Unit 5

LINEAR PROGRAMMING

Graphical Method - *Procedure:*

Step 1: If given problem is in language forms then change it into equation (or inequalities) form.

i.e. objective function:

$$\max (\min z) = c_1x_1 + c_2x_2 + \ldots\ldots + c_nx_n$$

where, $c = [c_1, c_2, c_3 \ldots\ldots\ldots\ldots c_n]_{1\times n}$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}_{n\times 1} = \text{Decision variable}$$

Subject to constraints

$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_nx_n (<, =, >) b_1$$

$$a_{21}x_1 + a_{12}x_2 + \ldots\ldots + a_nx_n (<, =, >) b_2$$

$$\vdots$$

$$a_{31}x_1 + a_{12}x_2 + \ldots\ldots + a_nx_n (<, =, >) b_n$$

Where,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n} = \text{coefficient matrix}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix}_{n\times 1} = \text{decision variable; } B = \begin{bmatrix} b_1 \\ b_2 \\ \cdot\cdot \\ \cdot \\ \cdot \\ b_n \end{bmatrix}_{m\times 1} = \text{Constant matrix}$$

In short form

Objective function: Max (Min) $z = CX$

Subject to constraint

$$AX (\leq, =, \geq) B$$

Step 2: Change the inequality into equation form.

Step 3: Find the boundary point form where the line passes by putting $x = 0$ to find y and $y = 0$ to find x.

Step 4: testing point $(x, y) = (0, 0)$. If inequality give true result then region covered by the inequality towards the origin.

If inequality give failure result then region covered by the inequality opposite to the origin.

In other word, if inequality is less than type ($\leq$)then region towards the origin and is greater than type ($\geq$) then region opposite to the origin.

Step 5: we find the common feasible region satisfied by all the inequalities with vertices.

Step 6: at least, put the value vertices in objective function and find maximum (or minimum) value.

EXERCISE 5.1

1. **Minimize:** $z = 45x_1 + 22.5x_2$

s.t. $-x_1 + x_2 \leq -5;\ 2x_1 + x_2 \geq 10;\ x_2 \geq 4,\ 10x_1 + 15x_2 \leq 150.$

Solution: We write given inequalities in equation form, we get

$-x_1 + x_2 = -5$ ....(i) $\qquad 2x_1 + x_2 = 10$ ......(ii)

$x_2 = 4$ ......(iii) $\qquad 10x_1 + 15x_2 = 150$ ......(iv)

From equation (i)

If $x_1 = 0$, $x_2 = -5$ and if $x_2 = 0$, $x_1 = 5$.

Hence eq^n. (i) passes through (0, -5) and (5, 0) inequality (i) cover region towards the origin.

From equation (ii)

If $x_1 = 0$, $x_2 = 10$ and if $x_2 = 0$, $x_1 = 5$.

Hence eq^n. (ii) passes through (0, 10) and (5, 0) inequality (ii) cover region opposite to the origin.

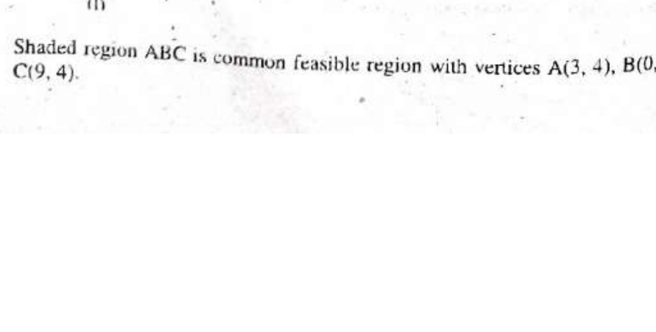
From equation (iii) we have $x_1 = 0$, $x_2 = 4$.

So the line passes through (0, 4) i.e. parallel to x_2-axis.

From equation (iv)

If $x_1 = 0$, $x_2 = 10$ and if $x_2 = 0$, $x_1 = 15$.

Hence equation (iv) passes through (0, 10) and (15, 0). The region cover by inequality (iv) towards the origin.



Shaded region ABC is common feasible region with vertices A(3, 4), B(0, 10) and C(9, 4).

To find minimum value we have

Corner point	Objective function Min. $Z = 45x_1 + 22.5x_2$
A(3, 4)	$Z = 45\times3 + 22.5\times4 = 225$
B(0, 10)	$Z = 45\times0 + 22.5\times10 = 225$
C(9, 4)	$Z = 45\times9 + 22.5\times4 = 495$

Hence min. Z = 225 on the segment from A(3, 4) to B(0, 10).

Minimize: $z = 5x_1 + 25x_2$

s.t. $-0.5x_1 + x_2 \leq 2,\ x_1 + x_2 \geq 2,\ -x_1 + 5x_2 \geq 5.$

ution: We write the given inequalities into equation form:

$-0.5x_1 + x_2 = 2$ ....(i)

$x_1 + x_2 = 2$ ....(ii)

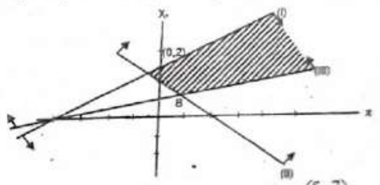
$-x_1 + 5x_2 = 5$ ....(ii)

From Eq^n. (i), put $x_1 = 0$, $x_2 = 2$ and at $x_2 = 0$, $x_1 = -4$. Thus equation (i) passes through the point (0, 2) and (-4, 0). Inequality (i) is $\leq$ type so it cover region towards the origin.

From equation (i), put $x_1 = 0$, $x_2 = 2$ and at $x_2 = 0$, $x_1 = 2$. Thus, equation (ii) passes through the point (0, 2) and (2, 0). Inequality (ii) is $\geq$ type so it cover region opposite to the origin.

From equation (iii), put $x_1 = 0$, $x_2 = 1$ and at $x_2 = 0$, $x_1 = -5$. Thus, equation (iii) passes through the point (0, 1) and (-5, 0). Inequality $\geq$ give region opposite to the region.

Now, plot the graph



shaded region is unbounded feasible region with cornor point A(0, 2) and B$\left(\frac{5}{6}, \frac{7}{6}\right)$

Corner point	Objective function Min. $Z = 5x_1 + 25x_2$
A(0, 2)	$Z = 5\times0 + 25\times2 = 50$
B$\left(\frac{5}{6}, \frac{7}{6}\right)$	$Z = 5\times\frac{5}{6} + 25\times\frac{7}{6} = \frac{200}{6} = \frac{100}{3}$

Hence the objective function is minimum at B$\left(\frac{5}{6}, \frac{7}{6}\right)$ with minimum value is $\frac{100}{3}$.

Maximize: $z = -10x_1 + 2x_2$

s.t. $x_1 \geq 0,\ x_2 \geq 0,\ -x_1 + x_2 \geq -1;\ x_1 + x_2 \leq 6,\ x_2 \leq 5.$

olution

We write given inequality into equation form

$x_1 - x_2 = 1$ ....(i)

$x_1 + x_2 = 6$ ....(ii)

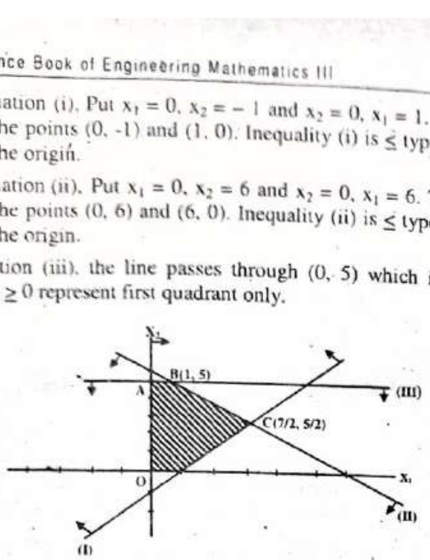
$x_2 = 5$ ....(iii)

$x_1 = 0,\ x_2 = 0$

From equation (i). Put $x_1 = 0$, $x_2 = -1$ and $x_2 = 0$, $x_1 = 1$. Thus, equation (i) passes through the points (0, -1) and (1, 0). Inequality (i) is $\leq$ type, so region covered by it towards the origin.

From equation (ii). Put $x_1 = 0$, $x_2 = 6$ and $x_2 = 0$, $x_1 = 6$. Thus, equation (ii) passes through the points (0, 6) and (6, 0). Inequality (ii) is $\leq$ type, so region covered by it towards the origin.

For equation (iii), the line passes through (0, 5) which is parallel to x_1-axis and $x_1 \geq 0$, $x_2 \geq 0$ represent first quadrant only.



The shaded region OABCD with vertices O(0, 0), A(0, 5), B(1, 5), $C\left(\frac{7}{2}, \frac{5}{2}\right)$, D(1, 0) to find the maximum value.

Corner point	Objective function Min. $Z = -10x_1 + 2x_2$
A(0, 5)	$Z = 0 + 10 = 10$
B(1, 5)	$Z = -10 + 10 = 0$
C(7/2, 5/2)	$Z = -10 \times 7/2 + 2 \times 5/2 = -30$
D(1, 0)	$Z = -10 \times 1 + 2 \times 0 = -10$

Hence the given function is maximum at A(0, 5) and maximum value is 10.

4. **Maximize: $z = 40x_1 + 88x_2$**

s.t. $2x_1 + 8x_2 \leq 60$; $5x_1 + 2x_2 \leq 60$, $x_1 \geq 0$, $x_2 \geq 0$.

Solution: Write the given inequality into equation form.

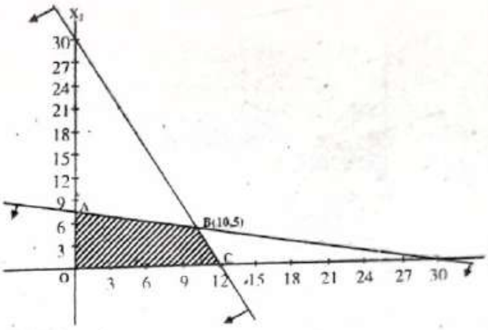
$2x_1 + 8x_2 = 60$ ....(i)

$5x_1 + 2x_2 = 60$ ....(ii)

From equation (i), put $x_1 = 0$, $x_2 = 15/2 = 7.5$ and put $x_2 = 0$, $x_1 = 30$. Then the equation (i) passes through the point (0, 7.5) and (30, 0). Inequality (i) is $\leq$ type, so region covered by (i) towards the origin.

From equation (ii), put $x_1 = 0$, $x_2 = 30$ and put $x_2 = 0$, $x_1 = 12$. Then the equation (ii) passes through the point (0, 30) and (12, 0). Inequality (ii) is of $\leq$ type, so region covered by (ii) towards the origin.

$x_1 \geq 0$, $x_2 \geq 0$ represents first quadrant only.



The shaded region is feasible region OABC with vertices O(0, 0), A(0, 7.5), B(10, 5) C(12, 0).

To find value, we have

Corner point	Objective function Min. $Z = 40x_1 + 88x_2$
O(0, 0)	$Z = 0$
A(0, 7.5)	$Z = 88 \times 7.5 = 660$
B(10, 5)	$Z = 40 \times 10 + 88 \times 5 = 840$ (max.)
C(12, 0)	$Z = 40 \times 12 + 88 \times 0 = 480$

Hence given objective function is maximum at B(10, 5) and maximum value is 840.

Maximize: $z = 5x_1 + 7x_2$

s.t. $x_1 + x_2 \leq 4$; $3x_1 + 8x_2 \leq 24$; $10x_1 + 7x_2 \leq 35$, $x_1 \geq 0$, $x_2 \geq 0$

by using graphical method.

tion: Write the given inequalities into equation form, we have

$x_1 + x_2 = 4$ .....(i)

$3x_1 + 8x_2 = 24$ ......(ii)

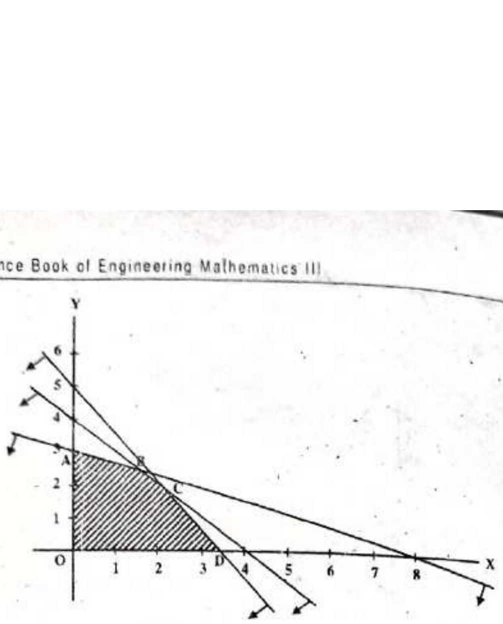
$10x_1 + 7x_2 = 35$ ....(iii)

From equation (i), put $x_1 = 0$, $x_2 = 4$ and put $x_2 = 0$, $x_1 = 4$. Then the equation (i) passes through the point (4, 0) and (0, 4).

From equation (ii), put $x_1 = 0$, $x_2 = 3$ and put $x_2 = 0$, $x_1 = 8$. Then the equation (ii) passes through (0, 3) and (8, 0).

From equation (iii), put $x_1 = 0$, $x_2 = 5$ and put $x_2 = 0$, $x_1 = 3.5$. Then the equation (iii) passes through the point (0, 5) and (3.5, 0).

All the inequalities are less then type so all inequalities cover region towards the origin.



The shaded region is feasible region OABCD with vertices O(0, 0), A(0, 3), $B\left(\frac{8}{5}, \frac{12}{5}\right)$, $C\left(\frac{7}{3}, \frac{5}{3}\right)$ and d(3.5, 0)

To find the value, we have

Corner point	Objective function Max. $Z = 5x_1 + 7x_2$
O(0, 0)	Z = 0
A(0, 3)	Z = 5×0 + 7×3 = 21
B(8/5, 12/5)	Z = 5×8/5 + 7×12/5 = 124/5 = 24.8 (Max.)
C(7/3, 5/3)	Z = 5×7/3 + 7×5/3 = 70/3 = 23.33
D(3.5, 0)	Z = 5×3.5 + 7×0 = 17.5

Hence, given objective function given maximizing value at B(8/5, 12/5) and maximum value is 24.8.

6. Minimize $z = 5x_1 + 3x_2$

s.t. $2x_1 + x_2 \geq 3$, $x_1 + x_2 \geq 2$; $x_1 \geq 0$, $x_2 \geq 0$, by using graphical method.

Solution: Write the given inequalities into equation

$2x_1 + x_2 = 3$ ....(i)

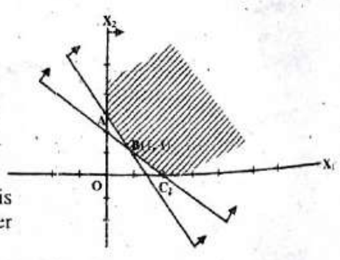
$x_1 + x_2 = 2$ ....(ii)

From equation (i), put $x_1 = 0$, $x_2 = 3a$ and put $x_2 = 0$, $x_1 = 3/2$. Then the equation (i) passes through the point (0, 3) and (3/2, 0).

From equation (ii), put $x_1 = 0$, $x_2 = 2$ and put $x_2 = 0$, $x_1 = 2$. Then the equation (ii) passes through the point (0, 2) and (2, 0).

Both inequality is ≥ type, so region represented by both opposite to the region.

$x_1, x_2 \geq 0$ represent first quadrant only.



After plotting graph, shaded region is unbounded feasible region with corner points A(0, 3), B(1, 1) and C(2, 0).

to find minimum value.

Corner point	Objective function Min. $Z = 5x_1 + 3x_2$
A(0, 3)	Z = 5.0 + 3.3 = 9
B(1, 1)	Z = 5.1 + 3.1 = 8 (min.
C(2, 0)	Z = 5.2 + 3.0 = 10

Hence given objective function give minimum value at B(1, 1) and minimum value is 8.

ilar questions for practice

Find maximum value of $z = 5x_1 + 3x_2$
subject to $3x_1 + 5x_2 \leq 15$, $5x_1 + 2x_2 \leq 10$, $x_1 \geq 0$, $x_2 \geq 0$.

Find maximum value of $z = x_1 + 6x_2$
subject to $x_1 + x_2 \geq 2$; $x_1 + x_2 \leq 3$; $x_1 \geq 0$, $x_2 \geq 0$.

Find maximum value of $z = 3x_1 + 2x_2$
subject to $x_1 + x_2 \leq 20$; $x_1 \leq 15$, $x_1 + 3x_2 \leq 45$, $-3x_1 + 5x_2 \leq 60$, $x_1 \geq 0$, $x_2 \geq 0$.

Optimize $z = 2x_1 + 5x_2$
subject to $x_1 + 3x_2 \geq 3$, $6x_1 + 5x_2 \leq 30$, $x_1 \geq 2$, $x_2 \geq 0$.

ution: Write the given inequalities into equation we have

$x_1 + 3x_2 = 3$ ......(i)

$6x_1 + 5x_2 = 30$ ......(ii)

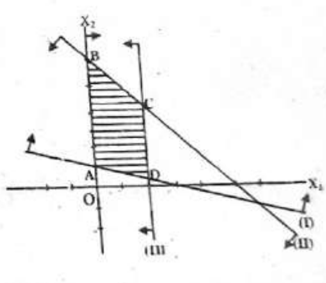
$x_1 = 2$ ....(iii)

From equation (i), if $x_1 = 0$ then $x_2 = 1$ and if $x_2 = 0$ then $x_1 = 3$. Thus, the equation (i) passes through (0, 1) and (3, 0). Inequality (i) is of ≥ type give region opposite to the origin.

From equation (ii), if $x_1 = 0$ then $x_2 = 6$ and if $x_2 = 0$ then $x_1 = 5$. Thus, the equation (ii) passes through the points (0, 6) and (5, 0) and the inequality (ii) has of ≤ type, give region towards the origin.

From equation (iii), if passes through (2, 0) which is parallel to x_2-axis. Inequality (iii) represent ≥ type give region opposite to the origin.

$x_2 \geq 0$ represent first quadrant only.



The shaded region is common feasible region with vertices A(0, 1), B(0, 6), $C\left(2, \frac{18}{5}\right)$, $D\left(2, \frac{1}{3}\right)$.

To find optimal value,

Corner point	Objective function Min. $Z = 2x_1 + 5x_2$
A(0, 1)	Z = 2.0 + 5.1 = 5 (min.)
B(0, 6)	Z = 2.0 + 6.1 = 6
C(2, 18/5)	Z = 2×2 + 5×18/5 = 22 (max.)
D(2, 1/3)	Z = 2×2 + 5 × 1/3 = 17/3

Hence the given function gives minimum value at A(0, 1) and minimum value is 5
Maximum value at C (2, 18/5) and maximum value is Z = 22.

11. Optimize $z = 2x_1 + x_2$

subject to $x_1 + 2x_2 \leq 10$, $x_1 + x_2 \geq 1$, $0 \leq x_2 \leq 4$, $x_1 \geq 0$.

Solution: Write the given inequality into equation form, we have

$x_1 + 2x_2 = 10$ ......(i)

$x_1 + x_2 = 1$ ....(ii)

$x_2 = 4$ ......(iii)

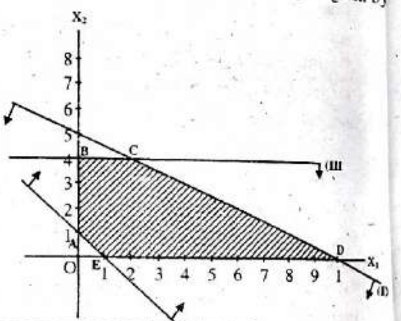
From equation (i), put $x_1 = 0$, $x_2 = 5$ and put $x_2 = 0$, $x_1 = 10$. Then the equation (i passes through the point (0, 5) and (10, 0). Inequality (i) is less than type, so regior given by it towards the origin.

From equation (ii), put $x_1 = 0$, $x_2 = 1$ and put $x_2 = 0$, $x_1 = 1$. Then the equation (ii passes through the point (1, 0) and (0, 1). Inequality (ii) is $\geq$ type, so region given by it opposite to the origin.

From equation (iii), the line $x_2 = 4$ is line passes through the point (0, 4) which is parallel to x_1-axis. Inequality $\leq$ type give region towards the origin.

Lastly $x_1 \geq 0$, $x_2 \geq 0$ represent first quadrant only.

shaded region ABCDE is common feasible region with co-ordinate A(0, 1), B(0, 4), C(2, 4), D(10, 0), E(1, 0) to find the optimal value.



Corner point	Objective function Min. $Z = 2x_1 + x_2$
A(0, 1)	$Z = 2\times0 + 1 = 1$ (min.)
B(0, 4)	$Z = 2\times0 + 4 = 4$
C(2, 4)	$Z = 2\times2 + 4 = 8$
D(10, 0)	$Z = 2\times10 + 0 = 20$ (max.)
E(1, 0)	$Z = 2\times1 + 0 = 2$

Hence objective function give minimum value 1 at A(0, 1) and maximum value 20 at D(10, 0).

12. Optimize $z = 15x_1 + 25x_2$

subject to $x_1 + x_2 \leq 8$, $2x_1 + x_2 \leq 9$, $3x_1 + x_2 \leq 12$, $x_1 \geq 0$, $x_2 \geq 0$.

Solution: Write the given inequality into equation form, we have

$x_1 + x_2 = 8$ ....(i)

$2x_1 + x_2 = 9$ ....(ii)

$3x_1 + x_2 = 12$ ......(iii)

From equation (i), put $x_1 = 0$, $x_2 = 8$ and put $x_2 = 0$, $x_1 = 8$. Thus the equation (i) passes through the point (0, 8) and (8, 0). Inequality is $\leq$ type, so region given by it towards the origin.

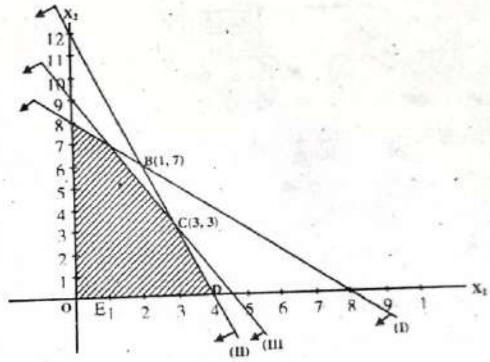
From equation (ii), put $x_1 = 0$, $x_2 = 9$ and put $x_2 = 0$, $x_1 = 9/2$. Thus the equation (ii) passes through the point (0, 9) and (9/2, 0). Inequality is $\leq$ type, region given by it towards the origin.

From equation (iii), put $x_1 = 0$, $x_2 = 12$ and put $x_2 = 0$, $x_1 = 4$. Thus the

Hence equation (iii) passes through the point (0, 12) and (4, 0). Inequality is $\leq$ type, region given by it towards the origin.

$x_1 \geq 0$, $x_2 \geq 0$ represent first quadrant only.

Plot the graph



The shaded region is common feasible region OABCD with co-ordinates O(0, 0), A(0, 8), B(1, 7), C(3, 3) and D(4, 0).

To find the optimal value, we have

Corner point	Objective function Min. $Z = 15x_1 + 25x_2$
O(0, 0)	$Z = 0$ (min.)
A(0, 8)	$Z = 15\times0 + 25\times8 = 200$ (max.)
B(1, 7)	$Z = 15\times1 + 25\times7 = 190$
C(3, 3)	$Z = 15\times3 + 25\times3 = 120$
D(4, 0)	$Z = 15\times4 + 25\times0 = 60$

Hence the given objective function given minimum value at O(0, 0) and maximum value 200 at A(0, 8).

EXAMPLE 5.2

1. Minimize: $x = 30x_1 + 20x_2$

s.t. $-x_1 + x_2 \leq 5$; $2x_1 + x_2 \leq 10$ by simplex method.

Solution: Given that, max: $z = 30x_1 + 20x_2$

Subject to $-x_1 + x_2 \leq 5$

$2x_1 + x_2 \leq 10$.

Introducing new variables x_3 and x_4 so that,

Maximize: $z - 30x_1 - 20x_2 = 0$

Subject to $-x_1 + x_2 + x_3 = 5$

$2x_1 + x_2 + x_4 = 10.$

The tabled form of above problem is,

	z	$x_1\downarrow$	x_2	x_3	x_4	Constant	ratio
R_1	1	−30	−20	0	0	0	0
R_2	0	−1	1	1	0	5	−ve value
R_3	0	2	1	0	1	10	← 5

Now, we have to maximize the function. So, we observe the negative entry in R_1. The first negative entry is −30. So, the column of x_1 is the pivot column and by ratio $\left(\text{ratio} = \frac{\text{constant}}{\text{pivot column}}\right)$, R_3 is the pivot row (row if least positive ratio). Therefore, 2 is the pivot point.

To eliminate the values of the pivot column rather than the pivot, apply, $R_1 \to R_1 + 15R_3$, $R_2 \to 2R_2 + R_3$ then the above table becomes,

	z	x_1	$x_2\downarrow$	x_3	x_4	Constant	ratio
R_1	1	0	−5	0	15	150	−ve value
R_2	0	0	3	2	1	20	← 6.66
R_3	0	2	1	0	1	10	10

Again, R_1 has negative entry and that is, −5. So, the column of x_2 is pivot column then by ratio, R_2 is the pivot row and pivot point is 3.

Now, applying $R_1 \to 3R_1 + 5R_2$, $R_3 \to 3R_3 - R_2$ then the above table becomes,

	z	x_1	x_2	x_3	x_4	Constant	ratio
R_1	3	0	0	10	50	550	
R_2	0	0	3	3	1	20	
R_3	0	6	0	−2	2	10	

Here R_1 has no negative entry. So the table gives optimal solution.

Assume the non_basic variables are zero i.e. $x_3 = 0 = x_4$.

Then by R_1, $3z = 550 \Rightarrow z = \frac{550}{3}$

by R_2, $3x_2 = 20 \Rightarrow x_2 = \frac{20}{3}$

by R_3, $6x_1 = 10 \Rightarrow x_1 = \frac{5}{3}$.

Thus, $\max(z) = \frac{550}{3}$ at $(x_1, x_2) = (5/3, 20/3)$.

2. Maximize $z = 2x_1 + x_2 + 3x_3$

s.t. $4x_1 + 3x_2 + 6x_3 \le 12$; by using simplex method.

Solution: Given that, max: $z = 2x_1 + x_2 + 3x_3$

Subject to $4x_1 + 3x_2 + 6x_3 \le 12.$

Introducing new variable x_4 so that,

Maximize: $z - 2x_1 - x_2 - 3x_3 = 0$

Subject to $4x_1 + 3x_2 + 6x_3 + x_4 = 12.$

The tabled form of above problem is,

	z	$x_1\downarrow$	x_2	x_3	x_4	Constant	ratio
R_1	1	−2	−1	−3	0	0	0
R_2	0	4	3	6	1	12	← 3

Now, we have to maximize the function. So, we observe the negative entry in R_1. The first negative entry is −2. So, the column of x_1 is the pivot column and by ratio $\left(\text{ratio} = \frac{\text{constant}}{\text{pivot column}}\right)$, R_3 is the pivot row (row if least positive ratio). Therefore, 3 is the pivot point.

To eliminate the values of the pivot column rather than the pivot, apply, $R_1 \to 2R_1 + R_2$ then the above table becomes,

	z	x_1	$x_2\downarrow$	x_3	x_4	Constant	ratio
R_1	2	0	1	0	1	12	
R_2	0	4	3	6	1	12	

Here R_1 has no negative entry. So the table gives optimal solution.

Assume the non_basic variables are zero i.e. $x_2 = 0 = x_4$.

Then by R_1, $2z = 12 \Rightarrow z = 6$

by R_2, $4x_1 + 6x_3 = 12$

$\Rightarrow x_1 = 3$ when $x_3 = 0$ and $x_3 = 2$ when $x_1 = 0$.

Thus, $\max(z) = 6$ at (x_1, x_2, x_3) = all points of the line segment from (3, 0, 0) to (0, 0, 6).

3. Minimize: $z = 5x_1 - 20x_2$

s.t. $-2x_1 + 10x_2 \le 5$, $2x_1 + 5x_2 \le 10$ by simplex method.

Solution: Given that, min: $z = 5x_1 - 20x_2$

Subject to $-2x_1 + 10x_2 \le 5$

$2x_1 + 5x_2 \le 10.$

Introducing new variables x_3 and x_4 so that,

Maximize: $z - 5x_1 + 20x_2 = 0$

Subject to $-2x_1 + 10x_2 + x_3 = 5$

$2x_1 + 5x_2 + x_4 = 10.$

The tabled form of above problem is,

	z	x_1	$x_2\downarrow$	x_3	x_4	Constant	ratio
R_1	1	−5	20	0	0	0	0
R_2	0	−2	10	1	0	5	← 0.5
R_3	0	2	5	0	1	10	2

Now, we have to minimize the function. So, we observe the positive entry in R_1. The first positive entry is 20 in R_1. So, the column of x_2 is the pivot column and by ratio $\left(\text{ratio} = \frac{\text{constant}}{\text{pivot column}}\right)$. R_2 is the pivot row (row if least positive ratio). Therefore, 10 is the pivot point.

To eliminate the values of the pivot column rather than the pivot, apply, $R_1 \to R_1 - 2R_2$, $R_3 \to 2R_3 - R_2$ then the above table becomes,

	z	x_1	x_2	x_3	x_4	Constant	ratio
R_1	1	−1	0	−2	0	−10	
R_2	0	−2	10	1	0	5	
R_3	0	6	0	−1	2	15	

Here R_1 has no positive entry. So the table gives optimal solution.

Assume the non_basic variables are zero i.e. $x_1 = 0 = x_3$.

Then by R_1, $z = -10 \Rightarrow z = -10$

by R_2, $10x_2 = 5 \Rightarrow x_2 = 0.5$

by R_3, $2x_4 = 15 \Rightarrow x_4 = 7.5$.

Thus, $\max(z) = -10$ at $(x_1, x_2) = (0, 0.5)$.

4. Maximize: $z = 40x_1 + 88x_2$

s.t. $2x_1 + 8x_2 \le 60$, $5x_1 + 2x_2 \le 60$, $x_1 \ge 0$, $x_2 \ge 0$.

Solution: Given that, max: $z = 40x_1 + 88x_2$

Subject to $2x_1 + 8x_2 \le 60$

$5x_1 + 2x_2 \le 60$.

Introducing new variables x_3 and x_4 so that,

Maximize: $z - 40x_1 - 88x_2 = 0$

Subject to $2x_1 + 8x_2 + x_3 = 60$

$5x_1 + 2x_2 + x_4 = 60$.

The tabled form of above problem is,

	z	$x_1\downarrow$	x_2	x_3	x_4	Constant	ratio
R_1	1	−40	−88	0	0	0	0
R_2	0	2	8	1	0	60	30
R_3	0	5	2	0	1	60	$\leftarrow$ 12

Now, we have to maximize the function. So, we observe the negative entry in R_1. The first negative entry is −40. So, the column of x_1 is the pivot column and by ratio $\left(\text{ratio} = \frac{\text{constant}}{\text{pivot column}}\right)$. R_3 is the pivot row (row if least positive ratio). Therefore, 5 is the pivot point.

To eliminate the values of the pivot column rather than the pivot, apply, $R_1 \to R_1 + 8R_3$, $R_2 \to 5R_2 - 2R_3$ then the above table becomes.

	z	x_1	$x_2\downarrow$	x_3	x_4	Constant	ratio
R_1	1	0	−72	0	8	480	−ve value
R_2	0	0	36	5	−2	180	$\leftarrow$ 5
R_3	0	5	2	0	1	60	30

Again, R_1 has negative entry and that is, −72. So, the column of x_2 is pivot column then by ratio, R_2 is the pivot row and pivot point is 36.

Now, applying $R_1 \to R_1 + 2R_2$, $R_3 \to 18R_3 - R_2$ then the above table becomes.

	z	x_1	x_2	x_3	x_4	Constant	ratio
R_1	1	0	0	10	4	840	
R_2	0	0	36	5	−2	180	
R_3	0	90	0	−5	20	900	

Here R_1 has no negative entry. So the table gives optimal solution.

Assume the non_basic variables are zero i.e. $x_3 = 0 = x_4$.

Then by R_1, $z = 840 \Rightarrow z = 840$

by R_2, $36x_2 = 180 \Rightarrow x_2 = 5$

by R_3, $90x_1 = 900 \Rightarrow x_1 = 10$.

Thus, $\max(z) = 840$ at $(x_1, x_2) = (10, 5)$.

Maximize: $z = 5x_1 + 10x_2$

s.t. $0 \le x_1 \le 5$, $x_1 + x_2 \le 6$, $0 \le x_2 \le 4$

Solution: Given that, max: $z = 5x_1 + 10x_2$

Subject to $x_1 \le 5$

$x_1 + x_2 \le 6$

$x_2 \le 4$.

Introducing new variables x_3, x_4 and x_5 so that,

Maximize: $z - 5x_1 - 10x_2 = 0$

Subject to $x_1 + x_3 = 5$

$x_1 + x_2 + x_4 = 6$

$x_2 + x_5 = 4$.

The tabled form of above problem is,

	z	$x_1\downarrow$	x_2	x_3	x_4	x_5	Constant	ratio
R_1	1	−5	−10	0	0	0	0	0
R_2	0	1	0	1	0	0	5	$\leftarrow$ 5
R_3	0	1	1	0	1	0	6	6
R_4	0	0	1	0	0	1	4	undefined

Now, we have to maximize the function. So, we observe the negative entry in R_1. The first negative entry is −5. So, the column of x_1 is the pivot column and by ratio $\left(\text{ratio} = \frac{\text{constant}}{\text{pivot column}}\right)$. R_2 is the pivot row (row if least positive ratio). Therefore, 1 is the pivot point.

To eliminate the values of the pivot column rather than the pivot, apply, $R_1 \to R_1 + 5R_2$, $R_3 \to R_3 - R_2$ then the above table becomes,

	z	x_1	$x_2 \downarrow$	x_3	x_4	x_5	Constant	ratio
R_1	1	0	−10	5	0	0	25	−ve value
R_2	0	1	0	1	0	0	5	undefined
R_3	0	0	1	−1	1	0	1	← 1
R_4	0	0	1	0	0	1	4	4

Again, R_1 has negative entry and that is, −10. So, the column of x_2 is pivot column then by ratio, R_3 is the pivot row and pivot point is 1.

Now, applying $R_1 \to R_1 + 10R_3$, $R_4 \to R_4 - R_3$ then the above table becomes,

	z	x_1	x_2	$x_3 \downarrow$	x_4	x_5	Constant	ratio
R_1	1	0	0	−5	10	0	35	−ve value
R_2	0	1	0	1	0	0	5	5
R_3	0	0	1	−1	1	0	1	−ve value
R_4	0	0	0	1	−1	1	3	← 3

Again, R_1 has negative entry and that is, −5. So, the column of x_3 is pivot column then by ratio, R_4 is the pivot row and pivot point is 1.

Now, applying $R_1 \to R_1 + 5R_4$, $R_2 \to R_2 - R_4$, $R_3 \to R_3 + R_4$ then the above table becomes,

	z	x_1	x_2	x_3	x_4	x_5	Constant	ratio
R_1	1	0	0	0	5	5	50	
R_2	0	1	0	0	1	−1	2	
R_3	0	0	1	0	0	1	4	
R_4	0	0	0	1	−1	1	3	

Here R_1 has no negative entry. So the table gives optimal solution.

Assume the non_basic variables are zero i.e. $x_4 = 0 = x_5$.

Then by R_1, $z = 50$ by R_2, $x_1 = 2$

by R_3, $x_2 = 4$ by R_4, $x_3 = 3$

Thus, $\max(z) = 50$ at $(x_1, x_2) = (2, 4)$.

6. Minimize: $z = 2x_1 - 10x_2$

s.t. $x_1 \geq 0, x_2 \geq 0, x_1 - x_2 \leq 4, 2x_1 + x_2 \leq 14, x_1 + x_2 \leq 9, -x_1 + 3x_2 \leq 15$.

Solution: Given that, min: $z = 2x_1 - 10x_2$

Subject to $x_1 - x_2 \leq 4$; $2x_1 + x_2 \leq 14$; $x_1 + x_2 \leq 9$; $-x_1 + 3x_2 \leq 15$.

Introducing new variables x_3, x_4, x_5 and x_6 so that,

Maximize: $z - 2x_1 + 20x_2 = 0$

Subject to
$$x_1 - x_2 + x_3 = 4$$
$$2x_1 + x_2 + x_4 = 14$$
$$x_1 + x_2 + x_5 = 9$$
$$-x_1 + 3x_2 + x_6 = 15.$$

The tabled form of above problem is,

	z	x_1	$x_2 \downarrow$	x_3	x_4	x_5	x_6	Constant	ratio
R_1	1	−2	10	0	0	0	0	0	0
R_2	0	1	−1	1	0	0	0	4	−ve value
R_3	0	2	1	0	1	0	0	14	14
R_4	0	1	1	0	0	1	0	9	9
R_5	0	−1	3	0	0	0	1	15	← 5

Now, we have to minimize the function. So, we observe the positive entry in R_1. The first positive entry is 10 in R_1. So, the column of x_2 is the pivot column and by ratio $\left(\text{ratio} = \frac{\text{constant}}{\text{pivot column}}\right)$, R_2 is the pivot row (row if least positive ratio).

Therefore, 5 is the pivot point.

To eliminate the values of the pivot column rather than the pivot, apply, $R_1 \to 3R_1 - 10R_5$, $R_2 \to 3R_2 + R_5$, $R_3 \to 3R_3 - R_5$, $R_4 \to 3R_4 - R_5$ then the above table becomes,

	z	$x_1 \downarrow$	x_2	x_3	x_4	x_5	x_6	Constant	ratio
R_1	3	4	0	0	0	0	−10	−150	−ve value
R_2	0	2	0	3	0	0	1	27	13.5
R_3	0	7	0	0	3	0	−1	27	3.85
R_4	0	4	0	0	0	3	−1	12	← 3
R_5	0	−1	3	0	0	0	1	15	−ve value

Again, R_1 has positive entry and that is, 4. So, the column of x_1 is pivot column then by ratio, R_4 is the pivot row and pivot point is 4.

Now, applying $R_1 \to R_1 - R_4$, $R_2 \to 2R_2 - R_4$, $R_3 \to 4R_3 - 7R_4$, $R_5 \to 4R_5 + R_4$ then the above table becomes,

	z	x_1	x_2	x_3	x_4	x_5	x_6	Constant	ratio
R_1	3	0	0	0	0	−3	−9	−162	
R_2	0	0	0	3	0	−3	3	42	
R_3	0	0	0	0	12	−21	3	24	
R_4	0	4	0	0	0	3	−1	12	
R_5	0	0	12	0	0	3	3	72	

Here R_1 has no positive entry. So the table gives optimal solution.

Assume the non_basic variables are zero i.e. $x_5 = 0 = x_6$.

Then by R_1, $3z = -162 \Rightarrow z = -54$

by R_2, $3x_3 = 42 \Rightarrow x_3 = 14$

by R_3, $12x_4 = 24 \Rightarrow x_4 = 2$

by R_4, $4x_1 = 12 \Rightarrow x_1 = 3$

by R_5, $12x_2 = 72 \Rightarrow x_2 = 6$.

Thus, $\min(z) = -54$ at $(x_1, x_2) = (3, 6)$.

7. Suppose that we produce x_1 batteries B_1 by process P_1 and x_2 by process P_2 and that we produce x_3 batteries B_2 by produces P_3 and x_4 by process P_4. Let the profit per battery be Rs 10 for B_1 and Rs 20 for B_2.

$12x_1 + 8x_2 + 6x_3 + 4x_4 \leq 120$ (machine hours)

$3x_1 + 6x_2 + 12x_3 + 24x_4 \leq 180$ (labor hours)

Solution: Let x_1 batteries B_1 by process P_1 and x_2 by process P_2 and that we produce x_3 batteries B_2 by produces P_3 and x_4 by process P_4. That is P_1 and P_2 produce batteries of type B_1 in the number quantity x_1 and x_2 respectively. Also, P_3 and P_4 produce batteries of type B_2 in the number quantity x_3 and x_4 respectively.

Let the profit per battery is Rs 10 for B_1 and Rs 20 for B_2.

Therefore the total profit on the trade of these batteries is $(x_1 + x_2)10 + (x_3 + x_4)20$ and we have to maximize the profit.

Thus, the objective function

$$\text{Max. } Z = (x_1 + x_2)10 + (x_3 + x_4)20$$

$$\text{subject to } 12x_1 + 8x_2 + 6x_3 + 4x_4 \leq 120$$

$$3x_1 + 6x_2 + 12x_3 + 24x_4 \leq 180$$

$$x_1, x_2 \geq 0$$

Introducing new variables x_5 and x_6 so that,

Maximize: $z - 10x_1 - 10x_2 - 20x_3 - 20x_4 = 0$

Subject to $12x_1 + 8x_2 + 6x_3 + 4x_4 + x_5 = 120$

$3x_1 + 6x_2 + 12x_3 + 24x_4 + x_6 = 180.$

The tabled form of above problem is,

	z	x_1	x_2	$x_3\downarrow$	x_4	x_5	x_6	Constant	ratio
R_1	1	−10	−10	−20	−20	0	0	0	0
R_2	0	12	8	6	4	1	0	120	20
R_3	0	3	6	12	24	0	1	180	← 15

Now, we have to maximize the function. So, we observe the negative entry in R_1. The greatest negative entry is −20 in R_1. So, the column of x_3 is the pivot column and by ratio $\left(\text{ratio} = \frac{\text{constant}}{\text{pivot column}}\right)$, R_3 is the pivot row (row if least positive ratio).

Therefore, 12 is the pivot point.

To eliminate the values of the pivot column rather than the pivot, apply, $R_1 \to 3R_1 + 5R_3$, $R_2 \to 2R_2 - R_3$ then the above table becomes,

	z	$x_1\downarrow$	x_2	x_3	x_4	x_5	x_6	Constant	ratio
R_1	3	−15	0	0	60	0	5	900	−ve value
R_2	0	21	10	0	−16	2	−1	60	← 2.86
R_3	0	3	6	12	24	0	1	180	60

Again, R_1 has negative entry and that is, −15. So, the column of x_1 is pivot column then by ratio, R_2 is the pivot row and pivot point is 21.

Now, applying $R_1 \to 7R_1 + 5R_2$, $R_3 \to 7R_3 - R_2$ then the above table becomes,

	z	x_1	x_2	x_3	x_4	x_5	x_6	Constant	ratio
R_1	21	0	50	0	340	10	30	6600	
R_2	0	21	10	0	−16	2	−1	60	
R_3	0	0	32	84	184	−2	8	1200	

Here R_1 has no negative entry. So the table gives optimal solution.

Assume the non_basic variables are zero i.e. $x_2 = x_4 = x_5 = 0 = x_6$.

Then by R_1. $21z = 6600 \Rightarrow z = \frac{2200}{7}$

by R_2. $21x_1 = 60 \Rightarrow x_1 = \frac{20}{7}$

by R_3, $84x_3 = 1200 \Rightarrow x_3 = \frac{100}{7}$

Thus, max(z) = 2200/7 at $x_1 = 20/7$, $x_2 = 0$, $x_3 = 100/7$, $x_4 = 0$.

EXERCISE 5.3

1. Maximize $z = 6x_1 + 12x_2$

s.t. $0 \leq x_1 \leq 4$, $0 \leq x_2 \leq 4$

$6x_1 + 12x_2 \leq 72$

Solution: Given that, max: $z = 6x_1 + 12x_2$

Subject to $x_1 \leq 4$

$6x_1 + 12x_2 \leq 72$

$x_2 \leq 4.$

Introducing new variables x_3, x_4 and x_5 so that.

Maximize: $z - 6x_1 - 12x_2 = 0$

Subject to $x_1 + x_3 = 4$

$6x_1 + 12x_2 + x_4 = 72$

$x_2 + x_5 = 4.$

The tabled form of above problem is,

	z	x_1	$x_2\downarrow$	x_3	x_4	x_5	Constant	ratio
R_1	1	−6	−12	0	0	0	0	0
R_2	0	1	0	1	0	0	4	undefined
R_3	0	6	12	0	1	0	72	6
R_4	0	0	1	0	0	1	4	← 4

Now, we have to maximize the function. So, we observe the negative entry in R_1. The greatest negative entry is −12 in R_1. So, the column of x_2 is the pivot column and by ratio $\left(\text{ratio} = \frac{\text{constant}}{\text{pivot column}}\right)$, R_4 is the pivot row (row if least positive ratio).

Therefore, 1 is the pivot point.

To eliminate the values of the pivot column rather than the pivot, apply, $R_1 \to R_1 + 12R_4$, $R_3 \to R_3 - 12R_4$ then the above table becomes.

	z	$x_1 \downarrow$	x_2	x_3	x_4	x_5	Constant	ratio
R_1	1	−6	0	0	0	12	48	−ve value
R_2	0	1	0	1	0	0	4	← 4
R_3	0	6	0	0	1	−12	24	4
R_4	0	0	1	0	0	1	4	undefined

Again, R_1 has negative entry and that is, −6. So, the column of x_1 is pivot column then by ratio, R_2 is the pivot row and pivot point is 1.

Now, applying $R_1 \to R_1 + 6R_2$, $R_3 \to R_3 - 6R_2$ then the above table becomes,

	z	x_1	x_2	x_3	x_4	x_5	Constant	ratio
R_1	1	0	0	6	0	12	72	
R_2	0	1	0	1	0	0	4	
R_3	0	0	0	−6	1	−12	0	
R_4	0	0	1	0	0	1	4	

Here R_1 has no negative entry. So the table gives optimal solution.

Assume the non_basic variables are zero i.e. $x_3 = 0 = x_5$.

Then by R_1, $z = 72$

by R_2, $x_1 = 4$

by R_3, x_4 = neglect the value

by R_4, $x_2 = 4$

Thus, $\max(z) = 72$ at $(x_1, x_2) = (4, 4)$.

2. Maximize the daily output in producing x_1 glass plates by a process P_1 and x_2 glass plates by a process P_2 subject to the constrains

$2x_1 + 3x_2 \le 130$ (labor hours)

$3x_1 + 8x_2 \le 300$ (machine hours)

$4x_1 + 2x_2 \le 140$ (raw material supply).

Solution: Let the daily output in producing x_1 glass plates by a process P_1 and x_2 glass plates by a process P_2.

Therefore the total output is $(x_1 + x_2)$ and we have to maximize the production.

Thus, the objective function

$$\text{Max. } z = x_1 + x_2$$

$$\text{subject to } 2x_1 + 3x_2 \le 130$$

$$3x_1 + 8x_2 \le 300$$

$$4x_1 + 2x_2 \le 140.$$

Introducing new variables x_3, x_4 and x_5 so that,

$$\text{Max. } z - x_1 - x_2 = 0$$

$$\text{subject to } 2x_1 + 3x_2 + x_3 = 130$$

$$3x_1 + 8x_2 + x_4 = 300$$

$$4x_1 + 2x_2 + x_5 = 140.$$

The tabled form of above problem is,

	z	$x_1 \downarrow$	x_2	x_3	x_4	x_5	Constant	ratio
R_1	1	−1	−1	0	0	0	0	0
R_2	0	2	3	1	0	0	130	65
R_3	0	3	8	0	1	0	300	100
R_4	0	4	2	0	0	1	140	← 35

Now, we have to maximize the function. So, we observe the negative entry in R_1. The greatest negative entry is −1 in R_1. So, the column of x_1 is the pivot column and by ratio $\left(\text{ratio} = \frac{\text{constant}}{\text{pivot column}}\right)$, R_4 is the pivot row (row if least positive ratio).

Therefore, 4 is the pivot point.

To eliminate the values of the pivot column rather than the pivot, apply, $R_1 \to 4R_1 + R_4$, $R_2 \to 2R_2 - R_4$, $R_3 \to 4R_3 - 3R_4$ then the above table becomes,

	z	x_1	$x_2 \downarrow$	x_3	x_4	x_5	Constant	ratio
R_1	4	0	−2	0	0	1	140	−ve value
R_2	0	0	4	2	0	−1	120	← 30
R_3	0	0	26	0	4	−3	780	30
R_4	0	4	2	0	0	1	140	70

Again, R_1 has negative entry and that is, −2. So, the column of x_2 is pivot column then by ratio, R_2 is the pivot row and pivot point is 4.

Now, applying $R_1 \to 2R_1 + R_2$, $R_3 \to 2R_3 - 13R_2$, $R_4 \to 2R_4 - R_2$ then the above table becomes,

	z	x_1	$x_2 \downarrow$	x_3	x_4	x_5	Constant	ratio
R_1	8	0	0	2	0	1	400	
R_2	0	0	4	2	0	−1	120	
R_3	0	0	0	−26	8	7	0	
R_4	0	8	0	−2	0	3	160	

Here R_1 has no negative entry. So the table gives optimal solution.

Assume the non_basic variables are zero i.e. $x_3 = 0 = x_5$.

Then by R_1, $8z = 400 \Rightarrow z = 50$

by R_2, $4x_2 = 120 \Rightarrow x_2 = 30$

by R_3, x_4 = neglect the value

by R_4, $8x_1 = 160 \Rightarrow x_1 = 20$

Thus, $\max(z) = 50$ at $(x_1, x_2) = (20, 30)$.

Maximize $z = 300x_1 + 500x_2$

s.t. $2x_1 + 8x_2 \le 60$; $4x_1 + 4x_2 \le 60$; $2x_1 + x_2 \le 30$

Solution: Given problem is

$$\text{Max. } z = 300x_1 + 500x_2$$

$$\text{s.t. } 2x_1 + 8x_2 \le 60;\ 4x_1 + 4x_2 \le 60;\ 2x_1 + x_2 \le 30.$$

Introducing new variables x_3, x_4 and x_5 so that,

$$\text{Max. } z - 300x_1 - 500x_2 = 0$$

subject to $2x_1 + 8x_2 + x_3 = 60$

$4x_1 + 4x_2 + x_4 = 60$

$2x_1 + x_2 + x_5 = 30.$

The tabled form of above problem is,

	z	x_1	$x_2 \downarrow$	x_3	x_4	x_5	Constant	ratio
R_1	1	–300	–500	0	0	0	0	0
R_2	0	2	8	1	0	0	60	← 7.5
R_3	0	4	4	0	1	0	60	15
R_4	0	2	1	0	0	1	30	30

Now, we have to maximize the function. So, we observe the negative entry in R_1. The greatest negative entry is –500 in R_1. So, the column of x_2 is the pivot column and by ratio $\left(\text{ratio} = \frac{\text{constant}}{\text{pivot column}}\right)$, R_2 is the pivot row (row if least positive ratio).

Therefore, 8 is the pivot point.

To eliminate the values of the pivot column rather than the pivot, apply, $R_1 \to 2R_1 + 125R_2$, $R_3 \to 2R_3 - R_2$, $R_4 \to 8R_4 - R_2$ then the above table becomes,

	z	$x_1 \downarrow$	x_2	x_3	x_4	x_5	Constant	ratio
R_1	2	–350	0	125	0	0	7500	–ve value
R_2	0	2	8	1	0	0	60	30
R_3	0	6	0	–1	2	0	60	← 10
R_4	0	14	0	–1	0	8	180	12.86

Again, R_1 has negative entry and that is, –350. So, the column of x_1 is pivot column then by ratio, R_3 is the pivot row and pivot point is 6.

Now, applying $R_1 \to 3R_1 + 175R_3$, $R_2 \to 3R_2 - R_3$, $R_4 \to 3R_4 - 7R_3$ then the above table becomes,

	z	x_1	x_2	x_3	x_4	x_5	Constant	ratio
R_1	6	0	0	200	350	0	33000	
R_2	0	0	24	4	–2	0	120	
R_3	0	6	0	–1	2	0	60	
R_4	0	0	0	4	–14	24	120	

Here R_1 has no negative entry. So the table gives optimal solution.

Assume the non_basic variables are zero i.e. $x_3 = 0 = x_4$.

Then by R_1, $6z = 33000 \Rightarrow z = 5500$

by R_2, $24x_2 = 120 \Rightarrow x_2 = 5$

by R_3, $6x_1 = 60 \Rightarrow x_1 = 10$

by R_4, x_5 = neglect the value

Thus, $\max(z) = 5500$ at $(x_1, x_2) = (10, 5)$.

4. **Maximize $f = 6x_1 + 6x_2 + 9x_3$**
subject to $x_j \geq 0$ {for j = 1, 2, 3, 4, 5}
and $x_1 + x_3 + x_4 = 1$, $x_2 + x_3 + x_5 = 1$.

Solution: Given problem is

Max. $f - 6x_1 + 6x_2 + 9x_3 = 0$

subject to $x_1 + x_3 + x_4 = 1$

$x_2 + x_3 + x_5 = 1$

for $x_j \geq 0$ {for j = 1, 2, 3, 4, 5}.

The tabled form of above problem is,

	f	x_1	x_2	$x_3 \downarrow$	x_4	x_5	Constant	ratio
R_1	1	–6	–6	–9	0	0	0	0
R_2	0	1	0	1	0	1	1	← 4
R_3	0	0	1	1	0	1	1	15

Now, we have to maximize the function. So, we observe the negative entry in R_1. The greatest negative entry is –9 in R_1. So, the column of x_3 is the pivot column and by ratio $\left(\text{ratio} = \frac{\text{constant}}{\text{pivot column}}\right)$, R_2 is the pivot row (row if least positive ratio).

Therefore, 1 is the pivot point.

To eliminate the values of the pivot column rather than the pivot, apply, $R_1 \to R_1 + 9R_2$, $R_3 \to R_3 - R_2$ then the above table becomes,

	f	x_1	$x_2 \downarrow$	x_3	x_4	x_5	Constant	ratio
R_1	1	3	–6	0	9	0	9	–ve value
R_2	0	1	0	1	0	1	1	undefined
R_3	0	–1	1	0	–1	1	0	← 0

Again, R_1 has negative entry and that is, –6. So, the column of x_2 is pivot column then by ratio, R_3 is the pivot row and pivot point is 1.

Now, applying $R_1 \to R_1 + 6R_3$ then the above table becomes,

	f	$x_1 \downarrow$	x_2	x_3	x_4	x_5	Constant	ratio
R_1	1	–3	0	0	3	6	9	–ve value
R_2	0	1	0	1	1	0	1	← 1
R_3	0	–1	1	0	–1	1	0	0

Again, R_1 has negative entry and that is, –3. So, the column of x_1 is pivot column then by ratio, R_3 is the pivot row and pivot point is 1.

Now, applying $R_1 \to R_1 + 6R_3$ then the above table becomes,

	f	x_1	x_2	x_3	x_4	x_5	Constant	ratio
R_1	1	0	0	3	6	6	12	
R_2	0	1	0	1	1	0	1	
R_3	0	0	1	1	0	1	1	

Here R_1 has no negative entry. So the table gives optimal solution.

Assume the non_basic variables are zero i.e. $x_3 = 0 = x_4 = x_5$.

Then by R_1, $f = 12$

by R_2, $x_1 = 1$ by R_3, $x_2 = 1$

Thus, $\max(f) = 12$ at $(x_1, x_2, x_3) = (1, 1, 0)$.

5. Maximize $f = 4x_1 + x_2 + 2x_3$

s.t. $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_1 + x_2 + x_3 \leq 1, x_1 + x_2 - x_3 \leq 0$.

Solution: Given problem is

$$\text{Max. } f = 4x_1 + x_2 + 2x_3$$

$$\text{s.t. } x_1 + x_2 + x_3 \leq 1;\ x_1 + x_2 - x_3 \leq 0.$$

Introducing new variables x_4 and x_5 so that,

$$\text{Max. } f - 4x_1 - x_2 - 2x_3 = 0$$

subject to $x_1 + x_2 + x_3 + x_4 = 1$

$x_1 + x_2 - x_3 + x_5 = 0.$

The tabled form of above problem is,

	f	x_1 ↓	x_2	x_3	x_4	x_5	Constant	ratio
R_1	1	−4	−1	−2	0	0	0	0
R_2	0	1	1	1	1	0	1	1
R_3	0	1	1	−1	0	1	0	← 0

Now, we have to maximize the function. So, we observe the negative entry in R_1. The greatest negative entry is −4 in R_1. So, the column of x_1 is pivot column and there arise degenerate condition on R_3. So, R_2 is the pivot row and 1 is the pivot point.

To eliminate the values of the pivot column rather than the pivot, apply, $R_1 \to R_1 + 4R_3$, $R_2 \to R_2 - R_3$ then the above table becomes,

	f	x_1	x_2	x_3 ↓	x_4	x_5	Constant	ratio
R_1	1	0	3	−6	0	4	0	0
R_2	0	0	0	2	1	−1	1	← 2
R_3	0	1	1	−1	0	1	0	0

Again, R_1 has negative entry and that is, −6. So, the column of x_3 is pivot column then by ratio by ratio $\left(\text{ratio} = \dfrac{\text{constant}}{\text{pivot column}}\right)$(we observe least positive ratio), R_2 is the pivot row and pivot point is 2.

Now, applying $R_1 \to R_1 + 3R_3$, $R_3 \to 2R_3 + R_2$ then the above table becomes,

	f	x_1	x_2	x_3	x_4	x_5	Constant	ratio
R_1	1	0	3	0	3	1	3	
R_2	0	0	0	2	1	−1	1	
R_3	0	2	2	0	1	1	1	

Here R_1 has no negative entry. So the table gives optimal solution.

Assume the non–basic variables are zero i.e. $x_2 = 0 = x_4 = x_5$.

Then by R_1, $f = 3$

by R_2, $2x_3 = 1 \Rightarrow x_3 = 0.5$

by R_3, $2x_1 = 1 \Rightarrow x_1 = 0.5$

Thus, $\max(f) = 3$ at $(x_1, x_2, x_3) = (0.5, 0, 0.5)$.

6. Maximize $f = -10x_1 + 2x_2$

s.t. $x_1 \geq 0, x_2 \geq 0; -x_1 + x_2 \geq -1, x_1 + x_2 \leq 6, x_2 \leq 5$.

Solution: Given problem is

$$\text{Max. } f = -10x_1 + 2x_2$$

$$\text{s.t. } -x_1 + x_2 \geq -1;\ x_1 + x_2 \leq 6;\ x_2 \leq 5.$$

Introducing new variables x_3, x_4 and x_5 so that,

$$\text{Max. } f + 10x_1 - 2x_2 = 0$$

subject to $x_1 - x_2 + x_3 = 1$

$x_1 + x_2 + x_4 = 6$

$x_2 + x_5 = 5.$

The tabled form of above problem is, + x_4

	f	x_1	x_2	x_3	x_4	x_5	Constant	ratio
R_1	1	10	−2	0	0	0	0	0
R_2	0	1	−1	1	0	0	1	−ve value
R_3	0	1	1	0	1	0	6	6
R_4	0	0	1	0	0	1	5	← 5

Now, we have to maximize the function. So, we observe the negative entry in R_1. The greatest negative entry is −2 in R_1. So, the column of x_2 is the pivot column and by ratio $\left(\text{ratio} = \dfrac{\text{constant}}{\text{pivot column}}\right)$, R_4 is the pivot row (row if least positive ratio).

Therefore, 1 is the pivot point.

To eliminate the values of the pivot column rather than the pivot, apply, $R_1 \to R_1 + 2R_4$, $R_2 \to R_2 + R_4$, $R_3 \to R_3 - R_4$ then the above table becomes,

	f	x_1	x_2	x_3	x_4	x_5	Constant	ratio
R_1	1	12	0	0	0	2	10	
R_2	0	1	0	1	0	1	6	
R_3	0	1	0	0	1	−1	1	
R_4	0	0	1	0	0	1	5	

Here R_1 has no negative entry. So the table gives optimal solution.

Assume the non–basic variables are zero i.e. $x_1 = 0 = x_5$.

Then by R_1, $f = 10$

by R_2, $x_3 = 6$ by R_3, $x_4 = 1$ by R_4, $x_2 = 5$

Thus, $\max(f) = 10$ at $(x_1, x_2) = (0, 5)$.

B. Construct the dual problem corresponding to each of the following linear programming problems.

1. Minimize $z = 6x_1 + 4x_2$

s.t. $2x_1 + x_2 \geq 1$; $6x_1 + 8x_2 \geq 3$; $x_1 \geq 0, x_2 \geq 0$.

Solution: Given that we have to minimize $Z = 6x_1 + 4x_2$

S.t. $2x_1 + x_2 \geq 1$; $6x_1 + 8x_2 \geq 3$; $x_1, x_2 \geq 0$.

The given problem is standard minimization problem with all constraint ≥ type.

x_1	x_2	Constant
2	1	1 (y_1)
6	8	3 (y_2)

Let y_1 and y_2 be the dual variable then

Max. $W = y_1 + 3y_2$

s.t. $2y_1 + 6y_2 \le 6$

$y_1 + 8y_2 \le 4$

$y_1, y_2 \ge 0$ is required dual.

2. **Maximize $z = 3x_1 + x_2$**

s.t. $x_1 + x_2 \le 1$; $2x_1 + 3x_2 \le 5$, $x_1 \ge 0$, $x_2 \ge 0$.

Solution: Given that we have to maximize $Z = 3x_1 + x_2$

s.t. $x_1 + x_2 \le 1$

$2x_1 + 3x_2 \le 5$

$x_1, x_2 \ge 0$

The given primal standard maximization type with au constraint $\le$ type.

x_1	x_2	Constant
1	1	1 (y_1)
2	3	5 (y_2)
3	1	

Let y_1 and y_2 be the dual variable, then its dual becomes

Min. $W = y_1 + 5y_2$

s.t. $y_1 + y_2 \ge 3$

$y_1 + 3y_2 \ge 1$

$y_1, y_2 \ge 0$.

3. **Maximize $z = x_1 - 2x_2 + 3x_3$**

s.t. $-2x_1 + x_2 + 3x_3 = 2$; $2x_1 + 3x_2 + 4x_3 = 1$, $x_1 \ge 0$, $x_2 \ge 0$, $x_3 \ge 0$.

Solution: Both constraint hold equality sign so dual variable are unrestricted in sign.

x_1	x_2	x_3	
-2	1	3	2 (y_1)
2	3	4	1 (y_2)
1	-2	3	

Let y_1 and y_2 be dual variables, then

Min. $Z = 2y_1 + y_2$

s.t. $-2y_1 + 2y_2 \ge 1$

$y_1 + 3y_2 \ge -2$

$3y_1 + 4y_2 \ge 3$

y_1 and y_2 is restricted in sign.

4. **Minimize $z = 3x_1 + 2x_2$**

s.t. $x_1 + 3x_2 = 4$; $2x_1 + x_2 = 3$, $x_1 \ge 0$, $x_2 \ge 0$.

Solution: Both constraint hold equality sign, so dual variable are unrestricted in sign.

x_1	x_2	
1	3	4 (y_1)
2	1	3 (y_2)
3	2	

Let y_1 and y_2 be dual variables then

Max. $z = 4y_1 + 3y_2$

s.t. $y_1 + 2y_2 \le 3$

$3y_1 + y_2 \le 2$

y_1, y_2 is unrestricted in sign.

Solve the following linear programming problems by using simplex method (Hint: by constructing duality)

Minimize $z = 2x_1 - 3x_2$

s.t. $2x_1 - x_2 - x_3 \ge 3$; $x_1 - x_2 + x_3 \ge 2$; $x_1 \ge 0$, $x_2 \ge 0$, $x_3 \ge 0$.

olution: The given problem is standard form of minimization problems. So

x_1	x_2	x_3	Constant
2	−1	−1	3 (y_1)
2	−1	1	2 (y_2)
2	−3	0	

Let y_1 and y_2 be dual variable, so

Max.: $W = 3y_1 + 2y_2$

s.t.: $2y_1 + y_2 \le 2$

$-y_1 - y_2 \le -3$

$-y_1 + y_2 \le 0$

$y_1, y_2 \ge 0$

Now, introducing new variables y_3, y_4 and y_5 so that,

Max. $W - 3y_1 - 2y_2 = 0$

subject to $2y_1 + y_2 + y_3 = 2$

$-y_1 - y_2 + y_4 = 3$

$-y_1 + y_2 + y_5 = 0$.

The tabled form of above problem is,

	W	$y_1 \downarrow$	y_2	y_3	y_4	y_5	Constant	ratio
R_1	1	−3	−2	0	0	0	0	0
R_2	0	2	1	1	0	0	2	$\leftarrow$ 1
R_3	0	−1	−1	0	1	0	3	−ve value
R_4	0	−1	1	0	0	1	0	undefined

Now, we have to maximize the function. So, we observe the negative entry in R_1. The greatest negative entry is −3 in R_1. So, the column of y_1 is pivot column and by ratio by ratio $\left(\text{ratio} = \frac{\text{constant}}{\text{pivot column}}\right)$(we observe least positive ratio), R_2 is the pivot row and pivot point is 2.

To eliminate the values of the pivot column rather than the pivot, apply, $R_1 \to 2R_1 + 3R_2$, $R_3 \to 2R_3 + R_2$, $R_4 \to 2R_4 + R_2$ then the above table becomes,

	W	y_1	$y_2 \downarrow$	y_3	y_4	y_5	Constant	ratio
R_1	2	0	–1	3	0	0	6	–ve value
R_2	0	2	1	1	0	0	2	2
R_3	0	0	–1	1	2	0	8	–ve value
R_4	0	0	3	1	0	2	2	$\leftarrow$ 1.5

Again, R_1 has negative entry and that is, –1. So, the column of y_2 is pivot column and by ratio R_4 is the pivot row and pivot point is 3.

Now, applying $R_1 \to 3R_1 + R_4$, $R_2 \to 3R_2 - R_4$, $R_3 \to 3R_3 + R_4$ then the above table becomes,

	W	y_1	$y_2 \downarrow$	y_3	y_4	y_5	Constant	ratio
R_1	6	0	0	10	0	2	20	
R_2	0	6	0	2	0	–2	4	
R_3	0	0	0	4	6	2	26	
R_4	0	0	3	1	0	2	2	

Here R_1 has no negative entry. So the table gives optimal solution.

Assume the non–basic variables are zero i.e. $y_3 = 0 = y_5$.

Then by R_1, $W = 10/3$; by R_2, $6y_1 = 4$

by R_3, $6y_4 = 26$ by R_4, $3y_2 = 2$

Thus, $\max(W) = 10/3$ at $(y_1, y_2) = (0.66, 0.66)$.

Therefore, $\min(z) = 10/3$ at $(x_1, x_2, x_3) = (5/3, 0, 1/3)$.

2. Minimize $z = 4x_1 + 3x_2$

s.t. $2x_1 + 3x_2 \geq 1$, $3x_1 + x_2 \geq 4$; $x_1 \geq 0$, $x_2 \geq 0$.

Solution: Given problem is standard form of minimization problem. So its dual is

x_1	x_2	Constant
2	3	1 (y_1)
3	1	4 (y_2)
4	3	

Let y_1 and y_2 be the dual variables then its dual is

Max. $Z = y_1 + 4y_2$

s.t. $2y_2 + 3y_2 \leq 4$

$3y_1 + y_2 \leq 3$

$y_1, y_2 \geq 0$

Similar to 1.

3. Minimize $z = 2x_1 + 9x_2 + x_3$

s.t. $x_1 + 4x_2 + 2x_3 \geq 5$; $3x_1 + x_2 + 2x_3 \geq 4$; $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$.

Solution: The given primal standard form of maximization. So, its dual is

x_1	x_2	x_3	Constant
1	4	2	5 (y_1)
3	1	2	4 (y_2)
2	9	1	

Let y_1 and y_2 be dual variable then

Max. $w = 5y_1 + 4y_2$

s.t. $y_1 + 3y_2 \leq 2$

$4y_1 + y_2 \leq 9$

$2y_1 + 2y_2 \leq 1$ $y_1, y_2 \geq 0$.

Solution: The given problem is standard form of minimization problems. So

x_1	x_2	x_3	Constant
2	–1	–1	3 (y_1)
2	–1	1	2 (y_2)
2	–3	0	

Let y_1 and y_2 be dual variable, so

Max.: $W = 3y_1 + 2y_2$

s.t.: $2y_1 + y_2 \leq 2$

$-y_1 - y_2 \leq -3$

$-y_1 + y_2 \leq 0$

$y_1, y_2 \geq 0$

Now, introducing new variables y_3, y_4 and y_5 so that,

Max. $W - 3y_1 - 2y_2 = 0$

subject to $2y_1 + y_2 + y_3 = 2$

$-y_1 - y_2 + y_4 = 3$

$-y_1 + y_2 + y_5 = 0$.

The tabled form of above problem is,

	W	$y_1 \downarrow$	y_2	y_3	y_4	y_5	Constant	ratio
R_1	1	–5	–4	0	0	0	0	0
R_2	0	1	3	1	0	0	2	2
R_3	0	4	1	0	1	0	9	2.25
R_4	0	2	2	0	0	1	1	$\leftarrow$ 0.5

Now, we have to maximize the function. So, we observe the negative entry in R_1. The greatest negative entry is –3 in R_1. So, the column of y_1 is pivot column and by ratio by ratio $\left(\text{ratio} = \frac{\text{constant}}{\text{pivot column}}\right)$(we observe least positive ratio). R_4 is the pivot row and pivot point is 2.

To eliminate the values of the pivot column rather than the pivot, apply, $R_1 \to 2R_1 + 5R_4$, $R_2 \to 2R_2 - R_4$, $R_3 \to R_3 - R_4$ then the above table becomes,

	W	y_1	y_2	y_3	y_4	y_5	Constant	ratio
R_1	2	0	1	0	0	5	5	
R_2	0	0	4	2	0	–1	3	
R_3	0	0	–3	0	1	–2	7	
R_4	0	2	2	0	0	1	1	

This implies

	W	y_1	y_2	y_3	y_4	y_5	Constant	ratio
R_1	1	0	1/2	0	0	5/2	5/2	
R_2	0	-0	4	2	0	−1	3	
R_3	0	0	−3	0	1	−2	7	
R_4	0	2	2	0	0	1	1	

Here R_1 has no negative entry. So the table gives optimal solution.

Assume the non–basic variables are zero i.e. $y_3 = 0 = y_5$.

Then by R_1, $\quad W = 5/2$ at (1/2, 0)

Thus, $\max(W) = 5/2$ at $(y_1, y_2) = (1/2, 0)$.

Therefore, $\min(z) = 5/2$ at $(x_1, x_2, x_3) = (0, 0, 5/2)$.

4. Minimize $z = 3x_1 + 2x_2$

s.t. $2x_1 + 4x_2 \geq 10$; $4x_1 + 2x_2 \geq 10$, $x_2 \geq 4$, $x_1 \geq 0$, $x_2 \geq 0$.

Solution: The given primal standard form of minimization problem. Its dual becomes

x_1	x_2	Constant
2	4	10 (y_1)
4	2	10 (y_2)
0	1	4 (y_3)
3	2	

Let y_1, y_2 and y_3 be its dual variable then its dual is

$\quad$ Max. $Z = 10y_1 + 10y_2 + 4y_3$

$\quad$ s.t. $\; 2y_1 + 4y_2 + 0.y_3 \leq 3$

$\quad\quad 4y_1 + 2y_2 + y_3 \leq 2$

$\quad\quad y_1, y_2, y_3 \geq 0$

Now, introducing new variables y_3, y_4 and y_5 so that,

$\quad$ Max. $\; W - 3y_1 - 2y_2 = 0$

subject to $\; 2y_1 + y_2 + y_3 = 2$

$\quad -y_1 - y_2 + y_4 = 3$

$\quad -y_1 + y_2 + y_5 = 0$.

The tabled form of above problem is,

	W	y_1 ↓	y_2	y_3	y_4	y_5	Constant	ratio
R_1	1	−10	−10	−4	0	0	0	0
R_2	0	2	4	0	1	0	3	1.5
R_3	0	4	2	1	0	1	2	← 0.5

Now, we have to maximize the function. So, we observe the negative entry in R_1. The greatest negative entry is −10 in R_1. So, the column of y_1 is pivot column and by ratio by ratio $\left(\text{ratio} = \frac{\text{constant}}{\text{pivot column}}\right)$(we observe least positive ratio), R_3 is the pivot row and pivot point is 4.

To eliminate the values of the pivot column rather than the pivot, apply, $R_1 \to 2R_1 + 5R_3$, $R_2 \to 2R_2 - R_3$ then the above table becomes,

	W	y_1	y_2 ↓	y_3	y_4	y_5	Constant	ratio
R_1	2	0	−10	−3	0	5	10	−ve value
R_2	0	0	6	−1	2	−1	4	← 0.66
R_3	0	4	2	1	0	1	2	1

Again, R_1 has negative entry and that is, −10. So, the column of y_2 is pivot column and by ratio R_2 is the pivot row and pivot point is 6.

Now, applying $R_1 \to 3R_1 + 5R_2$, $R_3 \to 3R_3 - R_2$ then the above table becomes,

	W	y_1	y_2	y_3 ↓	y_4	y_5	Constant	ratio
R_1	6	0	0	−14	10	10	50	−ve value
R_2	0	0	6	−1	2	−1	4	−ve value
R_3	0	12	0	4	−2	4	2	$\frac{-}{2}$

Again, R_1 has negative entry and that is, −14. So, the column of y_3 is pivot column and by ratio R_3 is the pivot row and pivot point is 4.

Now, applying $R_1 \to 2R_1 + 7R_3$ then the above table becomes,

	W	y_1	y_2	y_3	y_4	y_5	Constant	ratio
R_1	12	84	0	0	6	48	114	
R_2	0	0	6	−1	2	−1	4	
R_3	0	12	0	4	−2	4	2	

That is

	W	y_1	y_2	y_3	y_4	y_5	Constant	ratio
R_1	1	7	0	0	½	4	19/2	
R_2	0	0	6	−1	2	−1	4	
R_3	0	12	0	4	−2	4	2	

Here R_1 has no negative entry. So the table gives optimal solution.

Assume the non–basic variables are zero i.e. $y_1 = y_3 = y_4 = 0 = y_5$.

Then by R_1, $\quad W = 19/2$

Thus, $\max(W) = 19/2$ at (0, 2/3, 0).

Therefore, $\min(z) = 19/2$ at $(x_1, x_2) = (1/2, 4)$.

5. Minimize $z = 10x_1 + 15x_2$

s.t. $x_1 + x_2 \geq 8$, $10x_1 + 6x_2 \geq 60$, $x_1 \geq 0$, $x_2 \geq 0$.

Solution: Given that, we have to minimize $Z = 10x_1 + 15x_2$

$\quad$ s.t. $x_1 + x_2 \geq 8$; $10x_1 + 6x_2 \geq 60$

$\quad\quad x_1, x_2 \geq 0$

Its dual becomes

x_1	x_2	Constant
1	1	8 (y_1)
10	6	60 (y_2)
10	15	

Let y_1 and y_2 be its dual variable, then

Max. $Z = 8y_1 + 60y_2$

s.t. $y_1 + 10y_2 \leq 10$

$y_1 + 6t_2 \leq 15$

$y_1, y_2 \geq 0$

Similar to 4.

6. **Minimize $z = 20x_1 + 30x_2$**

s.t. $x_1 + 4x_2 \geq 8, x_1 + x_2 \geq 5; 2x_1 + x_2 \geq 7, x_1 \geq 0, x_2 \geq 0.$

Solution:

x_1	x_2	Constant
1	4	8 (y_1)
1	1	5 (y_2)
2	1	7 (y_3)
20	30	

Let y_1, y_2 and y_3 be the dual variables, then its dual become

Max. $Z = 8y_1 + 5y_2 + 7y_3$

s.t. $y_1 + y_2 + 2y_3 \leq 20$

$4y_1 + y_2 + y_3 \leq 30$

$y_1, y_2, y_3 \geq 0$

Similar to 5.