Exercise 10.2

1. **Verify Euler's theorem for the following functions:**

(i) $u = ax^2 + 2hxy + by^2$ **[2007 Fall; 2008 Fall –Short]**

Solution: Let, $u = ax^2 + 2hxy + by^2$

Set x as tx and y as ty then

$$u(tx, ty) = a(tx)^2 + 2h(tx)(ty) + b(ty)^2$$
$$= t^2 (ax^2 + 2hxy + by^2)$$
$$= t^2u.$$

This shows that u is homogeneous function of degree (n) = 2.

Then we wish to show $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u.$

Here differentiating u partially w. r. t. 'x' and 'y' then,

$$\frac{\partial u}{\partial x} = 2ax + 2hy + 0 \quad \text{and} \quad \frac{\partial u}{\partial y} = 2hx + 2by$$

So, $$x\frac{\partial u}{\partial x} = 2ax^2 + 2hxy \quad \text{and} \quad y\frac{\partial u}{\partial y} = 2hxy + 2by^2$$

Now,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2ax^2 + 2hxy + 2hxy + 2by^2$$
$$= 2(ax^2 + 2hxy + by^2) = 2u$$

Hence, Euler's theorem is verified.

(ii) $u = (x^2 + y^2)^{1/3}$

Solution: Let, $u = (x^2 + y^2)^{1/3}$

Set x as tx and y as ty then

$$u(tx, ty) = \{(tx)^2 + (ty)^2\}^{1/3}$$
$$= t^{2/3} (x + y^2)^{1/3} = t^{2/3} . u$$

This shows that u is homogeneous function of degree (n) = $\frac{2}{3}$.

Then we wish to show $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2}{3}u.$

Here differentiating u partially w. r. t. 'x' and 'y' then,

$$\frac{\partial u}{\partial x} = \frac{1}{3}(x^2+y^2)^{-2/3} \times 2x \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{1}{3}(x^2+y^2)^{-2/3} \times 2y$$

$$= \frac{2x}{3}(x^2+y^2)^{-2/3} \qquad = \frac{2y}{3}(x^2+y^2)^{-2/3}$$

So,

$$x\frac{\partial u}{\partial x} = \frac{2x^2}{3}(x^2+y^2)^{-2/3} \quad \text{and} \quad y\frac{\partial u}{\partial y} = \frac{2y^2}{3}(x^2+y^2)^{-2/3}$$

Now,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2x^2}{3}(x^2+y^2)^{-2/3} + \frac{2y^2}{3}(x^2+y^2)^{-2/3}$$

$$= \frac{2}{3}(x^2+y^2)^{-2/3}(x^2+y^2) = \frac{2}{3}(x^2+y^2)^{1/3} = \frac{2}{3}u$$

Hence the Euler's theorem is verified.

(iii) $\mathbf{u = x^n \tan^{-1}\left(\frac{y}{x}\right)}$

Solution: Let, $u = x^n \tan^{-1}\left(\frac{y}{x}\right)$

Set x as tx and y as ty then

$$u(tx, ty) = (tx)^n \tan^{-1}\left(\frac{ty}{tx}\right) = t^n x^n \tan^{-1}\left(\frac{y}{x}\right) = t^n.u$$

This shows that u is homogeneous function of degree (n) = n.

Then we wish to show $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$

Here differentiating u partially w. r. t. 'x' and 'y' then,

$$\frac{\partial u}{\partial x} = nx^{n-1}\tan^{-1}\left(\frac{y}{x}\right) + x^n \frac{1}{1+\frac{y^2}{x^2}} \times \left(-\frac{y}{x^2}\right) \quad \text{And} \quad \frac{\partial u}{\partial y} = x^n \times \frac{1}{1+\frac{y^2}{x^2}} \times \frac{1}{x}$$

$$= nx^{n-1}\tan^{-1}\left(\frac{y}{x}\right) - \frac{x^n y}{x^2+y^2} \qquad = \frac{x^{n+1}}{x^2+y^2}$$

So,

$$x\frac{\partial u}{\partial x} = nx^n \tan^{-1}\left(\frac{y}{x}\right) - \frac{x^{n+1}}{x^2+y^2} \quad \text{And} \quad y\frac{\partial u}{\partial y} = \frac{yx^{n1}}{x^2+y^2}$$

Now,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n \tan^{-1}\left(\frac{y}{x}\right) - \frac{x^{n+1}y}{x^2+y^2} + \frac{yx^{n+1}}{x^2+y^2}$$

$$= nx^n \tan^{-1}\left(\frac{y}{x}\right)$$

$$= n.u$$

Hence, the Euler's theorem is verified.

(iv) $\mathbf{u = x\,f\left(\frac{y}{x}\right)}$

Solution: Let, $u = x\,f\left(\frac{y}{x}\right)$

Set x as tx and y as ty then

$$u(tx, ty) = tx\,f\left(\frac{ty}{tx}\right) = t\,x\,f\left(\frac{y}{x}\right) = tu$$

This shows that u is homogeneous function of degree (n) = 1.

Then we wish to show $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1.u = u.$

Here differentiating u partially w. r. t. 'x' and 'y' then,

$$\frac{\partial u}{\partial x} = f\left(\frac{y}{x}\right) + x\,.\,f'\left(\frac{y}{x}\right) \times \left(-\frac{y}{x^2}\right) = f\left(\frac{y}{x}\right) - \frac{y}{x}f'\left(\frac{y}{x}\right)$$

And $$\frac{\partial u}{\partial y} = x\,f'\left(\frac{y}{x}\right).\frac{1}{x} = f'\left(\frac{y}{x}\right)$$

Now,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\,f\left(\frac{y}{x}\right) - y\,f'\left(\frac{y}{x}\right) + y\,f'\left(\frac{y}{x}\right)$$

$$= 1\,.\,x\,f'\left(\frac{y}{x}\right) = 1\,.\,u$$

Hence, Euler's theorem is verified.

(v) $\mathbf{u = \frac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}}$ **[2008 Spring Q. No. 2(b) OR]**

Solution: Let, $u = \frac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}$

Set x as tx and y as ty then

$$u(tx, ty) = \frac{(tx)^{1/4}+(ty)^{1/4}}{(tx)^{1/5}+(ty)^{1/5}}$$

$$= \frac{t^{1/4}(x^{1/4}+y^{1/4})}{t^{1/5}(x^{1/5}+y^{1/5})} = t^{1/4-1/5}\frac{(x^{1/4}+y^{1/4})}{(x^{1/5}+y^{1/5})} = t^{1/20}.\,u$$

This shows that u is homogeneous function of degree (n) $= \frac{1}{20}$.

Then we wish to show $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{u}{20}.$

Here differentiating u partially w. r. t. 'x' and 'y' then,

$$\frac{\partial u}{\partial x} = \frac{(x^{1/5}+y^{1/5})\frac{1}{4}x^{-3/4} - (x^{1/4}+y^{1/4})\frac{1}{5}x^{-4/5}}{(x^{1/5}+y^{1/5})^2}$$

And $$\frac{\partial u}{\partial y} = \frac{(x^{1/5}+y^{1/5})\frac{1}{4}x^{-3/4} - (x^{1/4}+y^{1/4})\frac{1}{5}x^{-4/5}}{(x^{1/5}+y^{1/5})^2}$$

Then,

$$x\frac{\partial u}{\partial x}=\frac{\frac{1}{4}x^{1/4}(x^{1/5}+y^{1/5})-\frac{1}{5}x^{1/5}(x^{1/4}+y^{1/4})}{(x^{1/5}+y^{1/5})^2}$$

And $$y\frac{\partial u}{\partial x}=\frac{\frac{1}{4}y^{1/4}(x^{1/5}+y^{1/5})-\frac{1}{5}x^{1/5}(x^{1/4}+y^{1/4})}{(x^{1/5}+y^{1/5})^2}$$

Now,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$$

$$=\frac{\frac{1}{4}x^{1/4}(x^{1/5}+y^{1/5})-\frac{1}{5}x^{1/5}(x^{1/4}+y^{1/4})+\frac{1}{4}y^{1/4}(x^{1/5}+y^{1/5})-\frac{1}{5}x^{1/5}(x^{1/4}+y^{1/4})}{(x^{1/5}+y^{1/5})^2}$$

$$=\frac{\frac{1}{4}(x^{1/5}+y^{1/5})(x^{1/4}+y^{1/4})-\frac{1}{5}(x^{1/4}+y^{1/4})(x^{1/5}+y^{1/5})}{(x^{1/5}+y^{1/5})^2}$$

$$=\frac{(x^{1/5}+y^{1/5})(x^{1/4}+y^{1/4})\left(\frac{1}{4}-\frac{1}{5}\right)}{(x^{1/5}+y^{1/5})^2}=\frac{1}{20}\cdot\frac{x^{1/4}+y^{1/4}}{x^{1/5}+y^{1/5}}=\frac{1}{20}\cdot u$$

Hence, the Euler's theorem is verified.

(vii) $\mathbf{u=\sin^{-1}\left(\frac{x}{y}\right)+\tan^{-1}\left(\frac{y}{x}\right)}$

Solution: Let, $u=\sin^{-1}\left(\frac{x}{y}\right)+\tan^{-1}\left(\frac{y}{x}\right)$

Set x as tx and y as ty then

$$u(tx, ty)=\sin^{-1}\left(\frac{tx}{ty}\right)+\tan^{-1}\left(\frac{ty}{tx}\right)$$

$$=\sin^{-1}\left(\frac{x}{y}\right)+\tan^{-1}\left(\frac{y}{x}\right)=t^0\left(\sin^{-1}\left(\frac{x}{y}\right)+\tan^{-1}\left(\frac{y}{x}\right)\right)$$

This shows that u is homogeneous function of degree (n) = 0.

Then we wish to show $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0.u=0.$

Here differentiating u partially w. r. t. 'x' and 'y' then,

$$\frac{\partial u}{\partial x}=\frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\times\frac{1}{y}+\frac{1}{1+\frac{y^2}{x^2}}\times\left(-\frac{y}{x^2}\right)=\left(\frac{1}{\sqrt{y^2-x^2}}-\frac{y}{x^2+y^2}\right)$$

And $$\frac{\partial u}{\partial y}=\frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\times\left(-\frac{x}{y^2}\right)+\frac{1}{1+\frac{y^2}{x^2}}\times\frac{1}{x}=\left(\frac{-x}{y\sqrt{y^2-x^2}}-\frac{x}{x^2+y^2}\right)$$

Then,

$$x\frac{\partial u}{\partial x}=\frac{x}{\sqrt{y^2-x^2}}-\frac{xy}{x^2+y^2}\quad\text{And}\quad y\frac{\partial u}{\partial y}=-\frac{x}{\sqrt{y^2-x^2}}+\frac{xy}{x^2+y^2}$$

Now,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{x}{\sqrt{y^2-x^2}}-\frac{xy}{x^2+y^2}-\frac{x}{\sqrt{y^2-x^2}}+\frac{xy}{x^2+y^2}$$

$$=0.u$$

$$=0.$$

Hence, the Euler's theorem is verified.

(viii) $\mathbf{u=x^3+y^3+z^3-3xyz}$

Solution: Let, $u=x^3+y^3+z^3-3xyz$

Set x as tx, y as ty and z as tz then

$$u(tx, ty, tz)=(tx)^3+(ty)^3+(tz)^3-3tx.\,ty.\,tz$$

$$=t^3(x^3+y^3+z^3-3xyz)$$

$$=t^3.\,u$$

This shows that u is homogeneous function of degree (n) = 3.

Then we wish to show $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3u.$

Here differentiating u partially w. r. t. 'x', 'y' and 'z' then,

$$\frac{\partial u}{\partial x}=3x^2-3yz,\quad\frac{\partial u}{\partial y}=3y^2-3xz\quad\text{and}\quad\frac{\partial u}{\partial z}=3z^2-3xy$$

Then,

$$x\frac{\partial u}{\partial x}=3x^3-3xyz,\quad y\frac{\partial u}{\partial y}=3y^3-3xyz\quad\text{and}\quad z\frac{\partial u}{\partial z}=3z^3-3xyz$$

Now,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=3x^3-3xyz+3y^3-3xyz+3z^3-3xyz$$

$$=3(x^3+y^3+z^3-3xyz)$$

$$=3u$$

Hence the Euler's theorem is verified.

(2) **If $u=\cos^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+\frac{1}{2}\cot u=0$.**

[1999, 2001 Q. No. 2(b) OR] [2008 Fall Q. No. 2(b)]

Solution: Let, $u=\cos^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}\Rightarrow\cos u=\frac{x+y}{\sqrt{x}+\sqrt{y}}$

Let, f = cos u

Differentiating f partially w. r. t. 'x' and 'y' then,

$$\frac{\partial f}{\partial x}=-\sin u\frac{\partial u}{\partial x},\quad\text{and}\quad\frac{\partial f}{\partial x}=-\sin u\frac{\partial u}{\partial y}$$

And, $f=\frac{x+y}{\sqrt{x}+\sqrt{y}}$

Set x as tx and y as ty then

$$f(tx, ty) = \frac{tx + ty}{\sqrt{tx} + \sqrt{ty}} = \frac{t(x+y)}{\sqrt{t}(\sqrt{x}+\sqrt{y})} = t^{1/2} f$$

This shows that f is homogeneous function of degree $(n) = \frac{1}{2}$.

Then, by Euler's theorem,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf \Rightarrow x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = \frac{1}{2}\tan u$$

$$\Rightarrow \sec^2 u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = \frac{1}{2}\tan u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\sin u\cos u$$

Thus, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{4}\sin 2u$.

(2) If $u = \tan^{-1}\frac{x^3+y^3}{x-y}$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$.

Solution: We have, $u = \tan^{-1}\frac{x^3+y^3}{x-y} \Rightarrow \tan u = \frac{x^3+y^3}{x-y}$

Let $f = \tan u$.

Differentiating f partially w.r.t. x and y, then

$$\frac{\partial f}{\partial x} = \sec^2 u\frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y} = \sec^2 u\frac{\partial u}{\partial y}$$

And, $f = \frac{x^3+y^3}{x-y}$

Also, replacing x by tx, y by ty then

$$f(tx, ty) = \frac{(tx)^3+(ty)^3}{tx-ty} = \frac{t^3(x^3+y^3)}{t(x-y)} = \frac{t^2(x^3+y^3)}{(x-y)} = t^2 f$$

This shows that f is homogeneous function of degree $(n) = 2$.

Then by Euler's theorem,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 2f \Rightarrow x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = 2\tan u$$

$$\Rightarrow \sec^2 u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = 2\tan u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\tan u}{\sec^2 u} = 2\frac{\sin u}{\cos u}\cos^2 u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$$

(3) If $u = \log\left(\frac{x^2+y^2}{x+y}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1$.

Solution: We have $e^u = \frac{x^2+y^2}{x+y}$

Let $f = e^u$

Differentiating f partially w.r.t. x and y, then

$$\frac{\partial f}{\partial x} = e^u\frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y} = e^u\frac{\partial u}{\partial y}$$

And, $f = \frac{x^2+y^2}{x+y}$

Replacing x by tx, y by ty then

$$f(tx, ty) = \frac{(tx)^2+(ty)^2}{tx+ty} = t\frac{x^2+y^2}{x+y} = tf$$

This shows that f is homogeneous function of degree $(n) = 1$.

Then by Euler's theorem,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 1f \Rightarrow xe^u\frac{\partial u}{\partial x} + ye^u\frac{\partial u}{\partial y} = e^u$$

$$\Rightarrow e^u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = e^u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1$$

(4) If $u = \sin^{-1}\left[\frac{x^2+y^2}{x+y}\right]$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u$.

Solution: Let, $u = \sin^{-1}\left[\frac{x^2+y^2}{x+y}\right] \Rightarrow \sin u = \frac{x^2+y^2}{x+y}$

Let $f = \sin u$.

Differentiating f partially w.r.t. x and y then

$$\frac{\partial f}{\partial x} = \cos u\frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y} = \cos u\frac{\partial u}{\partial y}$$

And, $f = \left[\frac{x^2+y^2}{x+y}\right]$

Replacing x by tx, y by ty then

$$f(tx, ty) = \frac{(tx)^2+(ty)^2}{tx+ty} = \frac{t^2(x^2+y^2)}{t(x+y)} = t\left(\frac{x^2+y^2}{x+y}\right)$$

This shows that f is homogeneous function of degree $(n) = 1$.

Then by Euler's theorem,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 1 \cdot f \Rightarrow x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = \sin u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u$$

(5) If $u = \sin^{-1}\frac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} + 3\tan u = 0$.

Solution: Let, $u = \sin^{-1}\dfrac{x+2y+3z}{\sqrt{x^8+y^8+z^8}} \Rightarrow \sin u = \dfrac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}$

Let, $f = \sin u$ and $f = \dfrac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}$.

For, $f = \sin u$.

Differentiating f partially w. r. t. 'x', 'y' and 'z' then,

$$\frac{\partial f}{\partial x} = \cos u\frac{\partial u}{\partial x}, \quad \frac{\partial f}{\partial y} = \cos u\frac{\partial u}{\partial y} \quad \text{and} \quad \frac{\partial f}{\partial z} = \cos u\frac{\partial u}{\partial z}$$

Also, $f = \dfrac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}$.

Set x as tx, y as ty and z as tz then

$$f(tx, ty, tz) = \frac{tx+2ty+3tz}{\sqrt{(tx)^8+(ty)^8+(tz)^8}} = \frac{t(x+2y+3z)}{t^4\sqrt{x^8+y^8+z^8}} = t^{-3}f$$

This shows that f is the homogeneous function of degree (n) = – 3

Then by Euler's theorem,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = -3.f$$

$$\Rightarrow x.\cos u\frac{\partial u}{\partial x} + y.\cos u\frac{\partial u}{\partial y} + z\cos u.\frac{\partial u}{\partial z} = -3\sin u$$

$$\Rightarrow \cos u\left(x.\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right) = -3\sin u$$

$$\Rightarrow x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} + 3\tan u = 0.$$

(7) If $u = \log\left[\dfrac{x^4+y^4}{x+y}\right]$ show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3$.

Solution: Let, $u = \log\left[\dfrac{x^4+y^4}{x+y}\right]. \Rightarrow e^u = \dfrac{x^4+y^4}{x+y}$

Let $f = e^u$ and $f = \dfrac{x^4+y^4}{x+y}$

For, $f = e^u$

Differentiating f partially w. r. t. 'x' and 'y' then,

$$\frac{\partial f}{\partial x} = e^u\frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y} = e^u\frac{\partial u}{\partial y}$$

Also, $f = \dfrac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}$.

Set x as tx and y as ty then

$$f(tx, ty) = \frac{(tx)^4+(ty)^4}{tx+ty} = \frac{t^4(x^4+y^4)}{t(x+y)} = t^3.f$$

This shows that f is the homogeneous function of degree (n) = 3.

Then by Euler's theorem,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 3f. \Rightarrow x\,e^u\frac{\partial u}{\partial x} + ye^u\frac{\partial u}{\partial y} = 3e^u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3.$$

(8) $u = \sqrt{x^2-y^2}\sin^{-1}\left(\dfrac{y}{x}\right)$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = u$.

Solution: Let, $u(x, y) = \sqrt{x^2-y^2}\sin^{-1}\left(\dfrac{y}{x}\right)$.

Set x as tx and y as ty then

$$u(tx, ty) = \sqrt{(tx)^2-(ty)^2}\sin^{-1}\left(\frac{ty}{tx}\right) = t\sqrt{x^2-y^2}\sin^{-1}\left(\frac{ty}{tx}\right) = t^1.u$$

This shows that u is the homogeneous function of degree (n) = 1.

Then by Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1.u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = u.$$

(9) If $u = \sqrt{y^2-x^2}\sin^{-1}\left(\dfrac{x}{y}\right) + \dfrac{x^2-y^2}{\sqrt{x^2+y^2}}$ show that $x\dfrac{\partial u}{\partial x} + y.\dfrac{\partial u}{\partial y} = u$.

Solution: Let, $u(x, y) = \sqrt{y^2-x^2}\sin^{-1}\left(\dfrac{x}{y}\right) + \dfrac{x^2-y^2}{\sqrt{x^2+y^2}}$

Set x as tx and y as ty then

$$u(tx, ty) = \sqrt{y^2-x^2}\sin^{-1}\left(\frac{x}{y}\right) + \frac{(tx)^2+(ty)^2}{\sqrt{(tx)^2+(ty)^2}}$$

$$= t\sqrt{y^2-x^2}\sin^{-1}\left(\frac{x}{y}\right) + \frac{t^2(x^2-y^2)}{t\sqrt{x^2+y^2}}$$

$$= t\left(\sqrt{y^2-x^2}\sin^{-1}\left(\frac{y}{x}\right) + \frac{x^2-y^2}{\sqrt{x^2+y^2}}\right) = t^1.u$$

This shows that u is the homogeneous function of degree (n) = 1.

So according to Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n.u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1.u = u.$$

(10) If $u = \cos\left[\dfrac{xy+yz+zx}{x^2+y^2+z^2}\right]$ show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 0$.

Solution: Let, $u = \cos\left[\dfrac{xy+yz+zx}{x^2+y^2+z^2}\right] \Rightarrow \cos^{-1}u = \dfrac{xy+yz+zx}{x^2+y^2+z^2}$

Let, $f = \cos^{-1}u$ and $f = \dfrac{xy + yz + zx}{x^2 + y^2 + z^2}$.

For, $f = \cos^{-1}u$.

Differentiating f partially w. r. t. 'x', 'y' and 'z' then,

$$\frac{\partial f}{\partial u} = -\frac{1}{\sqrt{1-u^2}}\frac{\partial u}{\partial x}, \quad \frac{\partial f}{\partial y} = -\frac{1}{\sqrt{1-u^2}}\frac{\partial u}{\partial y} \text{ and } \frac{\partial f}{\partial z} = -\frac{1}{\sqrt{1-u^2}}\frac{\partial u}{\partial z}$$

Also, $f = \dfrac{xy + yz + zx}{x^2 + y^2 + z^2}$

Set x as tx, y as ty and z as tz then

$$f(tx, ty) = \frac{tx.\ ty + ty.\ tz + tz.\ tx}{(tx)^2 + (ty)^2 + (tz)^2} = \frac{t^2(xy + yz + zx)}{t^2(x^2 + y^2 + z^2)} = t^0.\ u$$

This shows that f is the homogeneous function of degree (n) = 0.

Then by Euler's theorem,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial x} = n.f.$$

$$\Rightarrow x\left(-\frac{1}{\sqrt{1-u^2}}\right)\frac{\partial u}{\partial x} + y\left(-\frac{1}{\sqrt{1-u^2}}\right)\frac{\partial u}{\partial y} + z\left(-\frac{1}{\sqrt{1-u^2}}\right)\frac{\partial u}{\partial z} = 0$$

$$\Rightarrow -\frac{1}{\sqrt{1-u^2}}\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right) = 0$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0.$$

(11) If $\sin u = \dfrac{x^2y^2}{x + y}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3\tan u$.

Solution: Let, $\sin u = \dfrac{x^2y^2}{x + y}$

Let, $f = \sin u$ and $f = \dfrac{x^2y^2}{x + y}$

For, f = sin u.

Differentiating f partially w. r. t. 'x' and 'y' then,

$$\frac{\partial f}{\partial x} = \cos u\frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y} = \cos u\frac{\partial u}{\partial y}$$

Also, $f = \dfrac{x^2y^2}{x + y}$

Set x as tx and y as ty then

$$f(tx, ty) = \frac{(tx)^2\,(ty)^2}{tx + ty} = \frac{t^4\,x^2\,y^2}{t(x + y)} = t^3\frac{(x^2y^2)}{x + y} = t^3.u$$

This shows that f is the homogeneous function of degree (n) = 3.

Then by Euler's theorem,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = x.f \Rightarrow x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = 3\sin u$$

$$\Rightarrow x.\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3\tan u.$$

(12) If $u = \sin^{-1}\left[\dfrac{x + y}{\sqrt{x} + \sqrt{y}}\right]$ show that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = -\dfrac{\sin u \cos 2u}{4\cos^3 u}$.

Solution: Let, $u = \sin^{-1}\left[\dfrac{x + y}{\sqrt{x} + \sqrt{y}}\right] \Rightarrow \sin u = \dfrac{x + y}{\sqrt{x} + \sqrt{y}}$

Let, $f = \sin u$ and so, $f = \dfrac{x + y}{\sqrt{x} + \sqrt{y}}$

Differentiating f partially w. r. t. 'x' and 'y' then,

$$\frac{\partial f}{\partial x} = \cos u\frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y} = \cos u\frac{\partial u}{\partial y}$$

Also, $f = \dfrac{x + y}{\sqrt{x} + \sqrt{y}}$

Set x as tx and y as ty then

$$f(tx, ty) = \frac{tx + ty}{\sqrt{tx} + \sqrt{ty}} = \frac{t(x + y)}{\sqrt{t}\,(\sqrt{x} + \sqrt{y})} = t^{1/2}u$$

This shows that f is the homogeneous function of degree (n) = $\frac{1}{2}$.

Then by Euler's theorem,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = n.f$$

$$\Rightarrow x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = \frac{1}{2}\sin u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u \qquad \text{........ (i)}$$

Differentiating above equation w. r. t. 'x' then,

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial y \partial x} = \frac{1}{2}\sec^2 u.\frac{\partial u}{\partial x}$$

Multiplying by x

$$x^2\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} + xy\frac{\partial^2 u}{\partial y \partial x} = \frac{1}{2}x\sec^2 u.\frac{\partial u}{\partial x} \qquad \text{........ (ii)}$$

Differentiating (i) equation w. r. t. 'y' then

$$x\frac{\partial^2 u}{\partial x \partial y} + y\frac{\partial^2 u}{\partial y^2} + 1.\frac{\partial u}{\partial y} = \frac{1}{2}\sec^2 u.\frac{1}{2}$$

Multiplying by y

$$xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + y\frac{\partial u}{\partial y} = \frac{1}{2}y\sec^2 u\frac{\partial u}{\partial y} \qquad \text{......(iii)}$$

Adding (ii) and (iii) then,

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial y \partial x} + y^2\frac{\partial^2 u}{\partial y^2} + \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = \frac{1}{2}\sec^2 u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial y\partial x} + y^2\frac{\partial^2 u}{\partial y^2} + \frac{1}{2}\tan u = \frac{1}{2}\sec^2 u\,\frac{1}{2}\tan u \quad \text{[using (i)]}$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial y\partial x} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{1}{2}\tan u\left(\frac{1}{2}\sec^2 u - 1\right)$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial y\partial x} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{1}{2}\tan u\left(\frac{1}{2\cos^2 u} - 1\right)$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial y\partial x} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{1}{2}\frac{\sin u}{\cos u}\left(\frac{1 - 2\cos^2 u}{2\cos^2 u}\right)$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial y\partial x} + y^2\frac{\partial^2 u}{\partial y^2} = -\frac{\sin u\cos 2u}{4\cos^3 u}$$

(13) If $u = (x^2 + y^2)^{1/3}$, show that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\frac{2u}{9}$

Solution: Let, $u = (x^2 + y^2)^{1/3}$

Set x as tx and y as ty then

$$u(tx, ty) = \{(tx)^2 + (ty)^2\}^{1/3}$$
$$= t^{2\times 1/3}(x^2 + y^2)^{1/3} = t^{2/3}u$$

This shows that u is the homogeneous function of degree $(n) = \frac{2}{3}$.

Then by Euler's theorem,

$$x.\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n.f.$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2}{3}u \quad \text{.........(i)}$$

Differentiating (i) partially w. r. t. 'x' then,

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial y}{\partial x\partial y} = \frac{2}{3}\frac{\partial u}{\partial x}$$

Multiplying by 'x'

$$x^2\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} + xy\frac{\partial^2 y}{\partial x\partial y} = \frac{2}{3}x\frac{\partial u}{\partial x} \quad \text{.........(ii)}$$

Again differentiating (i) partially w. r. t. 'y' then,

$$x\frac{\partial^2 y}{\partial x\partial y} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = \frac{2}{3}\frac{\partial u}{\partial y}$$

Multiplying by 'y'

$$xy\frac{\partial^2 y}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + y\frac{\partial u}{\partial y} = \frac{2}{3}\frac{\partial u}{\partial y} \quad \text{.........(iii)}$$

Adding (ii) and (iii) then

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2}{3}\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right]$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + \frac{2}{3}u = \frac{2}{3}\times\frac{2}{3}u$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{4}{9}u - \frac{2}{3}u$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{4u - 6u}{9}$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\frac{2u}{9}.$$

(14) If $u = \tan^{-1}\left(\frac{y^2}{x}\right)$ show that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \sin 2u.\sin^2 u$

Solution: Let, $\tan u = \frac{y^2}{x}$

Let $F = \tan u$ and so, $f = \frac{y^2}{x}$

Differentiating f partially w. r. t. 'x' and 'y' then,

$$\frac{\partial f}{\partial x} = \sec^2 u\frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y} = \sec^2 u\frac{\partial u}{\partial y}$$

Also, $f = \frac{y^2}{x}$

Set x as tx and y as ty then

$$f(tx, ty) = \frac{(ty)^2}{tx} = \frac{t^2y^2}{tx} = t\left(\frac{y^2}{x}\right) = t^1u$$

Here u is the homogeneous function of degree (n) = 1.

Then by Euler's theorem

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = n.f.$$

$$\Rightarrow x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = 1\tan u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{\sin u}{\cos u}\times\cos^2 u = \frac{1}{2}\sin 2u \quad \text{........ (i)}$$

Differentiating (i) partially w. r. t. 'x' then,

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x\partial y} = \frac{2}{2}\cos 2u\frac{\partial u}{\partial x}$$

Multiplying by x, we get

$$x^2\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} + xy\frac{\partial^2 u}{\partial x\partial y} = x\cos 2u\frac{\partial u}{\partial x} \quad \text{....... (ii)}$$

Again, differentiating (i) partially w. r. t. 'y' then,

$$x^2\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = \frac{2}{2}\cos 2u\frac{\partial u}{\partial y}$$

Multiplying by y

$$xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + y\frac{\partial u}{\partial y} = y\cos 2u\frac{\partial u}{\partial y} \quad \text{....... (iii)}$$

Adding (ii) and (iii)

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\cos 2u\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

$$\Rightarrow\ x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+\frac{1}{2}\sin^2 u=\cos 2u\cdot\frac{1}{2}\sin 2u$$

$$\Rightarrow\ x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{1}{2}\sin^2(\cos 2u-1)$$

$$\Rightarrow\ x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{1}{2}\sin 2u(\cos 2u-1)$$

$$\Rightarrow\ x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{1}{2}[1-\sin^2 u-1]=-\frac{1}{2}\sin 2u.\ 2\sin^2 u$$

$$\Rightarrow\ x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=-\sin 2u.\ \sin^2 u.$$

(15) Find $\frac{dw}{dt}$ where (i) $w=x^3-y^3,\ x=\frac{1}{t+1},\ y=\frac{1}{t+1}$

(ii) $w=r^2-s\tan v,\ r=\sin^2 t,\ s=\cos t,\ v=4t.$

Solution:

(i) Let, $w=x^3-y^3,\ x=\frac{1}{t+1},\ y=\frac{1}{t+1}$

Differentiating f partially w. r. t. 'x' and 'y' then,

$$\frac{\partial w}{\partial x}=3x^2\ \frac{dx}{dt}=-\frac{1}{(t+1)^2},\ \frac{(t+1)1-t.1}{(t+1)^2}$$

$$\frac{\partial w}{\partial y}=-3y^2$$

We have,

$$\frac{dw}{dt}=\frac{\partial w}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial w}{\partial y}\cdot\frac{dy}{dt}$$

$$=3x^2\times-\frac{1}{(t+1)^2}+\frac{1}{(t+1)^2}\times(-3y^2)$$

$$=-\frac{3}{(t+1)^2}\cdot\frac{1}{(t+1)^2}+\frac{1}{(t+1)^2}\times-\frac{3t^2}{(t+1)^2}$$

$$=\frac{-3(1+t^2)}{(t+1)^4}.$$

(ii) Let, $w=r^2-s\tan v,\ r=\sin^2 t,\ s=\cos t,\ v=4t.$

Differentiating w partially w. r. t. 'r', 's' and 'v' then,

$$\frac{\partial w}{\partial r}=2r\frac{dr}{dt}=2\sin t\cos t\qquad\frac{\partial w}{\partial s}=-\tan v\frac{ds}{dt}=-\sin t\qquad\frac{\partial w}{\partial v}=s\sec^2 v\frac{dv}{dt}=4$$

We have,

$$\frac{dw}{dt}=\frac{\partial w}{\partial r}\cdot\frac{dr}{dt}+\frac{\partial w}{\partial s}\cdot\frac{ds}{dt}+\frac{\partial w}{\partial v}\cdot\frac{dv}{dt}$$

$$=2r.\ 2\sin t.\ \cos t+(-\tan v)(-\sin t)+(-\sec^2 v).4$$

$$=4r\sin t.\ \cos t+\tan v.\ \sin t-4s\sec^2 v$$

$$=4\sin^2 t.\ \sin t\cos t+\tan 4t.\ \sin t-4\cos t.\sec^2 4t$$

$$\Rightarrow\ \frac{dw}{dt}=4\sin^3 t.\ \cos t+\tan 4t.\sin t-4\cos t.\ \sec^2 4t.$$

(16) Find $\frac{dz}{dx}$ it $z=(y+x)e^{xy},\ y=\frac{1}{x^2}$

Solution: Let, $z=(y+x)e^{xy},\ y=\frac{1}{x^2}$

Differentiating z partially w. r. t. 'x' then,

$$\frac{dz}{dx}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial x}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial x}\qquad\text{.......(i)}$$

Since,

$$z=(y+x)e^{xy},\ y=\frac{1}{x^2}.$$

Then,

$$\frac{\partial z}{\partial x}=(0+1)e^{xy}+(y+x)y.e^{xy}=e^{xy}(1+xy+y^2)$$

And,
$$\frac{\partial z}{\partial y}=(1+0)e^{xy}+(y+x)xe^{xy}=e^{xy}(1+xy+x^2)$$

Also,
$$\frac{dy}{dx}=-\frac{2}{x^3}$$

Now, (i) becomes,

$$\frac{dz}{dx}=\frac{\partial z}{\partial x}\cdot\frac{dx}{dx}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dx}$$

$$=e^{xy}(1+xy+y^2)+e^{xy}(1+xy+x^2)\times-\frac{2}{x^3}$$

$$=e^{xy}\left\{(1+xy+y^2)-\frac{2}{x^3}(1+xy+x^2)\right\}$$

$$=e^{xy}\left(1+xy+y^2-\frac{2}{x^3}(1+xy+x^2)\right)$$

$$=e^{1/x}\left(1+x\cdot\frac{1}{x^2}+\left(\frac{1}{x^2}\right)^2-\frac{2}{x^3}-\frac{2}{x^3}-\frac{2}{x^2.x^2}-\frac{2}{x}\right)$$

$$=\left(1+\frac{1}{x}+\frac{1}{x^4}-\frac{2}{x^3}-\frac{2}{x^4}-\frac{2}{x}\right)e^{1/x}$$

$$\Rightarrow\ \frac{dz}{dx}=\left(1-\frac{1}{x}-\frac{2}{x^3}-\frac{1}{x^4}\right)e^{1/x}$$

(17) If $z=f(x,y)$ and if $x=e^u+e^{-v},\ y=e^{-u}-e^v$ prove that $\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}=x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}$.

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Solution: Let, $z = f(x, y)$ and if $x = e^u + e^{-v}$, $y = e^{-u} - e^v$

$$\frac{\partial x}{\partial u} = e^u, \quad \frac{\partial x}{\partial v} = -e^{-v}, \quad \frac{\partial y}{\partial u} = -e^{-u}, \quad \frac{\partial y}{\partial v} = -e^v$$

Now,

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = \left(\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}\right) - \left(\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}\right)$$

$$= \left(\frac{\partial z}{\partial x}e^u + \frac{\partial z}{\partial y}\times -e^{-u}\right) - \left(\frac{\partial z}{\partial x} - e^{-v} + \frac{\partial z}{\partial y}\times -e^v\right)$$

$$= \frac{\partial z}{\partial x}(e^u + e^{-v}) - \frac{\partial z}{\partial y}(e^{-u} - e^{-v})$$

$$= \frac{\partial z}{\partial x}\cdot x - \frac{\partial z}{\partial y}\cdot y$$

Thus, $$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

(18) Find $\frac{dy}{dx}$ is the following cases using $\frac{dy}{dx} = -\frac{f_x}{f_y}$.

(i) $x^{2/3} + y^{2/3} = a^{2/3}$

Solution: Let, $f(x, y) = x^{2/3} + y^{2/3} - a^{2/3} = 0$

Differentiating f partially w. r. t. 'x' and 'y' then,

$$f_x = \frac{2}{3}x^{-1/3} \quad \text{and} \quad f_y = \frac{2}{3}y^{-1/3}$$

Now,

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = \frac{-\frac{2}{3}x^{-1/3}}{\frac{2}{3}y^{-1/3}} = -\frac{y^{1/3}}{x^{1/3}} = -\left(\frac{y}{x}\right)^{1/3}$$

(ii) $x^p y^q = (x + y)^{p+q}$

Solution: Let, $f(x, y) = x^p y^q - (x + y)^{p+q} = 0$

Differentiating f partially w. r. t. 'x' and 'y' then,

$$f_x = y^q p.x^{p-1} - (p + q)(x + y)^{p+q-1} - (p + q)(x + y)^{p+q-1}$$

$$f_y = x^p.qy^{q-1} - (p + q)(x + y)^{p+1-1} = qx^py^{q-1} - (p + q)(x + y)^{p+q-1}$$

Now,

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = \frac{(p + q)(x + y)^{p+q-1} - py^qx^{p-1}}{qx^py^{q-1} - (p + q)(x + y)^{p+q-1}}$$

$$= \frac{(p + q)(x + y)^{p+q}(x + y)^{-1} - px^py^q.x^{-1}}{qx^py^qy^{-1} - (p + q)(x + y)^{p+q}(x + y)^{-1}}$$

$$= \frac{\frac{(p + q)x^py^q}{(x + y)} - \frac{px^py^q}{x}}{\frac{x^py^q}{y} - \frac{(p + q)(x^py^q)}{(x + y)}}$$

$$= \frac{x^py^q\{(p + q)x - p(x + y)\}}{x(x + y)} \times \frac{y(x + y)}{x^py^q\{q(x + y) - y(p + q)\}}$$

$$= \frac{y(px + qx - px - qy)}{x(qx + qy - py - qy)} = \frac{y}{x}\frac{(qx - py)}{(qx - py)} = \frac{y}{x}.$$

$$\frac{dy}{dx} = \frac{y}{x}.$$

(iii) $(\tan x)^y + (y)^{\tan x} = 0$

Solution: Let, $f(x, y) = (\tan x)^y + (y)^{\tan x}$

Differentiating f partially w. r. t. 'x' and 'y' then,

$$f_x = y(\tan x)^{y-1}\sec^2 x + y^{\tan x}.\log y.\sec^2 x.$$

$$f_y = (\tan x)^y \log\tan x + \tan x\, y^{\tan x - 1}$$

$$\begin{cases} \frac{d}{dx}a^x = a^x \log x \\ \frac{d}{dx}(x^n) = nx^{n-1} \end{cases}$$

Now,

$$\frac{dy}{dx} = -\frac{f_x}{f_y}$$

$$= -\frac{y(\tan x)^{y-1}.\sec^2 x + y^{\tan x}.\log y \sec^2 x}{(\tan x)^y \log\tan x + \tan x\, y^{\tan x - 1}}$$

$$= -\frac{\sec^2 x\,[y(\tan x)^{y-1} + \log y.\,y^{\tan x}]}{(\tan x)^y \log\tan x + \tan x\, y^{\tan x - 1}}$$

(iv) $x^y = y^x$

Solution: Let, $f(x, y) = x^y - y^x = 0$

Differentiating f partially w. r. t. 'x' and 'y' then,

$$\begin{cases} \frac{d}{dx}a^x = a^x \log x \\ \frac{d}{dx}(x^n) = nx^{n-1} \end{cases}$$

$$f_x = y\,x^{y-1} - y^x \log y$$

$$f_y = x^y\log x - xy^{x-1}$$

Now,

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{(yx^{y-1} - y^x\log y)}{x^y\log x - xy^{x-1}}.$$

(v) $x^y + y^x = a$

Solution: Let, $f(x, y) = x^y + y^x - a = 0.$

Differentiating f partially w. r. t. 'x' and 'y' then,

$$f_x = yx^{y-1} + y^x\log y \quad \text{and} \quad f_y = x^y\log x + xy^{x-1}$$

Now,

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{(yx^{y-1} + y^x\log y)}{x^y\log x + xy^{x-1}}.$$

(19) If $u = \sin^{-1}(x - y)$, $x = 3t$, $y = 4t^3$, show that $\frac{du}{dt} = \frac{3}{\sqrt{1 - t^2}}$.

Solution: Let, $u = \sin^{-1}(x - y)$, $x = 3t$, $y = 4t^3$

Differentiating u partially w. r. t. 'x' and 'y' then,

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-(x-y)^2}} \times 1 = \frac{1}{\sqrt{1-(x-y)^2}}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-(x-y)^2}} \times -1 = -\frac{1}{\sqrt{1-(x-y)^2}}$$

Then,

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} \qquad \left[\frac{dx}{dt} = 3,\ \frac{dy}{dt} = 12t^2\right]$$

$$= \frac{1}{\sqrt{1-(x-y)^2}} \times 3$$

$$= -\frac{1}{\sqrt{1-(x-y)^2}} = \frac{3 - 12t^2}{\sqrt{1 - 9t^2 + 24t^4 - 16t^6}}$$

$$= \frac{3 - 12t^2}{\sqrt{1 - t^2 - 8t^2 + 8t^4 + 16t^4 - 16t^6}}$$

$$= \frac{3 - 13t^2}{\sqrt{1(1-t^2) - 8t^2(1-t^2) + 16t^4(1-t^2)}}$$

$$= \frac{3 - 12t^2}{\sqrt{(1-t^2)(1-8t^2+16t^4)}}$$

$$= \frac{3(1-4t^2)}{\sqrt{(1-t^2)(1-4t^2)^2}}$$

$$= \frac{3(1-4t^2)}{\sqrt{1-t^2}\,(1-4t^2)} = \frac{3}{\sqrt{1-t^2}}.$$

Thus, $\frac{du}{dt} = \frac{3}{\sqrt{1-t^2}}$.

(20) If $u = x^2 + y^2$, $x = at^2$, $y = 2at$ show that $\frac{du}{dt} = 4a^2t\,(t^2 + 2)$.

Solution: Let, $u = x^2 + y^2$, $x = at^2$, $y = 2at$

Differentiating u partially w. r. t. 'x' and 'y' then,

$$\frac{\partial u}{\partial x} = 2x.\frac{dx}{dt} \quad \text{and} \quad \frac{\partial u}{\partial y} = 2y\,\frac{dy}{dt}$$

Next, we have, $x = at^2$, $y = 2at$.

Differentiating partially w. r. t. 't' then,

$$\frac{dx}{dt} = 2at \quad \text{and} \quad \frac{dy}{dt} = 2a$$

Now,

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} = 2x.\frac{dx}{dt} + 2y\,\frac{dy}{dt}$$

$$= 2x \times 2at + 2y \times 2a$$
$$= 2at^2 \times 2at + 2 \times 2at \times 2a$$
$$= 4a^2t^3 + 8a^2t$$
$$= 4a^2t\,(t^2 + 2).$$

Thus, $\frac{du}{dt} = 4a^2t\,(t^2 + 2)$.

(21) If $u = x^2 + y^2 + z^2$, $x = e^{2t}$, $y = e^{2t}\cos 3t$, $z = e^{2t}\sin 3t$, show that $\frac{du}{dt} = 8e^{4t}$

Solution: Let, $u = x^2 + y^2 + z^2$

Differentiating u partially w. r. t. 'x', 'y' and 'z' then,

$$\frac{\partial u}{\partial x} = 2x, \qquad \frac{\partial u}{\partial y} = 2y \quad \& \quad \frac{\partial u}{\partial z} = 2z$$

Also, let, $x = e^{2t}$, $y = e^{2t}\cos 3t$, $z = e^{2t}\sin 3t$

Differentiating partially w. r. t. 't' then,

$$\frac{dx}{dt} = 2e^{2t}, \qquad \frac{dy}{dt} = \cos 3t.\ 2e^{2t} + e^{2t}\,3\,(-\sin 3t)$$
$$= 2e^{2t}\cos 3t - 3e^{2t}\sin 3t$$

$$\frac{dz}{dt} = 2e^{2t}\sin 3t + 3e^{2t}\cos 3t.$$

Then, $\frac{\partial u}{\partial x}\cdot\frac{dx}{dt} = 2x.\ 2e^{2t} = 4e^{4t}$

$$\frac{\partial u}{\partial y}\cdot\frac{dy}{dt} = 2e^{2t}.\cos 3t\,(2e^{2t}\cos 3t - 3e^{2t}\sin 3t)$$
$$= 4e^{4t}\cos^2 3t - 6e^{4t}\sin 3t.\cos 3t.$$

$$\frac{\partial u}{\partial z}\cdot\frac{dz}{dt} = 2e^{2t}\sin 3t\,(2e^{2t}\sin 3t + 3e^{2t}\cos 3t)$$
$$= 4e^{4t}\sin^2 3t + 6e^{4t}\sin 3t.\cos 3t$$

Now,

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}$$
$$= 4e^{4t} + 4e^{4t}\cos^2 3t - 6e^{4t}\sin 3t.\cos 3t + 4e^{4t}\sin^2 3t + 6e^{4t}\sin 3t.\cos 3t$$
$$= 4e^{4t} + 4e^{4t}(\cos^2 3t + \sin^2 3t)$$
$$= 4e^{4t} + 4e^{4t}$$
$$= 8e^{4t}.$$

(22) If $u = \sin\frac{x}{y}$, $x = e^t$, $y = t2$, show that $\frac{du}{dt} = \frac{e^t(t-2)}{t^3}\cos\left(\frac{e^t}{t^2}\right)$

Solution: Let, $u = \sin\frac{x}{y}$, $x = e^t$, $y = t^2$

Differentiating u partially w. r. t. 'x' and 'y' then,

$$\frac{\partial u}{\partial x} = \cos\frac{x}{y}\cdot\frac{1}{y} = \frac{1}{y}\cos\frac{x}{y}. \qquad \frac{dx}{dt} = e^t$$

And $\frac{\partial u}{\partial y} = -\frac{x}{y^2}\cos\frac{x}{y}$ $\qquad$ $\frac{dy}{dt} = 2t$

Now,

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} = \frac{1}{y}\cos\frac{x}{y}\cdot e^t + \left(-\frac{x}{y^2}\right)\cos\frac{x}{y}\times 2t$$
$$= \frac{e^t}{t^2}\cos\frac{e^t}{t^2} - \frac{e^t}{t^3}\cdot\cos\frac{e^t}{t^2}\times 2t$$
$$= \cos\frac{e^t}{t^2}\left(\frac{e^t}{t^2} - \frac{2e^t}{t^3}\right) = \frac{e^t}{t^3}(t-2)\cos\left(\frac{e^t}{t^2}\right).$$

Thus, $\frac{du}{dt} = \frac{e^t}{t^3}(t-2)\cos\left(\frac{e^t}{t^2}\right)$.

(23) If $u = f(r, s)$, $r = x + y$, $s = x - y$, show that: $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2\frac{\partial u}{\partial r}$.

Solution: Let, $u = f(r, s)$, $r = x + y$, $s = x - y$.

Differentiating partially then,

$$\frac{\partial u}{\partial r} = f'(r, s)\, s \qquad \text{and} \qquad \frac{\partial u}{\partial s} = f'(r, s)\, r$$

Since, $r = x + y$, $s = x - y$.

Differentiating partially then,

$$\frac{\partial r}{\partial x} = 1, \quad \frac{\partial r}{\partial y} = 1, \quad \frac{\partial s}{\partial x} = 1, \quad \frac{\partial s}{\partial y} = 1.$$

Now,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y}$$
$$= f'(r, s).s.1 + f'(r, s).r + f'(r, s).s.1 + f'(r, s).r.(-1)$$
$$= f'(r, s)(s + r + s - r)$$
$$= 2s\, f'(r, s).$$
$$= 2\frac{\partial u}{\partial r}.$$

(24) If $z = e^{ax + by} f(ax - by)$, show that $b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz$

Solution: Let, $\quad z = e^{ax + by} f(ax - by)$

Differentiating partially then,

$$\frac{\partial z}{\partial x} = e^{ax + by}\times a\, f(ax - by) + f'(ax - by)\, a\, e^{ax + by}$$
$$= a\, e^{ax + by} f(ax - by) + ae^{ax + by} f'(ax - by)$$
$$= ae^{ax + by}\{f(ax - by) + f'(ax - by)\}$$

And, $\frac{\partial z}{\partial y} = e^{ax + by}\times b\, f(ax - by) + f'(ax - by)\times - be^{ax + by}$

$$= be^{ax+by}\{f(ax - by) - f'(ax - by)\}$$

Now,

$b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = b[ae^{ax + by}\{f(ax-by) + f'(ax-by)\}] + a[be^{ax + by}\{f(ax-by) + f'(ax-by)\}]$

$$= abe^{ax + by}\{f(ax - by) + f'(ax - by) + f(ax - by) - f'(ax - by)\}$$
$$= 2ab\, e^{ax + by} f(ax - by)$$
$$= 2abz.$$

(25) If $u = f(r, s)$, $r = x + at$, $s = y + bt$ show that $\frac{\partial u}{\partial t} = a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y}$

Solution: Let, $u = f(r, s)$, $r = x + at$, $s = y + bt$

Differentiating partially then,

$$\frac{\partial u}{\partial r} = f'(r, s), \quad \frac{\partial u}{\partial s} = f'(r, s), \quad \frac{\partial r}{\partial t} = a, \quad \frac{\partial s}{\partial t} = b.$$

$$\frac{\partial u}{\partial x} = f'(x + at, y + bt)(y + bt).1 = f'(r, s).s$$

$$\frac{\partial u}{\partial y} = f'(x + at, y + bt)(x + at).1 = f'(r, s).r$$

Now,

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial t} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial t} = f'(r, s).s.a + f'(r, s).r.b.$$
$$= f'(r, s)[as + br]$$

And,

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = a[f'(r, s).s] + b[f'(r, s).r]b$$
$$= f'(r, s)[as + br]$$
$$= \frac{\partial u}{\partial t}.$$

(26) If $z = x^2y$ and $x^2 + xy + y^2 = 1$ show that $\frac{dz}{dx} = 2xy - \frac{x^2(2x + y)}{(x + 2y)}$

Solution: Let, $z = x^2y$.

Differentiating partially then,

$$\frac{\partial z}{\partial x} = 2xy \qquad \text{and} \qquad \frac{\partial z}{\partial y} = x^2$$

And, $\quad f = x^2 + xy + y^2 - 1 = 0.$

Differentiating w. r. t. x then,

$$f_x = 2x + y \qquad \text{and} \qquad f_y = x + 2y$$

So, $\quad \frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{2x + y}{x + 2y}$

Now,

$$\frac{dz}{dx} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dx}$$

$$= 2xy.1 - x^2\left(\frac{2x+y}{x+2y}\right)$$

(27) If $x = e^r\cos\theta$, $y = e^r\sin\theta$ then

(28) If $w = f(x, y)$, $x = r\cos\theta$, $y = r\sin\theta$, show that $\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2$.

Solution: Let, $w = f(x, y)$, $x = r\cos\theta$, $y = r\sin\theta$.

Since we have,

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r} \quad \text{and} \quad \frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

Here, $x = r\cos\theta$ and $y = r\sin\theta$. Then,

$$\frac{\partial x}{\partial r} = \cos\theta = \frac{x}{r} \quad \text{and} \quad \frac{\partial y}{\partial r} = \sin\theta = \frac{y}{r}$$

Also, $\frac{\partial x}{\partial \theta} = -r\sin\theta = -y$ and $\frac{\partial y}{\partial \theta} = r\cos\theta = x$

Then,

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{x}{r} + \frac{\partial w}{\partial y}\cdot\frac{y}{r} \quad \text{and} \quad \frac{\partial w}{\partial \theta} = -y\frac{\partial w}{\partial x} + x\frac{\partial w}{\partial y}$$

So that,

$$\left(\frac{\partial w}{\partial r}\right)^2 = \left(\frac{\partial w}{\partial x}\right)^2\cdot\frac{x^2}{r^2} + \left(\frac{\partial w}{\partial y}\right)^2\cdot\frac{y^2}{r^2} + \frac{2xy}{r^2}\cdot\frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y}$$

$$\text{and } \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \frac{y^2}{r^2}\cdot\left(\frac{\partial w}{\partial x}\right)^2 + \frac{x^2}{r^2}\cdot\left(\frac{\partial w}{\partial y}\right)^2 - \frac{2xy}{r^2}\cdot\frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y}$$

Now,

$$\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial w}{\partial x}\right)^2\left(\frac{x^2}{r^2} + \frac{y^2}{r^2}\right) + \left(\frac{\partial w}{\partial y}\right)^2\left(\frac{y^2}{r^2} + \frac{x^2}{r^2}\right)$$

$$= \left(\frac{\partial w}{\partial x}\right)^2\left(\frac{x^2+y^2}{r^2}\right) + \left(\frac{\partial w}{\partial y}\right)^2\cdot\left(\frac{x^2+y^2}{r^2}\right)$$

$$= \left(\frac{\partial w}{\partial x}\right)^2\left(\frac{r^2}{r^2}\right) + \left(\frac{\partial w}{\partial y}\right)^2\cdot\left(\frac{r^2}{r^2}\right)$$

$$= \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 \qquad [\because w = f].$$

Thus, $\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2$.

(29) If $u = xe^yz$, where $y = \sqrt{a^2 - x^2}$, $z = \sin^2 x$, show that $\frac{du}{dx} = e^y\left[z - \frac{x^2z}{\sqrt{a^2 - x^2}} + x\sin 2x\right]$

Solution: Let, $u = xe^yz$, $y = \sqrt{a^2 - x^2}$, $z = \sin^2 x$.

Differentiating partially then,

$$\frac{\partial u}{\partial x} = e^yz, \qquad \frac{\partial u}{\partial y} = xe^yz, \qquad \frac{\partial u}{\partial z} = xe^y$$

$$\frac{dy}{dx} = \frac{-2x}{2\sqrt{a^2 - x^2}} = \frac{-x}{\sqrt{a^2 - x^2}} \quad \text{and} \quad \frac{dz}{dx} = 2\sin x.\cos x = \sin 2x$$

Now,

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dx}$$

$$= e^yz + xe^yz \times -\frac{x}{\sqrt{a^2 - x^2}} + xe^y.\sin 2x$$

$$= ey\left[z - \frac{x^2z}{\sqrt{a^2 - x^2}} + x\sin 2x\right]$$

(30) If $u = \sin(x^2 + y^2)$, where $a^2x^2 + b^2y^2 = c^2$, show that $\frac{du}{dx} = \frac{2(b^2 - a^2)x}{b^2}\cos(x^2 + y^2)$

Solution: Let, $u = \sin(x^2 + y^2)$, and $a^2x^2 + b^2y^2 = c^2$

Differentiating partially then,

$$\frac{\partial u}{\partial x} = 2x\cos(x^2 + y^2), \qquad \frac{\partial u}{\partial y} = 2y\cos(x^2 + y^2)$$

And, $f = a^2x^2 + b^2y^2 - c^2 = 0$

Differentiating partially then,

$f_x = 2a^2x \qquad f_y = 2b^2y$

Then, $\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{2a^2x}{2b^2y} = -\frac{a^2x}{b^2y}$

Now,

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$$

$$= 2x\cos(x^2 + y^2).1 + 2y\cos(x^2 + y^2)\left(-\frac{a^2x}{b^2y}\right)$$

$$= 2x\cos(x^2 + y^2) - 2\cos(x^2 + y^2)\left(-\frac{a^2x}{b^2}\right).$$

$$= 2x\cos(x^2 + y^2)\left(1 - \frac{a^2}{b^2}\right)$$

$$= 2x\cos(x^2 + y^2)\frac{(b^2 - a^2)}{b^2}.$$

(31) Find $\frac{dz}{dt}$ if $z = x \log y, x = t^2, y = e^t$

Solution: Let, $z = x \log y, x = t^2, y = e^t$

Differentiating partially then,

$$\frac{\partial z}{\partial x} = \log y, \quad \frac{\partial z}{\partial y} = \frac{x}{y}, \quad \frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = e^t$$

Now,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$= \log (y)\ 2t + \frac{x}{y} e^t = \log (e^t) \cdot 2t + \frac{t^2}{e^t} \cdot e^t = t \cdot 2t + t^2 = 2t^2 + t^2 = 3t^2.$$

Thus, $\frac{dz}{dt} = 3t^2$.

(32) If $x\sqrt{1 - y^2} + y\sqrt{1 - x^2} = a$, show that $\frac{d^2y}{dx^2} = \frac{-a}{(1 - x^2)^{3/2}}$

Solution: Let, $x\sqrt{1 - y^2} + y\sqrt{1 - x^2} = a$. ... (i)

Differentiating w. r. t. x,

$$\sqrt{1 - y^2}\ 1 + x \times \frac{-2y}{2\sqrt{1 - y^2}} \cdot \frac{dy}{dx} + \frac{dy}{dx} \cdot \sqrt{1 - x^2} + y \cdot \frac{-2x}{2\sqrt{1 - x^2}} = 0$$

$$\Rightarrow \sqrt{1 - y^2} - \frac{xy}{\sqrt{1 - y^2}} \frac{dy}{dx} + \frac{dy}{dx}\sqrt{1 - x^2} - \frac{xy}{\sqrt{1 - y^2}} = 0$$

$$\Rightarrow \frac{dy}{dx}\left[\sqrt{1 - x^2} - \frac{xy}{\sqrt{1 - y^2}}\right] = \frac{-xy}{\sqrt{1 - x^2}} - \sqrt{1 - y^2}$$

$$\Rightarrow \frac{dy}{dx}\left[\frac{\sqrt{1 - x^2}\sqrt{1 - y^2} - xy}{\sqrt{1 - y^2}}\right] = \frac{xy - \sqrt{1 - y^2}\sqrt{1 - x^2}}{\sqrt{1 - x^2}}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\sqrt{1 - y^2}\,(\sqrt{1 - x^2} \cdot \sqrt{1 - y^2} - xy)}{\sqrt{1 - x^2}\,(\sqrt{1 - x^2}\sqrt{1 - y^2} - xy)}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\sqrt{1 - y^2}}{\sqrt{1 - x^2}} \qquad \text{........ (ii)}$$

Again differentiating w. r. t. (ii) x,

$$\frac{d^2y}{dx^2} = \frac{-\left[\sqrt{1 - x^2} \times \frac{-2y}{2\sqrt{1 - y^2}} \times \frac{dy}{dx} - \sqrt{1 - y^2} \times \frac{-2x}{2\sqrt{1 - x^2}}\right]}{(1 - x^2)}$$

$$= \frac{-\left[\frac{-y\sqrt{1 - x^2}}{\sqrt{1 - y^2}} \frac{dy}{dx} + x\sqrt{1 - y^2} / \sqrt{1 - x^2}\right]}{(1 - x^2)}$$

$$= \frac{\left[\frac{y\sqrt{1 - x^2}}{\sqrt{1 - y^2}} \times -\frac{\sqrt{1 - y^2}}{\sqrt{1 - x^2}} - \frac{x\sqrt{1 - y^2}}{\sqrt{1 - x^2}}\right]}{(1 - x^2)} \qquad \text{Using eq}^n\text{. (ii)}$$

$$= \frac{-[y\sqrt{1 - x^2} + x\sqrt{1 - y^2}]}{(1 - x^2)^{3/2}}$$

$$= -\frac{a}{(1 - x^2)^{3/2}} \qquad \text{[using eq}^n\text{. (i)]}$$

(33) If $u = x(x + y) + y(x + y)$ proved that: $\frac{\partial^2 u}{\partial x^2} - \frac{2\partial^2 u}{\partial x . \partial y} + \frac{\partial^2 u}{\partial y^2}$.

Solution: Let, $u = x(x + y) + y(x + y)$

$$\Rightarrow u = x^2 + xy + xy + y^2 = x^2 + 2xy + y^2$$

Differentiating partially then,

$$\frac{\partial u}{\partial x} = 2x + 2y, \qquad \frac{\partial u}{\partial y} = 2x + 2y$$

Again,

$$\frac{\partial^2 u}{\partial x^2} = 2, \qquad \frac{\partial^2 u}{\partial x\, \partial y} = 2 \qquad \text{and} \qquad \frac{\partial^2 u}{\partial y^2} = 2.$$

Now,

$$\frac{\partial^2 u}{\partial x^2} - \frac{2\partial^2 u}{\partial x . \partial y} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 \times 2 + 2 = 4 - 4 = 0.$$

(34) If $u = \tan^{-1}\frac{y}{x}$ show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

Solution: Let, $u = \tan^{-1}\frac{y}{x}$

Differentiating partially then,

$$\frac{\partial u}{\partial x} = \frac{1}{1 + \frac{y^2}{x^2}} \times -\frac{y}{x^2} = -\frac{y}{x^2} \times \frac{x^2}{x^2 + y^2} = -\frac{y}{x^2 + y^2}$$

And,

$$\frac{\partial u}{\partial y} = \frac{1}{1 + \frac{y^2}{x^2}} \times \frac{1}{x} = \frac{1}{x} \times \frac{x^2}{x^2 + y^2} = \frac{x}{x^2 + y^2}$$

Again,

$$\frac{\partial^2 u}{\partial x^2} = \frac{-y}{(x^2 + y^2)^2} \times -1 \times 2x = \frac{2xy}{(x^2 + y^2)^2} \qquad \text{......(i)}$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{x}{(x^2 + y^2)^2} \times 2y = -\frac{2xy}{(x^2 + y^2)^2}$$

Now,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{2xy}{(x^2 + y^2)^2} - \frac{2xy}{(x^2 + y^2)^2} = 0$$

(35) If $y = f(x + ct) + q(x - ct)$ show that $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$

Solution: Let, $y = f(x + ct) + q(x - ct)$

Differentiating partially then,

$$\frac{\partial y}{\partial t} = f'(x + ct) \times c + q'(x - ct) \times -c$$

$$= cf'(x + ct) - cq'(x - ct)$$

And, $\frac{\partial y}{\partial x} = f'(x + ct).1 + q'(x - ct).1$

$$= f'(x + ct) + q'(x - ct)$$

Also,

$$\frac{\partial^2 y}{\partial t^2} = c^2 f''(x + ct) + c^2 q''(x - ct)$$

$$= c^2 \{f''(x + ct) + q''(x - ct)\} \qquad \text{........ (i)}$$

And, $\frac{\partial^2 y}{\partial x^2} = f'(x + ct) + q''(x - ct)$ ..... (ii)

Now, from (i) and (ii),

$$\frac{\partial^2 y}{\partial t^2} = c^2 \cdot \frac{\partial^2 y}{\partial x^2}$$

(36) If $u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{x}{y}$ show that $x \cdot \frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$.

Solution: Let, $u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{x}{y}$

Differentiating partially then,

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1 - \frac{x^2}{y^2}}} \times \frac{1}{y} + \frac{1}{1 + \frac{x^2}{y^2}} \times \frac{1}{y}$$

$$= \frac{1}{\sqrt{y^2 - x^2}} + \frac{y}{x^2 + y^2}$$

Multiplying by x,

$$x\frac{\partial u}{\partial x} = \frac{x}{\sqrt{y^2 - x^2}} + \frac{xy}{x^2 + y^2} \qquad \text{...... (i)}$$

Now,

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1 - \frac{x^2}{y^2}}} \times \left(-\frac{x}{y^2}\right) + \frac{1}{1 + \frac{x^2}{y^2}} \times \left(-\frac{x}{y^2}\right)$$

$$= \frac{-x}{y\sqrt{y^2 - x^2}} - \frac{x}{(x^2 + y^2)}$$

Multiplying by 'y'

$$y\frac{\partial u}{\partial y} = \frac{-xy}{y\sqrt{y^2 - x^2}} - \frac{xy}{x^2 + y^2}$$

$$= -\frac{-x}{\sqrt{y^2 - x^2}} - \frac{xy}{x^2 + y^2} \quad \text{..... (ii)}$$

Adding (i) and (ii)

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$$

(38) If $z = \frac{\cos y}{x}$ and $x = u^2 - v$, $y = e^v$ show that $\frac{\partial z}{\partial v} = \frac{\cos y - e^v x \sin y}{x^2}$

Solution: Let, $z = \frac{\cos y}{x}$ and $x = u^2 - v$, $y = e^v$

Differentiating partially then,

$$\frac{\partial z}{\partial x} = \left(-\frac{\cos y}{x^2}\right), \quad \frac{\partial z}{\partial y} = \left(-\frac{\sin y}{x}\right), \quad \frac{\partial x}{\partial v} = -1, \quad \frac{\partial y}{\partial v} = e^v.$$

Now,

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}$$

$$= \left(-\frac{\cos y}{x^2}\right)\frac{\partial x}{\partial v} + \left(-\frac{\sin y}{x}\right)\frac{\partial y}{\partial v} = \left(-\frac{\cos y}{x^2}\right)(-1) + \left(-\frac{\sin y}{x}\right) \times e^v$$

$$= \frac{\cos y}{x^2} - \frac{e^v \sin y}{x}$$

$$= \frac{\cos y - e^v x \sin y}{x^2}$$

Thus, $\frac{\partial z}{\partial v} = \frac{\cos y - e^v x \sin y}{x^2}$.