Exercise 6.6

Solve by the method of variation of parameters.

(1) $y' - \frac{2}{x}y = x^2 \cos 3x$

Solution: Given that, $y' - \frac{2}{x}y = x^2 \cos 3x$

$$\Rightarrow \frac{dy}{dx} - \frac{2}{x}y = x^2 \cos 3x \qquad \ldots\ldots (i)$$

Compare equation (i) with $\frac{dy}{dx} + Py = Q$ then we get,

$$P = -\frac{2}{x}, \qquad Q = x^2 \cos 3x$$

Now, corresponding homogeneous equation of (i) is

$$\frac{dy}{dx} - \frac{2}{x}y = 0 \Rightarrow \frac{dy}{dx} = \frac{2y}{x} \Rightarrow \frac{dy}{y} = 2\frac{dx}{x}$$

Integrating we get,

$$\log y = 2 \log x + \log c \Rightarrow \log y = \log (c\, x^2) \Rightarrow y = cx^2$$

Set, $v(x) = x^2$ then, $y = cv$.

Now, using formula,

$$y = v\left(\int \frac{Q}{v} dx + c\right) \Rightarrow y = x^2\left(\int \frac{x^2 \cos 3x}{x^2} dx + c\right)$$

$$\Rightarrow y = x^2\left(\frac{\sin 3x}{3} + c\right).$$

(2) $y' - y = x$

Solution: Given that, $\frac{dy}{dx} - y = x$ ...... (i)

Compare equation (i) with $\frac{dy}{dx} + Py = Q$ then we get,

$P = -1$ and $Q = x$

Corresponding homogeneous equation of (i) is

$$\frac{dy}{dx} - y = 0 \Rightarrow \frac{dy}{dx} = y \Rightarrow \frac{dy}{y} = dx$$

Integrating we get,

$$\log y = x + \log c \Rightarrow y = ce^{(x)}$$

Set, $v(x) = e^x$ then $y = cv$.

Now, using formula,

$$y = v\left(\int \frac{Q}{V} dx + c\right) \Rightarrow y = e^x\left(\int \frac{x}{e^x} dx + c\right)$$

$$\Rightarrow y = e^x (\int xe^{-x} dx + c)$$

$$\Rightarrow y = e^x(-xe^{-x} - e^x + c)$$

$$\Rightarrow y = -x - 1 + ce^x$$

$$\Rightarrow y = ce^x - x - 1.$$

(3) $xy' - 2y = x^4$

Solution: Given that, $x\frac{dy}{dx} - 2y = x^4 \Rightarrow \frac{dy}{dx} - \frac{2y}{x} = x^3$ ...... (i)

Comparing equation (i) with $\frac{dy}{dx} + Py = Q$ then we get,

$P = -\frac{2}{x}$ and $Q = x^3$

Corresponding homogeneous equation of (i) is

$$\frac{dy}{dx} - \frac{2y}{x} = 0 \Rightarrow \frac{dy}{dx} = \frac{2y}{x} \Rightarrow \frac{dy}{y} = 2\frac{dx}{x}$$

Integrating we get,

$$\log y = 2\log x + \log c \Rightarrow \log y = \log x^2 + \log c$$

$$\Rightarrow \log y = \log cx^2 \Rightarrow y = cx^2$$

Set, $v = x^2$ then $y = cv$.

Now, using formula,

$$y = v\left(\int \frac{Q}{V} dx + c\right) \Rightarrow y = x^2\left(\int \frac{Q}{V} dx + c\right)$$

$$\Rightarrow y = x^2\left(\frac{x^2}{2} + c\right) \Rightarrow y = \frac{x^4}{2} + cx^2$$

(4) $y' - 2xy = -2x$

Solution: Given that, $\frac{dy}{dx} - 2xy = -2x$ ...... (i)

Comparing equation (i) with $\frac{dy}{dx} + Py = Q$ then we get,

$P = -2x$ and $Q = -2x$.

Corresponding homogeneous equation of (i) is

$$\frac{dy}{dx} - 2xy = 0 \Rightarrow \frac{dy}{dx} = 2xy \Rightarrow \frac{dy}{y} = 2x\,dx.$$

Integration we get,

$$\log y = x^2 + \log c \Rightarrow y = ce^{x^2}$$

Set, $v(x) = e^{x^2}$ then we get, $y = cv$

Now, using formula,

$$y = v\left\{\int \frac{Q}{V} dx + c\right\} \Rightarrow y = e^{x^2}\left\{\int \frac{-2x}{e^{x^2}} dx + c\right\}$$

$$\Rightarrow y = e^{x^2}\{-\int(2xe^{-x^2} + c)\}$$

$$\Rightarrow y = e^{x^2}(e^{-x^2} + c)$$

$$\Rightarrow y = 1 + ce^{x^2}$$

(5) $y' + 3y = e^{-2x}$

Solution: Given that, $\frac{dy}{dx} + 2y = e^{-2x}$ ...... (i)

Comparing equation (i) with $\frac{dy}{dx} + Py = Q$ then we get,

$P = 2$ and $Q = e^{-2x}$

Corresponding homogeneous equation of (i) is

$$\frac{dy}{dx} + 2y = 0 \Rightarrow \frac{dy}{y} = -2dx$$

Integrating we get,

$$\log y = -2x + \log c \Rightarrow y = ce^{-2x}$$

Set, $v(x) = e^{-2x}$ then, $y = cv$.

Now, using formula,

$$y = v\left\{\int \frac{Q}{V} dx + c\right\} \Rightarrow y = e^{-2x}\left\{\int \frac{e^{-2x}}{e^{-2x}} dx + c\right\} \Rightarrow y = e^{-2x}(x + c).$$