Exercise 6.4

1. **Solve the following differential equation:**

(i) $y' + 2y = 4x$

Solution: Given that, $y' + 2y = 4x$ ...... (i)

Comparing (i) with the equation $y' + Py = Q$ then we get,

$P = 2$ and $Q = 4x$.

Then the integrating factor of (i) is,

$$\text{I.F.} = e^{\int p dx} = e^{\int 2dx} = e^{2x}$$

Now, multiplying (i) by I.F. and then taking integration both sides, so that (i) becomes,

$$\boxed{y \times \text{I.F.} = \int Q \times \text{I.F. } dx + c}$$

$$y \times e^{2x} = \int 4x \times e^{2x} dx + c$$

$$\Rightarrow \quad y \times e^{2x} = 4x \frac{e^{2x}}{2} - 4\frac{e^{2x}}{4} + c$$

$$\Rightarrow \quad y \times e^{2x} = 2xe^{2x} - e^{2x} + c$$

$$\Rightarrow \quad y = 2x - 1 + ce^{-2x}$$

(ii) $y' - y = 3$

Solution: Given that, $y' - y = 3$ ...... (i)

Comparing (i) with the equation $y' + Py = Q$ then we get,

$P = -1$ and $Q = 3$.

Then the integrating factor of (i) is,

$$\text{I.F.} = e^{\int p dx} = e^{-\int 1 dx} = e^{-x}$$

Now, multiplying (i) by I.F. and then taking integration both sides, so that (i) becomes,

$$\boxed{y \times \text{I.F.} = \int Q \times \text{I.F.}\ dx + c}$$

$$\Rightarrow \quad y \times e^{-x} = \int 3 \times e^{-x}\, dx + c$$

$$\Rightarrow \quad ye^{-x} = -3e^{-x} + c \qquad \Rightarrow y = -3 + ce^{x}.$$

(iii) $y' + 2y = 6e^x$

Solution: Given that, $\quad y' + 2y = 6e^x$ ...... (i)

Comparing (i) with the equation $y' + Py = Q$ then we get,

$P = 2$ and $Q = 6e^x$.

Then the integrating factor of (i) is,

$$\text{I.F.} = e^{\int p dx} = e^{\int 2 dx} = e^{2x}$$

Now, multiplying (i) by I.F. and then taking integration both sides, so that (i) becomes,

$$\boxed{y \times \text{I.F.} = \int Q \times \text{I.F.}\ dx + c}$$

$$y \times e^{2x} = \int 6e^x \times e^{2x}\, dx + c$$

$$\Rightarrow \quad y \times e^{2x} = 6\int e^{3x}\, dx + c \Rightarrow \quad y \times e^{2x} = 6\frac{e^{3x}}{3} + c$$

$$\Rightarrow \quad y = 2e^x + ce^{-2x}$$

(iv) $y' + y \cot x = 2 \cos x$ **[2008 Spring Q. No. 4(a)]**

Solution: Given that, $\quad y' + y \cot x = 2 \cos x$ ...... (i)

Comparing (i) with the equation $y' + Py = Q$ then we get,

$P = \cot x$ and $Q = 2 \cos x$.

Then the integrating factor of (i) is,

$$\text{I.F.} = e^{\int p dx} = e^{\int \cot x} = e^{\log(\sin x)} = \sin x$$

Now, multiplying (i) by I.F. and then taking integration both sides, so that (i) becomes,

$$\boxed{y \times \text{I.F.} = \int Q \times \text{I.F.}\ dx + c}$$

$$\Rightarrow \quad y \times \sin x = \int 2 \cos x \sin x\, dx + c$$

$$\Rightarrow \quad y \times \sin x = \int \sin 2x\, dx \ \ c$$

$$\Rightarrow \quad y \times \sin x = -\frac{\cos 2x}{2} + c$$

$$\Rightarrow \quad 2y \sin x + \cos 2x + c.$$

(v) $y' + ky = e^{-kx}$

Solution: Given that, $\quad y' + ky = e^{-kx}$ ...... (i)

Comparing (i) with the equation $y' + Py = Q$ then we get,

$P = k$ and $Q = e^{-kx}$.

Then the integrating factor of (i) is,

$$\text{I.F.} = e^{\int p dx} = e^{k\int dx} = e^{kx}$$

Now, multiplying (i) by I.F. and then taking integration both sides, so that (i) becomes,

$$\boxed{y \times \text{I.F.} = \int Q \times \text{I.F.}\ dx + c}$$

$$y \times e^{kx} = \int e^{-kx} \times e^{kx}\, dx + c$$

$$\Rightarrow \quad y \times e^{kx} = \int dx + c \quad \Rightarrow \quad ye^{kx} = x + c$$

$$\Rightarrow \quad y = (x + c)\, e^{-kx}$$

(vi) $y' + 2y \tan x = \sin x$

Solution: Given that, $\quad y' + 2y \tan x = \sin x$ ...... (i)

Comparing (i) with the equation $y' + Py = Q$ then we get,

$P = 2 \tan x$ and $Q = \sin x$.

Then the integrating factor of (i) is,

$$\text{I.F.} = e^{\int p dx} = e^{\int 2\tan x\, dx} = e^{2\log \sec x} = \log^{\sec^2} = \sec^2 x$$

Now, multiplying (i) by I.F. and then taking integration both sides, so that (i) becomes,

$$\boxed{y \times \text{I.F.} = \int Q \times \text{I.F.}\ dx + c}$$

$$y \times \sec^2 x = \int \sin x \times \sec^2 x\, dx + c$$

$$\Rightarrow \quad y \sec^2 x = \int \tan x \sec x\, dx + c \quad \Rightarrow y \sec^2 x = \sec x + c$$

$$\Rightarrow y \sec^2 x - \sec x = c$$

(vii) $xy' - 2y = x^3 e^x$

Solution: Given that, $\quad xy' - 2y = x^3 e^x$ ...... (i)

Comparing (i) with the equation $y' + Py = Q$ then we get,

$P = \dfrac{-2}{x}$ and $Q = x^2 e^x$

Then the integrating factor of (i) is,

$$\text{I.F.} = e^{\int p dx} = e^{-\int \frac{2}{x} dx} = e^{-2\log x} = e^{\log (x)^{-2}} = \frac{1}{x^2}$$

Now, multiplying (i) by I.F. and then taking integration both sides, so that (i) becomes,

$$\boxed{y \times I.F. = \int Q \times I.F.\, dx + c}$$

$$y \times \frac{1}{x^2} = \int x^2 e^x \times \frac{1}{x^2}\, dx + c \quad \Rightarrow \frac{y}{x^2} = e^x + c$$

$$\Rightarrow y = x^2e^x + cx^2$$

(viii) $x^2y' + 2xy + \sinh 3x$

Solution: Given that, $x^2y' + 2xy + \sinh 3x$

$$\Rightarrow \quad y' + \frac{2}{x}y = \frac{1}{x^2}\sinh 3x \qquad \text{...... (i)}$$

Comparing (i) with the equation $y' + Py = Q$ then we get,

$$P = \frac{2}{x} \text{ and } Q = \frac{1}{x^2}\sinh 3x$$

Then the integrating factor of (i) is,

$$I.F. = e^{\int pdx} = e^{\int \frac{2}{x}dx} = e^{2\log x} = e^{\log x^2} = x^2$$

Now, multiplying (i) by I.F. and then taking integration both sides, so that (i) becomes,

$$\boxed{y \times I.F. = \int Q \times I.F.\, dx + c}$$

$$y \times x^2 = \int \frac{1}{x^2}\sin h\, 3x \times x^2\, dx + c$$

$$\Rightarrow y \times x^2 = \frac{\cos h\, 3x}{3} + c \quad \Rightarrow 3yx^2 = \frac{e^{3x} + e^{-3x}}{2} + c$$

$$\Rightarrow 6x^2y = (e^{3x} + e^{-3x}) + c.$$

(ix) $(1 + x)\frac{dy}{dx} - xy = 1 - x$

Solution: Given that, $(1 + x)\frac{dy}{dx} - xy = 1 - x$

$$\Rightarrow \quad \frac{dy}{dx} - \frac{x}{(1 + x)}y = \frac{(1 - x)}{(1 + x)} \qquad \text{...... (i)}$$

Comparing (i) with the equation $y' + Py = Q$ then we get,

$$P = -\frac{x}{1 + x} \text{ and } \qquad Q = \frac{(1 - x)}{(1 + x)}$$

Then the integrating factor of (i) is,

$$I.F. = e^{\int pdx} = e^{-\int \frac{x}{1+x}}$$

$$= e^{-\int \frac{1+x-1}{1+x}}$$

$$= e^{-\int\left(1 - \frac{1}{1+x}\right)dx} = e^{-(x - \log(1+x))} = e^{\log(1+x)}\, e^{-x} = (1 + x)\, e^{-x}$$

Now, multiplying (i) by I.F. and then taking integration both sides, so that (i) becomes,

$$\boxed{y \times I.F. = \int Q \times I.F.\, dx + c}$$

$$y \times e^{-(x - \log(1+x))} = \int \frac{1 - x}{1 + x} \times e^{(\log(1+x) - x)}\, dx + c$$

$$\Rightarrow \quad y \times (1 + x)e^{-x} = \int \frac{1 - x}{1 + x} \times (1 + x)\, e^{-x}\, dx + c$$

$$\Rightarrow \quad y \times (1 + x)\, e^{-x} = \int (e^{-x} - xe^{-x})\, dx + c$$

$$\Rightarrow \quad y \times (1 + x)e^{-x} = -e^{-x} - (-xe^{-x}\ e^{-x}) + c$$

$$\Rightarrow \quad y \times (1 + x)e^{-x} = -e^{-x} + xe^{-x} + e^{-x} + c$$

$$\Rightarrow \quad y(1 + x) = x + ce^x$$

(x) $(1 - x^2)\frac{dy}{dx} - xy = 1$

Solution: Given that, $(1 - x^2)\frac{dy}{dx} - xy = 1.$

$$\Rightarrow \quad \frac{dy}{dx} - \frac{x}{1 - x^2}y = \frac{1}{1 - x^2} \qquad \text{...... (i)}$$

Comparing (i) with the equation $y' + Py = Q$ then we get,

$$P = -\frac{x}{1 + x} \qquad \text{and} \qquad Q = \frac{1}{1 - x^2}$$

Then the integrating factor of (i) is,

$$I.F. = e^{\int pdx} = e^{-\int \frac{x}{1-x^2}dx} = e^{\frac{1}{2}\int \frac{-2x}{1-x^2}dx} = e^{\frac{1}{2}\log(1-x^2)} = e^{\log(1-x^2)^{1/2}} = \sqrt{(1 - x^2)}$$

Now, multiplying (i) by I.F. and then taking integration both sides, so that (i) becomes,

$$\boxed{y \times I.F. = \int Q \times I.F.\, dx + c}$$

$$y \times (1 - x^2)^{1/2} = \int \frac{1}{(1 - x^2)} \times (1 - x^2)^{1/2}\, dx + c$$

$$\Rightarrow \quad y \times (1 - x^2)^{1/2} = \int (1 - x^2)^{-1/2}\, dx + c$$

$$\Rightarrow \quad y \times (1 - x^2)^{1/2} = \int \frac{1}{\sqrt{(1 - x^2)}}\, dx + c \qquad \left[\text{diff. of } \sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}\right]$$

$$\Rightarrow \quad y \times \sqrt{(1 - x^2)} = \sin^{-1} a + c.$$

(xi) $\cosh x\, dy + (y \sinh x + e^x)\, dx = 0$.

Solution: Given that, $\cosh x\, dy + (y \sinh x + e^x)\, dx = 0$

$$\Rightarrow \frac{dy}{dx} + \frac{(y \sin hx + e^x)}{\cos hx} = 0$$

$$\Rightarrow \frac{dy}{dx} + y \tanh x = \frac{-e^x}{\cos hx} \quad \text{...... (i)}$$

Comparing (i) with the equation $y' + Py = Q$ then we get,

$$P = \tanh x \quad \text{and} \quad Q = \frac{-e^x}{\cos hx}$$

Then the integrating factor of (i) is,

$$\text{I.F.} = e^{\int p dx} = e^{\int \tan hx\, dx} = e^{\log \cos h x} = \cosh x$$

Now, multiplying (i) by I.F. and then taking integration both sides, so that (i) becomes,

$$\boxed{y \times \text{I.F.} = \int Q \times \text{I.F.}\, dx + c}$$

$$y \times \cos hx = \int \frac{-e^x}{\cos hx} \times \cosh x\, dx + c$$

$$\Rightarrow y \times \cosh x = -e^x + c$$

$$\Rightarrow y \cosh x = c - e^x$$

(xii) $(x - 2y)\, dy + y\, dx = 0$.

Solution: Given that, $(x - 2y)\, dy + y\, dx = 0$.

$$\Rightarrow \frac{dx}{dy} = -\frac{x - 2y}{y}$$

$$\Rightarrow \frac{dx}{dy} + \frac{x}{y} = 2 \quad \text{...... (i)}$$

Comparing (i) with the equation $y' + Py = Q$ then we get,

$$P = \frac{1}{y} \quad \text{and} \quad Q = 2$$

Then the integrating factor of (i) is,

$$\text{I.F.} = e^{\int p dx} = e^{\int \frac{1}{y} dy}\ e^{\log y} = y$$

Now, multiplying (i) by I.F. and then taking integration both sides, so that (i) becomes,

$$\boxed{y \times \text{I.F.} = \int Q \times \text{I.F.}\, dx + c}$$

$$x \times y = \int 2y + c$$

$$\Rightarrow xy = y^2 + c.$$

(xiii) $x\, dy + y\, dx = y\, dy$

Solution: Given that, $x\, dy + y\, dx = y\, dy$

$$\Rightarrow \frac{x\, dy}{y\, dy} + \frac{y\, dx}{y\, dy} = 1$$

$$\Rightarrow \frac{dx}{dy} + \frac{x}{y} = 1 \quad \text{...... (i)}$$

Comparing (i) with the equation $y' + Py = Q$ then we get,

$$P = \frac{1}{y} \quad \text{and} \quad Q = 1.$$

Then the integrating factor of (i) is,

$$\text{I.F.} = e^{\int p dx} = e^{\int \frac{1}{y} dy} = e^{\log y} = y$$

Now, multiplying (i) by I.F. and then taking integration both sides, so that (i) becomes,

$$\boxed{y \times \text{I.F.} = \int Q \times \text{I.F.}\, dx + c}$$

$$x \times y = \int y\, dy + c$$

$$\Rightarrow xy = \frac{y^2}{2} + c$$

$$\Rightarrow x = \frac{y}{2} + \frac{c}{y}$$

(xiv) Repeated to (xi)

(xv) Repeated to (xii)

(xvi) $(x^2 + 1)\frac{dy}{dx} + 2xy = x^2$

Solution: Given that, $(x^2 + 1)\frac{dy}{dx} + 2xy = x^2$

$$\Rightarrow \frac{dy}{dx} + \frac{2xy}{(x^2 + 1)} = \frac{x^2}{(x^2 + 1)} \quad \text{...... (i)}$$

Comparing (i) with the equation $y' + Py = Q$ then we get.

$$P = \frac{2x}{(x^2 + 1)} \quad \text{and} \quad Q = \frac{x^2}{x^2 + 1}$$

Then the integrating factor of (i) is,

$$\text{I.F.} = e^{\int p dx} = e^{\int \frac{2x}{x^2 + 1} dx} = e^{\log(x^2 + 1)} = (x^2 + 1)$$

Now, multiplying (i) by I.F. and then taking integration both sides, so that (i) becomes,

$$\boxed{y \times \text{I.F.} = \int Q \times \text{I.F.}\, dx + c}$$

$$y \times (x^2+1) = \int \frac{a^2}{(x^2+1)} \times (x^2+1)\,dx + c$$

$$\Rightarrow \quad y(x^2+1) = \frac{x^3}{3} + c$$

(xvii) $(1+x^3)\frac{dy}{dx} + 6x^2y = 1 + x^2$

Solution: Given that, $(1+x^3)\frac{dy}{dx} + 6x^2y = 1 + x^2$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{6x^2y}{1+x^3} = \frac{1+x^2}{1+x^3} \qquad \text{...... (i)}$$

Comparing (i) with the equation $y' + Py = Q$ then we get,

$$P = \frac{6x^2}{1+x^2} \quad \text{and} \quad Q = \frac{1+x^2}{1+x^3}$$

Then the integrating factor of (i) is,

$$\text{I.F.} = e^{\int p dx} = e^{\int \frac{6x^2}{1+x^3}dx} = e^{2\int \frac{3x^2}{1+x^3}dx} = e^{2\log(1+x^3)} = e^{\log(1+x^3)^2} = (1+x^3)^2$$

Now, multiplying (i) by I.F. and then taking integration both sides, so that (i) becomes,

$$\boxed{y \times \text{I.F.} = \int Q \times \text{I.F.}\, dx + c}$$

$$\Rightarrow \quad y \times (1+3^3)^2 = \int \frac{1+x^2}{(1+x^3)} \times (1+x^3)^2\, dx + c$$

$$\Rightarrow \quad y(1+x^3)^2 = \int (1 + x^3 + x^2 + x^5)\, dx + c$$

$$\Rightarrow \quad y(1+x^3)^2 = \left(x + \frac{x^4}{4} + \frac{x^3}{3} + \frac{x^6}{6}\right) + c.$$

(xix) $x\frac{dy}{dx} + y = e^x - xy$

Solution: Given that, $x\frac{dy}{dx} + y = e^x - xy \Rightarrow \frac{dy}{dx} + \frac{y}{x} = \frac{e^x}{x} - y$

$$\Rightarrow \frac{dy}{dx} + y\left(\frac{1}{x} + 1\right) = \frac{e^x}{x} \qquad \text{...... (i)}$$

Comparing (i) with the equation $y' + Py = Q$ then we get,

$$P = \frac{1}{x} + 1 \quad \text{and} \quad Q = \frac{e^x}{x}$$

Then the integrating factor of (i) is,

$$\text{I.F.} = e^{\int p dx} = e^{\int\left(\frac{1}{x}+1\right)dx} = e^{(\log x + x)} = e^{\log x}\,.\,e^x$$

Now, multiplying (i) by I.F. and then taking integration both sides, so that (i) becomes,

$$\boxed{y \times \text{I.F.} = \int Q \times \text{I.F.}\, dx + c}$$

$$\Rightarrow \quad y \times xe^x = \int \frac{e^x}{x} \times xe^x\, dx + c \quad \Rightarrow \quad yxe^x = \frac{e^{2x}}{2} + c$$

$$\Rightarrow \quad xy = \frac{e^x}{2} + ce^{-x}$$

(xx) $ye^y\, dy = (y^2 + 2xe^y)\, dy$

Solution: Given that

$$ye^y\, dx = (y^3 + 2x\, e^y)\, dy \qquad \text{........(1)}$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{y^3 + 2xe^y}{ye^y} = y^2e^{-y} + x\left(\frac{2}{y}\right)$$

$$\Rightarrow \quad \frac{dx}{dy} + x\left(\frac{-2}{y}\right) = y^2e^{-y} \qquad \text{........(2)}$$

This is linear differential equation is x whose integrating factor is

$$\text{I.F.} = e^{\int -2/y\, dy} = e^{-2\log y} = \frac{1}{y^2}$$

Now, multiplying (2) by I.F. and then integrating we get,

$$x.\frac{1}{y^2} = \int y^2.\, e^{-y}\,.\,\frac{1}{y^2}\, dy + c$$

$$= \int e^{-y}\, dy + c = \frac{e^{-y}}{-1} + c = (c - e^{-y})$$

$$\Rightarrow \quad x = y^2(c - e^{-y})$$

(xxi) $\frac{dy}{dx} = \frac{2y \log y + y - x}{y}$

Solution: Given that, $\frac{dy}{dx} = \frac{2y \log y + y - x}{y}$

$$= 2\log y + 1 - \frac{x}{y}$$

$$\Rightarrow \quad \frac{dx}{dy} + \frac{x}{y} = (2\log y + 1) \qquad \text{...... (i)}$$

Comparing (i) with the equation $y' + Py = Q$ then we get,

$$P = \frac{1}{y} \quad \text{and} \quad Q = 2\log y + 1$$

Then the integrating factor of (i) is,

$$\text{I.F.} = e^{\int p dx} = e^{\int \frac{1}{y}dy} = e^{\log y} = y$$

Now, multiplying (i) by I.F. and then taking integration both sides, so that (i) becomes,

$$\boxed{y \times \text{I.F.} = \int Q \times \text{I.F. } dx + c}$$

$$x \times y = \int (2y \log y + 1)\, y\, dy + c$$

$$\Rightarrow \quad xy = \int (2y \log y + y)\ dy + c$$

$$\Rightarrow \quad xy = \int 2y \log y\, dy + \int y\, dy + c$$

$$\Rightarrow \quad xy = \log y \int 2y\, dy - \int \left[\frac{d \log y}{dy} \int 2y\, dy\right] dy + \int y\, dy + c$$

$$\Rightarrow \quad xy = \log y \times 2\frac{y^2}{2} - \int \frac{1}{y} \times 2\frac{y^2}{2}\, dy + \frac{y^2}{2} + c$$

$$\Rightarrow \quad xy = y^2 \log y - \frac{y^2}{2} + \frac{y^2}{2} + c$$

$$\Rightarrow \quad x = y \log y + \frac{c}{y}$$

(xxii) $x^2 \dfrac{dy}{dx} = 3x^2 - 2xy + 1$

Solution: Given that, $x^2 \dfrac{dy}{dx} = 3x^2 - 2xy + 1$

$$\Rightarrow \quad \frac{dy}{dx} = 3 - \frac{2y}{x} + \frac{1}{x^2}$$

$$\Rightarrow \quad \frac{dy}{dx} + \left(\frac{2}{x}\right) y = \left(3 + \frac{1}{x^2}\right) \quad \ldots\ldots \text{(i)}$$

Comparing (i) with the equation $y' + Py = Q$ then we get,

$$P = \frac{2}{x} \qquad \text{and} \qquad Q = \left(3 + \frac{1}{x^2}\right)$$

Then the integrating factor of (i) is,

$$\text{I.F.} = e^{\int p dx} = e^{2\int \frac{1}{x} dx} = e^{2\log x} = e\log^{x^2} = x^2$$

Now, multiplying (i) by I.F. and then taking integration both sides, so that (i) becomes,

$$\boxed{y \times \text{I.F.} = \int Q \times \text{I.F. } dx + c}$$

$$y \times x^2 = \int \left(3 + \frac{1}{x^2}\right) x^2\, dx + c \Rightarrow y \times x^2 = \int (3x^2 + 1)\, dx + c$$

$$\Rightarrow y \times x^2 = 3 \times \frac{x^3}{3} + x + c$$

$$\Rightarrow y = x + \frac{1}{x} + \frac{c}{x^2}$$

2. Solve the following initial value problems.

(i) $x^2y + 2xy - x + 1 = 0,\ y(1) = 0$

Solution: Given that, $x^2y + 2xy - x + 1 = 0$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{2}{x}y - \frac{1}{x} + \frac{1}{x^2} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} + \left(\frac{2}{x}\right) y = \left(\frac{1}{x} - \frac{1}{x^2}\right) \quad \ldots\ldots \text{(i)}$$

and $\quad y(1) = 0 \quad \ldots\ldots$ (ii)

Comparing (i) with the equation $y' + Py = Q$ then we get,

$$P = \frac{2}{x} \qquad \text{and} \qquad Q = \frac{1}{x} - \frac{1}{x^2}$$

Then the integrating factor of (i) is,

$$\text{I.F.} = e^{\int p dx} = e^{2\int \frac{1}{x} dx} = e^{2\log x} = e\log^{x^2} = x^2$$

Now, multiplying (i) by I.F. and then taking integration both sides, so that (i) becomes,

$$\boxed{y \times \text{I.F.} = \int Q \times \text{I.F. } dx + c}$$

$$y \times x^2 = \int \left(\frac{1}{x} - \frac{1}{x^2}\right) x^2\, dx + c$$

$$\Rightarrow \quad yx^2 = \int (x - 1)\, dx + c \ \Rightarrow x^2 y = \left(\frac{x^2}{2} - x\right) + c$$

$$\Rightarrow y = \frac{1}{2} - \frac{1}{x} + \frac{c}{x^2} \quad \ldots\ldots \text{(iii)}$$

Using (ii), then (iii) gives, $0 = \frac{1}{2} - \frac{1}{1} + \frac{c}{1^2} \Rightarrow 1 - \frac{1}{2} = c \ \Rightarrow c = \frac{1}{2}$

Now (iii) becomes,

$$y = \frac{1}{2} - \frac{1}{x} + \frac{1}{2x^2}$$

(ii) $y' + y = (x + 1)^2,\ y(0) = 0$

Solution: Given that, $y' + y = (x+1)^2 \ \Rightarrow \dfrac{dy}{dx} + y = (x+1)^2$

$$\Rightarrow \frac{dy}{dx} + y = x^2 + 2x + 1 \quad \ldots\ldots \text{(i)}$$

And, $y(0) = 0 \quad \ldots\ldots$ (ii)

Comparing (i) with the equation $y' + Py = Q$ then we get,

$$P = 1 \qquad \text{and} \qquad Q = (x+1)^2$$

Then the integrating factor of (i) is,

$$\text{I.F.} = e^{\int p dx} = e^{\int dx} = e^x$$

Now, multiplying (i) by I.F. and then taking integration both sides, so that (i) becomes,

$$\boxed{y \times \text{I.F.} = \int Q \times \text{I.F. } dx + c}$$

$$y \times e^x = \int (x+1)^2 e^x \, dx + c$$

$$\Rightarrow \quad ye^x = (x+1)^2 e^x - 2(x+1)e^{dx} + 2e^x + c \quad \text{...... (iii)}$$

Using (ii), then (iii) gives,

$$0 \,.\, e^0 = (0+1)^2 e^0 = 2(0+1) e^0 + 2e^0 = c$$

$$\Rightarrow \quad 0 = 1 - 2 + 2 + c$$

$$\Rightarrow \quad c = -1$$

Now (iii) becomes,

$$ye^x = (x+1)^2 e^x - 2(x+1) e^x + 2e^x - 1$$

$$\Rightarrow \quad y = (x+1)^2 - 2(x+1) + 2 - e^x$$

$$\Rightarrow \quad y = x^2 + 2x + 1 - 2x - 2 + 2 - e^{-x}$$

$$\Rightarrow \quad y = x^2 + 1 - e^{-x}$$

(iii) $xy' - 3y - x^4(e^x + \cos x) - 2x^2, \; y(\pi) - \pi^3 e^\pi + 2\pi^2$

Solution: Given that, $\quad xy' - 3y - x^4(e^x + \cos x) - 2x^2$

$$\Rightarrow \quad x\frac{dy}{dx} - 3y = x^4(e^x + \cos x) - 2x^2$$

$$\Rightarrow \quad \frac{dy}{dx} - \frac{3}{x}y = x^3(e^x + \cos x) - 2x \quad \text{...... (i)}$$

And, $y(\pi) - \pi^3 e^\pi + 2\pi^2$ .... (ii)

Comparing (i) with the equation $y' + Py = Q$ then we get,

$$P = -\frac{3}{x} \quad \text{and} \quad Q = x^3(e^x + \cos x) - 2x$$

Then the integrating factor of (i) is,

$$\text{I.F.} = e^{\int p dx} = e^{-3\int \frac{1}{x} dx} = e^{-3\log x} = e^{\log x^{-3}} = \frac{1}{x^3}$$

Now, multiplying (i) by I.F. and then taking integration both sides, so that (i) becomes,

$$\boxed{y \times \text{I.F.} = \int Q \times \text{I.F. } dx + c}$$

$$y \times \frac{1}{x^3} = \int \{x^3(e^x + \cos x) - 2x\} \frac{1}{x^3} dx + c$$

$$\Rightarrow \quad \frac{y}{x^3} = \int \left(e^x + \cos x - \frac{2}{x^2}\right) dx + c$$

$$\Rightarrow \quad \frac{y}{x^3} = e^x + \sin x + 2x^{-1} + c$$

$$\Rightarrow \quad y = x^3(e^x + \sin x + 2x^{-1} + c) \quad \text{...... (iii)}$$

Using (ii), then (iii) gives,

$$\pi^3 e^\pi + 2\pi^2 = \pi^3(e^\pi + \sin \pi + 2\pi^{-1} + c)$$

$$\Rightarrow \quad \pi^3 e^\pi + 2\pi^2 = e^\pi \pi^3 + 2\pi^2 + \pi^3 c \;\Rightarrow\; c = 0$$

Now (iii) becomes,

$$y = x^3\left(e^x + \sin x + \frac{2}{x}\right).$$

(iv) $y' + \frac{y}{x} = x^2, \; y(1) = 1$

Solution: Given that, $\quad y' + \frac{y}{x} = x^2$ ...... (i)

And, $\quad y(1) = 1$ .... (ii)

Comparing (i) with the equation $y' + Py = Q$ then we get,

$$P = \frac{1}{x} \quad \text{and} \quad Q = x^2$$

Then the integrating factor of (i) is,

$$\text{I.F.} = e^{\int p dx} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$$

Now, multiplying (i) by I.F. and then taking integration both sides, so that (i) becomes,

$$\boxed{y \times \text{I.F.} = \int Q \times \text{I.F. } dx + c}$$

$$y \times x = \int x^2 + x \, dx + c$$

$$\Rightarrow \quad xy = \frac{x^4}{4} + c \quad \text{...... (i)}$$

Using (ii), then (iii) gives,

$$1 \times 1 = \frac{1}{4} + c \;\Rightarrow\; 1 = \frac{1}{4} + c \;\Rightarrow\; c = \frac{3}{4}$$

Now (iii) becomes,

$$xy = \frac{x^4}{4} + \frac{3}{4} \;\Rightarrow\; 4xy = x^4 + 3.$$

(v) $\frac{dy}{dx} + 4y = 20,\ y(0) = 2$

Solution: Given that, $\frac{dy}{dx} + 4y = 20$ ...... (i)

And, $y(0) = 2$ ...... (ii)

Comparing (i) with the equation $y' + Py = Q$ then we get,

$P = 4$ and $Q = 20$

Then the integrating factor of (i) is,

$$\text{I.F.} = e^{\int p dx} = e^{4\int dx} = e^{4x}$$

Now, multiplying (i) by I.F. and then taking integration both sides, so that (i) becomes,

$$\boxed{y \times \text{I.F.} = \int Q \times \text{I.F.}\ dx + c}$$

$$ye^{4x} = \int 20\, e^{4x}\, dx + c \Rightarrow ye^{4x} = 20\frac{e^{4x}}{4} + c$$

$$\Rightarrow ye^{4x} = 5e^{4x} + c \quad ...... (iii)$$

Using (ii), then (iii) gives,

$$2 \times e^{4\times 0} = 5e^{4\times 0} + c \Rightarrow 2 = 5 + c \Rightarrow c = -3$$

Now (iii) becomes,

$$ye^{4x} = 5e^{4x} - 3$$

$$\Rightarrow y = 5 - 3e^{-4x}$$

(vi) $\frac{dy}{dx} = 2(y-1)\tanh 2x,\ y(0) = 4$

Solution: Given that, $\frac{dy}{dx} = 2(y-1)\tanh 2x$

$$\Rightarrow \frac{dy}{dx} - 2y\tanh 2x = -2\tanh 2x \quad ...... (i)$$

And $y(0) = 4$ ...... (ii)

Comparing (i) with the equation $y' + Py = Q$ then we get,

$P = -2\tanh 2x$ and $Q = -2\tanh 2x$

Then the integrating factor of (i) is,

$$\text{I.F.} = e^{\int p dx} = e^{-2\int \tanh 2x} = e^{-2\frac{\log \cosh 2x}{2}} = e^{\log(\cos h\, 2x)^{-1}} = \frac{1}{(\cos h\, 2x)}$$

Now, multiplying (i) by I.F. and then taking integration both sides, so that (i) becomes,

$$\boxed{y \times \text{I.F.} = \int Q \times \text{I.F.}\ dx + c}$$

$$y \times \frac{1}{(\cos h\, 2x)} = \int -2\tan 2x \times \frac{1}{\cos h\, 2x}\, dx + c$$

$$\Rightarrow \frac{y}{(\cos h\, 2x)} = \int \frac{-2\sin h\, 2x}{(\cos h\, 2x)^2}\, dx + c \quad ...... (iii)$$

Let, $I = \int \frac{-2\sin h\, 2x}{(\cos h\, 2x)^2}\, dx$

Put, $v = \cos h\, 2x$ then $\frac{dv}{dx} = -2\sin h\, 2x \Rightarrow dv = -2\sin h\, 2x\, dx$.

Then, $I = \int \frac{dv}{v^2} = \int v^{-2}\, dv = \frac{v^{-1}}{-1} = \frac{(\cos h\, 2x)^{-1}}{-2}$

So that (iii) becomes,

$$\frac{y}{(\cos h\, 2x)} = \frac{2(\cos h\, 2x)^{-1}}{2} + c$$

$$\Rightarrow y = 1 + c(\cosh 2x) \quad ...... (iv)$$

Using (ii), then (iii) gives,

$$4 = 1 + c(\cosh 0) \Rightarrow c = 4 - 1 = 3$$

Now (iv) becomes,

$$y = 1 + 3\cosh 2x$$

(vii) $\frac{dy}{dx} + 3y = \sin x,\ y\left(\frac{\pi}{2}\right) = 0.3$

Solution: Given that, $\frac{dy}{dx} + 3y = \sin x$ ...... (i)

And, $y\left(\frac{\pi}{2}\right) = 0.3$ ...... (ii)

Comparing (i) with the equation $y' + Py = Q$ then we get,

$P = 3$ and $Q = \sin x$

Then the integrating factor of (i) is,

$$\text{I.F.} = e^{\int p dx} = e^{3\int dx} = e^{3x}$$

Now, multiplying (i) by I.F. and then taking integration both sides, so that (i) becomes,

$$\boxed{y \times \text{I.F.} = \int Q \times \text{I.F.}\ dx + c}$$

$$ye^{3x} = \int \sin x\, e^{3x}\, dx + c \quad ...... (iii)$$

Let, $I = \int \sin x\, e^{3x}\, dx + c$

$$\Rightarrow I = \frac{e^{3x}}{9+1}[3\sin x - \cos x] + c$$

$$\Rightarrow I = \frac{e^{3x}}{10}[3\sin x - \cos x] + c$$

Then (iii) becomes,

$$ye^{3x} = \frac{e^{3x}}{10}[3\sin x - \cos x] + c$$

$$\Rightarrow \quad y = \frac{1}{10}[3\sin x - \cos x] + c\,e^{-3x} \qquad \text{...... (iv)}$$

Using (ii), then (iv) gives,

$$0.3 = \frac{1}{10}[3 \times 1 - 0] + c \;\Rightarrow\; c = 0.$$

Now (iv) becomes,

$$y = \frac{1}{10}[3\sin x - \cos x].$$

(viii) $\frac{dy}{dx} = (1 + y^2),\ y(0) = 0$

Solution: Given that, $\frac{dy}{dx} = 1 + y^2 \Rightarrow \frac{dy}{1 + y^2} = dx$

Integrating we get, $\tan^{-1}(y) = x + c$ ...... (i)

Also, given that, $y(0) = 0$ then (i) gives,

$$\tan^{-1}(0) = 0 + c \Rightarrow 0 = c.$$

Now, (i) becomes,

$$\tan^{-1} y = x \;\Rightarrow\; y = \tan x.$$

(ix) $\frac{dy}{dx} + y\cot x = 4x\cos x,\ y\left(\frac{\pi}{2}\right) = 0$

Solution: Given that, $\frac{dy}{dx} + y\cot x = 4x\,\text{cosec}\,x$ ...... (i)

And, $y\left(\frac{\pi}{2}\right) = 0$ ...... (ii)

Comparing (i) with the equation $y' + Py = Q$ then we get,

$P = \cot x$ and $Q = 4x\,\text{cosec}\,x$

Then the integrating factor of (i) is,

$$\text{I.F.} = e^{\int p dx} = e^{\int \cot x\, dx} = e^{\log \sin x} = \sin x$$

Now, multiplying (i) by I.F. and then taking integration both sides, so that (i) becomes,

$$\boxed{y \times \text{I.F.} = \int Q \times \text{I.F.}\, dx + c}$$

$$y \times \sin x = \int 4x\,\text{cosec}\,x \times \sin x\, dx + c$$

$$\Rightarrow \quad y\sin x = 2\int 2x\, dx + c$$

$$\Rightarrow \quad y\sin x = 2x^2 + c \qquad \text{...... (iii)}$$

Using (ii), then (iii) gives,

$$0\sin\frac{\pi}{2} = 2\left(\frac{\pi}{2}\right)^2 + c \;\Rightarrow\; c = \frac{-\pi^2}{2}$$

Now (iii) becomes,

$$y\sin x = 2x^2 - \frac{\pi^2}{2}$$

(x) $\frac{dy}{dx} + y\cot x = 5e^{\cos x},\ y\left(\frac{\pi}{2}\right) = -4$

Solution: Given that, $\frac{dy}{dx} + y\cot x = 5e^{\cos x}$ ...... (i)

And, $y\left(\frac{\pi}{2}\right) = -4$ ...... (ii)

Comparing (i) with the equation $y' + Py = Q$ then we get,

$P = \cot x$ and $Q = 5e^{\cos x}$

Then the integrating factor of (i) is,

$$\text{I.F.} = e^{\int p dx} = e^{\int \cot x\, dx} = e^{\log \sin x} = \sin x$$

Now, multiplying (i) by I.F. and then taking integration both sides, so that (i) becomes,

$$\boxed{y \times \text{I.F.} = \int Q \times \text{I.F.}\, dx + c}$$

$$y \times \sin x = \int 5e^{\cos x} \times \sin x\, dx + c \qquad \text{....... (iii)}$$

Put $u = \cos x$ then $\frac{du}{dx} = -\sin x \Rightarrow -du = \sin x\, dx$. Then (iii) becomes,

$$y\sin x = -5\int e^u\, du + c = -5e^u + c = -5e^{\cos x} + c$$

$$\Rightarrow \quad y\sin x + 5e^{\cos x} = c \qquad \text{...... (iv)}$$

Using (ii), then (iv) gives,

$$-4\sin\frac{\pi}{2} + 5e^{\cos\left(\frac{\pi}{2}\right)} = c \;\Rightarrow\; -4 + 5e^0 = c \;\Rightarrow\; c = 1$$

Now (iii) becomes,

$$y\sin x + 5e^{\cos x} = 1.$$

(xi) $\frac{dy}{dx} - y\tan x = 3e^{-\sin x},\ y(0) = 4$

Solution: Given that, $\frac{dy}{dx} - y\tan x = 3e^{-\sin x}$ ...... (i)

And, $y(0) = 4$ ...... (ii)

Comparing (i) with the equation $y' + Py = Q$ then we get,

$$P = -\tan x \qquad \text{and} \qquad Q = 3e^{-\sin x}$$

Then the integrating factor of (i) is,

$$\text{I.F.} = e^{\int p\,dx} = e^{\int -\tan x} = e^{-\log \sec x} = e^{\log(\sec x)^{-1}} = (\sec x)^{-1} = \frac{1}{\sec x} = \cos x$$

Now, multiplying (i) by I.F. and then taking integration both sides, so that (
becomes,

$$\boxed{y \times \text{I.F.} = \int Q \times \text{I.F.}\, dx + c}$$

$$y \times \cos x = \int 3e^{-\sin x} \cos x\, dx + c$$

Put, $u = -\sin x$, then $\dfrac{du}{dx} = -\cos x \Rightarrow -du = \cos x\, dx$. So that,

$$y \cos x = -\int 3e^{u}\, du + c$$

$$= -3e^{u} + c = -3e^{-\sin x} + c$$

$$\Rightarrow \quad y \cos x = -3e^{-\sin x} + c \qquad \ldots\ldots \text{(iii)}$$

Using (ii), then (iii) gives,

$$4 \cos 0 = -3e^{-\sin 0} + c \Rightarrow 4 + 3 = c \Rightarrow c = 7$$

Now (iii) becomes,

$$y \cos x = 7 - 3e^{-\sin x}$$