Tangent plane:

The tangent plane at a given point of a sphere is the locus of lines through that point which touches the sphere at that point.

Equation of tangent plane to the sphere at (α, β, γ)

The equation of tangent plane to the sphere, $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

at the point (α, β, γ) be,

$$x\alpha + y\beta + z\gamma + u(x + \alpha) + v(y + \beta) + w(z + \gamma) + d = 0.$$

Exercise 4.1

1. **Find the equation of the sphere whose centre (3, -4, 5) and radius 7.**

Solution: Given that, the sphere has,

Centre $(\alpha, \beta, \delta) = (3, -4, 5)$ and Radius $r = 7$

Now the equation of sphere is,

$$(x - \alpha)^2 + (y - \beta)^2 + (z - \delta)^2 = r^2$$

$$\Rightarrow (x - 3)^2 + (y + 4)^2 + (z - 5)^2 = 7^2$$

$$\Rightarrow x^2 + y^2 + z^2 - 6x + 8y - 10z + 1 = 0.$$

2. **Find the centre and radius of sphere $x^2 + y^2 + z^2 + 2x - 3y - 4z - 12 = 0$.**

Solution: We know the equation of a sphere having centre at (α, β, δ) and the radius r is,

$$(x - \alpha)^2 + (y - \beta)^2 + (z - \delta)^2 = r^2 \quad \text{...... (i)}$$

Since the given equation is,

$$x^2 + y^2 + z^2 + 2x - 3y - 4z - 12 = 0$$

$$\Rightarrow (x^2 + 2x + 1) + \left(y^2 - 2.y.\frac{3}{2} + \left(\frac{3}{2}\right)^2\right) + (z^2 - 2.z.2 + 4) = 12 + 1 + \left(\frac{3}{2}\right)^2 + 4$$

$$\Rightarrow (x + 1)^2 + \left(y - \frac{3}{2}\right)^2 + (z - 2)^2 = \frac{77}{4}$$

Comparing it with equation (i) then we get

$$(\alpha, \beta, \delta) = \left(-1, \frac{3}{2}, 2\right) \text{ and } r = \left(\frac{\sqrt{77}}{2}\right).$$

Thus the centre is $\left(-1, \frac{3}{2}, 2\right)$ and radius is $\left(\frac{\sqrt{77}}{2}\right)$.

3. **Find the equation of sphere passing the points (0, 0, 0), (a, 0, 0), (0, b, 0), (0, 0, c). Find it centre and radius.**

Solution: Since the equation of sphere in general form is,

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad \text{...... (i)}$$

having centre at $(-u, -v, -w)$ and radius is $\sqrt{u^2 + v^2 + w^2 - d}$.

Since, it passes through the point (0, 0, 0) then (i) gives $d = 0$.

Also, (i) passes through (a, 0, 0) then (i) gives,

$$a^2 + 2ua = 0 \Rightarrow a(a + u) = 0 \Rightarrow u = \frac{-a}{2} \quad \text{...... (ii)}$$

Similarly (i) passes through (0, b, 0) then $v = \frac{-b}{2}$ ...... (iii)

Also, (i) passes through (0, 0, c) then $w = \frac{-c}{2}$ ...... (iv)

Now, putting the value of u, v, w, d in equation (i) then it becomes,

$$x^2 + y^2 + z^2 + 2.\frac{-a}{2}.x + 2.\frac{-b}{2}.y + 2.\frac{-c}{2}.z + d = 0$$

$$\Rightarrow x^2 + y^2 + z^2 - ax - by - cz = 0.$$

This is the equation of required sphere whose centre is at

$$(-u, -v, -w) = \left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$$

And, radius is, $r = \sqrt{u^2 + v^2 + w^2 - d} = \sqrt{\frac{a^2}{4} + \frac{b^2}{4} + \frac{c^2}{4}}$.

4. **Find the equation of the sphere through (1, 0, 0), (0, 1, 0), (0, 0, 1) and its centre lies on the plane $x + y + z = 6$.**

Solution: Since the equation of sphere in general form is,

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad \text{...... (i)}$$

having centre at $(-u, -v, -w)$ and radius is $\sqrt{u^2 + v^2 + w^2 - d}$.

Since, it passes through the point (1, 0, 0) then (i) gives

$$1 + 2u + d = 0 \Rightarrow u = -\frac{(d+1)}{2} \quad \text{...... (ii)}$$

Similarly (i) passes through (0, 1, 0) then $v = -\frac{(d+1)}{2}$ ...... (iii)

Also, (i) passes through (0, 0, 1) then $w = -\frac{(d+1)}{2}$ ...... (iv)

Since the centre of the sphere lies on the plane $x + y + z = 6$. So,

$$-u - v - w = 6 \quad \Rightarrow \quad \frac{(d+1)}{2} + \frac{(d+1)}{2} + \frac{(d+1)}{2} = 6$$

$$\Rightarrow \quad 3d + 3 = 12 \quad \Rightarrow \quad d + 1 = 4 \quad \Rightarrow \quad d = 3.$$

Therefore,

$$u = -\frac{(d+1)}{2} = -\left(\frac{3+1}{2}\right) = -2.$$

So, $v = w = -2$.

Now, putting the value of u, v, w, d in equation (i) then it becomes,

$$x^2 + y^2 + z^2 - 4x - 4y - 4z + 3 = 0.$$

This is the equation of required sphere.

5. Obtain the equation of sphere through (1, 0, 0), (0, 1, 0), (0, 0, 1) and its radius as small as possible. **[2008 Fall Q. No. 1(b)]**

Solution: Since the equation of sphere in general form is,

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \qquad \text{...... (i)}$$

having centre at (-u, −v, -w) and radius is $\sqrt{u^2 + v^2 + w^2 - d}$.

Since, it passes through the point (1, 0, 0) then (i) gives

$$1 + 2u + d = 0 \quad \Rightarrow \quad u = -\frac{(d+1)}{2} \qquad \text{...... (ii)}$$

Similarly (i) passes through (0, 1, 0) then $\quad v = -\frac{(d+1)}{2} \qquad$ ...... (iii)

Also, (i) passes through (0, 0, 1) then $\quad w = -\frac{(d+1)}{2} \qquad$ ...... (iv)

So, the radius of the sphere is,

$$r = \sqrt{u^2 + v^2 + w^2 - \cancel{d^2}\, d}$$

$$= \sqrt{\left(-\left(\frac{d+1}{2}\right)\right)^2 + \left(-\left(\frac{d+1}{2}\right)\right)^2 + \left(-\left(\frac{d+1}{2}\right)\right)^2 - d}$$

$$= \sqrt{3\left(\frac{d+1}{2}\right)^2 - d} = \sqrt{3\left(\frac{d^2 + 2d + 1 - 12d}{4}\right)} = \sqrt{\frac{3d^2 + 2d +}{4}}$$

$$\Rightarrow \quad r^2 = \frac{3d^2 + 2d + 3}{4}.$$

Since the sphere is minimum if it is a point sphere. Therefore, derivative of $r^2 = 0$.

$$\Rightarrow \quad d\left(\frac{3d^2+2d+3}{4}\right) = d(r^2)$$

$$\Rightarrow \quad \frac{6d+2}{4} = 0 \quad \Rightarrow 6d = -2 \quad \Rightarrow d = -\frac{1}{3}$$

Then,

$$u = v = w = -\left(\frac{d+1}{2}\right) = -\left(\frac{\frac{-1}{3}+1}{2}\right) = -\left(\frac{-1+3}{6}\right) = \frac{-1}{3}$$

Now, putting the value of u, v, w, d in equation (i) then it becomes,

$$x^2 + y^2 + z^2 - 2.\frac{1}{3}.x - 2.\frac{1}{3}.y - 2.\frac{1}{3}.z - \frac{1}{3} = 0$$

$$\Rightarrow \quad 3x^2 + 3y^2 + 3z^2 - 2x - 2y - 2z - 1 = 0$$

$$\Rightarrow \quad 3(x^2 + y^2 + z^2) - 2(x - y - z) - 1 = 0$$

This is the equation of required sphere.

6. A plane passes through a fixed (a, b, c) and cuts the axes in A, B, C. Prove that the locus of the locus of the centre of the sphere OABC is $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 2$

[2000 (OR); 2004 Spring (OR); 2004 Fall Q. No. 1(b)]

Solution: Given that the plane cut the axes at A, B, C.

Let coordinate of A = (l, 0, 0), B = (0, m, 0) and C = (0, 0, n)

Let the sphere contains the origin so its general equation be,

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz = 0 \qquad \text{......... (i)}$$

it passes through A, B, C

$$l^2 + 2ul = 0 \quad \Rightarrow \quad u = \frac{-l}{2}$$

Similarly, $\quad v = \frac{-m}{2} \quad$ and $\quad w = \frac{-n}{2}$

Now, the centre of (i) is $(-u, -v, -w) = \left(\frac{l}{2}, \frac{m}{2}, \frac{n}{2}\right)$.

Since the equation of plane ABC in intercept form is,

$$\frac{x}{l} + \frac{y}{m} + \frac{z}{n} = 1$$ where l, m, n be the intercept made by plane.

Given that the plane passes through the point (a, b, c). So,

$$\frac{a}{l} + \frac{b}{m} + \frac{c}{n} = 1 \qquad \text{........... (ii)}$$

Let $(-u = x_1, -v = y_1, -w = z_1)$ be centre of sphere OABC then,

$$x_1 = \frac{l}{2}, \quad y_1 = \frac{m}{2}, \quad z_1 = \frac{n}{2}$$

$$\Rightarrow \quad l = 2x_1, \quad m = 2y_1 \quad n = 2z_1$$

Then the equation (ii) becomes,

$$\frac{a}{2x_1} + \frac{b}{2y_1} + \frac{c}{2z_1} = 1 \quad \Rightarrow \quad \frac{a}{x_1} + \frac{b}{y_1} + \frac{c}{z_1} = 2$$

Hence, locus of centre is $\frac{a}{x_1} + \frac{b}{y_1} + \frac{c}{z_1} = 2$.

2. Find the equation of sphere which passes through the points (1, 2, 3) and (2, 3, 4) and has its centre on the line x = y = z.

Solution:

The equation of sphere is

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad \ldots\ldots (i)$$

Since the equation passes through the points (1, 2, 3) and (2, 3, 4) then,

$$1^2 + 2^2 + 3^2 + 2u \times 1 + 2v \times 2 + 2w \times 4 + d = 0$$

$$\Rightarrow \quad 2u + 4v + 6w + d + 14 = 0 \quad \ldots\ldots (ii)$$

And, $2^2 + 3^2 + 4^2 + 2u\times2 + 2v\times3 + 2w\times4 + d = 0$

$$\Rightarrow \quad 4u + 6v + 8w + d + 29 = 0 \quad \ldots\ldots (iii)$$

Now, subtracting (ii) from (iii) then we get,

$$2u + 2v + 2w + 15 = 0 \quad \ldots\ldots (iv)$$

Since the centre of equation (i) lies on the line x = y = z. So,

$$-u = -v = -w$$

$$\Rightarrow \quad u = v = w$$

Therefore, the equation (iv) becomes,

$$2u + 2u + 2u + 15 = 0 \quad \Rightarrow \quad 6u = -15 \quad \Rightarrow \quad u = -\frac{5}{2}$$

Thus, $u = v = w = -\frac{5}{2}$

So, (ii) gives,

$$2\times\left(-\frac{5}{2}\right) + 4\times\left(-\frac{5}{2}\right) + 6\times\left(-\frac{5}{2}\right) + d + 14 = 0$$

$$\Rightarrow \quad -5 - 10 - 15 + d + 14 = 0$$

$$\Rightarrow \quad d = 16$$

Therefore, (i) becomes,

$$x^2 + y^2 + z^2 + 2.\left(-\frac{5}{2}\right).x + 2.\left(-\frac{5}{2}\right).y + 2.\left(-\frac{5}{2}\right).z + 16 = 0$$

$$\Rightarrow \quad x^2 + y^2 + z^2 - 5(x + y + z) + 16 = 0.$$

Exercise 4.2

1. Find the equation of sphere described in the join of (4, 5, 6) and (2, 3, 4) as a diameter.

Solution: Since we know that the equation of sphere joining the point (x_1, y_1, z_1) and (x_2, y_2, z_2) as the ends of its diameter is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$$

Given that,

$$(x_1, y_1, z_1) = (4, 5, 6) \quad \text{and} \quad (x_2, y_2, z_2) = (2, 3, 4)$$

Then the equation sphere be,

$$(x - 4)(x - 2) + (y - 5)(y - 3) + (z - 6)(z - 4) = 0$$

$$\Rightarrow \quad x^2 - 4x - 2x + 8 + y^2 - 3y - 5y + 15 + z^2 - 4z - 6z + 24 = 0$$

$$\Rightarrow \quad x^2 + y^2 + z^2 - 6x - 8y - 10z + 47 = 0$$

2. Find the equation of sphere through the circle $x^2 + y^2 + z^2 = 9$, $2x + 3y + 4z = 5$ and (1, 2, 3) find its centre and radius.

Solution: Given that the equation of sphere through the circle $x^2 + y^2 + z^2 = 9$, $2x + 3y + 4z = 5$ is

$$x^2 + y^2 + z^2 - 9 + \lambda(2x + 3y + 4z - 5) = 0 \quad \ldots\ldots (i)$$

Since the equation (i) passes through (1, 2, 3), so,

$$1^2 + 2^2 + 3^2 - 9 + \lambda(2\times1 + 3\times2 + 4\times3 - 5) = 0$$

$$\Rightarrow \quad 1 + 4 + 9 - 9 + \lambda(2 + 6 + 12 - 5) = 0 \quad \Rightarrow \quad 15\lambda = -5$$

$$\Rightarrow \quad \lambda = \frac{-1}{3}$$

Then equation (i) becomes,

$$x^2 + y^2 + z^2 - 9 - \frac{1}{3}(2x + 3y + 4z - 5) = 0$$

$$\Rightarrow \quad 3x^2 + 3y^2 + 3z^2 - 27 - 2x - 3y - 4z + 5 = 0$$

$$\Rightarrow \quad x^2 + y^2 + z^2 - \frac{2}{3}x - y - \frac{4}{3}z - \frac{22}{7} = 0 \quad \ldots\ldots (ii)$$

Since, the centre and radius of (ii) is,

$$\text{Centre} = \left(\frac{\text{coeff. of x}}{-2}, \frac{\text{coeff. of y}}{-2}, \frac{\text{coeff. of z}}{-2}\right)$$

$$= \left(\frac{\frac{-2}{3}}{-2}, \frac{-1}{-2}, \frac{\frac{-4}{3}}{-2}\right) = \left(\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\right)$$

And, $\text{Radius} = \sqrt{u^2+v^2+w^2 - d}$

$$= \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{2}{3}\right)^2 - \left(\frac{-22}{3}\right)}$$

$$= \sqrt{\frac{1}{4} + \frac{1}{2} + \frac{2}{3} + \frac{22}{3}} = \sqrt{\frac{293}{36}}$$

Thus centre is $\left(\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\right)$ and radius is $\sqrt{\frac{293}{36}}$.

3. Find the equation of the sphere one of whose great circles is $x^2 + y^2 + z^2 = 4$, $x^2 + y^2 + z^2 - 2x + 4y - 6z = 11$. **[2009 Spring Q. No. 1(b)]**

Solution: The given circles are

$$x^2 + y^2 + z^2 = 4 \qquad \text{...... (i)}$$

$$x^2 + y^2 + z^2 - 2x + 4y - 6z = 11 \qquad \text{...... (ii)}$$

Subtracting (ii) from (i) then

$$2x - 4y + 6z + 7 = 0 \qquad \text{...... (iii)}$$

Then equation of sphere through given circle is

$$x^2 + y^2 + z^2 - 4 + \lambda\,(2x - 4y + 6z + 7) = 0 \qquad \text{...... (iv)}$$

If the given circle is a great circle of (iv), then the centre of (iv) lies on the plane (iii).

That is, the centre of (iv) $(-\lambda, 2\lambda, -3\lambda)$ lies on (iii). Therefore,

$$2\times(-\lambda) - 4 \times 2\lambda + 6\times(-3\lambda) + 7 = 0$$

$$\Rightarrow -2\lambda - 8\lambda - 18\lambda = -7 \Rightarrow -28\lambda = -7 \Rightarrow \lambda = \frac{1}{4}$$

Now equation (iv) becomes,

$$4\,(x^2 + y^2 + z^2) + 2x - 4y + 6z - 9 = 0.$$

4. Find the equation to a sphere which passes through the point (1, -2, 3) and the circle $z = 0, x^2 + y^2 = 9$.

Solution: The equation of sphere through the circle $z = 0$, $x^2 + y^2 = 9$ is

$$x^2 + y^2 + z^2 - 9 + \lambda z = 0 \qquad \text{...... (i)}$$

since equation (i) passes through the point (1, -2, 3) is

$$1^2 + (-2)^2 + (3)^2 - 9 + \lambda 3 = 0$$

$$\Rightarrow \quad 1 + 4 + 9 - 9 + \lambda 3 = 0 \Rightarrow 3\lambda = -5 \Rightarrow \lambda = \frac{-5}{3}$$

Now, equation (i) becomes,

$$x^2 + y^2 + z^2 - 9 - \frac{5}{3}z = 0$$

$$\Rightarrow 3(x^2 + y^2 + z^2) - 5z - 27 = 0$$

5. Show that the two circles $x^2 + y^2 + z^2 - y + 2z = 0 = x - y - z - 2$ and $x^2 + y^2 + z^2 + x - 3y + z - 5 = 0 = 2xy + 4z - 1$ lie on the same sphere and find its equation.

Solution: The given circles are

$$x^2 + y^2 + z^2 - y + 2z = 0 = x - y + z - 2 \qquad \text{..... (i)}$$

$$x^2 + y^2 + z^2 - x - 3y + z - 5 = 0 = 2xy + 4z - 1 \qquad \text{..... (ii)}$$

Then the equation of circle passing through circle (i) and (ii) be,

$$x^2 + y^2 + z^2 - y + 2z + \lambda_1(x - y + z - 2) = 0$$

$$\Rightarrow x^2 + y^2 + z^2 + \lambda_1 x + (-1 - \lambda_1)y + (2 + z)\lambda_1 - 2\lambda_1 = 0 \qquad \text{...... (iii)}$$

And,

$$x^2 + y^2 + z^2 - x - 3y + z - 5 + \lambda_2(2x - y + 4z - 1) = 0$$

$$\Rightarrow x^2 + y^2 + z^2 + (-1+2\lambda_2)x + (-3-\lambda_2)y + (1+4\lambda_2)z + (-5-\lambda_2) = 0 \text{.......}$$

(iv)

The two circle (i) & (ii) lie on same sphere. So, for some values of λ_1 & λ_2 the spheres (iii) & (iv) are identical. Therefore, comparing the terms on (iii) & (iv) we get,

$$\lambda_1 = -1 + 2\lambda_2, \quad -1 - \lambda_1 = -3 - \lambda_2, \quad 2 + \lambda_1 = 1 + 4\lambda_2, \quad -2\lambda_1 = -5 - \lambda_2$$

Solving first two equations we get,

$$\lambda_1 = 3 \quad \text{and} \quad \lambda_2 = 1$$

The value of λ_1 & λ_2 also satisfies the first two equation hence sphere (iii) & (iv) are identical. So, from (iii) we get the equation of sphere is

$$x^2 + y^2 + z^2 - y + 2z + 3(x - y + z - 2) = 0$$

$$\Rightarrow x^2 + y^2 + z^2 - y + 2z + 3x - 3y + 3z - 6 = 0$$

$$\Rightarrow x^2 + y^2 + z^2 + 3x - 4y + 5z - 6 = 0.$$

This is the equation of required sphere.

6. A sphere parses through the circle $x^2 + y^2 + z^2 = 4$, $x^2 + y^2 + z^2 - 2x + 4y - 6z - 11 = 0$ and its centre lies on the plane $x + 2y - 3z = 12$ find its equation.

Solution: Given circle is,

$$x^2 + y^2 + z^2 = 4,\ x^2 + y^2 + z^2 - 2x + 4y - 6z - 11 = 0 \qquad \text{..... (i)}$$

Subtracting second equation from first of (i) then,

$$2x - 4y + 6z + 7 = 0 \qquad \text{..... (ii)}$$

The equation of sphere through given circle is

$$x^2 + y^2 + z^2 - 4 + \lambda(2x - 4y + 6z + 7) = 0 \qquad \text{..... (iii)}$$

Clearly, the centre of (iii) is, $\left(\frac{2\lambda}{-2}, \frac{-4\lambda}{-2}, \frac{6\lambda}{-2}\right) \Rightarrow (-\lambda, 2\lambda, -3\lambda)$

Since the centre $(-\lambda, 2\lambda, -3\lambda)$ lies on the plane $x + 2y - 3z = 12$. So,

$$-\lambda + 2\cdot 2\lambda - 3\cdot(-3\lambda) - 12 = 0$$

$$\Rightarrow 12\lambda = 12$$

$$\Rightarrow \lambda = 1$$

Now, equation (iii) becomes,

$$x^2 + y^2 + z^2 - 4 + 1(2x - 4y + 6z + 7) = 0$$

$$\Rightarrow x^2 + y^2 + z^2 + 2x - 4y + 6z + 3 = 0$$

This is the equation of required sphere.

7. Find the centre and radius of the circle in which the sphere $x^2 + y^2 + z^2 - 8x + 4y + 8z - 45 = 0$ is cut by plane $x - 2y + 2z = 3$.

Solution: Given that the equation of sphere is

$$x^2 + y^2 + z^2 - 8x + 4y + 8z - 45 = 0$$

whose centre is, $\left(\frac{-8}{-2}, \frac{4}{-2}, \frac{8}{-2}\right) = (4, -2, -4)$

and the radius is, $r = \sqrt{4^2+(-2)^2+(-4)^2-(-45)}$

$= \sqrt{16+4+16+45} = 9.$

Since, the perpendicular distance from (x_1, y_1, z_1) to plane $ax + by + cz + d = 0$ is,

$$\left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right|$$

So, the perpendicular distance OM, from O(4, -2, -4) to plane $x - 2y + 2z = 3$ is

$$OM = \left|\frac{1\times4 - 2\times(-2) + 2\times(-4) - 3}{\sqrt{1^2+(-2)^2+2^2}}\right|$$

$$= \left|\frac{4+4-8-3}{\sqrt{1+4+4}}\right|$$

$$= \left|\frac{-3}{3}\right| = 1.$$

Hence, the radius of circle OM is,

$$\sqrt{r^2 - OM^2} = \sqrt{9^2 - 1^2} = \sqrt{81-1} = \sqrt{80} = 4\sqrt{5}.$$

Since the direction ratios of the plane $x - 2y + 2z = 3$ are 1, -2, 2. So the equation of the line through O and is normal to the plane is,

$$\frac{x-4}{1} = \frac{y+2}{-2} = \frac{z+4}{2} = r \text{ (suppose)}$$

Then, any point of the line is, $(r + 4, -2r - 2, 2r - 4)$. Since the point also touches the plane $x - 2y + 2x + 2z = 3$. So,

$$r + 4 - 2(-2r - 2) + 2(2r - 4) = 3$$

$$\Rightarrow r + 4 + 4r + 4 + 4r - 8 = 3$$

$$\Rightarrow 9r = 3 \Rightarrow r = \frac{1}{3}$$

Hence, the coordinate of M is, $\left(\frac{1}{3}+4, -2\times\frac{1}{3} - 2, 2\times\frac{1}{3} - 4\right) = \left(\frac{13}{3}, \frac{-8}{3}, \frac{-10}{3}\right)$.

Thus centre of the circle is $\left(\frac{13}{3}, \frac{-8}{3}, \frac{-10}{3}\right)$ and radius is $4\sqrt{5}$.

8. Find the equation of sphere through the circle $x^2 + y^2 = 4$, $z = 0$ and is cut by the plane $x + 2y + 2z = 0$ in a circle of radius 3.

[2008 Fall (OR); 2010 Spring; 2011 Fall Q. No. 1(b)]

Solution: Given circle is

$$x^2 + y^2 + z^2 = 4, z = 0$$

$$\Rightarrow x^2 + y^2 + z^2 - 4 + \lambda z = 0 \quad \text{.......... (i)}$$

its centre is, $\left(\frac{0}{-2}, \frac{0}{-2}, \frac{-\lambda}{2}\right) = \left(0, 0, \frac{-\lambda}{2}\right)$

and, its radius is, $r = \sqrt{0^2 + 0^2 + \left(\frac{-\lambda}{2}\right)^2 - (-4)} = \sqrt{\frac{\lambda^2}{4} + 4}$

Here, from the right angled triangle in the figure.

$$ON^2 = OM^2 + MN^2$$

$$\Rightarrow \left(\sqrt{\frac{\lambda^2}{4} + 4}\right)^2 = \left(\frac{0 + 2\times0 + 2\times\frac{-\lambda}{2}}{\sqrt{\lambda^2 + 2^2 + 2^2}}\right)^2 + 3^2$$

$$\Rightarrow \frac{\lambda^2}{4} + 4 = \frac{\lambda^2}{3^2} + 4 \Rightarrow \frac{\lambda^2}{4} - \frac{\lambda^2}{9} = 9 - 4$$

$$\Rightarrow \frac{9\lambda^2 - 4\lambda^2}{36} = 36 \Rightarrow 5\lambda^2 = 5\times36 \Rightarrow \lambda = \pm 6$$

Then equation (i) becomes

$$x^2 + y^2 + z^2 - 4 + 6z = 0 \quad \text{and} \quad x^2 + y^2 + z^2 - 4 - 6z = 0.$$

9. Find the equation of the sphere which passes through the circle $x^2 + y^2 + z^2 - 6x - 2z + 5 = 0$, $y = 0$ and touch the plane $3y + 4z + 5 = 0$.

Solution: Given circle is,

$$x^2 + y^2 + z^2 - 6x - 2z + 5 = 0, y = 0 \quad \text{...... (i)}$$

So, the equation of the sphere that passes through (i) is,

$$x^2 + y^2 + z^2 - 6x - 2z + 5 + \lambda y = 0 \quad \text{...... (ii)}$$

Then clearly the centre of the sphere (ii) is,

$$\left(\frac{-6}{-2}, \frac{\lambda}{-2}, \frac{-2}{-2}\right) = \left(3, \frac{-\lambda}{2}, 1\right)$$

And, the radius of the sphere (ii) is,

$$r = \sqrt{9 + \left(\frac{\lambda}{-2}\right)^2 + 1^2 - 5} = \sqrt{9 + \frac{\lambda^2}{4} + 1 - 5} = \sqrt{5 + \frac{\lambda^2}{4}}$$

Since the length of perpendicular from centre to the plane $3y + 4z + 5 = 0$ is equal to the radius of the sphere. So,

$$\frac{0\times3+3\times\frac{-\lambda}{2}+4\times1+5}{\sqrt{3^2+4^2}}=\sqrt{\frac{\lambda^2}{4}+5}$$

$$\Rightarrow \frac{\left(\frac{-3\lambda}{2}+9\right)^2}{25}=\frac{\lambda^2}{4}+5 \Rightarrow \frac{(-3\lambda+18)^2}{100}=\frac{\lambda^2}{4}+5$$

$$\Rightarrow 9\lambda^2-108\lambda+324=\frac{100\lambda^2}{4}+500$$

$$\Rightarrow 36\lambda^2-432\lambda+1296=100\lambda^2+2000$$

$$\Rightarrow 64\lambda^2+432\lambda+704=0$$

$$\Rightarrow \lambda^2+6.75\lambda+11=0$$

$$\Rightarrow \lambda=\frac{-6.75\pm\sqrt{(6.75)^2-4\cdot1\cdot11}}{2\cdot1}$$

$$\Rightarrow \lambda=\frac{-6.75\pm1.25}{2} \Rightarrow \lambda=-4,-\frac{11}{4}$$

Now, equation (i) becomes,

$$x^2+y^2+z^2-6x-2z+5-4y=0$$

and $x^2+y^2+z^2-6x-2z+5-\frac{11}{4}y=0$

$$\Rightarrow 4(x^2+y^2+z^2)-24x-8z+20-11y=0.$$

10. A sphere has a points (0, 0, 0), (4, 5, 6) at opposite ends of a diameter. Find the equation of the sphere having the intersection of the circle with the plane $x+y+z=3$ is a great circle.

Solution: Since we know that the equation of sphere joining the point (x_1, y_1, z_1) and (x_2, y_2, z_2) as the ends of its diameter is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$$

Given that,

$(x_1, y_1, z_1)=(0, 0, 0)$ and $(x_2, y_2, z_2)=(4, 5, 6)$

Then the equation sphere be,

$$(x-0)(x-4)+(y-0)(y-5)+(z-0)(z-6)=0$$

$$\Rightarrow x^2+y^2+z^2-4x-5y-6z=0 \quad \text{..... (i)}$$

The equation of sphere passing through the circle $x^2+y^2+z^2-4x-5y-6z=0$, $x+y+z=3$ is,

$$x^2+y^2+z^2-4x-5y-6z+\lambda(x+y+z-3)=0 \quad \text{.... (ii)}$$

Clearly the centre of (ii) is, $\left(\frac{-4+\lambda}{-2}, \frac{-5+\lambda}{2}, \frac{-6+\lambda}{-2}\right)$

Since the circle is great circle centre of (ii) lies on the plane $x+y+z=3$

$$\left(\frac{-4+\lambda}{-2}\right)+\left(\frac{-5+\lambda}{2}\right)+\left(\frac{-6+\lambda}{-2}\right)=3$$

$$\Rightarrow \frac{-4+\lambda-5+\lambda-6+\lambda}{-2}=3 \Rightarrow 3\lambda-15=-6$$

$$\Rightarrow 3\lambda=9$$

$$\Rightarrow \lambda=3$$

Then the equation (ii) becomes,

$$x^2+y^2+z^2-4x-5y-6z+3(x+y+z-3)=0$$

$$\Rightarrow x^2+y^2+z^2-x-2y-3z-9=0.$$

Exercise 4.3

1. **Find the equation tangent plane of sphere $x^2+y^2+z^2+2x-6y+1=0$ at (1, 2, –2).**

Solution: Given sphere is,

$$x^2+y^2+z^2+2x-6y+1=0$$

Comparing above equation with equation $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ then we get,

$u=1$, $v=-3$, $w=0$, $d=1$

Since we have the equation of tangent plane at (α, β, δ) to the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ is,

$$x\alpha+y\beta+z\delta+u(x+\alpha)+v(y+\beta)+w(z+\delta)+d=0.$$

Then the equation of the tangent plane to $x^2+y^2+z^2+2x-6y+1=0$ at (1, 2, -2) is,

$$x+2y-2z+u(x+1)+v(y+2)+w(z-2)+d=0.$$

$$\Rightarrow x+2y-2z+1(x+1)-3(y+2)+0(z-2)+1=0.$$

$$\Rightarrow x+2y-2z+x-3y-6-4=0.$$

This is the equation of required tangent plane.

2. **Find the equation of the sphere which passes through the circle $x^2+y^2+z^2-2x+2z=2$, $y=0$ and touches the plane $y-z=7$.**

Solution: Since the equation of the sphere which passing through the given circle $x^2+y^2+z^2-2x+2z=2$, $y=0$ is,

$$x^2+y^2+z^2-2x+2z-2+\lambda y=0 \quad \text{...... (i)}$$

Comparing (i) with the equation $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ then the centre of (i) is, $(-u, -v, -w) = \left(1, \frac{\lambda}{2}, -1\right)$.

And, the radius is, $r = \sqrt{u^2 + v^2 + w^2 - d}$

$$= \sqrt{1^2 + \left(\frac{\lambda}{-2}\right)^2 + (-1)^2 - (-2)} = \sqrt{4 + \frac{\lambda^2}{4}}$$

Since the length of perpendicular from centre to the plane $y - z - 7 = 0$ is equal to the radius of the sphere. So,

$$\frac{0 \cdot 1 + 1 \cdot \frac{\lambda}{-2} - 1 \cdot (-1) - 7}{\sqrt{1^2 + (-1)^2}} = \sqrt{4 + \frac{\lambda^2}{4}}$$

$$\Rightarrow -\left(\frac{\lambda + 12}{2\sqrt{2}}\right) = \sqrt{4 + \frac{\lambda^2}{4}}$$

Squaring both side

$$\frac{(\lambda+12)^2}{8} = 4 + \frac{\lambda^2}{4} \Rightarrow \frac{(\lambda+12)^2}{8} = \frac{16 + \lambda^2}{4}$$

$$\Rightarrow \lambda^2 + 24\lambda + 144 = 32 + 2\lambda^2$$

$$\Rightarrow \lambda^2 - 24\lambda - 112 = 0$$

So, $\lambda = \dfrac{24 \pm \sqrt{(24)^2 - 4 \cdot 1 \cdot (-112)}}{2 \cdot 1} = \dfrac{24 \pm 32}{2} = (28, -4)$.

Now, equation (i) becomes

$$x^2 + y^2 + z^2 - 2x + 28y + 2z - 2 = 0$$

$$\Rightarrow x^2 + y^2 + z^2 - 2x - 4y + 2z - 2 = 0.$$

3. Find the centre and radius of the sphere given by $x^2 + y^2 + z^2 - 2x + 4y - 6z = 11$ and find the point at which the plane $3y + 4z - 31 = 0$ touches the sphere.

Solution: Given that the equation of sphere is

$$x^2 + y^2 + z^2 - 2x + 4y - 6z - 11 = 0$$

whose centre is at, $\left(\frac{-2}{-2}, \frac{4}{-2}, \frac{-6}{-2}\right) = (1, -2, 3)$.

And its radius is, $r = \sqrt{1^2 + (-2)^2 + 3^2 - (-11)} = \sqrt{1 + 4 + 9 + 11} = 5$.

Thus the centre of the sphere is $(1, -2, 3)$ and radius $r = 5$.

For second part, since the equation of line passing through $(1, -2, 3)$ and perpendicular to the plane $3y + 4z - 31 = 0$ is,

$$\frac{x - 1}{0} = \frac{y + 2}{3} = \frac{z - 3}{4} = r \text{ (Suppose)}$$

Then	$x - 1 = 0$	$y + 2 = 3r$	$z - 3 = 4r$
	$\Rightarrow x = 1$	$\Rightarrow y = 3r - 2$	$\Rightarrow z = 4r + 3$

So, any point on the plane is,

$$0 \times 1 + 3(3r - 2) + 4(4r + 3) - 31 = 0$$

$$\Rightarrow 9r - 6 + 16r + 12 - 31 = 0 \Rightarrow 25r = 25$$

$$\Rightarrow r = 1.$$

So that, $x = 1$, $y = 1$ and $z = 7$.

Thus, at the point $(1, 1, 7)$ the given sphere touches the given plane.

4. Find the value of k will the plane $x - 2y - 2z = k$ touches the sphere $x^2 + y^2 + z^2 - 2x + 4y - 6z + 5 = 0$.

Solution: Given sphere is,

$$x^2 + y^2 + z^2 - 2x + 4y - 6z + 5 = 0. \qquad \text{....... (i)}$$

Comparing (i) with $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ then the centre of (i) is,

$$(-u, -v, -w) = (1, -2, 3)$$

and the radius of (i) is, $r = \sqrt{u^2 + v^2 + w^2 - d}$

$$= \sqrt{1^2 + (-2)^2 + 3^2 - 5} = \sqrt{1 + 4 + 9 - 5} = 3.$$

Since the length of perpendicular from centre to the plane $x - 2y - 2z = k$ is equal to the radius of the sphere. So,

$$\pm \frac{1 \times 1 - 2 \times (-2) - 2 \times 3 - k}{\sqrt{1^2 + (-2)^2 + 2^2}} = 3 \Rightarrow \pm \frac{1 + 4 - 6 - k}{\sqrt{9}} = 3$$

$$\Rightarrow \pm(-1 - k) = 9$$

When +ve sign	&	where –ve sign.
$k = -10$		$1 + k = 9 \Rightarrow k = 8$

Thus, the value of k is 8 or -10.

5. Find the equation of the spheres passing through the circle $x^2 + z^2 + - 6x - 2z + 5 = 0$, $y = 0$ and touching the plane $3y + 4z + 5 = 0$.

Solution: Since the equation of sphere passes through the circle $x^2 + z^2 + - 6x - 2z + 5 = 0$, $y = 0$ is

$$x^2 + z^2 + - 6x - 2z + 5 + \lambda y = 0. \qquad \text{....... (i)}$$

Comparing (i) with $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ then the centre of (i) is,

$$(-u, -v, -w) = \left(3, \frac{-\lambda}{2}, 1\right)$$

and the radius of (i) is, $r = \sqrt{3^2 + \left(\frac{-\lambda}{-2}\right)^2 + 1^2 - 5} = \sqrt{5 + \frac{\lambda^2}{4}}$

Since the length of perpendicular from centre to the plane $3y + 4z + 5 = 0$ is equal to the radius of the sphere. So,

$$\frac{0 \cdot 3 + 3 \cdot \frac{-\lambda}{2} + 4 \cdot 1 + 5}{\sqrt{3^2 + 4^2}} = \sqrt{5 + \frac{\lambda^2}{4}}$$

$$\Rightarrow \frac{\left(9-\frac{3\lambda}{2}\right)^2}{25} = (5+\frac{\lambda^2}{4})$$

$$\Rightarrow \frac{(18-3\lambda)^2}{25\cdot 4} = \frac{20+\lambda^2}{4}$$

$$\Rightarrow \frac{324-108\lambda+9\lambda^2}{25} = \frac{20+\lambda^2}{4}$$

$$\Rightarrow 1296 - 432\lambda + 36\lambda^2 = 500+25\lambda^2$$

$$\Rightarrow 16\lambda^2 - 108\lambda + 176 = 0$$

$$\Rightarrow \lambda = \frac{108 \pm \sqrt{108^2 - 4(16)(176)}}{32}$$

$$= \frac{108 \pm \sqrt{11664 - 11264}}{32} = \frac{108 \pm \sqrt{400}}{32} = \frac{108 \pm 20}{32}$$

Therefore, taking positive sign, $\lambda = \frac{128}{32} = 4$,

And taking negative sign, $\lambda = \frac{88}{32} = \frac{11}{4}$.

Now, (i) becomes,

$$x^2 + y^2 + z^2 - 6x + 4y - 2z + 5 = 0$$

and $x^2 + y^2 + z^2 - 6x + \frac{11}{4}y - 2z + 5 = 0$

$$\Rightarrow 4(x^2 + y^2 + z^2) - 24x + 11y - 8z + 20 = 0.$$

This is the equation of required spheres.

6. Show that the plane $2x - 2y + z + 12 = 0$ touches the sphere $x^2 + y^2 + z^2 - 2x - 4y + 2z = 3$ and find he point of contact.

Solution: Given sphere is

$$x^2 + y^2 + z^2 - 2x - 4y + 2z = 3 \qquad \text{........(i)}$$

Comparing (i) with the equation $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ then we get,

$$u = -1, \quad v = -2, \quad w = 1 \quad \text{and} \quad d = -3.$$

Then the centre of (i) is

$$(-u, -v, -w) = (1, 2, -1)$$

and radius of (i) is

$$r = \sqrt{u^2 + v^2 + w^2 - d} = \sqrt{1+4+1+3} = \sqrt{9} = 3$$

Also, given plane is

$$2x - 2y + z + 12 = 0 \qquad \text{........(ii)}$$

Then the perpendicular distance from the centre of sphere (1, 2, -1) to (ii) is,

$$d = \left|\pm \frac{2\times 1 - 2\times 2 - 1 + 12}{\sqrt{4+4+1}}\right| = \left|\pm \frac{2-4-1+12}{\sqrt{9}}\right| = \left|\pm \frac{9}{3}\right| = 3$$

Thus, the radius of sphere is equal to the distance. That means the plane (ii) touches the sphere (i).

Since we know that the direction cosines of the line perpendicular to (ii) are 2, -2, 1.

Since the line through the centre of (i) and touches to (ii) is always normal to (ii). Therefore, the equation of line through the centre (1, 2, -1) and having direction cosines 2, -2, -1 be

$$\frac{x-1}{x} = \frac{y-2}{-2} = \frac{z+1}{1} = r(\text{say}) \qquad \text{........(iii)}$$

This gives, $x = 2r + 1$, $y = 2 - 2r$ and $z = r - 1$

So, $(2r + 1, 2 - 2r, r - 1)$ be general point of (iii).

So, it satisfies (ii). That is

$$2(2r+1) - 2(2-2r) + (r-1) + 12 = 0$$

$$\Rightarrow 4r + 2 - 4 + 4r + r - 1 + 12 = 0 \quad \Rightarrow 9r + 9 = 0 \Rightarrow r = -1$$

Then, $x = -1$, $y = 4$, $z = -2$

Thus, (-1, 4, -2) be the point of contact of (i) and (ii).

7. Find the equations to the sphere which touches the sphere $x^2 + y^2 + z^2 - x + 3y + 2z - 3 = 0$ at the point (1, 1, -1) and passes through the origin.

Solution: Given that the equation of a sphere is

$$x^2 + y^2 + z^2 - x + 3y + 2z - 3 = 0 \qquad \text{.......(i)}$$

Comparing it with $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ then we get,

$$u = -\frac{1}{2}, \quad v = \frac{3}{2}, \quad w = 1 \quad \text{and} \quad d = -3$$

Then the equation of tangent plane to (i) and passing through (1, 1, -1) be

$$(1 + 1/2)(x-1) + (1 + 3/2)(y-1) + (-1+1)(z+1) = 0$$

$$\Rightarrow \frac{1}{2}(x-1) + \frac{5}{2}(y-1) = 0$$

$$\Rightarrow x + 5y - 6 = 0 \qquad \text{.........(ii)}$$

Therefore, equation of sphere through the point circle of (i) and the plane (ii) is

$$(i) + k(ii) = 0$$

i.e. $(x^2 + y^2 + z^2 - x + 3y + 2z - 3) + k(x + 5y - 6) = 0$ ....(iii)

Since this sphere passes through the origin. So,

$$-3 - 6k = 0 \quad \Rightarrow k = -\frac{1}{2}$$

Then (iii) becomes,

$$2(x^2 + y^2 + z^2 - x + 3y + 2z - 3) - (x + 5y - 6) = 0$$

$$\Rightarrow \quad 2(x^2 + y^2 + z^2) - 3x + y + 4z = 0$$

This is the equation of required sphere.

8. Find the equations of the sphere through the circle $x^2 + y^2 + z^2 = 1$, $2x + 4y + 5z = 6$ and touches the plane $z = 0$. **[2009 Fall Q. No. 1(b)]**

Solution: The equation of sphere through the given circle $x^2 + y^2 + z^2 = 1$, $2x + 4y + 5z = 6$ be

$$(x^2 + y^2 + z^2 - 1) + k(2x + 4y + 5z - 6) = 0$$

$$\Rightarrow \quad x^2 + y^2 + z^2 + 2kx + 4ky + 5kz - (1 + 6k) = 0 \quad \text{......(i)}$$

Comparing (i) with the general equation of sphere $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ then we get

$$v = k, \quad v = 2k, \quad w = \frac{5k}{2} \quad \text{and} \quad d = -(1 + 6k).$$

Then the centre of (i) be

$$(-u, -v, -w) = (-k, -2k, -5k/2)$$

and radius of (i) be

$$r = \sqrt{u^2 + v^2 + w^2 - d}$$

$$= \sqrt{k^2\left(1 + 4 + \frac{25}{4}\right) + 1 + 6k} = \frac{1}{2}\sqrt{45k^2 + 24k + 4}$$

Since the sphere (i) touches the plane $z = 0$. That means the length of z-coordinate is equal to the radius of the sphere.

i.e. $\dfrac{5k}{2} = \dfrac{1}{2}\sqrt{45k^2 + 24k + 4}$

$$\Rightarrow \quad 25k^2 = 45k^2 + 24k + 4$$

$$\Rightarrow \quad 20k^2 + 24k + 4 = 0$$

$$\Rightarrow \quad 5k^2 + 6k + 1 = 0 \quad \Rightarrow \quad (5k + 1)(k + 1) = 0 \Rightarrow k = -1, -1/5$$

Therefore (i) become,

$$x^2 + y^2 + z^2 - 2x - 4y - 5z + 5 = 0 \quad \text{and} \quad 5(x^2 + y^2 + z^2) - 2x - 4y - 5z + 1 = 0.$$

These are equation of required spheres.

9. Find the equations to the tangent planes to the sphere $x^2 + y^2 + z^2 - 2x + 4y - 6z + 10 = 0$ which passes through $\dfrac{x+3}{14} = \dfrac{y+1}{-3} = \dfrac{z-5}{4}$. Find also the angle between these planes.

Solution: Given that the equation of a sphere is

$$x^2 + y^2 + z^2 - 2x + 4y - 6z + 10 = 0 \quad \text{...... (i)}$$

Comparing (i) with $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ then the centre of (i) is,

$$(-u, -v, -w) = (1, -2, 3).$$

and the radius of (i) is, $r = \sqrt{1^2 + (-2)^2 + 3^2 - 10} = \sqrt{4} = 2$

And, given line is,

$$\frac{x+3}{14} = \frac{y+1}{-3} = \frac{z-5}{4} \quad \text{....... (ii)}$$

$$\Rightarrow \quad \frac{x+3}{14} = \frac{z-5}{4} \quad \text{and} \quad \frac{x+3}{14} = \frac{y+1}{-3}$$

$$\Rightarrow \quad 4x + 12 = 14z - 70 \qquad \Rightarrow \quad -3x - 9 = 14y + 14$$

$$\Rightarrow \quad 4x - 14z + 82 = 0 \qquad \Rightarrow \quad 3x + 14y + 23 = 0$$

$$\Rightarrow \quad 2x - 7z + 41 = 0$$

Now, equation of a plane passing through line (ii) is

$$2x - 7z + 41 + k(3x + 14y + 23) = 0$$

$$\Rightarrow \quad (2 + 3k)x + 14ky - 7z + 23k + 41 = 0 \quad \text{......... (iii)}$$

If plane (iii) is tangent to the given sphere (i) then the length of perpendicular from centre to the plane (iii) is equal to the radius of sphere.

i.e. $\pm \dfrac{(2 + 3k).1 + 14k(-2) - 7\times 3 + 23k + 41}{\sqrt{(2+3k)^2 + (14k)^2 + (-7)^2}} = 2$

$$\Rightarrow \quad \frac{2 + 3k - 28k - 21 + 23k + 41}{\sqrt{4 + 12k + 9k^2 + 196k^2 + 49}} = 2$$

$$\Rightarrow \quad \frac{(22 - 2k)^2}{53 + 12k + 205k^2} = 4 \Rightarrow \quad (11 - k)^2 = 53 + 12k + 205k^2$$

$$\Rightarrow \quad 121 - 22k + k^2 = 53 + 12k + 205k^2$$

$$\Rightarrow \quad 204k^2 + 34k - 68 = 0$$

$$\Rightarrow \quad 34(6k^2 + k - 2) = 0$$

$$\Rightarrow \quad 6k^2 + 4k - 3k - 2 = 0$$

$$\Rightarrow \quad 2k(3k + 2) - 1(3k + 2) = 0$$

$$\Rightarrow \quad (3k + 2)(2k - 1) = 0$$

This gives, $\quad k = \dfrac{-2}{3}, \dfrac{1}{2}$

Putting the values of k in equation (iii) then we get,

$$2x - 7z + 41 - \frac{2}{3}(3x + 14y + 23) = 0$$

$$\Rightarrow \quad 6x - 21z + 123 - 6x - 28y - 46 = 0$$

$$\Rightarrow \quad -28y - 21z + 77 = 0$$

$$\Rightarrow \quad -7(4y + 3z - 11) = 0 \quad \Rightarrow \quad 4y + 3y - 11 = 0.$$

And, $\quad 2x - 7z + 41 + \dfrac{1}{2}(3x + 14y + 23) = 0$

$$\Rightarrow \quad 4x - 14z + 82 + 3x + 14y + 23 = 0$$

$$\Rightarrow \quad 7x + 14y - 14z + 105 = 0$$

$$\Rightarrow \quad x + 2y - 2z + 15 = 0$$

Thus, the required equation of tangent planes are $4y + 3y - 11 = 0$ and $x + 2y - 2z + 15 = 0$.

Let θ be angle between tangents to the planes then

$$\cos\theta = \frac{0\times1+4\times2+3\times(-2)}{\sqrt{4^2+3^2}\sqrt{1^2+2^2+(-2)^2}}$$

$$\Rightarrow \cos\theta = \frac{2}{\sqrt{25}\sqrt{9}} = \frac{2}{5\times3} = \frac{2}{15}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{2}{15}\right)$$

10. Find the equation of sphere having its centre (3, 4, 5) and touching the line $\frac{x-1}{1} = \frac{y+2}{2} = \frac{z+2}{-2}$.

Solution: Given line is $\frac{x-1}{1} = \frac{y+2}{2} = \frac{z+2}{-2} = r$ (say)

So, any point of the line is $(r + 1, 2r - 2, -2r - 2)$.

Given that the line touches the sphere.

Suppose that the point $(r + 1, 2r - 2, -2r - 2)$ is the common point of line and sphere.

Also, given that the centre of the sphere is (3, 4, 5). So, let the direction ratios of the line joining the centre of the sphere and $(r + 1, 2r - 2, -2r - 2)$ be,

$$r + 1 - 3,\ 2r - 2 - 4,\ -2r - 2 - 5.$$

i.e. $r - 2,\ 2r - 6,\ -2r - 7$

Since the line joining the centre and the point $(r + 1, 2r - 2, -2r - 2)$ is perpendicular to the given line, so,

$$(r-2).1+(2r-6).2+(-2r-7)(-2) = 0$$

$$\Rightarrow r - 2 + 4r - 12 + 4r + 14 = 0$$

$$\Rightarrow 9r = 0$$

$$\Rightarrow r = 0$$

Therefore the foot of the perpendicular to the line is,

$$(r+1, 2r-2, -2r-2)$$
$$= (0+1, 2\times0-2, -2\times0-2)$$
$$= (1, -2, -2)$$

Since the line joining the centre and the point (1, -2, -2) is the radius of the sphere and its length is

$$\sqrt{(3-1)^2+(4+2)^2+(5+2)^2} = \sqrt{4+36+49} = \sqrt{89}$$

Now, the equation of a sphere having centre (3, 4, 5) and radius $\sqrt{89}$ is,

$$(x-3)^2 + (y-4)^2 + (z-5)^2 = (\sqrt{89})^2$$

$$\Rightarrow x^2 - 6x + 9 + y^2 - 8y + 16 + z^2 - 10z + 25 = 89$$

$$\Rightarrow x^2 + y^2 + z^2 - 6x - 8y - 10z = 89 - 50$$

11. **Find the equations to the tangent planes to the sphere $x^2 + y^2 + z^2 + 6x - 2z + 1 = 0$ which passes through $\frac{16-x}{1} = z = \frac{2y+30}{3}$. Find also the angle between these planes.**

Solution: Given that the equation of a sphere is,

$$x^2 + y^2 + z^2 + 6x - 2z + 1 = 0$$

Comparing (i) with $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ then the centre of (i) is,

$$(-u, -v, -w) = (-3, 0, 1)$$

and the radius of (i) is, $r = \sqrt{9+0+1-1} = 3$

Also, given line is,

$$\frac{16-x}{1} = z = \frac{2y+30}{3}$$

This gives,

$$\frac{16-x}{1} = z, \quad \text{and} \quad z = \frac{2y+30}{3}$$

$$\Rightarrow x + z - 16 = 0 \qquad \Rightarrow 2y - 3z + 30 = 0$$

Then the equation of a plane passing through the given line is,

$$x + z - 16 + k(2y - 3z + 30) = 0$$

$$\Rightarrow x + 2ky + (1-3k)z - 16 + 30k = 0 \qquad \text{....... (i)}$$

If plane (i) is tangent to the given sphere then the length of perpendicular from centre of the sphere to the plane (i) is equal to the radius of the sphere.

i.e. $$\pm\frac{-3+2k\times0+(1-3k)\times1-16+30k}{\sqrt{1^2+(2k)^2+(1-3k)^2}} = 3$$

$$\Rightarrow \pm\frac{-3+1-3k-16+30k}{\sqrt{1+4k^2+1-6k+9k^2}} = 3$$

$$\Rightarrow \pm\frac{-18+27k}{\sqrt{13k^2-6k+2}} = 3 \quad \Rightarrow (-6+9k)^2 = 13k^2 - 6k + 2$$

$$\Rightarrow 36 - 108k + 81k^2 - 13k^2 + 6k - 2 = 0$$

$$\Rightarrow 68k^2 - 102k + 34 = 0$$

$$\Rightarrow 2k^2 - 3k + 1 = 0$$

$$\Rightarrow 2k^2 - 2k - k + 1 = 0$$

$$\Rightarrow 2k(k-1) - 1(k-1) = 0$$

$$\Rightarrow (k-1)(2k-1) = 0$$

$$\Rightarrow k = 1, \ \frac{1}{2}$$

Putting the value of k in equation (i), we get,

$$x + z - 16 + 1(2y - 3z + 30) = 0$$

$$\Rightarrow x + z - 16 + 2y - 3z + 30 = 0$$

$$\Rightarrow x + 2y - 2z + 14 = 0$$

And,

$$x + z - 16 + \frac{1}{2}(2y - 3z + 30) = 0$$

$$\Rightarrow \quad 2x + 2z - 32 + 2y - 3z + 30 = 0$$

$$\Rightarrow \quad 2x + 2y - z - 2 = 0$$

Hence, the equation of tangent planes to (i) are

$$x + 2y - 2z + 14 = 0 \text{ and } 2x + 2y - z - 2 = 0.$$

OTHER QUESTIONS FROM SEMESTER END EXAMINATION

1999; 2001; 2008 Spring Q. No. 1(b) OR

A sphere S has points (0, 1, 0), (3, -5, 2) as opposite ends of a diameter. Find the equation of the sphere having the intersection of the sphere S with the plane $5x - 2y + 4z + 7 = 0$ as a great circle.

Solution: Since we know that the equation of sphere joining the point (x_1, y_1, z_1) and (x_2, y_2, z_2) as the ends of its diameter is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$$

Given that,

$$(x_1, y_1, z_1) = (0, 1, 0) \quad \text{and} \quad (x_2, y_2, z_2) = (3, -5, 2)$$

Then the equation sphere be,

$$(x - 0)(x - 3) + (y - 3)(y + 5) + (z - 0)(z - 2) = 0$$

$$\Rightarrow \quad x^2 + y^2 + z^2 - 3x + 2y - 2z - 15 = 0 \qquad \text{..... (i)}$$

The equation of sphere passing through the circle $x^2 + y^2 + z^2 - 3x + 2y - 2z - 15 = 0$, $5x - 2y + 4z + 7 = 0$ is,

$$x^2 + y^2 + z^2 - 3x + 2y - 2z - 15 + \lambda(5x - 2y + 4z + 7) = 0 \qquad \text{.... (ii)}$$

Clearly the centre of (ii) is, $\left(\frac{-3 + 5\lambda}{-2}, \frac{2 - 2\lambda}{-2}, \frac{-2 + 4\lambda}{-2}\right)$

Since the circle is great circle centre of (ii) lies on the plane $5x - 2y + 4z + 7 = 0$ then

$$\frac{-3 + 5\lambda}{-2} + \frac{2 - 2\lambda}{-2} + \frac{-2 + 4\lambda}{-2} + 7 = 0.$$

$$\Rightarrow \quad -3 + 5\lambda + 2 - 2\lambda - 2 + 4\lambda - 14 = 0 \Rightarrow 7\lambda - 17 = 0 \Rightarrow \lambda = \frac{17}{7}$$

Then the equation (ii) becomes,

$$x^2 + y^2 + z^2 - 3x + 2y - 2z - 15 + \frac{17}{7}(5x - 2y + 4z + 7) = 0$$

$$\Rightarrow \quad 7(x^2 + y^2 + z^2) - 64x - 20y - 54z + 34 = 0.$$

This is the equation of required sphere.

1999; 2001 Q. No. 1(b) OR

A sphere of radius k passes through the origin and meets the axes in A, B, C. Prove that the centroid of the triangle ABC lies on the sphere $9(x^2 + y^2 + z^2) = 4k^2$.

Solution: Since the equation of sphere through the origin be,

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz = 0 \qquad \text{...... (i)}$$

Let A(a, 0, 0), B(a, b, 0) and C(0, 0, c) be points sphere at the axes. Then,

$$a^2 + 2ua = 0 \qquad b^2 + 2vb = 0 \qquad \text{and } c^2 + 2wc = 0$$

$$\Rightarrow a(a + 2u) = 0 \qquad \Rightarrow b(b + 2v) = 0 \qquad \Rightarrow c(c + 2w) = 0$$

$$\Rightarrow u = -\frac{1}{2} \qquad \Rightarrow v = -\frac{1}{2} \qquad \Rightarrow w = -\frac{1}{2}$$

Also, given that the radius of sphere (i) is

$$r = \sqrt{u^2 + v^2 + w^2} = k$$

So, $$u^2 + v^2 + w^2 = k^2 \Rightarrow \frac{a^2}{4} + \frac{b^2}{4} + \frac{c^2}{4} = k^2$$

$$\Rightarrow a^2 + b^2 + c^2 = 4k^2 \qquad \text{...... (ii)}$$

Let, (x_1, y_1, z_1) be the centroid of triangle ABC. Then,

$$x_1 = \frac{a+0+0}{3} = \frac{a}{3}, \qquad y_1 = \frac{b+0+0}{3} = \frac{b}{3}, \qquad z_1 = \frac{c+0+0}{3} = \frac{c}{3}$$

$$\Rightarrow a = 3x_1, \qquad \Rightarrow b = 3y_1 \qquad \Rightarrow c = 3z_1$$

Then (ii) becomes,

$$(3x_1)^2 + (3y_1)^2 + (3z_1)^2 = 4k^2$$

$$\Rightarrow \quad 9(x_1^2 + y_1^2 + z_1^2) = 4k^2$$

Hence, the centroid of ABC lies in the sphere $9(x^2 + y^2 + z^2) = 4k^2$.

2002; 2004 Spring Q. No. 1(b)

Find the equation of the sphere having the circle, $x^2 + y^2 + z^2 = 9$, $x - 2y + 2z = 5$ as a great circle. Determine its radius and centre.

Solution: The given circles are

$$x^2 + y^2 + z^2 = 9 \qquad \text{...... (i)}$$

$$x - 2y + 2z = 5 \qquad \text{...... (ii)}$$

Then equation of sphere through given circle is

$$x^2 + y^2 + z^2 - 9 + \lambda(x - 2y + 2z - 5) = 0 \qquad \text{...... (iii)}$$

If the given circle is a great circle of (iii), then the centre of (iii) lies on the plane (ii).

That is, the centre of (iii) $(-\lambda/2, \lambda, -\lambda)$ lies on (iii). Therefore,

$$(-\lambda/2) - 2\lambda - 2\lambda = 5$$

$$\Rightarrow \; -\lambda - 4\lambda - 4\lambda = 10 \;\Rightarrow\; \lambda = \frac{-10}{9}$$

Now equation (iii) becomes,

$$9(x^2 + y^2 + z^2) - 10x + 20y - 20z + 41 = 0 \qquad \ldots\ldots (iv)$$

Then comparing (iv) with $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz = 0$ then we get,

(a) Centre of (iv) is, $(-u, -v, -w) = (5, -10, 10)$.

(b) Radius of (iv) is, $r = \sqrt{u^2 + v^2 + w^2 - d} = \sqrt{25 + 100 + 100 - 41} = \sqrt{184}$.

Thus the equation of sphere is (iv) and its centre and radius are (5, -10, 10) and $\sqrt{184}$ respectively.

Similar Question for Practice from Final Exam:

2000 Q. No. 1(b)

Find the equation of the sphere having the circle $x^2 + y^2 + z^2 = 9$, $x - 2y + 2z = 5$ as a great circle.

2002 Q. No. 1(b)

Find the equation of a sphere for which the circle, $x^2 + y^2 + z^2 + 7y - 2z + 2 = 0$, $2x + 3y + 4z = 8$ is a great circle.

2007 Fall Q. No. 1(b)

Find the equation of the sphere for which the circle, $x^2 + y^2 + z^2 + 7y - 2z + 2 = 0$, $2x + 3y + 4z = 8$ is a great circle. Determine its centre and radius.

OTHER QUESTIONS

2003 Fall Q. No. 1(b)

Find the equation of the sphere through the circle $x^2 + y^2 + z^2 = 9$, $x + 3y + 4z - 2 = 0$ and the point (2, 2, 3). Also determine the radius and centre of the sphere.

Solution: Given that the equation of sphere through the circle $x^2 + y^2 + z^2 = 9$, $x + 3y + 4z = 2$ is

$$x^2 + y^2 + z^2 - 9 + \lambda(x + 3y + 4z - 2) = 0 \qquad \ldots\ldots (i)$$

Since the equation (i) passes through (2, 2, 3), so,

$$2^2 + 2^2 + 3^2 - 9 + \lambda\,(2 + 3 \times 2 + 4 \times 3 - 2) = 0$$

$$\Rightarrow \; 4 + 4 + 9 - 9 + \lambda\,(2 + 6 + 12 - 2) = 0 \Rightarrow 18\lambda = -8 \;\Rightarrow\; \lambda = \frac{-4}{9}$$

Then equation (i) becomes,

$$9(x^2 + y^2 + z^2 - 9) - 4(x + 3y + 4z - 2) = 0$$

$$\Rightarrow \quad 9(x^2 + y^2 + z^2) - 4x - 12y - 16z - 73 = 0 \qquad \ldots\ldots (ii)$$

Then comparing (iv) with $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz = 0$ then we get;

(a) Centre of (iv) is, $(-u, -v, -w) = \left(\frac{2}{9}, \frac{6}{9}, \frac{8}{9}\right)$.

(b) Radius of (iv) is, $r = \sqrt{u^2 + v^2 + w^2 - d}$

$$= \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{2}{3}\right)^2 - \left(\frac{-22}{3}\right)}$$

$$= \sqrt{\frac{1}{4} + \frac{1}{2} + \frac{2}{3} + \frac{22}{3}} = \sqrt{\frac{293}{36}}$$

Thus the equation of sphere is (ii) and its centre and radius are $\left(\frac{2}{9}, \frac{6}{9}, \frac{8}{9}\right)$ and $\sqrt{\frac{293}{36}}$ respectively.

2006 Fall Q. No. 1(b)

Find the equation of the tangent plane to the sphere $x^2 + y^2 + z^2 + 2x + 4y + 6z - 7 = 0$ which intersect the line $6x - 3y - 23 = 0 = 3z + 2$.

Solution: Given that the equation of sphere is

$$x^2 + y^2 + z^2 + 2x - 4y + 6z - 7 = 0 \qquad \ldots\ldots (1)$$

Comparing (1) with $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ then we get,

$$u = 1, \quad v = 2, \quad w = 3 \quad \text{and} \quad d = -7.$$

Then the centre of (1) is,

$$(-u, -v, -w) = (-1, -2, -3)$$

And radius of (1) is,

$$r = \sqrt{u^2 = v^2 + w^2 - d} = \sqrt{1 + 4 + 9 + 7} = \sqrt{21}$$

Also, given line is

$$6x - 3y - 23 = 0 = 3z + 2 \qquad \ldots\ldots\ldots (2)$$

So, the equation of plane through the line (2) is

$$(6x - 3y - 23) + \lambda(3z + 2) = 0 \qquad \ldots\ldots\ldots (3)$$

If the plane (3) is a tangent plane to the sphere (i) then the perpendicular distance from the centre of (1) to it is equal to the length of radius of the sphere (1). Therefore,

$$\sqrt{21} = \frac{-6 + 6 - 9\lambda}{\sqrt{36 + 9 + 9\lambda^2}} = \frac{-9\lambda}{3\sqrt{5 + \lambda^2}} = \frac{3\lambda}{\sqrt{\lambda^2 + 5}}.$$

$$\Rightarrow \quad 21 = \frac{9\lambda^2}{\lambda^2 + 5}$$

$$\Rightarrow \quad 21\lambda^2 + 105 = 9\lambda^2 \;\Rightarrow\; \lambda^2 = -\frac{105}{12}$$

This gives imaginary value. So, the tangent plane to (1) with given condition is impossible.

2006 Spring Q. No. 1(b)

Find the equation of tangent planes to the sphere $x^2 + y^2 + z^2 - 4x + 2y - 6z + 5 = 0$ which are parallel to the plane $2x + y - z = 0$.

Solution: Given sphere is

$$x^2 + y^2 + z^2 - 4x + 2y - 6z + 5 = 0 \qquad \ldots\ldots (1)$$

Comparing it with $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ then we get,

$$u = -2, \quad v = 1, \quad w = -3 \quad \text{and} \quad d = 5.$$

Then the centre of (1) is

$$(-u, -v, -w) = (2, -1, 3)$$

and length of radius of (1) is

$$r = \sqrt{u^2 + v^2 + w^2 - d} = \sqrt{4 + 1 + 9 - 5} = \sqrt{9} = 3$$

Given that the tangent plane to (1) is parallel to the plane $2x + y - z = 0$.

So, the equation of tangent plane to (1) is

$$2x + y - z + \lambda = 0 \qquad \text{....... (2)}$$

Since the length of radius of (1) is equal to the perpendicular distance from the centre of (1) to its tangent plane (3). That is,

$$3 = \pm \frac{2(2) + (-1) - 3 + \lambda}{\sqrt{4 + 1 + 1}} = \pm \frac{4 - 1 - 3 + \lambda}{\sqrt{6}} = \pm \frac{\lambda}{\sqrt{6}}$$

$$\Rightarrow x = \pm 3\sqrt{6}$$

Therefore (2) becomes,

$$2x + y - z \pm 3\sqrt{6} = 0$$

This is the equation of required tangent plane to (1) and is parallel to $2x + y - z = 0$.

SHORT QUESTIONS

2008 Spring: Find the equation of the sphere whose centre is (1, 3, 2) and radius 3.

Solution: Given that, the sphere has,

$$\text{Centre } (\alpha, \beta, \delta) = (1, 3, 2) \quad \text{and} \quad \text{Radius } r = 3.$$

Now the equation of sphere is,

$$(x - \alpha)^2 + (y - \beta)^2 + (z - \delta)^2 = r^2$$

$$\Rightarrow (x - 1)^2 + (y - 3)^2 + (z - 2)^2 = 3^2$$

$$\Rightarrow x^2 + y^2 + z^2 - 2x - 6y - 4z + 5 = 0.$$

2009 Fall: Find Center and radius of a sphere $x^2 + y^2 + z^2 + 4x - 6y + 8z = 10$.

Solution: Given sphere is

$$x^2 + y^2 + z^2 + 4x - 6y + 8z = 10 \qquad \text{........(1)}$$

Since the sphere,

$$x^2 + y^2 + z^2 + 2ux + 2vy + 20wz + d = 0 \qquad \text{........(2)}$$

has centre at $(-u, -v, -w)$ and radius $r = \sqrt{u^2 + v^2 + w^2 - d}$

Comparing (1) and (2) we get,

$$u = 2, \quad v = -3, \quad w = 4, \quad d = -10.$$

Therefore, the centre of (1) is (-2, 3, -4) and the radius of (1) is

$$r = \sqrt{4 + 9 + 16 - 10} = \sqrt{19}.$$

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