# Exercise 8.5

**A. Calculate the following convolutions by integrating:**

**(1) 1 * 1.**

**Solution:** Let f(t) * g(t) = 1 * 1. So, f(t) = 1, g (t) = 1.

Then the convolution of 1*1 is,

$$1*1 = f*g = \int_0^t f(t-u)\, g(u)\, du$$

$$= \int_0^t 1.1\, du = \int_0^t du = [u]_0^t = t.$$

Thus, 1*1 = t.

**(2) $e^t * e^{-t}$.**

**Solution:** Let f(t) * g(t) = $e^t * e^{-t}$. So, f(t) = $e^t$, g(t) = $e^{-t}$.

Then the convolution of $e^t * e^{-t}$ is

$$e^t * e^{-t} = f(t) * g(t) = \int_0^t f(t-u)\, g(u)\, du$$

$$= \int_0^t e^{t-u}\, e^{-u}\, du$$

$$= \int_0^t e^{t-2u}\,du = e^t \int_0^t e^{-2u}\,du$$

$$= e^t\left[\frac{e^{-2u}}{-2}\right]_0^t = \frac{e^t}{-2}(e^{-2t}-1) = \frac{e^t - e^{-t}}{2} = \sinh t$$

Thus, $e^t * e^{-t} = \sinh t$.

**3. sin wt * cos wt**

**Solution:** Let $f(t) * g(t) = \sin wt * \cos wt$. Then $f(t) = \sin wt$, $g(t) = \cos wt$.

Then,

$$f(t) * g(t) = \int_0^t f(t-u)\,g(u)\,du$$

$$= \int_0^t \sin w(t-u) \cos wu\,du$$

$$= \int_0^t (\sin wt \cos wu - \sin wu \cos wt) \cos wu\,du$$

$$= \sin wt \int_0^t \cos^2 wu\,du - \cos wt \int_0^t \sin wu \cos wu\,du$$

$$= \sin wt \int_0^t \left(\frac{1+\cos 2wu}{2}\right) du - \cos wt \int_0^t \frac{\sin 2wu}{2}\,du$$

$$= \frac{\sin wt}{2}\left[u + \frac{\sin 2wu}{2w}\right]_0^t - \frac{\cos wt}{2}\left[-\frac{\cos 2wu}{2w}\right]_0^t$$

$$= \frac{\sin wt}{2}\left(t + \frac{\sin 2wt}{2w}\right) + \frac{\cos wt}{4w}(\cos 2wt - 1)$$

$$= \frac{t}{2}\sin wt + \frac{1}{4w}\left(\sin wt \sin 2wt + \cos wt \cos 2wt - \frac{\cos 2t}{4w}\right)$$

$$= \frac{t}{2}\sin wt - \frac{\cos wt}{4w} + \frac{1}{4w}\cos(2wt - wt)$$

$$= \frac{t}{2}\sin wt - \frac{\cos wt}{4w} + \frac{\cos wt}{4w} = \frac{t}{2}\sin wt$$

Thus, $\sin wt * \cos wt = \frac{t}{2}\sin wt$.

**4. $t*e^t$.**

**Solution:** Let $f(t) * g(t) = t*e^t$. Then $f(t) = t$, $g(t) = e^t$

Now,

$$t*e^t = f(t) * g(t) = \int_0^t f(t-u)\,g(u)\,du$$



$$= \int_0^t (t-u)\,e^u\,du$$

$$= t\int_0^t e^u\,du - \int_0^t u\,e^u\,du$$

$$= t\,[e^u]_0^t - [ue^u - (1)\,e^u]_0^t$$

[∵ applying successive integration for 2nd integral]

$$= t(e^t - 1) - [(te^t - e^t) - (0-1)]$$

$$= te^t - t - te^t + e^t - 1$$

$$= e^t - t - 1$$

Thus, $t*e^t = e^t - t - 1$.

**5. $u(t-3) * e^{-2t}$.**

**Solution:** Let $f(t) * g(t) = u(t-3) * e^{-2t}$. So, $f(t) = u(t-3)$, $g(t) = e^{-2t}$.

Then, $f(t) = 1$ for $t \geq 3$

Now,

$$u(t-3) * e^{-2t} = \int_0^t u(t-3)\,e^{-2(t-T)}\,dT$$

$$= \int_3^t e^{-2(t-T)}\,dT \qquad \left[\because u(t-a) = \begin{cases} 1 \text{ for } t > a \\ 0 \text{ for } t < a \end{cases}\right.$$

$$= e^{-2t}\int_3^t e^{2T}\,dT = e^{-2t}\left[\frac{e^{2T}}{2}\right]_3^t = \frac{1}{2}e^{-2t}(e^{2t} - e^6) \quad \text{for } t > 3$$

$$= \frac{1}{2}[1 - e^{-2(t-3)}] \quad \text{for } t > 3$$

$$= \frac{1}{2}[1 - e^{-2(t-3)}]\;u(t-3)$$

**6. t * 1.**

**Solution:** Let $f(t) * g(t) = t * 1$. So, $f(t) = t$, $g(t) = 1$.

Then,

$$t*1 = f(t) * g(t) = \int_0^t f(t-u)\,g(u)\,du$$

$$= \int_0^t (t-u)\,du = \left[t.u - \frac{u^2}{2}\right]_0^t = t^2 - \frac{t^2}{2} = \frac{t^2}{2}$$

Thus, $t * 1 = \frac{t^2}{2}$.

**7. sint * sint.**

**Solution:** Let $f(t) * g(t) = \sin t * \sin t$. So, $f(t) = \sin t$, $g(t) = \sin t$.

Now,

$$\sin t * \sin t = f(t) * g(t) = \int_0^t f(t-u)\, g(u)\, du$$

$$= \int_0^t \sin(t-u) \sin u\, du$$

$$= \int_0^t (\sin t \cos u - \cos t \sin u) \sin u\, du$$

$$= \sin t \int_0^t \cos u \sin u\, du - \cos t \int_0^t \sin^2 u\, du$$

$$= \sin t \int_0^t \frac{\sin 2u}{2} du - \cos t \int_0^t \left(\frac{1-\cos 2u}{2}\right) du$$

$$= \frac{\sin t}{2}\left[-\frac{\cos 2u}{2}\right]_0^t - \frac{\cos t}{2}\left[u - \frac{\sin 2u}{2}\right]_0^t$$

$$= \frac{\sin t}{4}(1-\cos 2t) - \frac{\cos t}{2}\left(t - \frac{\sin 2t}{2}\right)$$

$$= \frac{\sin t}{4} - \frac{t\cos t}{2} + \frac{1}{4}(-\sin t \cos 2t + \cos t \sin 2t)$$

$$= \frac{\sin t}{4} - \frac{t\cos t}{2} + \frac{1}{4}\sin(2t - t)$$

$$= \frac{\sin t}{4} - \frac{t\cos t}{2} + \frac{\sin t}{4}$$

$$= -\frac{t\cos t}{2} + \frac{\sin t}{2}$$

Thus, $\sin t * \sin t = -\frac{t\cos t}{2} + \frac{\sin t}{2}$.

**B. Calculate the following inverse transform by convolution:**

**(1)** $\frac{1}{s^2}$

**Solution:** Let $\mathcal{L}\{f*g\} = \frac{1}{s^2} = \frac{1}{s} \cdot \frac{1}{s}$



Since by the table, $\mathcal{L}\{1\} = \frac{1}{s}$

So,

$$\mathcal{L}\{f*g\} = \mathcal{L}\{1\} \cdot \mathcal{L}\{1\}$$

Now,

$$\mathcal{L}^{-1}\{\mathcal{L}\{1\} \cdot \{1\}\} = (f*g)(t) = \int_0^t f(t-u)\, g(u)\, du = \int_0^t 1\, du = t.$$

**2.** $\frac{1}{s(s^2+4)}$

**Solution:** Let $\mathcal{L}\{f*g\} = \frac{1}{s(s^2+4)} = \frac{1}{s} \cdot \frac{1}{s^2+4}$

Since, $\mathcal{L}\{1\} = \frac{1}{s}$ and $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$

So, $\mathcal{L}\{f*g\} = \mathcal{L}\{1\} \cdot \frac{1}{2}\mathcal{L}\{\sin 2t\} = \mathcal{L}\{1\} \cdot \mathcal{L}\left\{\frac{\sin 2t}{2}\right\}$

Now,

$$\mathcal{L}^{-1}\left\{\mathcal{L}\{1\} \cdot \mathcal{L}\left\{\frac{\sin 2t}{2}\right\}\right\} = (f*g)(t)$$

$$= \int_0^t f(t-u)\, g(u)\, du$$

$$= \int_0^t 1 \cdot \frac{\sin 2u}{2} du = \frac{1}{2}\left[-\frac{\cos 2u}{2}\right]_0^t = \frac{1}{4}(1-\cos 2t)$$

**3.** $\frac{1}{(s^2+1)^2}$

**Solution:** Let $\mathcal{L}\{f*g\} = \frac{1}{(s^2+1)^2} = \frac{1}{(s^2+1)} \cdot \frac{1}{(s^2+1)}$

Since we have, $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$

So, $\mathcal{L}\{f*g\} = \mathcal{L}\{\sin t\} \cdot \mathcal{L}\{\sin t\}$

Now,

$$\mathcal{L}\{\mathcal{L}\{\sin t\} \cdot \mathcal{L}\{\sin t\}\} = (f*g)(t)$$

$$= \int_0^t f(t-u)\, g(u)\, du$$

$$= \int_0^t \sin(t-u) \sin u\, du$$

$$= \int_0^t (\sin t \cos u - \cos t \sin u) \sin u \, du$$

$$= \sin t \int_0^t \cos u \sin u \, du - \cos t \int_0^t \sin^2 u \, du$$

$$= \frac{\sin t}{2} \int_0^t \sin 2u \, du - \frac{\cos t}{2} \int_0^t (1 - \cos 2u) \, du$$

$$= \frac{\sin t}{2}\left[\frac{-\cos 2u}{2}\right]_0^t - \frac{\cos t}{2}\left[u - \frac{\sin 2u}{2}\right]_0^t$$

$$= \frac{\sin t}{4}(1 - \cos 2t) - \frac{\cos t}{2}\left(t - \frac{\sin 2t}{2}\right)$$

$$= \frac{\sin t}{4} - \frac{t\cos t}{2} + \frac{1}{4}(-\sin t \cos 2t + \cos t \sin 2t)$$

$$= \frac{\sin t}{4} - \frac{t\cos t}{2} + \frac{1}{4}\sin(2t - t)$$

$$= \frac{\sin t}{4} - \frac{t\cos t}{2} + \frac{\sin t}{4}$$

$$= -\frac{t\cos t}{2} + \frac{\sin t}{2}.$$

**4.** $\frac{5}{(s^2+\pi^2)}$

**Solution:** Let $\mathcal{L}\{f*g\} = \frac{5}{(s^2+\pi^2)} = 5\left(\frac{1}{s^2+\pi^2}\right)\left(\frac{1}{s^2+\pi^2}\right)$

Since we have, $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$

So,

$$\mathcal{L}\{f*g\} = \frac{5}{\pi^2}\mathcal{L}\{\sin \pi t\}\,\mathcal{L}\{\sin \pi t\} = \frac{\pi^2}{5}\int_0^t f(t-u)\,g(u)\,du$$

$$= \frac{\pi^2}{5}\int_0^t \sin(t-u)\pi \sin \pi u \, du \quad \text{.........(1)}$$

Here,

$$\sin(t-u)\pi \sin \pi u = (\sin \pi t \cos \pi u - \sin \pi u \cos \pi t)\,.\,\sin \pi u$$

$$= \sin \pi t \sin \pi u \cos \pi u - \cos \pi t \sin^2 \pi u$$

$$= \sin \pi t\,.\,\frac{\sin 2\pi u}{2} - \cos \pi t\left(\frac{1 - \cos 2\pi u}{2}\right)$$



$$= \frac{\sin \pi t}{2}\,.\,\sin 2\pi u - \frac{\cos \pi t}{2}(1 - \cos 2\pi u)$$

Then (1) gives,

$$= \frac{\pi^2}{5}\left[\frac{\sin \pi t}{2}\int_0^t \sin 2\pi u \, du - \frac{\cos \pi t}{2}\int_0^t (1 - \cos 2\pi u)\, du\right]$$

$$= \frac{\pi^2}{5}\left[\frac{\sin \pi t}{2}\left(-\frac{\cos 2\pi u}{2\pi}\right)_0^t - \frac{\cos \pi t}{2}\left(u - \frac{\sin 2\pi u}{2\pi}\right)_0^t\right]$$

$$= \frac{\pi^2}{5}\left[\frac{\sin \pi t}{4\pi}(1 - \cos 2\pi t) - \frac{\cos \pi t}{2}\left(t - \frac{\sin 2\pi t}{2\pi}\right)\right]$$

$$= \frac{\pi^2}{5}\left[\frac{\sin \pi t}{4\pi} - \frac{t\cos \pi t}{2} + \frac{1}{4\pi}(-\sin \pi t \cos 2\pi t + \cos \pi t \sin 2\pi t)\right]$$

$$= \frac{\pi^2}{5}\left[\frac{\sin \pi t}{4\pi} - \frac{t\cos \pi t}{2} + \frac{\sin \pi t}{4\pi}\right]$$

$$= \frac{\pi^2}{5}\left[\frac{\sin \pi t}{2\pi} - \frac{t\cos \pi t}{2}\right]$$

Thus,

$$\mathcal{L}^{-1}\left\{\frac{5}{(s^2+\pi^2)^2}\right\} = \frac{\sin \pi t}{2\pi} - \frac{t\cos \pi t}{2}.$$

**5.** $\frac{w}{s^2(s^2+w^2)}$

**Solution:** Let $\mathcal{L}\{f*g\} = \frac{w}{s^2(s^2+w^2)} = \frac{1}{s^2}\cdot\frac{w}{s^2+w^2}$

Since, $\mathcal{L}\{t\} = \frac{1}{s^2}$ and $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$

So,

$$\mathcal{L}\{f*g\} = \mathcal{L}\{t\}\,\mathcal{L}\{\sin wt\}$$

Then,

$$\mathcal{L}^{-1}\{\mathcal{L}\{t\}\,\mathcal{L}\{\sin wt\}\} = (f*g)(t)$$

$$= \int_0^t f(t-u)\,g(u)\,du$$

$$= \int_0^t (t-u)\sin wu\,du$$

$$= \left[(t-u)\left(-\frac{\cos wu}{w}\right) - (-1)\left(-\frac{\sin wu}{w^2}\right)\right]_0^t$$

[∵ applying successive integration]

$$= \left(0 - \frac{\sin wt}{w^2}\right) - \left(-\frac{t}{w} - 0\right)$$

$$= \frac{wt - \sin wt}{w^2}$$

Thus,

$$\mathcal{L}^{-1}\left\{\frac{w}{s^2(s^2+w^2)}\right\} = \frac{wt - \sin wt}{w^2}.$$

## OTHER QUESTIONS FROM SEMESTER END EXAMINATION

**1999-Q. No. 6(a)(OR); 2001 Q. No. 6a(OR)**

**Define unit step function $u_a(t)$ and then find the Laplace transformation of**

$$f(t), \text{ where } f(t) = \begin{bmatrix} 1 & \text{if} & 0 < t < \pi \\ 0 & \text{if} & \pi < t < 2\pi \\ \sin & \text{if} & t > 2\pi \end{bmatrix}$$

**Solution: First Part:** See definition p.

**Second Part:** Given that

$$f(t) = \begin{cases} 1 & \text{for } 0 < t < \pi \\ 0 & \text{for } \pi < t < 2\pi \\ \sin t & \text{for } t > 2\pi \end{cases} \qquad \text{......(i)}$$

Then,

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt = \left(\int_0^\pi + \int_\pi^{2\pi} + \int_{2\pi}^\infty\right) f(t)\, e^{-st}\, dt$$

$$= \int_0^\pi e^{-st}\, dt + 0 + \int_{2\pi}^\infty \sin t\, e^{-st}\, dt \quad [\because \text{using (i)}]$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\pi + \left[\frac{e^{-st}}{s^2+1}\{(-s)\sin t - \cos t\}\right]_{2\pi}^\infty$$

$$= \frac{1 - e^{-\pi s}}{s} + \frac{e^{-2\pi s}}{s^2+1}\cos 2\pi \quad [\because e^{-\infty} = 0,\ \sin 2\pi = 0]$$

Thus,

$$\mathcal{L}\{f(t)\} = \frac{1 - e^{-\pi s}}{s} + \frac{e^{-2\pi s}}{s^2+1} \qquad [\because \cos 2\pi = 1]$$

**2000 Q. No. 6(a)(OR)**

**Find (i) $\mathcal{L}$\{cosh at cosat\} and (ii) $\mathcal{L}\{e^{-2}\sin n\pi t\}$.**

**Solution:**

(i) Let $f(t) = \cosh at \cos at = \frac{e^{at} + e^{-at}}{2} \cdot \cos at = \frac{1}{2}[e^{at}\cos at + e^{-at}\cos at]$

Since we have,



$$\mathcal{L}\{e^{at} f(t)\} = (\mathcal{L}\{f(t)\})_{s \to s-a} \quad \text{and} \quad \mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$$

Now,

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\cos hat \cos at\} = \frac{1}{2}[\mathcal{L}\{e^{at}\cos at\} + \mathcal{L}\{e^{-at}\cos at\}]$$

$$= \frac{1}{2}[[\mathcal{L}\{\cos at\}]_{s \to s-a} + [\mathcal{L}\{\cos at\}]_{s \to s+a}]$$

$$= \frac{1}{2}\left[\left(\frac{s}{s^2+a^2}\right)_{s \to s-a} + \left(\frac{s}{s^2+a^2}\right)_{s \to s+a}\right]$$

$$= \frac{1}{2}\left[\frac{s-a}{(s-a)^2+a^2} + \frac{s+a}{(s+a)^2+a^2}\right]$$

(ii) Given that, $f(t) = e^{-2t}\sin n\pi t$

Since we have,

$$\mathcal{L}\{e^{at} f(t)\} = [\mathcal{L}\{f(t)\}]_{s \to s-1} \quad \text{and} \quad \mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$$

Now,

$$\mathcal{L}f(t)\} = \mathcal{L}\{e^{-2t}\sin n\pi t\} = (\mathcal{L}\{\sin n\pi t\})_{s \to s+2}$$

$$= \left(\frac{n\pi}{s^2+n^2\pi^2}\right)_{s \to s+2} = \frac{n\pi}{(s+2)^2+n^2\pi^2}$$

**2002 Q. No. 6(a) OR**

**Prove the following: $L(t\cosh at) = \frac{s^2+a^2}{(s^2-a^2)^2}$**

**Solution:** We have find the Laplace transform of t cosh at.

Since we have, $\cosh at = \frac{e^{at} + e^{-at}}{2}$

$$\mathcal{L}\{e^{at} f(t)\} = (\mathcal{L}\{f(t)\})_{s \to s-a} \quad \text{and} \quad \mathcal{L}\{1\} = \frac{1}{s}$$

Now,

$$\mathcal{L}\{t\cosh at\} = \frac{1}{2}[\mathcal{L}\{te^{at}\} + \mathcal{L}\{te^{-at}\}]$$

$$= \frac{1}{2}[(\mathcal{L}\{t\})_{s \to s-a} + (\mathcal{L}\{t\})_{s \to s+a}]$$

$$= \frac{1}{2}\left[\left(\frac{1}{s}\right)_{s \to s-a} + \left(\frac{1}{s}\right)_{s \to s+a}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] = \frac{1}{2}\left(\frac{2s}{s^2-a^2}\right) = \frac{s}{s^2-a^2}$$

**2002 Q. No. 6(a)**

**If f(t) is continuous for $t \geq 0$ and L\{f(t)\} exists for some k and M. $|f(t)| \leq Me^{-kt}$ for $t \geq 0$ and $f'(t)$ is piece wise continuous on every finite interval in the range $t \geq 0$ then show that the Laplace transform of $f'(t)$ exists when $s > k$ and hence prove that following $L\{f'(t)\} = sL\{f(t)\} - f(0)$.**

**Solution:** See theorem in theoretical part.

**2002 Q. No. 6(a)**

**If $f(t) = \sin^2 t$, find Laplace transform of f(t).**

**Solution:** Let $f(t) = \sin^2 t = \frac{1 - \cos 2t}{2}$

Since we have,

$$\mathcal{L}\{1\} = \frac{1}{s} \quad \text{and} \quad \mathcal{L}\{\cos at\} = \frac{s}{s^2 + a^2}$$

Now,

$$\mathcal{L}\{f(t)\} = \frac{1}{2}\{\mathcal{L}\{1\} - \mathcal{L}\{\cos 2t\}\}$$

$$= \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right) = \frac{1}{2}\left(\frac{s^2 + 4 - s^2}{s(s^2+4)}\right) = \frac{2}{s(s^2+4)}$$

Thus, $\mathcal{L}\{\text{Sin}^2 t\} = \frac{2}{s(s^2+4)}$

**2003 Fall Q. No. 6(a)**

**Find the Laplace inverse of: (i) $\frac{4}{s^2 - 2s - 3}$ (ii) $\frac{e^{-3s}}{(s-1)^3}$**

**Solution:**

(i) Let. $\mathcal{L}\{f(t)\} = \frac{4}{s^2 - 2s - 3}$ .......(i)

Here,

$$\frac{4}{s^2 - 2s - 3} = \frac{4}{(s-3)(s+1)}$$

$$= \frac{A}{s-3} + \frac{B}{s+1} = \frac{A(s+1) + B(s-3)}{(s-3)(s+1)} = \frac{(A+B)s + (A - 3B)}{(s-3)(s+1)}$$

This gives, $4 = (A + B)s + (A - 3B)$

Comparing the like terms from both sides then,

$A + B + 0$ and $A - 3B = 4$

Solving we get, $A = -1, B = 1$

Then (i) becomes,

$$\mathcal{L}\{f(t)\} = -\frac{1}{s-3} + \frac{1}{s+1} \quad \text{......... (ii)}$$

Since we have, $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$. So, (ii) gives,

$$\mathcal{L}\{f(t)\} = -\mathcal{L}\{e^{3t}\} + \mathcal{L}\{e^{-t}\} = \mathcal{L}\{e^{-t} - e^{3t}\}$$

**Thus,**

$$f(t) = e^{-t} - e^{3t}$$

**(ii)** Let $\mathcal{L}\{f(t)\} = \frac{e^{-3s}}{(s-1)^3}$

Since we have,

$$\mathcal{L}\{e^{at} f(t)\} = (L\{f(t)\})_{s \to s-a} \quad \text{........(i)}$$



$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \quad \text{.......(ii)}$$

$$\text{and } \mathcal{L}\{f(t-a)\, u_a(t)\} = e^{-as}\, \mathcal{L}\{f(t)\} \quad \text{........(iii)}$$

Now,

$$\frac{e^{-3s}}{(s-1)^3} = e^{-3s} \cdot \frac{1}{(s-1)^3}$$

$$\Rightarrow \mathcal{L}\{f(t)\} = e^{-3s}\left(\frac{1}{s^3}\right)_{s \to s-1} = e^{-3s}\left(L\left\{\frac{t^2}{2!}\right\}\right)_{s \to s-1} \quad [\because \text{using (2)}]$$

$$= e^{-3s}\, \mathcal{L}\left\{\frac{e^t t^2}{2}\right\} \quad [\because \text{using (1)}]$$

$$= \frac{1}{2}\mathcal{L}\{e^{(t-3)}(t-3)^2 u_3(t)\} \quad [\because \text{using (3)}]$$

$$= \mathcal{L}\left\{\frac{1}{2} e^{(t-3)}(t-3)^2 u_3(t)\right\}$$

Thus, $f(t) = \frac{1}{2}\{e^{(t-3)}(t-3)^2 u_3(t)\}$.

**2004 Spring Q. No. 6(a)**

**Define Laplace Transform. State and prove first-shifting theorem of Laplace transform. Use it to prove: $L\{e^t.t\} = \frac{1}{(s-1)^2}$**

**Solution: First Part see definition, and the theorem.**

**Problem Part:** Given function is, $te^t$

Since we have,

$$\mathcal{L}\{e^{at} f(t)\} = (\mathcal{L}\{f(t)\})_{s \to s-a} \text{ and } \quad \mathcal{L}\{t\} = \frac{1}{s^2}$$

Now,

$$\mathcal{L}\{te^t\} = (\mathcal{L}\{t\})_{s \to s-1} = \left(\frac{1}{s^2}\right)_{s \to s-1} = \frac{1}{(s-1)^2}$$

**2004 Spring Q. No. 6(b), 2006 Spring Q. No. 6(b)**

**Find f(t) if $L\{f(t) = F(s) = \log\left(\frac{s+a}{s+b}\right)$.**

**Solution:** Let $\mathcal{L}\{f(t)\} = F(s) = \log\left(\frac{s+a}{s+b}\right)$

So, $F'(s) = \left(\frac{s+b}{s+a}\right) = \frac{b-a}{(s+a)(s+b)}$

Since we have,

$$\mathcal{L}\{t\, f(t)\} = -F'(s)$$

So, $\mathcal{L}\{t\, f(t)\} = \frac{a-b}{(s+a)(s+b)}$ .......(i)

Here,

$$\frac{a-b}{(s+a)(s+b)} = \frac{A}{s+a} + \frac{B}{s+b} = \frac{A(s+b) + B(s+a)}{(s+a)(s+b)}$$

$$\Rightarrow (a-b) = (A+B)s + (Ab + Ba)$$

Equating the like terms then,

$$A + B = 0 \quad \text{and} \quad Ab + Ba = a - b$$

Solving we get, $A = -1$, $B = 1$.

Then (i) becomes,

$$\mathcal{L}\{t\,f(t)\} = \frac{1}{s+b} - \frac{1}{s+a} = L\{e^{-bt}\} - L\{e^{-at}\} \qquad \left[\because L\{e^{at}\} = \frac{1}{s-a}\right]$$

$$= L\{e^{-bt} - e^{-at}\}$$

Thus, $f(t) = \dfrac{e^{-bt} - e^{-at}}{t}$

**2004 Fall Q. No. 6(a)**

**State and prove second shifting theorem of Laplace transform. Hence evaluate $L^{-1}\left\{\frac{e^{-2s}}{s^2}\right\}$.**

**Solution: First Part: See theorem.**

**Problem Part:** Given function is, $\dfrac{e^{-2s}}{s^2}$

Since we have,

$$\mathcal{L}\{f(t-a)\,u_a(t)\} = e^{-as}\,\mathcal{L}\{f(t)\} \qquad \text{....... (1)}$$

$$\text{and} \quad \mathcal{L}\{t\} = \frac{1}{s^2}. \qquad \text{....... (2)}$$

Now,

$$\frac{e^{-2s}}{s^2} = e^{-2s}\,\mathcal{L}\{t\} \qquad [\because \text{using (2)}]$$

$$= \mathcal{L}\{f(t-2)\,u_2(t)\} \qquad [\because \text{using (1)}]$$

This gives,

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\} = f(t-2)\,u_2(t)$$

**2006 Fall Q. No. 6(a); 2010 Spring Q. No. 6(a); 2011 Fall Q. No. 6(a)**

**Evaluate the following: (i) $L(t^2 \cos\omega t)$ (ii) $L^{-1}\left(\dfrac{1}{s^2(s^2+\omega^2)}\right)$**

**Solution: (i) See Exercise 8.4 – 1(iii)**

**(ii)** Given function is

$$\frac{1}{s^2(s^2+w^2)} = \frac{1}{w^2}\left[\frac{1}{s^2} - \frac{1}{s^2+w^2}\right] \qquad \text{........ (i)}$$

Since we have,

$$\mathcal{L}\{t\} = \frac{1}{s^2} \quad \text{and} \quad \mathcal{L}\{\sin wt\} = \frac{w}{s^2+w^2}$$

So (i) becomes,

$$\frac{1}{s^2(s^2+w^2)} = \frac{1}{w^2}\left[L\{t\} - \frac{1}{w}L\{\sin wt\}\right]$$

$$= \mathcal{L}\left\{\frac{1}{w^3}(wt - \sin wt)\right\}$$



Thus, $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2(s^2+w^2)}\right\} = \dfrac{1}{w^3}(wt - \sin wt)$

**2006 Spring Q. No. 6(a)**

**Find the Laplace transform of $t^2 \sin 2t$**

**Solution:** Let $f(t) = t^2 \sin 2t$.

Since we have,

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}(L\{f(t)\}) \quad \text{and} \quad \mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$$

Now,

$$\mathcal{L}\{t^2 \sin 2t\} = (-1)^2 \frac{d^2}{ds^2}(\mathcal{L}\{\sin 2t\})$$

$$= \frac{d^2}{ds^2}\left(\frac{2}{s^2+4}\right)$$

$$= \frac{d}{ds}\left(\frac{-4s}{(s^2+4)^2}\right)$$

$$= -4\left(\frac{(s^2+4)^2 - 2(s^2+4)\,2s^2}{(s^2+4)^4}\right) = -4\left(\frac{s^2+4-4s^2}{(s^2+4)^3}\right) = 4\,\frac{(3s^2-4)}{(s^2+4)^3}$$

Thus, $\mathcal{L}\{t^2 \sin 2t\} = \dfrac{4(3s^2-4)}{(s^2+4)^3}$

**2006 Spring Q. No. 6(c)**

**If $L[f(t)] = F(s)$, then show that $L[tf(t)] = -\dfrac{d}{ds}[F(s)]$. Using it evaluate $L[t^2 \cos 3t]$.**

**Solution:** First Part: See P.

Second Part: See solution of 2006 fall Q. No. 6(a(i)) with replacing $w$ by 3.

**2007 Fall Q. No. 6(b)**

**Find the Laplace transform of (i) $t^2 \cos wt$ (ii) $\cosh at \cos at$**

**Solution:** (i) See Exercise 8.4 – 1(iii). **(ii)** See question from 2000.

**2008 Fall Q. No. 6(a)**

**Define Laplace transform of any function. Find $L(e^{at}\cos wt)$ and $L(e^{at}\sin wt)$**

**Solution: See definition, P.**

**Problem Part:** Given function is $e^{at}\cos wt$ and $e^{at}\sin wt$

Since we have,

$$\mathcal{L}\{e^{at} f(t)\} = (\mathcal{L}\{f(t)\})_{s \to s-a}$$

$$\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2} \quad \text{and} \quad \mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$$

Now,

$$\mathcal{L}\{e^{at}\cos wt\} = (\mathcal{L}\{\cos wt\})_{s \to s-a}$$

$$= \left(\frac{s}{s^2+w^2}\right)_{s \to s-a} = \frac{s-a}{(s-a)^2+w^2}$$

and $\mathcal{L}\{e^{at}\sin wt\} = (\mathcal{L}\{\sin wt\})_{s\to s-a}$

$$= \left(\frac{w}{s^2+w^2}\right)_{s\to s-a} = \frac{w}{(s-a)^2+w^2}$$

**2008 Spring Q. No. 6(a)**

**Evaluate the following: (i) $\mathcal{L}(t\cos wt)$ (ii) $\mathcal{L}^{-1}\left(\frac{1}{4s+s^2}\right)$.**

Solution: (i) See question from 1999 (ii) See Exercise 8.2 – 2(vii).

**2009 Spring Q. No. 6(a)**

**Define Laplace transform. State and prove first shifting theorem of Laplace transform. Using it find Laplace transform of sinh at cosbt.**

**Solution: See Definition and see theorem.**

**Problem Part:** Given function is

$$\sinh at \cos bt = \left(\frac{e^{at}-e^{-at}}{2}\right)\cos bt = \frac{1}{2}(e^{at}\cos bt - e^{-at}\cos bt)$$

Since we have,

$$\mathcal{L}\{e^{at}f(t)\} = (\mathcal{L}\{f(t)\})_{s\to s-a} \quad \text{and} \quad \mathcal{L}\{\cos bt\} = \frac{s}{s^2+b^2}$$

Now,

$$\mathcal{L}\{\sinh at\cos bt\} = \frac{1}{2}\mathcal{L}\{e^{at}\cos bt - e^{-at}\cos bt\}$$

$$= \frac{1}{2}[\{(\mathcal{L}\{\cos bt\})_{s\to s-a} - (\mathcal{L}\{\cos bt\})_{s\to s+a}]$$

$$= \frac{1}{2}\left[\left(\frac{s}{s^2+b^2}\right)_{s\to s-a} - \left(\frac{s}{s^2+b^2}\right)_{s\to s+a}\right]$$

$$= \frac{1}{2}\left[\frac{s-a}{(s-a)^2+b^2} - \frac{s+a}{(s+a)^2+b^2}\right]$$

**2009 Spring Q. No. 6(b)**

**Define unit step function. State and prove second shifting theorem of Laplace transform. Find $\mathcal{L}^{-1}\left(\frac{e^{-2s}s}{s^2+1}\right)$.**

**Solution: See Definition, and see theorem.**

**Problem Part:** Given function is, $\frac{se^{-2s}}{s^2+1}$

Since we have,

$$\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2} \quad \text{and} \quad \mathcal{L}\{f(t-a)\,u_a(t)\} = e^{-as}L\{f(t)\}$$

Now,

$$\frac{se^{-2s}}{s^2+1} = e^{-2s}\cdot\frac{s}{s^2+1} = e^{-2s}\mathcal{L}\{\cos t\} = \mathcal{L}\{\cos(t-2)\,u_2(t)\}$$

Thus,

$$\mathcal{L}^{-1}\left\{\frac{se^{-2s}}{s^2+1}\right\} = \cos(t-2)\,u_2(t)$$



**2009 Fall Q. No. 6(a)**

**Define Laplace transform. State and prove first shifting theorem of Laplace transform. Use it to evaluate: $\mathcal{L}(e^{-\lambda t}\cos\beta t)$.**

**Solution: See Definition and see theorem.**

**Second Part:** See a part of Exercise 8.3 Q. No. 1(iv).

**2009 Fall Q. No. 6(a) OR**

**Solve: (i) $\mathcal{L}\{\sinh t\cos t\}$ (ii) $\mathcal{L}^{-1}\{\log\left(1+\frac{\omega^2}{s^2}\right)$.**

**Solution:** (i) See Exercise 8.3 -1(ix) (ii) See Exercise 8.4 – 2(iii).

## Application of Laplace transform

**2001 Q. No. 6(b); 2007 Fall Q. No. 6(a); 2008 Fall Q. No. 6(b)**

**Using the method of Laplace transformation solve the following initial value problem. $9y'' - 6y' + y = 0$, $y(0) = 3$, $y'(0) = 1$.**

**Solution:** Given equation is

$$9y'' - 6y' + y = 0 \quad \text{.......(i)}$$

$$\text{with} \quad y(0) = 3,\ y'(0) = 1 \quad \text{......(ii)}$$

Then taking Laplace transform on both sides of (i) then,

$$9\mathcal{L}\{y''\} - 6\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{0\}$$

$$\Rightarrow 9\{s^2\bar{y} - sy(0) - y'(0)\} - 6\{s\bar{y} - y(0)\} = \bar{y} = 0 \quad \text{for } \bar{y} = \mathcal{L}\{y\}$$

[$\because$ using the relation of L.T. of derivative of a function]

$$\Rightarrow 9\{s^2\bar{y} - 2s - 1\} - 6(s\bar{y} - 3) + \bar{y} = 0 \quad [\because \text{using (ii)}]$$

$$\Rightarrow (9s^2 - 6s + 1)\bar{y} = 27s - 9$$

$$\Rightarrow \bar{y} = 9\left(\frac{3s-1}{9s^2-6s+1}\right) = 9\left(\frac{3s-1}{(3s-1)^2}\right) = \frac{9}{3s-1}$$

$$\Rightarrow \bar{y} = \frac{3}{s-\frac{1}{3}} = 3\mathcal{L}\{e^{t/3}\}. \quad \left[\because L\{e^{at}\} = \frac{1}{s-a}\right]$$

$$\Rightarrow \mathcal{L}\{y\} = \mathcal{L}\{3e^{t/3}\}$$

Taking inverse Laplace transform on both sides then,

$$y = 3e^{t/3}$$

**2002 Q. No. 6(b)**

**Solve the following by the method of Laplace transformation. $y'' + 2y' + y = e^{-t}$, $y(0) = -1$, $y'(0) = 1$**

**Solution:** Given that,

$$y'' + 2y' + y = e^{-t} \quad \text{.......(i)}$$

with $y(0) = -1$ and $y'(0) = 1$ ......(ii)

Taking L.T. on (i) then,

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{e^{-t}\}$$

$$\Rightarrow [s^2\bar{y} - sy(0) - y'(0)] + 2[s\bar{y} - y(0)] + \bar{y} = \frac{1}{s+1}$$

$$\Rightarrow (s^2\bar{y} + s - 1) + 2(s\bar{y} + 1) + \bar{y} = \frac{1}{s+1}$$

$$\Rightarrow (s^2 + 2s + 1)\bar{y} - s + 1 = \frac{1}{s+1}$$

$$\Rightarrow (s^2 + 2s + 1)\bar{y} = \frac{1}{s+1} - (s+1) = -\frac{s^2 + 2s}{s+1}$$

$$\Rightarrow (s+1)^2\bar{y} = -\frac{s^2 + 2s}{s+1}$$

$$\Rightarrow \bar{y} = -\frac{s^2 + 2s}{(s+1)^3} \quad \text{.........(iii)}$$

Here,

$$\frac{s^2 + 2s}{(s+1)^3} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)^2} \quad \text{.......(iv)}$$

$$= \frac{A(s+1)^2 + B(s+1) + C}{(s+1)^3}$$

$$\Rightarrow s^2 + 2s = A^2 + (2A + B)s + (A + B + C)$$

Equating the like terms then,

$A = 1$, $2A + B = 2$ and $A + B + C = 0$

Solving we get, $A = 1$, $B = 0$ and $C = -1$

Then (iv) becomes,

$$\frac{s^2 + 2s}{(s+1)^3} = \frac{1}{s+1} - \frac{1}{(s+1)^3}$$

$$= \mathcal{L}\{e^{-t}\} - \left(\frac{1}{s^3}\right)_{s\to s+1} \qquad \left[\because L\{e^{at}\} = \frac{1}{s-a}\right]$$

$$= \mathcal{L}\{e^{-t}\} - \left(L\left\{\frac{t^2}{2!}\right\}\right)_{s\to s+1} \qquad \left[\because L\{t^n\} = \frac{n!}{s^{n+1}}\right]$$

$$= \mathcal{L}\{e^{-t}\} - \mathcal{L}\left\{\frac{t^2 e^{-t}}{2}\right\} \qquad [\because \mathcal{L}\{e^{at} f(t)\} = (\mathcal{L}\{f(t)\})_{s\to s-a}]$$

$$= \mathcal{L}\left\{e^{-t}\left(1 - \frac{t^2}{2}\right)\right\}$$

Therefore, (iii) becomes,

$$\mathcal{L}\{y\} = -\mathcal{L}\left\{e^{-t}\left(1 - \frac{t^2}{2}\right)\right\}$$

$$\Rightarrow \quad y = e^{-t}\left(\frac{t^2}{2} - 1\right)$$



**2 Q. No. 5(b)**

**Solution:** Given that,

$$y'' + 2y' - 3y = 6e^{-2t} \quad \text{.........(i)}$$

$$y(0) = 2, \quad y'(0) = -14 \quad \text{........(ii)}$$

Taking Laplace transform on both sides of (i) then,

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} - 3\mathcal{L}\{y\} = 6\mathcal{L}\{e^{-2t}\}$$

$$\Rightarrow s^2\mathcal{L}\{y\} - sy(0) - y'(0) + 2[s\mathcal{L}\{y\} - y(0)] - 3\mathcal{L}\{y\} = 6 \cdot \frac{1}{s+2}$$

$$\Rightarrow (s^2 + 2s - 3)\mathcal{L}\{y\} - 2s + 14 - 4 = \frac{6}{s+2} \qquad [\because \text{using (ii)}]$$

$$\Rightarrow (s^2 + 2s - 3)\mathcal{L}\{y\} = \frac{6}{s+2} + 2s - 10 = \frac{2s^2 - 6s - 14}{s+2}$$

$$\Rightarrow \mathcal{L}\{y\} = 2 \cdot \frac{s^2 - 3s - 7}{(s+2)(s^2 + 2s - 3)} \quad \text{.......(iii)}$$

Here,

$$\frac{s^2 - 3s - 7}{(s+2)(s^2 + 2s - 3)} = \frac{s^2 - 3s - 7}{(s+2)(s+3)(s-1)}$$

$$= \frac{A}{s+2} + \frac{B}{s+3} + \frac{C}{s-1}$$

$$= \frac{A(s+3)(s-1) + B(s+2)(s-1) + C(s+2)(s+3)}{(s+2)(s+3)(s-1)}$$

$$\Rightarrow s^2 - 3s - 7 = A(s+3)(s-1) + B(s+2)(s-1) + C(s+2)(s+3)$$

$$= A(s^2 + 2s - 3) + B(s^2 + s - 2) + C(s^2 + 5s + 6)$$

Equating the like terms in both sides then

$A + B + C = 1$, $2A + B + 5C = -3$, $-3A - 2B + 6C = -7$

Solving we get,

$$A = \frac{8}{3}, \; B = -\frac{1}{2}, \; C = -\frac{7}{6}$$

Then (iii) becomes,

$$\mathcal{L}\{y\} = 2\left(\frac{8}{3} \cdot \frac{1}{s+2} - \frac{1}{2} \cdot \frac{1}{s+3} - \frac{7}{6} \cdot \frac{1}{s-1}\right)$$

Taking inverse Laplace transform then,

$$y = \frac{16}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} - \frac{7}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\}$$

$$= \frac{16}{3}e^{-2t} - e^{-3t} - \frac{7}{3}e^{t}$$

**2003 Fall Q. No. 6(b)**

**Solve by using Laplace transform.** $y'' + 6y' + 8y = e^{-3t} - e^{-5t}$

**Solution:** The question is missing the initial condition, so the problem can not solve by Laplace transform.

**2003 Fall Q. No. 5(b)**

$y'' + 4y' + 4y = \sin t,\ y(0) = 1,\ y'(0) = 3.$

**Solution:** Given that,

$$y'' + 4y' + 4y = \sin t \qquad \text{....... (i)}$$

$$y(0) = 1,\ y'(0) = 3 \qquad \text{...... (ii)}$$

Taking Laplace transform on (i) then,

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = \mathcal{L}\{\sin t\}$$

$$\Rightarrow \quad [s^2\mathcal{L}\{y\} - sy(0) - y'(0)] + 4[s\mathcal{L}\{y\} - y(0)] + 4\mathcal{L}\{y\} = \frac{1}{s^2+1}$$

$$\Rightarrow \quad (s^2 + 4s + 4)\,\mathcal{L}\{y\} - s - 3 - 4 = \frac{1}{s^2+1}$$

$$\Rightarrow \quad \mathcal{L}\{y\} = \frac{1}{s^2+4s+4}\left[\frac{1}{s^2+1} + s + 7\right]$$

$$= \frac{1}{s^2+4s+4}\left(\frac{s^3+7s^2+s+7}{s^2+1}\right)$$

$$= \frac{s^3+7s^2+s+7}{(s^2+1)(s+2)^2} \qquad \text{...... (iii)}$$

Here,

$$\frac{s^3+7s^2+s+7}{(s^2+1)(s+2)^2} = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{Cs+D}{s^2+1}$$

$$= \frac{A(s+2)(s^2+1) + B(s^2+1) + (Cs+D)(s+2)^2}{(s^2+1)(s+2)^2}$$

$$\Rightarrow \quad s^3 + 7s^2 + s + 7 = A(s^3 + 2s^2 + s + 2) + B(s^2 + 1) + C(s^3 + 4s^2 + 4s) + D(s^2 + 4s + 4)$$

Equating the coefficient of like terms then

$$A + C = 1, \qquad A + B + 4C + D = 7,$$

$$A + 4C + 4D = 1, \qquad 2A + B + 4D = 7$$

Solving we get,

$$A = \frac{25}{29},\quad B = \frac{165}{29},\quad C = \frac{4}{29},\quad D = -\frac{3}{29}$$

Then (iii) becomes,

$$\mathcal{L}\{y\} = \frac{25}{29}\cdot\frac{1}{s+2} + \frac{165}{29}\,\frac{1}{(s+2)^2} + \frac{4}{29}\cdot\frac{s}{s^2+1} - \frac{3}{29}\cdot\frac{1}{s^2+1}$$

$$= \frac{25}{29}\left(\frac{1}{s}\right)_{s\to s+2} + \frac{165}{29}\left(\frac{1}{s}\right)_{s\to s+2} + \frac{4}{29}\cdot\frac{s}{s^2+1} - \frac{3}{29}\cdot\frac{1}{s^2+1}$$

Taking inverse Laplace transform then

$$y = \frac{25}{29}\,\mathcal{L}^{-1}\left\{\left(\frac{1}{s}\right)_{s\to s+2}\right\} + \frac{165}{29}\,\mathcal{L}^{-1}\left\{\left(\frac{1}{s^2}\right)_{s\to s+2}\right\} + \frac{4}{29}\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - \frac{3}{29}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$



$$= \frac{25}{29}(e^{-2t}.1) + \frac{165}{29}\left(e^{-2t}\,\frac{t}{1!}\right) + \frac{4}{29}\cos t - \frac{3}{29}\sin t$$

$$= \frac{1}{29}[e^{-2t}(25 + 165t) + 4\cos t - 3\sin t]$$

**2007 Fall Q. No. 5(b)**

$y'' + 4y' + 5y' = 0,\ y(0) = 2,\ y'(0) = -3$

**Solution:** Given that,

$$y'' + 4y' + 5y' = 0 \qquad \text{........ (i)}$$

$$y(0) = 2,\ y'(0) = -3 \qquad \text{....... (ii)}$$

Here, the auxiliary equation of (i) is

$$m^2 + 4m + 5 = 0$$

$$\Rightarrow \quad m = \frac{-4 \pm \sqrt{16-20}}{2} = \frac{-4 \pm 2i}{2} = -2 \pm i$$

So solution of (i) be

$$y = e^{-2x}(A\cos x + B\sin x) \qquad \text{........(iii)}$$

Since $y(0) = 2$ then (iii) gives,

$$2 = e^0(A\,.\,1 + B.\,0) \Rightarrow A = 2$$

Also, differentiating (iii) then,

$$y' = -2e^{-2x}(A\cos x + B\sin x) + e^{-2x}(-A\sin x + B\cos x) \qquad \text{.......(iv)}$$

Since we have $y'(0) = -3$ then (iv) gives

$$-3 = -2A + B \ \Rightarrow\ B = -3 + 4 \qquad [\because A = 2]$$

$$\Rightarrow\ B = 1.$$

Thus (iii) becomes

$$y = e^{-2x}(2\cos x + \sin x)$$

## SHORT QUESTIONS

**1999; 2001 : If $\mathcal{L}\{f(t)\} = F(s)$ then $\mathcal{L}\{e^{at}f(t)\}$ = ........**

**Solution:** Let $\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty e^{at} f(t)\,e^{-st}\,dt = F(s-a) = (F(s))_{s\to s-a}$

Thus,

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a) = (F(s))_{s\to s-a}$$

**2000: Find $\mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2 + n^2\pi^2}\right\}$**

**Solution:** Since, we have,

$$\mathcal{L}\{e^{at}f(t)\} = (L\{f(t)\})_{s\to s-a} \Rightarrow \quad \mathcal{L}^{-1}\{\mathcal{L}\{f(t)\}\}_{s\to s-a})$$

$$\Rightarrow\ e^{at}f(t)$$

Now,

$$\mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+n^2\pi^2}\right\} = \mathcal{L}^{-1}\left\{\left(\frac{s}{s^2+(n\pi)^2}\right)_{s\to s+2}\right\}$$
$$= \mathcal{L}^{-1}\{(\mathcal{L}\{\cos n\pi t\})_{s\to s+2}\}$$
$$= e^{-2t}\cos n\pi t$$

**1999; 2001: Find $\mathcal{L}^{-1}\left\{\frac{\pi}{(s+2)^2+\pi^2}\right\}$**

**Solution:** Since we have,

$$\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2} \Rightarrow \mathcal{L}^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin at$$

and

$$\mathcal{L}\{e^{at} f(t)\} = (\mathcal{L}\{f(t)\})_{s\to s-a} \Rightarrow \mathcal{L}^{-1}\{\mathcal{L}\{f(t)\}_{s\to s-a}\} = e^{at} f(t)$$

Now,

$$\mathcal{L}^{-1}\left\{\frac{\pi}{(s+2)^2+\pi^2}\right\} = \mathcal{L}^{-1}\left(L\left\{\frac{\pi}{s^2+\pi^2}\right\}\right)_{s\to s+2)}$$
$$= \mathcal{L}^{-1}\{(\mathcal{L}\{\sin \pi t\})_{s\to s+2}\}$$
$$= e^{-2t}\sin \pi t.$$

**2002: Find the Laplace transform of $t^n e^{at}$.**

**Solution:** Since we have,

$$\mathcal{L}\{e^{at} f(t)\} = (\mathcal{L}\{f(t)\})_{s\to s-a} \quad \text{by first shifting property}$$

and $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

Now,

$$\mathcal{L}\{t^n e^{at}\} = (\mathcal{L}\{t^n\})_{s\to s-a}$$
$$= \left(\frac{n!}{s^{n+1}}\right)_{s\to s-a} = \frac{n!}{(s-a)^{n+1}}$$

**2004 Spring: Solve by using Laplace transform: $y'' + \pi^2 y = 0$, $y(0) = 2$, $y'(0) = 0$.**

**Solution:** Given that

$$y'' + \pi^2 y = 0 \quad \text{......(i)}$$
$$y(0) = 2,\ y' = 0 \quad \text{.....(ii)}$$

Taking Laplace transform of (i) then,

$$\mathcal{L}\{y''\} + \pi^2 \mathcal{L}\{y\} = \mathcal{L}\{0\}$$
$$\Rightarrow s^2\mathcal{L}\{y\} - sy(0) - y'(0) + \pi^2\mathcal{L}\{y\} = 0$$
$$\Rightarrow (s^2+\pi^2)\mathcal{L}\{y\} - 2s = 0 \quad [\because \text{using (ii)}]$$
$$\Rightarrow \mathcal{L}\{y\} = \frac{2s}{s^2+\pi^2} \quad \text{......(iii)}$$

Since we have, $\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$. Then (iii) becomes,

$$\mathcal{L}\{y\} = 2\,\mathcal{L}\{\cos \pi t\} = \mathcal{L}\{2\cos \pi t\}$$
$$\Rightarrow y = 2\cos \pi t$$



**2006 Fall : Find the Laplace transform of $te^{2t}$.**

**Solution:** Since we have,

$$\mathcal{L}\{e^{at} f(t)\} = (\mathcal{L}\{f(t)\})_{s\to s-a} \quad \text{by first shifting property}$$

and $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

Now,

$$\mathcal{L}\{te^{2t}\} = (\mathcal{L}\{t\})_{s\to s-2}$$
$$= \left(\frac{1!}{s^{1+1}}\right)_{s\to s-2} = \frac{1}{(s-2)^2}$$

**2006 Spring; 2009 Spring: Find the Laplace transform of t sinat.**

**Solution:** Since we have,

$$\mathcal{L}\{tf(t)\} = -\frac{d}{ds}(\mathcal{L}\{f(t)\} \quad \text{and} \quad \mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$$

Now,

$$\mathcal{L}\{t\sin at\} = -\frac{d}{ds}(\mathcal{L}\{\sin at\})$$
$$= -\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right) = \left(\frac{-2as}{(s^2+a^2)^2}\right) = \frac{2as}{(s^2+a^2)^2}$$

Thus, $\mathcal{L}\{t\sin at\} = \frac{2as}{(s^2+a^2)^2}$

**2007 Fall: Find inverse Laplace transform of $\frac{s+4}{s^2-4}$.**

**Solution:** Since we have,

$$\mathcal{L}\{\sin hat\} = \frac{a}{s^2-a^2} \quad \text{and} \quad \mathcal{L}\{\cos hat\} = \frac{s}{s^2-a^2}$$

Now,

$$\frac{s+4}{s^2-4} = \frac{s}{s^2-4} + \frac{4}{s^2-4} = \mathcal{L}\{\cos h\,2t\} + \mathcal{L}\{\sin h\,2t\}$$
$$= \mathcal{L}\{\cos h\,2t + \sin h\,2t\}$$
$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{s+4}{s^2-4}\right\} = \cosh 2t + \sin h\,2t$$

**2008 Spring: Find Laplace transform of $te^t$.**

**Solution:** Since we have,

$$\mathcal{L}\{tf(t)\} = -\frac{d}{ds}(\mathcal{L}\{f(t)\}) \quad \text{and} \quad \mathcal{L}\{e^t\} = \frac{1}{s-1}$$

Now,

$$\mathcal{L}\{te^t\} = -\frac{d}{ds}(\mathcal{L}\{e^t\})$$
$$= -\frac{d}{ds}\left(\frac{1}{s-1}\right) = -\left(\frac{-1}{(s-1)^2}\right) = \frac{1}{(s-1)^2}$$

Thus,

$$\mathcal{L}\{te^t\} = \frac{1}{(s-1)^2} W$$

**2009 Fall: Find Laplace transform of $f(t) = \frac{\sin 2t}{t}$**

**Solution:** Since we have,

$$\mathcal{L}\{t\, f(t)\} = -\frac{d}{ds}(\mathcal{L}\{f(t)\}) \quad \text{and} \quad \mathcal{L}\{\sin at\} = \frac{a}{s^2 + a^2}$$

Let,

$$f(t) = \frac{\sin 2t}{t} \quad \Rightarrow t\, f(t) = \sin 2t$$

Taking Laplace transform on both sides

$$\mathcal{L}\{t\, f(t)\} = \mathcal{L}\{\sin 2t\} = \frac{2}{s^2 + 4}$$

$$\Rightarrow \quad -\frac{d}{ds}(\mathcal{L}\{f(t)\}) = \frac{2}{s^2 + 4}$$

Integrating w. r. t. s then,

$$-\mathcal{L}\{f(t)\} = \int \frac{2}{s^2 + 4}\, ds$$

$$= 2 \cdot \frac{1}{2} \tan^{-1}\left(\frac{s}{2}\right) = \tan^{-1}\left(\frac{s}{2}\right).$$

$$\Rightarrow \quad \mathcal{L}\left\{\frac{\sin 2t}{t}\right\} = -\tan^{-1}\left(\frac{s}{2}\right)$$

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