A differential equation of the form $y'' + Py' + Q = 0$, is known as homogeneous equation of second order. Where, P and Q are constants or function of independent variables of the equaton.

Note: If the differential equation is not homogeneous then it is called non-homogenous.

Exact differential equation

A differential equation of the form, $M\,dx + N\,dy = 0$, is called differential equation if $\frac{dM}{dy} = \frac{dN}{dX}$, is satisfied.

Integrating factor:

If a differential equation is not exact, some time the equation may be exact if the equation is multiplied by a function. Such function is called an integrating factor (I. F.).

Linear differential equation:

A differential equation of the form, $y' + Py = Q$ where, P and Q are constant or function of independent variables of the equation, is called a linear differential equation of first order.

Note: If $Q = 0$ in above equation, then the equation is called a homogeneous linear differential equation.

A differential equation of the form $y'' + Py' + Qy = R$ where, P, Q and R are constant or function or function of independent variable of the equation is called homogenous linear differential equation of second order.

Bernoulli's equations:

A differential equation of the form $y' + Py = Qy^n$, where, P and Q are constant or function of independent variable of the equation is called a Bernoulli's equation of first order.

Exercise 6.1

A. Find the general solution of the following:

(i) $y' - 2y + a = 0$

Solution: Given equation is

$$y' - 2y + a = 0 \Rightarrow \frac{dy}{dx} = 2y - a$$

$$\Rightarrow \frac{2\,dy}{2y - a} = 2dx$$

Integrating both sides then,

$$\log(2y - a) = 2x + \log(c)$$

$$\Rightarrow \log\left(\frac{2y - a}{c}\right) = 2x \Rightarrow \frac{2y - a}{c} = e^{2x} \Rightarrow y = \frac{c}{2}e^{2x} + \frac{a}{2}$$

(ii) (x logx) y'

Solution: Given equation is

$$(x\log x)\,y' = y \Rightarrow y' = \frac{y}{x\log x}$$

$$\Rightarrow \frac{dy}{y} = \frac{dx}{x\log x}$$

Taking integration on both sides then,

$$\int\frac{dy}{y} = \int\frac{dx}{x\log x}$$

Put $\log(x) = t$ then $\frac{1}{x}dx = dt$. So,

$$\int\frac{dy}{y} = \int\frac{dt}{t}$$

$$\Rightarrow \log(y) = \log(t) + \log(c)$$

$$\Rightarrow \log(y) = \log(ct) \Rightarrow y = ct \Rightarrow y = c.\log(x)$$

(iii) $(1 + x)y\,dx + (1 + y)x\,dy = 0$

Solution: Given equation is

$$(1 + x)y\,dx + (1 + y)x\,dy = 0$$

$$\Rightarrow \frac{(1 + x)\,dx}{x} + \frac{(1 + y)dy}{y} = 0 \quad [\because \text{dividing both sides by } xy]$$

$$\Rightarrow \left(\frac{1}{x} + 1\right)dx + \left(\frac{1}{y} + 1\right)dy = 0$$

Taking integration on both sides then,

$$\int\left(\frac{1}{x} + 1\right)dx + \int\left(\frac{1}{y} + 1\right)dy = 0$$

$$\Rightarrow x + \log(x) + y + \log(y) = C$$

$$\Rightarrow x + y + \log(xy) = C.$$

(iv) tany dx + tanx dy = 0

Solution: Given equation is

$$\tan y\,dx + \tan x\,dy = 0$$

$$\Rightarrow \frac{dx}{\tan x} + \frac{dy}{\tan y} = 0 \quad [\because \text{dividing both sides by } \tan x \tan y]$$

$$\Rightarrow \frac{\cos x\, dx}{\sin x} + \frac{\cos y}{\sin y} dy = 0$$

Taking integration on both sides then,

$$\int \frac{\cos x}{\sin x} dx + \frac{\cos y}{\sin y} dy = 0$$

$$\Rightarrow \log(\sin x) + \log(\sin y) = c_1$$

$$\Rightarrow \log(\sin x \,.\, \sin y) = c_1$$

$$\Rightarrow \sin x \,.\, \sin y = e^{c_1} = c \quad \text{(say)}$$

(v) y' = y tanh x

Solution: Given equation is

$$y' = y \tanh x \quad \Rightarrow \frac{dy}{dx} = y \tanh x$$

$$\Rightarrow \frac{dy}{y} = \tanh x\, dx = \frac{\sinh x}{\cosh x} . dx$$

Taking integration on both sides then,

$$\int \frac{dy}{y} = \int \frac{\sinh x}{\cosh x} . dx$$

$$\Rightarrow \log(y) = \log(\cosh x) + \log(c)$$

$$\Rightarrow y = c \cosh x.$$

(vi) $y' = 2x^{-1}\sqrt{y-1}$

Solution: Given equation is

$$y' = 2x^{-1}\sqrt{y-1} \Rightarrow \frac{dy}{dx} = \frac{2\sqrt{y-1}}{x}$$

$$\Rightarrow \frac{dy}{2\sqrt{y-1}} = \frac{dx}{x}$$

Taking integration on both sides then

$$\int \frac{dy}{2\sqrt{y-1}} = \int \frac{dx}{x}$$

Put $y - 1 = t^2$ then $dy = 2t\, dt$. Therefore,

$$\int \frac{2\, t dt}{2\sqrt{t^2}} = \int \frac{dx}{x} \Rightarrow \int dt = \int \frac{dx}{x}$$

$$\Rightarrow t = \log(x) + c$$

$$\Rightarrow \sqrt{y-1} = \log(x) + c$$

$$\Rightarrow y - 1 = (\log(x) + c)^2$$

$$\Rightarrow y = 1 + (\log(x) + c)^2$$

(vii) $e^{x-y}\, dx + e^{y-x}\, dy = 0$. **[2010 Spring- Short]**

Solution: Given equation is

$$e^{x-y}\, dx + e^{y-x}\, dy = 0$$

$$\Rightarrow \frac{e^x}{e^y} dx + \frac{e^y}{e^x} dy = 0$$

$$\Rightarrow e^{2x}\, dx + e^{2y}\, dy = 0$$

Taking integration on both sides then,

$$\int e^{2x}\, dx + \int e^{2y}\, dy = 0$$

$$\Rightarrow \frac{e^{2x}}{2} + \frac{e^{2y}}{2} = c_1 \quad \Rightarrow e^{2x} + e^{2y} = 2c_1 = c \quad \text{(say)}$$

(viii) $x \cos y\, dy = (x\, e^x \log(x) + e^x)\, dx$

Solution: Given equation is

$$x \cos y\, dy = (x\, e^x \log(x) + e^x)\, dx$$

$$\Rightarrow \cos y\, dy = \frac{x\, e^x \log(x) + e^x}{x} dx$$

$$= \left[e^x \log(x) + \frac{e^x}{x}\right] dx$$

Taking integration on both sides then,

$$\int \cos y\, dy = \int e^x \left[\log(x) + \frac{1}{x}\right] dx$$

$$\Rightarrow \sin y = e^x \log(x) + c \qquad \left[\because \int e^x [f(x) + f'(x)]\, dx = e^x f(x) + c\right]$$

(ix) $y\, dx = (1 + e^x)\, dy$

Solution: Given equation is

$$y\, dx = (1 + e^x)\, dy$$

$$\Rightarrow \frac{dx}{1 + e^x} = \frac{dy}{y}$$

$$\Rightarrow \frac{e^{-x}}{(e^{-x} + 1)} dx = \frac{dy}{y}$$

Taking integration on both sides then,

$$\int \frac{e^{-x}}{(e^{-x} + 1)} dx = \int \frac{dy}{y}$$

[illegible]

[illegible]

(iv) [illegible]

Solution: Given equation is

[illegible]

Taking integration on both sides, we get

[illegible]

(v) [illegible] $= 0$

Solution: Given equation is

[illegible]

Taking integration on both sides,

[illegible]

2. Solve the following initial value problems.

(i) [illegible]

Solution: Given equation is

[illegible]

Thus,

[illegible]

Taking integration on both sides, we get

[illegible]

Since $y(0) = 0$. So (1) gives

[illegible]

Then (1) becomes

[illegible]

(ii) [illegible] $y(0) = 2$

Solution: Given equation is

[illegible]

Integrating we get,

[illegible]

Since $y(0) = 2$, then (1) gives

[illegible]

So, then (1) becomes

[illegible]

(iii) [illegible]

Solution: Given equation is

[illegible]

Here,

$$2xy' = 3y \Rightarrow 2\frac{dy}{y} = \frac{3}{x}dx$$

Integrating we get,

$$2\log(y) = 3\log(x) + \log c$$

$$\Rightarrow y^2 = x^3.c \quad ....(1)$$

Since $y(1) = 4$. Then (1) gives,

$$(4)^2 = (1)^3.c \Rightarrow c = 16.$$

Therefore, (1) becomes,

$$y^2 = 16x^3$$

$$\Rightarrow y = 4x^{3/2}$$

(iv) $xy.y' = y + 2$, $y(2) = 0$

Solution: Given equation is

$$xy.y' = y + 2 \text{ with } y(2) = 0$$

Here, $xy.y' = y + 2 \Rightarrow \left(\frac{y}{y+2}\right)dy = \frac{dx}{x}$

$$\Rightarrow \left(1 - \frac{2}{y+2}\right)dy = \frac{dx}{x}$$

Integrating we get,

$$y - 2\log(y+2) = \log(x) + \log c$$

$$= \log(cx) \quad ....(1)$$

Since we have, $y(2) = 0$. Then (1) gives

$$0 - 2\log(2) = \log(2c)$$

$$\Rightarrow \log(4)^{-1} = \log(2c) \Rightarrow \frac{1}{4} = 2c \Rightarrow c = \frac{1}{8}.$$

Therefore (1) becomes,

$$y - 2\log(y+2) = \log\left(\frac{x}{8}\right).$$

(v) $Li + RI = 0$, $I(0) = I_0$ whre $i = \frac{dI}{dt}$, (L & R are constants)

Solution: Given equation is

$$Li + RI = 0 \quad \text{with } I(0) = I_0 \text{ whre } i = \frac{dI}{dt}, \quad \text{(L \& R are constants)}$$

Here, $Li + RI = 0 \Rightarrow L\frac{dI}{dt} + RI = 0$

$$\Rightarrow \frac{L\,dI}{I} + R\,dt = 0$$

Integrating we get,

$$L\log(I) + Rt = c \quad .....(1)$$

Since we have,

$I(0) = I_0$. Then (1) gives

$$L\log(I_0) + R.0 = c$$

$$\Rightarrow c = L\log(I_0)$$

Therefore, (1) becomes,

$$L\log(I) + Rt = L\log(I_0)$$

$$\Rightarrow \log(I) + \frac{R}{L}t = \log I_0 \quad \Rightarrow \log(I) - \log(I_0) = -\frac{R}{L}t$$

$$\Rightarrow \log\left(\frac{I}{I_0}\right) = \frac{R}{L}t$$

$$\Rightarrow I = I_0\, e^{-(Rt/L)}$$

(vi) $dr\sin\theta = 2r\cos\theta\, d\theta$, $r\left(\frac{\pi}{2}\right) = 2$

Solution: Given equation is

$$dr\sin\theta = 2r\cos\theta\, d\theta \quad \text{with } r\left(\frac{\pi}{2}\right) = 2$$

Here, $dr.\sin\theta = 2r\cos\theta\, d\theta$

$$\Rightarrow \frac{dr}{r} = 2.\frac{\cos\theta}{\sin\theta}d\theta$$

Integrating we get,

$$\log(r) = 2\log(\sin\theta) + \log(c)$$

$$= \log(\sin^2\theta) + \log(c)$$

$$\Rightarrow r = c\sin^2\theta \quad .....(1)$$

Since we have, $r\left(\frac{\pi}{2}\right) = 2$. So, (1) gives

$$2 = c\sin^2\left(\frac{\pi}{2}\right) = 2 \Rightarrow c = 2$$

Thus, (1) becomes,

$$r = 2\sin^2\theta.$$

C. Show that the given function is a solution of given differential equation. (Here a, b, c are arbitrary constants).

(1) $y = c\,e^{-x} + x^2 - 2x$

Solution: Given equation is

$$y = c\,e^{-x} + x^2 - 2x \quad .....(1)$$

Then we have to show (1) is solution of

$y' + y = x^2 - 2$ .....(2)

Here, by (1),

$y = ce^{-x} + x^2 - 2x$

So, $y' = -ce^{-x} + 2x - 2$

Then,

$y' + y = -ce^{-x} + 2x - 2 + ce^{-x} + x^2 - 2x$

$= x^2 - 2$

$\Rightarrow$ $y' + y = x^2 - 2$. Thus, (2) is satisfied.

This shows, (1) is solution of (2).

2. $y = e^x + ax^2 + bx + c$

Solution: Given equation is

$y = e^x + ax^2 + bx + c$ .....(1)

So, $y' = e^x + 2ax + b$

And, $y'' = e^x + 2a$

Also, $y''' = e^x$ .....(2)

Thus, (2) shows that (1) is the solution of (2).

3. $x^2 + y^2 = 1$

Solution: Given equation is

$x^2 + y^2 = 1$ ......(1)

Then we shall show that (1) satisfies the equation

$x + y\,y' = 0$ .....(2)

Here

$x^2 + y^2 = 1$

Differentiating w. r. t. x, then

$2x + 2y.y' = 0$

$\Rightarrow$ $x + yy' = 0$

This shows that (1) satisfies (2). So, (1) is the solution of (2).

4. $y = ce^{-2x} + 14$

Solution: Given equation is

$y = ce^{-2x} + 14$ . ...(1)

Then we have to show (1) is the solution of

$y' + 2y = 2.8$ .....(2)

Here, $y = ce^{-2x} + 1.4$

So, $y' = -2ce^{-2x}$

Then,

$y' + 2y = -2c\,e^{-2x} + 2ce^{-2x} + 2.8$

$\Rightarrow$ $y' + 2y = 2.8$

This proves that y is the solution of equation (2)

5. $y = cx^3$

Solution: Given equation is

$y = cx^3$ ......(1)

Then we have to show (1) is the solution of

$xy' = 3y$ ....(2)

Here, $y = cx^3$

So, $y' = 3cx^2$

Then, $xy' = 3cx.x^2 = 3cx^3 = 3y$

$\Rightarrow$ $xy' = 3y$

This shows that (1) is the solution of (2).

6. $x^2 + 4y^2 = c$

Solution: Given equation is

$x^2 + 4y^2 = c$ ....(1)

Then we have to show (1) is the solution of

$4yy' + x = 0$ ......(2)

Here,

$x^2 + 4y^2 = c$

So, $2x + 8y.y' = 0 \Rightarrow x + 4yy' = 0$

$\Rightarrow 4yy' + x = 0$

This shows that (1) is the solution of (2).

). Solve the following initial value problems.

1) $xy' + y = 0$, $y(2) = -2$

Solution: Given equation is

$xy' + y = 0$ with $y(2) = -2$

Here,

$$xy' + y = 0 \Rightarrow x\frac{dy}{dx} + y = 0$$

$$\Rightarrow \frac{dy}{y} + \frac{dx}{x} = 0$$

Integrating we get,

$\log(y) + \log(x) = \log(c)$

$\Rightarrow \quad xy = c \quad$ ....(i)

Since we have y(2) = - 2. So,

$$2(-2) = c \Rightarrow c = -4$$

Then (i) becomes,

$$xy = -4 \Rightarrow xy + 4 = 0$$

This is required solution.

2. $\mathbf{e^x y' = 2(x + 1)\, y^2,\ y(0) = \frac{1}{6}}$

Solution: Given equation is

$$e^x y' = 2(x + 1)\, y^2 \qquad \text{with } y(0) = \frac{1}{6}$$

Here,

$$e^x y' = 2(x + 1)\, y^2 \quad \Rightarrow \quad \frac{dy}{y^2} = \frac{2(x + 1)}{e^x}\, dx$$

$$\Rightarrow \quad y^{-2}\, dy = 2e^{-x}\, (1 + x)\, dx$$

Integrating we get,

$$\frac{y^{-1}}{-1} = 2 \int e^{-x}\, (x + 1)\, dx$$

$$\Rightarrow \quad -\frac{1}{y} = 2 \left[(x + 1)\frac{e^{-x}}{-1} - (1)\frac{e^{-x}}{(-1)^2}\right] + c$$ [∵ applying integrating parts]

$$= -2\, e^{-x}(x + 1 + 1) + c$$

$$= -2e^{x}(x + 2) + c \qquad \text{...... (1)}$$

Since we have $y(0) = \frac{1}{6}$ then (1) gives,

$$-6 = -2e^0\,(0 + 2) + c \Rightarrow -6 = -4 + c \quad \Rightarrow c = -2$$

Therefore (i) becomes,

$$-\frac{1}{y} = -2e^{-x}\,(x + 2) - 2$$

$$\Rightarrow \quad y = \frac{1}{2\, e^{-x}(x + 2) + 2}$$

3. $\mathbf{y' \cosh^2 x - \sin^2 y = 0,\ y(0) = \frac{\pi}{2}}$

Solution: Given equation is

$$y' \cosh^2 x - \sin^2 y = 0 \qquad \text{with } y(0) = \frac{\pi}{2}$$

Here,

$$y' \cosh^2 x - \sin^2 y = 0 \quad \Rightarrow \quad \frac{dy}{\sin^2 y} - \frac{dx}{\cosh^2 x} = 0$$

$$\Rightarrow \quad \operatorname{cosec}^2 y \, dy - \operatorname{sech}^2 x \, dx = 0$$

Integrating we get,

$$-\cot y - \tanh x = c \quad ....(1)$$

Since we have, $y(0) = \frac{\pi}{2}$. Then (i) gives,

$$-\cot\left(\frac{\pi}{2}\right) - \tanh(0) = c \quad \Rightarrow \quad 0 - 0 = c \quad \Rightarrow c = 0$$

Thus (1) becomes,

$$-\cot y - \tanh x = 0$$

$$\Rightarrow \quad \cot y = -\tanh x.$$

4. $\mathbf{y' = -\frac{y}{x}, y(1) = 1}$

Solution: Given equation is

$$y' = -\frac{y}{x} \quad \text{with } y(1) = 1$$

Here, $\quad y' = -\frac{y}{x}$

$$\Rightarrow \quad \frac{dy}{y} = -\frac{dx}{x}$$

Integrating we get,

$$\log(y) = -\log(x) + \log(c)$$

$$\Rightarrow \quad \log(xy) = \log(c)$$

$$\Rightarrow \quad xy = c \quad .....(1)$$

Since we have $y(1) = 1$. So (1) gives,

$$1 . 1 = c \quad \Rightarrow c = 1$$

Then (1) becomes,

$$xy = 1 \quad \Rightarrow \quad y = \frac{1}{x}$$