Exercise 3.1

1. **Find the value of k, such that the lines $\frac{x-1}{2}=\frac{y-3}{4k}=\frac{z}{2}$ and $\frac{x-2}{2k}=\frac{y-1}{3}=\frac{2-1}{4}$ are perpendicular.**

Solution: Given lines are

$$\frac{x-1}{2}=\frac{y-3}{4k}=\frac{z}{2} \text{ and } \frac{x-2}{2k}=\frac{y-1}{3}=\frac{2-1}{4} \qquad \text{...... (i)}$$

Comparing the lines with $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$ then we get,

$a_1 = 2,\quad b_1 = 4k,\quad c_1 = 2$ and $a_2 = 2k,\quad b_2 = 3,\quad c_2 = 4.$

Let θ be the angle between the given lines then,

$$\cos\theta = \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}} \qquad \text{...... (ii)}$$

Given that $\theta = 90^\circ$ so $\cos 90^\circ = 0$ then (ii) gives,

$$a_1a_2+b_1b_2+c_1c_2 = 0 \qquad \text{...... (i)}$$

$$\Rightarrow \quad 2\times 2k + 4k\times 3 + 2\times 4 = 0$$

$$\Rightarrow \quad 4k + 12k + 8 = 0 \quad \Rightarrow \quad 16k = -8 \Rightarrow k = \frac{-1}{2}.$$

Thus for $k = \frac{-1}{2}$, the given lines (i) are perpendicular to each other.

2. **Find the distance of the point (1, –3, 5) from the plane 3x – 2y + 6z = 15 along a line with direction cosines proportional to (2, 1, -2).**

Solution: The equation of line passing through point (1, –3, 5) and having direction ratio (2, 1, –2) is

$$\frac{x-1}{2}=\frac{y+3}{1}=\frac{z-5}{-2} = r \text{ (Suppose)} \qquad \text{...... (i)}$$

So, the general point of the line (i) is, $(x = 2r + 1, y = r - 3, z = -2r + 5)$.

Since, the line (i) is perpendicular to the plane. So, this point also lies on the plane

$3x - 2y + 6z = 15$ then,

$$3(2r+1) - 2(r-3) + 6(2r+5) = 15$$

$$\Rightarrow \quad 6r + 3 - 2r + 6 - 12r + 30 = 15$$

$$\Rightarrow \quad -8r = -24$$

$$\Rightarrow \quad r = 3$$

Therefore, the point is, $(x = 2\times 3 + 1 = 7, y = 3 - 3 = 0, z = -2\times 3 + 5 = -1)$.

i.e. (7, 0, -1).

Thus, point of intersection is (7, 0, -1).

Now, distance between (1, -3, 5) and (7, 0, -1) is

$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} = \sqrt{(7-1)^2 + (0+3)^2 + (-1-5)^2}$$

$$= \sqrt{6^2 + 3^2 + (-6)^2}$$

$$= \sqrt{36 + 9 + 36} = \sqrt{81} = 9.$$

Thus, the required distance is 9 units.

3. Find the points in which the line $\frac{x+1}{-1} = \frac{y-12}{5} = \frac{z-7}{2}$ out the surface $11x^2 - 5y^2 + z^2 = 0$.

Solution: Given line is,

$$\frac{x+1}{-1} = \frac{y-12}{5} = \frac{z-7}{2} = r \text{ (Suppose)} \qquad \text{...... (i)}$$

So, the general point of the line (i) is, $(-r-1, 5r+12, 2r+7)$.

If this point lies on the surface $11x^2 - 5y^2 + z^2 = 0$. Then,

$$11(-r-1)^2 - 5(5r+12)^2 + (2r+7)^2 = 0$$

$$\Rightarrow \quad 11(r^2 + 2r + 1) - 5(25r^2 + 120r + 144) + 4r^2 + 28r + 49 = 0$$

$$\Rightarrow \quad 11r^2 + 22r + 11 - 125r^2 - 600r - 720 + 4r^2 + 28r + 49 = 0$$

$$\Rightarrow \quad -110r^2 - 550r - 660 = 0$$

$$\Rightarrow \quad r^2 + 5r + 6 = 0$$

$$\Rightarrow \quad r = -2, -3$$

Then the point becomes, (1, 2, 3) at $r = -2$ and (2, -3, 1) at $r = -3$.

Thus, the required points are (1, 2, 3) and (2, -3, 1).

4. Find the point where the line joining (1, -3, 4), (9, 2, –1) cuts the plane $x - y + 2z = 3$.

Solution: Since the equation of line joining the given points (1, -3, 4) and (9, 2, -1) is

$$\boxed{\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}}$$

$$\frac{x-1}{9-1} = \frac{y-3}{3+3} = \frac{z-4}{-1-4}$$

$$\Rightarrow \quad \frac{x-1}{8} = \frac{y-3}{6} = \frac{z-4}{-5} = r \text{ (suppose)} \qquad \text{...... (i)}$$

So, the general point of the line (i) is, $(x = 8r + 1, y = 6r - 3, z = -5r + 4)$.

Since the line (i) cuts the plane $x - y + 2z = 3$, so the point $(8r + 1, 6r - 3, -5r + 4)$ lies on the plane $x - y + 2z = 3$. Therefore,

$$8r + 1 - (6r - 3) + 2(-5r + 4) = 3$$

$$\Rightarrow \quad 8r + 1 - 6r - 3 - 10r + 8 = 3 \quad \Rightarrow \quad -8r = -9 \quad \Rightarrow \quad r = \frac{9}{8}$$

Then,

$$x = 8 \times \frac{9}{8} + 1 = 10, \quad y = 6 \times \frac{9}{8} - 3 = \frac{27}{4} - 3 = \frac{15}{4},$$

$$z = -5 \times \frac{9}{8} + 4 = -\frac{45+32}{8} = \frac{-13}{8}$$

Thus, at the point $(x, y, z) = (10, \frac{15}{4}, \frac{-13}{8})$ the line cuts the plane.

5. Find the distance of the point (-1, -5, -10) from the point of intersection of line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ and the plane $x - y + z = 5$.

Solution: Given line is,

$$\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12} = r \text{ (suppose)} \qquad \text{...... (i)}$$

Given that the line (i) intersects the plane $x - y + z = 5$. So, the general point of the line (i) is, $(3r + 2, 4r - 1, 12r + 2)$ lies on the plane $x - y - z = 5$. Then,

$$3r + 2 - (4r - 1) + (12r + 2) = 5$$

$$\Rightarrow \quad 3r + 2 - 4r + 1 + 12r + 2 = 5 \quad \Rightarrow \quad 11r = 0 \quad \Rightarrow \quad r = 0.$$

Therefore, the point is $(x.0 + 2, 4.0 - 1, 12.0 + 2)$ i.e. (2, -1, 2).

Now, the distance between the point P(2, -1, 2) and M(-1, -5, -10) be

$$PM = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$= \sqrt{(2+1)^2 + (-1+5)^2 + (2+10)^2} = \sqrt{9 + 16 + 144} = \sqrt{169} = 13.$$

Thus, the distance of the point (-1, -5, -10) from the point of intersection of line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ and the plane $x - y + z = 5$ is 13 units.

6. Find the two points on the line $\frac{x-2}{1} = \frac{y+3}{2} = \frac{z+5}{2}$ either side of (2, -3, -5) and at a distance 3 from it. **[2008 Fall Q. No. 1(a)]**

Solution: Given line is,

$$\frac{x-2}{1} = \frac{y+3}{2} = \frac{z+5}{2} = r \text{ (Suppose)} \qquad \text{...... (i)}$$

So, the general point of the line (i) is, $(x = r + 2, y = 2r - 3, z = 2r - 5)$

Given that the distance between the line (i) and the point (2, -3, -5) be 3. So, the distance between the point $(r + 2, 2r - 3, 2r - 5)$ and (2, -3, -5) is 3.

Therefore,

$$3 = \sqrt{(2 - r - 2)^2 + (-3 - 2r + 3)^2 + (-5 - 2r + 5)^2}$$

$$\Rightarrow \quad 3 = \sqrt{r^2 + 4r^2 + 4r^2}$$

$$\Rightarrow \quad 3 = \sqrt{9r^2} \quad \Rightarrow \quad 9r^2 = 9 \quad \Rightarrow \quad r^2 = 1 \quad \Rightarrow \quad r = \pm 1.$$

Then at $r = 1$, the general point is, $(x, y, z) = (3, -1, -3)$.

Then at $r = -1$, the general point is, $(x, y, z) =$

Thus the required points are (3, -1, -3) and (1, -5, -7).

7. Find the distance of the point (1, -2, 3) from the plane $x - y + z = 5$ measured parallel to $\frac{x}{2} = \frac{y}{3} = \frac{z}{-6}$.

Solution: Since the equation of the line through (1, -2, 3) and parallel to line $\frac{x}{2} = \frac{y}{3} = \frac{z}{-6}$ is

$$\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-3}{-6} = r \text{ (Suppose)} \qquad \ldots\ldots (i)$$

So, the general point of the line (i) is, $(x = 2r + 1, y = 3r - 2, z = -6r + 3)$

Since the line (i) is parallel to given line and the given line and given plane $x - y + z = 5$ are parallel, so the line (i) touches the plane. Therefore, the point $(2r + 1, 3r - 2, -6r + 3)$ lies on the plane $x - y + z = 5$. that is,

$$2r + 1 - (3r - 2) + (-6r + 3) = 5$$

$$\Rightarrow 2r + 1 - 3r + 2 - 6r + 3 = 5 \Rightarrow -7r = -1 \Rightarrow r = \frac{1}{7}$$

Then, $x = 2r + 1 = 2\times\frac{1}{7} + 1 = \frac{9}{7}$, $\quad y = 3r - 2 = 3\times\frac{1}{7} - 2 = \frac{-11}{7}$

and $\quad z = -6r + 3 = -6\times\frac{1}{7} + 3 = \frac{15}{7}$

Thus, the common point of the plane and the line through the point (1, -2, 3) is $(x, y, z) = (\frac{9}{7}, \frac{-11}{7}, \frac{15}{7})$

Now, the distance between point (1, -2, 3) and $(\frac{9}{7}, \frac{-11}{7}, \frac{15}{7})$ is

$$d = \sqrt{\left(\frac{9}{7} - 1\right)^2 + \left(\frac{-11}{7} + 2\right)^2 + \left(\frac{15}{7} - 3\right)^2}$$

$$= \sqrt{\frac{4}{49} + \frac{9}{49} + \frac{36}{49}} = 1.$$

8. Find the equation to the line passing through (-1, -2, -3) and the perpendicular to each of the lines $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ and $\frac{x+2}{4} = \frac{y+3}{5} = \frac{z+y}{6}$.

Solution: We have the equation of the line passing through (-1, 2, -3) and having direction ratio (a, b, c) be,

$$\frac{x+1}{a} = \frac{y+2}{b} = \frac{z+3}{c} \qquad \ldots\ldots (i)$$

Given that the line (i) perpendicular to line $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ and $\frac{x+2}{4} = \frac{y+3}{5} = \frac{z+y}{6}$. So,

$$3a + 4b + 5c = 0$$
$$4a + 5b + 6c = 0$$

Solving by multiplication we get,

$$\frac{a}{4\times6 - 5\times5} = \frac{b}{5\times4 - 6\times3} = \frac{c}{3\times5 - 4\times4}$$

$$\Rightarrow \frac{a}{-1} = \frac{b}{2} = \frac{c}{-1} = \lambda$$

Thus, $a = -\lambda$, $b = 2\lambda$, $c = -\lambda$.

Then, equation (i) becomes

$$\frac{x+1}{-\lambda} = \frac{x+2}{2\lambda} = \frac{z+3}{-\lambda}$$

$$\Rightarrow \frac{x+1}{1} = \frac{y+2}{-2} = \frac{z+3}{1}$$

This is the equation of required line.

9. Show that the equation of the perpendicular from the point (1, 6, 3) to the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ is $\frac{x-1}{0} = \frac{y-6}{-3} = \frac{z-3}{2}$ and the foot of the perpendicular is (1, 3, 5) and the length of perpendicular is $\sqrt{13}$.

Solution: Given line is,

$$\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3} = r \text{ (say)} \qquad \ldots\ldots (i)$$

So, the general point of the line (i) is, $(r, 2r + 1, 3r + 2)$

Then, the direction ratio between line joining point P(1, 6, 3) and M(r, 2r + 1, 3r + 2) is,

$r - 1, 2r + 1 - 6, 3r + 2 - 3 \quad$ i.e. $r - 1, 2r - 5, 3r - 1$.

Since the line PM is perpendicular to line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ so,

$$\boxed{a_1a_2 + b_1b_2 + c_1c_2 = 0}$$
$$(r - 1)\times1 + (2r - 5)\times 2 + (3r - 1)\times 3 = 0$$
$$\Rightarrow r - 1 + 4r - 10 + 9r - 3 = 0$$
$$\Rightarrow 14r = 14$$
$$\Rightarrow r = 1$$

Then, $x = 1$, $y = 3$, $z = 5$.

And, the direction ratio of the perpendicular line is

$(r-1, 2r-5, 3r-1) = (1-1, 2-5, 3-1) = (0, -3, 2)$.

Then the equation of the line through (1, 6, 3) and having direction ratios 0, -3, 2 be,

$$\frac{x-1}{0} = \frac{y-6}{-3} = \frac{z-3}{2}.$$

Now, distance of perpendicular line,

$$PM = \sqrt{(1-1)^2 + (6-3)^2 + (3-5)^2} = \sqrt{13}$$

Thus, distance of the perpendicular line is $\sqrt{13}$.

10. Find the equation to the line through (-1, 3, 2) and perpendicular to the plane $x + 2y + 2z = 3$, the length of perpendicular and the co-ordination of its foot.

[2007 Fall; 2009 Spring Q. No. 1(a)]

Solution: Since the equation of the line PM (perpendicular) passing through (-1, 3, 2) and perpendicular to the plane is $x + 2y + 2z = 3$.

$$\frac{x+1}{1}=\frac{y-3}{2}=\frac{z-2}{2}=r \text{ (suppose)} \qquad \text{...... (i)}$$

So, the general point of the line (i) is, $(r-1, 2r+3, 2r+2)$ which also lies on plane
$x+2y+2z=3$. So,

$$r-1+2(2r+3)+2(2r+2)=3$$
$$\Rightarrow r-1+4r+6+4r+4=3$$
$$\Rightarrow 9r=-6$$
$$\Rightarrow r=\frac{-2}{3}$$

Then, $r-1=\frac{-2}{3}-1=\frac{-5}{3}$, $\quad 2r+3=2\times\frac{-2}{3}+3=\frac{5}{3}$, $\quad 2r+2=2\times\frac{-2}{3}+2=\frac{2}{3}$

Thus, the coordinate of the perpendicular foot is $\left(\frac{-5}{2}, \frac{5}{3}, \frac{2}{3}\right)$

Now, the length of distance between point $\left(\frac{-5}{2}, \frac{5}{3}, \frac{2}{3}\right)$ and $(-1, 3, 2)$ is

$$\sqrt{\left(-1+\frac{5}{3}\right)^2+\left(3-\frac{5}{3}\right)^2+\left(2-\frac{2}{3}\right)^2}$$
$$=\sqrt{\frac{4}{9}+\frac{16}{9}+\frac{16}{9}}=\sqrt{\frac{36}{9}}=2 \text{ unit.}$$

Thus, the equation of the line is $\frac{x+1}{1}=\frac{y-3}{2}=\frac{z-2}{2}$, the length of perpendicular is 2 units and the coordinate of the foot is $\left(\frac{-5}{2}, \frac{5}{3}, \frac{2}{3}\right)$.

11. Find the image of the point P (1, 3, 4) in the plane $2x-y+z+3=0$.

Solution: The equation of line passing through (1, 3, 4) and perpendicular to the plane $2x-y+z+3=0$ is,

$$\frac{x-1}{2}=\frac{y-3}{-1}=\frac{z-4}{1}=r \text{ (suppose)} \qquad \text{...... (i)}$$

Then the general point of (i) is, $(x, y, z)=(2r+1, 3-r, r+4)$.

Since, if the point $Q(2r+1, 3-r, r+4)$ is the image of P then the middle point R lies on the plane $2x-y+z+3=0$.

Here, the coordinate of R is,

$$R\left(\frac{2r+1+1}{2}, \frac{3-r+3}{2}, \frac{r+4+4}{2}\right)=R\left((r+1), 3-\frac{r}{2}, 4+\frac{r}{2}\right)$$

Since the point lies on the plane $2x-y+z+3=0$, so,

$$2(r+1)-3+\frac{r}{2}+4+\frac{r}{2}+3=0.$$
$$\Rightarrow 2r+2+4+r=0.$$
$$\Rightarrow r=-2.$$

Then the coordinate of Q is,

$$(2r+1, 3-r, r+4)=(-4+1, 3+2, -2+4)$$
$$=(-3, 5, 2)$$

Thus, the image of the point P (1, 3, 4) in the plane $2x-y+z+3=0$ be $(-3, 5, 2)$.

Exercise 3.2

1. Change the equation $x+y+z+1=0$, $4x+y-2z+2=0$ in symmetrical form.

Solution: Given equation of line is,

$$x+y+z+1=0=4x+y-2z+2. \qquad \text{...... (i)}$$

Then we have to change (i) in symmetrical form.
Put $z=0$ then,

$$x+y+1=0 \quad \text{and} \quad 4x+y+2=0 \qquad \text{...... (ii)}$$

Solving these equations we get,

$$x=\frac{-1}{3}, \quad \text{and} \quad y=\frac{1}{3}-1=\frac{-2}{3}$$

Thus, the point on the line is $\left(\frac{-1}{3}, \frac{-2}{3}, 0\right)$.

Let the direction ratio is a, b, c. Then, (ii) gives,

$$a+b+c=0$$
$$4a+b-2c=0$$

Solving by cross multiplication,

$$\frac{a}{-2-1}=\frac{b}{4+2}=\frac{c}{1-4}$$
$$\Rightarrow \frac{a}{-3}=\frac{b}{6}=\frac{c}{-3}=\lambda \text{ (assume)}$$

Then, $a=-2\lambda$, $\quad b=6\lambda$, $\quad c=-3\lambda$.

So, the equation of line passing through $\left(\frac{-1}{3}, \frac{-2}{3}, 0\right)$ and having the direction ratio
$(a=-3\lambda, b=6\lambda, c=-3\lambda)$ be,

$$\frac{x+\frac{1}{3}}{-3\lambda}=\frac{y+\frac{2}{3}}{6\lambda}=\frac{z-0}{-3\lambda} \Rightarrow \frac{3x+1}{-9}=\frac{3y+2}{18}=\frac{z}{-3}$$
$$\Rightarrow \frac{3x+1}{3}=\frac{3y+2}{-6}=\frac{z}{1}$$

This is the equation of line (i) in symmetrical form.

2. Find the equation of the line plane through (-1, 1, -1) and perpendicular to the line $x-2y+z=4$, $4x+3y-z+4=0$.

Solution: Let the equation of plane is,

$$ax+by+cz+d=0 \qquad \text{...... (i)}$$

Since equation (i) passing through $(-1, 1, -1)$

$$-a+b-c+d=0 \qquad \text{...... (ii)}$$

Then subtract (ii) to (i) then

$$ax + by + cz + d - (-a + b - c + d) = 0$$

$$\Rightarrow \quad a(x+1) + b(y-1) + c(z+1) = 0 \quad \text{...... (iii)}$$

Since equation (iii) is perpendicular to the line $x - 2y + z = 4$, $4x + 3y - z + 4 = 0$.

Then,

$$a - 2b + c = 0$$
$$4a + 3b - c = 0$$

So, by cross multiplication

$$\frac{a}{2-3} = \frac{b}{4+1} = \frac{c}{3+8}$$

$$\Rightarrow \quad \frac{a}{-1} = \frac{b}{5} = \frac{c}{11} = \lambda \text{ (say)}$$

This gives, $a = -\lambda$, $b = 5\lambda$, $c = 11\lambda$. Putting the value of a, b, c in equation (iii) then,

$$-\lambda(x+1) + 5\lambda(y-1) + 11\lambda(z+1) = 0$$

$$\Rightarrow \quad -x - 1 + 5y - 5 + 11z + 11 = 0$$

$$\Rightarrow \quad x - 5y - 11z = 5.$$

This is the equation of required plane.

3. Find the equation of the line passing through (2, 3, 4) parallel to the line $x - 2y + z = 4$, $4x + 3y - z + 4 = 0$.

Solution: Let the equation of the line passing through (2, 3, 4) and the direction ratio (a, b, c) be

$$\frac{x-2}{a} = \frac{y-3}{b} = \frac{z-4}{c} \quad \text{......... (i)}$$

Given that the line through (2, 3, 4) is parallel to the given line $x - 2y + z = 4$, $4x + 3y - z + 4 = 0$. Then the direction ratios of (i) satisfies the condition of perpendicularity with the coefficient of given line. So,

$$a - 2b + c = 0$$
$$4a + 3b - c = 0$$

Solving by cross multiplication

$$\frac{a}{2-3} = \frac{b}{4+1} = \frac{c}{3+8} = \lambda \text{ (suppose)}$$

$$\Rightarrow \quad a = -\lambda, \quad b = 5\lambda \quad \text{and } c = 11\lambda.$$

Putting the value of a, b, c in equation (i)

$$\frac{x-2}{-\lambda} = \frac{y-3}{5\lambda} = \frac{z-4}{11\lambda}$$

$$\Rightarrow \quad \frac{x-2}{-1} = \frac{y-3}{5} = \frac{z-4}{11}$$

This is the equation of required line.

4. Find the angle between the lines in which the plane $x - y + z = 5$ is cut by the planes $2x + y - z = 3$ and $2x + y + 3z - 1 = 0$.

Solution: Given planes are,

$$2x + y - z = 3 \quad \text{........ (i)}$$
$$2x + 2y + 3z - 1 = 0 \quad \text{......... (ii)}$$

Given that the plane $x - y + z = 5$ cuts the planes (i) and (ii). Let direction ratio of line (i) is a_1, b_1 c_1 and direction ratio of line (ii) is a_2, b_2, c_2. So,

$$a_1 - b_1 + c_1 = 0 \qquad a_2 - b_2 + c_2 = 0$$
$$2a_1 + b_1 - c_1 = 0 \qquad 2a_2 + 2b_2 + 2c_2 = 0$$

Solving by cross multiplication we get,

$$\frac{a_1}{1-1} = \frac{b_1}{2+1} = \frac{c_1}{1+2} \qquad \frac{a_2}{-3-2} = \frac{b_2}{2-3} = \frac{c_2}{2+2}$$

$$\Rightarrow \quad \frac{a_1}{0} = \frac{b_1}{3} = \frac{c_1}{3} \qquad \Rightarrow \quad \frac{a_2}{-5} = \frac{b_2}{-1} = \frac{c_2}{4}$$

Thus, $(a_1, b_1, c_1) = (0, 3, 3)$ Thus, $(a_2, b_2, c_2) = (-5, -1, 4)$

Let θ be the angle between the lines then,

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$= \frac{0\times-5 + 3\times-1 + 3\times4}{\sqrt{0^2 + 3^2 + 3^2}\sqrt{(-5)^2 + (-1)^2 + 4^2}}$$

$$= \frac{-3 + 12}{\sqrt{9+9}\sqrt{25 + 1 + 16}} = \frac{9}{\sqrt{18\times42}} = \frac{3}{\sqrt{84}} = \frac{3}{2\sqrt{21}}$$

$$\theta = \cos^{-1}\left(\frac{3}{2\sqrt{21}}\right).$$

Thus, the angle between the given lines is $\cos^{-1}\left(\frac{3}{2\sqrt{21}}\right)$.

5. Proved that the lines $x = ay + b$, $z = cy + d$ and $x = a^1y + b^1$, $z = a^1y + d^1$ are perpendicular it $aa^1 + cc^1 + 1 = 0$.

Solution: Given line is,

$$x = ay + b, z = cy + d$$

$$\Rightarrow \quad y = \frac{x-b}{a}, y = \frac{z-d}{c}$$

This implies,

$$\frac{x-b}{a} = \frac{y-0}{1} = \frac{z-d}{c}$$

And, another given line is,

$$x = a^1y + b^1, z = c^1y + d^1$$

$$\Rightarrow \quad y = \frac{x-b^1}{a^1}, y = \frac{z-d^1}{c}$$

This implies,

$$\frac{x-b^1}{a^1} = \frac{y-0}{1} = \frac{z-d^1}{c^1}$$

The line (i) and (ii) are perpendicular only if,

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

$$\Rightarrow \quad aa^1 + 1.1 + cc^1 = 0$$

$$\Rightarrow \quad aa^1 + cc^1 + 1 = 0.$$

6. **Proved that the lines $x = -2y + 7$, $z = 3y + 10$ and $x = 5y - 1$, $z = 3y - 6$ are perpendicular to each other.**

Solution: Given line is,

$$x = -2y + 7, \ z = 3y + 10$$

$$\Rightarrow \quad -2y = x - 7, \ 3y = z - 10$$

$$\Rightarrow \quad y = \frac{x-7}{-2}, \ y = \frac{z-10}{3}$$

This implies, $\frac{x-7}{-2} = \frac{y-0}{1} = \frac{z-10}{3}$ ...... (i)

And, another given line is,

$$x = 5y - 1, z = 3y - 6 \Rightarrow 5y = x + 1, 3y = z + 6$$

$$\Rightarrow \quad y = \frac{x+1}{5}, \ y = \frac{z+6}{3}$$

This implies, $\frac{x+1}{5} = \frac{y-0}{1} = \frac{z+6}{3}$ ...... (ii)

The line (i) and (ii) are perpendicular only if,

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

$$\Rightarrow \quad -2\times5 + 1\times1 + 3\times3 = 0$$

$$\Rightarrow \quad -10 + 1 + 9 = 0$$

$$\Rightarrow \quad 0 = 0.$$

This proves that the lines are perpendicular to each other.

7. **Find the co-ordinate of the foot of the perpendicular from the origin on the straight line given by $x + 2y + 3z + 4 = 0$, $x + y + z + 1 = 0$.**

Solution: Given line is the intersection of two planes,

$$x + 2y + 3z + 4 = 0 \quad \text{and} \quad x + y + z + 1 = 0$$

Put $z = 0$, then,

$$x + 2y + 4 = 0 \quad \text{and} \quad x + y + 1 = 0$$

Solving these two planes we get,

$$y = -3, \ x = 2$$

Therefore, the given line passes through the point (2, -3, 0).

Also, let direction ratio is a, b, c then,

$$a + 2b + 3c + 4 = 0 \quad \text{and} \quad a + b + c + 1 = 0$$

Solving by cross multiplication

$$\frac{a}{2-3} = \frac{b}{3-1} = \frac{c}{2-1} = \lambda \ \text{(Suppose)},$$

$$\Rightarrow \quad a = -\lambda, \ b = 2\lambda, \ c = \lambda$$

Now, the equation of line passing through (2, –3, 0) and having direction ratio ($-\lambda$, 2λ, $-\lambda$) is,

$$\frac{x-2}{-\lambda} = \frac{y+3}{2\lambda} = \frac{z}{-\lambda}$$

$$\Rightarrow \quad \frac{x-2}{1} = \frac{y+3}{-2} = \frac{z}{1} = r \ \text{(say)} \quad \text{...... (i)}$$

Therefore, the general point of the line is

$$x = r + 2, \ y = -2r - 3, \ z = r$$

Then the direction ratio between point (0, 0, 0) and $(r + 2, -2r - 3, r)$ is

$$r + 2 + 0, -2r - 3 - 0, r - 0 = (r + 2, -2r - 3, r)$$

Since the direction ratios of the line joining origin and the point of (i), is perpendicular to line (i). So,

$$(r + 2)\times1 + (-2r - 3)\times(-2) + r\times1 = 0$$

$$\Rightarrow \quad r + 2 + 4r + 6 + r = 0$$

$$\Rightarrow \quad r = -\frac{6}{8} = -\frac{4}{3}$$

Then,

$$x = r + 2 = -\frac{4}{3} + 2 = \frac{2}{3}, \quad y = -2r - 3 = -2\times(-\frac{4}{3}) - 3 = \frac{8}{3} - 3 = -\frac{1}{3}$$

and $z = -\frac{4}{3}$

Thus, $(x, y, z) = \left(\frac{2}{3}, \frac{-1}{3}, \frac{-4}{3}\right)$.

Hence the coordinate of the foot of the perpendicular from the origin on the given line is, $\left(\frac{2}{3}, \frac{-1}{3}, \frac{-4}{3}\right)$.

Exercise 3.3

1. **Find the value of k such that the line $\frac{x-2}{2} = \frac{y+3}{5} = \frac{z-5}{k}$ is parallel to the plane $2x - 3y + z = 3$.**

Solution: Given that the line $\frac{x-2}{2} = \frac{y+3}{5} = \frac{z-5}{k}$ is parallel to the plane $2x-3y + z = 3$.

Then, $\boxed{al + bm + cn = 0}$

i.e. $2\times2 + 5\times(-3) + k\times1 = 0$

$\Rightarrow 4 - 15 + k = 0 \quad \Rightarrow k = 11.$

Thus, for k= 11 the given line and the plane are in the form of parallel.

2. **Find the equation of the plane parallel to the line $x - 2 = \frac{y-1}{3} = \frac{z-3}{2}$ containing (0, 0, 0) and (-3, 1, 2).**

Solution: Let us consider the equation plane is $ax + by + cz + d = 0$ ... (i)

Let, the plane passes through (0, 0, 0), then d = 0. Then, (i) becomes,

$$ax + by + cz = 0 \quad \text{.... (ii)}$$

Also, the plane (i) passes through (-3, 1, 2) then (ii) gives,

$$-3a + b + 2c = 0 \quad \text{.... (iii)}$$

And, given that the plane is parallel to the line $\frac{x-2}{1} = \frac{y-1}{3} = \frac{z-3}{2}$. Then,

$$\boxed{al + bm + cn = 0}$$

$$a + 3b + 2c = 0 \quad \text{..... (iv)}$$

Solving the equation (iii) and (iv), by cross multiplication

$$\frac{a}{2-6} = \frac{b}{2+6} = \frac{c}{-9-1} = \lambda$$

$$\Rightarrow a = -4\lambda, b = 8\lambda, c = 10\lambda$$

Now, equation (ii) becomes

$$-4\lambda x + 8\lambda y - 10\lambda = 0$$

$$\Rightarrow 2x - 4y + 5z = 0.$$

This is equation of required plane.

3. **Find the equation of a plane containing the line $\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-3}{4}$ and is perpendicular to the plane $x + 2y + z = 12$.**

Solution: Given line is,

$$\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-3}{4} \quad \text{...... (i)}$$

Let the equation of plane containing the line is (i) is,

$$a(x-1) + b(y+1) + c(z-3) = 0 \quad \text{...... (ii)}$$

Since the direction ratios of the line is normal to the line. So, it is also normal to (ii). Then, $2a + (-1)b + 4c = 0$

$$\Rightarrow 2a - b + 4c = 0 \quad \text{...... (iii)}$$

And, given that the plane (ii) is perpendicular to the plane $x + 2y + z = 12$ then,

$$a \times 1 + b \times 2 + 1 \times c = 0$$

$$\Rightarrow a + 2b + c = 0 \quad \text{...... (iv)}$$

Solving equation (iii) and (iv), by cross multiplication,

$$\frac{a}{-1-8} = \frac{b}{4-2} = \frac{c}{4+1} = \lambda \text{ (Suppose)}$$

$$a = -9\lambda, b = 2\lambda, c = 5\lambda$$

Now equation (ii) becomes,

$$-9\lambda(x-1) + 2\lambda(y+1) + 5\lambda(z-3) = 0$$

$$\Rightarrow -9(x-1) + 2(y+1) + 5(z-3) = 0$$

$$\Rightarrow -9x + 9 + 2y + 2 + 5z - 15 = 0$$

$$\Rightarrow 9x - 2y - 5z + 4 = 0.$$

Thus, the equation of required plane is $9x - 2y - 5z + 4 = 0$.

If plane $a_1x + b_1y + c_1z + d_1 = 0$ & $a_2x + b_2y + c_2z + d_2 = 0$ are perpendicular. Then, $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

4. **Find the equation of the plane through (2, -3, 1) normal to the line joining (3, 4, -1) and (2, -1, 5).**

Solution: Let equation of plane is

$$ax + by + cz + d = 0 \quad \text{...... (i)}$$

Since equation (i) passing through the point (2, -3, 1), then

$$2a - 3b + c + d = 0 \quad \text{...... (ii)}$$

Equation of line joining (3, 4, -1) and (2, -1, 5) is

$$\frac{x-3}{3-2} = \frac{y-4}{4-(-1)} = \frac{z-(-1)}{-1-5}$$

$$\Rightarrow \frac{x-3}{1} = \frac{y-4}{5} = \frac{z+1}{-6}$$

If the line is perpendicular to the plane then it is parallel to the normal of the plane.

Then,

$$\boxed{\frac{a}{l} = \frac{b}{m} = \frac{c}{n} = k}$$

$$\frac{a}{1} = \frac{b}{5} = \frac{c}{-6} = k$$

$$\Rightarrow a = k, b = 5k, c = -6k$$

Now, subtracting (ii) from (ii) then

$$ax + by + cz + d - (2a - 3b + c + d) = 0$$

$$\Rightarrow ax + by + cz + d - 2a + 3b - c - d = 0$$

$$\Rightarrow a(x-2) + b(y+3) + c(z-1) = 0 \quad \text{...... (iii)}$$

Putting the value of a, b and c in equation (iii) then

$$k(x-2) + 5k(y+3) - 6k(z-1) = 0$$

$$\Rightarrow x - 2 + 5y + 15 - 6z + 6 = 0$$

$$\Rightarrow x + 5y - 6z + 19 = 0.$$

This is the equation of required plane.

5. **Find the equation of the line through (2, -1, -1) is parallel to the plane $4x + y + z + 2 = 0$ and is perpendicular to the line $2x + y = 0 = x - z + 5$.**

Solution: The equation of line through (2, -1, -1) is

$$\frac{x-2}{a}=\frac{y+1}{b}=\frac{z+1}{c} \quad \text{...... (i)}$$

where, a, b and c are the direction ratio of the line.

Since the equation (i) is parallel to the plane $4x + y + z + 2 = 0$. Then,

$$4a + b + c = 0 \quad \text{...... (ii)}$$

Again, given a line,

$$2x + y = 0 = x - z + 5$$

This gives, $x = -\frac{y}{2}$ and $x = z - 5$

Therefore, $\frac{x}{1}=\frac{-y}{2}=\frac{z-5}{1}$ ...... (iii)

Given that the line (i) is perpendicular to (iii) and (i) is parallel to (ii). So, the line (iii) is perpendicular the plane (ii).

$$\boxed{a_1a_2+b_1b_2+c_1c_2=0}$$

$$a\times 1 + (-2)\times b + c\times 1 = 0$$

$$\Rightarrow a - 2b + c = 0 \quad \text{....... (iv)}$$

From equation (ii) and (iv)

$$\frac{a}{1+2}=\frac{b}{1-4}=\frac{c}{-8-1}=\lambda$$

$$\Rightarrow a = 3\lambda,\ b = -3\lambda, \text{ and } c = -9\lambda$$

Putting the value of a, b and c in equation (i) then

$$\frac{x-2}{3\lambda}=\frac{y+1}{-3\lambda}=\frac{z+1}{-9\lambda}$$

$$\Rightarrow \frac{x-2}{1}=\frac{y+1}{-1}=\frac{z+1}{-3}$$

This is the equation of required line.

6. Find the equation of the straight line lying in the plane $x - 2y + 4z - 51 = 0$ and intersecting the straight line $\frac{x-2}{3}=\frac{y+1}{4}=\frac{z-6}{7}$ at right angle.

7. Find the equation of line through (α, β, δ) parallel, to the plane $lx + my + nz = P$, $l_1x + m_1y + n_1z = P$.

Solution: The equation of line passing through (α, β, δ) and having the direction ratio (a, b, c) is

$$\frac{x-\alpha}{a}=\frac{y-\beta}{b}=\frac{z-\delta}{c} \quad \text{......... (i)}$$

Since equation (i) is parallel to the plane $lx + my + nz = P$

Then, $al + bm + cn = 0$ .... (ii)

Similarly, equation (i) is parallel to $l_1x + m_1y + n_1z = P$

Then, $al_1 + bm_1 + cn_1 = 0$ ....(iii)

Solving equation (ii) and (iii) we get,

$$\frac{a}{mn_1 - m_1n}=\frac{b}{l_1n - ln_1}=\frac{c}{lm_1 - l_1m}=\lambda \text{ (Say)}$$

$$\Rightarrow a = \lambda(mn_1 - m_1n), \quad b = \lambda(l_1n - ln_1), \quad c = \lambda(lm_1 - l_1m)$$

Now, equation (i) becomes

$$\frac{x-\alpha}{\lambda(mn_1 - m_1n)}=\frac{y-\beta}{\lambda(l_1n - ln_1)}=\frac{z-\delta}{\lambda(lm_1 - l_1m)}$$

$$\Rightarrow \frac{x-\alpha}{(mn_1 - m_1n)}=\frac{y-\beta}{(l_1n - ln_1)}=\frac{z-\delta}{(lm_1 - l_1m)}$$

This is the equation of required line.

8. Find the equations to the planes through the line $2x + 3y - 5z - 4 = 0 = 3x - 4y + 3z - 6$, parallel to the coordinate axes.

Solution: The equation of plane passes through the lines, $2x + 3y - 5z - 4 = 0 = 3x - 4y + 3z - 6$ is

$$2x + 3y - 5z - 4 + \lambda(3x - 4y + 3z - 6) = 0 \quad \text{....... (i)}$$

$$\Rightarrow 2x + 3y - 5z - 4 + 3x\lambda - 4y\lambda + 3z\lambda - 6\lambda = 0$$

$$\Rightarrow x(2 + 3\lambda) + y(3 - 4\lambda) + z(-5 + 5\lambda) - 4 - 6\lambda = 0$$

If the equation (i) is parallel to the x-axis (1, 0, 0) then,

$$al + bm + cn = 0$$

$$\Rightarrow (2 + 3\lambda)\times 1 + (3 - 4\lambda\times 1 + (-5 + 5\lambda)\times 0 = 0$$

$$\Rightarrow 2 + 3\lambda = 0$$

$$\Rightarrow \lambda = \frac{-2}{3}$$

Then, the equation (i) becomes

$$2x + 3y - 5z - 4 + \frac{-2}{3}(3x - 4y + 5z - 6) = 0$$

$$\Rightarrow 6x + 9y - 15z - 12 - 6x + 8y - 10z + 12 = 0$$

$$\Rightarrow 17y - 25z = 0$$

Again the equation (i) is parallel to the y-axis (0, 1, 0) then,

$$(2 + 3\lambda)\times 0 + (3 - 4\lambda)\times 1 + (-5 + 5\lambda)\times 0 = 0$$

$$\Rightarrow 3 - 4\lambda = 0$$

$$\Rightarrow \lambda = \frac{3}{4}$$

Then, equation (i) becomes

$$2x + 3y - 5z - 4 + \frac{3}{4}(3x - 4y + 5z - 6) = 0$$

$$\Rightarrow 8x + 12y - 20z - 16 + 9x - 12y + 15z - 18 = 0$$

$$\Rightarrow 17x - 5z - 34 = 0$$

Similarly, the equation (i) is parallel to the z-axis (0, 0, 1) then

$$(2 + 3\lambda)\times 0 + (3 - 4\lambda)\times 0 + (-5 + 5\lambda)\times 1 = 0$$

$$\Rightarrow -5 + 5\lambda = 0$$

$$\Rightarrow \lambda = 1$$

Then, equation (i) becomes

$$2x + 3y - 5z - 4 + 3x - 4y + 5z - 6 = 0$$

$$\Rightarrow \quad 5x - y - 10 = 0.$$

Thus the equations of required planes are $17y - 25z = 0$, $17x - 5z - 34 = 0$, $5x - y - 10 = 0$.

9. Find the equation of plane through the line $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ parallel to y-axis. **[2004 Spring Q. No. 1(a)]**

Solution: Given that the equation of a line is,

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} \quad \text{...... (i)}$$

Clearly, the line passes through the point (1, 2, 3).

If a plane passes through the line (i) then the plane also passes through the point (1, 2, 3). Now, the equation of a plane through the point and having direction ratios a, b, c be

$$a(x-1) + b(y-2) + c(z-3) = 0 \quad \text{..... (ii)}$$

Since the direction ratios of the line is normal to itself. So, by the condition of perpendicularity,

$$2a + 3b + 4c = 0 \quad \text{...... (iii)}$$

Given that the plane (ii) is parallel to y-axis. So, the condition of parallelism gives,

$$a\times0 + b\times1 + c\times0 = 0$$

$$\Rightarrow \quad b = 0 \quad \text{...... (iv)}$$

Solving equation (iii) and (iv) we get,

$$\frac{a}{0-4} = \frac{b}{0-0} = \frac{c}{2-0}$$

This gives, (a, b, c) = (-4, 0, 2).

Now, equation (ii) becomes

$$-4(x-1) + 0(y-2) + 2(z-3) = 0$$

$$\Rightarrow \quad -4x + 4 + 2z - 6 = 0$$

$$\Rightarrow \quad -4x + 2z - 2 = 0$$

$$\Rightarrow \quad 2x - z + 1 = 0.$$

Thus, the equation of required plane is $2x - z + 1 = 0$.

10. Find the equation of the plane through the points (2, 2, 1), (1, -2, 3) and parallel to the line joining the points (2, 1, -3), (-1, 5, -8).

Solution: Let the equation of the plane through the points (2, 2, 1) is,

$$a(x-2) + b(y-2) + c(z-1) = 0 \quad \text{...... (i)}$$

Also, the equation (i) passes through the points (1, -2, 3) then,

$$a(1-2) + b(-2-2) + c(3-1) = 0$$

$$\Rightarrow \quad -a - 4b + 2c = 0$$

$$\Rightarrow \quad a + 4b - 2c = 0 \quad \text{...... (ii)}$$

The equation of line joining points (2, 1, -3) and (-1, 5, -8) is

$$\frac{x-2}{2-(-1)} = \frac{y-1}{1-5} = \frac{z-(-3)}{-3-(-8)}$$

$$\Rightarrow \quad \frac{x-2}{3} = \frac{y-1}{-4} = \frac{z+3}{5} \quad \text{...... (iii)}$$

Given that the plane (i) to is parallel to the line (iii). So, by the condition of parallelism,

$$3a - 4b + 5c = 0 \quad \text{...... (iv)}$$

Solving equation (ii) and (iv) we get

$$\frac{a}{20-8} = \frac{b}{-6-5} = \frac{c}{-4-12} = \lambda$$

$$\Rightarrow \quad a = 12\lambda,\ b = -11\lambda,\ \&\ c = -16\lambda$$

Now equation (iii) becomes

$$12\lambda(x-2) - 11\lambda(y-2) - 16\lambda(z-1) = 0$$

$$\Rightarrow \quad 12x - 24 - 11y + 22 - 16z + 16 = 0$$

$$\Rightarrow \quad 12x - 11y - 16z + 14 = 0.$$

Thus the equation of required plane is $12x - 11y - 16z + 14 = 0$.

Exercise 3.4

1. Prove that the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ are coplanar. Also, obtain the equation of plane containing them. **[2003 Fall Q. No. 19a)]**

Solution: Given line is,

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} \quad \text{...... (i)}$$

Then the line (i) passes through the point $(x_1, y_1, z_1) = (1, 2, 3)$ and having the direction ratio $(l_1, m_1, n_1) = (2, 3, 4)$.

Also, given line is,

$$\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5} \quad \text{...... (ii)}$$

Then the line (i) passes through the point $(x_2, y_2, z_2) = (2, 3, 4)$ and having the direction ratio $(l_2, m_2, n_2) = (3, 4, 5)$.

Now, the lines (i) and (ii) are coplanar if

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

$$\text{Now,} \quad \begin{vmatrix} 2-1 & 3-2 & 4-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$$

$$= 1(15-16) - 1(10-12) + 1(8-9)$$

$$= -1 + 2 - 1 = 0$$

This shows that the lines are coplanar.

And, the equation of plane containing the lines (i) and (ii) is,

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

$$\Rightarrow \begin{vmatrix} x - 1 & y - 2 & z - 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 0$$

$\Rightarrow \quad (x-1)(15 - 16) - (y - 2)(10 - 12) + (z - 3)(8 - 9) = 0$

$\Rightarrow \quad - x + 1 + 2y - 4 - z + 3 = 0$

$\Rightarrow \quad x - 2y + z = 0.$

Thus, the equation of the plane containing the lines be $x - 2y + z = 0$.

2. Show that the lines $\frac{x-5}{4} = \frac{y-7}{4} = \frac{z+3}{-5}$ and $\frac{x-8}{7} = y - 4 = \frac{z-5}{3}$ are coplanar. Find their common point and equation of plane in which they lie.

Solution: Given line is,

$$\frac{x-5}{4} = \frac{y-7}{4} = \frac{z+3}{-5} \qquad \text{...... (i)}$$

Then the line (i) passes through the point $(x_1, y_1, z_1) = (5, 7, -3)$ and having the direction ratio $(l_1, m_1, n_1) = (4, 4, -5)$.

Also, given line is,

$$\frac{x-8}{7} = \frac{y-4}{1} = \frac{z-5}{3} \qquad \text{...... (ii)}$$

Then the line (ii) passes through the point $(x_2, y_2, z_2) = (8, 4, 5)$ and having the direction ratio $(l_2, m_2, n_2) = (7, 1, 3)$.

Now, the lines (i) and (ii) are coplanar if

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

Now, $\begin{vmatrix} 8-5 & 4-7 & 5+3 \\ 4 & 4 & -5 \\ 7 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 3 & -3 & 8 \\ 4 & 4 & -5 \\ 7 & 1 & 3 \end{vmatrix}$

$= 3(12 + 5) - (-3)(12 + 35) + 8(4 - 28)$

$= 51 + 141 - 192$

$= 0$

This shows that the lines are coplanar.

For common point, let,

$$\frac{x-5}{4} = \frac{y-7}{4} = \frac{z+3}{-5} = r \text{ (say)} \qquad \text{...... (iii)}$$

So, the general point of (iii) is, $(x_1, y_1, z_1) = (4r + 5, 4r + 7, -5r - 3)$ ... (*)

And,

$$\frac{x-8}{7} = \frac{y-4}{1} = \frac{z-5}{3} = r^1 \text{ (say)} \qquad \text{...... (iv)}$$

So, the general point of (iv) is, $(x_2, y_2, z_2) = (7r^1 + 8, r^1 + 4, 3r^1 + 5)$ ... (**)

Since the lines (iii) and (iv) meets at a point. So, the general points (*) and (**) are identical. That is,

$x_1 = x_2, \quad y_1 = y_2, \quad z_1 = z_2$

Then, $4r + 5 = 7r^1 + 8$ ..... (a)

$4r + 7 = r^1 + 4$ ..... (b)

$-5r - 3 = 3r^1 + 5$ ..... (c)

Solving (a) and (b) we get,

$r^1 = -1$

So that, $(7r^1 + 8, r^1 + 4, 3r^1 + 5) = (-7 + 8, -1 + 4, -3 + 5) = (1, 3, 2)$.

Thus (1, 3, 2) be the common point of (i) and (ii).

And, the equation of plane containing the lines (i) and (ii) is,

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

$$\Rightarrow \begin{vmatrix} x - 5 & y - 7 & z + 3 \\ 4 & 4 & -5 \\ 7 & 1 & 3 \end{vmatrix} = 0$$

$\Rightarrow \quad (x - 5)(12 + 5) - (y - 7)(12 + 35) + (z + 3)(4 - 28) = 0$

$\Rightarrow \quad (x - 5).17 - (y - 7).47 + (z + 3).(-24) = 0$

$\Rightarrow \quad 17x - 85 - 47y + 329 - 24z - 72 = 0$

$\Rightarrow \quad 17x - 47y - 24z + 172 = 0.$

Thus, the equation of the plane containing the lines be $17x - 47y - 24z + 172 = 0$.

coplanar condition.

3. Show that the lines $x - 1 = 2y - 4 = 3z$ and $3x - 5 = 4y - 9 = 3z$ meet in a point and the equation of the plane is which they lie is $3x - 8y + 3z + 13 = 0$.

Solution: Given lines is,

$x - 1 = 2y - 4 = 3z$

$$\Rightarrow \frac{x-1}{6} = \frac{2y-6}{6} = \frac{3z}{6}$$

$$\Rightarrow \frac{x-1}{6} = \frac{y-3}{3} = \frac{z-0}{2} \qquad \text{...... (i)}$$

Then the line (i) passes through the point $(x_1, y_1, z_1) = (1, 2, 0)$ and having the direction ratio $(l_1, m_1, n_1) = (6, 3, 2)$

And $3x - 5 = 4y - 9 = 3z$

$$\frac{3x-5}{12} = \frac{4y-9}{12} = \frac{3z}{12}$$

$$\Rightarrow \frac{x-\frac{5}{3}}{4}=\frac{y-\frac{9}{4}}{3}=\frac{z-0}{4}$$

Then the line (ii) passes through the point $(x_2, y_2, z_2) = \left(\frac{5}{3}, \frac{9}{4}, 0\right)$ and having the direction ratio $(l_2, m_2, n_2) = (4, 3, 4)$.

If the line (i) and line (ii) meet at a point then they should be coplanar.

That is,

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

Here,

$$\begin{vmatrix} \frac{5}{3}-1 & \frac{9}{4}-2 & 0-0 \\ 6 & 3 & 2 \\ 4 & 3 & 4 \end{vmatrix} = \begin{vmatrix} \frac{2}{3} & \frac{1}{4} & 0 \\ 6 & 3 & 2 \\ 4 & 3 & 4 \end{vmatrix}$$

$$=\frac{2}{3}(12-6)-\frac{1}{4}\times(24-8)+0$$

$$=\frac{2}{3}\times 6-\frac{1}{4}\times 16 = 4-4=0.$$

This shows that the lines meet at a point.

And the equation of plane containing the lines be,

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

$$\Rightarrow \begin{vmatrix} x-1 & y-2 & z-0 \\ 6 & 3 & 2 \\ 4 & 3 & 4 \end{vmatrix} = 0.$$

$\Rightarrow (x-1)(12-6)-(y-2)(24-8)+z(18-12)=0$

$\Rightarrow 6(x-1)-16(y-2)+6z=0$

$\Rightarrow 6x-6-16y-32+6z=0$

$\Rightarrow 6x-16y+6z-38=0$

$\Rightarrow 3x-8y+3z+13=0.$

This shows that the plane $3x-8y+3z+13=0$ contains the lines (i) and (ii).

4. Show that the lines $\frac{x+4}{3}=\frac{y+6}{5}=\frac{1-z}{2}$ and $3x-2y+z+5=0=2x+3y+4z-4$ are coplanar. Find their point of intersection and the plane in which they lie.

Solution: Given lines are

$$\frac{x+4}{3}=\frac{y+6}{5}=\frac{1-z}{2} \quad \text{...... (i)}$$

And, $3x-2y+z+5=0=2x+3y+4z-4$ ...... (ii)

First we set (ii) in symmetrical form. For this set $z=0$, then

$3x-2y+z+5=0$ ..... (*)

$2x+3y+4z-4=0$ ..... (**)

Solving (*) and (**) we get,

$$x=-\frac{7}{13} \text{ and } y=\frac{22}{13}$$

Therefore the line (ii) passes through the point $\left(\frac{-7}{13}, \frac{22}{13}, 0\right)$.

Let direction ratio of lines is (a, b, c) of (ii) then

$3a-2b+c=0$ .... (a)

$2a+3b+4c=0$ .... (b)

Solving equation (a) and (b), by cross multiplication we get,

$$\frac{a}{-8-3}=\frac{b}{2-12}=\frac{c}{9+4}$$

$\Rightarrow (a, b, c) = (-11, -10, 13)$

Now, equation of the line passes through the point $\left(\frac{-7}{13}, \frac{22}{13}, 0\right)$ and have direction ratio $(-11, -10, 13)$ is

$$\frac{x+\frac{7}{13}}{-11}=\frac{y-\frac{22}{13}}{-10}=\frac{z-0}{13} \quad \text{...... (iii)}$$

Since the line (i) passes through the point $(x_1, y_1, z_1) = (-4, -6, 1)$ and having the direction ratio $(l_1, m_1, n_1) = (3, 5, -2)$.

And, Then the line (ii) passes through the point $(x_2, y_2, z_2) = \left(\frac{-7}{13}, \frac{22}{13}, 0\right)$ and having the direction ratio $(l_2, m_2, n_2) = (-11, -10, 13)$.

Now, the lines (i) and (ii) are coplanar if

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

Now,

$$\begin{vmatrix} \frac{-7}{13}+14 & \frac{22}{13}+6 & 0-1 \\ 3 & 5 & -2 \\ -11 & -10 & 13 \end{vmatrix} = \begin{vmatrix} \frac{45}{13} & \frac{100}{13} & -1 \\ 3 & 5 & -2 \\ -11 & -10 & 13 \end{vmatrix}$$

$$=\frac{45}{13}(65-20)-\frac{100}{13}(39-22)-1(-30+55)$$

$$=\frac{45}{13}\times 45-\frac{100}{13}\times 17-25=0$$

Hence the given lines are coplanar.

For point of intersection, let,

$$\frac{x+4}{3} = \frac{y+6}{5} = \frac{1-z}{2} = r \text{ (Suppose)} \qquad \ldots\ldots \text{(iv)}$$

Then the general point of (iv) is, $(x = 3r - 4,\ y = 5r - 6,\ z = 1 - 2r)$.

And,

$$\frac{x+\frac{7}{13}}{-11} = \frac{y-\frac{22}{13}}{-10} = \frac{z-0}{13} = r^1 \qquad \ldots\ldots \text{(v)}$$

Then the general point of (v) is, $(x_2 = -11r^1 - \frac{7}{13},\ y_2 = -10r^1 + \frac{22}{13},\ z_2 = 13r^1)$

For point of intersection, at least one point of the line (iv) and (v) is same.

So, $x_1 = x_2$, $y_1 = y_2$ and $z_1 = z_2$

That is, $3r - 4 = -11r^1 - \frac{7}{13}$ ...... (a)

$5r - 6 = -10r^1 + \frac{22}{13}$ ...... (b)

$-2r + 1 = 13r^1$ .... (c)

Solving the equation (a) and (b) we get,

$$r^1 = -\frac{3}{13}$$

Then (c) gives, $-2r = 13 \times (-\frac{3}{13}) - 1 \quad \Rightarrow \quad r = 2$

Therefore the point of intersection is $(3r - 4, 5r - 6, -2r + 1) = (2, 4, -3)$.

And the equation of plane containing the lines be,

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

$$\Rightarrow \begin{vmatrix} x+4 & y+6 & z-1 \\ 3 & 5 & -2 \\ -11 & -10 & 13 \end{vmatrix} = 0$$

$\Rightarrow (x + 4)(65 - 20) - (y + 6)(39 - 22) + (z - 1)(-35 + 55) = 0$

$\Rightarrow 45(x + 4) - 17(y + 6) + 25(z - 1) = 0$

$\Rightarrow 45x + 180 - 17y - 102 + 25z - 25 = 0$

$\Rightarrow 45x - 17y + 25z + 53 = 0.$

This is the equation of required plane that contains the given line lines (i) and (ii).

5. Prove that the lines $\frac{x+1}{1} = \frac{y+1}{2} = \frac{z+1}{3}$ and $x + 2y + 3z - 8 = 0 = 2x + 3y + 4z - 11$ are coplanar and find the point it intersection and the equation of the plane containing them.

Solution: Given lines are

$$\frac{x+1}{1} = \frac{y+1}{2} = \frac{z+1}{3} \qquad \ldots\ldots \text{(i)}$$

And, $x + 2y + 3z - 8 = 0 = 2x + 3y + 4z - 11$ ...... (ii)

First we set (ii) in symmetrical form. For this set $z = 0$, then

$x + 2y - 8 = 0$ .... (*)

$2x + 3y - 11 = 0$ ..... (**)

Solving (*) and (**) we get,

$x = -2$ and $y = 5$.

Therefore the line (ii) passes through the point $(-2, 5, 0)$.

Let direction ratio of lines is (a, b, c) of (ii) then

$a + 2b + 3c = 0$ ...... (a)

$2a + 3b + 4c = 0$ .... (b)

Solving equation (a) and (b), by cross multiplication we get,

$$\frac{a}{8-9} = \frac{b}{6-4} = \frac{c}{3-4} = \lambda$$

$\Rightarrow \quad a = -\lambda,\ b = 2\lambda,\ c = -\lambda$

Now, equation of the line passes through the point $(-2, 5, 0)$ and have direction ratio $(-\lambda, 2\lambda, -\lambda)$ is

$$\frac{x+2}{-\lambda} = \frac{y-5}{2\lambda} = \frac{z-0}{-\lambda}$$

$$\Rightarrow \quad \frac{x+2}{-1} = \frac{y-5}{2} = \frac{z-0}{-1} \qquad \ldots\ldots \text{(iii)}$$

Since the line (i) passes through the point $(x_1, y_1, z_1) = (-1, -1, -1)$ and having the direction ratio $(l_1, m_1, n_1) = (1, 2, 3)$.

And, Then the line (ii) passes through the point $(x_2, y_2, z_2) = (-2, 5, 0)$ and having the direction ratio $(l_2, m_2, n_2) = (-1, 2, -1)$.

Now, the lines (i) and (ii) are coplanar if

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

Now,

$$\begin{vmatrix} -2-(-1) & 5-(-1) & 0-(-1) \\ 1 & 2 & 3 \\ -1 & 2 & -1 \end{vmatrix} = \begin{vmatrix} -1 & 6 & 1 \\ 1 & 2 & 3 \\ -1 & 2 & -1 \end{vmatrix}$$

$= -1(-2 - 6) - 6(-1 + 3) + 1(2 + 2)$

$= 8 - 12 + 4 = 0.$

Hence the given lines are coplanar.

For point of intersection, let,

$$\frac{x+1}{1} = \frac{y+1}{2} = \frac{z+1}{3} = r \text{ (Suppose)} \qquad \ldots\ldots \text{(iv)}$$

Then the general point of (iv) is, $(r - 1, 2r - 1, 3r - 1)$.

And,

$$\frac{x+2}{-1} = \frac{y-5}{2} = \frac{z-0}{-1} = r^1 \qquad \ldots\ldots \text{(v)}$$

Then the general point of (v) is, $(-r^1 - 2, 2r^1 + 5, -r^1)$.

For point of intersection, at least one point of the line (iv) and (v) is same.

So, $x_1 = x_2$, $y_1 = y_2$ and $z_1 = z_2$

That is, $r - 1 = -r^1 - 2$ ...... (a)

$2r - 1 = 2r^1 + 5$ ...... (b)

$3r - 1 = -r^1$ .... (c)

Solving the equation (a) and (b) we get,

$r = 1$.

Therefore the point of intersection is $(r - 1, 2r - 1, 3r - 1) = (0, 1, 2)$.

And the equation of plane containing the lines be,

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

$$\Rightarrow \begin{vmatrix} x + 1 & y + 1 & z + 1 \\ 1 & 2 & 3 \\ -1 & 2 & -1 \end{vmatrix} = 0$$

$\Rightarrow$ $(x + 1)(-2 - 6) - (y + 1)(-1 + 3) + (z + 1)(2 + 2) = 0$

$\Rightarrow$ $-8x - 8 - 2y - 2 + 4z + 4 = 0$

$\Rightarrow$ $-8x - 2y + 4z - 6 = 0$

$\Rightarrow$ $4x + y - 2z + 3 = 0$

This is the equation of required plane that contains the given line lines (i) and (ii).

6. Find the equation of the plane, containing $\frac{x-1}{2} = \frac{y-1}{2} = \frac{z-3}{3}$ and $\frac{x-2}{3} = \frac{y-1}{4} = \frac{z-4}{5}$.

Solution: Given lines are

$$\frac{x-1}{2} = \frac{y-1}{2} = \frac{z-3}{3} \quad \text{...... (i)}$$

And $$\frac{x-2}{3} = \frac{y-1}{4} = \frac{z-4}{5} \quad \text{...... (ii)}$$

Then the general point of (i) is, $(x_1, y_1, z_1) = (1, 1, 3)$ and its direction ratio is, $(l_1, m_1, n_1) = (2, 2, 3)$.

And, the general point of (ii) is, $(x_2, y_2, z_2) = (2, 1, 4)$ and its direction ratio is, $(l_2, m_2, n_2) = (3, 4, 5)$.

Now, equation of plane containing (i) and (ii) is,

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x - 1 & y - 1 & z - 3 \\ 2 & 2 & 3 \\ 3 & 4 & 5 \end{vmatrix} = 0$$

$\Rightarrow$ $(x - 1)(10 - 12) - (y - 1)(10 - 9) + (z - 3)(8 - 6) = 0$

$\Rightarrow$ $-2(x - 1) - 1(y - 1) + 2(z - 3) = 0$

$\Rightarrow$ $-2x + 2 - y + 1 + 2z - 6 = 0$

$\Rightarrow$ $-2x - y + 2z - 3 = 0$

$\Rightarrow$ $2x + y - 2z + 3 = 0$.

This is the equation of required plane that contains the given line lines (i) and (ii).

7. Show that the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $4x - 3y + 1 = 0 = 5x - 3z + 2$ are coplanar. Also find their point of contact. [2009 Fall Q. No. 1(a) OR]

Solution: Given lines are

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} \quad \text{...... (i)}$$

And, $4x - 3y + 1 = 0 = 5x - 3z + 2$ ...... (ii)

The equation can be written as,

$4x - 3y + 1 = 0$ and $5x - 3z + 2 = 0$

$\Rightarrow 4x = 3y - 1$ $\Rightarrow 5x = 3z - 2$

$\Rightarrow x = \frac{3y-1}{4}$ $\Rightarrow x = \frac{3z-2}{5}$

Then the line (ii) can be written in the symmetrical form as,

$$\frac{x-0}{1} = \frac{3y-1}{4} = \frac{3z-2}{5}$$

$$\Rightarrow \frac{x}{1} = \frac{y - \frac{1}{3}}{\frac{4}{3}} = \frac{z - \frac{2}{3}}{\frac{5}{3}} \quad \text{...... (iii)}$$

Since the line (i) passes through the point $(x_1, y_1, z_1) = (1, 2, 3)$ and having the direction ratio $(l_1, m_1, n_1) = (2, 3, 4)$.

And, Then the line (ii) passes through the point $(x_2, y_2, z_2) = \left(0, \frac{1}{3}, \frac{2}{3}\right)$ and having the direction ratio $(l_2, m_2, n_2) = \left(1, \frac{4}{3}, \frac{5}{3}\right)$.

Now, the lines (i) and (ii) are coplanar if

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

Now,

$$\begin{vmatrix} 0-1 & \frac{1}{3}-2 & \frac{2}{3}-3 \\ 2 & 3 & 4 \\ 1 & \frac{4}{3} & \frac{5}{3} \end{vmatrix} = \begin{vmatrix} -1 & \frac{-5}{3} & \frac{-7}{3} \\ 2 & 3 & 4 \\ 1 & \frac{4}{3} & \frac{5}{3} \end{vmatrix}$$

$$= -1\left(\frac{15}{3}-\frac{16}{3}\right) + \frac{5}{3}\left(\frac{10}{3}-4\right) - \frac{7}{3}\left(\frac{8}{3}-3\right)$$

$$= \frac{1}{3} - \frac{10}{9} + \frac{7}{9} = \frac{3-10+7}{9} = 0$$

Hence, the given lines are coplanar.

For point of intersection, let,

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} = r \text{ (Suppose)} \qquad \ldots\ldots \text{(iv)}$$

Then the general point of (iv) is, $(2r+1, 3r+2, 4r+3)$.

And,

$$\frac{x}{1} = \frac{y-\frac{1}{3}}{\frac{4}{3}} = \frac{z-\frac{2}{3}}{\frac{5}{3}} = r^1 \qquad \ldots\ldots \text{(v)}$$

Then the general point of (v) is, $(r^1, \frac{4r^1}{3}+\frac{1}{3}, \frac{5r^1}{3}+\frac{2}{3})$.

For point of intersection, at least one point of the line (iv) and (v) is same.

So, $x_1 = x_2$, $y_1 = y_2$ and $z_1 = z_2$

That is, $2r + 1 = r_1$ ...... (a)

$$3r + 2 = \frac{4r}{3} + \frac{1}{3} \qquad \ldots\ldots \text{(b)}$$

$$4r + 3 = \frac{5r^1}{3} + \frac{2}{3} \qquad \ldots\ldots \text{(c)}$$

Solving the equation (a) and (b) we get,

$r^1 = -1$ and $r = -1$

Therefore the point of contact is,

$(2r+1, 3r+2, 4r+3) = (-1, -1, -1)$.

Exercise 3.5

1. **Find the magnitude and equation of the line of shortest distance between the line of shortest distance the lines $\frac{x-8}{3} = \frac{y+9}{-16} = \frac{z-10}{7}$ and $\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$.**

Solution: Given line is,

$$\frac{x-8}{3} = \frac{y+9}{-16} = \frac{z-10}{7} \qquad \ldots\ldots \text{(i)}$$

The general point of line (i) is, $(x_1, y_1, z_1) = (8, -9, 10)$ and the direction ratio of (i) are $(l_1, m_1, n_1) = (3, -16, 7)$.

Another given line is,

$$\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5} \qquad \ldots\ldots \text{(ii)}$$

The general point of line (ii) is, $(x_2, y_2, z_2) = (15, 29, 5)$ and the direction ratio of (ii) are $(l_2, m_2, n_2) = (3, 8, -5)$.

Let l, m, n be the direction ratio of shortest distance between (i) and (ii).

Since the line of shortest distance is perpendicular to both lines (i) and (ii), so

$$3l - 16m + 7n = 0$$
$$3l + 8m - 5n = 0$$

Solving these equations by method of cross multiplication,

$$\frac{l}{80-56} = \frac{m}{21+15} = \frac{n}{24+48} \Rightarrow \frac{l}{24} = \frac{m}{36} = \frac{n}{72}$$

$$\Rightarrow \frac{l}{2} = \frac{m}{3} = \frac{n}{6} = k \text{ (suppose)}$$

$$\Rightarrow l = 2k,\ m = 3k,\ n = 6k$$

Since we have,

$$l^2 + m^2 + n^2 = 1 \Rightarrow (2k)^2 + (3k)^2 + (6k)^2 = 1$$
$$\Rightarrow 49k^2 = 1$$
$$\Rightarrow k = \frac{1}{7} \text{ (taking the +ve sign only)}$$

Then, $l = \frac{2}{7}, m = \frac{3}{7}, n = \frac{6}{7}$

Now, the length of shortest distance (SD) is,

$$SD = (x_2 - x_1)\,l + (y_2 - y_1)\,m + (z_2 - z_1)\,n$$
$$= (15-8)\frac{2}{7} + (29+9)\frac{3}{7} + (5-10)\frac{6}{7}$$
$$= 7\times\frac{2}{7} + 38\times\frac{3}{7} - 5\times\frac{6}{7} = \frac{14+114+-30}{7} = \frac{98}{7} = 14$$

And, the equation of shortest distance be,

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix} = 0 = \begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ l_1 & m_2 & n_2 \\ l & m & n \end{vmatrix}.$$

$$\Rightarrow \begin{vmatrix} x-8 & y+9 & z-10 \\ 3 & -16 & 7 \\ \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \end{vmatrix} = 0 = \begin{vmatrix} x-15 & y-29 & z-5 \\ 3 & 8 & -5 \\ \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \end{vmatrix}$$

$$\Rightarrow \quad (x-8)\left(\frac{-96}{7}-3\right) - (y+9)\left(\frac{18}{7}-2\right) + (z-10)\left(\frac{9}{7}-\frac{32}{7}\right) = 0 = (x-15)\left(\frac{48}{7}+\frac{15}{7}\right) - (y-29)\left(\frac{18}{7}+\frac{10}{7}\right) + (z-5)\left(\frac{9}{7}-\frac{16}{7}\right)$$

$$\Rightarrow \quad (x-8)(-96-21) - (y+9)(18-14) + (z-9)(9+32) = 0 = (x-15)\frac{63}{7} - (y-29)\frac{28}{7} + (z-5)\left(\frac{-7}{7}\right)$$

$$\Rightarrow \quad -117(x-8) - 4(y+9) + 41(z-10) = 0 = (x-15)9 - (y-29)4 + (z-5)(-1).$$

$$\Rightarrow \quad 117x + 4y - 41z - 490 = 0 = 9x - 4y - z - 135 + 116 + 5.$$

$$\Rightarrow \quad 117x + 4y - 41z - 490 = 0 = 9x - 4y - z - 14.$$

Thus, the magnitude of the shortest distance is 14 unit and the equation of the line of shortest distance is, $117x + 4y - 41z - 490 = 0 = 9x - 4y - z - 14$.

2. Show that the shortest distance between the lines $\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$ and $\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$ is $3\sqrt{30}$. Find out the equation of the line of shortest distance.

Solution: Given line is,

$$\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1} \quad \ldots\ldots\ldots \text{(i)}$$

The general point of line (i) is, $(x_1, y_1, z_1) = (3, 8, 3)$ and the direction ratio of (i) are $(l_1, m_1, n_1) = (3, -1, 1)$.

Another given line is,

$$\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4} \quad \ldots\ldots\ldots \text{(ii)}$$

The general point of line (ii) is, $(x_2, y_2, z_2) = (-3, -7, 6)$ and the direction ratio of (ii) are $(l_2, m_2, n_2) = (-3, 2, 4)$.

Let l, m, n be the direction ratio of shortest distance between (i) and (ii).

Since the line of shortest distance is perpendicular to both lines (i) and (ii), so

$$3l - m + n = 0$$

$$-3l + 2m + 4n = 0$$

Solving by cross multiplication, we get,

$$\frac{l}{-4-2} = \frac{m}{-3-12} = \frac{n}{6-3} \Rightarrow \frac{l}{-6} = \frac{m}{-15} = \frac{n}{3}$$

$$\Rightarrow \frac{l}{-2} = \frac{m}{-5} = \frac{n}{1} = k$$

This gives, $l = -2k$, $m = -5k$, $n = k$.

Since we have,

$$l^2 + m^2 + n^2 = 1 \Rightarrow 4k^2 + 25k^2 + k^2 = 1$$

$$\Rightarrow 30k^2 = 1$$

$$\Rightarrow k = \frac{1}{\sqrt{30}} \quad \text{(taking the +ve positive sign)}$$

Then, $l = \frac{-2}{\sqrt{30}}$, $m = \frac{-5}{\sqrt{30}}$, $n = \frac{1}{\sqrt{30}}$.

Now, the length of shortest distance (SD) is,

$$SD = (x_2 - x_1)\, l + (y_2 - y_1)\, m + (z_2 - z_1)\, n$$

$$= (-3-3)\times\frac{-2}{\sqrt{30}} + (-7-8)\times\frac{-5}{\sqrt{30}} + (6-3)\times\frac{1}{\sqrt{30}}$$

$$= \frac{12}{\sqrt{30}} + \frac{75}{\sqrt{30}} + \frac{3}{\sqrt{30}}$$

$$= \frac{12+75+3}{\sqrt{30}} = \frac{90}{\sqrt{30}} = 3\sqrt{30}$$

And, the equation of shortest distance be,

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix} = 0 = \begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ l_1 & m_2 & n_2 \\ l & m & n \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} x-3 & y-8 & z-3 \\ 3 & -1 & 1 \\ \frac{-2}{\sqrt{30}} & \frac{-5}{\sqrt{30}} & \frac{1}{\sqrt{30}} \end{vmatrix} = 0 = \begin{vmatrix} x+3 & y+7 & z-6 \\ -3 & 2 & 4 \\ \frac{-2}{\sqrt{30}} & \frac{-5}{\sqrt{30}} & \frac{1}{\sqrt{30}} \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} x-3 & y-8 & z-3 \\ 3 & -1 & 1 \\ -2 & -5 & 1 \end{vmatrix} = 0 = \begin{vmatrix} x+3 & y+7 & z-6 \\ -3 & 2 & 4 \\ -2 & -5 & 1 \end{vmatrix}$$

$$\Rightarrow (x-3)(-1+5) - (y-8)(3+2) + (z-3)(-15-2) = 0 = (x+3)(2+20) - (y+7)(-3+8) + (z-6)(15+4)$$

$$\Rightarrow (x-3)\,4 - (y-8)\,5 + (z-3)(-17) = 0 = (x+3)(22) - (y+7)\,5 + (z-6)\,19.$$

$$\Rightarrow 4x - 5y - 17z - 12 + 40 + 51 = 0 = 22x - 5y + 19z + 66 - 35 - 114.$$

$$\Rightarrow 4x - 5y - 17z + 79 = 0 = 22x - 3y + 19z - 83.$$

Thus, the magnitude of the shortest distance is $3\sqrt{30}$ unit and the equation of the line of shortest distance is, $4x - 5y - 17z + 79 = 0 = 22x - 3y + 19z - 83$.

3. **Find the shortest distance between the lines $x = y + 4 = \frac{z}{3}$, $\frac{x-1}{3} = \frac{y}{z} = z$.**

Solution: Given lines are,

$$\frac{x-0}{1} = \frac{y+4}{1} = \frac{z-0}{3} \quad \text{...... (i)}$$

And $$\frac{x-1}{3} = \frac{y-0}{2} = \frac{z-0}{1} \quad \text{...... (ii)}$$

Comparing the lines with $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ then we get

(a) the line (i) passes through the point $(x_1, y_1, z_1) = (0, -4, 0)$ and its direction ratio be $(l, m, n) = (1, 1, 3)$.

(b) the line (ii) passes through the point $(x_1, y_1, z_1) = (1, 0, 0)$ and its direction ratio be $(l, m, n) = (3, 2, 1)$.

Let l, m, n be the direction ratio of the line which is shortest distance between (i) and (ii), is perpendicular to the lines (i) and (ii).

Then,

$$l + m + 3n = 0$$
$$3l + 2m + n = 0$$

Solving by cross – multiplication

$$\frac{l}{1-6} = \frac{m}{9-1} = \frac{n}{2-3} \Rightarrow \frac{l}{-5} = \frac{m}{8} = \frac{n}{-1} = k \text{ (suppose)}$$
$$\Rightarrow l = 5k,\ m = 8k,\ n = -k$$

We have, $l^2 + m^2 + n^2 = 1$. So,

$$(-5k)^2 + (8k)^2 + (-k)^2 = 1 \Rightarrow 25k^2 + 64k^2 + k^2 = 1$$
$$\Rightarrow 90k^2 = 1 \Rightarrow k = \pm\frac{1}{\sqrt{90}}$$

Taking the positive sign,

$$l = \frac{-5}{\sqrt{90}},\ m = \frac{8}{\sqrt{90}},\ n = \frac{-1}{\sqrt{90}}$$

Now, the shortest distance between (i) and (ii) be

$$\text{S.D.} = (x_2 - x_1)\,l + (y_2 - y_1)\,m + (z_2 - z_1)\,n$$
$$= (1-0) \times \frac{-5}{\sqrt{90}} + (0+4)\frac{8}{\sqrt{90}} + (0-0)\frac{-1}{\sqrt{90}}$$
$$= \frac{-5}{\sqrt{90}} + \frac{32}{\sqrt{90}}$$
$$= \frac{27}{\sqrt{90}} = \frac{27}{3\sqrt{10}} = \frac{9}{\sqrt{10}}.$$

4. **Find the shortest distance between the lines $\frac{x-5}{3} = \frac{7-y}{16} = \frac{z-3}{7}$ and $\frac{x-9}{3} = \frac{y-13}{8} = \frac{15-z}{5}$ find also the equation of shortest distances.**

[2008 Spring Q. No. 1(a)]

Solution: Given that,

$$\frac{x-5}{3} = \frac{7-y}{16} = \frac{z-3}{7} \quad \text{...... (i)}$$

Clearly the line (i) passes through the point $(x_1, y_1, z_1) = (5, 7, 3)$ and it has the direction ratio $(l_1, m_1, n_1) = (3, -16, 7)$.

Also, given line is,

$$\frac{x-9}{3} = \frac{y-13}{8} = \frac{15-z}{5} \quad \text{...... (ii)}$$

Clearly the line (ii) passes through the point $(x_2, y_2, z_2) = (9, 13, 15)$ and it has the direction ratio $(l_2, m_2, n_2) = (3, 8, -5)$

Let l, m, n be the direction ratio of the line which is shortest distance between (i) and (ii), is perpendicular to the lines (i) and (ii).

Then,

$$3l - 16m + 7n = 0$$
$$3l + 8m - 5n = 0$$

Solving by cross multiplication, we get

$$\frac{l}{80-56} = \frac{m}{21+15} = \frac{n}{24+48} \Rightarrow \frac{l}{24} = \frac{m}{36} = \frac{n}{72} = k$$
$$\Rightarrow l = 24k,\ m = 36k,\ n = 72k$$

We know that,

$$l^2 + m^2 + n^2 = 1 \Rightarrow (24k)^2 + (36k)^2 + (72k)^2 = 1$$
$$\Rightarrow 576\,k^2 + 1296\,k^2 + 5184\,k^2 = 1$$
$$\Rightarrow 7056\,k^2 = 1$$
$$\Rightarrow k = \frac{1}{84} \quad \text{,taking +ve sign only.}$$

So that, $l = 24 \times \frac{1}{84} = \frac{2}{7}$, $m = 36 \times \frac{1}{84} = \frac{3}{7}$, $n = 72 \times \frac{1}{84} = \frac{6}{7}$

Now, the shortest distance between (i) and (ii) be

$$\text{S.D.} = (x_2 - x_1)\,l + (y_2 - y_1)\,m + (z_2 - z_1)\,n$$
$$= (9-5) \times \frac{2}{7} + (13-7) \times \frac{3}{7} + (15-3) \times \frac{6}{7}$$
$$= \frac{8 + 18 + 72}{7} = \frac{98}{7} = 14.$$

Thus the shortest distance between the lines is 14 units.

Next for equation of shortest distance,

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix} = 0 = \begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ l_2 & m_2 & n_2 \\ l & m & n \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} x-5 & y-7 & z-3 \\ 3 & -16 & 7 \\ 2/7 & 3/7 & 6/7 \end{vmatrix} = 0 = \begin{vmatrix} x-9 & y-13 & z-15 \\ 3 & 8 & -5 \\ 2/7 & 3/7 & 6/7 \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} x-5 & y-7 & z-3 \\ 3 & -16 & 7 \\ 2 & 3 & 6 \end{vmatrix} = 0 = \begin{vmatrix} x-9 & y-13 & z-15 \\ 3 & 8 & -5 \\ 2 & 3 & 6 \end{vmatrix}$$

$\Rightarrow$ $(x-5)(-96-21) - (y-7)(18-14) + (z-3)(9+32) = 0 = (x-9)(48+15) - (y-13)(18+10) + (z-15)(9-16)$

$\Rightarrow$ $(x-5)(-117) - (y-7)4 + (z-3)41 = 0 = (x-9)9 - (y-13)4 + (z-15)(-1)$

$\Rightarrow$ $-117(x-5) - 4(y-7) + 41(z-3) = 0 = (x-9)9 - (y-13)4 - (z-15)$.

$\Rightarrow$ $-117x - 4y + 41z + 490 = 0 = 9x - 4y - z - 14$.

5. Find the magnitude and equation of the shortest distance between the lines $\frac{x}{2} = \frac{y}{-3} = \frac{z}{1}$ and $\frac{x-2}{3} = \frac{y-1}{-5} = \frac{z+2}{2}$. **[2009 Fall Q. No. 1(a)]**

Solution: Given that,

$$\frac{x}{2} = \frac{y}{-3} = \frac{z}{1} \qquad \text{...... (i)}$$

Clearly the line (i) passes through the point $(x_1, y_1, z_1) = (0, 0, 0)$ and it has the direction ratio $(l_1, m_1, n_1) = (2, 1, -2)$.

Also, given line is,

$$\frac{x-2}{3} = \frac{y-1}{-5} = \frac{z-(-2)}{2} \qquad \text{...... (ii)}$$

Clearly the line (ii) passes through the point $(x_2, y_2, z_2) = (2, 1, -2)$ and it has the direction ratio $(l_2, m_2, n_2) = (3, -5, 2)$

Let l, m, n be the direction ratio of the line which is shortest distance between (i) and (ii), is perpendicular to the lines (i) and (ii).

Then,

$$2l - 3m + n = 0$$
$$3l - 5m + 2n = 0$$

Solving by cross multiplication we get,

$$\frac{l}{-6+5} = \frac{m}{3-4} = \frac{n}{-10+9} \Rightarrow \frac{l}{-1} = \frac{m}{-1} = \frac{n}{-1} = k$$

$$\Rightarrow l = -k,\ m = -k,\ n = -k$$

We know that,

$$l^2 + m^2 + n^2 = 1 \Rightarrow k^2 + k^2 + k^2 = 1 \Rightarrow k = \pm\frac{1}{\sqrt{3}}$$

Now, taking negative sign (because the direction ratios have negative sign)

$$l = -\frac{1}{\sqrt{3}},\ m = -\frac{1}{\sqrt{3}},\ n = -\frac{1}{\sqrt{3}}$$

Now, the shortest distance between (i) and (ii) be

$$\text{S.D.} = (x_2 - x_1)\,l + (y_2 - y_1)\,m + (z_2 - z_1)\,n$$
$$= (2-0) \times -\frac{1}{\sqrt{3}} + (1-0) \cdot -\frac{1}{\sqrt{3}} + (-2-0) \times -\frac{1}{\sqrt{3}}$$
$$= \frac{-2}{\sqrt{3}} - \frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}} = -\frac{1}{\sqrt{3}} \text{ (neglecting -ve sign)}$$

Thus the shortest distance between the lines is $\frac{1}{\sqrt{3}}$ units.

Next for equation of shortest distance,

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix} = 0 = \begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ l_2 & m_2 & n_2 \\ l & m & n \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} x-0 & y-0 & z-0 \\ 2 & -3 & 1 \\ \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{vmatrix} = 0 = \begin{vmatrix} x-2 & y-1 & z+2 \\ 3 & -5 & 2 \\ \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{vmatrix}$$

$$\Rightarrow -\frac{1}{\sqrt{3}} \begin{vmatrix} x-0 & y-0 & z-0 \\ 2 & -3 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0 = -\frac{1}{\sqrt{3}} \begin{vmatrix} x-2 & y-1 & z+2 \\ 3 & -5 & 2 \\ 1 & 1 & 1 \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} x & y & z \\ 2 & -3 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0 = \begin{vmatrix} x-2 & y-1 & z+2 \\ 3 & -5 & 2 \\ 1 & 1 & 1 \end{vmatrix}$$

$\Rightarrow$ $x(-3-1) - y(2-1) + z(2+3) = 0 = (x-2)(-5-2) - (y-1)(3-2) + (z+2)(3+5)$.

$\Rightarrow$ $-4x - y + 5z = 0 = -7x - y + 8z + 31$

$\Rightarrow$ $4x + y - 5z = 0 = 7x + y - 8z - 31$.

This is the equation of the required line.

6. Find the length and equation of the shortest distance between the lines $x - y + z = 0 = 2x - 3y + 4z$ and $x + y + 2z - 3 = 0 = 2x + 3y + 3z - 4$

Solution: Given that,

$$x - y + z = 0 = 2x - 3y + 4z$$

When, z = 0 then $\quad x - y = 0 \qquad$ ...... (i)

and $\quad 2x - 3y = 0 \qquad$ ...... (ii)

Solving equation (i) and (ii) then we get,

$$y = 0 \text{ and } x = 0$$

Thus the point be, $(x_1, y_1, z_1) = (0, 0, 0)$.

Let direction ratio of the line is, (a, b, c). Then,

$$a - b + c = 0 \qquad \text{...... (iii)}$$

$$2a - 3b + 4c = 0 \qquad \text{...... (iv)}$$

Solving by cross multiplication

$$\frac{a}{-4+3} = \frac{b}{2-4} = \frac{c}{-3+2} \Rightarrow \frac{a}{-1} = \frac{b}{-2} = \frac{c}{-1}$$

$$\Rightarrow (a, b, c) = (-1, -2, -1)$$

Now the equation of line passing through (0, 0, 0) and having direction ratios -1, -2, -1 be,

$$\frac{x-0}{-1} = \frac{y-0}{-2} = \frac{z-0}{-1} \qquad \text{...... (v)}$$

Another given line is,

$$x + y + 2z - 3 = 0 = 2x + 3y + 3z - 4$$

When, z = 0 then we get,

$$x + y - 3 = 0 \qquad \text{...... (vi)}$$

$$2x + 3y - 4 = 0 \qquad \text{...... (vii)}$$

Solving (vi) and (vii) we get,

$$x = 5 \text{ and } y = -2.$$

Thus the point be, $(x_2, y_2, z_2) = (5, -2, 0)$.

Let direction ratio of the line is, (l, m, n). Then,

$$l + m + 2n = 0 \qquad \text{...... (*)}$$

$$2l + 3m + 3n = 0 \qquad \text{...... (**)}$$

Solving above equation by cross multiplication we get,

$$\frac{l}{3-6} = \frac{m}{4-3} = \frac{n}{3-2} \Rightarrow \frac{l}{-3} = \frac{m}{1} = \frac{n}{1}$$

$$\Rightarrow (l, m, n) = (-3, 1, 1)$$

Thus the equation of line passing through (5, -2, 0) and having direction ratios -3, 1, 1 be,

$$\frac{x-5}{-3} = \frac{y+2}{1} = \frac{z-0}{1} \qquad \text{...... (viii)}$$

From line (v) and (viii), $(x_1, y_1, z_1) = (0, 0, 0)$ and $(l_1, m_1, n_1) = (-1, -2, -1)$.

Also, $(x_2, y_2, z_2) = (5, -2, 0)$ and $(l_2, m_2, n_2) = (-3, 1, 1)$.

Let l, m, n, be the direction ratio of the line that measure the shortest distance between the given lines. So,

$$-l - 2m - n = 0 \qquad \text{...... (ix)}$$

$$-3l + m + n = 0 \qquad \text{...... (x)}$$

From equation (ix) and (x) by cross multiplication

$$\frac{l}{-2+1} = \frac{m}{3+1} = \frac{n}{-1-6} \Rightarrow \frac{l}{-1} = \frac{m}{4} = \frac{n}{-7} = k$$

$$\Rightarrow l = -k, \; m = 4k \text{ and } n = -7k.$$

Since we have,

$$l^2 + m^2 + n^2 = 1 \Rightarrow (-k)^2 + 4k)^2 + (-7k)^2 = 1$$

$$\Rightarrow k^2 + 16k^2 + 49k^2 = 1$$

$$\Rightarrow 66k^2 = 1 \Rightarrow k = \pm\frac{1}{\sqrt{66}} \text{ (taking the +ve sign).}$$

Thus, $l = \frac{-1}{\sqrt{66}}, \; m = \frac{4}{\sqrt{66}}, \; n = \frac{-7}{\sqrt{66}}$

Now, the shortest distance between (i) and (ii) be

$$\text{S.D.} = (x_2 - x_1)\, l + (y_2 - y_1)\, m + (z_2 - z_1)\, n$$

$$= (5 - 0) \times \frac{-1}{\sqrt{66}} + (-2-0) \times \frac{4}{\sqrt{66}} + 0 \times \frac{-7}{\sqrt{66}}$$

$$= -\frac{5}{\sqrt{66}} - \frac{8}{\sqrt{66}} = -\frac{13}{\sqrt{66}} = \frac{13}{\sqrt{66}} \text{ (neglecting -ve sign)}$$

Thus the shortest distance between the lines is $\frac{13}{\sqrt{66}}$ units.

Next for equation of shortest distance,

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix} = 0 = \begin{vmatrix} x - x_2 & y - y_2 & z - z_2 \\ l_2 & m_2 & n_2 \\ l & m & n \end{vmatrix}$$

$$\Rightarrow \frac{1}{\sqrt{66}} \begin{vmatrix} x & y & z \\ -1 & -2 & -1 \\ -1 & 4 & -7 \end{vmatrix} = 0 = \frac{1}{\sqrt{66}} \begin{vmatrix} x-5 & y+2 & z-0 \\ -3 & 1 & 1 \\ -1 & 4 & -7 \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} x & y & z \\ -1 & -2 & -1 \\ -1 & 4 & -7 \end{vmatrix} = 0 = \begin{vmatrix} x-5 & y+2 & z-0 \\ -3 & 1 & 1 \\ -1 & 4 & -7 \end{vmatrix}$$

$$\Rightarrow x(14+4) - y(7-1) + z(-4-2) = 0 = (x-5)(-7-4) - (y+2)(21+1) + z(-12+1)$$

$$\Rightarrow 18x - 6y - 6z = 0 = -11.(x-5) - 22.(y+2) - 11z$$

$$\Rightarrow 18x - 6y - 6z = 0 = -11x + 55 - 22y - 44 - 11z$$

$$\Rightarrow 18x - 6y - 6z = 0 = -11x - 22y - 11z + 11$$

$$\Rightarrow 3x - y - z = 0 = x + 2y + z - 1$$

This is the equation of the required line.

OTHER QUESTIONS FROM SEMESTER END EXAMINATION

1999; 2001 Q. No. 1(a)

Find the shortest distance between the lines, $\frac{x-3}{2} = \frac{y-4}{3} = \frac{z-5}{4}$ and $\frac{x-4}{3} = \frac{y-5}{4} = \frac{z-7}{5}$.

Solution: Given that,

$$\frac{x-3}{2} = \frac{y-4}{3} = \frac{z-5}{4} \quad \text{...... (i)}$$

Clearly the line (i) passes through the point $(x_1, y_1, z_1) = (3, 4, 5)$ and it has the direction ratio $(l_1, m_1, n_1) = (2, 3, 4)$.

Also, given line is,

$$\frac{x-4}{3} = \frac{y-5}{4} = \frac{z-7}{5} \quad \text{...... (ii)}$$

Clearly the line (ii) passes through the point $(x_2, y_2, z_2) = (4, 5, 7)$ and it has the direction ratio $(l_2, m_2, n_2) = (3, 4, 5)$

Let l, m, n be the direction ratio of the line which is shortest distance between (i) and (ii), is perpendicular to the lines (i) and (ii).

Then,

$$2l + 3m + 4n = 0$$
$$3l + 4m + 5n = 0$$

Solving by cross multiplication, we get

$$\frac{l}{15-16} = \frac{m}{12-10} = \frac{n}{8-9} \Rightarrow \frac{l}{-1} = \frac{m}{2} = \frac{n}{-1} = k$$

$$\Rightarrow l = -k, m = 2k, n = -k$$

We know that,

$$l^2 + m^2 + n^2 = 1 \Rightarrow (-k)^2 + (2k)^2 + (-k)^2 = 1$$
$$\Rightarrow 6k^2 = 1$$
$$\Rightarrow k = \frac{1}{\sqrt{6}} \quad \text{,taking +ve sign only.}$$

So that, $l = \frac{1}{\sqrt{6}}$, $m = -2\frac{1}{\sqrt{6}}$, $n = \frac{1}{\sqrt{6}}$

Now, the shortest distance between (i) and (ii) be

$$\text{S.D.} = (x_2 - x_1)\, l + (y_2 - y_1)\, m + (z_2 - z_1)\, n$$
$$= (4-3) \times \left(\frac{1}{\sqrt{6}}\right) + (5-4) \times \frac{-2}{\sqrt{6}} + (7-5) \times \left(\frac{1}{\sqrt{6}}\right)$$
$$= \frac{1-2+2}{\sqrt{6}} = \frac{1}{\sqrt{6}}.$$

Thus the shortest distance between the lines is $\frac{1}{\sqrt{6}}$ units.

2002 Q. No. 1(a)

Find the distance from the point (3, 4, 5) to the point where the line $\frac{x-3}{1} = \frac{y-4}{2} = \frac{z-5}{2}$ meets the plane $x + y + z = 2$.

Solution: The given line is

$$\frac{x-3}{1} = \frac{y-4}{2} = \frac{z-5}{2} = r \text{ (say)} \quad \text{............(1)}$$

So, the general point of (1) is

$$(r + 3, 2r + 4, 2r + 5) \quad \text{.........(2)}$$

Given that the line (1) meets the plane $x + y + z = 2$. Then the point (2) is the common point of (1) and (2). Therefore,

$$r + 3 + 2r + 4 + 2r + 5 = 2.$$
$$\Rightarrow 5r + 12 = 2 \Rightarrow r = -2$$

So the point of intersection of the line and the plane is

$$(-2 + 3, -4 + 4, -4 + 5) \Rightarrow (1, 0, 1)$$

Now, length of (3, 4, 5) from (1, 0, 1) is

$$d = \sqrt{(3-1)^2 + (4-0)^2 + (5-1)^2}$$
$$= \sqrt{4 + 16 + 16} = \sqrt{36} = 6.$$

2002 Q. No. 1(a)

Find the equation of the plane containing the line $\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$ and parallel to the line $\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$.

Solution: Given line is

$$\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1} \quad \text{.........(1)}$$

Clearly, the line (1) passes through he point (x_1, y_1, z_1).

Then the plane containing the line (1) passes through the point (x_1, y_1, z_1).

Now, the equation of plane through the point (x_1, y_1, z_1) is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0 \quad \text{........(2)}$$

So, $al_1 + bm_1 + cn_1 = 0$ ....(3)

Given that the line (1) is parallel to the line

$$\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2} \quad \text{..........(4)}$$

So, $al_2 + bm_2 + cn_2 = 0$ ....(5)

Solving (3) and (5) we get,

$$\frac{a}{m_1n_2 - m_2n_1} = \frac{b}{n_1l_2 - n_2l_1} = \frac{c}{l_1m_2 - l_2m_1}$$

Thus (2) becomes,

$$(m_1n_2 - m_2n_1)(x - x_1) + (n_1l_2 - n_2l_1)(y - y_1) + (l_1m_2 - l_2m_1)(z - z_1) = 0$$

This is the equation of required plane that contains (1) and is parallel to (2).

2002 Q. No. 1(a) OR

Find shortest distance between the lines $\frac{x-1}{2} = \frac{y-4}{3} = \frac{z-5}{4}$ and $\frac{x-4}{3} = \frac{y-5}{4} = \frac{z-6}{5}$. Also find the equation of the shortest distance.

Solution: Given that,

$$\frac{x-1}{2} = \frac{y-4}{3} = \frac{z-5}{4} \qquad \ldots\ldots (i)$$

Clearly the line (i) passes through the point $(x_1, y_1, z_1) = (1, 4, 5)$ and it has the direction ratio $(l_1, m_1, n_1) = (2, 3, 4)$.

Also, given line is,

$$\frac{x-4}{3} = \frac{y-5}{4} = \frac{z-6}{5} \qquad \ldots\ldots (ii)$$

Clearly the line (ii) passes through the point $(x_2, y_2, z_2) = (4, 5, 6)$ and it has the direction ratio $(l_2, m_2, n_2) = (3, 4, 5)$

Let l, m, n be the direction ratio of the line which is shortest distance between (i) and (ii), is perpendicular to the lines (i) and (ii).

Then,

$$2l + 3m + 4n = 0$$
$$3l + 4m + 5n = 0$$

Solving by cross multiplication, we get

$$\frac{l}{15-16} = \frac{m}{12-10} = \frac{n}{8-9} \Rightarrow \frac{l}{-1} = \frac{m}{2} = \frac{n}{-1} = k$$

$$\Rightarrow \quad l = -k,\ m = 2k,\ n = -k$$

We know that,

$$l^2 + m^2 + n^2 = 1 \ \Rightarrow\ (-k)^2 + (2k)^2 + (-k)^2 = 1$$
$$\Rightarrow\ 6k^2 = 1$$
$$\Rightarrow\ k = \frac{1}{\sqrt{6}} \qquad \text{,taking +ve sign only.}$$

So that, $\quad l = \frac{1}{\sqrt{6}}, \qquad m = -2\frac{1}{\sqrt{6}}, \qquad n = \frac{1}{\sqrt{6}}$

Now, the shortest distance between (i) and (ii) be

$$S.D. = (x_2 - x_1)\,l + (y_2 - y_1)\,m + (z_2 - z_1)\,n$$

$$= (4-1)\times\left(\frac{1}{\sqrt{6}}\right) + (5-4)\times\frac{-2}{\sqrt{6}} + (6-5)\times\left(\frac{1}{\sqrt{6}}\right)$$

$$= \frac{3+1+1}{\sqrt{6}} = \frac{5}{\sqrt{6}}.$$

Thus the shortest distance between the lines is $\frac{5}{\sqrt{6}}$ units.

Next for equation of shortest distance,

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix} = 0 = \begin{vmatrix} x - x_2 & y - y_2 & z - z_2 \\ l_2 & m_2 & n_2 \\ l & m & n \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} x-1 & y-4 & z-5 \\ 2 & 3 & 4 \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{vmatrix} = 0 = \begin{vmatrix} x-4 & y-5 & z-6 \\ 3 & 4 & 5 \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} x-1 & y-4 & z-5 \\ 2 & 3 & 4 \\ 1 & -2 & 1 \end{vmatrix} = 0 = \begin{vmatrix} x-4 & y-5 & z-6 \\ 3 & 4 & 5 \\ 1 & -2 & 1 \end{vmatrix}$$

$\Rightarrow \quad (x-1)(3+8) - (y-4)(2-4) + (z-5)(-4-3) = 0 = (x-4)(4+10) - (y-5)(3-5) + (z-6)(-6-4).$

$\Rightarrow \quad (x-1)11 - (y-4)(-2) + (z-5)(-7) = 0 = (x-4)14 - (y-5)(-2) + (z-6)(-10)$

$\Rightarrow \quad 11x + 2y - 7z - 11 - 8 + 35 = 0 = 14x + 2y - 10z - 56 - 10 + 60.$

$\Rightarrow \quad 11x + 2y - 7z + 16 = 0 = 14x + 2y - 10z - 6.$

This is he equation of required line.

Similar Question for Practice from Final Exam:

2003 Fall Q. No. 1(a) OR

Find the magnitude and the equation of S.D. between $\frac{x}{2} = \frac{y}{-3} = \frac{z}{1}$ and $\frac{x-2}{3} = \frac{y-1}{-5} = \frac{z+2}{2}$

2007 Fall Q. No. 1(a) OR

Find the shortest distance between the lines $\frac{x-3}{2} = \frac{y-4}{-3} = \frac{z-5}{4}$ and $\frac{x-4}{3} = \frac{y-5}{4} = \frac{z-7}{5}$. Also find the equation of shortest distance.

2009 Spring Q. No. 1(a) OR

Find the shortest distance between the lines $x = y + 4 = \frac{z}{3}$ and $\frac{x-1}{3} = \frac{y}{2} = z$. Find also the equation of shortest distance.

2010 Spring Q. No. 1(a) OR

Find the shortest distance between the lines, $\frac{x-5}{3} = \frac{7-y}{16} = \frac{z-3}{7}$ and $\frac{x-9}{3} = \frac{y-13}{3} = \frac{15-z}{5}$. Also find the equation of shortest distance.

OTHER QUESTIONS

2011 Fall Q. No. 1(a) OR

Define shortest distance between two skew lines in space. Find the length and equation of shortest distance between the lines $\frac{x}{2} = \frac{y}{-3} = \frac{z}{1}$ and $\frac{x-2}{3} = \frac{y-1}{-5} = \frac{z+2}{2}$.

Solution: See definition of S.D. and for second part process as above.

2004 Fall Q. No. 1(a)

Find the equation of the plane through the points (1, 0, -1) and (3, 2, 2) and parallel to the line: $x - 1 = \frac{1-y}{2} = \frac{z-2}{3}$.

Solution: Given line is

$$\frac{x-1}{1} = \frac{y-1}{-2} = \frac{z-2}{3} \quad \text{.........(1)}$$

Since the equation of plane through (1, 0, -1) is

$$a(x-1) + b(y-0) + c(z+1) = 0 \quad \text{........(2)}$$

Given that the plane also passes through (3, 2, 2). Then (2) gives,

$$a(2-1) + 2b + c(2+1) = 0$$

$$\Rightarrow a + 2b + 3c = 0 \quad \text{.........(3)}$$

Also, given that the plane (2) is parallel to the line (1). So,

$$a - 2b + 3c = 0 \quad \text{...........(4)}$$

Solving the equations (3) and (4) then,

$$\frac{a}{6+6} = \frac{b}{3-3} = \frac{c}{-2-2}$$

$$\Rightarrow \frac{a}{12} = \frac{b}{0} = \frac{c}{-4}$$

Then (2) becomes,

$$12(x-1) + 0(y-0) - 4(z+1) = 0$$

$$\Rightarrow 12x - 4z - 16\ 0$$

$$\Rightarrow 3x - z - 4 = 0$$

This is the equation of required plane.

2006 Spring Q. No. 1(a) OR

Find the length of the shortest distance between the lines $\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$; $2x - 3y + 27 = 0, 2y - z + 20 = 0$.

Solution: Given lines are

$$\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1} \quad \text{........(1)}$$

$$\text{and } 2x - 3y + 27 = 0 = 2y - z + 20 \quad \text{.......(2)}$$

Then the equation of plane through (2) is

$$(2x - 3y + 28) + \lambda(2y - y + 2) = 0 \quad \text{.......(3)}$$

Since the lines (1) and (2) are parallel to each other, otherwise they will meet at a point that gives the length of shortest distance is zero which is impossible. So, we should have the line (i) is parallel to the line (2). Therefore, (1) is perpendicular to the plane (3). So that, by using condition of perpendicularity.

$$2(3) + (2\lambda - 3)(-1) + (-\lambda) 1 = 0$$

$$\Rightarrow 6 - 2\lambda + 3 - \lambda = 0 \Rightarrow -3\lambda + 9 = 0 \Rightarrow \lambda = 3$$

Then (3) gives us

$$(2x - 3y + 27) + 3(2y - z + 2) = 0$$

$$\Rightarrow 2x + 3y - 3z + 87 = 0 \quad \text{.........(4)}$$

Clearly the line (1) passes through the point (3, 8, 3). Now, the length of the line of shortest distance from (1) to the line (2) is same as the distance from (3, 8, 3) to the plane (4). So, the perpendicular distance from (3, 8, 3) to (4) is,

$$d = \frac{2(3) + 3(8) - 3(3) + 87}{\sqrt{4+9+9}}$$

$$= \frac{6 + 18 - 9 + 87}{\sqrt{22}} = \frac{108}{\sqrt{22}}$$

2006 Fall Q. No. 1(a) OR

Prove that the lines $\frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8}$ and $\frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7}$ intersect. Find also their point of intersection and plane through them.

Solution: Given lines are

$$\frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8} = r \text{ (say)} \quad \text{........(1)}$$

$$\text{and } \frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7} = r \text{ (say)} \quad \text{........(2)}$$

Then the general point of (1) is $(2r + 1, -3r - 1, 8r - 10)$ and of (2) is, $(r' + 4, -4r' - 3, 7r' - 1)$.

If the lines are intersect then they should have at least a common point. So,

$$2r + 1 = r' + 4, \qquad -3r - 1 = -4r' - 3 \qquad 8r - 10 = 7r' - 1$$

$$\Rightarrow r' = 2r - 3, \qquad \Rightarrow 3r = 4r' + 2 \qquad \Rightarrow 8r = 7r' + 9$$

Solving first two equations, we get

$$r = 2 \quad \text{and} \quad r' = 1.$$

Then the third equations becomes

$$8(2) = 7(1) + 9 \Rightarrow 16 = 7 + 9$$

$$\Rightarrow 16 = 16$$

This proves that (i) and (ii) intersect each other.

And, the general point of (i) with r = 2, we get, (5, -7, 6).

This is the point of contact of (1) and (2).

Also, the equation of plane containing the line (1) and (2) is

$$\begin{vmatrix} x-1 & y+1 & z+10 \\ 2 & -3 & 8 \\ 1 & -4 & 7 \end{vmatrix} = 0$$

$$\Rightarrow (x-1)(-21+32) - (y+1)(14-8) + (z+10)(-8+3) = 0$$

$$\Rightarrow 11(x-1) - 6(y+1) - 5(z+10) = 0$$

$$\Rightarrow 11x - 6y - 5z - 67 = 0$$

This is the equation of required plane.

SHORT QUESTIONS

2003 Fall: Write down the equations of the line through (2, 1, 3) and (4, 2, 4).

Solution: Since we have the equation of line through (x_1, y_1, z_1) and (x_2, y_2, z_2) be

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}.$$

So, the equation of line through (2, 1, 3) and (4, 2, 4) be

$$\frac{x-2}{4-2} = \frac{y-1}{2-1} = \frac{z-3}{4-3}$$

$$\Rightarrow \frac{x-2}{2} = y - 1 = z - 3$$

2010 Spring: Find the equation of the line through (1, 2, 3) and normal to the plane $2x + 3y - z = 4$.

Solution: Since we have the equation of line through (x_1, y_1, z_1) and is normal to the plane $ax + by + cz + d = 0$ be

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

So, the equation of line through (1, 2, 3) and normal to the plane $2x + 3y - z = 4$ be

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{-1}$$

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