3}{7} - \frac{6}{5} + 3 + 6 = \frac{15-42}{35} + 9 = \frac{-27+315}{35} = \frac{288}{35}.$$

Thus, the work done by F is $\frac{288}{35}$.

E. Find the work done in moving a particle in the force field $\vec{F} = 3x^2\vec{i} + (2xz - y)\vec{j} + z\vec{k}$ along the curve defined by $x^2 = 4y$, $3x^3 = 8z$ from $x = 0$ to $x = 2$.

[2010 Spring Q.No. 6(a) OR]

Solution: Given that,

$$\vec{F} = 3x^2\vec{i} + (2xz - y)\vec{j} + z\vec{k} \quad ....(i)$$

This force moves along the curve

$$x^2 = 4y,\ 3x^3 = 8z \quad \text{for } 0 \le x \le 2 \quad ....(ii)$$
$$\Rightarrow y = \frac{x^2}{4},\ z = \frac{3x^3}{8}.$$

Then (i) becomes,

$$\vec{F} = 3x^2\vec{i} + \left(2x.\frac{3x^3}{8} - \frac{x^2}{4}\right)\vec{j} + \frac{3x^3}{8}\vec{k}$$
$$= 3x^2\vec{i} + \frac{3x^4 - x^2}{4}\vec{j} + \frac{3x^2}{8}\vec{k}$$

Since we know that,

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = x\vec{i} + \frac{x^2}{4}\vec{j} + \frac{3x^3}{8}\vec{k}$$

Then,

$$d\vec{r} = \left(\vec{i} + \frac{x}{2}\vec{j} + \frac{9x^2}{8}\vec{k}\right)dx$$

So that,

$$\vec{F}.d\vec{r} = \left(3x^2\vec{i} + \left(\frac{3x^4 - x^2}{4}\right)\vec{j} + \left(\frac{3x^2}{8}\right)\vec{k}\right).\left(\vec{i} + \frac{x}{2}\vec{j} + \frac{9x^2}{8}\vec{k}\right)dx$$
$$= \left(3x^2 + \frac{3x^5}{8} - \frac{x^3}{8} + \frac{27x^5}{64}\right)dx$$

Now, work done by force $\vec{F}$ along the curve (ii) is,

$$\int_C \vec{F}.d\vec{r} = \int_0^2 \left(3x^2 + \frac{3x^5}{8} - \frac{x^3}{8} + \frac{27x^5}{64}\right)dx$$
$$= \left[x^3 + \frac{3x^6}{48} - \frac{x^4}{32} + \frac{27x^6}{64 \times 6}\right]_0^2$$
$$= \left[x^3 + \frac{x^6}{16} - \frac{x^4}{32} + \frac{9x^6}{128}\right]_0^2$$
$$= 8 + \frac{64}{16} - \frac{16}{32} + \frac{9 \times 64}{128} = 8 + 4 - \frac{1}{2} + \frac{9}{2} = 16.$$

Thus, the work–done by $\vec{F}$ is 16.

F. Evaluate $\int_C \vec{F}d\vec{r}$

i. where $\vec{F} = x^2y^2\vec{i} + y\vec{j}$ and the curve C is $y^2 = 4x$ in the xy plane from (0, 0) to (4, 4).

Solution: Given that the force is,

$$\vec{F} = x^2y^2\vec{i} + y\vec{j} \quad .... (i)$$

And the curve c is, $y^2 = 4x$ .... (ii)

Then, $\vec{F} = \left(\frac{y^2}{4}\right)y^2\vec{i} + y\vec{j} = \frac{y^4}{4}\vec{i} + y\vec{j}$

Since, $\vec{r} = x\vec{i} + y\vec{j} = \frac{y^2}{4}\vec{i} + y\vec{j}$

Then, $d\vec{r} = \left(\frac{y}{2}\vec{i} + \vec{j}\right)dy$

Therefore,

$$\vec{F}.d\vec{r} = \left(\frac{y^2}{4}\vec{i} + y\vec{j}\right).\left(\frac{y}{2}\vec{i} + \vec{j}\right)dy$$
$$= \left(\frac{y^3}{8} + y\right)dy = \left(\frac{y^3 + 8y}{8}\right)dy.$$

Given that the force $\vec{F}$ moves along the curve c from (0, 0) to (4, 4). Then,

$$\int_C \vec{F}.d\vec{r} = \int_0^4 \left(\frac{y^3 + 8y}{8}\right)dy$$
$$= \frac{1}{8}\left[\frac{y^4}{4} + 4y^2\right]_0^4$$
$$= \frac{1}{8}\left[\frac{(4)^4}{4} + 4.(4)^2\right] = \frac{1}{8}[64 + 64] = \frac{128}{8} = 16$$

Thus, $\int_C \vec{F}.d\vec{r} = 16.$

(ii) $\vec{F} = (x^2 + y^2)\vec{i} + (x^2 - y^2)\vec{j}$ **and c is the curve** $y^2 = x$ **joining (0, 0) and (1, 1)**

Solution: Similar to (i).

(iii) $\vec{F} = \cos y\,\vec{i} - x \sin y\,\vec{j}$ **and c is the curve** $y = \sqrt{1 - x^2}$ **in the xy plane from (1, 0) to (0, 1).**

Solution: Let, $\vec{F} = \cos y\,\vec{i} - x\sin y\,\vec{j}$.

And it moves along $y = \sqrt{1 - x^2}$ which is a half range circle having radius $r = 1$.

Since, $\vec{r} = x\vec{i} + y\vec{j}$. So, $d\vec{r} = dx\,\vec{i} + dy\,\vec{j}$.

Then,

$$\int_c \vec{F}.d\vec{r} = \int_c (\cos y\,\vec{i} - x\sin\vec{j}).(dx\,\vec{i} + dy\,\vec{j})$$

$$= \int_c (\cos y\, dx - x\sin y\, dy)$$

$$= \int_c d(x\cos y)$$

$$= \int_1^0 d(x\cos\sqrt{1-x^2}) = \left[x\cos\sqrt{1-x^2}\right]_1^0 = -1$$

(iv) $\vec{F} = \sin y\,\vec{i} + x(1 + \cos y)\,\vec{j}$ **and c is the curve** $x^2 + y^2 = a^2, z = 0$.

Solution: Given that $\vec{F} = \sin y\,\vec{i} + x(1 + \cos y)\,\vec{j}$ .... (i)

That moves along c: $x^2 + y^2 = a^2$, $z = 0$

Since we have, $\vec{r} = \vec{i} + \vec{j}$

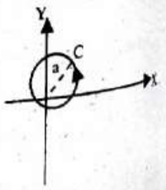
So, $d\vec{r} = dx\,\vec{i} + dy\,\vec{j}$

Then $\vec{F}.d\vec{r} = (\sin y\,\vec{i} + x(1 + \cos y\,\vec{j})).(dx\,\vec{i} + dy\,\vec{j})$

$= \sin y\, dx + x(1 + \cos y)\, dy$

Now,

$$\int_c \vec{F}.d\vec{r} = \int_c [d(x\sin y) + x\,dy]$$



Since the path is circular curve. So, its parametric form is,

$x = a\cos t$, $y = a\sin t$

And t varies from $t = 0$ to $t = 2\pi$.

Therefore,

$$\int_c \vec{F}.d\vec{r} = \int_0^{2\pi} [d\,(a\cos t\sin(a\sin t)) + a\cos t\; a\, d(\sin t)$$

$$= a\,[\cos t\sin(a\sin t)]_0^{2\pi} + a^2\int_0^{2\pi}\cos t\cos t\,dt$$

$$= a\,[\cos 2\pi\sin(a\sin 2\pi) - \cos 0\sin(a\sin 0) + a^2\int_0^{2\pi}\left(\frac{1+\cos 2t}{2}\right)dt$$

$$= 0 + \frac{a^2}{2}\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi}$$

$$= \frac{a^2}{2}\times 2\pi \qquad [\because \sin 2\pi = 0 = \sin 0]$$

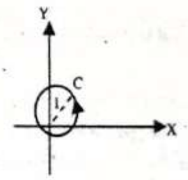
$$= a^2\pi.$$

(v) $\vec{F} = -\frac{y}{x^2+y^2}\vec{i} + \frac{x}{x^2+y^2}\vec{j}$, **where c is the circle** $x^2 + y^2 = 1$ **in the z–plane described in the anticlockwise direction.**

Solution: Given that,

$$\vec{F} = \int_c \left(\frac{-y\vec{i} + x\vec{j}}{x^2+y^2}\right)(dx\,\vec{i} + dy\,\vec{j})$$

$$= \int_c \left(\frac{1}{x^2+y^2}\right)(-y\,dx + x\,dy)$$



Since the path is a circular path with radius $r = 1$. So, its parametric form is,

$x = \cos t$, $y = \sin t$

And the variable t varies from $t = 0$ to $t = 2\pi$. Therefore,

$$\int_c F.d\vec{r} = \int_0^{2\pi}\left(\frac{1}{\cos^2 t + \sin^2 t}\right)[-\sin t\; d(\cos t) + \cos t\; d(\sin t)]$$

$$= \int_0^{2\pi}(\sin^2 t + \cos^2 t)dt = \int_0^{2\pi} dt = [t]_0^{2\pi} = 2\pi.$$

(vi) $\vec{F} = (2x - y + z)\vec{i} + (x + y - z^2)\vec{j} + (3x - 2y + 4z)\vec{k}$, **around the circle** $x^2 + y^2 = a^2, z = 0$.

Solution: Similar to (iv).

(vii) $\vec{F} = yz\,\vec{i} + (xz + 1)\vec{j} + xy\,\vec{k}$, **and c is the any path from (1, 0, 0) to (2, 1, 4)**

Solution: Given that,

$$\vec{F} = yz\,\vec{i} + (xz + 1)\vec{j} + xy\,\vec{k}.$$

And the path is from (1, 0, 0) to (2, 1, 4).

Here, $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

Now,

$$\int_0^{2\pi} \vec{F}\cdot d\vec{r} = \int_{(1,0,0)}^{(2,1,4)} (yz\,\vec{i} + (xz+1)\,\vec{j} + xy\,\vec{k})\cdot(dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k})$$

$$= \int_{(1,0,0)}^{(2,1,4)} [yz\,dx + (xz+1)\,dy + xy\,dz]$$

$$= \int_{(1,0,0)}^{(2,1,4)} d(xyz) + \int_{(1,0,0)}^{(2,1,4)} dy = [xyz]_{(1,0,0)}^{(2,1,4)} + [y]_{(1,0,0)}^{(2,1,4)}$$

$$= [2.1.4 - 1.0.0] + (1-0)$$

$$= 8 + 1$$

$$= 9.$$

G. Find $\int_c \vec{F}\cdot d\vec{r}$ where, $\vec{F} = y^2\vec{i} + 2xy\,\vec{j}$ from O(0, 0) to P(1, 1) in each of the following cases: **a. along the straight line OP.**

b. along the parabola $y^2 = x$.

c. along the x–axis from $x = 0$ to $x = 1$ and then along the line $x = 1$, from $y = 0$ to $y = 1$.

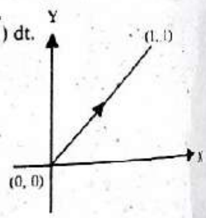
Solution: Here, $\vec{F} = y^2\vec{i} + 2xy\,\vec{j}$ and applied from O(0, 0) to P(1, 1).

(a) The path is a straight line OP. Here, O(0, 0) and P(1, 1). So, the equation of straight line is $x = y$.

Set, $x = t$ then $y = t$. Also, t moves from $t = 0$ to $t = 1$.

Since, $\vec{r} = x\vec{i} + y\vec{j} = t\vec{i} + t\vec{j}$. So, $d\vec{r} = (\vec{i} + \vec{j})\,dt$.

Now,



$$\int_c \vec{F}\cdot d\vec{r} = \int_c (y^2\vec{i} + 2xy\,\vec{j})\,d\vec{r}$$

$$= \int_0^1 (t^2\vec{i} + 2t^2\vec{j})\cdot(\vec{i} + \vec{j})dt$$

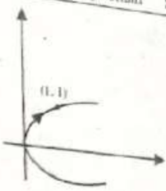
$$= \int_0^1 (t^2 + 2t^2)\,dt = \int_0^1 (3t^2)\,dt = [t^3]_0^1 = 1.$$

(b)

The path is a parabola $y^2 = x$ from O(0, 0) to P(1, 1).

Set, $y = t$ then $x = t^2$. So that t moves from $t = 0$ to $t = 1$.

Now,



$$\int_c \vec{F}\cdot d\vec{r} = \int_0^1 (y^2\vec{i} + 2xy\,\vec{j})\cdot(dx\,\vec{i} + dy\,\vec{j})$$

$$= \int_0^1 (y^2\vec{i} + 2xy\,\vec{j})\cdot(2t\,\vec{i} + \vec{j})dt$$

$$= \int_0^1 (2t^3 + 2t^3)\,dt$$

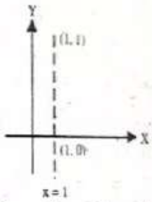
$$= \int_0^1 (4t^3)\,dt = [t^4]_0^1 = 1.$$

(c) Given that the path of the force $\vec{F}$ is along x–axis from $x = 0$ to $x = 1$ and then along the line $x = 1$ from $y = 0$ to $y = 1$.

When the force moves along x–axis, $y = 0$. So, $dy = 0$

When the force moves along $x = 1$, $x = 1$. So, $dx = 0$

Now,



$$\int_c \vec{F}\cdot d\vec{r} = \int_{x=0}^{1} \vec{F}\cdot d\vec{r} + \int_{y=0}^{1} \vec{F}\cdot d\vec{r}$$

$$= \int_{x=0}^{1} (y^2\vec{i} + 2xy\,\vec{j})\cdot(dx\,\vec{i} + dy\,\vec{j}) + \int_{y=0}^{1} (y^2\vec{i} + 2xy\,\vec{j})\cdot(dx\,\vec{i} + dy\,\vec{j})$$

$$= 0 + \int_{y=0}^{1} (y^2\vec{i} + 2xy\,\vec{j})\cdot(0\,\vec{i} + dy\,\vec{j})$$

$$= \int_{y=0}^{1} 2y\,dy = [y^2]_0^1 = 1.$$

H. If $\vec{F} = (2xy - z)\vec{i} + yz\,\vec{j} + x\vec{k}$ evaluate $\int_c \vec{F}\cdot d\vec{r}$ along the curve c, where

a. c is the curve $x = t$, $y = 2t$, $z = t^2 - 1$, with t increasing from 0 to 1.

b. c consists of two straight line from the origin to the point (1, 0, –1) and from (1, 0, –1) to the point (2, 3, –3).

Solution: Given that, $\vec{F} = (2xy - z)\,\vec{i} + yz\,\vec{j} + x\,\vec{k}$

(a) And the path of $\vec{F}$ is, $x = t$, $y = 2t$, $z = t^2 - 1$. When t moves from $t = 0$ to $t = 1$.

Since, $\vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k} = t\,\vec{i} + 2t\,\vec{j} + (t^2 - 1)\,\vec{k}$

So, $d\vec{r} = dt\,\vec{i} + 2dt\,\vec{j} + 2t\,dt\,\vec{k} = (\vec{i} + 2\vec{j} + 2t\,\vec{k})\,dt.$

Then,

$$\vec{F}.d\vec{r} = [(2(t)(2t) - (t^2-1))\vec{i} + (2t(t^2-1))\vec{j} + t\vec{k}].(\vec{i} + 2\vec{j} + 2t\vec{k})\,dt$$
$$= [(4t^2 - t^2 + 1)\vec{i} + (2t^3 - 2t)\vec{j} + t\vec{k}].(\vec{i} + 2\vec{j} + 2t\vec{k})\,dt$$
$$= (4t^2 - t^2 + 1 + 4t^3 - 4t + 2t^2)\,dt$$
$$= (4t^3 + 5t^2 - 4t + 1)\,dt$$
$$= \int_0^1 (4t^3 + 5t^2 - 4t + 1)\,dt = \left[t^4 + \frac{5t^3}{3} - 2t^2 + t\right]_0^1$$
$$= 1 + \frac{5}{3} - 2 + 1 = \frac{5}{3}.$$

(b) Let C_1 be the line segment from (0, 0, 0) to (1, 0, −1). So, equation of C_1 is

$$\frac{x-1}{1-0} = \frac{y-0}{0-0} = \frac{z+1}{-1-0} = t \text{ (say)}$$

i.e. $x = t + 1,\ y = 0,\ z = -t - 1$.

So, $dx = dt,\ dy = 0,\ dz = -dt$.

Since $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$. Then $d\vec{r} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$.

So that,

$$\vec{F}.d\vec{r} = ((2xy - z)\vec{i} + yz\vec{j} + x\vec{k}).(dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k})$$
$$= (2xy - z)dx + yzdy + xdz$$

Here the integral along C_1 is,

$$\int_{C_1} \vec{F}.d\vec{r} = \int_{C_1} ((2xy - z)dx + yzdy + xdz)$$

$$= \int_{C_1} [(t+1)dt + (t+1)(-dt)] = \int_{C_1} (t + 1 - t - 1)\,dt = \int_{C_1} 0\,dt = 0$$

Also, C_2 be the line segment from (1, 0, −1) to (2, 3, −3).

The equation of line C_2 is,

$$\frac{x-1}{2-1} = \frac{y-0}{3-0} = \frac{z+1}{-3+1} = 4$$

i.e. $x = 4 + 1,\ y = 3u,\ z = -2u - 1$

Then $dx = du$, $dy = 3du$ and $dz = -2du$.

Here,

$$\vec{F}.d\vec{r} = (2xy - z, yz, x).(dx, dy, dz)$$
$$= (2(u+1)(3u) + 2u + 1, 3u(-2u-1), u+1)(du, 3du, -2du)$$
$$= \{(6u^2 + 6u + 2u + 1, -6u^2 - 3u, u + 1).(1, 3, -2)\}\,du$$
$$= (6u^2 + 8u + 1 - 18u^2 - 9u - 2u - 2)\,du$$
$$= (-12u^2 - 3u - 1)\,du$$

Also, y moves from 0 to 3. So, y = 3u gives u moves from u = 0 to u = 1 along C_2.

So,

$$\int_{C_1} \vec{F}.d\vec{r} = \int_0^1 (-12u^2 - 3u - 1)\,du$$
$$= \left[-4u^3 - \frac{3u^2}{2} - u\right]_0^1 = -4 - \frac{3}{2} - 1 = -\frac{13}{2}$$

Now, given that C consists two lines C_1 and C_2. So,

$$\int_C \vec{F}.d\vec{r} = \int_{C_1} \vec{F}.d\vec{r} + \int_{C_2} \vec{F}.d\vec{r} = 0 - \frac{13}{2} = -\frac{13}{2}$$

1. **Evaluate the line integral along c**

a. $\int_c (6x^2y\,dx + xy\,dy)$, **where C is the graph of $y = x^3 + 1$ from (−1, 0) to (1, 2).**

Solution: Given integral is,

$$I = \int_c (6x^2 y dx + xy\,dy)$$

And the path of integration is c that moves along $y = x^3 + 1$ from (−1, 0) to (1, 2).

Set $x = t$ then $y = t^3 + 1$. Then t varies from t = −1 to t = 1.

Then,

$$I = \int_{-1}^{1} [6(t^2)(t^3 + 1)\,dt + t(t^3 + 1)(3t^2\,dt)]$$
$$= \left[t^6 + 2t^3 + \frac{3t^7}{7} + \frac{3t^4}{4}\right]_{-1}^{1}$$
$$= \left(1 + 2 + \frac{3}{7} + \frac{3}{4}\right) - \left(1 - 2 - \frac{3}{7} + \frac{3}{4}\right)$$
$$= 3 + \frac{3}{7} + \frac{3}{4} + 1 + \frac{3}{7} - \frac{3}{4} = 4 + \frac{6}{7} = \frac{28+6}{7} = \frac{34}{7}.$$

b. $\int_c [(x - y)\,dx + xdy]$, **where C is the graph of $y^2 = x$ from (4, −2) to (4, 2).**

Solution: Similar to (a).

1. **Evaluate** $\int_c [(xz\,dx + (y - z)\,dy + x\,dz]$, **if c is the graph of $x = e^t$, $y = e^{-t}$, $z = e^{2t}$, $0 \le t \le 1$.**

Solution: Given that,

$$I = \int_c [xz\,dx + (y + z)\,dy + x\,dz]$$

And the path of integration c is along the graph of $x = e^t$, $y = e^{-t}$, $z = e^{2t}$ for $0 \le t \le 1$.

Then, $dx = e^t\,dt$, $dy = -e^{-t}\,dt$, $dz = 2e^{2t}dt$.

Now,

$$I = \int_0^1 [e^t e^{2t} e^t\,dt + (e^{-t} + e^{2t})(-e^{-t}\,dt) + e^t\,2e^{2t}\,dt]$$

$$= \int_0^1 (e^{4t} - e^{-2t} - e^t + 2e^{3t})\,dt$$

$$= \left[\frac{e^{4t}}{4} - \frac{e^{-2t}}{-2} - \frac{e^t}{1} + \frac{2e^{3t}}{3}\right]_0^1$$

$$= \left(\frac{e^4}{4} + \frac{e^{-2}}{2} - e + \frac{2e^3}{3}\right) - \left(\frac{1}{4} + \frac{1}{2} - 1 + \frac{2}{3}\right)$$

$$= \frac{1}{12}(3e^4 + 8e^3 - 12e + 6e^{-2}) - \left(\frac{3 + 6 - 12 + 8}{12}\right)$$

$$= \frac{1}{12}(3e^4 + 8e^3 - 12e + 6e^{-2} - 5).$$

K. Evaluate $\int_c [(x + y + z)\,dx + (x - 2y + 3z)\,dy + (2x + y - z)\,dz]$, where c is the curve from (0, 0, 0) to (2, 3, 4) if

a. C consists of three line segments the first parallel to the x–axis the second parallel to the y–axis and the third parallel to z–axis.

b. C consists of three line segments the first parallel to the z–axis the second parallel to the x–axis and the third is parallel to the y–axis.

c. C is the line segments.

Solution: Given that,

$$I = \int_c [(x + y + z)dx + (x - 2y + 3z)\,dy + (2x + y - z)\,dz]$$

Where, the curve varies from (0, 0, 0) to (2, 3, 4).

(a) Given that the movement of the curve is along the line parallel to x– axis i.e. from (0, 0, 0) to (2, 0, 0), then along the line parallel to y–axis i.e. from (2, 0, 0) to (2, 3, 0) and then along the line parallel to z–axis i.e. from (2, 3, 0) to (2, 3, 4).

Therefore,

$$I = \left[\int_{(0,0,0)}^{(2,0,0)} + \int_{(2,0,0)}^{(2,3,0)} + \int_{(2,3,0)}^{(2,3,4)}\right][(x + y + z)dx + (x - 2y + 3z)dy + (2x + y - z)dz]$$

$$= \int_0^2 x\,dx + \int_0^3 (2 - 2y)\,dy + \int_0^4 (4 + 3 - z)\,dz$$

$$= \left[\frac{x^2}{2}\right]_0^2 + [2y - y^2]_0^3 + \left[7z - \frac{z^2}{2}\right]_0^4$$

$$= 2 + (6 - 9) + (28 - 8)$$

$$= 2 - 3 + 20$$

$$= 19.$$

(b) Given that the movement of the curve is along the line parallel to z–axis i.e. from (0, 0, 0) to (0, 0, 4), then along the line parallel to y–axis i.e. from (2, 0, 4) to (2, 3, 4).

Therefore,

$$I = \int_{(0,0,0)}^{(0,0,4)} [(x + y + z)dx + (x - 2y + 3z)dy + (2x + y - z)dz] +$$

$$\int_{(0,0,0)}^{(2,0,4)} [(x + y + z)dx + (x - 2y + 3z)dy + (2x + y - z)dz] +$$

$$\int_{(2,0,4)}^{(2,3,4)} [(x + y + z)dx + (x - 2y + 3z)dy + (2x + y - z)dz]$$

$$= \int_0^4 (-z)dz + \int_0^2 (x + 4)dx + \int_0^3 (2 - 2y + 12)dy$$

$$= \left[\frac{-z^2}{2}\right]_0^4 + \left[\frac{x^2}{2} + 4x\right] + [14y - y^2]_0^3$$

$$= -8 + (2 + 8) + (42 - 9)$$

$$= 2 + 33$$

$$= 35.$$

(c) Given that the movement of the path curve is along the line segments.

Since c moves fro, (0, 0, 0) to (2, 3, 4). So, $x = 2t$, $y = 3t$, $z = 4t$ for $0 \le t \le 1$.

Now,

$$I = \int_0^1 [2t + 3t + 4t)\,2dt + (2t - 6t + 12t)\,3dt + (4t + 3t - 4t)\,4dt]$$

$$= \int_0^1 [18t + 24t + 12t]\,dt = \int_0^1 (54t)\,dt = [27t^2]_0^1 = 27.$$

L. Evaluate $\int_c (xyz)\,ds$, if c is the line segments from (0, 0, 0) to (1, 2, 3).

Solution: Given that $I = \int_c (xyz)\,dx.$

And the curve is the line segments from (0, 0, 0) to (1, 2, 3).

Set $x = t$ then $y = 2t$, $z = 3t$. Then t varies from $t = 0$ to $t = 1$.

Since the position vector of the path is

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = t\vec{i} + 2t\vec{j} + 3t\vec{k}$$

Then, $d\vec{r} = \left(\vec{i} + 2\vec{j} + 3\vec{k}\right) dt$.

Since we know that,

$$\frac{ds}{dt} = \sqrt{\frac{d\vec{r}}{dt} \cdot \frac{d\vec{r}}{dt}} = \sqrt{(\vec{i} + 2\vec{j} + 3\vec{k}).(\vec{i} + 2\vec{j} + 3\vec{k})}$$
$$= \sqrt{1 + 4 + 9} = \sqrt{14}$$

Now,

$$I = \int_0^1 t.2t.3t\sqrt{14}\, dt = 6\sqrt{14}\int_0^1 t^3\, dt$$
$$= 6\sqrt{14}\left[\frac{t^4}{4}\right]_0^1 = \frac{6\sqrt{14}}{4} = \frac{3}{2}\sqrt{14}.$$

M. If the force at (x, y) is $\vec{F} = xy^2\vec{i} + x^2y\vec{j}$ find the work done by $\vec{F}$ along the curves in (J) (c).

Solution: Since the work done by force $\vec{F}$ is $\int_c \vec{F}.d\vec{r}$.

Solution is similar to the solution J.

N. The force at a point (x, y, z) in three dimensional is given by $\vec{F} = y\vec{i} + z\vec{j} + x\vec{k}$. Find the work done by $\vec{F}$ along the twisted cubic $x = t$, $y = t^2$, $z = t^3$ from (0, 0, 0) to (2, 4, 8).

Solution: Given that, $\vec{F} = y\vec{i} + z\vec{j} + x\vec{k}$.

And the force $\vec{F}$ works along $x = t$, $y = t^2$, $z = t^3$ from (0, 0, 0) to (2, 4, 8). Thus, t varies from $t = 0$ to $t = 2$. Also, we have the position vector of the curve is,

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = (t\vec{i} + t^2\vec{j} + t^3\vec{k})\, dt$$

Now, the work done by $\vec{F}$ is,

$$\int_c \vec{F}.d\vec{r} = \int_0^2 (t^2\vec{i} + t^3\vec{j} + t\vec{k})(t^2\vec{i} + 2t\vec{j} + 3t^2\vec{k})$$
$$= \int_0^2 (t^2 + 2t^4 + 3t^3)\, dt = \left[\frac{t^3}{3} + \frac{2t^5}{5} + \frac{3t^4}{4}\right]_0^2$$
$$= \frac{8}{3} + \frac{64}{5} + \frac{48}{4}$$
$$= \frac{8}{3} + \frac{64}{5} + 12 = \frac{40 + 192 + 180}{15} = \frac{412}{15}$$

Thus, the work done by $\vec{F}$ along the given curve is $\frac{412}{15}$.

O. Show that following vectors are conservative field

(i) $\vec{F} = \text{Cosy}\,\vec{i} - x\,\text{Siny}\,\vec{j} - \text{Cosz}\,\vec{k}$

(ii) $\vec{F} = (y + \sin z)\,\vec{i} + x\,\vec{j} + x\cos z\,\vec{k}$ (iii) $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$

(iv) $\vec{F} = (2xy^2 + yz)\,\vec{i} + (2x^2y + xz + 2yz^2)\,\vec{j} + (2y^2z + xy)\,\vec{k}$

Solution: Given that,

$$\vec{F} = \text{Cosy}\,\vec{i} - x\,\text{Siny}\,\vec{j} - \text{Cosz}\,\vec{k}$$

Then, $\vec{F}$ is conservative is curl $\vec{F} = 0$

Here,

$$\text{Curl}\,\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ \text{cosy} & -x\,\text{Siny} & -\text{cosz} \end{vmatrix}$$
$$= 0\,\vec{i} - 0\,\vec{j} + (-\text{Siny} + \text{Siny})\,\vec{k} = 0.$$

This shows that $\vec{F}$ is Conservative.

Solution: (ii) – (iv) – Similar to (i).

P. Show that following vectors are conservative and find ϕ such that $\vec{F} = \nabla\phi$.

i) $\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$ ii) $\vec{F} = yz\vec{i} + xz\vec{j} + xy\vec{k}$

iii) $\vec{F} = (x^2 - yz)\,\vec{i} + (y^2 - zx)\,\vec{j} + (z^2 - xy)\,\vec{k}$

Solution: Given that, $\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$

Then, $\vec{F}$ is conservative if curl $\vec{F} = 0$.

Here,

$$\text{Curl}\,\vec{F} = \nabla.\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ x & y & z \end{vmatrix} = 0\vec{i} + 0\vec{j} + 0\vec{k} = 0$$

This shows that $\vec{F}$ is conservative.

Then, $\vec{F}$ can be written as $\vec{F} = \nabla\varnothing$. So,

$$\nabla\varnothing.d\vec{r} = \left(\frac{\delta\varnothing}{\delta x}\vec{i} + \frac{\delta\varnothing}{\delta y}\vec{j} + \frac{\delta\varnothing}{\delta y}\vec{j}\right).(dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k})$$
$$= \frac{\delta\varnothing}{\delta x}dx + \frac{\delta\varnothing}{\delta y}dy + \frac{\delta\varnothing}{\delta y}dz = d\varnothing.$$

$$\Rightarrow\ d\varnothing = \vec{F}.d\vec{r} = (x\vec{i} + y\vec{j} + z\vec{k})(dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k})$$
$$= xdx + ydy + zdz$$

Integrating we get,

$$\varnothing = \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} + c \quad\Rightarrow\quad \varnothing = \frac{1}{2}(x^2 + y^2 + z^2) + c$$

Solution: (ii) – (iii) – Similar to (i).

Q. Show that the vector $\vec{F} = (y \sin z - \sin x)\vec{i} + (x \sin z + 2yz)\vec{j} + (xy \cos z + y^2)\vec{k}$ is irrotational and find a function and such that $\vec{F} = \nabla\phi$.

Solution: Given that,

$$\vec{F} = (y \text{ Sin}z - \text{Sin}x)\vec{i} + (x \text{ Sin}z + 2yz)\vec{j} + (xy \text{ Cos}z + y^2)\vec{k}$$

Then, $\vec{F}$ is Irrotational if curl $\vec{F} = 0$.

Here,

$$\text{Curl } \vec{F} = \nabla\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ y \text{ Sin}z - \text{Sin}x & x \text{ Sin}z + 2yz & xy \text{ Cos}z + y^2 \end{vmatrix}.$$

$$= (x \text{ Cos}z + 2y - x \text{ Cos}z - 2y)\vec{i} - (y \text{ Cos}z - y \text{ Cos}z)\vec{j} + (\text{Sin}z - \text{Sin}z)\vec{k}$$

$$= 0\vec{i} + 0\vec{j} + 0\vec{k} = \vec{0}$$

This shows that $\vec{F}$ is irroratational.

Then we can write as $\vec{F} = \nabla\varnothing$. So,

$$\vec{F}.d\vec{r} = \nabla\varnothing.d\vec{r}$$

$$\Rightarrow \vec{F}.(dx\vec{i} + dy\vec{j} + dz\vec{k}) = \left(\frac{\delta\varnothing}{\delta x}\vec{i} + \frac{\delta\varnothing}{\delta y}\vec{j} + \frac{\delta\varnothing}{\delta y}\vec{j}\right)(dx\vec{i} + dy\vec{j} + dz\vec{k})$$

$$\Rightarrow (y \text{ Sin}z - \text{Sin}x)dx + (x\text{Sin}z + 2yz)\,dy + (xy \text{ Cos}z + y^2)\,dz = \frac{\delta\varnothing}{\delta x}dx + \frac{\delta\varnothing}{\delta y}dy + \frac{\delta\varnothing}{\delta y}dz = d\varnothing.$$

$$\Rightarrow d\varnothing = (y \text{ Sin}z\, dx + x \text{ Sin}z\, dy + xy \text{ Cos}z\, dz) - \text{Sin}x\, dx + (2yz + y^2dz)$$

$$= d(xy \text{ Sin}z) + d(\text{Cos}x) + d(y^2z)$$

$$= d(xy \text{ Sin}z + \text{Cos}x + y^2z).$$

Integrating we get,

$$\varnothing = xy \text{ Sin}z + \text{Cos}x + y^2z + C.$$

For Exercise 4.6

Process to make the value under integral sign as under differentiation. If the integral is of type, $I = \int_a^b (F_1dx + F_2dy + F_3dz)$

And if the integral is exact. Then,

$$I = \int_a^b d\left[\int F_1dx + \int(\text{terms free from } x \text{ in } F_2)\,dy + \int(\text{terms free from } x \text{ and } y \text{ in } F_3)\,dz\right]$$