# Exercise 8.3

**1. Find the Laplace transform of the following:**

**(i) $e^{-2t} \cos t$**

**Solution:** Given function is, $e^{-2t} \cos t$

Since by first shifting theorem

$$\mathcal{L}\{e^{at} f(t)\} = (\mathcal{L}\{f(t)\})_{s \to s-a} \quad \text{and} \quad \mathcal{L}\{\cos t\} = \frac{s}{s^2+1}$$

Now,

$$\mathcal{L}\{e^{-2t} \cos t\} = (\mathcal{L}\{\cos t\})_{s \to s+2}$$

$$= \left(\frac{s}{s^2+1}\right)_{s \to s+2} = \frac{s+2}{(s+2)^2+1}.$$

Thus, $\mathcal{L}\{e^{-2t} \cos t\} = \dfrac{s+2}{(s+2)^2+1}$

**(ii) $\sin 2t\, u_\pi(t)$**

**Solution:** Given function is, $\sin 2t\, u_\pi(t)$

Since by second shifting theorem

$$\mathcal{L}\{f(t-a)\,u_a(t)\} = e^{-as}\,\mathcal{L}\{f(t)\} \quad \text{and} \quad \mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$$

Now,

$$\mathcal{L}\{\sin 2t\, u_\pi(t)\} = \mathcal{L}\{\sin(\pi-2t)\, u_\pi(t)\}$$

$$= \mathcal{L}\{-\sin(2t-\pi)\, u_\pi(t)\} \qquad [\because \sin(-\theta) = -\sin\theta]$$

$$= -\mathcal{L}\{\sin(2t-\pi)\, 4_\pi(t)\}$$

$$= -e^{-\pi s}\,\mathcal{L}\{\sin 2t\} = -e^{-\pi s}\frac{2}{s^2+4} = -\frac{2e^{-\pi s}}{s^2+4}$$

Thus, $\mathcal{L}\{\sin 2t\, u_\pi(t)\} = -\dfrac{2e^{-\pi s}}{s^2+4}$

**(iii) $e^{-at}(A\cos\beta t + B\sin\beta t)$**

**Solution:** Given function is, $e^{-at}(A\cos\beta t + B\sin\beta t)$

Now, Laplace transform of the function is

$$\mathcal{L}\{e^{-at}(A\cos\beta t + B\sin\beta t)\} = A\,\mathcal{L}\{e^{-at}\cos\beta t\} + B\,\mathcal{L}\{e^{-at}\sin\beta t\} \quad \text{......(i)}$$

Since we have by first shifting theorem

$$\mathcal{L}\{e^{at}f(t)\} = (\mathcal{L}\{f(t)\})_{s\to s-a}$$

and, $\mathcal{L}\{\sin at\} = \dfrac{a}{s^2+a^2}$, $\quad \mathcal{L}\{\cos at\} = \dfrac{s}{s^2+a^2}$

Them (1) becomes,

$$\mathcal{L}\{e^{-at}(A\cos\beta t + B\sin\beta t)\} = A(\mathcal{L}\{\cos\beta t\})_{s\to s+a} + B(\mathcal{L}\{\sin\beta t\})_{s\to s+a}$$

$$= A\cdot\left(\frac{s}{s^2+b^2}\right)_{s\to s+a} + B\left(\frac{b}{s^2+b^2}\right)_{s\to s+a}$$

$$= A\cdot\left(\frac{s+a}{(s+a)^2+b^2}\right) + B\left(\frac{b}{(s+a)^2+b^2}\right)$$

$$= \frac{A(s+a)+Bb}{(s+a^2)+b^2}$$

Thus, $\mathcal{L}\{e^{-at}(A\cos\beta t + B\sin\beta t)\} = \dfrac{A(s+a)+Bb}{(s+a^2)+b^2}$.

**(iv) $\left\{\cosh 2t + \frac{1}{2}\sin h\, 2t\right\}$**

**Solution:** Given function is,

$$e^t\left\{\cosh 2t + \frac{1}{2}\sin h\, 2t\right\} = e^t\left[\frac{e^{2t}+e^{-2t}}{2} + \frac{1}{2}\cdot\frac{e^{2t}-e^{-2t}}{2}\right]$$

$$= \frac{e^t}{4}(2e^{2t} + 2e^{-2t} + e^{2t} - e^{-2t})$$

$$= \frac{e^t}{4}(3e^{2t} + e^{-2t})$$

$$= \frac{3}{4}\cdot e^{3t} + \frac{1}{4}e^{-t}$$

Then the Laplace transform of the function is



$$\mathcal{L}\{e^t(\cos h\, 2t + \frac{1}{2}\sin h\, 2t)\} = \frac{3}{4}\mathcal{L}\{e^{3t}\} + \frac{1}{4}\mathcal{L}\{e^{-t}\} \quad \text{......(1)}$$

Since we have,

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

So, (1) becomes,

$$\mathcal{L}\{e^t(\cosh 2t + \frac{1}{2}\sinh 2t)\} = \frac{3}{4}\cdot\left(\frac{1}{s-3}\right) + \frac{1}{4}\left(\frac{1}{s+1}\right)$$

$$= \frac{3s+3+s-3}{4(s-3)(s+1)}$$

$$= \frac{4s}{(4(s-3)(s+1))}$$

$$= \frac{s}{(s-3)(s+1)} = \frac{s}{s^2-2s-3} = \frac{s}{(s-1)^2-4}$$

Thus, $\mathcal{L}\{e^t(\cosh 2t + \frac{1}{2}\sinh 2t)\} = \dfrac{s}{(s-1)^2-4}$.

**(v) $e^{-3t}u_2(t)$**

**Solution:** Given function is

$$e^{-3t}u_2(t) = e^{-3(t-2)-6}u_2(t) = e^{-6}e^{-3(t-2)}u_2(t)$$

Then Laplace transform of the function is

$$\mathcal{L}\{e^{-3t}u_2(t)\} = \mathcal{L}\{e^{-6}e^{-3(t-2)}u_2(t)\}$$

$$= e^{-6}\,L\{e^{-3(t-2)}u_2(t)\} \quad \text{........(1)}$$

Since the second shifting theorem given is

$$\mathcal{L}\{f(t-a)\,u_a(t)\} = e^{-as}\mathcal{L}\{f(t)\} \quad \text{and} \quad \mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

Then (1) becomes,

$$\mathcal{L}\{e^{-3t}u_2(t)\} = e^{-6}\cdot e^{-2s}\mathcal{L}\{e^{-3t}\}$$

$$= e^{-(2s+6)}\cdot\frac{1}{s+3} = \frac{e^{-2(s+3)}}{s+3}.$$

Thus, $\mathcal{L}\{e^{-3t}u_2(t)\} = \dfrac{e^{-2(s+3)}}{s+3}$

**(vi) $t^2e^{-3t}$**

**Solution:** Given function is, $t^2e^{-3t}$

Since by first shifting theorem we have,

$$\mathcal{L}\{e^{at}f(t)\} = (\mathcal{L}\{f(t)\})_{s\to s-1} \quad \text{and} \quad \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

Now, the Laplace transform of (1) is

$$\mathcal{L}\{t^2e^{-3t}\} = (\mathcal{L}\{t^2\})_{s\to s+3}$$

$$= \left(\frac{2!}{s^3}\right)_{s\to s+3} = \frac{s}{(s+3)^3}$$

Thus, $\mathcal{L}\{t^2e^{-3t}\} = \dfrac{s}{(s+3)^3}$

**(vii)** $5e^{2t}$ **sinh 2t** ........(i)

**Solution:** Given function is, $5e^{2t}\sinh 2t$

Since the first shifting theorem given that

$$\mathcal{L}\{e^{at}f(t)\} = (\mathcal{L}\{f(t)\})_{s\to s-a} \quad \text{and} \quad \mathcal{L}\{\sinh at\} = \frac{a}{s^2 - a^2}$$

Now, the Laplace transform of (1) is

$$\mathcal{L}\{5e^{at}\sinh 2t\} = 5\,\mathcal{L}\{e^{2t}\sinh 2t\}$$

$$= 5(\mathcal{L}\{\sinh 2t\})_{s\to s-2}$$

$$= 5\left(\frac{2}{s^2-4}\right)_{s\to s-2} = \frac{10}{(s-2)^2 - 4}$$

Thus, $\mathcal{L}\{5e^{at}\sinh 2t\} = \dfrac{10}{(s-2)^2-4}$.

**(viii) sinht cost**

**Solution:** Given function is

$$\sinh t\cos t = \left(\frac{e^t - e^{-t}}{2}\right).\cos t \qquad ......(1)$$

Since the first shifting theorem given that,

$$\mathcal{L}\{e^{at}f(t)\} = (\mathcal{L}\{f(t)\})_{s\to s-a} \quad \text{and} \quad \mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$$

Now, the Laplace transform of (1) is

$$\mathcal{L}\{\sin ht\cos t\} = \frac{1}{2}[\mathcal{L}\{e^t \cot\} - \mathcal{L}\{e^{-t}\cos t\}]$$

$$= \frac{1}{2}[(L\{\cos t\})_{s\to s-1} - (L\{\cos t\})_{s\to s+1}]$$

$$= \frac{1}{2}\left[\left(\frac{s}{s^2+1}\right)_{s\to s-1} - \left(\frac{s}{s^2+1}\right)_{s\to s+1}\right]$$

$$= \frac{1}{2}\left[\frac{s-1}{(s-1)^2+1} - \frac{s+1}{(s+1)^2+1}\right]$$

$$= \frac{1}{2}\left[\frac{s-1}{s^2-2s+2} - \frac{s+1}{s^2+2s+2}\right]$$

$$= \frac{1}{2}\left[\frac{(s-1)(s^2+2s+2) - (s+1)(s^2-2s+2)}{s^4+2s^3+2s^2-2s^3-4s^2-4s+2s^2+4s+4}\right]$$

$$= \frac{1}{2}\left[\frac{s^3+2s^2+2s-s^2-2s-2-s^3+2s^2-2s-s^2+2s-2}{s^4+4}\right]$$

$$= \frac{1}{2}\left[\frac{2s^2-4}{s^4+4}\right] = \frac{s^2-2}{s^4+4}$$

Thus, $\mathcal{L}\{\sin ht\cos t\} = \dfrac{s^2-2}{s^4+4}$

**(ix) sinat sinbt**

**Solution:** Given function is

$$\sin at\sin bt = \frac{1}{2}[\cos(a-b)t - \cos(a+b)t]$$



Since we have,

$$\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$$

Now, Laplace transform of given function is

$$\mathcal{L}\{\sin at\sin bt\} = \frac{1}{2}[\mathcal{L}\{\cos(a-b)t\} - \mathcal{L}\{\cos(a+b)t\}]$$

$$= \frac{1}{2}\left[\frac{s}{s^2+(a-b)^2} - \frac{s}{s^2+(a+b)^2}\right]$$

$$= \frac{s}{2}\cdot\frac{\{s^2+(a+b)^2\} - \{s^2+(a-b)^2\}}{[s^2+(a+b)^2][s^2+(a-b)^2]}$$

$$= \frac{2}{2}\cdot\frac{4ab}{[s^2+(a+b)^2][s^2+(a-b)^2]}$$

$$= \frac{2abs}{[s^2+(a+b)^2][s^2+(a-b)^2]}$$

**(x)** $e^{-2t}$ **sin4t**

**Solution:** Given function is, $e^{-2t}\sin 4t$

By first shifting theorem we have,

$$\mathcal{L}\{e^{at}f(t)\} = (\mathcal{L}\{f(t)\})_{s\to s-a} \quad \text{and,} \quad \mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$$

Now,

$$\mathcal{L}\{e^{-2t}\sin 4t\} = (\mathcal{L}\{\sin 4t\})_{s\to s+2}$$

$$= \left(\frac{4}{s^2+16}\right)_{s\to s+2} = \frac{4}{(s+2)^2+16} = \frac{4}{s^2+4s+2}$$

Thus, $\mathcal{L}\{e^{-2t}\sin 4t\} = \dfrac{4}{s^2+4s+2}$

**(xi)** $t^3e^{-3t}$

**Solution:** Given function is, $t^3e^{-3t}$

Since the first shifting theorem gives

$$\mathcal{L}\{e^{at}f(t)\} = (\mathcal{L}\{f(t)\})_{s\to s-a} \quad \text{and} \quad \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

Now,

$$\mathcal{L}\{t^3e^{-3t}\} = (L\{t^3\})_{s\to s+3} = \left(\frac{3!}{s^4}\right)_{s\to s+3} = \frac{6}{(s+3)^4}$$

Thus, $\mathcal{L}\{t^3e^{-3t}\} = \dfrac{6}{(s+3)^4}$

**(xii)** $e^{-3t}$ **(2 cos 5t – 3sin 5t)**

**Solution:** Given function is, $e^{-3t}(2\cos 5t - 3\sin 5t)$

Since we have,

$\mathcal{L}\{e^{at}f(t)\} = (\mathcal{L}\{f(t)\})_{s\to s-a}$ by first shifting theorem.

$$\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2} \quad \text{and} \quad \mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}.$$

Now,

$$\mathcal{L}\{e^{-3t}(2\cos 5t - 3\sin 5t)\}$$
$$= (\mathcal{L}\{2\cos 5t - 3\sin 5t\})_{s\to s+3}$$
$$= (2\,\mathcal{L}\{\cos 5t\} - 3\,\mathcal{L}\{\sin 5t\})_{s\to s+3}$$
$$= \left(2\cdot\frac{s}{s^2+25} - 3\cdot\frac{5}{s^2+25}\right)_{s\to s+3}$$
$$= \left(\frac{2s-15}{s^2+25}\right)_{s\to s+3} = \frac{2(s+3)-15}{(s+3)^2+25} = \frac{2s+6-15}{s^2+6s+9+25} = \frac{2s-9}{s^2+6s+34}$$

Thus, $\mathcal{L}\{e^{-3t}(2\cos 5t - 3\sin 5t)\} = \dfrac{2s-9}{s^2+6s+34}$

**(xiii) $e^{-t}(\sin 2t - 2t\cos 2t)$**

**Solution:** Given function is, $e^{-t}(\sin 2t - 2t\cos 2t)$

Since we have,

$\mathcal{L}\{e^{at} f(t)\} = (\mathcal{L}\{f(t)\})_{s\to s-a}$, by first shifting theorem.

$$\mathcal{L}\{t\,f(t)\} = -F'(s) = -\frac{d}{ds}(\mathcal{L}\{f(t)\})$$

$$\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2} \quad \text{and} \quad \mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$$

Now,

$$\mathcal{L}\{e^{-t}(\sin 2t - 2t\cos 2t)\} = (\mathcal{L}\{\sin 2t - 2t\cos 2t\})_{s\to s+1}$$
$$= (\mathcal{L}\{\sin 2t\} - 2\,\mathcal{L}\{t\cos 2t\})_{s\to s+1}$$
$$= \left(L\{\sin 2t\} - 2\left(-\frac{d}{ds}L\{\cos 2t\}\right)\right)_{s\to s+1}$$
$$= \left(\frac{2}{s^2+4} + 2\frac{d}{ds}\left(\frac{s}{s^2+4}\right)\right)_{s\to s+1}$$
$$= \left(\frac{2}{s^2+4} + 2\cdot\frac{s^2+4-2s^2}{(s^2+4)^2}\right)_{s\to s+1}$$
$$= \left(\frac{2}{s^2+4} + \frac{4-s^2}{(s^2+4)^2}\right)_{s\to s+1}$$
$$= 2\cdot\left(\frac{s^2+4+4-s^2}{(s^2+4)^2}\right)_{s\to s+1}$$
$$= 16\left(\frac{1}{(s^2+4)^2}\right)_{s\to s+1}$$
$$= \frac{16}{((s+1)^2+4)^2} = \frac{16}{(s^2+2s+5)^2}$$

Thus, $\mathcal{L}\{e^{-t}(\sin 2t - 2t\cos 2t)\} = \dfrac{16}{(s^2+2s+5)^2}$.

**(xiv) $(t-1)\,u(t-1)$**

**Solution:** Given function is, $(t-1)\,u(t-1)$

Since we have, by second shifting theorem,

$$\mathcal{L}\{f(t-a)\,u_a(t)\} = \mathcal{L}\{f(t-a)\,u(t-a)\} = e^{-as}\,\mathcal{L}\{f(t)\}$$

and $\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}$



Now,

$$\mathcal{L}\{(t-a)\,u(t-1)\} = e^{-s}\,\mathcal{L}\{t\}$$
$$= e^{-s}\left(\frac{1!}{s^2}\right) = \frac{e^{-s}}{s^2}.$$

Thus, $\mathcal{L}\{(t-a)\,u(t-1)\} = \dfrac{e^{-s}}{s^2}$

**(xv) $t^2.\,u(t-1)$**

**Solution:** Given function is, $t^2.\,u(t-1)$

Now,

$$\mathcal{L}\{t^2u(t-1)\} = \mathcal{L}\{(t-1+1)^2\,u(t-1)\}$$
$$= \mathcal{L}\{((t-1)^2 + 2(t-1) + 1)\,u(t-1)\}$$
$$= \mathcal{L}\{(t-1)^2\,u(t-1)\} + 2\,\mathcal{L}\{(t-1)\,u(t-1)\} + \mathcal{L}\{u(t-1)\}$$
$$= e^{-s}\,\mathcal{L}\{t^2\} + 2e^{-s}\,\mathcal{L}\{t\} + \int_1^{\infty} e^{-st}\,dt$$
$$= e^{-s}\left(\frac{2!}{s^3}\right) + 2e^{-s}\left(\frac{1!}{s^2}\right) + \left[\frac{e^{-st}}{-s}\right]_1^{\infty} = e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right)$$

Thus, $\mathcal{L}\{t^2u(t-1)\} = e^{-s}\left(\dfrac{2}{s^3} + \dfrac{2}{s^2} + \dfrac{1}{s}\right)$.

**(xvi) $4u(t-\pi)\cos t$**

**Solution:** Given function is, $4u(t-\pi)\cos t$

Now,

$$\mathcal{L}\{4u(t-\pi)\cos t\} = 4\,\mathcal{L}\{\cos t.\,u(t-\pi)\}$$
$$= -4\,\mathcal{L}\{\cos(t-\pi)\,u(t-\pi)\}$$
$$= -4\,e^{-\pi s}\,\mathcal{L}\{\cos t\} = -4\,e^{-\pi s}\cdot\frac{s}{s^2+1} = \frac{-4se^{-\pi s}}{s^2+1}$$

Thus, $\mathcal{L}\{4u(t-\pi)\cos t\} = \dfrac{-4se^{-\pi s}}{s^2+1}$

**2. Find f(t) if $\mathcal{L}\{f(t)\}$ equals:**

(i) $\dfrac{n\pi}{(s+2)^2 + n^2\pi^2}$

**Solution:** Let

$$\mathcal{L}\{f(t)\} = \frac{n\pi}{(s+2)^2 + n^2\pi^2} = \left(\frac{n\pi}{s^2 + 9n^2\pi}\right)_{s\to s+2}$$

Since, $\mathcal{L}\{\sin at\} = \dfrac{a}{s^2+a^2}$ and $\mathcal{L}\{e^{at} f(t)\} = (\mathcal{L}\{f(t)\})_{s\to s-a}$

Therefore,

$$\mathcal{L}\{f(t)\} = \left(\frac{n\pi}{s^2 + (n\pi)^2}\right)_{s\to s+2}$$

$$= (\mathcal{L}\{\sin n\pi t\})_{s\to s+2} = \mathcal{L}\{e^{-2t}\sin n\pi t\}$$

Thus, $f(t) = e^{-2t}\sin n\pi t$.

**(ii)** $\dfrac{s}{(s+3)^2+1}$

**Solution:** Let.

$$\mathcal{L}\{f(t)\} = \frac{s}{(s+3)^2+1} = \frac{s+3-3}{(s+3)^2+1}$$

$$= \frac{s+3}{(s+3)^2+1} - 3\cdot\frac{1}{(s+3)^2+1}$$

$$= \left(\frac{s}{s^2+1}\right)_{s\to s+3} - 3\left(\frac{1}{s^2+1}\right)_{s\to s+3}$$

Since we have,

$$\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2},\quad \mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2} \text{ and } \mathcal{L}\{e^{at}f(t)\} = \mathcal{L}\{f(t)\}_{s\to s-a}$$

Then,

$$\mathcal{L}\{f(t)\} = \frac{s}{(s+3)^2+1} = (\mathcal{L}\{\cos t\})_{s\to s+3} - 3(\mathcal{L}\{\sin t\})_{s\to s+3}$$

$$= \mathcal{L}\{e^{-3t}\cos t\} - 3\,\mathcal{L}\{e^{-3t}\sin t\}$$

$$= \mathcal{L}\{e^{-3t}(\cos t - 3\sin t)\}$$

Thus, $f(t) = e^{-3t}(\cos t - 3\sin t)$.

**(iii)** $\dfrac{e^{-2s}}{s-3}$

**Solution:** Let,

$$\mathcal{L}\{f(t)\} = \frac{e^{-2s}}{s-3}.$$

Since we have,

$$\mathcal{L}\{f(t-a)\,u_a(t)\} = e^{-as}\mathcal{L}\{f(t)\} \quad \text{and} \quad \mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

Then,

$$\mathcal{L}\{f(t)\} = \frac{e^{-2s}}{s-3} = e^{-2s}\cdot\frac{1}{s-3} = e^{-2s}\mathcal{L}\{e^{3t}\} = \mathcal{L}\{e^{3(t-2)}u_2(t)\}$$

Thus, $f(t) = e^{3(t-2)}u_2(t)$

**(iv)** $\dfrac{se^{-as}}{s^2-w^2}$ **for a > 0**

**Solution:** Let, $\mathcal{L}\{f(t)\} = \dfrac{se^{-as}}{s^2-w^2}$ for $a > 0$

Since we have,

$$\mathcal{L}\{f(t-a)\,u_a(t)\} = e^{-as}\mathcal{L}\{f(t)\} \quad \text{and} \quad \mathcal{L}\{\cosh at\} = \frac{s}{s^2-a^2}$$

Now,



$$\mathcal{L}\{f(t)\} = e^{-as}\cdot\frac{s}{s^2-w^2} = e^{-as}\cdot\mathcal{L}\{\cosh wt\} = \mathcal{L}\{\cosh w(t-a)\,u_a(t)\}$$

Thus, $f(t) = \cosh w(t-a)\,u_a(t)$

**(v)** $\dfrac{e^{-\pi s}}{s^2+1}$

**Solution:** Let, $\mathcal{L}\{f(t)\} = \dfrac{e^{-\pi s}}{s^2+1}$

Since we have,

$$\mathcal{L}\{f(t-a)\,u_a(t)\} = e^{-as}\mathcal{L}\{f(t)\} \quad \text{and} \quad \mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$$

Now,

$$\mathcal{L}\{f(t)\} = e^{-\pi s} = e^{-\pi s}\mathcal{L}\{\sin t\} = \mathcal{L}\{\sin(t-\pi)\,u_\pi(t)\}$$

Thus,

$$f(t) = \sin(t-\pi)\,u_\pi(t) = \sin(-(\pi-t))\,u_\pi(t)$$

$$= -\sin(\pi-t)\,u_\pi(t)$$

$$= -\sin t\,u_\pi(t)$$

**(vi)** $\dfrac{e^{-\pi s}}{s^2}$

**Solution:** Let, $\mathcal{L}\{f(t)\} = \dfrac{e^{-\pi s}}{s^2}$

Since we have,

$$\mathcal{L}\{f(t-a)\,u_a(t)\} = e^{-as}\mathcal{L}\{f(t)\} \quad \text{and} \quad \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

Now,

$$\mathcal{L}\{f(t)\} = e^{-\pi s}\cdot\frac{1}{s^2} = e^{-\pi s}\cdot\frac{1}{1!}\mathcal{L}\{t\} = \mathcal{L}\{(t-\pi)\,u_\pi(t)\}$$

Thus, $f(t) = (t-\pi)\,u_\pi(t)$

**(vii)** $\dfrac{e^{-as}}{s}$

**Solution:** Let, $\mathcal{L}\{f(t)\} = \dfrac{e^{-as}}{s}$

Now,

$$\mathcal{L}\{f(t)\} = e^{-as}\cdot\frac{1}{s} = e^{-as}\frac{1}{0!}\mathcal{L}\{t^0\} = e^{-as}\mathcal{L}\{1\} = \mathcal{L}\{u_a(t)\}$$

Thus, $f(t) = u_a(t)$

**(vii)** $\dfrac{e^{-s} + e^{-2s} - 3e^{-3s} + e^{6s}}{s^2}$

**Solution:** Let, $\mathcal{L}\{f(t)\} = \dfrac{e^{-s} + e^{-2s} - 3e^{-3s} + e^{6s}}{s^2}$

Since we have,

$$\mathcal{L}\{f(t-a)\,u_a(t)\} = e^{-as}\mathcal{L}\{f(t)\} \quad \text{and} \quad \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}.$$

Now,

$$\mathcal{L}\{f(t)\} = \frac{e^{-s} + e^{-2s} - 3e^{-3s} + e^{-6s}}{s^2}$$

$$= e^{-s}.\frac{1}{s^2} + e^{-2s}.\frac{1}{s^2} - 3.e^{-3s}.\frac{1}{s^2} + e^{-6s}.\frac{1}{s^2}$$

$$= e^{-s}.\mathcal{L}\{t\} + e^{-2s}\mathcal{L}\{t\} - 3e^{-2s}\mathcal{L}\{t\} + e^{-6s}\mathcal{L}\{t\}$$

$$= \mathcal{L}\{(t-1)\,u_1(t)\} + \mathcal{L}\{(t-2)\,u_2(t)\} - 3\mathcal{L}\{(t-3)\,u_3(t)\} + \mathcal{L}\{(t-6)\,u_6(t)\}$$

$$= \mathcal{L}\{(t-1)\,u_1(t) + (t-2)\,u_2(t) - 3(t-3)\,u_3(t) + (t-6)\,u_6(t)\}$$

Thus,

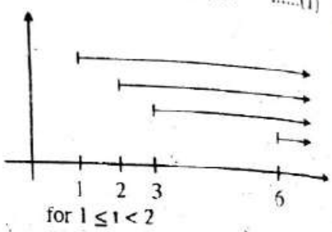
$$f(t) = (t-1)\,u_1(t) + (t-2)\,u_2(t) - 3(t-3)\,u_3(t) + (t-6)\,u_6(t) \quad .....(1)$$

Since we have,

$$u_a(t) = \begin{cases} 1 & \text{for } t \geq a \\ 0 & \text{for } t < a \end{cases}$$

Then,

So from (1), the figure informs,




$$f(t) = \begin{cases} t-1 & \text{for } 1 \leq t < 2 \\ (t-1)\,t(t-2) & \text{for } 2 \leq t < 3 \\ (t-1)\,t(t-2)\,t\{-3\,(t-3)\} & \text{for } 3 \leq t < 6 \\ (t-1)\,t(t-2)\,t\{-3\,(t-3)\}\,t(t-6) & \text{for } t \geq 6 \end{cases}$$

$$\Rightarrow \quad f(t) = \begin{cases} 0 & \text{for } t < 1 \\ t-1 & \text{for } 1 \leq t < 2 \\ 2t-3 & \text{for } 2 \leq t < 3 \\ 6-t & \text{for } 3 \leq t < 6 \\ 0 & \text{for } t \geq 6 \end{cases}$$

**(ix)** $\dfrac{1}{(s+1)^2}$

**Solution:** Given function is, $\dfrac{1}{(s+1)^2}$

Since we have,

$$\mathcal{L}\{e^{at} f(t)\} = (\mathcal{L}\{f(t)\})_{s\to s-a} \quad \text{and} \quad \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

Now,

$$\mathcal{L}\{f(t)\} = \frac{1}{(s+1)^2} = \left(\frac{1}{s^2}\right)_{s\to s+1} = \left(\frac{1}{1!}L\{t^1\}\right)_{s\to s+1} = (\mathcal{L}\{t\})_{s\to s+1} = \mathcal{L}\{e^{-t}t\}$$

Thus, $f(t) = te^{-t}$.

**(x)** $\dfrac{3}{s^2 + 6s + 18}$

**Solution:** Given function is

$$\frac{3}{s^2 + 6s + 18} = \frac{3}{(s+3)^2 + 3^2}$$

Since, we have



$$\mathcal{L}\{\sin at\} = \frac{a}{s^2 + a^2} \quad \text{and} \quad \mathcal{L}\{e^{at} f(t)\} = (\mathcal{L}\{f(t)\})_{s\to s-a}$$

Now,

$$\mathcal{L}\{f(t)\} = \frac{3}{(s+3)^2 + 3^2} = \left(\frac{3}{s^2 + 3^2}\right)_{s\to s+3} = (\mathcal{L}\{\sin 3t\})_{s\to s+3} = \mathcal{L}\{e^{-3t}\sin 3t\}$$

Thus, $f(t) = e^{-3t}\sin 3t$

(xi) $\dfrac{3}{\left(s+\frac{1}{2}\right)^2 + 1}$

**Solution:** Given function is

$$\mathcal{L}\{f(t)\} = \frac{3}{\left(s+\frac{1}{2}\right)^2 + 1} = 3\left(\frac{1}{s^2+1}\right)_{s\to s+1/2}$$

$$= 3\,\mathcal{L}\{e^{-t/2}.\sin t\} = \mathcal{L}\{3e^{-t/2}\sin t\}$$

Thus, $f(t) = 3e^{-t/2}\sin t$

(xii) $\dfrac{e^{-3s}}{s^3}$

**Solution:** Given function is, $\mathcal{L}\{f(t)\} = \dfrac{e^{-3s}}{s^3}$

Since we have,

$$\mathcal{L}\{f(t-a)\,u_a(t)\} = e^{-as}\mathcal{L}\{f(t)\} \quad \text{and} \quad \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

Now,

$$\mathcal{L}\{f(t)\} = e^{-3s}.\frac{1}{s^3} = e^{-3s}.\frac{1}{2!}\mathcal{L}\{t^2\} = \frac{1}{2}\mathcal{L}\{(t-3)^2\,u_3(t)\}$$

Thus, $f(t) = \frac{1}{2}(t-3)^2\,u(t-3)$

(xiii) $\dfrac{3(1 - e^{-\pi s})}{s^2 + 9}$

**Solution:** Given function is

$$\mathcal{L}\{f(t)\} = \frac{3(1 - e^{-\pi s})}{s^2 + 9} = \mathcal{L}\{\sin 3t\} - e^{-\pi s}\,L\{\sin 3t\}$$

$$= \mathcal{L}\{\sin 3t\} - \mathcal{L}\{\sin 3(t-\pi)\,u_\pi(t)\}$$

$$= \mathcal{L}\{\sin 3t - \sin 3(t-\pi)\,u_\pi(t)\}$$

Thus, $f(t) = \sin 3t - \sin 3(t-\pi)\,u_\pi(t)$.

(xiv) $\dfrac{se^{-2s}}{s^2 + \pi^2}$

**Solution:** Given function is, $\mathcal{L}\{f(t)\} = \dfrac{se^{-2s}}{s^2 + \pi^2}$

Since we have, $\mathcal{L}\{f(t-a)\,u_a(t)\} = e^{-as}\mathcal{L}\{f(t)\}$ and $\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$

Now,

$$\mathcal{L}\{f(t)\} = e^{-2s}\frac{s}{s^2+\pi^2} = e^{-2s}\mathcal{L}\{\cos \pi t\} = \mathcal{L}\{\cos \pi(t-2)\,.\,u_2(t)\}$$

Thus,

$$\begin{aligned} f(t) &= \cos\pi(t-2)\,u_2(t) \\ &= \cos \pi(2-t)\,u(t-2) \\ &= \cos(2\pi - \pi t)\,u(t-2) \\ &= \cos \pi t\; u(t-2). \end{aligned}$$

3. **Solve the following initial value problem:**

(i) $y'' + 2y' + 2y = 0,\ y(0) = 0,\ y'(0) = 1$ **[2000 Q. No. 6(b)]**

**Solution:** Given that,

$$y'' + 2y' + 2y = 0 \quad \ldots (i)$$
$$y(0) = 0,\ y'(0) = 1 \quad \ldots (ii)$$

Taking Laplace transform both side of (i) then,

$$\mathcal{L}(y'') + 2\mathcal{L}(y') + 2\mathcal{L}(y) = \mathcal{L}(0)$$
$$\Rightarrow s^2\mathcal{L}(y) - sy(0) - y'(0) + 2s\mathcal{L}(y) - 2y(0) + 2\mathcal{L}(y) = 0$$
$$\Rightarrow s^2\mathcal{L}(y) - 0 - 1 + 2s\,\mathcal{L}(y) - 0 + 2\mathcal{L}(y) = 0 \quad \text{[using (ii)]}$$
$$\Rightarrow \mathcal{L}(y)\,(s^2 + 2s + 2) = 1$$
$$\Rightarrow \mathcal{L}(y) = \frac{1}{s^2 + 2y + 2} = \frac{1}{(s+1)^2 + 1^2}$$

This gives,

$$y = \mathcal{L}^{-1}\left[\frac{1}{(s+1)^2 + (1)^2}\right] = e^{-t}\sin t$$

(ii) $y'' + 4y' + 5y = 0,\ y(0) = 1,\ y'(0) = 2$

**Solution:** Given that,

$$y'' + 4y' + 5y = 0 \quad \ldots (i)$$
$$y(0) = 1,\ y'(0) = 2 \quad \ldots (ii)$$

Taking Laplace transform both side of (i) then,

$$\mathcal{L}(y'') + 4\mathcal{L}(y'') + 5\mathcal{L}(y) = 0$$
$$\Rightarrow s^2\mathcal{L}(y) - sy(0) - y'(0) + 4s\mathcal{L}(y) - 4y(0) + 5\mathcal{L}(y) = 0$$
$$\Rightarrow s^2\mathcal{L}(y) - 0 - 1 + 2s\,\mathcal{L}(y) - 0 + 2\mathcal{L}(y) = 0 \quad \text{[using (ii)]}$$
$$\Rightarrow \mathcal{L}(y)\,(s^2 + 2s + 2) = 1$$
$$\Rightarrow \mathcal{L}(y) = \frac{1}{s^2 + 2y + 2} = \frac{1}{(s+1)^2 + 1^2}$$

This gives,

$$y = \mathcal{L}^{-1}\left[\frac{1}{(s+1)^2 + (1)^2}\right] = e^{-t}\sin t$$

(iii) $y'' - 2y' + 10y = 0,\ y(0) = 3,\ y'(0) = 3$ **[2004 Fall Q. No. 6(b)]**

**Solution:** Given that,

$$y'' - 2y' + 10y = 0 \quad \ldots (i)$$



$$y(0) = 3,\ y'(0) = 3 \quad \ldots (ii)$$

Taking Laplace transform both side of (i) then,

$$\mathcal{L}(y'') - 2\mathcal{L}(y') + 10\mathcal{L}(y) = \mathcal{L}(0)$$
$$\Rightarrow s^2\,\mathcal{L}(y) - sy(0) - y'(0) - 2s\mathcal{L}(y) + 2y(0) + 10\mathcal{L}(y) = 0$$
$$\Rightarrow s^2\,\mathcal{L}(y) - s \times 3 - 3 - 2s\mathcal{L}(y) + 2 \times 3 + 10\mathcal{L}(y) = 0 \quad \text{[using (ii)]}$$
$$\Rightarrow \mathcal{L}(y)\,(s^2 - 2s + 10) = 3s - 3$$
$$\Rightarrow \mathcal{L}(y) = \frac{3s-3}{s^2 - 2s + 10} = \frac{3s-3}{(s-1)^2 + 3^2} = \frac{3(s-1)}{(s-1)^2 + 3^2}$$

This gives,

$$y = \mathcal{L}^{-1}\left[\frac{3(s-1)}{(s-1)^2 + 3^2}\right]$$
$$\Rightarrow y = 3e^{t}\cos 3t.$$

(iv) $4y'' + 8y' + 5y = 0,\ y(0) = 0,\ y'(0) = 1$ **[2008 Spring Q. No. 5(b) OR]**

**Solution:** Given that,

$$4y'' + 8y' + 5y = 0 \quad \ldots (i)$$
$$y(0) = 0,\ y'(0) = 1 \quad \ldots (ii)$$

Taking Laplace transform both side of (i) then,

$$4\,\mathcal{L}(y'') + 8\mathcal{L}(y') = 5\mathcal{L}(y) = \mathcal{L}(0)$$
$$\Rightarrow 4s^2\mathcal{L}(y) - 4sy(0) - 4y''(0) = 8s\mathcal{L}(y) - 8y(0) + 5\mathcal{L}(y) = 0$$
$$\Rightarrow 4s^2\mathcal{L}(y) - 0 - 4 \times 1 + 8\,s\mathcal{L}(y) - 0 + 5\mathcal{L}(y) = 0 \quad \text{[using (ii)]}$$
$$\Rightarrow \mathcal{L}(y)\,(4s^2 + 8s + 5) = 4$$

This gives,

$$y = \mathcal{L}^{-1}\left(\frac{4}{4s^2 + 8s + 5}\right) = \mathcal{L}^{-1}\left(\frac{1}{s^2 + 2s + \frac{5}{4}}\right) = 2\mathcal{L}^{-1}\left[\frac{\frac{1}{2}}{(s+1)^2 + \left(\frac{1}{2}\right)^2}\right]$$
$$= 2e^{-t}\sin\frac{1}{2}t.$$

$$\Rightarrow y = 2e^{-t}\sin\frac{1}{2}t.$$

(v) $y'' - 2y' + 2y = 0,\ y(0) = 1,\ y'(0) = 1$

**Solution:** Given that,

$$y'' - 2y' + 2y = 0 \quad \ldots (i)$$
$$y(0) = 1,\ y'(0) = 1 \quad \ldots (ii)$$

Taking Laplace transform both side of (i) then,

$$\mathcal{L}(y'') - 2\mathcal{L}(y') + 2\mathcal{L}(y) = \mathcal{L}(0)$$
$$\Rightarrow s^2\mathcal{L}(y) - sy(0) - y'(0) - 2s\mathcal{L}(y) + 2y(0) = 2\mathcal{L}(y) = 0$$
$$\Rightarrow s^2\mathcal{L}(y) - s \times 1 - 1 - 2s\mathcal{L}(y) + 2 \times 1 + 2\mathcal{L}(y) = 0$$

This gives,

$$y = \mathcal{L}^{-1}\left[\frac{(s-1)}{(s-2s+2)}\right] = \mathcal{L}^{-1}\left[\frac{(s-1)}{(s-1)^2 + 1^2}\right] = e^{t}\cos t$$

(vi) $y'' + 4y' + 3y = e^{-t}, y(0) = y'(0) = 1$

**Solution:** Given that,

$$y'' + 4y' + 3y = e^{-t} \quad \text{... (i)}$$
$$y(0) = y'(0) = 1 \quad \text{... (ii)}$$

Taking Laplace transform both side of (i) then,

$$\mathcal{L}(y'') + 4\mathcal{L}(y') + 3\mathcal{L}(y) = \mathcal{L}(e^{-t})$$

$$\Rightarrow \quad s^2\mathcal{L}(y) - sy(0) - y'(0) + 4s\,\mathcal{L}(y) - 4y(0) + 3\,\mathcal{L}(y) = \frac{1}{s+1}$$

$$\Rightarrow \quad s^2\mathcal{L}(y) - s - 1 + 4s\,\mathcal{L}(y) - 4 + 3\,\mathcal{L}(y) = \frac{1}{s+1}$$

$$\Rightarrow \quad \mathcal{L}(y)(s^2 + 4s + 3) = \frac{1}{s+1} + s + 5 = \frac{1 + s(s+1) + 5(s+1)}{(s+1)}$$

$$\Rightarrow \quad \mathcal{L}(y) = \mathcal{L}^{-1}\frac{1 + s^2 + s + 5s + 5}{(s+1)(s^2+4s+3)} = \frac{s^2 + 6s + 6}{(s+1)(s+1)(s+3)} = \frac{s^2+6s+6}{(s+1)^2(s+3)}$$

This gives,

$$y = \mathcal{L}^{-1}\left[\frac{s^2+6s+6}{(s+3)(s+1)^2}\right] \quad \text{...... (iii)}$$

Let

$$\frac{s^2+6s+6}{(s+3)(s+1)^2} = \frac{A}{(s+3)} + \frac{B}{(s+1)} + \frac{C}{(s+1)^2}$$

$$\Rightarrow \quad \frac{s^2+6s+6}{(s+3)(s+1)^2} = \frac{A(s+1)^2 + B(s+1)(s+3) + C(s+3)}{(s+3)(s+1)^2}$$

$$\Rightarrow \quad s^2 + 6s + 6 = As^2 + 2AS + A + Bs^2 + 4Bs + 3B + Cs + 3c$$

$$\Rightarrow \quad s^2 + 6s + 6 = s^2(A+B) + s(2A + 4B + C) + (A + 3B + 3C)$$

Equating coefficient of s and the constant term on both sides then we get,

$$A + B = 1, \quad 2A + 4B + C = 6 \quad \text{and} \quad A + 3B + 3C = 6$$

Solving we get,

$$A = -\frac{3}{4}, B = \frac{7}{4}, C = \frac{1}{2}$$

Now, (iii) becomes,

$$y = \mathcal{L}^{-1}\left[-\frac{3}{4}\frac{1}{(s+3)} + \frac{7}{4}\frac{1}{(s+1)} + \frac{1}{2}\frac{1}{(s+1)^2}\right]$$

**(vii) $x'' + 2x' + 5x = e^{-t}\sin t,\ x(0) = x'(0) = 1$.** **[2009 Fall Q. No. 6(b) OR]**

**Solution:** Given that,

$$x'' + 2x' + 5x = e^{-t}\sin t \quad \text{... (i)}$$
$$x(0) = x'(0) = 1 \quad \text{... (ii)}$$

Taking Laplace transform both side of (i) then,

$$\mathcal{L}(x'') + 2\,\mathcal{L}(x') + 5\,\mathcal{L}(x) = \mathcal{L}(e^{-t}\sin t)$$

$$\Rightarrow \quad s^2\mathcal{L}(x) - sx(0) - x'(0) + 2s\,\mathcal{L}(x) - 2x(0) + 5\,\mathcal{L}(x) = [\mathcal{L}(\sin t)]_{s\to s+1}$$

$$\Rightarrow \quad s^2\mathcal{L}(x) - s \times 0 - 1 + 2s\,\mathcal{L}(x) - 0 + 5\,\mathcal{L}(x) = \frac{1}{(s+1)^2 + 1}$$

$$\Rightarrow \quad \mathcal{L}(x)(s^2 + 2s + 5) = \frac{1}{s^2+2s+2} + 1 = \frac{1 + s^2 + 2s + 2}{(s^2+2s+2)}$$



$$\Rightarrow \quad \mathcal{L}(x) = \frac{s^2 + 2s + 3}{(s^2+2s+2)(s^2+2s+5)}$$

This gives,

$$x = \mathcal{L}^{-1}\left[\frac{(s^2+2s+3)}{(s^2+2s+2)(s^2+2s+5)}\right] \quad \text{........ (iii)}$$

Let,

$$\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)} = \frac{As+B}{(s^2+2s+2)} + \frac{Cs+D}{(s^2+2s+5)}$$
$$= \frac{(As+B)(s^2+2s+5) + (Cs+D)(s^2+2s+2)}{(s^2+2s+2)(s^2+2s+5)}$$

$$\Rightarrow \quad s^2 + 2s + 3 = As^3 + 2As^2 + 5As + Bs^2 + 2Bs + 5B + Cs^3 + 2Cs^2 + 2Cs + Ds^2 + 2Ds + 2D$$

$$\Rightarrow \quad s^2 + 2s + 3 = s^3(A + C) + s^2(2A + B + 2C + D) + s(5A + 2B + 2C + 2D) + (5B + 2D)$$

Equating coefficient of s and the constant term on both sides then we get,

$$A + C = 0, \quad 2A + B + 2C + D = 1,$$
$$5A + 2B + 2C + 2D = 2, \quad 5B + 2D = 3$$

Solving we get,

$$A = 0, B = \frac{1}{3}, C = 0, D = \frac{2}{3}$$

Now (iii) becomes,

$$x = \mathcal{L}^{-1}\left[\frac{\frac{1}{3}}{s^2+2s+2} + \frac{\frac{2}{3}}{s^2+2s+5}\right]$$
$$= \mathcal{L}^{-1}\left[\frac{1}{3}\left\{\frac{1}{(s+1)^2+1^2}\right\} + \frac{1}{3}\left\{\frac{2}{(s+1)^2+2^2}\right\}\right]$$
$$= \frac{1}{3}e^{-t}\sin t + \frac{1}{3}e^{-t}\sin 2t$$
$$= \frac{1}{3}e^{-t}(\sin t + \sin 2t).$$

**(viii) $y'' - 2y' + y = e^t,\ y(0) = 2,\ y'(0) = -1$** **[2009 Spring Q. No. 6(b) OR]**

**Solution:** Given that,

$$y'' - 2y' + y = e^t \quad \text{... (i)}$$
$$y(0) = 2, y'(0) = -1 \quad \text{... (ii)}$$

Taking Laplace transform both side of (i) then,

$$\mathcal{L}(y'') - 2\mathcal{L}(y') + \mathcal{L}(y) = \mathcal{L}(e^t)$$

$$\Rightarrow \quad s^2\mathcal{L}(y) - 5y(0) - y'(0) - y'(0) - 2s\mathcal{L}(y) + 2y(0) + 2(y) = \frac{1}{s-1}$$

$$\Rightarrow \quad s^2\mathcal{L}(y) - 2s + 1 - 2s\mathcal{L}(y) + 4 + \mathcal{L}(y) = \frac{1}{(s-1)}$$

$$\Rightarrow \quad \mathcal{L}(y)(s^2 - 2s + 1) = \frac{1}{(s-1)} + 2s - 5 = \frac{1 + (2s-5)(s-1)}{(s-1)}$$

$$\Rightarrow \mathcal{L}(y) = \frac{2s^2 - 2s - 5s + 5 + 1}{(s-1)(s-1)^2} = \frac{2s^2 - 7s + 6}{(s-1)^3}$$

This gives,

$$y = \mathcal{L}^{-1}\left[\frac{2s^2 - 7s + 6}{(s-1)^3}\right] \qquad \text{.......... (iii)}$$

Let,

$$\frac{2s^2 - 7s + 6}{(s-1)^3} = \frac{A}{(s-1)} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)^3}$$

$$= \frac{A(s-1)^2 + B(s-1) + C}{(s-1)^3}$$

$$= \frac{As^2 - 2As + A + Bs - B + C}{(s-1)^3}$$

$$\Rightarrow 2s^2 - 7s + 6 = As^2 + s(B - 2A) + (A - B + C)$$

Equating coefficient of s and the constant term on both sides then we get,

$A = 2, \quad B - 2A = -7, \quad A - B + C = 6.$

Solving we get,

$A = 2$, $B = -3$ and $C = 1$.

Now (iii) becomes,

$$y = \mathcal{L}^{-1}\left[\frac{2}{(s-1)} - \frac{3}{(s-1)^2} + \frac{1}{(s-1)^3}\right]$$

$$= 2e^t - 3e^t.\, t + \frac{1}{2}e^t t^2$$

$$= e^t\left(2 - 3t + \frac{1}{2} + t^2\right).$$

**(ix) $(D^2 - 2D + 2)\,x = 0,\ x(0) = 1, x'(0) = 1$**

**Solution:** Given that,

$$(D^2 - 2D + 2)\,x = 0 \qquad \text{... (i)}$$

$$x(0) = 1,\ x'(0) = 1 \qquad \text{... (ii)}$$

Since, (i) can be written as,

$$\left(\frac{d^2}{dt^2} - 2\frac{d}{dt} + 2\right)x = 0$$

$$\Rightarrow \frac{d^2x}{dt^2} - 2\frac{dx}{dt} + 2x = 0$$

$$\Rightarrow x'' - 2x' + 2x = 0$$

**Same to Q. (v) with replacing x by y.**

**(x) $y'' + 4y' + 3y = e^{-t},\ y(0) = y'(0) = 1.$** **[2006 spring Q. No. 6(b) OR]**

**Solution:** Given that,

$$y'' + 4y' + 3y = e^{-t} \qquad \text{... (i)}$$

$$y(0) = y'(0) = 1 \qquad \text{... (ii)}$$

Taking Laplace transform both side of (i) then,

$$\mathcal{L}(y'') + 4\mathcal{L}(y') + 3\mathcal{L}(y) = \mathcal{L}(e^{-t})$$

$$\Rightarrow s^2\mathcal{L}(y) - sy(0) - y'(0) + 4s\,\mathcal{L}(y) - 4y(0) + 3\,\mathcal{L}(y) = \frac{1}{s+1}$$



$$\Rightarrow s^2\mathcal{L}(y) - s - 1 + 4s\,\mathcal{L}(y) - 4 + 3\,\mathcal{L}(y) = \frac{1}{s+1}$$

$$\Rightarrow \mathcal{L}(y)\,(s^2 + 4s + 3) = \frac{1}{s+1} + (s+5)$$

This gives,

$$y = \mathcal{L}^{-1}\left[\frac{1 + (s+5)(s+1)}{(s+1)(s^2+4s+3)}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{1 + s^2 + s + 5s + 5}{(s+1)\{(s+2)^2 - 1\}}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{s^2 + 6s + 6}{(s+1)(s^2+4s+3)}\right] = \mathcal{L}^{-1}\left[\frac{s^2 + 6s + 6}{(s+3)(s+1)^2}\right] \qquad \text{...... (iii)}$$

Let,

$$\frac{s^2 + 6s + 6}{(s+3)(s+1)^2} = \frac{A}{(s+3)} + \frac{B}{s+1} + \frac{c}{(s+1)^2}$$

$$\Rightarrow \frac{s^2 + 6s + 6}{(s+3)(s+1)^2} = \frac{A(s+1)^2 + B(s+1)(s+3) + C(s+3)}{(s+1)^2(s+3)}$$

$$\Rightarrow s^2 + 6s + 6 = As^2 + 2As + A + Bs^2 + 4Bs + 3B + Cs + 3C$$

$$\Rightarrow s^2 + 6s + 6 = s^2(A+B) + s(2A + 4B + C) + (A + 3B + 3C)$$

Equating coefficient of s and the constant term on both sides then we get,

$A + B = 1, \quad 2A + 4B + c = 6, \quad A + 3B + 3C = 6$

Solving we get, $A = -\frac{3}{4}, B = \frac{7}{4}, C = \frac{1}{2}$

Now (iii) becomes,

$$y = \mathcal{L}^{-1}\left[-\frac{3}{4}\frac{1}{(s+3)} + \frac{7}{4}\frac{1}{s+1} + \frac{1}{2}\frac{1}{(s+1)^2}\right]$$

$$= -\frac{3}{4}e^{-3t} + \frac{7}{4}e^{-t} + \frac{1}{2}e^{-t}.\, t.$$

**(xi) $x'' - 3x' + 2x = 1 - e^{2t},\ x(0) = 1, x'(0) = 0$**

**Solution:** Given that,

$$x'' - 3x' + 2x = 1 - e^{2t} \qquad \text{......(i)}$$

$$x(0) = 1,\ x'(0) = 0 \qquad \text{......(ii)}$$

Taking Laplace transform both side of (i) then,

$$\mathcal{L}(x'') - 3\,\mathcal{L}(x') + 2(x) = \mathcal{L}(1 - e^{2t})$$

$$\Rightarrow s^2\mathcal{L}(x) - sx(0) - x'(0) - 3s\,\mathcal{L}(x) + 3x(0) + 2\,\mathcal{L}(x) = \left(\frac{1}{s} - \frac{1}{s-2}\right)$$

$$\Rightarrow s^2\mathcal{L}(x) - s - 3s\,\mathcal{L}(x) + 3 + 2\,\mathcal{L}(x) = \frac{s - 2 - s}{s(s-2)}$$

$$\Rightarrow \mathcal{L}(x)\,(s^2 - 3s + 2) = -\frac{2}{s(s-2)} + s - 3$$

$$\Rightarrow \mathcal{L}(x) = \frac{-2 + s(s-2)(s-3)}{s(s-2)(s^2 - 3s + 2)} = \frac{-2 + (s^2 - 2s)(s-3)}{s(s-2)(s-2)(s-1)}$$

This gives,

$$x = \mathcal{L}^{-1}\left[\frac{-2 + s^3 - 3s^2 - 2s^2 + 6s}{s(s-1)(s-2)^2}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{s^3 - 5s^2 + 6s - 2}{s(s-1)(s-2)^2}\right] \qquad \text{........ (iii)}$$

Let,

$$\frac{s^3 - 5s^2 + 6s - 2}{s(s-1)(s-2)^2} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2} + \frac{D}{(s-2)^2}$$

$$= \frac{A(s-1)(s-2)^2 + Bs(s-2)^2 + Cs(s-1)(s-2) + Ds(s-1)}{s(s-1)(s-2)^2}$$

$$\Rightarrow \quad s^3 - 5s^2 + 6s - 2 = A(s-1)(s^2 - 4s + 4) + Bs(s^2 - 4s + 4) + Cs(s^2 - 3s + 2) + Ds^2 - Ds$$

$$\Rightarrow \quad s^3 - 5s^2 + 6s - 2 = A(s^3 - 4s^2 + 4s - s^2 + 4s - 4) + s^3B - 4Bs^2 + 4Bs + Cs^3 - 3Cs^2 + 2Cs + Ds^2 - Ds$$

$$\Rightarrow \quad s^3 - 5s^2 + 6s - 2 = As^3 - 5As^2 + 8As - 4A + s^3B - 4Bs^2 + 4Bs + Cs^3 - 3Cs^2 + 2Cs + Ds^2 - Ds$$

$$\Rightarrow \quad s^3 - 5s^2 + 6s - 2 = s^3(A + B + C) + s^2(D - 5A - 4B - 3C) + s(8A + 4B + 2C - D) - 4A$$

Equating coefficient of s and the constant term on both sides then we get,

$$A + B + C = 1, \qquad D - 5A - 4B - 3C = -5, \qquad 8A + 4B + 2C - D = 6$$

Solving we get,

$$A = \frac{1}{2}, B = 0, C = \frac{1}{2}, D = -1$$

Now (iii) becomes,

$$x = \mathcal{L}^{-1}\left[\frac{1}{2}\frac{1}{s} + \frac{1}{2}\frac{1}{(s-2)} - \frac{1}{(s-2)^2}\right]$$

$$\Rightarrow \quad x = \left(\frac{1}{2} + \frac{e^{2t}}{2} - e^{2t}.\, t\right)$$

**(xii) $y'' + y' - 2y = t$, $y(0) = 1$, $y'(0) = 0$.**

**Solution:** Given that,

$$y'' + y' - 2y = t \qquad \text{......(i)}$$

$$y(0) = 1,\ y'(0) = 0 \qquad \text{...... (ii)}$$

Taking Laplace transform both side of (i) then,

$$\mathcal{L}(y'') + \mathcal{L}(y') - 2\,\mathcal{L}(y) = \mathcal{L}(t)$$

$$\Rightarrow \quad s^2\mathcal{L}(y) - sy(0) - y'(0) + s\,\mathcal{L}(y) - y(0) - 2\,\mathcal{L}(y) = \frac{1}{s^2}$$

$$\Rightarrow \quad s^2\mathcal{L}(y) - 0 + s\,\mathcal{L}(y) - 1 - 2\,\mathcal{L}(y) = \frac{1}{s^2}$$

$$\Rightarrow \quad \mathcal{L}(y)(s^2 + s - 2) = \frac{1}{s^2} + 1 + s$$

This gives,

$$y = \mathcal{L}^{-1}\left[\frac{1 + s^3 + s^2}{s^2(s^2 + s - 2)}\right] = \mathcal{L}^{-1}\left[\frac{1 + s^2 + s^3}{s^2(s+2)(s-1)}\right] \qquad \text{........ (iii)}$$

Let,



$$\frac{1 + s^2 + s^3}{s^2(s-1)(s+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{s+2}$$

$$= \frac{As(s-1)(s+2) + B(s-1)(s+2) + C\,s^2(s+2) + Ds^2(s-1)}{s^2(s-1)(s+2)}$$

$$\Rightarrow \quad 1 + s^2 + s^3 = As(s^2 + 2s - s - 2) + B(s^2 + 2s - s - 2) + Cs^3 + 2Cs^2 + Ds^3 - Ds^2$$

$$\Rightarrow \quad 1 + s^2 + s^3 = As^3 + As^2 - 2As + Bs^2 + Bs - 2B + Cs^3 + 2Cs^2 + Ds^3 - Ds^2$$

$$\Rightarrow \quad 1 + s^2 + s^3 = s^3(A + C + D) + s^2(A + B + 2C - D) + s(-2A + B) + (-2B)$$

Equating coefficient of s and the constant term on both sides then we get,

$$A + C + D = 1, \qquad A + B + 2C - D = 1, \qquad -2A + B = 0$$

Solving we get,

$$A = -\frac{1}{4}, B = -\frac{1}{2}, C = 1, D = \frac{1}{4}$$

Now (iii) becomes,

$$y = \mathcal{L}^{-1}\left[\frac{1}{4}\frac{1}{s} - \frac{1}{2}\frac{1}{s^2} + \frac{1}{(s-1)} + \frac{1}{4}\frac{1}{(s+2)}\right]$$

$$= -\frac{1}{4} - \frac{1}{2}t + e^t + \frac{1}{4}e^{-2t}$$

$$= e^t + \frac{1}{4}(e^{-2t} - 1) - \frac{t}{2}$$

**(xiii) $x'' + 5x' + 6x = 5et$, $x(0) = 2$, $x'(0) = 1$**

**Solution:** Given that,

$$x'' + 5x' + 6x = 5et \qquad \text{...... (i)}$$

$$x(0) = 2,\ x'(0) = 1 \qquad \text{...... (ii)}$$

Taking Laplace transform both side of (i) then,

$$\mathcal{L}(x'') + 5\mathcal{L}(x') + 6\mathcal{L}(x) = 5\mathcal{L}(e^t)$$

$$\Rightarrow \quad s^2\mathcal{L}(x) - sx(0) - x'(0) + 5s\mathcal{L}(x) - 5x(0) + 6\mathcal{L}(x) = \frac{5}{s-1}$$

$$\Rightarrow \quad s^2\,\mathcal{L}(x) - 2s - 1 + 5s\,\mathcal{L}(x) - 10 + 6\,\mathcal{L}(x) = \frac{5}{s-1}$$

$$\Rightarrow \quad \mathcal{L}(x)(s^2 + 5s + 6) = \frac{5}{s-1} + 2s + 11$$

$$\Rightarrow \quad \mathcal{L}(x)(s^2 + 5s + 6) = \frac{5 + (2s + 11)(s-1)}{(s-1)}$$

$$\Rightarrow \quad \mathcal{L}(x)(s^2 + 5s + 6) = \frac{5 + 2s^2 - 2s + 11s - 11}{(s-1)}$$

$$\Rightarrow \quad x = \mathcal{L}^{-1}\left[\frac{2s^2 + 9s - 6}{(s-1)(s+2)(s+3)}\right] \qquad \text{........ (iii)}$$

Let,

$$\frac{2s^2 + 9s - 6}{(s-1)(s+2)(s+3)} = \frac{A}{s-1} + \frac{B}{s+2} + \frac{C}{s+3}$$

$$= \frac{A(s+2)(s+3) + B(s-1)(s+3) + C(s-1)(s+2)}{(s-1)(s+2)(s+3)}$$

$$\Rightarrow \quad 2s^2 + 9s - 6 = A(s^2 + 5s + 6) + B(s^2 + 2s - 3) + C(s^2 + s - 2)$$

$+ 2Bs - 3B + Cs^2 + Cs - 2C$

$\Rightarrow \quad 2s^2 + 9s - 6 = As^2 + 5As + 6A + Bs^2 + 2Bs - 3B + Cs^2 + Cs - 2C$

$\Rightarrow \quad 2s^2 + 9s - 6 = s^2(A + B + C) + s(5A + 2B + C) + (6A - 3B - 2C)$

Equating coefficient of s and the constant term on both sides then we get,

$A + B + C = 2, \quad 5A + 2B + C = 9, \quad 6A - 3B - 2C = -6$

Solving we get, $A = \frac{5}{12}, B = \frac{16}{3}, C = -\frac{15}{4}$

Now (iii) becomes,

$$x = \mathcal{L}^{-1}\left[\frac{5}{12}\frac{1}{s-1} + \frac{16}{3}\frac{1}{s+2} - \frac{15}{4}\frac{1}{s+3}\right]$$

$$= \frac{5}{12}e^{t} + \frac{16}{3}e^{-2t} - \frac{15}{4}e^{-3t}$$

**(xiv) $x'' - x = a\cosh t, x(0) = x'(0) = 0$.**

**Solution:** Given that,

$$x'' - x = a\cosh t \qquad \ldots\ldots (i)$$

$$x(0) = x'(0) = 0. \qquad \ldots\ldots (ii)$$

Taking Laplace transform both side of (i) then,

$$\mathcal{L}(x'') - \mathcal{L}(x) = a\,\mathcal{L}(\cosh t)$$

$$\Rightarrow \quad s^2\mathcal{L}(x) - s\,x(0) - x'(0) - \mathcal{L}(x) = a\frac{s}{s^2 - 1^2}$$

$$\Rightarrow \quad s^2\mathcal{L}(x) - 0 - 0 - \mathcal{L}(x) = a\frac{s}{s^2 - 1} \qquad \text{[Using (ii)]}$$

$$\Rightarrow \quad \mathcal{L}(x)(s^2 - 1) = \frac{as}{(s^2 - 1)}$$

This gives, $x = a\,\mathcal{L}^{-1}\left[\frac{s}{(s^2 - 1)^2}\right]$

$$= a\,\mathcal{L}^{-1}\left[\frac{s}{(s^2 - 1)^2}\right]$$

$$= \frac{a}{2}\mathcal{L}^{-1}\left[\frac{2s}{(s^2 - 1)^2}\right] = \frac{a}{2} \times t\sinh t \qquad [\because \mathcal{L}(t\sinh at) = \frac{2s}{(s^2 - a)^2}]$$

$$= \frac{1}{2}at\sinh t$$

**(xv) $x'' - x' - 2x = 20\sin 2t, x(0) = -1, x'(0) = 2$**

**Solution:** Given that,

$$x'' - x' - 2x = 20\sin 2t \qquad \ldots\ldots (i)$$

$$x(0) = -1, x'(0) = 2 \qquad \ldots\ldots (ii)$$

Taking Laplace transform both side of (i) then,

$$\mathcal{L}(x'') - \mathcal{L}(x') - 2\mathcal{L}(x) = 20\,\mathcal{L}(\sin 2t)$$

$$\Rightarrow \quad s^2\mathcal{L}(x) - sx(0) - x'(0) - s\mathcal{L}(x) + x(0) - 2\mathcal{L}(x) = 20 \times \frac{2}{s^2 + 4}$$

$$\Rightarrow \quad s^2\mathcal{L}(x) + s - 2 - s\mathcal{L}(x) - 1 - 2\mathcal{L}(x) = \frac{40}{s^2 + 4}$$



$$\Rightarrow \quad \mathcal{L}(x)(s^2 - s - 2) = \frac{40}{s^2 + 4} - s + 3$$

$$\Rightarrow \quad \mathcal{L}(x)(s^2 - s - 2) = \frac{40 - s(s^2 + 4) + 3(s^2 + 4)}{s^2 + 4}$$

$$= \frac{40 - s^3 - 4s + 3s^2 + 12}{s^2 + 4}$$

$$\Rightarrow \quad \mathcal{L}(x)(s^2 - s - 2) = \frac{-s^3 - 3s^2 - 4s + 52}{s^2 + 4}$$

This gives, $x = \mathcal{L}^{-1}\left[\frac{-s^3 + 3s^2 - 4s + 52}{(s^2 + 4)(s^2 - s - 2)}\right]$

$$\Rightarrow \quad x = \mathcal{L}^{-1}\left[\frac{-s^3 + 3s^2 - 4s + 52}{(s^2 + 4)(s - 2)(s + 1)}\right] \qquad \ldots\ldots (iii)$$

Let,

$$\frac{-s^3 + 3s^2 - 4s + 52}{(s^2 + 4)(s - 2)(s + 1)} = \frac{A}{s + 1} + \frac{B}{s - 2} + \frac{Cs + D}{s^2 + 4}$$

$$= \frac{A(s-2)(s^2 + 4) + B(s + 1)(s^2 + 4) + (Cs + D)(s^2 - s - 2)}{(s + 1)(s - 2)(s^2 + 4)}$$

$$\Rightarrow \quad -s^3 + 3s^2 - 4s + 52 = A(s^3 + 4s - 2s^2 - 8) + B(s^3 + 4s + s^2 + c) + Cs^3 - Cs^2 - 2Cs + Ds^2 - 2D - DS$$

$$\Rightarrow \quad -s^3 + 3s^2 - 4s + 52 = As^3 + 4As - 2As^2 - 8A + Bs^3 + 4Bs + Bs^2 + 4B + Cs^3 - Cs^2 - 2Cs + Ds^2 - Ds - 2D$$

$$\Rightarrow \quad -s^3 + 3s^2 - 4s + 52 = s^3(A + B + C) + s^2(-2A + B - C + D) + s(4A + 4B - 2C - D) + (-8A + 4B - 2D)$$

Equating coefficient of s and the constant term on both sides then we get,

$A + B + C = -1, \qquad -2A + B - C + D = 3$

$4A + 4B - 2C - D = -4 \qquad -8A + 4B - 2D = 52$

Solving we get,

$A = -4, B = 2, C = 1, D = -6$

Now (iii) becomes,

$$x = \mathcal{L}^{-1}\left[-4\frac{1}{s + 1} + 2\frac{1}{s - 2} + \frac{s - 6}{s^2 + 4}\right]$$

$$= -4e^{-t} + 2e^{2t} + \cos 2 - 3\sin 2t.$$

**(xvi) $y'' + 2y' + 2y = \frac{17}{2}\sin 5t, y(0) = 2, y'(0) = -4$**

**Solution:** Given that,

$$y'' + 2y' + 2y = \frac{17}{2}\sin 5t \qquad \ldots\ldots (i)$$

$$y(0) = 2, y'(0) = -4 \qquad \ldots\ldots (ii)$$

Taking Laplace transform both side of (i) then,

$$\mathcal{L}(y'') + 2\mathcal{L}(y') + 2\mathcal{L}(y) = \frac{17}{2}\mathcal{L}(\sin 5t)$$

$$s^2\mathcal{L}(y) - s\,y(0) - y'(0) + 2s\,\mathcal{L}(y) - 2y(0) + 2\,\mathcal{L}(y) = \frac{17}{2}\cdot\frac{5}{s^2+25}$$

$$\Rightarrow \quad s^2\mathcal{L}(y) - 2s + 4 + 2s\,\mathcal{L}(y) - 4 + 2\,\mathcal{L}(y) = \frac{85}{2}\cdot\frac{1}{(s^2+25)}$$

$$\Rightarrow \quad \mathcal{L}(y)(s^2 + 2s + 2) = \frac{85}{2(s^2+25)} + 2s$$

This implies, $y = \mathcal{L}^{-1}\left[\dfrac{85 + 2s(2s^2+50)}{2(s^2+25)(s^2+2s+2)}\right]$

$$= \frac{1}{2}\mathcal{L}^{-1}\left[\frac{4s^3 + 100s + 85}{(s^2+25)(s^2+2s+2)}\right] \quad \text{...... (iii)}$$

Let,

$$\frac{4s^3 + 100s + 85}{(s^2+25)(s^2+2s+2)} = \frac{As+B}{s^2+25} + \frac{Cs+D}{(s^2+2s+2)}$$

$$\Rightarrow \quad \frac{4s^3 + 100s + 85}{(s^2+25)(s^2+2s+2)} = \frac{(As+B)(s^2+2s+2) + (Cs+D)(s^2+25)}{(s^2+25)(s^2+2s+2)}$$

This gives,

$$4s^3 + 100s + 85 = As^3 + 2As^2 + 2As + Bs^2 + 2Bs + 2B + Cs^3 + 25Cs + Ds^2 + 25D$$

$$\Rightarrow \quad 4s^3 + 100s + 85 = s^3(A + C) + s^2(2A + B + D) + s(2A + 2B + 25C) + (2B + 25D)$$

Equating coefficient of s and the constant term on both sides then we get,

$A + C = 4,$ $\qquad 2A + B + C = 0,$

$2A + 2B + 25C = 100,$ $\qquad 2B + 25D = 85$

Solving we get,

$A =$ $\qquad B = -\dfrac{3910}{1208}$

Now, (iii) becomes,

$$y = e^{-t}\left(4\cos 2t - \frac{1}{2}\sin 2t\right)$$

$$= -2\cos 2t + \frac{1}{2}\sin 2t$$

**(xvii)** $\quad$ **$y'' + y' - 2y = 3\cos 3t - 11\sin 3t$, $y(0) = 0$, $y'(0) = 6$.**

**Solution:** Given that,

$$y'' + y' - 2y = 3\cos 3t - 11\sin 3t \quad \text{...... (i)}$$

$$y(0) = 0,\ y'(0) = 6 \quad \text{...... (ii)}$$

Taking Laplace transform both side of (i) then,

$$\mathcal{L}(y'') + \mathcal{L}(y') - 2\,\mathcal{L}(y) = 3\,\mathcal{L}(\cos 3t) - 11\,\mathcal{L}(\sin 3t)$$

$$\Rightarrow \quad s^2\mathcal{L}(y) - sy(0) - y'(0) + s\,\mathcal{L}(y) - y(0) - 2\,\mathcal{L}(y) = 3\cdot\frac{s}{s^2+9} - 11\times\frac{3}{s^2+9}$$

$$\Rightarrow \quad s^2\mathcal{L}(y) - 0 - 6 + s\,\mathcal{L}(y) - 0 - 2\,\mathcal{L}(y) = \frac{3s-33}{s^2+9}$$

$$\Rightarrow \quad \mathcal{L}(y)(s^2 + s - 2) = \frac{3s-33}{s^2+9} + 6$$

$$\Rightarrow \quad \mathcal{L}(y)(s^2 + s - 2) = \frac{3s - 33 + 6s^2 + 54}{s^2+9}$$



This gives, $\quad y = \mathcal{L}^{-1}\left[\dfrac{6s^2 + 3s + 21}{(s^2+9)(s+2)(s-1)}\right] \quad$ ...... (iii)

Here,

$$\frac{6s^2 + 3s + 21}{(s^2+9)(s+2)(s-1)} = \frac{A}{(s-1)} + \frac{B}{(s+2)} + \frac{Cs+D}{s^2+9}$$

$$= \frac{A(s+2)(s^2+9) + B(s-1)(s^2+9) + (Cs+D)(s^2+s-2)}{(s-1)(s+2)(s^2+9)}$$

This gives,

$$6s^2 + 3s + 21 = A(s^3 + 9s + 2s^2 + 18) + B(s^3 + 9s - s^2 - 9) + Cs^3 + Cs^2 - 2Cs + Ds^2 + Ds - 2D$$

$$\Rightarrow \quad 6s^2 + 3s + 21 = s^3(A + B + C) + s^2(2A - B + C + D) + s(9A + 9B - 2C + D) + (18A - 9B - 2D).$$

Equating coefficient of s and the constant term on both sides then we get,

$A + B + C = 0,$ $\qquad 2A - B + C + D = 6,$

$9A + 9B - 2C + D = 3,$ $\qquad 18A - 9B - 2D = 21.$

Solving we get,

$A = 1, B = -1, C = 0, D = 3.$

Now (iii) becomes,

$$y = \mathcal{L}^{-1}\left[\frac{1}{(s-1)} + \frac{(-1)}{(s+2)} + \frac{3}{s^2+9}\right]$$

$$= e^{t} - e^{-2t} + \sin 3t.$$

**(xviii)** $\quad$ **$x'' + x = t\cos 2t$, $x(0) = x'(0) = 0$.**

**Solution:** Given that,

$$x'' + x = t\cos 2t \quad \text{...... (i)}$$

$$x(0) = x'(0) = 0 \quad \text{...... (ii)}$$

Taking Laplace transform both side of (i) then,

$$\mathcal{L}(x'') + \mathcal{L}(x) = \mathcal{L}(t\cos 2t)$$

$$\Rightarrow \quad s^2\mathcal{L}(x) - s\,x(0) - x'(0) + \mathcal{L}(x) = \mathcal{L}(t\cos 2t)$$

$$\Rightarrow \quad s^2\mathcal{L}(x) - 0 - 0 + \mathcal{L}(x) = \mathcal{L}(t\cos 2t)$$

$$\Rightarrow \quad \mathcal{L}(x)(s^2 + 1) = \mathcal{L}(t\cos 2t) \quad \text{...... (iii)}$$

Since we have,

$$\mathcal{L}\{t\,f(t)\} = -F'(s) \quad \text{where } F(s) = \mathcal{L}\{f(t)\}$$

and $\mathcal{L}\{\cos wt\} = \dfrac{s}{s^2 + w^2}$

Now,

$$\mathcal{L}\{t\cos 2t\} = -\frac{d}{ds}(\mathcal{L}\{\cos 2t\})$$

$$= -\frac{d}{ds}\left(\frac{s}{s^2+4}\right) = -\frac{s^2 + 4 - 2s.s.}{(s^2+4)^2} = \frac{s^2 - 4}{(s^2+4)^2}.$$

Therefore, (iii) becomes,

$$\mathcal{L}(x)(s^2+1) = \frac{s^2-4}{(s^2+4)^2}$$

This gives, $\quad x = \mathcal{L}^{-1}\left[\frac{(s^2-4)}{(s^2+4)^2(s^2+1)}\right] \quad$ ....... (iv)

Let
$$\frac{(s^2-4)}{(s^2+4)^2(s^2+1)} = \frac{A}{s^2+1} + \frac{B}{(s^2+4)} + \frac{C}{(s^2+4)^2}$$
$$\Rightarrow \quad \frac{(s^2-4)}{(s^2+4)^2(s^2+1)} = \frac{A(s^2+4)^2 + B(s^2+4)(s^2+1) + C(s^2+1)}{(s^2+1)(s^2+4)^2}$$

This implies,
$$s^2 - 4 = As^4 + 8As^2 + 16A + Bs^4 + 5Bs^2 + 4B + Cs^2 + C$$
$$\Rightarrow \quad s^2 - 4 = s^4(A+B) + s^2(8A + 5B + C) + (16A + 4B + C)$$

Equating coefficient of s and the constant term on both sides then we get,

$A + B = 0, \qquad 8A + 5B + C = 1, \qquad 16A + 4B + C = -4$

Solving we get,
$$A = -\frac{5}{9},\ B = \frac{5}{9},\ C = \frac{8}{3}$$

Now (iv) becomes,
$$x = \mathcal{L}^{-1}\left[-\frac{5}{9}\times\frac{1}{(s^2+1)} + \frac{5}{9}\cdot\frac{1}{(s^2+4)} + \frac{8}{3}\cdot\frac{1}{(s^2+4)^2}\right]$$
$$= -\frac{5}{9}\sin t + \frac{5}{18}\frac{2}{s^2+2^2} + \frac{8}{3}\times\frac{1}{2\times 2^3}(\sin 2t + 2t\cos 2t)$$
$$= -\frac{5}{9}\sin t + \frac{5}{18}\sin 2t + \frac{1}{6}\sin 2t - \frac{1}{3}\cos 2t$$
$$= -\frac{5}{9}\sin t + \frac{4}{9}\sin 2t - \frac{1}{3}t\cos 2t$$
$$= \frac{1}{9}[4\sin 2t - 5\sin t - 3t\cos 2t].$$

**(xix)** $\quad y''' + y'' = 6t^2 + 4 \qquad y(0) = 0 = y''(0) = 0,\ y'(0) = 2$

**Solution:** Given that,
$$y''' + y'' = 6t^2 + 4 \qquad \text{...... (i)}$$
$$y(0) = 0 = y''(0) = 0,\ y'(0) = 2 \qquad \text{...... (ii)}$$

Taking Laplace transform both side of (i) then,
$$\mathcal{L}(y''') + \mathcal{L}(y'') = 6\,\mathcal{L}(t^2) + 4\,\mathcal{L}(1)$$
$$\Rightarrow \quad s^3\mathcal{L}(y) - s^2y(0) - s\,y'(0) - y''(0) + s^2\mathcal{L}(y) - sy(0) - y'(0) = \frac{6\times 2}{s^3} + \frac{4}{s}$$
$$\Rightarrow \quad s^3\mathcal{L}(y) - 0 - 2s - 0 + s^2\mathcal{L}(y) - 0 - 2 = \frac{12}{s^3} + \frac{4}{s} \qquad \text{[Using (ii)]}$$
$$\Rightarrow \quad \mathcal{L}(y)(s^3 + s^2) = \frac{12}{s^3} + \frac{4}{s} + 2s + 2$$
$$\Rightarrow \quad \mathcal{L}(y) = \frac{12 + 4s^2 + 2s^4 + 2s^3}{s^3(s^3 + s^2)}$$



This implies $\quad y = \mathcal{L}^{-1}\left[\frac{2s^4 + 2s^3 + 4s^2 + 12}{s^5(s+1)}\right] \quad$ ..... (iii)

Let,
$$\frac{2s^4 + 2s^3 + 4s^2 + 12}{s^5(s+1)} = \frac{A}{s+1} + \frac{B}{s} + \frac{C}{s^2} + \frac{D}{s^3} + \frac{E}{s^4} + \frac{F}{s^5}$$
$$= \frac{As^5 + Bs^4(s+1) + Cs^3(s+1) + Ds^2(s+1) + Es(s+1) + F(s+1)}{s^5(s+1)}$$

This gives,
$$2s^4 + 2s^3 + 4s^2 + 12 = s^5(A+B) + s^4(B+C) + s^3(C+D) + s^2(D+E) + s(E+F)$$

Equating coefficient of s and the constant term on both sides then we get,

Solving we get,

$A + B = 0, \qquad B + C = 2 \qquad C + D = 2$

$D + E = 4 \qquad E + F = 0 \qquad F = 12$

Solving we get,

$A = -16,\ B = 16,\ C = -14,\ D = 16,\ E = -12,\ F = 12.$

Now (iii) becomes,
$$y = \mathcal{L}^{-1}\left[\frac{-16}{s+1} + \frac{16}{s} + \frac{(-14)}{s^2} + \frac{16}{s^3} + \frac{(-12)}{s^4} + \frac{12}{s^5}\right]$$
$$= \left[-16e^{-t} + 16 - 14t + 8t^2 - 2t^3 + \frac{1}{2}t^4\right]$$
$$\Rightarrow \quad y = \left[\frac{t^4}{2} - 2t^3 + 8t^2 - 14t + 16(1 - e^{-t})\right]$$

**4. (i) Find the Laplace transform of** $f(t) = \begin{cases} \sin wt & \text{for } 0 < t < \pi/w \\ 0 & \text{for other wise} \end{cases}$

**Solution:** Let
$$f(t) = \begin{cases} \sin wt & \text{for } 0 < t < \pi/w \\ 0 & \text{for other wise} \end{cases}$$

Now, the Laplace transform of f(t) is
$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)\,e^{-st}\,dt$$
$$= \int_0^{\pi/w} e^{-st}\sin wt\,dt + \int_{\pi/w}^\infty 0\,.\,e^{-st}\,dt$$
$$= \left[\frac{e^{-st}}{(-s)^2 + w^2}\{(-s)\sin wt - w\cos wt\}\right]_0^{\pi/w} + 0$$
$$= \frac{e^{-\pi/s}}{s^2 + w^2}[(-s)\sin\pi - w\cos\pi] - \frac{1}{s^2 + w^2}[(-s)\sin 0 - w\,.\cos 0]$$
$$= \frac{e^{-\pi s/w}}{s^2 + w^2}\,.\,w + \frac{w}{s^2 + w^2} = \frac{w}{s^2 + w^2}(1 + e^{-\pi s/w}).$$

Thus, $\mathcal{L}\{f(t)\} = \frac{w}{s^2 + w^2}(1 + e^{-\pi s/w})$.

**(ii) Find the Laplace transform of** $f(t) = \begin{cases} e^t & \text{for } 0 < x < 1 \\ 0 & \text{for otherwise} \end{cases}$

**Solution:** Let,

$$f(t) = \begin{cases} e^t & \text{for } 0 < x < 1 \\ 0 & \text{for otherwise} \end{cases}$$

Now, Laplace transform of f(t) is

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)\,e^{-st}\,dt = \int_0^1 e^t e^{-st} dt + \int_0^\infty 0.\,e^{-st}\,dt$$

$$= \int_0^1 e^{-(s-1)t}\,dt + 0$$

$$= \left[\frac{e^{-(s-1)t}}{-(s-1)}\right]_0^1 = \frac{e^{-(s-1)} - 1}{-(s-1)} = \left(\frac{1 - e^{1-s}}{s-1}\right).$$

Thus, $\mathcal{L}\{f(t)\} = \left(\frac{1 - e^{1-s}}{s-1}\right)$.

**(iii) Find the Laplace transform of** $f(t) = \begin{cases} 10\cos\pi t & \text{for } 0 < t \le 2 \\ 0 & \text{for other wise} \end{cases}$

**Solution:** Let

$$f(t) = \begin{cases} 10\cos\pi t & \text{for } 0 < t \le 2 \\ 0 & \text{for other wise} \end{cases}$$

Now, Laplace transform of f(t) is

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)\,e^{-st}\,dt$$

$$= \int_0^2 10\cos\pi t\, e^{-st} dt + \int_2^\infty 0\,e^{-st}\,dt$$

$$= 10\left[\frac{e^{-st}}{(-s)^2 + \pi^2}\{(-s)\cos\pi t + \pi\sin\pi t\}\right]_0^2 + 0$$

$$= \frac{10e^{-2s}}{s^2 + \pi^2}(-s\cos 2\pi + \pi\sin 2\pi) - \frac{10}{s^2 + \pi^2}(-s\cos 0 + \pi\sin 0)$$

$$= \frac{10}{s^2 + \pi^2}(-se^{-2s} + s) = \frac{10s}{s^2 + \pi^2}(1 - e^{-2s}).$$

Thus, $\mathcal{L}\{f(t)\} = \frac{10s}{s^2 + \pi^2}(1 - e^{-2s})$.