Exercise 8.2

1. Prove the following results using theorems.

(i) $\mathcal{L}\{t \cos wt\} = \frac{s^2 - w^2}{(s^2 + w^2)^2}$

Solution: Since we have,

$$\mathcal{L}\{t\, f(t)\} = -F'(s) \quad \text{where } F(s) = \mathcal{L}\{f(t)\}$$

$$\text{and } \mathcal{L}\{\cos wt\} = \frac{s}{s^2 + w^2}$$

Now,

$$\mathcal{L}\{t \cos wt\} = -\frac{d}{ds}(\mathcal{L}\{\cos wt\})$$

$$= -\frac{d}{ds}\left(\frac{s}{s^2 + w^2}\right) = -\frac{s^2 + w^2 - 2s.s.}{(s^2 + w^2)^2} = \frac{s^2 - w^2}{(s^2 + w^2)^2}.$$

Thus, $\mathcal{L}\{t \cos wt\} = \frac{s^2 - w^2}{(s^2 + w^2)^2}$

(ii) $\mathcal{L}\{t \sin wt\} = \frac{2ws}{(s^2 + w^2)^2}$

Solution: Since we have,

$$\mathcal{L}\{t\, f(t)\} = -F'(s) \quad \text{where } F(s) = \mathcal{L}\{f(t)\}$$

$$\text{and } \mathcal{L}\{\sin wt\} = \left(\frac{w}{s^2 + w^2}\right)$$

Now,

$$\mathcal{L}\{t \sin wt\} = -\frac{d}{ds}(\mathcal{L}\{\sin wt\})$$

$$= -\frac{d}{ds}\left(\frac{w}{s^2 + w^2}\right) = -\frac{0 - w.\,2s}{(s^2 + w^2)^2} = \frac{2ws}{(s^2 + w^2)^2}.$$

Thus, $\mathcal{L}\{t \sin wt\} = \frac{2ws}{(s^2 + w^2)^2}$

(iii) $\mathcal{L}\{t \cosh at\} = \frac{s^2 + a^2}{(s^2 - a^2)^2}$.

Solution: Since we have,

$$\mathcal{L}\{t\, f(t)\} = -F'(s) = -\frac{d}{ds}(\mathcal{L}\{f(t)\}) \quad \text{and, } \mathcal{L}\{\cosh at\} = \frac{s}{s^2 - a^2}$$

Now,

$$\mathcal{L}\{t \cosh at\} = -\frac{d}{ds}(\mathcal{L}\{\cosh at\})$$

$$= -\frac{d}{ds}\left(\frac{s}{s^2 - a^2}\right)$$

$$= -\frac{s^2 - a^2 - s.\,2s}{(s^2 - a^2)^2} = \frac{s^2 + a^2}{(s^2 - a^2)^2}$$

Thus, $\mathcal{L}\{t \cosh at\} = \frac{s^2 + a^2}{(s^2 - a^2)^2}$.

(iv) $\mathcal{L}\{t \sinh at\} = \frac{2as}{(s^2 - a^2)^2}$. [2000 – Short]

Solution: Since we have,

$$\mathcal{L}\{t\, f(t)\} = -F'(s) = -\frac{d}{ds}(\mathcal{L}\{f(t)\})$$

$$\text{and,} \quad \mathcal{L}\{\sinh at\} = \frac{a}{s^2 - a^2}$$

Now,

$$\mathcal{L}\{t \sinh at\} = -\frac{d}{ds}(\mathcal{L}\{\sinh at\})$$

$$= -\frac{d}{ds}\left(\frac{a}{s^2 - a^2}\right) = -\frac{0 - a.\,2s}{(s^2 - a^2)^2} = \frac{2as}{(s^2 - a^2)^2}$$

Thus, $\mathcal{L}\{t \sinh at\} = \frac{2as}{(s^2 - a^2)^2}$.

(v) $\mathcal{L}^{-1}\left\{\frac{1}{(s^2 + w^2)^2}\right\} = \frac{1}{2w^3}(\sin wt - wt \cos wt)$

Solution: Since we have,

$$\mathcal{L}\{t\, f(t)\} = -F'(s) = -\frac{d}{ds}(\mathcal{L}\{f(t)\});$$

$$\mathcal{L}\{\sin wt\} = \frac{w}{s^2 + w^2} \quad \text{and } \mathcal{L}\{\cos wt\} = \frac{s}{s^2 + w^2}$$

Now,

$$\mathcal{L}\left\{\frac{1}{2w^3}(\sin wt - wt \cos wt)\right\}$$

$$= \frac{1}{2w^3}[\mathcal{L}\{\sin wt\} - w\,\mathcal{L}\{t\cos wt\}]$$

$$= \frac{1}{2w^3}\left[\frac{w}{s^2 + w^2} - w\left(-\frac{d}{ds}(L\{\cos wt\})\right)\right]$$

$$= \frac{w}{2w^3}\left[\frac{1}{s^2 + w^2} + \frac{d}{ds}\left(\frac{s}{s^2 + w^2}\right)\right]$$

$$= \frac{1}{2w^2}\left[\frac{1}{s^2 + w^2} + \frac{s^2 + w^2 - s.\,2s}{(s^2 + w^2)^2}\right]$$

$$= \frac{1}{2w^2}\left[\frac{1}{s^2 + w^2} + \frac{w^2 - s^2}{(s^2 + w^2)^2}\right]$$

$$= \frac{1}{2w^2}\left[\frac{s^2 + w^2 + w^2 - s^2}{(s^2 + w^2)^2}\right] = \frac{1}{2w^2}\left[\frac{2w^2}{(s^2 + w^2)^2}\right] = \frac{1}{s^2 + w^2}^2$$

$$\Rightarrow \quad \frac{1}{s^2 + w^2}^2 = \mathcal{L}\left\{\frac{1}{2w^3}(\sin wt - wt \cos wt)\right\}$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{1}{(s^2+w^2)^2}\right\} = \frac{1}{2w^2}(\sin wt - wt\cos wt)$$

(vi) $\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+w^2)^2}\right\} = \frac{1}{2w}(\sin wt + \text{cot coswt})$

Solution: Since we have,

$$\mathcal{L}\{t f(t)\} = -F'(s) = -\frac{d}{ds}(\mathcal{L}\{f(t)\})$$

$$\mathcal{L}\{\sin wt\} = \frac{w}{s^2+w^2} \quad \text{and} \quad \mathcal{L}\{\cos wt\} = \frac{s}{s^2+w^2}$$

Then,

$$\mathcal{L}\left\{\frac{1}{2w}(\sin wt + \text{cot coswt})\right\}$$

$$= \frac{1}{2w}[\mathcal{L}\{\sin wt\} + w\,\mathcal{L}\{t\cos wt\}]$$

$$= \frac{1}{2w}\left[\frac{w}{s^2+w^2} + w.\left(\frac{s^2-w^2}{(s^2+w^2)^2}\right)\right]$$

$$= \frac{w}{2w}\left(\frac{s^2+w^2+s^2-w^2}{(s^2+w^2)^2}\right) = \frac{1}{2}\left(\frac{2s^2}{(s^2+w^2)^2}\right) = \frac{s^2}{(s^2+w^2)^2}$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+w^2)^2}\right\} = \frac{1}{2w}(\sin wt + \text{cot coswt})$$

(vii) $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+w^2)^2}\right\} = \frac{t\sin wt}{2w}$

Solution: Since we have,

$$\mathcal{L}\{t f(t)\} = -F'(s) = -\frac{d}{ds}(\mathcal{L}\{f(t)\}) \quad \text{and} \quad \mathcal{L}\{\sin wt\} = \frac{w}{s^2+w^2}$$

Now,

$$\mathcal{L}\left\{\frac{t\sin wt}{2w}\right\} = \frac{1}{2w}\mathcal{L}\{t\sin wt\} = \frac{1}{2w}\left[-\frac{d}{ds}(L\{\sin wt\})\right]$$

$$= \frac{1}{2w}\left[-\frac{d}{ds}\left(\frac{w}{s^2+w^2}\right)\right]$$

$$= \frac{1}{2w}\left[-\left(\frac{0-w.2s}{(s^2+w^2)^2}\right)\right]$$

$$= \frac{1}{2w}\left[\frac{2ws}{(s^2+w^2)^2}\right] = \frac{s}{(s^2+w^2)^2}$$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{s}{(s^2+w^2)^2}\right\} = \frac{t\sin wt}{2w}$$

2. Find f(t), it L{f(t)} equals the following:

(i) $\frac{1}{s^2+s}$

Solution: Let, $\mathcal{L}\{f(t)\} = \frac{1}{s^2+s} \Rightarrow f(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2+s}\right)$ ....(i)

Here,

$$\frac{1}{s^2+s} = \frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} = \frac{A(s+1)+Bs}{s(s+1)} \quad \text{... (ii)}$$

$$\Rightarrow 1 = s(A+B) + A$$

Equating coefficient of s and the constant term on both sides then we get,

$$A + B = 0 \quad \text{and} \quad A = 1.$$

This gives, $A = 1$ and $B = -1$.

Now from (i) and (ii) becomes,

$$f(t) = \mathcal{L}^{-1}\left(\frac{1}{s} - \frac{1}{s+1}\right) = (1-e^{-t})$$

(ii) $\frac{1}{4s+s^3}$ **[2002-short]**

Solution: Let, $\mathcal{L}\{f(t)\} = \frac{1}{4s+s^3}$

$$\Rightarrow f(t) = \mathcal{L}^{-1}\left(\frac{1}{4s+s^3}\right) \quad \text{......(i)}$$

Here,

$$\frac{1}{4s+s^3} = \frac{1}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4} \quad \text{........(ii)}$$

$$\Rightarrow \frac{1}{s(s^2+4)} = \frac{A(s^2+4)+(Bs+C)s}{s(s^2+4)}$$

$$\Rightarrow 1 = As^2 + 4A + Bs^2 + Cs$$

$$\Rightarrow 1 = s^2(A+B) + Cs + 4A$$

Equating coefficient of s and the constant term on both sides then we get.

$$A + B = 0, \quad C = 0 \quad \text{and} \quad 4A = 1$$

Solving we get, $A = \frac{1}{4}, B = -\frac{1}{4}$ and $C = 0$.

Now (ii) becomes,

$$\frac{1}{4s+s^3} = \frac{1}{4s} - \frac{s}{4(s^2+4)} = \frac{1}{4}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

Therefore (i) becomes,

$$f(t) = \frac{L^{-1}}{4}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

$$\Rightarrow f(t) = \frac{1}{4}(1-\cos 2t)$$

(iii) $\frac{1}{s}\left(\frac{s-a}{s+a}\right)$

Solution: Let, $\mathcal{L}\{f(t)\} = \frac{s-a}{s(s+a)}$

$$\Rightarrow f(t) = \mathcal{L}^{-1}\left\{\frac{s-a}{s(s+a)}\right\} \quad \text{.......(i)}$$

Here, $\frac{s-a}{s(s+a)} = \frac{A}{s} + \frac{B}{s+a}$ ....(ii)

$$\Rightarrow \frac{s-a}{s(s+a)} = \frac{A(s+a)+Bs}{s(s+a)} = \frac{As+Aa+Bs}{s(s+a)}$$

$$\Rightarrow \quad s - a = s(A + B) + Aa$$

Equating coefficient of s and the constant term on both sides then we get,

$$A + B = 1, \quad Aa = -a$$

This gives, $A = -1, B = 2$.

Now, equation (i) and (ii) becomes

$$f(t) = \mathcal{L}^{-1}\left(\frac{2}{s+1} - \frac{1}{s}\right) = 2\mathcal{L}^{-1}\left(\frac{1}{s+a}\right) - \mathcal{L}^{-1}\left(\frac{1}{s}\right)$$

$$= (2e^{-at} - 1).$$

(iv) $\dfrac{8}{s^4 - 4s^2}$

Solution: Let, $\mathcal{L}\{f(t)\} = \dfrac{8}{s^4 - 4s^2} \quad \Rightarrow \quad f(t) = \mathcal{L}^{-1}\left(\dfrac{8}{s^4 - 4s^2}\right)$ ....(i)

Let,

$$\frac{8}{s^4 - 4s^2} = \frac{8}{s^2(s^2-4)} = \frac{As + B}{s^2} + \frac{Cs + D}{s^2 - 4} \quad \text{.........(ii)}$$

$$\Rightarrow \quad \frac{8}{s^4 - 4s^2} = \frac{(As + B)(s^2 - 4) + (Cs + d)s^2}{s^2(s^2-4)}$$

This gives, $8 = As^3 - 4As + Bs^2 - 4B + Cs^3 + Ds^2$

$$\Rightarrow \quad 8 = s^3(A + C) + s^2(B + D) - 4As - 4B$$

Equating coefficient of like term on both sides then we get,

$$A + C = 0, \quad B + D = 0, \quad -4A = 0, \quad -4B = 0$$

Solving we get, $A = 0, B = -2, C = 0, D = 2$.

Now, equation (i) and (ii) becomes,

$$f(t) = \mathcal{L}^{-1}\left(\frac{-2}{s^2} + \frac{2}{s^2-4}\right)$$

$$= -2\,\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) + \mathcal{L}^{-1}\left(\frac{2}{s^2 - 2^2}\right) = -2t + \sinh 2t - 2t.$$

Thus, $f(t) = -2t + \sinh 2t - 2t$.

(v) $\dfrac{1}{s^2}\left(\dfrac{s+1}{s^2+1}\right)$

Solution: Let,

$$\mathcal{L}\{F(t)\} = \frac{1}{s^2}\left(\frac{s+1}{s^2+1}\right) \quad \Rightarrow \quad f(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2}\left(\frac{s+1}{s^2+1}\right)\right) \quad \text{........ (i)}$$

Let,

$$\frac{1}{s^2}\left(\frac{s+1}{s^2+1}\right) = \frac{As+B}{s^2} + \frac{Cs+D}{s^2+1} = \frac{(As+B)(s^2+1) + (Cs+D)}{s^2(s^2+1)}$$

This implies,

$$s + 1 = As^3 + As + Bs^2 + B + Cs^3 + Ds^2$$

$$\Rightarrow \quad s + 1 = s^3(A + C) + s^2(B + D) + As + B.$$

Equating coefficient of like term on both sides then we get,

$$A + C = 0, \quad B + D = 0, \quad A = 1, \quad B = 1.$$

Solving we get,

$$A = 1, B = 1, C = -1, D = -1.$$

Then the equation (i) becomes

$$f(t) = \mathcal{L}^{-1}\left[\frac{s+1}{s^2} - \frac{s+1}{s^2+1}\right] = \mathcal{L}^{-1}\left[\frac{1}{s} + \frac{1}{s^2} - \frac{s}{s^2+1} - \frac{1}{s^2+1}\right]$$

$$= (1 + t - \cos t - \sin t).$$

Thus, $f(t) = (1 + t - \cos t - \sin t)$.

(vi) $\dfrac{1}{s^4 - 2s^3}$

Solution: Let,

$$\mathcal{L}\{f(t)\} = \frac{1}{s^3(s-2)} \quad \Rightarrow \quad f(t) = \mathcal{L}^{-1}\left(\frac{1}{s^3(s-2)}\right) \quad \text{....... (i)}$$

Let,

$$\frac{1}{s^3(s-2)} = \frac{A}{s-2} + \frac{Bs^2 + Cs + D}{s^3} \quad \text{...... (ii)}$$

$$\Rightarrow \quad \frac{1}{s^3(s-2)} = \frac{As^3 + (Bs^2 + Cs + D)(s-2)}{s^3(s-2)}$$

This gives, $1 = As^3 + Bs^3 + Cs^2 + Ds - 2Bs^2 - 2Cs - 2D$

$$\Rightarrow \quad 1 = s^3(A + B) + s^2(C - 2B) + s(D - 2C) - 2D.$$

Equating coefficient of like term on both sides then we get,

$$A + B = 0, \quad C - 2B = 0, \quad D - 2C = 0, \quad -2D = 1$$

Solving we get, $A = \frac{1}{8}, B = \frac{-1}{8}, C = \frac{-1}{4}, D = \frac{1}{2}$.

Now, equation (i) and (ii) becomes,

$$f(t) = \mathcal{L}^{-1}\left[\frac{1}{8(s-2)} + \frac{-s^2 - 2s - 4}{8s^3}\right] = \mathcal{L}^{-1}\left[\frac{1}{8(s-2)} - \frac{1}{8s} - \frac{1}{4s^2} - \frac{1}{2s^3}\right]$$

$$= \frac{1}{8}(e^{2t} - 1 - 2t - 2t^2)$$

Thus, $f(t) = \frac{1}{8}(e^{2t} - 1 - 2t - 2t^2)$.

(vii) $\dfrac{1}{s^2 + 4s}$

Solution: Let, $\mathcal{L}\{f(t)\} = \dfrac{1}{s^2 + 4s} \quad \Rightarrow \quad f(t) = \mathcal{L}^{-1}\left[\dfrac{1}{s^2+4s}\right]$ ...... (i)

Let, $\dfrac{1}{s^2 + 4s} = \dfrac{A}{s} + \dfrac{B}{s + 4s}$ ...... (ii)

$$\Rightarrow \quad \frac{1}{s^2 + 4s} = \frac{As + 4A + Bs}{s(s+4)}$$

This implies, $1 = s(s + 4) + 4A$

Equating coefficient of like term on both sides then we get,

$$A + B = 0, \quad 4A = 1,$$

Solving we get, $A = \frac{1}{4}$ and $B = \frac{-1}{4}$

Now, equation (i) and (ii) becomes

$$f(t) = \mathcal{L}^{-1}\left[\frac{1}{4s} - \frac{1}{4(s+4)}\right] = \left(\frac{1}{4}\times 1 - \frac{1}{4}e^{-4t}\right) = \frac{1}{4}(1 - e^{-4t})$$

$$\Rightarrow f(t) = \frac{1}{4}(1 - e^{-4t}).$$

(viii) $\frac{1}{s(s^2+w^2)}$

Solution: Let, $\mathcal{L}\{f(t)\} = \frac{1}{s(s^2+w^2)} \Rightarrow f(t) = \mathcal{L}^{-1}\left[\frac{1}{s(s^2+w^2)}\right]$ .... (i)

Let,

$$\frac{1}{s(s^2+w^2)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 4s} \quad \text{...... (ii)}$$

$$\Rightarrow \frac{1}{s(s^2+w^2)} = \frac{A(s^2+w^2) + (Bs + C)s}{s(s^2+w^2)}$$

This gives, $1 = As^2 + Aw^2 + Bs^2 + Cs$

$$\Rightarrow 1 = s^2(A+B) + Cs + Aw^2$$

Equating coefficient of like term on both sides then we get,

$$A + B = 0, \quad C = 0, \quad Aw^2 = 1$$

Then, $A = \frac{1}{w^2}$, $B = -\frac{1}{w^2}$ And $C = 0$.

Now equation (i) and (ii) becomes,

$$f(t) = \mathcal{L}^{-1}\left[\frac{1}{w^2 s} - \frac{s}{w^2(s^2+w^2)}\right]$$

$$= \frac{1}{w^2}L^{-1}\left(\frac{1}{s}\right) - \frac{1}{w^2}\mathcal{L}^{-1}\left[\frac{s}{s^2+w^2}\right] = \frac{1}{w^2} - \frac{1}{w^2}\cos wt = \frac{1}{w^2}(1 - \cos wt)$$

Thus, $f(t) = \frac{1}{w^2}(1 - \cos wt)$

(ix) $\frac{1}{s^3 - s}$

Solution: Let, $\mathcal{L}\{f(t)\} = \frac{1}{s^3 - s} \Rightarrow f(t) = \mathcal{L}^{-1}\left(\frac{1}{s^3 - s}\right)$ .... (i)

Let,

$$\frac{1}{s^3 - s} = \frac{1}{s(s^2-1)} = \frac{A}{s} - \frac{Bs + C}{s^2 - 1} = \frac{A(s^2-1) - (Bs + C)s}{s(s^2-1)}$$

This implies, $1 = As^2 - A - Bs^2 - Cs = s^2(A - B) - Cs - A$

Equating coefficient of like term on both sides then we get,

$$A - B = 0, \quad C = 0, \quad -A = 1$$

Solving we get, $A = -1$, $B = -1$ and $C = 0$.

Then, the equation (i) becomes,

$$f(t) = \mathcal{L}^{-1}\left[-\frac{1}{s} + \frac{s}{s^2 - 1}\right]$$

$$= -1 + \cosh t = \cosh t - 1.$$

(x) $\frac{9(s+1)}{s^2(s^2+9)}$

Solution: Let, $\mathcal{L}\{f(t)\} = \frac{9(s+1)}{s^2(s^2+9)} \Rightarrow f(t) = \mathcal{L}^{-1}\left[\frac{9(s+1)}{s^2(s^2+9)}\right]$ ..... (i)

Let,

$$\frac{9(s+1)}{s^2(s^2+9)} = \frac{As + B}{s^2} + \frac{Cs + D}{s^2 + 9} = \frac{(As + B)(s^2 + 9) + (Cs + D)s^2}{s^2(s^2+9)}$$

$$\Rightarrow \frac{9(s+1)}{s^2(s^2+9)} = \frac{As^3 + 9As + Bs^2 + 9B + Cs^3 + Ds^2}{s^2(s^2+9)}$$

This implies, $9s + 9 = s^3(A + C) + s^2(B + D) + 9As + 9B$

Equating coefficient of like term on both sides then we get,

$$A + C = 0, \quad B + D = 0, \quad 9A = 9, \quad 9B = 9.$$

Solving we get,

$$A = 1, B = 1, C = -1, D = -1.$$

Now, equation (i) becomes

$$f(t) = \mathcal{L}^{-1}\left[\frac{s+1}{s^2} - \frac{s+1}{s^2+9}\right] = \mathcal{L}^{-1}\left[\frac{1}{s} + \frac{1}{s^2} - \frac{s}{s^2+9} - \frac{1}{s^2+9}\right]$$

$$= \left(1 + t - \cos 3t - \frac{1}{3}\sin 3t\right).$$

Thus, $f(t) = \left(1 + t - \cos 3t - \frac{1}{3}\sin 3t\right)$

3. Solve the following initial value problem using the Laplace transformation:

(i) $4y' + \pi^2 y = 0$, $y(0) = 2$, $y'(0) = 0$.

Solution: Given that,

$$4y' + \pi^2 y = 0 \quad \text{...... (i)}$$

$$y(0) = 2, \ y'(0) = 0 \quad \text{...... (ii)}$$

Taking Laplace transform both sides

$$4\mathcal{L}(y'') + \pi^2\mathcal{L}(y) = \mathcal{L}(0)$$

$$\Rightarrow 4\{s^2\mathcal{L}(y) - sy(0) - y'(0) + \pi^2\mathcal{L}(y) = 0$$

$$\Rightarrow 4\{s^2\mathcal{L}(y) - s\times 2 - 0\} + \pi^2\mathcal{L}(y) = 0 \quad \text{[using (ii)]}$$

$$\Rightarrow 4s^2\mathcal{L}(y) - 8s + \pi^2\mathcal{L}(y) = 0$$

$$\Rightarrow \mathcal{L}(y)(4s^2 + \pi^2) = 8s$$

$$\Rightarrow \mathcal{L}(y) = \frac{8s}{4s^2 + \pi^2}$$

This gives,

$$y = \mathcal{L}^{-1}\left[\frac{8s}{4s^2 + \pi^2}\right]$$

$$= \mathcal{L}^{-1}\left\{\frac{8}{4}\left(\frac{s}{s^2 + \frac{\pi^2}{4}}\right)\right\} = 2\mathcal{L}^{-1}\frac{s}{\left[s^2 + \left(\frac{\pi}{2}\right)^2\right]} = 2\cos\frac{\pi}{2}t$$

Thus, $y = 2\cos\frac{\pi}{2}t$

(ii) $y'' + 2y' - 8y = 0, y(0) = 1, y'(0) = 8$

Solution: Given that,

$$y'' + 2y' - 8y = 0 \quad \text{...... (i)}$$
$$y(0) = 1, y'(0) = 8 \quad \text{...... (ii)}$$

Taking Laplace transform both side

$$\mathcal{L}(y'') + 2\mathcal{L}(y') - 8\mathcal{L}(y) = \mathcal{L}(0)$$
$$\Rightarrow s^2\mathcal{L}(y) - sy(0) - y'(0) + 2\{s\mathcal{L}(y) - y(0)\} - 8\mathcal{L}(y) = 0$$
$$\Rightarrow s^2\mathcal{L}(y) - s\times 1 - 8 + 2s\mathcal{L}(y) - 2\times 1 - 8\mathcal{L}(y) = 0 \quad \text{[using (ii)]}$$
$$\Rightarrow \{\mathcal{L}(y)\}(s^2 + 2s - 8) - s - 10 = 0$$
$$\Rightarrow \mathcal{L}(y) = \frac{s+10}{s^2 + 2s - 8}$$

This gives,

$$y = \mathcal{L}^{-1}\left(\frac{s+10}{s^2 + 2s - 8}\right) = \mathcal{L}^{-1}\left\{\frac{s+10}{s^2 + 4s - 2s - 8}\right\}$$
$$= \mathcal{L}^{-1}\left(\frac{s+10}{(s+4)(s-2)}\right) \quad \text{....... (iii)}$$

Let,

$$\frac{s+10}{(s+4)(s-2)} = \frac{A}{s+4} + \frac{B}{s-2}$$
$$= \frac{A(s-2) + B(s+4)}{(s+4)(s-2)}$$

This implies,

$$s + 10 = As - 2A + Bs + 4B$$
$$\Rightarrow s + 10 = s(A + B) + (4B - 2A)$$

Equating coefficient of like term on both sides then we get,

$A + B = 1$, $4B - 2A = 10$.

Solving we get, $A = 1$ and $B = 2$.

Now, equation (iii) becomes

$$y = \mathcal{L}^{-1}\left(-\frac{1}{(s+4)} + \frac{2}{(s-2)}\right) = -e^{-4t} + 2e^{2t}$$
$$= 2e^{2t} - e^{-4t}$$

Thus, $y = 2e^{2t} - e^{-4t}$

(iii) $y'' - ky' = 0, y(0) = 2, y'(0) = k.$

Solution: Given that,

$$y'' - ky' = 0 \quad \text{...... (i)}$$
$$y(0) = 2, y'(0) = k \quad \text{...... (ii)}$$

Taking Laplace transform on both sides of (i) then

$$\mathcal{L}(y'') - k(y') = \mathcal{L}(0)$$
$$\Rightarrow s^2\mathcal{L}(y) - sy(0) - y'(0) - k\{s\mathcal{L}(y) - y(0)\} = 0$$
$$\Rightarrow s^2\mathcal{L}(y) - s\times 2 - k - ks\mathcal{L}(y) + 2k = 0 \quad \text{[using (ii)]}$$
$$\Rightarrow \mathcal{L}(y)(s^2 - ks) = 2s - k$$
$$\Rightarrow \mathcal{L}(y) = \frac{2s-k}{s^2 - ks} = \frac{s(2s-k)}{s(s-k)}$$

This gives,

$$y = \mathcal{L}^{-1}\left\{\frac{2s-k}{s(s-k)}\right\} \quad \text{....... (iii)}$$

Let,

$$\frac{2s-k}{s(s-k)} = \frac{A}{s} + \frac{B}{s-k}$$
$$\Rightarrow \frac{2s-k}{s(s-k)} = \frac{As - Ak + Bs}{s(s-k)}$$

This implies, $2s - k = As - Ak + Bs = (A + B)s - Ak.$

Equating coefficient of like term on both sides then we get.

$A + B = 2$, $-Ak = -k$

Then, $A = 1$, $B = 1$

Now, equation (iii) becomes

$$y = \mathcal{L}^{-1}\left\{\frac{1}{s} + \frac{1}{s-k}\right\} = 1 + e^{kt}$$

(iv) $y'' + w^2y = 0, y(0) = A, y'(0) = B$

Solution: Given that,

$$y'' + w^2y = 0 \quad \text{...... (i)}$$
$$y(0) = A, y'(0) = B \quad \text{...... (ii)}$$

Taking Laplace transform on both sides of (i) then

$$\mathcal{L}(y'') + w^2\mathcal{L}(y) = \mathcal{L}(0)$$
$$\Rightarrow s^2\mathcal{L}(y) - sy(0) - y'(0) + w^2\mathcal{L}(y) = 0$$
$$\Rightarrow s^2\mathcal{L}(y) - sA - B + w^2\mathcal{L}(y) = 0$$
$$\Rightarrow \mathcal{L}(y)(s^2 + w^2) = sA + B$$
$$\Rightarrow \mathcal{L}(y) = \frac{sA + B}{s^2 + w^2} \quad \text{....... (iii)}$$

This gives,

$$y = \mathcal{L}^{-1}\left(\frac{sA}{s^2 + w^2} + \frac{B}{s^2 + w^2}\right) = A\,\mathcal{L}^{-1}\left(\frac{s}{s^2 + w^2}\right) + \frac{B}{w}\mathcal{L}^{-1}\left(\frac{w}{s^2 + w^2}\right)$$
$$= A\cos wt + \frac{B}{w}\sin wt$$

Thus, $y = A\cos wt + \frac{B}{w}\sin wt$.

(v) $y' + 3y = 10\sin t, y(0) = 0$

Solution: Given that,

$$y' + 3y = 10\sin t \quad \text{...... (i)}$$
$$y(0) = 0 \quad \text{...... (ii)}$$

Taking Laplace transform on both sides of (i) then

$$\mathcal{L}(y') + 3\mathcal{L}(y) = 10\mathcal{L}(\sin t)$$
$$\Rightarrow s\mathcal{L}(y) + 3\mathcal{L}(y) = 10\frac{1}{s^2+1}$$
$$\Rightarrow \mathcal{L}(y)(s+3) = \frac{10}{s^2+1}$$

$$\Rightarrow \quad \mathcal{L}(y) = \frac{10}{(s^2+1)(s+3)}$$

This gives,

$$y = \mathcal{L}^{-1}\left(\frac{10}{(s^2+1)(s+3)}\right) \quad \ldots\ldots \text{(iii)}$$

Let,

$$\frac{10}{(s^2+1)(s+3)} = \frac{As+B}{s^2+1} + \frac{C}{s+3}$$

$$= \frac{(As+B)(s+3)+C(s^2+1)}{(s^2+1)(s+3)}$$

This implies,

$$10 = As^2 + 3As + Bs + 3B + Cs^2 + C$$

$$\Rightarrow \quad 10 = s^2(A+C) + s(3A+B) + (3B+C)$$

Equating coefficient of like term on both sides then we get,

$A + C = 0$ .... (a) $\qquad 3A + B = 0$ .... (b)

$3B + C = 10$ .... (c)

Solving these equations we get,

$A = -1, \quad B = 3, \quad C = 1.$

Now, equation (iii) becomes

$$y = \mathcal{L}^{-1}\left\{\frac{-s+3}{s^2+1} + \frac{1}{s+3}\right\} = \mathcal{L}^{-1}\left(-\frac{s}{s^2+1} + \frac{3}{s^2+1} + \frac{1}{s+3}\right)$$

$$= -\cos t + 3\sin t + e^{-3t}$$

Thus, $\quad y = -\cos t + 3\sin t + e^{-3t}$

(vi) $y' + 0.2y = 0.01\,t,\ y(0) = -0.25$

Solution: Given that,

$$y' + 0.2y = 0.01\,t \quad \ldots\ldots \text{(i)}$$

$$y(0) = -0.25 \quad \ldots\ldots \text{(ii)}$$

Taking Laplace transform on both sides of (i) then

$$\mathcal{L}(y') + 0.2\,\mathcal{L}(y) = 0.01\,\mathcal{L}(t)$$

$$\Rightarrow \quad s\,\mathcal{L}(y) + y(o) + 0.2\,\mathcal{L}(y) = \frac{0.01}{s^2}$$

$$\Rightarrow \quad \mathcal{L}(y)\,(s+0.2) - 0.25 = \frac{0.01}{s^2}$$

$$\Rightarrow \quad \mathcal{L}(y) = \left(\frac{0.01}{s^2} + 0.25\right)\frac{1}{(s+0.2)} = \left(\frac{0.01}{s^2(s+0.2)} + \frac{0.25}{(s+0.2)}\right)$$

This gives,

$$y = \mathcal{L}^{-1}\left\{\frac{0.01}{s^2(s+0.2)} + \frac{0.25}{(s+0.2)}\right\} = \left\{\frac{0.01+0.25s^2}{s^2(s+0.2)}\right\}$$

Let,

$$\frac{0.01+0.25s^2}{s^2(s+0.2)} = \frac{As+B}{s^2} + \frac{C}{s+0.2}$$

$$= \frac{(As+B)(s+0.2)+Cs^2}{s^2(s+0.2)}$$

This gives,

$$0.01 + 0.25s^2 = As^2 + 0.2As + Bs + 0.2B + Cs^2$$

$$\Rightarrow \quad 0.01 + 0.25s^2 = s^2(A+C) + s(0.2A+B) + 0.2B$$

Equating coefficient of like term on both sides then we get,

$A + C = 0.25, \qquad 0.2A + B = 0, \qquad 0.2B = 0.01$

Solving we get, $\quad A = -0.25,\ B = 0.05$ and $C = 0$

Now, equation (iii) becomes

$$y = \mathcal{L}^{-1}\left\{\frac{0.05-0.25s}{s^2}\right\} = \mathcal{L}^{-1}\left(\frac{0.05}{s^2} - \frac{0.25}{s}\right)$$

$$= 0.05t - 0.25$$

Thus, $\quad y = 0.05t - 0.25.$

(vii) $y'' + ay' - 2a^2y = 0,\ y(0) = 6,\ y'(0) = 0$

Solution: Given that,

$$y'' + ay' - 2a^2y = 0 \quad \ldots\ldots \text{(i)}$$

$$y(0) = 6,\ y'(0) = 0 \quad \ldots\ldots \text{(ii)}$$

Taking Laplace transform on both sides of (i) then

$$\mathcal{L}(y'') + a\,\mathcal{L}(y') - 2a^2\mathcal{L}(y) = \mathcal{L}(0)$$

$$\Rightarrow \quad s^2\mathcal{L}(y) - s\,y(0) - y'(0) + a\,[s\,\mathcal{L}(y) - y(0)] - 2a^2\,\mathcal{L}(y) = 0$$

$$\Rightarrow \quad s^2\mathcal{L}(y) - 6s - 0 + as\,\mathcal{L}(y) - 6a - 2a^2\,\mathcal{L}(y) = 0$$

$$\Rightarrow \quad \mathcal{L}(y)\,(s^2 + as - 2a^2) = 6s + 6a$$

This gives,

$$y = \mathcal{L}^{-1}\left[\frac{6(s+a)}{s^2+as-2a^2}\right] \quad \ldots\ldots \text{(i)}$$

Let,

$$\frac{6(s+a)}{s^2+as-2a^2} = \frac{6(s+a)}{s^2+2as-as-2a^2} = \frac{6(s+a)}{s(s+a)-a(s+2a)} = \frac{6(s+a)}{(s+2a)(s-a)}$$

Then,

$$\frac{6(s+a)}{(s+2a)(s-a)} = \frac{A}{s+2a} + \frac{B}{s-a} = \frac{A(s-a)+B(s+2a)}{(s+2a)(s-a)}$$

This implies,

$$6s + 6a = As - Aa + Bs + 2aB$$

$$\Rightarrow \quad 6s + 6a = s(A+B) + 2aB - Aa$$

Equating coefficient of like term on both sides then we get,

$A + B = 6 \quad$ and $\quad 2aB - Aa = 6a.$

Solving these equations we get,

$A = 2, \quad B = 4.$

Now, equation (iii) becomes

$$y = \mathcal{L}^{-1}\left[\frac{2}{s+2a} + \frac{4}{s-a}\right] = 2e^{-2at} + 4e^{at}$$

Thus, $\quad y = 2e^{-2at} + 4e^{at}.$

(viii) $y'' - 4y' + 3y = 6t - 8,\ y(0) = 0,\ y'(0) = 0$

Solution: Given that,

$$y'' - 4y' + 3y = 6t - 8 \qquad \ldots\ldots (i)$$
$$y(0) = 0,\ y'(0) = 0 \qquad \ldots\ldots (ii)$$

Taking Laplace transform on both sides of (i) then

$$\mathcal{L}(y'') - 4\mathcal{L}(y') + 3\mathcal{L}(y) = \mathcal{L}(6t - 8)$$

$$\Rightarrow\ s^2\mathcal{L}(y) - sy(0) - y'(0) - 4[s\mathcal{L}(y) - sy(0)] + 3\mathcal{L}(y) = \left(\frac{6}{s^2} - \frac{8}{s}\right)$$

$$\Rightarrow\ \mathcal{L}(y)(s^2 - 4s + 3) = \frac{6 - 8s}{s^2}$$

$$\Rightarrow\ \mathcal{L}(y) = \frac{6 - 8s}{s^2(s^2 - 4s + 3)}$$

This gives,

$$y = \mathcal{L}^{-1}\left(\frac{6 - 8s}{s^2(s^2 - 4s + 3)}\right) \qquad \ldots\ldots (i)$$

Let,

$$\frac{6 - 8s}{s^2(s^2 - 4s + 3)} = \frac{6 - 8s}{s^2(s^2 - 3s - s + 3)} = \frac{6 - 8s}{s^2(s - 3)(s - 1)} = \frac{A}{s - 3} + \frac{B}{s - 1} + \frac{Cs + D}{s^2}$$

So,

$$\frac{6 - 8s}{s^2(s^2 - 4s + 3)} = \frac{A(s - 1)s^2 + B(s - 3)s^2 + (Cs + D)(s - 3)(s - 1)}{s^2(s - 1)(s - 3)}$$

This implies,

$$6 - 8s = As^3 - As^2 + Bs^3 - 3Bs^2 + Cs^3 - 4Cs^2 - 3Cs + Ds^2 - 4Ds + 3D$$

$$\Rightarrow\ 6 - 8s = s^3(A + B + C) + s^2(D - A - 3B - 4C) + s(3C - 4D) + 3D$$

Equating coefficient of like term on both sides then we get,

$$A + B + C = 0 \qquad D - A - 3B - 4C = 0$$
$$3C - 4D = -8 \qquad 3D = 6$$

Solving these equations we get,

$$A = -1, \quad B = 1, \quad C = 0, \quad D = 2.$$

Now, equation (iii) becomes

$$y = L^{-1}\left[\frac{1}{s - 1} - \frac{1}{s - 3} + \frac{2}{s^2}\right]$$
$$= e^t - e^{3t} + 2t.$$

Thus, $y = e^t - e^{3t} + 2t.$

(ix) $y'' + 2y' - 3y = 6e^{-2t}$ $y(0) = 2, y'(0) = -14$

Solution: Given that,

$$y'' + 2y' - 3y = 6e^{-2t} \qquad \ldots\ldots\ldots(i)$$
$$y(0) = 2,\ y'(0) = -14 \qquad \ldots\ldots(ii)$$

Taking Laplace transform of (i) then,

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} - 3\mathcal{L}\{y\} = 6\mathcal{L}\{e^{-2t}\}$$

$$\Rightarrow\ [s^2\mathcal{L}\{y\} - sy(0) - y'(0)] + 2[s\mathcal{L}\{y\} - y(0)] - 3\mathcal{L}\{y\} = 6\cdot\frac{1}{s + 2}$$

$$\Rightarrow\ \mathcal{L}\{y\}[s^2 + 2s - 3] - 2s + 14 - 4 = \frac{6}{s + 2} \qquad \text{[using (ii)]}$$

$$\Rightarrow \quad \mathcal{L}\{y\}(s^2 + 2s - 3) = \frac{6}{s+2} + 2s - 10$$

$$= \frac{6 + 2s^2 - 10s + 4s - 20}{s+2} = \frac{2s^2 - 6s - 14}{s+2}$$

$$\Rightarrow \quad \mathcal{L}\{y\} = \frac{2(s^2 - 3s - 14)}{(s+2)(s^2 + 2s - 3)} \qquad ....(3)$$

Here,

$$\frac{s^2 - 3s - 7}{(s+2)(s^2 + 2s - 3)} = \frac{s^2 - 3s - 7}{(s+2)(s+3)(s-1)} = \frac{A}{s+2} + \frac{B}{s+3} + \frac{C}{s-1}$$

This gives,

$$\begin{aligned} s^2 - 3s - 7 &= A(s+3)(s-1) + B(s+2)(s-1) + C(s+2)(s-3) \\ &= A(s^2 + 2s - 3) + B(s^2 + s - 2) + C(s^2 - S - 6) \\ &= (A + B + C)s^2 + (2A + B - C)s + (-3A - 2B - 6C) \end{aligned}$$

Comparing the like terms from both sides we get,

$A + B + C = 1,\quad 2A + B - C = -3$ and $-3A - 2B - 6C = -7$

Solving we get,

$$A = -1,\ B = \frac{1}{2},\ C = \frac{3}{2}$$

Then (3) becomes,

$$\mathcal{L}\{y\} = -2\left[\frac{1}{s+2} - \frac{1}{2}\left(\frac{1}{s+3}\right) - \frac{3}{2}\left(\frac{1}{s-1}\right)\right]$$

Taking inverse Laplace transform then

$$y = -2\left[e^{-2t} - \frac{1}{s}e^{-3t} - \frac{3}{2}e^{t}\right]$$