OTHER QUESTIONS FROM SEMESTER END EXAMINATION

First Order Differential Equation

1999 Q. No. 4(a); 2001 Q. No. 4(a)

Show that the differentiation equation: sin hx cosydx – coshx siny dy = 0 is exact and solve it.

Solution: Given equation is

$$\sinh x \cos y dx - \cosh x \sin y \, dy = 0 \qquad \text{.......(i)}$$

Comparing (i) with Mdx + Ndy = 0 then we get,

$$M = \sinh x \cos y \text{ and } N = -\cosh x \sin y$$

Then, $\frac{\partial M}{\partial y} = -\sinh x \sin y$ and $\frac{\partial N}{\partial x} = -\sinh x \sin y$

Thus, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. So, the equation (i) is exact.

Therefore solution of (i) is

$$\int M dx + \int (\text{terms of N free from x})\, dy = C$$

$\Rightarrow \int \sinh x \cos y \, dx + \int 0 \, dy = C$ [$\because$ N has no term which is not included x]

$\Rightarrow \cos y \int \sinh x \, dx = C$

$\Rightarrow \cos y \cosh x = C$

This is required solution of (i)

2000 Q. No. 4(a); 2007 Fall Q. No. 4(a)

Solve the differential equation $y' + \frac{y}{x} = \frac{y^2}{x}$.

Solution: Given equation is

$$y' + \frac{y}{x} = \frac{y^2}{x} \Rightarrow \frac{1}{y^2} y' + \frac{1}{xy} = \frac{1}{x} \qquad \text{............(i)}$$

Put $\frac{1}{y} = u$ then $-\frac{1}{y^2} y' = u'$. Then (i) becomes,

$$-u' + \frac{u}{x} = \frac{1}{x} \Rightarrow u' - \frac{u}{x} = -\frac{1}{x} \qquad \text{............(ii)}$$

This is a linear differential equation of first order. Its integrating factor is

$$\text{I.F.} = e^{\int \left(-\frac{1}{x}\right) dx} = e^{-\log x} = e^{\log (x^{-1})} = x^{-1}$$

Now, multiplying (2) by IF and then integrating w.r. to x then we get

$$u.\, x^{-1} = \int \left(-\frac{1}{x}\right)(x^{-1})\, dx + C = -\int x^{-2} dx + C = -\frac{x^{-3}}{-3} + C = \frac{1}{3x^3} + C$$

$\Rightarrow \frac{x^{-1}}{y} = \frac{1}{3x^3} + C$

$\Rightarrow \frac{1}{xy} = \frac{1}{3x^3} + C \Rightarrow 3x^2 = y(1 + 3Cx^2)$

2002 Set I & II; 2004 Spring; 2011 Fall Q. No. 4(a)

Solve $\frac{dy}{dx} + \frac{y}{x} \log y = \frac{y}{x^2} (\log y)^2$.

Solution: Given that,

$$\frac{dy}{dx} + \frac{y \log y}{x} = \frac{y (\log y)^2}{x^2}$$

$$\Rightarrow \frac{1}{y (\log y)^2} \frac{dy}{dx} + \frac{1}{x \log y} = \frac{1}{x^2} \qquad \text{...... (i)}$$

Put $\frac{1}{\log y} = u$ then, $\frac{-1}{y (\log y)^2} \frac{dy}{dx} = \frac{du}{dx}$, so (i) becomes,

$$\frac{-du}{dx} + \frac{1}{x} u = \frac{1}{x^2}$$

$$\Rightarrow \frac{du}{dx} - \frac{u}{x} = \frac{-1}{x^2} \qquad \text{...... (ii)}$$

This is a linear differential equation of first order whose integrating factor is,

$$\text{I.F.} = e^{-\int 1/x \, dx} = e^{-\log x} = x^{-1}$$

Now, multiplying both sides of (ii) by I F and then integrating we get,

$$u.x^{-1} = \int \frac{x^{-1}}{x^2} dx + \frac{c}{2}.$$

$$\Rightarrow \frac{u}{x^4} = -\int x^{-3} dx + \frac{x}{2} = -\frac{x^{-2}}{-2} + \frac{c}{2} = \frac{1}{2x^2} \frac{c}{2}$$

$$\Rightarrow 2u = \frac{1}{x^2} + c$$

$$\Rightarrow \frac{2}{\log y} = \frac{1}{x^2} + c.$$

This is the solution of given equation.

2004 Fall; 2006 Fall Q. No. 4(a)

Solve: y' + y tanx = sec x

Solution: Given equation is

$$y' + y \tan x = \sec x \qquad \text{.........(i)}$$

This is first order linear differential equation of first order.

Comparing (i) with $y' + Py = Q$ then we get.

$$P = \tan x, \qquad Q = \sec x$$

So, the integrating factor of (i) is

$$\text{I.F.} = e^{\int P dx} = e^{\int \tan x\, dx} = e^{\log(\sec x)} = \sec x$$

Now, multiplying (i) by I.F. and then taking integration w. r. t. x then,

$$y.\sec x = \int \sec^2 x\, dx + C$$

$$= \tan x + C$$

$$\Rightarrow \quad y = \sin x + C.\cos x$$

This is required solution of (i).

2006 Spring Q. No. 4(a)

Define the first order linear differential equations with suitable example and solve: $x\dfrac{dy}{dx} + y = y^2 \log x$.

Solution: See the definition.

For problem, see Q. 6, Exercise 6.5.

2008 Fall Q. No. 4(a)

Define order and degree of the differential equation with suitable example. Check exactness condition of the differential equation: $(2\cos y + 4x^2)\, dx = x \sin y\, dy$, if it is not exact find integrating factor (IF) and then solve it by using IF.

Solution: See the definition.

For problem, see Q. A(vi), Exercise 6.3.

2008 Fall; 2010 Spring Q. No. 4(a)

Solve: $\dfrac{dy}{dx} + \dfrac{\sin 2y}{x} = x^3 \cos^2 y$.

Solution: Give differential equation is,

$$y' + \frac{\sin 2y}{x} = x^3 \cos^2 y \Rightarrow \sec^2 y.y' + \frac{2\sin y \cos y}{\cos^2 y.\ x} = x^3$$

$$\Rightarrow \quad \sec^2 y.y' + 2\tan y \frac{1}{x} = x^3 \qquad \ldots\ldots \text{(i)}$$

Put $\tan y = u$ then $\sec^2 y.\ y' = u'$, then (i) becomes

$$u' + \frac{2u}{x} = x^3 \qquad \ldots\ldots \text{(ii)}$$

This is a linear differential equation of first order is a whose integrating factor is

$$\text{I.F.} = e^{\int 2/x\, dx} = e^{2\log x} = x^2$$

Now, multiplying on both sides of (ii) by I.F. and then integrating we get,

$$u.\ x^2 = \int x^3.x^2 dx + \frac{c}{6} = \frac{x^6}{6} + \frac{c}{6}$$

$$\Rightarrow \quad 6x^2 \tan y = x^6 + c.$$

This is the solution of given equation.

2009 Fall Q. No. 4(a)

Define order and degree of ordinary differential equation. Solve the initial value problem: $y' + \dfrac{y}{x} = x^2$; $y(1) = 0$.

Solution: For the first part see definition.

For second Part: Given equation is

$$y' + \frac{y}{x} = x^2 \qquad \ldots\ldots\text{(i)}$$

$$\text{with} \qquad y(1) = 0 \qquad \ldots\ldots\text{(ii)}$$

Clearly, the equation (i) is a linear differential equation of first order.

Comparing (i) with $y' + Py = Q$ then we get

$$P = \frac{1}{x} \qquad \text{and} \qquad Q = x^2$$

So, the integrating factor of (i) is

$$\text{I.F.} = e^{\int P dx} = e^{\int dx/x} = e^{\log x} = x$$

Now, multiplying (i) by I.F. and then taking integration w. r. t. x, then,

$$y.\ x = \int x^3\, dx + C = \frac{x^4}{4} + C \qquad \ldots\ldots\text{(iii)}$$

And by (ii), we have $y(1) = 0$. Then (iii) gives,

$$0 = \frac{1}{4} + C \Rightarrow C = -\frac{1}{4}$$

Then (iii) becomes,

$$y.x = \frac{x^4}{4} - \frac{1}{4}$$

$$\Rightarrow \quad 4xy = x^4 - 1.$$

This is required solution of (i).

SHORT QUESTIONS

2002: Solve: $\dfrac{dx}{1+x^2} + \dfrac{dy}{1+y^2} = 0$.

Solution: Here,

$$\frac{dx}{1+x^2} + \frac{dy}{1+y^2} = 0$$

Integrating we get,

$$\tan^{-1}(x) + \tan^{-1}(y) = C_1$$

$$\Rightarrow \tan^{-1}\left(\frac{x+y}{1-xy}\right) = C_1 \Rightarrow \frac{x+y}{1-xy} = \tan C_1$$

$$\Rightarrow x + y = C(1 - xy) \qquad \text{for } C = \tan(C_1)$$

$$\Rightarrow y(1 + Cx) = C - x$$

$$\Rightarrow y = \frac{C - x}{1 + Cx}$$

2002: What is meant by integrating factor. Write down the condition for the differential equation Mdx + Ndy = 0 to be exact.

Solution: See the definition.

See the condition for exactness.

2003 Fall: Find integrating factor of $\frac{dy}{dx} + \frac{y}{x} = x$.

Solution: Given equation is

$$\frac{dy}{dx} + \frac{y}{x} = x \Rightarrow y' + \frac{y}{x} = x \qquad \text{...... (i)}$$

Comparing (i) with $y' + Py = Q$ then

$$P = \frac{1}{x} \quad \text{and} \quad Q = x$$

Now, the integrating factor of (i) is

$$\text{I.F.} = e^{\int P dx} = e^{\int dx/x} = e^{\log(x)} = x.$$

Similar Question for Practice from Final Exam:

2000: Show that: $\frac{1}{x^2 + y^2}$ is an integrating factor of x dy – y dx = $\frac{1}{x^2}$

2006 Fall: Find integrating factor of $\frac{dy}{dx}$ + cotx y = cosx

2006 Spring: Find integrating factor of $\frac{dy}{dx}$ + y tanx = secx.

2009 Fall: Find integrating factor of $(xy^3 + y)\, x + 2(x^2y^2 + x + y^4)\, dy = 0$

OTHER SHORT QUESTIONS

2004 Fall: Show that the equation 2(y sin2x + cos2x) dx = cos2x dy is exact.

Solution: Given that,

$$2(y \sin 2x + \cos 2x)\, dx = \cos 2x\, dy \qquad \text{...... (i)}$$

Comparing above equation with Mdx + Ndy = 0 then,

$$M = 2(y \sin 2x + \cos 2x) \quad \text{and} \quad -\cos 2x$$

So. $\frac{\partial M}{\partial y} = 2 \sin 2x$ and $\frac{\partial N}{\partial x} = 2 \sin 2x$

Thus, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. So, (i) is exact.

2004 Fall: Solve: $x\sqrt{1+y^2}\, dx + y\sqrt{1+x^2}\, dy = 0$.

Solution: Given that,

$$x\sqrt{1+y^2}\, dx + y\sqrt{1+x^2}\, dy = 0$$

$$\Rightarrow \frac{y}{\sqrt{1+y^2}}\frac{dy}{dx} + \frac{x}{\sqrt{1+x^2}} = 0 \qquad \text{...... (i)}$$

Put, $1 + y^2 = u^2$ then $2y\frac{dy}{dx} = 2u\frac{du}{dx} \Rightarrow y\frac{dy}{dx} = u\frac{du}{dx}$. So. (i) become,

$$\frac{u}{u}\frac{du}{dx} + \frac{x}{\sqrt{1+x^2}} = 0$$

$$\Rightarrow \frac{du}{dx} + \frac{x}{\sqrt{1+x^2}} = 0 \Rightarrow 1 + \frac{x}{\sqrt{1+x^2}}\frac{dx}{du} = 0 \qquad \text{....... (ii)}$$

Put $1 + x^2 = v^2$ then, $x\frac{dx}{du} = v\frac{dv}{du}$ so, equation (ii) becomes,

$$1 + \frac{v}{v}\frac{dx}{du} = 0 \Rightarrow 1 + \frac{dv}{du} = 0$$

$$\Rightarrow du + dv = 0$$

Integrating we get,

$$u + v = c$$

$$\Rightarrow \sqrt{1+x^2} + \sqrt{1+y^2} = c$$

2007 Fall: Solve: $\cos^2 x \frac{dy}{dx} + y = \tan x$.

Solution: Here,

$$\cos^2 x \frac{dy}{dx} + y = \tan x$$

$$\Rightarrow \frac{dy}{dx} + \sec^2 x\, y = \sec^2 x \tan x \qquad \text{...... (i)}$$

This is a linear differential equation of first order whose I.F. is,

$$\text{I.F.} = e^{\int \sec^2 x\, dx} = e^{\tan x}$$

Now, multiplying (i) by I.F. and then integrating, we get,

$$y.e^{\tan x} = \int e^{\tan x} \sec^2 x \tan x \, dx + c$$

Put $\tan x = u$ then $\sec^2 x \, dx = du$. So,

$$y.e^{\tan x} = \int e^4 u \, du + c = 4e^4 - e^4 + c = \tan x \, e^{\tan x} - e^{\tan x} + c$$

$$\Rightarrow \quad y = \tan x - 1 + c \, e^{\tan x}$$

2008 Fall: Solve $(x + 1)y' = x(y^2 + 1)$.

Solution: Here,

$$(x + 1) y' = x (y^2 + 1)$$

$$\Rightarrow \quad \frac{dy}{1 + y^2} = \frac{x}{x + 1} dx = \left(1 - \frac{1}{x + 1} dx\right)$$

Integrating,

$$\text{Tan}^{-1}(y) = x - \log (x + 1) + c$$

2009 Spring: Solve: $\frac{dy}{dx} = (y - x)^2$.

Solution: Here, $\frac{dy}{dx} = (y - x)^2$

Put $y - x = u$, then, $\frac{dy}{dx} = \frac{dy}{dx}$. Then,

$$\frac{du}{dx} = u^2$$

$$\Rightarrow \quad u^{-2} du = dx$$

Integrating we get,

$$\frac{u^{-1}}{-1} = x + c \Rightarrow \frac{1}{y - x} = -(x + c)$$

$$\Rightarrow \frac{1}{y - x} + x + c = 0$$