OTHER QUESTIONS FROM SEMESTER END EXAMINATION

2002 Q. No. 2(b)

State Euler's theorem for partial derivative of homogeneous function of two variables. If $u = \sin^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{1}{2}\tan u$.

Solution: First Part: See the statement of Euler's theorem for partial derivative of homogeneous function of two variables, P.

Second Part:

Let, $u = \sin^{-1}\dfrac{x+y}{\sqrt{x}+\sqrt{y}} \Rightarrow \sin u = \dfrac{x+y}{\sqrt{x}+\sqrt{y}}$

Let, $f = \sin u$

Differentiating f partially w. r. t. 'x' and 'y' then,

$$\frac{\partial f}{\partial x} = \cos u \frac{\partial u}{\partial x}, \qquad \text{and} \qquad \frac{\partial u}{\partial x} = \cos u \frac{\partial u}{\partial y}$$

And, $f = \dfrac{x + y}{\sqrt{x} + \sqrt{y}}$

Set x as tx and y as ty then

$$f(tx, ty) = \frac{tx + ty}{\sqrt{tx} + \sqrt{ty}} = \frac{t(x + y)}{\sqrt{t}\,(\sqrt{x} + \sqrt{y})} = t^{1/2}.f$$

This shows that f is homogeneous function of degree (n) $= \frac{1}{2}$.

Then by Euler's theorem,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = \frac{1}{2}.f \;\Rightarrow\; x \cos u \frac{\partial u}{\partial x} + y \cos u \frac{\partial u}{\partial y} = \frac{1}{2}\sin u$$

$$\Rightarrow \cos u \left(x\frac{\partial u}{\partial x} + y\frac{1}{2}\right) = \frac{1}{2}\sin u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{1}{2} = \frac{1}{2}\tan u$$

Thus, $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{1}{2}\tan u.$

2003 Fall Q. No. 2(b)

State and prove Euler's theorem for a homogeneous function of two variable of degree n and hence if $v = \dfrac{xy}{x + y}$, show that $x\dfrac{\partial v}{\partial x} + y\dfrac{\partial v}{\partial y} = v$.

Solution: First Part: See the statement and its prove of Euler's theorem for partial derivative of homogeneous function of two variables.

Second Part: Let $v = \dfrac{xy}{x + y}$

Set x by tx and y by ty then

$$v = \frac{t^2 xy}{t(x + y)} = t\left(\frac{xy}{x + y}\right)$$

This shows that v is a homogeneous function of degree (n) = 1.

Then by Euler's theorem,

$$x\frac{\partial v}{\partial v} + y\frac{\partial v}{\partial y} = nv \;\Rightarrow\; x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = v$$

2004 Spring Q. No. 2(b)

What is homogeneous function? State and prove Euler's theorem on homogeneous function of two variables.

Solution: First Part: See definition of homogeneous function.

Second Part: See the statement and its prove of Euler's theorem for partial derivative of homogeneous function of two variables.

2006 Fall Q. No. 2(b)

State and prove Eulers theorem for homogeneous function of two variables with degree n. If $\sin u = \dfrac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$.

Solution: First Part: See the statement and its prove of Euler's theorem for partial derivative of homogeneous function of two variables.

Second Part: Let $f = \sin u = \dfrac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}$. For $f = \sin u$.

Differentiating f w. r. to x and y then

$$\frac{\partial f}{\partial x} = \cos u \frac{\partial u}{\partial x} \qquad \text{and} \qquad \frac{\partial f}{\partial y} = \cos u \frac{\partial u}{\partial y}$$

Also, for $f = \dfrac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}$, set x by tx and y by ty.

So, $$f = \frac{t^{1/2}(\sqrt{x} - \sqrt{y})}{t^{1/2}(\sqrt{x} + \sqrt{y})} = t^0\left(\frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}\right)$$

This shows that f is homogeneous of degree (n) = 0.

Then by Euler's theorem,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf \;\Rightarrow\; \cos u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = 0.\,f = 0$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$$

2006 Spring Q. No. 2(b)

Define partial derivative of a function at a point. If $u = \log\sqrt{x^2 + y^2 + z^2}$, show that: $(x^2 + y^2 + z^2)\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}\right) = 1$.

Solution: First Part: See definition of partial derivative of a function.

Second Part: See Exercise 10.1 Q. No. 6.

2008 Spring Q. No. 2(b)

Define homogeneous function of two variables. Verify Euler's theorem for $u = x^n \sin(y/x)$.

Solution: First Part: See definition of homogeneous function. P.

Second Part: Let $u = x^n \sin\left(\dfrac{y}{x}\right)$

Set x as tx and y as ty then

$$u(tx, ty) = (tx)^n \sin\left(\frac{ty}{tx}\right) = t^n x^n \sin\left(\frac{y}{x}\right) = t^n.u$$

This shows that u is homogeneous function of degree (n) = n.

Then we wish to show $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$.

Here differentiating u partially w. r. t. 'x' and 'y' then,

$$\frac{\partial u}{\partial x} = nx^{n-1}\sin\left(\frac{y}{x}\right) + x^n\cos\left(\frac{y}{x}\right)\times\left(-\frac{y}{x^2}\right) \qquad \text{And } \frac{\partial u}{\partial y} = x^n \times \cos\left(\frac{y}{x}\right)\times\frac{1}{x}$$

$$= nx^{n-1}\sin\left(\frac{y}{x}\right) - x^{n-2}y\cos\left(\frac{y}{x}\right) \qquad = x^{n-1}\cos\left(\frac{y}{x}\right)$$

So,

$$x\frac{\partial u}{\partial x} = nx^n\sin\left(\frac{y}{x}\right) - x^{n-1}y\cos\left(\frac{y}{x}\right) \qquad \text{And} \qquad y\frac{\partial u}{\partial y} = x^{n-1}y\cos\left(\frac{y}{x}\right)$$

Now,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n\sin\left(\frac{y}{x}\right) - x^{n-1}y\cos\left(\frac{y}{x}\right) + x^{n-1}y\cos\left(\frac{y}{x}\right)$$

$$= nx^n\sin\left(\frac{y}{x}\right)$$

$$= n.u$$

Hence, the Euler's theorem is verified.

2009 Fall Q. No. 2(a)

State and prove Euler's theorem for a homogeneous function of two variable of degree n and hence if $u = \tan^{-1}\left[\frac{x^3+y^3}{x+y}\right]$, $x \neq y$. Show that: $x\frac{\partial u}{\partial u} + y\frac{\partial u}{\partial y} = \sin 2u$.

Solution: First Part: See the statement and its prove of Euler's theorem for partial derivative of homogeneous function of two variables.

Second Part: See Exercise 10.2 Q. No. 3.

2008 Spring Q. No. 2(a); 2011 Fall Q. No. 2(b)

State and prove Euler's theorem for homogeneous function of two variables. If $U = \log\left(\frac{x^2+y^2}{x+y}\right)$, prove that: $x\frac{\partial U}{\partial 2x} + y\frac{\partial U}{\partial y} = 1$.

Solution: First Part: See the statement and its prove of Euler's theorem for partial derivative of homogeneous function of two variables.

Second Part: See Exercise 10.2 Q. No. 4.

SHORT QUESTIONS:

1999, 2001: If $u = \tan^{-1}\frac{y}{x}$ then $\partial u = \ldots$

Solution: Let $u = \tan^{-1}\left(\frac{y}{x}\right)$. Then,

$$\frac{\partial u}{\partial x} = \frac{1}{1+(y/x)^2}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{1}{1+(y/x)^2}\left(\frac{1}{x^2}\right) = \frac{1}{x^2+y^2}$$

Now,

$$du = \frac{\partial u}{\partial x}\cdot dx + \frac{\partial u}{\partial y}\cdot dy$$

$$= \frac{1}{x^2+y^2}(-y\,dx + dy).$$

2004 Spring; 2008 Spring: Find $\frac{dy}{dx}$ if $x^3 + y^3 - 3axy = 0$ (using partial derivative).

2006 Fall: If $u = x^2 + y^2 + z^2$, show that: $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2u$.

Solution: Here, $u = x^2 + y^2 + z^2$

Replace x by tx, y by ty and z by tz. Then,

$$u = t^2(x^2 + y^2 + z^2)$$

This shows that u is a homogeneous function of order (n) = 2.

Then by Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu \Rightarrow x\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2u.$$

2006 Spring: If $v = x^3 + y^3 + z^3$ than verify that $x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + z\frac{\partial v}{\partial z} = 3v$.

Solution: Here, $v = x^3 + y^3 + z^3$

Replace x by tx, y by ty and z by tz. Then,

$$u = t^3(x^3 + y^3 + z^3)$$

This shows that u is a homogeneous function of order (n) = 3.

Then by Euler's theorem

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + z\frac{\partial v}{\partial z} = nu \Rightarrow x\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} + z\frac{\partial v}{\partial z} = 3v.$$

2009 Spring: If $V = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$, then find the value of $x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} + z\frac{\partial V}{\partial z}$.

Solution: Let, $v = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$

Replace x by tx, y by ty and z by tz. Then

$$v = \frac{tx}{ty} + \frac{ty}{tz} + \frac{tz}{tx} = t^0\left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x}\right)$$

This shows that v is homogeneous of order (n) = 0. Then by Euler's theorem.

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{y} + z\frac{\partial v}{\partial z} = nv \Rightarrow x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + z\frac{\partial v}{\partial z} = 0.$$

2009 Fall: If $f(x, y, z) = \frac{x}{y} + \frac{y}{z} + \frac{x}{x}$, then show that: $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = 0$.

Solution: See 2009 Spring with replacing V by f.

2010 Spring, 2011 Fall: Verify the Eulers' theorem for f(x, y) = x/y.

Solution: Let $f = \frac{x}{y}$

Set x by tx and y by ty. Then,

$$f = \frac{tx}{ty} = t^{\circ}\left(\frac{x}{y}\right)$$

This shows that f is homogeneous of order (n) = 0.

Then by Euler's theorem,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf = 0 \qquad \text{........ (i)}$$

And we have, $f = \frac{x}{y}$

So, $\frac{\partial f}{\partial x} = \frac{y}{y^2}$ And $\frac{\partial f}{\partial y} = -\frac{x}{y^2}$

Now,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = \frac{xy}{y^2} - \frac{xy}{y^2} = 0$$

This verifies (i).

Thus, f verifies the Euler's theorem.

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