Exercise 9.2

1. Sketch the region bounded by the graph of the equation and find its area using one or more double integral.

(i) $y = \frac{1}{x^2}, y = -x^2, x = 1, x = 2.$

Solution: Here,

For, $y = \frac{1}{x^2}$

x	0	1	2	-1	-2	+ ½
u	∞	1	¼	1	¼	4

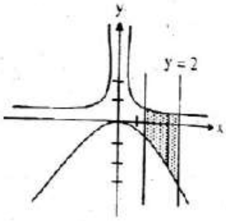
Since the curve $x^2 = -y$ is a parabola that has vertex at (0, 0) and has line of symmetry x = 0. So, the parabola is down openward. And, line x = 1 and x = 2 are parallel o the y-axis.

On he bases the region of integration is as shown in the figure.

Here the required region is the shaded part in the corresponding figure.

Now, taking vertical strip,

$$\text{Area} = \int_1^2 \int_{-x^2}^{1/x^2} dy\, dx$$



$$= \int_1^2 [y]_{-x^2}^{1/x^2} dx$$

$$= \int_1^2 \left(\frac{1}{x^2} + x^2\right) dx$$

$$= \left[-\frac{1}{x} + \frac{x^3}{3}\right]_1^2 = \left(-\frac{1}{2} + \frac{8}{3} + 1 - \frac{1}{3}\right) = \left(\frac{-3 + 16 + 6 - 2}{6}\right) = \frac{17}{6}$$

(ii) $y^2 = -x, x - y = 4, y = -1, y = 2$

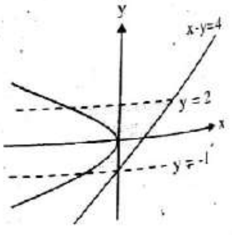
Solution: Here, $y^2 = -x$, $x - y = 4$, $y = -1$, $y = 2$.

Since the curve $y^2 = -x$ is a parabola that has vertex at (0, 0) and has line of symmetry y = 0. So, the parabola is left openward. And, the line x − y = 4 passes through he point (4, 0) and (0, -4)

Also, the line y = -1 and y = 2 are parallel o the x-axis.

On he bases the region of integration is as shown in the figure.

Here the required region is the shaded part in the corresponding figure.



Taking horizontal strip

$$\text{Area} = \int_{-1}^{2} \int_{-y^2}^{y+4} dx\, dy$$

$$= \int_{-1}^{2} [x]_{-y^2}^{y+4} dy = \int_{-1}^{2} (y + 4 + y^2)\, dy$$

$$= \left[\frac{y^2}{2} + 4y + \frac{y^3}{3}\right]_{-1}^{2}$$

$$= \left(\frac{4}{2} + 8 + \frac{8}{3}\right) - \left(\frac{1}{2} - 4 - \frac{1}{3}\right)$$

$$= \left(\frac{12 + 48 + 16}{6} - \frac{3 - 24 - 2}{6}\right) = \frac{76 + 23}{6} = \frac{99}{6} = \frac{33}{2}$$

Thus, area of the region is $\frac{33}{2}$.

(iii) $y = x, y = 3x, x + y = 4.$

Solution: Here, $y = x$, $y = 3x$, $x + y = 4$.

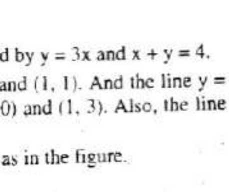
Here the region of integration is bounded by y = x, and by y = 3x and x + y = 4.

Since the line y = x passes through the points (0, 0) and (1, 1). And the line y = 3x is a straight line that passes through the point (0, 0) and (1, 3). Also, the line x + y = 4 passes through the points (4, 0) and (0, 4).

On the bases of these boundaries the region is sketch as in the figure.

Taking vertical strip

$$\text{Area} = \int_{R1}\int + \int_{R2}\int$$



$$= \int_0^1 \int_x^{3x} dy\, dx + \int_1^2 \int_x^{4-x} dy\, dx$$

$$= \int_0^1 [y]_x^{3x} dx \int_1^2 [y]_x^{4-x} dx$$

$$= \int_0^1 (3x - x)\, dx + \int_1^2 (4 - x - x)\, dx$$

$$= \int_0^1 2x\, dx + \int_1^2 (4 - 2x)\, dx$$

$$= [x^2]_0^1 + [4x - x^2]_1^2 = 1 + (8 - 4 - 4 + 1) = 2.$$

Thus, area of the region is 2.

(iv) $y = e^x, y = \sin x, x = -\pi, x = \pi$

Solution: Here,

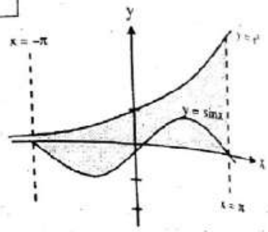
For, $y = e^x$

x	0	1	2	-1
y	1	e	e^2	1/e

For, $y = \sin x$

x	0	$\pi/2$	π	$-\pi$	$-\pi/2$
y	0	1	0	0	-1

Taking vertical strip



$$\text{Area} = \int_{-\pi}^{\pi}\int_{\sin x}^{e^x} dy\,dx$$

$$= \int_{-\pi}^{\pi} [y]_{\sin x}^{e^x} dx = \int_{-\pi}^{\pi} (e^x - \sin x)\,dx$$

$$= [e^x + \cos x]_{-\pi}^{\pi} = e^{\pi} - 1 - e^{-\pi} + 1 = (e^{\pi} - e^{-\pi}).$$

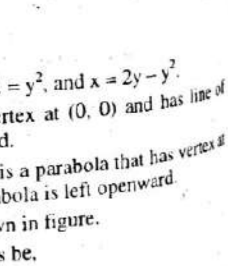
Thus, area of the region is $(e^{\pi} - e^{-\pi})$.

(v) The y-axis, the line $y = 2x$, and the line $y = 4$.

Solution: Given that the required region is bounded by y-axis, the line $y = 2x$, and the line $y = 4$.

Since the line $y = 2x$ passes through the points (0, 0) and (1, 2). And the line $y = 4$ is a straight line that is parallel to x-axis. Also, the region is bounded by y-axis. On the bases of these boundaries the region is sketch as in the figure.

Now, taking horizontal strip



$$\text{Area} = \int_0^4\int_0^{y/2} dx\,dy = \int_0^4 [x]_0^{y/2} dy$$

$$= \int_0^4 \frac{y}{2} dy$$

$$= \left[\frac{y^2}{4}\right]_0^4 = \frac{16}{4} = 4.$$

Thus the area of the region is 4.

(vi) $x = y^2, x = 2y - y^2$.

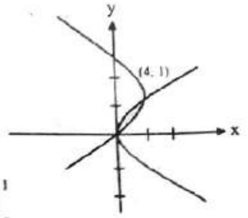
Solution: Here the region of integration is bounded by $x = y^2$, and $x = 2y - y^2$.

Since the curve $y^2 = x$ is a parabola that has vertex at (0, 0) and has line of symmetry $y = 0$. So, the parabola is right openward.

Also, the curve $x = 2y - y^2 \Rightarrow (y-1)^2 = -(x-1)$ is a parabola that has vertex at (1, 1) and has line of symmetry $y = 1$. So, the parabola is left openward.

On these bases the region of integration is as shown in figure.

Now, the area of the region bounded by the curves be,



$$\text{Area} = \int_0^1\int_{y^2}^{2y-y^2} dx\,dy$$

$$= \int_0^1 [x]_{y^2}^{2y-y^2} dy$$

$$= \int_0^1 (2y - y^2 - y^2)\,dy = \left[\frac{2y^2}{2} - \frac{2y^3}{3}\right]_0^1$$

$$= \left(\frac{2}{2} - \frac{2}{3}\right) = \frac{1}{3}$$

Thus, the area of the region is $\frac{1}{3}$.

(vii) – **(viii) Similar as above, left for practice.**

2. Sketch the solid in the first octant bounded by the curves and find its volume.

(i) $x^2 + z^2 = 9, y = 2x, y = 0, z = 0$

Solution: Given curves are $x^2 + z^2 = 9$, $y = 2x$, $y = 0$, $z = 0$.

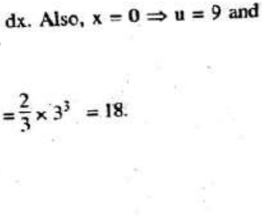
In xy plane, $z = 0$. And, $x^2 = 9 \Rightarrow x = \pm 3$.

Also, the line $y = 2x$ passes through (0, 0) and (1, 2).

The base of the figure is shown as in figure.

The region of integration in the first octant xy plane bounded by above line and curves be, R: $0 \le x \le 3, 0 \le y \le 2x$.

Now, taking vertical strip,



$$\text{Volume} = \int_0^3\int_0^{2x} \sqrt{9 - x^2}\,dy\,dx$$

$$= \int_0^3 \sqrt{9 - x^2}\,[y]_0^{2x}\,dx$$

$$= \int_0^3 2x\sqrt{9 - x^2}\,dx$$

Put, $u = 9 - x^2$ then $\frac{du}{dx} = -2x \Rightarrow -du = 2x\,dx$. Also, $x = 0 \Rightarrow u = 9$ and $x = 3 \Rightarrow u = 0$. then,

$$\text{Volume} = \int_9^0 -\sqrt{u}\,du = -\left[\frac{2}{3} \times u^{3/2}\right]_9^0 = \frac{2}{3} \times 3^3 = 18.$$

Thus the volume of the solid is 18 cubic units.

(ii) $2x + y + z = 4, x = 0, y = 0, z = 0.$

Solution: Given curves are $2x + y + z = 4$, $x = 0$, $y = 0$, $z = 0$

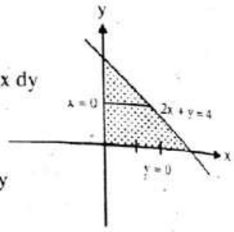
In xy plane, $z = 0$. Then $2x + y = 4$ which is passing through the point (2, 0) and (0, 4).

Also, the line $y = 0$, $x = 0$ are the axes.

The base of the figure is shown as in figure.

The region of integration in the first octant xy-plane bounded by above line and curves be, R: $0 \le y \le 4,\ 0 \le x \le \frac{4-y}{2}$.

Now, taking horizontal strip,



$$\text{Volume} = \iint dx\,dy = \int_0^4 \int_0^{(4-y)/2} (4 - 2x - y)\,dx\,dy$$

$$= \int_0^4 [4x - x^2 - yx]_0^{(4-y)/2} dy$$

$$= \int_0^4 [4x - x^2 - yx]_0^{(4-y)/2} dy$$

$$= \int_0^4 \left\{4\frac{(4-y)}{2} - \left(\frac{4-y}{2}\right)^2 - y\left(\frac{4-y}{2}\right)\right\} dy$$

$$= \int_0^4 \left\{8 - 2y - \frac{(16 - 8y + y^2)}{4} - \frac{(4y - y^2)}{2}\right\} dy$$

$$= \frac{1}{4}\int_0^4 (32 - 8y - 16 + 8y - y^2 - 8y + 2y^2)\,dy$$

$$= \frac{1}{4}\int_0^4 (y^2 - 8y + 16)\,dy$$

$$= \frac{1}{4}\left[\frac{y^3}{3} - \frac{8y^2}{2} + 16y\right]_0^4 = \frac{1}{4}\left[\frac{64}{3} - 64 + 64\right] = \frac{16}{3}$$

Thus the volume of the solid is $\frac{16}{3}$ cubic units.

(iii) $z = x^2 + y^2, y = 4 - x^2, x = 0, y = 0, z = 0.$

Solution: Given curves are $z = x^2 + y^2$, $y = 4 - x^2$, $x = 0$, $y = 0$, $z = 0$.

In xy-plane $z = 0$.

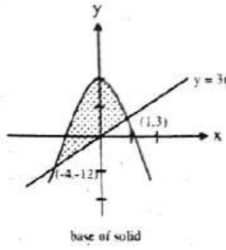
And, the region is bounded by the curve $y = 4 - x^2$ this is a parabola having vertex at (0, 4), line of symmetry $x = 0$ and down openward.

Also, the region is bounded by $x = 0$ and $y = 0$.

On these bases the base of the region is shown as in figure.

The region of integration in the first octant xy plane bounded by above parabola and lines be, R: $0 \le y \le 4,\ 0 \le x \le \sqrt{4-y}$.

Now, taking horizontal component



$$\text{Volume} = \iint_R dx\,dy = \int_0^4 \int_0^{\sqrt{4-y}} (x^2 + y^2)\,dx\,dy$$

$$= \int_0^4 \left[\frac{x^3}{3} + y^2x\right]_0^{\sqrt{4-y}} dy$$

$$= \int_0^4 \left\{\frac{(4-y)^{3/2}}{3} + \sqrt{(4-y)}\,y^2\right\} dy$$

Put, $u = 4 - y$ then $\frac{du}{dy} = -1 \Rightarrow -du = dy$. Also, $y = 0 \Rightarrow u = 4$; $y = 4 \Rightarrow u = 0$.

Then the above part becomes,

$$\text{Volume} = -\int_4^0 \left\{\frac{u^{3/2}}{3} + u^{1/2}(4-u)^2\right\} du$$

$$= -\int_4^0 \left\{\frac{u^{3/2}}{3} + u^{1/2}(16 - 8u + u^2)^2\right\} du$$

$$= -\int_4^0 \left(\frac{u^{3/2}}{3} + 16u^{1/2} - 8u^{3/2} + u^{5/2}\right) du$$

$$= -\frac{1}{3}\int_4^0 (48u^{1/2} - 23u^{3/2} + 3u^{5/2})\,du$$

$$= -\frac{1}{3}\left[48 \times \frac{2}{3}u^{3/2} - 23 \times \frac{2}{5}u^{5/2} + 3 \times \frac{2}{7}u^{7/2}\right]_4^0$$

$$= \frac{1}{3}\left[32 \times u^{3/2} - \frac{46}{5}4^{5/2} + \frac{6}{7}4^{7/2}\right]$$

$$= \frac{1}{3}\left[32 \times 2^{2\times3/2} - \frac{46}{5}2^{2\times5/2} + \frac{6}{7}2^{2\times7/2}\right]$$

$$= \frac{1}{3}\left[32 \times 8 - \frac{46}{5}32 + \frac{6}{7} + 128\right]$$

$$= \frac{1}{3}\left[\frac{8960 - 10304 + 8340}{35}\right] = \frac{2496}{105} = \frac{832}{35}.$$

Thus, the volume the solid is $\frac{832}{35}$.

(iv) $z = x^3$, $x = 4y^2$, $16y = x^2$, $z = 0$

Solution: Given curves are $z = x^3$, $x = 4y^2$, $16y = x^2$, $z = 0$.

In xy plane $z = 0$.

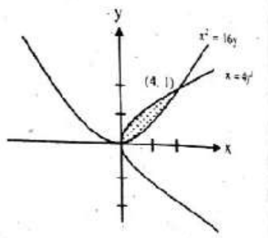
And, the region is bounded by the curve $x = 4y^2$ this is a parabola having vertex at (0, 0), line of symmetry $y = 0$ and right openward.

Also, the region is bounded by the curve $16y = x^2$ this is a parabola having vertex at (0, 0), line of symmetry $x = 0$ and up openward.

On these bases the base of the region is shown as in figure.

The region of integration in xy plane bounded by parabolas be, R: $0 \le x \le 4$, $0 \le y \le \sqrt{x/4}$.

Now, taking vertical strip,



$$\text{Volume} = \iint z \, dy \, dx = \int_0^4 \int_{x^2/16}^{\sqrt{x/4}} x^3 \, dy \, dx$$

$$= \int_0^4 x^3 \, [y]_{x^2/16}^{\sqrt{x/4}} dx$$

$$= \frac{1}{4}\int_0^4 x^3\left[\frac{\sqrt{x}}{2} - \frac{x^2}{16}\right] dx$$

$$= \int_0^4 \left(\frac{x^{7/2}}{2} - \frac{x^5}{16}\right) dx$$

$$= \left[\frac{1}{2} \times \frac{2}{9} x^{9/2} - \frac{1}{16} \times \frac{1}{6} x^6\right]_0^4$$

$$= \left[\frac{1}{9} 2^{2\times 9/2} - \frac{1}{96} 4^6\right]$$

$$= \frac{512}{9} - \frac{4096}{96}$$

$$= \frac{512}{9} - \frac{256}{6} = \frac{1024 - 768}{18} = \frac{256}{18} = \frac{128}{9}$$

Thus, the volume of the solid is cubic units.

(v) $z = x^2 + 4$, $y = 4 - x^2$, $x + y = 2$, $z = 0$

Solution: Given curves are $z = x^2 + 4$, $y = 4 - x^2$, $x + y = 2$, $z = 0$.

In xy plane, $z = 0 \Rightarrow x^2 + 4 = 0$.

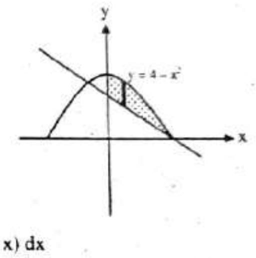
And, the region is bounded by the curve $y = 4 - x^2 \Rightarrow x^2 = -(y - 4)$. This is a parabola having vertex at (0, 4), line of symmetry $x = 0$ and down open-ward.

Also, the region is bounded by the line $x + y = 2$ passes through the points (0, 2) and (2, 0).

On these bases the base of the region is shown as in figure.

The region of integration in xy-plane bounded by parabola and lines be, R: $0 \le x \le 2$, $2 - x \le y \le (4 - x^2)$.

Now, taking vertical strip,



$$\text{Volume} = \iint z \, dy \, dx = \int_0^2 \int_{2-x}^{(4-x^2)} (x^2 + 4) \, dy \, dx$$

$$= \int_0^2 (x^2 + 4) \, [y]_{2-x}^{4-x^2} \, dx$$

$$= \int_0^2 (x^2 + 4)(4 - x^2 - 2 + x) \, dx$$

$$= \int_0^2 (4x^2 - x^4 - 2x^2 + x^3 + 16 - 4x^2 - 8 + 4x) \, dx$$

$$= \int_0^2 (-x^4 + x^3 - 2x^2 + 4x + 8) \, dx$$

$$= \left[\frac{x^5}{5} + \frac{x^4}{4} - \frac{2x^3}{3} + \frac{4x^2}{2} + 8x\right]_0^2$$

$$= \left(\frac{-32}{5} + \frac{16}{4} - \frac{16}{3} + 8 + 16\right)$$

$$= \frac{-96 + 60 - 80 + 120 + 240}{15} = \frac{244}{15}.$$

Thus, the volume of the solid is $\frac{244}{15}$.

3. **Find the volume of the solid whose base is the region in the xy plane that is bounded by the parabola $y = 4 - x^2$ and the line $y = 3x$, while the top of the solid is bounded by the plane $z = x + 4$.**

[2006 Spring; 2007 Fall; 2009 Fall; 2011 Fall Q. No. 3(b)]

Solution:

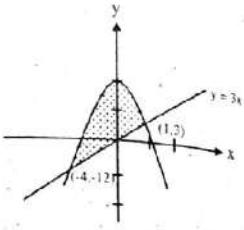
Given that the parabola $y = 4 - x^2$ and the line $y = 3x$ made a solid in xy-plane which is bounded in the top by a plane $z = x + 4$.

By solving $y = 4 - x^2$ and $y = 3x$, we get the base has common limits (1, 3) and (−4, −12).

Now, integrate the plane z over the region of base then.

$$V = \int_{x=-4}^{1} \int_{y=3x}^{4-x^2} (z)\, dy\, dx$$

$$= \int_{x=-4}^{1} \int_{y=3x}^{4-x^2} (x+4)\, dy\, dx$$



base of solid

$$= \int_{x=-4}^{1} (x+4)\, [y]_{3x}^{4-x^2}\, dx$$

$$= \int_{x=-4}^{1} (x+4)(4-x^2-3x)\, dx$$

$$= \int_{x=-4}^{1} (-8x - x^3 - 7x^2 + 16)\, dx$$

$$= \left[-4x^2 - \frac{x^4}{4} - \frac{7x^3}{3} + 16x\right]_{-4}^{1}$$

$$= \left(-4 - \frac{1}{4} - \frac{7}{3} + 16\right) - \left(-64 - 64 + \frac{448}{3} - 64\right)$$

$$= \left(12 - \frac{31}{12}\right) - \left(-192 + \frac{448}{3}\right) = \frac{144 - 31 + 2304 - 1792}{12} = \frac{625}{12}$$

Thus, volume of the solid is $\frac{625}{12}$ cubic units.

4. **Find the volume in the first octant bounded by the co-ordinate planes, the cylinder $x^2 + y^2 = 4$ and the plane $z + y = 3$.** **[2008 Fall Q. No. 3(b)]**

Solution:

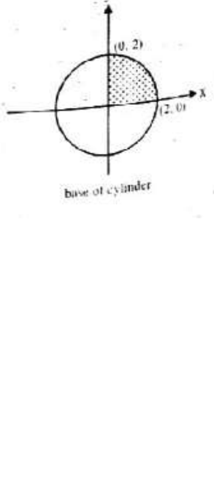
Given that, we have observe the volume in the first octant bounded by the coordinate planes, cylinder $x^2 + y^2 = 4$ and the plane $z + y = 3$.

From the figure it is clear that $z = 3 - y$ is to be integrated over the first quadrant of the circle $x^2 + y^2 = 4$.

For the base, limits for y are $y = 0$, and $y = \sqrt{4 - x^2}$ and limits for x are $x = 0$ and $x = 2$.

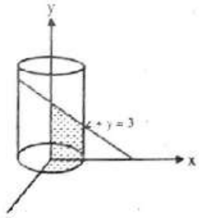
Now, volume of the solid is

$$V = \int_{x=0}^{2} \int_{y=0}^{\sqrt{4-x^2}} z\, dy\, dx$$



base of cylinder

$$= \int_{x=0}^{2} \int_{y=0}^{\sqrt{4-x^2}} (3-y)\, dy\, dx$$



$$= \int_{x=0}^{2} \left[3y - \frac{y^2}{2}\right]_0^{\sqrt{4-x^2}} dx$$

$$= \int_{x=0}^{2} \left(3\sqrt{4-x^2} - \frac{4-x^2}{2}\right) dx$$

$$= \left[3\left(\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right) - \frac{1}{2}\left(4x - \frac{x^3}{3}\right)\right]_0^2$$

$$= 3\left[0 + 2\sin^{-1}\left(\frac{2}{2}\right) - \frac{1}{2}\left(8 - \frac{8}{3}\right)\right] - 3[0 + 2\sin^{-1}0] + \frac{1}{2} . 0$$

$$= 6\sin^{-1}(1) - \frac{4}{3} \qquad [\because \sin^{-1} 0 = 0]$$

$$= 6 . \frac{\pi}{2} - \frac{4}{3} = 3\pi - \frac{4}{3}$$

Thus, volume of the solid is $3\pi - \frac{4}{3}$ cubic units.

5. **Find the volume of the solid in the first octant bounded by the co-ordinate planes, the plane $x = 3$, and the parabolic cylinder $z = 4 - y^2$.**

Solution

Given that the volume of the solid is restricted in the first octant bounded by coordinate planes, the plane $x = 3$ and the parabolic cylinder $z = 4 - y^2$.

Here, when $z = 0$ then $y = \pm 2$ that projected in xy-plane. Now, volume of the solid is

$$V = \int_{x=0}^{3} \int_{y=0}^{2} z\, dy\, dx = \int_{x=0}^{3} \int_{y=0}^{2} (4 - y^2)\, dy\, dx$$

$$= \int_{x=0}^{3} \left[4y - \frac{y^3}{3}\right]_0^2 dx$$

$$= \int_{x=0}^{3} \left(8 - \frac{8}{3}\right) dx = \frac{16}{3} \int_{x=0}^{3} dx = \frac{16}{3} [x]_0^3 = 16$$

Thus, volume of the solid is 16 cubic units.

6. **Find the volume of the prism whose base is the triangle in the xy-plane bounded by the x-axis and the line $y = x$, $x = 1$ and top lies in the plane $z = f(x, y) = 3 - x - y$.**

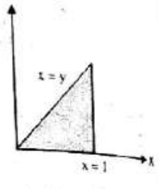
Solution

Given that the prism has base is the triangle in xy-plane that bounded by x-axis, $y = x$ and $x = 1$. And the top of the prism is bounded by $z = 3 - x - y$.

Thus, the limits of the prism for y are $y = 0$ and $y = x$.

And, the limits for x are $x = 0$ to $x = 1$.

Now, the volume generated by prism be,



base of prism.

$$V = \int_{x=0}^{1}\int_{y=0}^{x} z\,dy\,dx = \int_{x=0}^{1}\int_{y=0}^{x} (3 - x - y)\,dy\,dx$$

$$= \int_0^1 \left[3y - xy - \frac{y^2}{2}\right]_0^x dx$$

$$= \int_0^1 \left(3x - x^2 - \frac{x^2}{2}\right) dx$$

$$= \int_0^1 \left(3x - \frac{3x^2}{2}\right) dx = \left[\frac{3x^2}{2} - \frac{x^3}{2}\right]_0^1$$

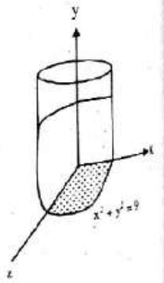
$$= \frac{3}{2} - \frac{1}{2} = 1.$$

Thus, volume of the prism is 1 cubic units.

7. **Find the volume V of the solid that lies in the first octant and is bounded by the three co-ordinate planes and the cylinder $x^2 + y^2 = 9$ and $y^2 + z^2 = 9$.**

Solution

Given that the solid lies is the first octant and it is bounded by $x^2 + y^2 = 9$ and $x^2 + z^2 = 9$. Here, these two cylinders have common portion whose volume is required. We need the volume in the first octant. For required volume, we integrate $z = \sqrt{9 - y^2}$ over the circle $x^2 + y^2 = 9$ in first quadrant. For this x varies from $y = 0$ to $y = 3$ and y varies from $x = 0$ to $x = \sqrt{9 - y^2}$. Then,



$$V = \int_0^3\int_0^{\sqrt{9-y^2}} z\,dx\,dy = \int_0^3\int_0^{\sqrt{9-y^2}} \sqrt{9-y^2}\,dx\,dy$$

$$= \int_0^3 \sqrt{9-y^2}\int_0^{\sqrt{9-y^2}} dx\,dy$$

$$= \int_0^3 \sqrt{9-y^2}\,[x]_0^{\sqrt{9-y^2}}\,dy$$

$$= \int_0^3 (9 - y^2)\,dy = \left[9y - \frac{y^3}{3}\right]_0^3 = 27 - \frac{27}{3} = 18$$

Thus, volume of the solid be 18 cubic units.

8. **Find the volume bounded by the xy-plane, the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 3$.** **[2008 Spring Q. No. 3(b)]**

Solution

Here, we have to determine the volume of the solid bounded by xy-plane, $x^2 + y^2 = 1$ and $x + y + z = 3$. For this we integrate the plane $z = 3 - x - y$ over $x^2 + y^2 = 1$ which is a circle with radius $r = 1$.

Set $x = r\cos\theta$, $y = r\sin\theta$ and the radius of circle is 1. So, $r = 1$. Moreover, the region is circle. So, the angle θ varies from 0 to 2π in the circle.

Also, $z = 3 - x - y = 3 - r\cos\theta - r\sin\theta$.

And, $dx\,dy = r\,dr\,d\theta$

Now, volume of the solid is

$$V = \iint z\,dx\,dy = \int_0^{2\pi}\int_0^1 z\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 (3r - r^2\cos\theta - r^2\sin\theta)\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[\frac{3r^2}{2} - \frac{r^3}{3}(\cos\theta + \sin\theta)\right]_0^1 d\theta$$

$$= \int_0^{2\pi} \left[\frac{3}{2} - \frac{1}{3}(\cos\theta + \sin\theta)\right] d\theta$$

$$= \left[\frac{3}{2}\theta - \frac{1}{3}(\sin\theta - \cos\theta)\right]_0^{2\pi}$$

$$= \left[\frac{3}{2}.2\pi - \frac{1}{3}(\sin 2\pi - \cos 2\pi)\right] - \left[0 - \frac{1}{3}(0 - 1)\right]$$

$$= 3\pi + \frac{1}{3} - \frac{1}{3} = 3\pi$$

Thus, the volume of the solid is 3π cubic units.

9. **Find the volume bounded by the xy-plane, the paraboloid $2z = x^2 + y^2$ and the cylinder $x^2 + y^2 = 4$.** **[2009 Spring; 2010 Spring Q. No. 3(b)]**

Solution

We have to generate the volume of the solid bounded by xy-plane, paraboloid $2z = x^2 + y^2$ and the cylinder $x^2 + y^2 = 4$.

For this, we integrate z over the circle $x^2 + y^2 = 4$ of radius r = 2.

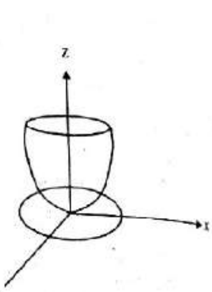
We change the integration in polar form as,

$x = r\cos\theta, \quad y = r\sin\theta$

Then, r = 0 to 2 and θ = 0 to θ = 2π.

Moreover, dxdy = rdr dθ

Now, volume of the solid is



$$V = \iint z\, dx\, dy = \int_0^{2\pi}\int_0^{2} \frac{r^2}{2} . r\, dr\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^{2} r^3\, dr\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left[\frac{r^4}{4}\right]_0^2 d\theta$$

$$= \frac{1}{8}\int_0^{2\pi}(16)\, d\theta = 2\int_0^{2\pi} d\theta = 2\,[\theta]_0^{2\pi} = 2\,.\,2\pi = 4\pi.$$

Thus, volume of the solid is 4π cubic units.

10. Find the volume of the region bounded by $z = x^2 + y^2$, $z = 0$, $x = -a$, $x = a$ and $y = -a$, $y = a$.

Solution

We have to determine the volume of the region bounded by $= x^2 + y^2$, $z = 0$, $x = -a$, $x = a$, $y = -a$, $y = a$.

Clearly, the given solid is uniform cylinder that has four symmetrical parts.

So, volume of the solid be

$$V = 4\int_{x=0}^{a}\int_{y=0}^{a} z\, dy\, dx$$

$$= 4\int_{x=0}^{a}\int_{y=0}^{a} (x^2 + y^2)\, dy\, dx$$

$$= 4\int_{x=0}^{a}\left[x^2y + \frac{y^3}{3}\right]_0^a dx$$

$$= 4\int_{x=0}^{a}\left[ax^2 + \frac{a^3}{3}\right]dx = 4\left[a\frac{x^3}{3} + \frac{a^3x}{3}\right]_0^a = 4\left[\frac{a^4}{3} + \frac{a^4x}{3}\right] = \frac{8a^4}{3}$$

Thus, the volume determine by the cylinder is $\frac{8a^4}{3}$ cubic units.

11. Proved that the volume enclosed between the cylinders $x^2 + y^2 = 2ax$ and $z^2 = 2ax$ is $\frac{64a^3}{15}$.

Solution

Given that the solid is enclosed by the cylinders $x^2 + y^2 = 2ax$ and $z^2 = 2ax$.

To find the required volume z is to be integrated over the circle $x^2 + y^2 = 2ax$ in xy-plane.

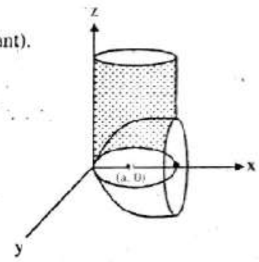
Also, $x^2 + y^2 = 2ax \Rightarrow x^2 - 2ax + a^2 + y^2 = a^2$

$\Rightarrow (x - a)^2 + (y - 0)^2 = a^2$

From the above equation, it is clear that radius of circle is a and centre lies at (a, 0).

Now, taking vertical strip,

Volume = $\iint z\, dy\, dx$ = 2 (volume in the first quadrant).



$$= 2\int_0^{2a}\int_0^{\sqrt{2ax - x^2}} \sqrt{2ax}\, dy\, dx$$

$$= 2\int_0^{2a} \sqrt{2ax}\,[y]_0^{\sqrt{2ax - x^2}}\, dx$$

$$= 2\int_0^{2a} \sqrt{2ax}\,.\sqrt{2ax - x^2}\, dx$$

$$= 2\int_0^{2a} \sqrt{x}\sqrt{2a}\sqrt{x}\sqrt{2a - x}\, dx$$

$$= 2\sqrt{2a}\int_0^{2a} x\sqrt{2a - x}\, dx$$

Put, $t = \sqrt{2a - x}$ then $2t\frac{dt}{dx} = -1 \Rightarrow 2t\, dt = -\, dx$. Also, $x = 0 \Rightarrow t = \sqrt{2a}$ and $x = 2a \Rightarrow t = 0$. Then,

$$= -2\sqrt{2a}\int_{\sqrt{2a}}^{0} (2a - t^2)t.\, 2tdt = -\,4\sqrt{2a}\int_{\sqrt{2a}}^{0} (2at^2 - t^4)\, dt$$

$$= -4\sqrt{2a}\left[2a\frac{t^3}{3} - \frac{t^5}{5}\right]_{\sqrt{2a}}^{0}$$

$$= -4\sqrt{2a}\left[-\frac{2a}{3}(\sqrt{2a})^3 + \frac{(\sqrt{2a})^5}{5}\right]$$

$$= -4\sqrt{2a}\left\{-\frac{4a^2\sqrt{2a}}{3} + \frac{4a^2\sqrt{2a}}{5}\right\}$$

$$= \frac{16a^2 \times 2a}{3} - \frac{16a^2 \times 2a}{5}$$

$$= \frac{32a^3}{3} - \frac{32a^3}{5} = \frac{32a^3(5-3)}{15} = \frac{64a^3}{15}.$$

Thus, the volume determine by the cylinder is $\frac{64a^3}{15}$ cubic units.

12. **Find the volume bounded by the plane $z = 0$, surface $z = x^2 + y^2 + 2$ and the cylinder $x^2 + y^2 = 4$.**

Solution

Since the solid is bounded by $z = 0$, $z = x^2 + y^2 + 2$, $x^2 + y^2 = 4$.

Here, the solid has volume in xy-plane. Clearly, the solid has four symmetrical parts. And the circle $x^2 + y^2 = 4$ has radius 2 and it moves form $\theta = 0$ to $\theta = 2\pi$.

Also, $x = r\cos\theta$, $y = r\sin\theta$ and $dx\,dy = r\,dr\,d\theta$

Now, volume of the solid is

$$V = \iint z\,dx\,dy$$

$$= \int_0^{2\pi}\int_0^{2} (r^2 + 2)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\frac{r^4}{4} + r^2\right]_0^2 d\theta$$

$$= \int_0^{2\pi}\left[\frac{16}{4} + 4\right] d\theta$$

$$= 8\int_0^{2\pi} d\theta = 8\,[\theta]_0^{2\pi} = 16\pi$$

Thus, volume of the solid is 16π.

13. **Find the volume under the plane $z = x + y$ and above the area cut from the first quadrant by the ellipse $4x^2 + 9y^2 = 36$.**

Solution

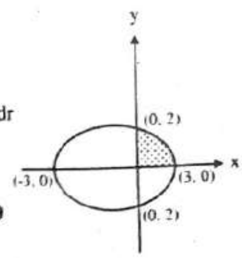
Given that the solid is bounded by, $z = x + y$, $4x^2 + 4y^2 = 36 \Rightarrow x^2 + y^2 = 9$.

Here, the solid has volume in xy-plane. Clearly, the solid has four symmetrical parts. And the circle $x^2 + y^2 = 9$ has radius 3 and it moves from $\theta = 0$ to $\theta = 2\pi$.

Moreover the radius moves from $r = 0$ to $r = 3$.

Also, $x = r\cos\theta$, $y = r\sin\theta$ and $dx\,dy = r\,dr\,d\theta$.

Now, volume of the solid is



$$V = \iint z\,dx\,dy = \int_0^{3}\int_0^{\pi/2} r(\cos\theta + \sin\theta)\,r\,d\theta\,dr$$

$$= \int_0^{\pi/2} (\cos\theta + \sin\theta)\left[\frac{r^3}{3}\right]_0^3 d\theta$$

$$= 9\int_0^{\pi/2} (\cos\theta + \sin\theta)\,d\theta$$

$$= 9\,[\sin\theta - \cos\theta]_0^{\pi/2}$$

$$= 9\left[\left(\sin\frac{\pi}{2} - \cos\frac{\pi}{2}\right) - (\sin 0 - \cos 0)\right]$$

$$= 9\int_0^{\pi/2} \{9\cos^3\theta + 12\sin\theta\,(1 - \sin^2\theta)\}\,d\theta$$

$$= \int_0^{\pi/2} (9\cos^3\theta + 12\sin\theta - 12\sin^3\theta)\,d\theta$$

$$= \int_0^{\pi/2}\left\{9\frac{(\cos3\theta + 3\cos\theta)}{4} + 12\sin\theta - 3(3\sin\theta - \sin3\theta)\right\}d\theta$$

$$= \int_0^{\pi/2}\left\{\frac{9}{4}(\cos3\theta + 3\cos\theta) + 12\sin\theta - 9\sin\theta + 3\sin3\theta\right\}d\theta$$

$$= \left[\frac{9}{4}\left(\frac{\sin3\theta}{3} + 3\sin\theta\right) - 12\cos\theta + 9\cos\theta - \frac{3\cos3\theta}{3}\right]_0^{\pi/2}$$

$$= \frac{9}{4}\left\{\frac{\sin\frac{3\pi}{2}}{3} + 3\sin\frac{\pi}{2}\right\} - 12\cos\frac{\pi}{2} + 9\cos\frac{\pi}{2} - \cos\frac{3\pi}{2} + 12\cos0^\circ - 9\cos0^\circ + \cos0^\circ$$

$$= \frac{9}{4} \times -\frac{1}{3} + \frac{27}{4} + 12 - 9 + 1$$

$$= -\frac{3}{4} + \frac{27}{4} + 4$$

$$= \frac{-3 + 27 + 16}{4}$$

$$= 10.$$

14. Find the volume bounded by the parabolic $x^2 + y^2 = az$, the cylinder $x^2 + y^2 = 2ay$ and the plane $z = 0$.

Solution

Given that the solid is bounded by the parabolic $x^2 + y^2 = az$, the cylinder $x^2 + y^2 = 2ay$ and the plane $z = 0$.

In xy plane, $z = 0 \Rightarrow x^2 + y^2 = 0$.

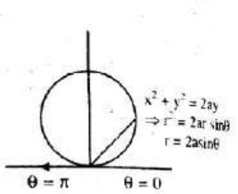
And, $x^2 + y^2 = 2ay$

$\Rightarrow x^2 + y^2 - 2ay + a^2 = a^2$

$\Rightarrow (x - 0)^2 + (y - a)^2 = a^2$

From the above equation it is clear that the centre of circle lies in (0, a) and radius is a. For required volume, we integrate $z = \left(\frac{x^2 + y^2}{a}\right)$ over the circle $x^2 + y^2 = 2ay$. For this y varies from $y = 0$ to $y = 2a$ and x varies from $x = 0$ to $x = \sqrt{2ay - y^2}$. Then,

$$\text{Volume} = \int_{\theta=0}^{\pi} \int_{r=0}^{2a\sin\theta} \frac{r^2}{a} \cdot r\, dr d\theta$$



$$= \frac{1}{a} \int_{\theta=0}^{\pi} \left[\frac{r^4}{4}\right]_0^{2a\sin\theta} d\theta$$

$$= \frac{1}{a} \int_{\theta=0}^{\pi} 4a^4 \sin^4\theta\, d\theta$$

$$= 4a^3 \times 2 \int_0^{\pi/2} \sin 4\theta\, d\theta$$

$$= 8a^3 \frac{\Gamma\frac{4+1}{2}\,\Gamma\frac{1}{2}}{2\Gamma\frac{4+0+2}{2}} = \frac{8a^3 \frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}\cdot\Gamma\frac{1}{2}}{2\Gamma 3}$$

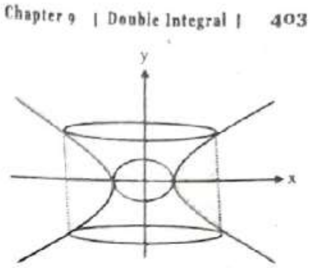
$$= \frac{3\pi a^3}{2}$$

15. Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the hyperboloid $x^2 + y^2 - z^2 = 1$.

Solution

Given that the solid is bounded by the cylinder $x^2 + y^2 = 4$, the hyperboloid $x^2 + y^2 - z^2 = 1$.

In xy-plane, $z = 0 \Rightarrow x^2 + y^2 = 1$.



$$\text{Total volume} = 8 \int_{\theta=0}^{\pi/2} \int_{r=1}^{2} \sqrt{(r^2 - 1)}\, r\, dr\, d\theta$$

$$= 8 \int_{\theta=0}^{\pi/2} \left[\frac{1}{3}(r^2 - 1)^{3/2}\right]_1^2 d\theta$$

$$= 8 \int_0^{\pi/2} \frac{1}{3}[3^{3/2} - 0]\, d\theta$$

$$= \frac{8.3\sqrt{3}}{3} \int_0^{\pi/2} d\theta = 8\sqrt{3} \times \frac{\pi}{2} = 4\sqrt{3}\pi.$$

16. Find the volume of the cylinder $x^2 + y^2 - 2ax = 0$ intercepted between the parabolic $x^2 + y^2 = 2az$ and the xy plane.

Solution

Given that the solid is bounded by the cylinder $x^2 + y^2 - 2ax = 0$ intercepted between the parabolic $x^2 + y^2 = 2az$ and the xy plane.

In xy-plane, $z = 0 \Rightarrow x^2 + y^2 = 0$.

Also given cylinder is,

$x^2 + y^2 - 2ax = 0$

$\Rightarrow x^2 - 2ax + a^2 + y^2 = a^2$

$\Rightarrow (x - a)^2 + y^2 = a^2$

For required volume, we integrate $2z = \left(\frac{x^2 + y^2}{a}\right)$ over the circle $x^2 + y^2 = 2ax$.

For this y varies from $x = 0$ to $x = 2a$ and x varies from $y = 0$ to $y = \sqrt{2ax - x^2}$. Then,

$$\text{Volume} = 2\iint z\, dy\, dx = 2\int_0^{2a} \int_0^{\sqrt{2ax - x^2}} x\, dy\, dx$$

$$= 2\int_0^{2a} \sqrt{2ax - x^2}\, dx$$

$$= \int_0^{2a} 2x\sqrt{2ax - x^2}\, dx$$

$$= \int_0^{2a} (2a - 2a + 2x)\sqrt{2ax - x^2}\, dx$$

$$= 2a\int_0^{2a} \sqrt{2ax - x^2}\, dx - \int_0^{2a} (2a - 2x)\sqrt{2ax - x^2}\, dx$$

$$= 2a\int_0^{2a} \sqrt{a^2 - (x^2 - 2ax + a^2)}\, dx - \left[\frac{2}{3}(2ax - x^2)^{3/2}\right]_0^{2a}$$

$$= 2a\int_0^{2a} \sqrt{a^2 - (x - a)^2}\, dx - 0 = 2a\left[\frac{x - a}{2}\sqrt{2ax - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x - a}{x}\right]_0^{2a}$$

$$= 2a\left[0 + \frac{a^2}{2}\sin^{-1} 1 + \frac{a^2}{2}\sin^{-1} 1\right]$$

$$= 2a\left[\frac{a^2}{2} \times \frac{\pi}{2} + \frac{a^2}{2} \times \frac{\pi}{2}\right]$$

$$= 2a \times \frac{2a^2\pi}{4}$$

$$= a^3\pi$$

Thus, the volume of the solid is $a^3\pi$ cubic units.