Exercise 6.7

A. **Find solutions of the following differential equation.**

(1) $y'' + 5y' + 6y = 0$

Solution: Given that, $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$

The auxiliary equation is,

$$m^2 + 5m + 6 = 0 \Rightarrow m^2 + 2m + 3m + 6 = 0\,]$$

$$\Rightarrow (m + 2)(m + 3) = 0 \Rightarrow m = -2, -3.$$

So, $m_1 = -2$, $m_2 = -3$.

So the solutions are,

$$y_1 = e^{mx} = e^{-2x} \quad \text{and} \quad y_2 = e^{mx} = e^{-3x}$$

(2) $y'' + 6y' + 9y = 0$

Solution: Given that, $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$

The auxiliary equation is,

$$m^2 + 6m + 9 = 0 \Rightarrow m^2 + 3m + 3m + 9 = 0$$

$$\Rightarrow m(m + 3) + 3(m + 3) = 0$$

$$\Rightarrow (m + 3)(m + 3) = 0 \Rightarrow m = -3, -3.$$

So, $m_1 = -3$, $m_2 = -3$.

It has real double root hence we obtained only one solution

$$y = {}^{mx} = e^{-3x}$$

(3) $y'' + y' = 0.$

Solution: Given that, $\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$

The auxiliary equation is

$$m^2 + m = 0 \Rightarrow m(m + 1) = 0 \Rightarrow m = 0, -1$$

So, $m_1 = 0$, $m_2 = -3$.

So the solutions are,

$$y_1 = e^0 = 1 \quad \text{and} \quad y_2 = e^{-x}$$

(4) $y'' + y' - 2y = 0.$

Solution: Given that, $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$

The auxiliary equation is,

$$m^2 + m - 2 = 0 \Rightarrow m^2 + 2m - m - 2 = 0$$
$$\Rightarrow m(m+2) - 1(m+2) = 0$$
$$\Rightarrow (m+2)(m-1) = 0 \Rightarrow m = 1, -2.$$

So, $m_1 = 1$, $m_2 = -2$.

So the solutions are,

$$y_1 = e^x \quad \text{and} \quad y_2 = e^{-2x}$$

(5) $y'' + w^2y = 0$.

Solution: Given that, $\frac{d^2y}{dx^2} + w^2y = 0$

The auxiliary equation is,

$$m^2 + w^2 = 0 \Rightarrow m^2 = -w^2 \Rightarrow m = iw, -iw.$$

Thus, m has two imaginary roots, so its solutions are,

$$y_1 = e^{-iwx} \quad \text{and} \quad y_2 = e^{iwx}$$

(6) $y'' - y' - 2y = 0$.

Solution: Given that, $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$

The auxiliary equation is,

$$m^2 - m - 2 = 0 \Rightarrow m^2 - 2m + m - 2 = 0$$
$$\Rightarrow m(m-2) + 1(m-2) = 0$$
$$\Rightarrow (m-2)(m+1) = 0 \Rightarrow m = 2, -1.$$

So, $m_1 = 2$, $m_2 = -1$.

So the solutions are,

$$y_1 = e^{2x} \quad \text{and} \quad y_2 = e^{-x}$$

(7) $y'' + 2y' - 3y = 0$.

Solution: Given that, $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 0$

The auxiliary equation is,

$$m^2 + 2m - 3 = 0 \Rightarrow m^2 + 3m - m - 3 = 0$$
$$\Rightarrow m(m+3) - 1(m+3) = 0$$
$$\Rightarrow (m+3)(m-1) = 0 \Rightarrow m = -3, 1$$

So, $m_1 = -3$, $m_2 = 1$.

So the solutions are,

$$y_1 = e^{-3x} \quad \text{and} \quad y_2 = e^{m_2x} = e^x.$$

(8) $y'' + y' + y = 0$

Solution: Given that, $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0.$

The auxiliary equation is,

$$m^2 + m + 1 = 0 \Rightarrow m^2 + 2m\frac{1}{2} + \left(\frac{1}{2}\right)^2 + \frac{3}{4} = 0$$
$$\Rightarrow \left(m + \frac{1}{2}\right)^2 = -\frac{3}{4}$$
$$\Rightarrow \left(m + \frac{1}{2}\right)^2 = i^2\left(\frac{\sqrt{3}}{2}\right)^2$$
$$\Rightarrow \left(m + \frac{1}{2}\right)^2 = \left(\pm i\frac{\sqrt{3}}{2}\right)^2 \Rightarrow m + \frac{1}{2} = \pm i\frac{\sqrt{3}}{2}$$

So, $m_1 = i\frac{\sqrt{3}}{2} - \frac{1}{2} = \frac{1}{2}(i\sqrt{3} - 1)$ and $m_2 = i\frac{\sqrt{3}}{2} - \frac{1}{2} = -\frac{1}{2}(i\sqrt{3} + 1)$

So the solutions are,

$$\left.\begin{aligned} y_1 &= e^{m_1x} = e^{\frac{1}{2}(i\sqrt{3}+1)x} \\ y_2 &= e^{m_2x} = e^{\frac{1}{2}(i\sqrt{3}+1)} \end{aligned}\right\}$$

(9) $y'' - 2y + 4y = 0$.

Solution: Given that, $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 4y = 0.$

The auxiliary equation is

$$m^2 - 2m + 4 = 0 \Rightarrow m^2 - 2.m.1 + (1)^2 + \sqrt{3}^2 = 0$$
$$\Rightarrow (m-1)^2 = -3$$
$$\Rightarrow (m-1)^2 = (\pm i\sqrt{3})^2 \Rightarrow m - 1 = \pm i\sqrt{3}$$

So, $m_1 = i\sqrt{3} + 1$ and $m_2 = 1 - i\sqrt{3}$

So the solutions are,

$$\left.\begin{aligned} y_1 &= e(1 + i\sqrt{3})x \\ y_2 &= e^{m}2^x = e(1 - i\sqrt{3}x) \end{aligned}\right\}$$

B. Find a differential equation of the form $y'' + ay' + by = 0$ for which the following functions are the solutions.

(i) e^{2x}, e^{-2x}

Solution: Given that the roots are, $m = -2, 2$

Its auxiliary equation is,

$$(m + 2)(m - 2) = 0 \Rightarrow m^2 - 4 = 0$$

So, its different equation is,

$$y'' - 4y = 0.$$

(ii) $e^{(2+1)x}, e^{(2-1)x}$

Solution: Given that the roots are, $m = (2 + 1), (2 - i)$

Its auxiliary equations is,

$$(m - 2)^2 = (\pm i)^2 \Rightarrow (m - 2)^2 = -1$$

$$\Rightarrow m^2 - 4m + 4 = 1$$

$$\Rightarrow m^2 - 4m + 5 = 0$$

So, its different equation is,

$$y'' - 4y' + 5y = 0.$$

(iii) $e^{-2x}, 1$

Solution: Given that the roots are, $m = -2, 0.$

Its auxiliary equation is,

$$(m + 2)(m - 0) = 0 \Rightarrow m(m + 2) = 0$$

$$\Rightarrow m^2 + 2m = 0.$$

So, its different equation is,

$$y'' + 2y' = 0.$$