Exercise 8.4

1. **Find the Laplace transform of the following functions:**

(i) **t cos 2t.**

Solution: Given function is, t cos 2t.

Since we have, $\mathcal{L}\{t\, f(t)\} = -\frac{d}{ds}(\mathcal{L}\{f(t)\})$ and $\mathcal{L}\{\cos at\} = \frac{s}{s^2 + a^2}$

Now,

$$\mathcal{L}\{t\cos 2t\} = -\frac{d}{ds}(\mathcal{L}\{\cos 2t\})$$

$$= -\frac{d}{ds}\left(\frac{s}{s^2+4}\right) = -\frac{s^2+4-2s^2}{(s^2+4)^2} = \frac{s^2-4}{(s^2+4)^2}$$

Thus, $\mathcal{L}\{t\cos 2t\} = \frac{s^2-4}{(s^2+4)^2}$

(ii) **t cosh t.** **[1999; 2001 Q. No. 4-a(i)]**

Solution: Given function is, t cosh t.

Since we have, $\mathcal{L}\{t\, f(t)\} = -\frac{d}{ds}(\mathcal{L}\{f(t)\})$ and $\mathcal{L}\{\cosh at\} = \frac{s}{s^2 - a^2}$

Now,

$$\mathcal{L}\{t\cosh t\} = -\frac{d}{ds}(\mathcal{L}\{\cosh t\})$$

$$= -\frac{d}{ds}\left(\frac{s}{s^2-1}\right) = -\frac{s^2-1-2s^2}{(s^2-1)^2} = \frac{s^2+1}{(s^2-1)^2}$$

Thus, $\mathcal{L}\{t\cosh t\} = \frac{s^2+1}{(s^2-1)^2}$

(iii) **$t^2\cos wt$** **[2006 Fall Q. No. 6(a-(i))]**

Solution: Given function is, $t^2\cos wt$

Since we have, $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}(\mathcal{L}\{f(t)\})$ and $\mathcal{L}\{\cos at\} = \frac{s}{s^2 + a^2}$

Now,

$$\mathcal{L}\{t^2\cos wt\} = (-1)^2\frac{d^2}{ds^2}(\mathcal{L}(\cos wt\})$$

$$= \frac{d^2}{ds^2}\left(\frac{s}{s^2+w^2}\right)$$

$$= \frac{d}{ds}\left(\frac{s^2+w^2-2s^2}{(s^2+w^2)^2}\right)$$

$$= \frac{d}{ds}\left(\frac{w^2-s^2}{(s^2+w^2)^2}\right)$$

$$= \frac{(s^2+w^2)^2(-2s) - (w^2-s^2)\, 2.\, (s^2+w^2)(2s)}{(s^2+w^2)^4}$$

$$= \frac{-2s\,[s^2 + w^2 - 2(w^2 - s^2)]}{(s^2 + w^2)^3} = \frac{-2s\,(3s^2 - w^2)}{(s^2 + w^2)^3} = \frac{2s\,(w^2 - 3s^2)}{(s^2 + w^2)^3}$$

Thus, $\mathcal{L}\{t^2 \cos wt\} = \frac{2s\,(w^2 - 3s^2)}{(s^2 + w^2)^3}$

(iv) t sinh 2t.

Solution: Given function is,

Since we have, $\mathcal{L}\{t\, f(t)\} = -\frac{d}{ds}(\mathcal{L}\{f(t)\})$ and $\mathcal{L}\{\sinh at\} = \frac{a}{s^2 - a^2}$

Now,

$$\mathcal{L}\{t \sinh 2t\} = -\frac{d}{ds}(\mathcal{L}\{\sinh 2t\})$$
$$= -\frac{d}{ds}\left(\frac{2}{s^2 - 4}\right) = -\frac{-4s}{(s^2 - 4)^2} = \frac{4s}{(s^2 - 4)^2}$$

Thus, $\mathcal{L}\{t \sinh 2t\} = \frac{4s}{(s^2 - 4)^2}$

(v) $t^2 e^t$

Solution: Given function is, $t^2 e^t$

Since we have, $\mathcal{L}\{e^{at} f(t)\} = (\mathcal{L}\{f(t)\})_{s \to s-a}$ and, $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

Now,

$$\mathcal{L}\{t^2 e^t\} = (\mathcal{L}\{t^2\})_{s \to s-1}$$
$$= \left(\frac{2!}{s^3}\right)_{s \to s-1} = \frac{2}{(s-1)^3}$$

Thus, $\mathcal{L}\{t^2 e^t\} = \frac{2}{(s-1)^3}$

(vi) $te^{-2t} \sin wt$

Solution: Given function is, $te^{-2t} \sin wt$

Since we have, $\mathcal{L}\{e^{at} f(t)\} = (\mathcal{L}\{f(t)\})_{s \to s-a}$

$\mathcal{L}\{tf(t)\} = -\frac{d}{ds}(\mathcal{L}\{f(t)\})$ and $\mathcal{L}\{\sin at\} = \frac{a}{s^2 + a^2}$

Now,

$$\mathcal{L}\{te^{-2t} \sin wt\} = (\mathcal{L}\{t \sin wt\})_{s \to s+2}$$
$$= \left(-\frac{d}{ds}(L\{\sin wt\})\right)_{s \to s+2}$$
$$= \left(-\frac{d}{ds}\left(\frac{w}{s^2 + w^2}\right)\right)_{s \to s+2} = \left(-\left(\frac{-2ws}{(s^2 + w^2)^2}\right)\right)_{s \to s+2}$$
$$\frac{2w\,(s+2)}{((s+2)^2 + w^2)^2}$$

Thus, $\mathcal{L}\{te^{-2t} \sin wt\} = \frac{2w\,(s+2)}{((s+2)^2 + w^2)^2}$

(vii) $t^2 e^{2t} \cos t$

Solution: Given function is, $t^2 e^{2t} \cos t$

Since we have, $\mathcal{L}\{e^{at} f(t)\} = (\mathcal{L}\{f(t)\})_{s \to s-a}$

$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}(\mathcal{L}\{f(t)\})$ and $\mathcal{L}\{\cos at\} = \frac{s}{s^2 + a^2}$

Now,

$$\mathcal{L}\{t^2 e^{-2t} \cos t\} = (\mathcal{L}\{t^2 \cos t\})_{s \to s+a}$$
$$= \left((-1)^2 \frac{d^2}{ds^2}(L\{\cos t\})\right)_{s \to s+2}$$
$$= \left(\frac{d^2}{ds^2}\left(\frac{s}{s^2 + 1}\right)\right)_{s \to s+2}$$
$$= \left(\frac{d}{ds}\left(\frac{1 - s^2}{(s^2 + 1)^2}\right)\right)_{s \to s+2}$$
$$= \left(\frac{(s^2 + 1)^2(-2s) - (1 - s^2) \cdot 2(s^2 + 1)\,2s}{(s^2 + 1)^4}\right)_{s \to s+2}$$
$$= \left(-2s\left(\frac{s^2 + 1 - 2 + 2s^2}{(s^2 + 1)^3}\right)\right)_{s \to s+2}$$
$$= \left(2s\left(\frac{1 - 3s^2}{(s^2 + 1)^3}\right)\right)_{s \to s+2} = 2(s+2)\left[\frac{1 - 3(s+2)^2}{((s+2)^2 + 1)^3}\right]$$

Thus, $\mathcal{L}\{t^2 e^{-2t} \cos t\} = \frac{2(s+2)\,\{1 - 3(s+2)^2\}}{\{(s+2)^2 + 1\}^3}$

(viii) $te^{-t} \cosh t$

Solution: Given function is, $te^{-t} \cosh t$

Since we have, $\mathcal{L}\{e^{at} f(t)\} = (\mathcal{L}\{f(t)\})_{s \to s-a}$

$\mathcal{L}\{tf(t)\} = -\frac{d}{ds}(\mathcal{L}f(t))$ and $\mathcal{L}\{\cosh at\} = \frac{s}{s^2 - a^2}$

Now,

$$\mathcal{L}\{te^{-t} \cosh t\} = (\mathcal{L}\{t \cosh t\})_{s \to s+1}$$
$$= \left(-\frac{d}{ds}(L\{\cosh t\})\right)_{s \to s+1}$$
$$= -\left(\frac{d}{ds}\left(\frac{s}{s^2 - 1}\right)\right)_{s \to s+1}$$
$$= \left(\frac{s^2 - 1 - 2s^2}{(s^2 - 1)^2}\right)_{s \to s+1} = -\left(\frac{s^2 + 1}{(s^2 - 1)^2}\right)_{s \to s+1} = \frac{(s+1)^2 + 1}{\{(s+1)^2 - 1\}^2}$$

Thus, $\mathcal{L}\{te^{-t} \cosh t\} = \frac{(s+1)^2 + 1}{\{(s+1)^2 - 1\}^2} = \frac{s^2 + 2s + 2}{(s^2 + 2s)^2} = \frac{s^2 + 2s + 2}{s^2\,(s+2)^2}$

(ix) t sin3t cos2t

Solution: Given function is, $t \sin 3t \cos 2t = \frac{1}{2}\, t\{\sin(3+2)t + \sin(3-2)t\}$

$$= \frac{1}{2}\, t\,\{\sin 5t + \sin t\}$$

Since we have, $\mathcal{L}\{tf(t)\} = -\frac{d}{ds}(\mathcal{L}\{f(t)\})$ and $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$

Now,

$$\mathcal{L}\left\{\frac{1}{2}t(\sin 5t + \sin t)\right\} = -\frac{1}{2}\frac{d}{ds}(\mathcal{L}\{\sin 5t\} + \mathcal{L}\{\sin t\})$$

$$= -\frac{1}{2}\frac{d}{ds}\left(\frac{5}{s^2+25} + \frac{1}{s^2+1}\right)$$

$$= -\frac{1}{2}\left[\frac{-10s}{(s^2+25)^2} + \frac{-2s}{(s^2+1)^2}\right] = \frac{5s}{(s^2+25)^2} + \frac{s}{(s^2+1)^2}$$

Thus, $\mathcal{L}\{t\sin 3t\cos 2t\} = \frac{5s}{(s^2+2t)^2} + \frac{s}{(s^2+1)^2}$.

(x) $t^2\cosh \pi t$

Solution: Given function is, $t^2\cosh \pi t$

Since we have, $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}(\mathcal{L}\{f(t)\})$ and $\mathcal{L}\{\cosh at\} = \frac{s}{s^2-a^2}$

Now,

$$\mathcal{L}\{t^2\cosh \pi t\} = (-1)^2\frac{d^2}{ds^2}(\mathcal{L}(\cosh \pi t))$$

$$= \frac{d^2}{ds^2}\left(\frac{s}{s^2-\pi^2}\right)$$

$$= \frac{d}{ds}\frac{s^2-\pi^2-2s^2}{(s^2-\pi^2)^2}$$

$$= -\frac{d}{ds}\left(\frac{s^2+\pi^2}{(s^2-\pi^2)^2}\right)$$

$$= -\frac{(s^2-\pi^2)^2(2s) - (s^2+\pi^2)\,2.\,(s^2-\pi^2)(2s)}{(s^2-\pi^2)^4}$$

$$= -\frac{2s\,[s^2-\pi^2-2(\pi^2+s^2)]}{(s^2-\pi^2)^3} = \frac{2s\,(s^2+3\pi^2)}{(s^2-\pi^2)^3}$$

Thus, $\mathcal{L}\{t^2\cos wt\} = \frac{2s\,(w^2-3s^2)}{(s^2+w^2)^3}$.

(xi) t cos wt. **[1999 Q. No. 4-a(ii)]**

Solution: Given function is, t cos wt.

Since we have, $\mathcal{L}\{t\,f(t)\} = -\frac{d}{ds}(\mathcal{L}\{f(t)\})$ and $\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$

Now,

$$\mathcal{L}\{t\cos wt\} = -\frac{d}{ds}(\mathcal{L}\{\cos wt\})$$

$$= -\frac{d}{ds}\left(\frac{s}{s^2+w^2}\right) = -\frac{s^2+w^2-2s^2}{(s^2+w^2)^2} = \frac{s^2-w^2}{(s^2+w^2)^2}$$

Thus, $\mathcal{L}\{t\cos 2t\} = \frac{s^2-w^2}{(s^2+w^2)^2}$.

(xii) $te^{-t}\sin t$

Solution: Given function is, $te^{-t}\sin t$.

Since we have, $\mathcal{L}\{e^{at}f(t)\} = (\mathcal{L}\{f(t)\})_{s\to s-a}$

$\mathcal{L}\{tf(t)\} = -\frac{d}{ds}(\mathcal{L}\{f(t)\})$ and $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$

Now,

$$\mathcal{L}\{te^{-t}\sin t\} = (\mathcal{L}\{t\sin t\})_{s\to s+2}$$

$$= \left(-\frac{d}{ds}(L\{\sin t\})\right)_{s\to s+2}$$

$$= \left(-\frac{d}{ds}\left(\frac{1}{s^2+1}\right)\right)_{s\to s+2} = \left(-\left(\frac{-2s}{(s^2+1)^2}\right)\right)_{s\to s+2} = \frac{2(s+2)}{((s+2)^2+1)^2}$$

Thus, $\mathcal{L}\{te^{-t}\sin t\} = \frac{2(s+2)}{((s+2)^2+1)^2}$

(xiii) $\frac{\cos 2t - \cos 3t}{t}$

Solution: Given function is $f(t) = \frac{\cos 2t - \cos 3t}{t} \Rightarrow t\,f(t) = \cos 2t - \cos 3t$ ......(1)

Since we have, $\mathcal{L}\{t(f(t)\} = -\frac{d}{ds}(\mathcal{L}\{f(t)\})$ and $\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$

Now, taking Laplace transform on both sides of (1) then,

$$\mathcal{L}\{t(f(t)\} = \mathcal{L}\{\cos 2t\} - \mathcal{L}\{\cos 3t\}$$

$$\Rightarrow -\frac{d}{ds}(\mathcal{L}\{f(t)\}) = \frac{s}{s^2+4} - \frac{s}{s^2+9}$$

$$\Rightarrow \frac{d}{ds}(\mathcal{L}(\{t(t)\}) = \frac{s}{s^2+9} - \frac{s}{s^2+4}$$

Taking integration on both sides w.r. to s then,

$$\mathcal{L}\{f(t)\} = \int\frac{s}{s^2+9}ds - \int\frac{s}{s^2+4}ds$$

$$= \frac{1}{2}[\log(s^2+9) - \log(s^2+4)] = \frac{1}{2}\log\left(\frac{s^2+9}{s^2+4}\right)$$

Thus, $\mathcal{L}\left\{\frac{\cos 2t - \cos 3t}{t}\right\} = \frac{1}{2}\log\left(\frac{s^2+9}{s^2+4}\right)$.

(xiv) $\frac{\sin ht}{t}$

Solution: Let $f(t) = \frac{\sin ht}{t} \Rightarrow t\,f(t) = \sin ht$.

Now, taking Laplace transform then,

$$-\mathcal{L}\{f(t)\} = \int\frac{1}{s^2-1}ds = \frac{1}{2}\log\left(\frac{s+1}{s-1}\right)$$

$$\Rightarrow \mathcal{L}\{f(t)\} = -\frac{1}{2}\log\left(\frac{s+1}{s-1}\right) = \frac{1}{2}\log\left(\frac{s+1}{s-1}\right)^{-1} = \frac{1}{2}\log\left(\frac{s-1}{s+1}\right).$$

Thus, $\mathcal{L}\left\{\frac{\sin ht}{t}\right\} = \frac{1}{2}\log\left(\frac{s-1}{s+1}\right)$.

(xv) $\frac{\sin^2 t}{t}$

Solution: Let $f(t) = \frac{\sin^2 t}{t} \Rightarrow tf(t) = \sin^2 t = \frac{1-\cos 2t}{2}$

Now, taking Laplace transform then

$$\mathcal{L}\{t(f(t)\} = \mathcal{L}\left\{\frac{1-\cos 2t}{2}\right\}$$

$$\Rightarrow -\frac{d}{ds}(\mathcal{L}\{f(t)\}) = \frac{1}{2}[\mathcal{L}\{1\} - \mathcal{L}\{\cos 2t\}]$$

$$\Rightarrow \frac{d}{ds}(\mathcal{L}\{f(t)\}) = \frac{1}{2}\{\mathcal{L}(\cos 2t) - \mathcal{L}\{1\}\}$$

$$= \frac{1}{2}\left(\frac{s}{s^2+4} - \frac{1}{s}\right)$$

And, taking integration w.r. to s then,

$$\mathcal{L}\{f(t)\} = \frac{1}{2}\left[\int \frac{s}{s^2+4}ds - \int\frac{ds}{s}\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}\log(s^2+4) - \log(s)\right]$$

$$= \frac{1}{4}[\log(s^2+4) - \log s^2] = \frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right).$$

Thus, $\mathcal{L}\left\{\frac{\sin^2 t}{t}\right\} = \frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right)$.

(xiv) $\frac{e^{-at} - e^{-bt}}{t}$

Solution: Let $f(t) = \frac{e^{-at} - e^{-bt}}{t} \Rightarrow t f(t) = e^{-at} - e^{-bt}$

Now, taking Laplace transform then

$$\mathcal{L}\{t f(t)\} = \mathcal{L}\{e^{-at}\} - \mathcal{L}\{e^{-bt}\}$$

$$\Rightarrow -\frac{d}{ds}(\mathcal{L}\{f(t)\}) = (\mathcal{L}\{\sin t\})_{s\to s+1} \quad [\because \text{ using first shifting theorem}]$$

$$= \left(\frac{1}{s^2+1}\right)_{s\to s+1} \quad \left[\because L\{\sin at\} = \frac{a}{s^2+a^2}\right]$$

$$= \frac{1}{(s+1)^2+1}$$

And, taking integration w.r. to s then,

$$-\mathcal{L}\{f(t)\} = \int\frac{ds}{(s+1)^2+1} = \tan^{-1}(s+1)$$

$$\Rightarrow \mathcal{L}\{f(t)\} = -\tan^{-1}(s+1).$$

Thus, $\mathcal{L}\left\{\frac{e^{-at}-e^{-bt}}{t}\right\} = -\tan^{-1}(s+1)$.

Find f(t) if $\mathcal{L}\{f(t)\}$ equals:

(i) $\frac{1}{(s+1)^2}$

Solution: Let $\mathcal{L}\{f(t)\} = \frac{1}{(s+1)^2} = \left(\frac{1}{s^2}\right)_{s\to s+1}$

Since we have, $\mathcal{L}\{e^{at} f(t)\} = (\mathcal{L}\{f(t)\})_{s\to s-a}$ and $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

So, $\mathcal{L}\{f(t)\} = \left(\frac{1}{1!}L\{t\}\right)_{s\to s+1} = \mathcal{L}\{e^{-t}t\}$

Thus, $f(t) = t\,e^{-t}$

(ii) $\frac{2s}{(s^2-4)^2}$

Solution: Let $\mathcal{L}\{f(t)\} = \frac{2s}{(s^2-4)^2}$

Since we have, $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\bar{s})\,d\bar{s} = 2\int_s^\infty \frac{\bar{s}}{(\bar{s}^2-4)^2}d\bar{s}$

Set, $\bar{s}^2 - 4 = u$ then $2\bar{s}\,d\bar{s} = du$. Also, $\bar{s} = s \Rightarrow u = s^2-4$ and $\bar{s} = \infty \Rightarrow u = \infty$.

Now,

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_{s^2-4}^{\infty}\frac{du}{u^2} = \left[-\frac{1}{u}\right]_{s^2-4}^{\infty} = \frac{1}{s^2-4}$$

Taking inverse Laplace transform then,

$$\frac{f(t)}{t} = \mathcal{L}^{-1}\left\{\frac{1}{s^2-4}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2-4}\right\} = \frac{1}{2}\sinh 2t$$

Thus, $f(t) = \frac{t}{2}\sinh 2t$.

(iii) $\log\left[1 + \frac{w^2}{s^2}\right]$

Solution: Let $\mathcal{L}\{f(t)\} = F(s) = \log\left[1+\frac{w^2}{s^2}\right]$

Then, $F'(s) = \frac{1}{\left[1+\frac{w^2}{s^2}\right]} \times -w^2 \times 2s^{-3} = -\frac{2w^2 s^{-3}}{s^2+w^2}\times s^2 = -\frac{2w^2}{s(s^2+w^2)}$

$$\Rightarrow -F'(s) = \frac{2w^2}{s(s^2+w^2)}$$

Since we have, $\mathcal{L}\{tf(t)\} = -\frac{d}{ds}(\mathcal{L}\{f(t)\}) = -F'(s)$.

So, $\mathcal{L}[t\{f(t)\}] = \frac{2w^2}{s(s^2+w^2)} \Rightarrow t f(t) = \mathcal{L}^{-1}\left[\frac{2w^2}{s(s^2+w^2)}\right]$ ....(i)

Let,

$$\frac{2w^2}{s(s^2+w^2)} = \frac{A}{s} + \frac{Bs+C}{s^2+w^2}$$

$$\Rightarrow \frac{2w^2}{s(s^2+w^2)} = \frac{As^2+Aw^2+Bs^2+Cs}{s(s^2+w^2)}$$

This gives, $2w^2 = s^2(A+B) + Cs + Aw^2$

Equating the like terms,

$A + B = 0$, $C = 0$ and $A = 2$.

Solving we get, $A = 2$, $B = -2$ and $C = 0$.

Now equation (i) becomes,

$$t\,f(t) = \mathcal{L}^{-1}\left[\frac{2}{s} - \frac{2s}{s^2+w^2}\right] = 2\,\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 2\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+w^2}\right\}$$

$$= 2\{1\} - 2\{\cos wt\}$$ [using table of Laplace transform]

$$= (2 - 2\cos wt)$$

$$= 2(1 - \cos wt)$$

Thus, $f(t) = \dfrac{2(1-\cos wt)}{t}$.

(iv) $\log \dfrac{s}{s-1}$

Solution: Let, $\mathcal{L}\{f(t)\} = F(s) = \log\left(\dfrac{s}{s-1}\right)$

Then, $F'(s) = \dfrac{1}{\dfrac{s}{(s-1)}} \times \dfrac{(s-1)\cdot 1 - s(1)}{(s-1)^2}$

$$= \frac{(s-1)}{s} \times \frac{s-1-s}{(s-1)^2} = -\frac{1}{s(s-1)}$$

$$\Rightarrow -F'(s) = \frac{1}{s(s-1)}$$

Since we have, $\mathcal{L}\{tf(t)\} = -\dfrac{d}{ds}(\mathcal{L}\{f(t)\}) = -F'(s)$.

So, $\mathcal{L}\{t\,f(t)\} = \dfrac{1}{s(s-1)} \Rightarrow t\,f(t) = \mathcal{L}^{-1}\left[\dfrac{1}{s(s-1)}\right]$ ... (i)

Let,

$$\frac{1}{s(s-1)} = \frac{A}{s} + \frac{B}{(s-1)} = \frac{A(s-1)+Bs}{s(s-1)}$$

This gives, $1 = As - A + Bs$

$\Rightarrow 1 = s(A+B) - A$

Equating the like terms then we get,

$A + B = 0$, $A = -1$.

Solving we get, $A = -1$ and $B = 1$.

Now equation (i) becomes,

$t\,f(t) = \mathcal{L}^{-1}\left[-\dfrac{1}{s} + \dfrac{1}{(s-1)}\right] = (-1 + e^t)$ [Using the table of Laplace transform]

$$\Rightarrow f(t) = \frac{e^t - 1}{t}$$

(v) $\cot^{-1}\dfrac{s}{w}$

Solution: Let, $\mathcal{L}\{f(t)\} = F(s) = \cot^{-1}\dfrac{s}{w}$

Then, $F'(s) = -\dfrac{1}{\left(\dfrac{s^2}{w^2}+1\right)} \times \dfrac{1}{w} = -\dfrac{w}{s^2+w^2}$

Since we have, $\mathcal{L}\{tf(t)\} = -\dfrac{d}{ds}(\mathcal{L}\{f(t)\}) = -F'(s)$.

So, $\mathcal{L}[\{t\,f(t)\}] = \dfrac{w}{s^2+w^2}$

$\Rightarrow t\,f(t) = \mathcal{L}^{-1}\left[\dfrac{w}{s^2+w^2}\right] = \sin wt$ [Using the Table of Laplace transform]

$$\Rightarrow f(t) = \frac{\sin wt}{t}$$

(vi) $\dfrac{s}{(s^2+1)^2}$

Solution: Let, $\mathcal{L}\{f(t)\} = F(s) = \dfrac{s}{(s^2+1)^2}$

Since we have,

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\bar{s})\,d(\bar{s}) = \int_s^\infty \frac{\bar{s}}{(\bar{s}^2+1)^2}\,d(\bar{s}) \quad \dots (1)$$

Put, $u = \bar{s}^2 + 1$ then $du = 2\bar{s}\,d(\bar{s}) \Rightarrow \dfrac{du}{2} = \bar{s}\,d(\bar{s})$. So, (1) becomes,

$$= \frac{1}{2}\int_s^\infty \frac{du}{u^2}$$

$$= -\frac{1}{2}\left[\frac{1}{u}\right]_s^\infty = -\frac{1}{2}\left[\frac{1}{\bar{s}^2+1}\right]_s^\infty = 0 + \frac{1}{2}\left(\frac{1}{s^2+1}\right)$$

$\Rightarrow \dfrac{f(t)}{t} = \dfrac{1}{2}\mathcal{L}^{-1}\left[\dfrac{1}{s^2+1}\right] = \dfrac{1}{2}\sin t$ [Using the table of Laplace transform]

$$\Rightarrow f(t) = \frac{t}{2}\sin t.$$

(vii) $\dfrac{s}{(s^2+a^2)^2}$

Solution: Let, $\mathcal{L}\{f(t)\} = F(s) = \dfrac{s}{(s^2+a^2)^2}$

We have,

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\bar{s})\,d(\bar{s}) = \int_s^\infty \frac{\bar{s}}{(\bar{s}^2+a^2)^2}\,d(\bar{s}) \quad \dots (1)$$

Put, $u = \bar{s}^2 + a^2$ then $\frac{du}{2} = \bar{s}\, d(\bar{s})$. So (1) becomes,

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \frac{du}{2u^2}$$

$$= -\frac{1}{2}\left[\frac{1}{u}\right]_s^\infty = -\frac{1}{2}\left[\frac{1}{(\bar{s}^2 + a^2)}\right]_s^\infty = 0 + \frac{1}{(\bar{s}^2 + a^2)}$$

$$\Rightarrow \frac{f(t)}{t} = \mathcal{L}^{-1}\left[\frac{1}{(\bar{s}^2 + a^2)}\right] = \frac{1}{a}\sin at \quad \text{[Using the table of Laplace transform]}$$

$$\Rightarrow f(t) = \frac{t}{a}\sin at.$$

(viii) $\frac{1}{(s-3)^3}$

Solution: Let, $\mathcal{L}\{f(t)\} = F(s) = \frac{1}{(s-3)^3}$

Since we have,

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\bar{s})\, d(\bar{s}) = \int_s^\infty \frac{1}{(\bar{s}-3)^3}\, d(\bar{s}) \quad \dots (1)$$

Put, $\bar{s} - 3 = u$ then $du = d\bar{s}$. So (1) becomes,

$$= \int_s^\infty \frac{1}{u^3}\, du = \left[-\frac{1}{2u^2}\right]_s^\infty = 0 + \frac{1}{2(s-3)^2}$$

$$\Rightarrow \frac{f(t)}{t} = \mathcal{L}^{-1}\left[\frac{1}{2(s-3)^2}\right] = \frac{1}{2}e^{3t}.\, t \quad \text{[Using the table of Laplace transform]}$$

$$\Rightarrow f(t) = \frac{1}{2}t^2 e^{3t}$$

(ix) $\frac{s^2 - \pi^2}{(s^2 + \pi^2)^2}$

Solution: Let, $\mathcal{L}\{f(t)\} = F(s) = \frac{s^2 - \pi^2}{(s^2 + \pi^2)^2}$

Then, $f(t) = \mathcal{L}^{-1}\left[\frac{s^2 - \pi^2}{(s^2 + \pi^2)^2}\right]$

$$= \mathcal{L}^{-1}\left[\frac{s^2}{(s^2 + \pi^2)^2}\right] - \pi^2 \mathcal{L}^{-1}\left[\frac{1}{(s^2 + \pi^2)^2}\right]$$

$$= \frac{1}{2\pi}(\sin \pi t + \pi t \cos \pi t) - \pi^2\left[\frac{1}{2\pi^3}(\sin \pi t - \pi t\cos \pi t)\right]$$

[Using the table of Laplace transform]

$$= \frac{1}{2\pi}\sin \pi t + \frac{t}{2}\cos \pi t - \frac{1}{2\pi}\sin \pi t + \frac{t}{2}\cos \pi t$$

$$= \frac{t\cos \pi t + t\cos \pi t}{2} = \frac{2t\cos \pi t}{2} = t\cos \pi t$$

Thus, $f(t) = t\cos \pi t$

(x) $\log\left[\frac{s^2 + 1}{(s-1)^2}\right]$

Solution: Let, $\mathcal{L}\{f(t)\} = F(s) = \log\left[\frac{s^2 + 1}{(s-1)^2}\right]$

$$\text{Then, } F'(s) = \frac{1}{\left[\frac{s^2+1}{(s-1)^2}\right]} \times \frac{(s-1)^2 . 2s - (s^2+1)2(s-1)}{(s-1)^4}$$

$$= \frac{(s-1)^2}{(s^2+1)} \times \frac{(s-1)\{2s(s-1) - 2(s^2+1)\}}{(s-1)^4}$$

$$= \frac{2s^2 - 2s - 2s^2 - 2}{(s^2+1)(s-1)} = -\frac{2(s+1)}{(s^2+1)(s-1)}$$

Since we have, $\mathcal{L}\{tf(t)\} = -\frac{d}{ds}(\mathcal{L}\{f(t)\}) = -F'(s)$.

So, $\mathcal{L}\{t\, f(t)\} = \frac{2(s+1)}{(s^2+1)(s-1)}$

$$\Rightarrow t\, f(t) = \mathcal{L}^{-1}\left[\frac{2(s+1)}{(s^2+1)(s-1)}\right] \quad \dots\dots (1)$$

Let,

$$\frac{2s+2}{(s^2+1)(s-1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1} = \frac{A(s^2+1) + (Bs+C)(s-1)}{(s-1)(s^2+1)}$$

This gives, $(2s+2) + As^2 + A + (Bs^2 - Bs + Cs - C)$

$\Rightarrow 2s + 2 = s^2(A+B) + s(C-B) + (A-C)$

Equating the like terms,

$A + B = 0,\quad C - B = 2,\quad A - C = 2.$

Solving we get,

$A = 2,\ B = -2$ and $C = 0$.

Now, equation (1) becomes

$$t\, f(t) = \mathcal{L}^{-1}\left[\frac{2}{(s-1)} - \frac{2s}{(s^2+1)}\right]$$

$$= (2e^t - 2\cos t) \quad \text{[Using the table of Laplace transform]}$$

$$\Rightarrow f(t) = \frac{1}{t}(2e^t - 2\cos t).$$

(xi) $\frac{s}{(s^2+4)^2}$

Solution: Let, $\mathcal{L}\{f(t)\} = F(s) = \frac{s}{(s^2+4)^2}$

We have,

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\bar{s})\, d(\bar{s}) = \int_s^\infty \frac{\bar{s}}{(\bar{s}^2+4)^2}\, d(\bar{s}) \quad \dots (1)$$

Let, $u = s^2 + 4$ then $du = 2s \Rightarrow \frac{du}{2} = s\,.\,ds$.

Therefore, (1) becomes,

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \frac{du}{u^2} = \frac{1}{2}\left[-\frac{1}{u}\right]_s^\infty = -\frac{1}{2}\left[\frac{1}{s^2+4}\right]_s^\infty = \frac{1}{2}\frac{1}{s^2+4}$$

$$\Rightarrow \quad \frac{f(t)}{t} = \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s^2+4}\right]$$

$$= \frac{1}{2\times 2}\sin 2t \qquad \text{[Using the table of Laplace transform]}$$

Thus, $f(t) = \frac{t}{4}\sin 2t$.

(xii) $\log\left(1 - \frac{a^2}{s^2}\right)$

Solution: Let, $\mathcal{L}\{f(t)\} = F(s) = \log\left(1 - \frac{a^2}{s^2}\right)$

Then, $F'(s) = \frac{1}{\left(\frac{s^2 - a^2}{s^2}\right)} \times \frac{2a^2}{s^3} = \frac{2}{s^2 - a^2} \times \frac{2a^2}{s^3} = \frac{2a^2}{s(s^2 - a^2)} = -\frac{-2a^2}{s(s^2 - a^2)}$

Since we have, $\mathcal{L}\{tf(t)\} = -\frac{d}{ds}(\mathcal{L}\{f(t)\}) = -F'(s)$.

So. $\mathcal{L}\{t\,f(t)\} = \frac{-2a^2}{s(s^2 - a^2)}$

$$\Rightarrow t\,f(t) = \mathcal{L}^{-1}\left[\frac{2a^2}{s(s^2 - a^2)}\right] \qquad .....(1)$$

Let,

$$\frac{2a^2}{s(s^2 - a^2)} = \frac{A}{s} + \frac{B}{s+a} + \frac{C}{(s-a)} = \frac{A(s^2 - a^2) + Bs(s-a) + Cs(s+a)}{s(s^2 - a^2)}$$

$$\Rightarrow \quad 2a^2 = As^2 - Aa^2 + Bs^2 - Bas + Cs^2 + Cas$$

$$= s^2(A + B + C) + s(Ca - Ba) - Aa^2$$

Equating the like terms,

$A + B + C = 0, \quad Ca - Ba = 0, \quad -Aa^2 = 2a^2$

Solving we get,

$A = -2, C = 1$ and $B = 1$.

Now equation (1) becomes,

$$t\,f(t) = -\mathcal{L}^{-1}\left[-\frac{2}{s} + \frac{1}{(s+a)} + \frac{1}{(s-a)}\right]$$

$$= -\mathcal{L}^{-1}\left\{-\frac{2}{s} + \frac{2s}{s^2 - a^2}\right\} = 2\,\mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s}{s^2 - a^2}\right\} = 2\,(1 - \cosh at)$$

Thus, $f(t) = \frac{2(1 - \cosh at)}{t}$

(xiii) $\log\left(1 + \frac{1}{s^2}\right)$

Solution: Let $\mathcal{L}\{f(t)\} = F(s) = \log\left(1 + \frac{1}{s^2}\right)$

So, differentiating w. r. t. s then,

$$F'(s) = \frac{1}{1 + \frac{1}{s^2}}\left(-\frac{2}{s^3}\right) = -\frac{2}{(s^2+1)^3}$$

$$\Rightarrow -F'(s) = \frac{2}{(s^2+1)^3}$$

Since we have, $\mathcal{L}\{t\,f(t)\} = -F'(s)$

So,

$$\mathcal{L}\{t\,f(t)\} = \frac{2}{s(s^2+1)} \qquad ....(1)$$

Since,

$$\frac{2}{s(s^2+1)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 1} = \frac{A(s^2+1) + (Bs + C)s}{s(s^2+1)}$$

$$\Rightarrow 2 = (A + B)\,s^2 + Cs + A$$

Comparing the coefficients of s^2, s and the constant terms then,

$A + B = 0, \quad C = 0, \quad A = 2$

Solving we get,

$A = 2, B = -2, C = 0$.

So, (1) becomes,

$$\mathcal{L}\{t\,f(t)\} = \frac{s}{2} - \frac{2s}{s^2+1}$$

$$= 2\,\mathcal{L}\{1\} - 2\mathcal{L}\{\cos t\} = 2\,\mathcal{L}\{1 - \cos t\}$$

$$\Rightarrow t\,f(t) = 2(1 - \cos t)$$

$$\Rightarrow f(t) = \frac{2(1 - \cos t)}{t}$$

(xiv) $\log\left(\frac{1+s}{s}\right)$

Solution: Let $\mathcal{L}\{f(t)\} = F(s) = \log\left(\frac{1+s}{s}\right) = \log\left(\frac{1}{s} + 1\right)$

So, differentiating w. r. t. s then,

$$F'(s) = \frac{1}{\frac{1}{s} + 1}\cdot\left(-\frac{1}{s^2}\right) = -\frac{1}{s(1+s)}$$

$$\Rightarrow -F'(s) = \frac{1}{s(1+s)}$$

Since we have, $\mathcal{L}\{t\,f(t)\} = -F'(s)$.

So,

$$\mathcal{L}\{t\,f(t)\} = \frac{1}{s(1+s)} \qquad ....(1)$$

Since,

$$\frac{1}{s(1+s)} = \frac{A}{s} + \frac{B}{1+s}$$

$$\Rightarrow 1 = A + (A + B)s$$

Comparing the coefficient of s and the constant terms then

$$A = 1, \quad A + B = 0$$

Solving we get, $A = 1, B = -1$.

Then (1) becomes,

$$\mathcal{L}\{t\, f(t)\} = \frac{1}{s} - \frac{1}{1+s}$$

$$= \mathcal{L}\{1\} - \mathcal{L}\{e^{-t}\} = \mathcal{L}\{1 - e^{-t}\}$$

$$\Rightarrow t\, f(t) = 1 - e^{-t}$$

$$\Rightarrow f(t) = \frac{1 - e^{-t}}{t}.$$

(xv) $\cot^{-1}(1+s)$

Solution: Let $\mathcal{L}\{f(t)\} = F(s) = \cot^{-1}(1+s)$

Differentiating w. r. t. s then,

$$F'(s) = -\frac{1}{1+(s+1)^2} = -\left(\frac{1}{s^2+1}\right)_{s\to s+1}$$

$$= -(\mathcal{L}\{\sin at\})_{s\to s+1} \qquad \left[\because L\{\sin at\} = \frac{a}{s^2+a^2}\right]$$

$$= -\mathcal{L}\{e^{-t}\sin t\} \qquad [\because \mathcal{L}\{e^{at} f(t)\} = (\mathcal{L}\{f(t)\})_{s\to s-a}]$$

Since we have, $\mathcal{L}\{t\, f(t)\} = -F'(s)$

So,

$$\mathcal{L}\{t\, f(t)\} = \mathcal{L}\{e^{-t}\sin t\}$$

$$\Rightarrow t\, f(t) = e^{-t}\sin t$$

$$\Rightarrow f(t) = \frac{e^{-t}\sin t}{t}.$$

(xvi) $\log\left(\frac{s(s+1)}{s^2+4}\right)$

Solution: Let $\mathcal{L}\{f(t)\} = F(s) = \log\left(\frac{s(s+1)}{s^2+4}\right)$

Differentiating w. r. to s then,

$$F'(s) = \frac{s^2+4}{s(s+1)}\left[\frac{(s^2+4)(2s+1) - (s^2+s)(2s)}{(s^2+4)^2}\right]$$

$$= \frac{(s^2+4)(2s+1) - (s^2+s)\,2s}{s(s+1)(s^2+4)}$$

$$= \frac{2s^3+s^2+8s+4-2s^3-2s^2}{s(s+1)(s^2+4)} = -\frac{s^2-8s-4}{s(s+1)(s^2+4)}$$

Here,

$$\frac{s^2-8s-4}{s(s+1)(s^2+4)} = \frac{A}{s} + \frac{B}{s+1} + \frac{Cs+D}{s^2+4}$$

$$= \frac{A(s+1)(s^2+4) + Bs(s^2+4) + (Cs+D)\,s(s+1)}{S(s+1)(s^2+4)}$$

$$\Rightarrow s^2 - 8s - 4 = (A+B+C)\,s^3 + (A+C+D)\,s^2 + (4A+4B+D)\,s + 4A$$

Comparing the coefficient of s^3, s^2, s and the constant term then,

$A + B + C = 0$, $A + C + D = 1$, $4A + 4B + D = -8$ and $4A = -4$.

Solving we get,

$A = -1, B = -1, C = 2, D = 0$

Then,

$$\frac{s^2 - 8s - 4}{s(s+1)(s^2+4)} = -\frac{1}{s} - \frac{1}{s+1} + \frac{2s}{s^2+4}$$

$$= -\mathcal{L}\{1\} - \mathcal{L}\{e^{-t}\} + \mathcal{L}\{\cos 2t\}$$

$$= -\mathcal{L}\{1 + e^{-t} - \cos 2t\}$$

Then, $F'(s) = \mathcal{L}\{1 + e^{-t} - \cos 2t\}$.

Since we have, $\mathcal{L}\{t\,f(t)\} = -F'(s)$

So,

$$\mathcal{L}\{t\,f(t)\} = -\mathcal{L}\{1 + e^{-t} - \cos 2t\}$$

$$\Rightarrow t\,f(t) = -1 - e^{-t} + \cos 2t$$

$$\Rightarrow f(t) = \frac{\cos 2t - 1 - e^{-t}}{t}.$$