Exercise 6.10

Find the general solution of the following differential equation

(1) $\mathbf{y''' - 2y'' - y' + 2y = 0.}$

Solution: Given that, $y''' - y'' - y' + 2y = 0$ ……. (i)

Its auxiliary equation is

$$m^3 - 2m^2 - m + 2 = 0$$

$$\Rightarrow \quad m^2(m-2) - 1(m-2) = 0$$

$$\Rightarrow \quad (m-2)(m^2-1) = 0$$

$$\Rightarrow \quad (m-2)(m-1)(m+1) = 0$$

$$\Rightarrow \quad m = 2, 1, -1$$

So the general solution of (i) is,

$$y(x) = c_1e^{2x} + c_2e^{x} + c_3e^{-x}$$

(2) $\mathbf{y''' - y' = 0}$

Solution: Given that, $y''' - y' = 0$ ……. (i)

Its auxiliary equation is $m^3 - m = 0$

$$m(m^2 - 1) = 0 \Rightarrow m = 0, 1, -1$$

So, the general solution of (i) is

$$y(x0 = c_1 + c_2e^x + c_3e^{-x}$$

(3) $y^{iv} - 5y'' + 4y = 0$

Solution: Given that, $y^{iv} - 5y'' + 4y = 0$ .... (i)

Its auxiliary equation is

$$m^4 - 5m^2 + = 0 \Rightarrow m^4 - 4m^2 - m^2 + 4 = 0$$

$$\Rightarrow m^2(m^2 - 4) - 1(m^2 - 4) = 0$$

$$\Rightarrow (m^2 - 4)(m^2 - 1) = 0$$

$$\Rightarrow (m - 2)(m + 2)(m - 1)(m + 1) = 0$$

$$\Rightarrow m = 2, -2, 2, -1$$

So, the general solution of (i) is,

$$y(x) = c_1e^{2x} + c_2e^{-2x} + c_3e^x + c_4e^{-4}$$

(4) $(d^4 + 4)y = 0$

Solution: Given that, $\frac{d^4y}{dx^4} + 4y = 0$ ...... (i)

Its auxiliary equation is

$$m^4 + 4 = 0 \quad \Rightarrow (m^2)^2 + (2)^2 = 0$$

$$\Rightarrow m^2 + 2)^2 - 2.m^2.2 = 0$$

$$\Rightarrow (m^2 + 2)^2 - (2m)^2 = 0$$

$$\Rightarrow (m^2 + 2m + 2)(m^2 - 2m + 2) = 0$$

for $m^2 + 2m + 2 = 0$

$$m = \frac{-2 \pm \sqrt{4 - 8}}{2} = \frac{-2 \pm \sqrt{4i}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

for $m^2 - 2m + 2 = 0$

$$m = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm \sqrt{4i}}{2} = \frac{2 + \pm 2i}{2} = 1 \pm i.$$

So, the general solution of (i) is,

$$y(x) = e^{-x}(A \cos x + D \sin x) + e^x (C \cos x + D \sin x).$$

(5) $(D^3 + 6d^2 + 11D + 6) = y = 0$

Solution: Given that, $\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} + 6y = 0$ ...... (i)

Its auxiliary equation is

$$m^3 + 6m^2 + 11m + 6 = 0$$
$$\Rightarrow \quad m^3 + m^2 + 5m^2 + 5m + 6m + 6 = 0$$
$$\Rightarrow \quad m^2(m + 1) + 5m(m + 1) + 6(m + 1) = 0$$
$$\Rightarrow \quad (m + 1)(m^2 + 5m + 6) = 0$$
$$\Rightarrow \quad (m + 1)(m^2 + 2m + 3m + 6) = 0$$
$$\Rightarrow \quad (m + 1)\{m(m + 2) + 3(m, + 2)\} = 0$$
$$\Rightarrow \quad (m + 1)(m + 2)(m + 3) = 0$$
$$\Rightarrow \quad m = -1, -2, -3$$

So the general solution of (i) is,

$$y(x) = c_1e^{-x} + c_2e^{-2x} + c_3e^{-3x}$$

(6) $y''' + 3y'' + 3y' - y = 0$

Solution: Given that, $y''' + 3y'' + 3y' - y = 0$ ...... (i)

Its auxiliary equation is

$$m^3 - 3m^2 + 3m - 1 = 0 \Rightarrow m^3 - m^2 - 2m^2 + 2m + m - 1 = 0$$
$$\Rightarrow m^2(m - 1) - 2m(m - 1) + 1(m - 1) = 0$$
$$\Rightarrow (m - 1)(m^2 - 2m + 1) = 0$$
$$\Rightarrow (m - 1)(m - 1)^2 = 0$$
$$\Rightarrow (m - 1)^3 = 0$$
$$\Rightarrow m = 1, 1, 1$$

So the general solution of (i) is,

$$y(x) = (c_1 + c_2x + c_3x^2)e^x$$

(7) $y^{iv} + 8y'' + 16y = 0$

Solution: Given that, $y^{iv} + 8y'' + 16y = 0$ .... (i)

Its auxiliary equation is,

$$m^4 + 8m^2 + 16 = 0 \quad \Rightarrow \quad (m^2)^2 + 2.m^2.4 + (4)^2 = 0$$
$$\Rightarrow \quad (m^2 + 4)^2 = 0$$

for $m^2 + 4 = 0 \Rightarrow m^2 = -4 \Rightarrow m^2 = (\pm 2i)^2 \Rightarrow m = 0 \pm 2i$

So the general solution of (i) is,

$$y(x) = (c_1 + c_2x)\cos 2x + (c_3 + c_4x)\sin 2x$$

(8) If $\dfrac{d^4x}{dt^4} = m^4x$

Solution: Given that, $x^{iv} - m^4x = 0.$ ...... (i)

Its auxiliary equations is,

$$u^4 - m^4 = 0$$
$$\Rightarrow \quad (u^2)^2 - (m^2)^2 = 0$$
$$\Rightarrow \quad (u^2 + m^2)(u^2 - m^2) = 0$$
$$\Rightarrow \quad u^2 + m^2 = 0 \qquad \text{and} \qquad u^2 - m^2 = 0$$
$$\Rightarrow \quad u^2 = (im)^2, \qquad \Rightarrow u^2 = m^2$$
$$\Rightarrow \quad u = \pm mi, \qquad \Rightarrow u = \pm m$$

So the general solution of (i) is,

$$x = c_1e^{mt} + c_2e^{-mt} + c_3\cos mt + c_4\sin mt$$

(9) $y''' - 7y' - 6y = 0$

Solution: Given that, $y''' - 7y' - 6y = 0$ .... (i)

Its auxiliary equation is,

$$m^3 - 7m - 6 = 0 \Rightarrow m^3 + m^2 - m^2 - m - 6m - 6 = 0$$
$$\Rightarrow m^2(m + 1) - m(m + 1) - 6(m + 1) = 0$$
$$\Rightarrow (m + 1)(m^2 - m - 6) = 0$$
$$\Rightarrow (m + 1))(m^2 - 3m + 2m - 6) = 0$$
$$\Rightarrow (m + 1)\{m(m-3) + 2(m - 3)\} = 0$$
$$\Rightarrow (m + 1)(m - 3)(m + 1) = 0$$
$$\Rightarrow m = -1, 3, -2$$

So, the general solution of (i) is,

$$y(x) = c_1e^{-x} + c_2e^{3x} + c_3e^{-2x}$$

(10) $y''' - 4y'' + 4y' = 0$

Solution: Given that, $y''' - 4y'' + 4y' = 0$ .... (i)

Its auxiliary equation is

$$m^3 - 4m^2 + 4m = 0 \quad \Rightarrow \quad m(m^2 - 4m + 4) = 0$$
$$\Rightarrow \quad m(m - 2)^2 = 0$$
$$\Rightarrow \quad m = 0, 2, 2$$

So the general solution of (i) is,

$$y(x) = c_1 + (c_2 + c_3x)e^{2x}$$