$$= -\iint_S \left(\frac{\partial F_1}{\partial y}\cos\theta - \frac{\partial F_1}{\partial z}\cdot\cos\beta\right) ds$$

$$= \iint_S \left(\frac{\partial F_1}{\partial z}\cos\theta - \frac{\partial F_1}{\partial y}\cdot\cos\theta\right) ds.$$

Similarly we can get

$$\oint_C F_2(x, y, z)dy = \iint_S \left(\frac{\partial F_2}{\partial x}\cos\theta - \frac{\partial F_2}{\partial z}\cdot\cos\alpha\right) ds$$

and $$\oint_C F_3(x, y, z)dz = \iint_S \left(\frac{\partial F_3}{\partial y}\cos\alpha - \frac{\partial F_3}{\partial x}\cdot\cos\beta\right) ds$$

Adding these we get,

$$\oint_C \vec{F}.d\vec{r} = \iint_S \text{curl}\,\vec{F}.\vec{n}\; ds.$$

This is the required form.

EXERCISE 4.1

1. If $\vec{r}_1 = t^2\vec{i} - t\vec{j} + (2t+1)\vec{k}$, $\vec{r}_2 = (2t-3)\vec{i} + \vec{j} - t\vec{k}$.

Find (i) $\frac{d}{dt}(\vec{r}_1.\vec{r}_2)$ [2002 – Short] (ii) $\frac{d}{dt}(\vec{r}_1\times\vec{r}_2)$ at t = 1.

Solution: Let $\vec{r}_1 = t^2\vec{i} - t\vec{j} + (2t+1)\vec{k}$ and $\vec{r}_2 = (2t-3)\vec{i} + \vec{j} - t\vec{k}$

Then,

$$\vec{r}_1.\vec{r}_2 = (t^2\vec{i} - t\vec{j} + (2t+1)\vec{k}).((2t-3)\vec{i} + \vec{j} - t\vec{k})$$
$$= (t^2, -t, 2t+1).(2t-3, 1, -t)$$
$$= 2t^3 - 3t^2 - t - 2t^2 - t$$
$$= 2t^3 - 5t^2 - 2t$$

and

$$\vec{r}_1\times\vec{r}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -t & 2t+1 \\ 2t-3 & 1 & -t \end{vmatrix}$$
$$= (t^2 - 2t - 1)\vec{i} + ((2t-3)(2t+1) + t^3)\vec{j} + (t^2 + t(2t-3))\vec{k}$$
$$= (t^2 - 2t - 1)\vec{i} + (4t^2 - 4t - 3 + t^3)\vec{j} + (3t^2 - 3t)\vec{k}$$

Now,

(i) $\frac{d}{dt}(\vec{r}_1.\vec{r}_2) = \frac{d}{dt}(2t^3 - 5t^2 - 2t) = 6t - 10t - 2$

At t = 1,

$$\frac{d}{dt}(\vec{r}_1.\vec{r}_2) = 6 - 10 - 2 = -6$$

(ii) $\frac{d}{dt}(\vec{r}_1\times\vec{r}_2) = \frac{d}{dt}(t^2 - 2t - 1, t^3 + 4t^2 - 4t - 3, 3t^2 - 3t)$

$$= (2t - 2, 3t^2 + 8t - 4, 6t - 3)$$

At t = 1,

$$\frac{d}{dt}(\vec{r}_1\times\vec{r}_2) = (2-2, 3+8-4, 6-3) = (0, 7, 3)$$

2. If $\vec{A} = t^2\vec{i} + (3t^2 - 2t)\vec{j} + \left(2t - \frac{1}{t}\right)\vec{k}$. Find $\left|\frac{d\vec{A}}{dt}\right|$ at t = 1.

Solution: Let

$$\vec{A} = t^2\vec{i} + (3t^2 - 2t)\vec{j} + \left(2t - \frac{1}{t}\right)\vec{k}$$

Then,

$$\frac{d\vec{A}}{dt} = 2t\,\vec{i} + (6t-2)\vec{j} + (2 + t^{-2})\vec{k}$$

So, $$\left|\frac{d\vec{A}}{dt}\right| = \sqrt{(2t)^2 + (6t-2)^2 + (2+t^{-2})^2}$$

at t = 1,

$$\left|\frac{d\vec{A}}{dt}\right| = \sqrt{2^2 + (6-2)^2 + (2+1)^2} = \sqrt{4+16+9} = \sqrt{29}$$

Thus,

$$\left|\frac{d\vec{A}}{dt}\right|_{at\,t=1} = \sqrt{29}.$$

3. If $\vec{r} = \vec{a}e^{nt} + \vec{b}e^{-nt}$, where $\vec{a}$ and $\vec{b}$ are constant vectors, show that $\frac{d^2\vec{r}}{dt^2} - n^2\vec{r} = 0$.

[2013 Fall Q. No. 6(a)] [2004 Spring Q.No. 3(a)]

Solution: Let $\vec{r} = \vec{a}\,e^{nt} + \vec{b}e^{-nt}$ for $\vec{a}$ and $\vec{b}$ are constant vectors.

Then,

$$\left|\frac{d\vec{r}}{dt}\right| = \vec{a}\,ne^{nt} + \vec{b}\,(-n)\,e^{-nt} = n[\vec{a}\,e^{nt} - \vec{b}\,e^{-nt}]$$

And $$\frac{d^2\vec{r}}{dt^2} = n[\vec{a}\,ne^{nt} - (-n)\,\vec{b}\,e^{-nt}] = n^2[\vec{a}\,e^{nt} + \vec{b}e^{-nt}] = n^2\vec{r}$$

$$\Rightarrow \frac{d^2\vec{r}}{dt^2} - n^2\vec{r} = 0.$$

4. If $\vec{r} = \text{cosnt}\,\vec{i} + \text{sinnt}\,\vec{j}$. Show that $\vec{r}\times\frac{d\vec{r}}{dt} = n\vec{k}$

Solution: Let $\vec{r} = \text{cosnt}\,\vec{i} + \text{sinnt}\,\vec{j} + 0\vec{k}$. Then,

$$\frac{d\vec{r}}{dt} = -\text{sinnt}.\,n\,\vec{i} + \text{cosnt}.\,n\,\vec{j} + o\,\vec{k}$$

Now,

$$\vec{r}\times\frac{d\vec{r}}{dt} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \text{cosnt} & \text{sinnt} & 0 \\ -\text{nsinnt} & \text{ncosnt} & 0 \end{vmatrix}$$

$$= 0\vec{i} + 0\vec{j} + (n\cos^2 nt + n\sin^2 nt)\vec{k} = n(\cos^2 nt + \sin^2 nt)\vec{k}$$

$$= n.1\,\vec{k} = n\vec{k}.$$

Thus, $\vec{r} \times \frac{d\vec{r}}{dt} = n\vec{k}$.

5. If $\vec{r}$ is a vector function of a scalar t and $\vec{a}$ is a constant vector, then differentiate with respect to t

(i) $\vec{r}\,.\,\vec{a}$ (ii) $\vec{r} \times \vec{a}$ (iii) $\vec{r} \times \frac{d\vec{r}}{dt}$ (iv) $\vec{r}\,.\,\frac{d\vec{r}}{dt}$

Solution: Let $\vec{r}$ be a vector function of scalar t and $\vec{a}$ be a constant vector. Then,

(i) Derivative of $\vec{r}\,.\,\vec{a}$ w. r. t. 't' be,

$$\frac{d}{dt}(\vec{r}\,.\,\vec{a}) = \frac{d\vec{r}}{dt}\,.\,\vec{a} + \vec{r}\,.\,\frac{d\vec{a}}{dt}$$

$$= \frac{d\vec{r}}{dt}\,.\,\vec{a} + \vec{r}\,.\,0 \quad [\because \vec{a} \text{ is a constant. So, } \frac{d\vec{a}}{dt} = 0.]$$

$$= \frac{d\vec{r}}{dt}\,.\,\vec{a}$$

(ii) Derivative of $\vec{r} \times \vec{a}$ be,

$$\frac{d}{dt}(\vec{r} \times \vec{a}) = \frac{d\vec{r}}{dt} \times \vec{a} + \vec{r} \times \frac{d\vec{a}}{dt}$$

$$= \frac{d\vec{r}}{dt} \times \vec{a} + \vec{r} \times 0 \quad [\because \vec{a} \text{ is a constant. So, } \frac{d\vec{a}}{dt} = 0]$$

$$= \frac{d\vec{r}}{dt} \times a$$

(iii) Derivative of $\vec{r} \times \frac{d\vec{r}}{dt}$ be,

$$\frac{d}{dt}\left(\vec{r} \times \frac{d\vec{r}}{dt}\right) = \frac{d\vec{r}}{dt} \times \frac{d\vec{r}}{dt} + \vec{r} \times \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right)$$

Since cross product of same vector is zero. So, $\frac{d\vec{r}}{dt} \times \frac{d\vec{r}}{dt} = 0$. Then,

$$= 0 + \vec{r} \times \frac{d^2\vec{r}}{dt^2} = \vec{r} \times \frac{d^2\vec{r}}{dt^2} = \vec{r} \times \frac{d^2\vec{r}}{dt^2}$$

(iv) Derivative of $\vec{r}\,.\,\frac{d\vec{r}}{dt}$ be,

$$\frac{d}{dt}\left(\vec{r}\,.\,\frac{d\vec{r}}{dt}\right) = \frac{d\vec{r}}{dt}\,.\,\frac{d\vec{r}}{dt} + \vec{r}\,.\,\frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right)$$

$$= \left(\frac{d\vec{r}}{dt}\right)^2 + \vec{r}\,.\,\frac{d^2\vec{r}}{dt^2}$$

6. Verify $\frac{d}{dt}(\vec{a} \times \vec{b}) = \vec{a} \times \frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt} \times \vec{b}$ and $\frac{d}{dt}(\vec{a}\,.\,\vec{b}) = \vec{a}\,.\,\frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt}\,.\,\vec{b}$ where $\vec{a} = 2t^3\vec{i} + t\vec{j} + 3t^2\vec{k}$ and $\vec{b} = 2t\vec{i} + 3\vec{j} + t^3\vec{k}$.

Solution: Let, $\vec{a} = (2t^3\vec{i} + t\vec{j} + 3t^2\vec{k}) = (2t^3, t, 3t^2)$

and $\vec{b} = 2t\vec{i} + 3\vec{j} + t^3\vec{k} = (2t, 3, t^3)$

Then,

$$\vec{a}\,.\,\vec{b} = 4t^4 + 3t + 3t^5$$

and
$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2t^3 & t & 3t^2 \\ 2t & 3 & t^3 \end{vmatrix} = (t^4 - 9t^2)\vec{i} + (6t^3 - 2t^6)\vec{j} + (6t^3 - 2t^2)\vec{k}$$

So that,

$$\frac{d\vec{a}}{dt} = (6t^2, 1, 6t); \qquad \frac{d\vec{b}}{dt} = (2, 0, 3t^2); \qquad \frac{d}{dt}(\vec{a}\,.\,\vec{b}) = 16t^3 + 3 + 15t^4$$

and
$$\frac{d}{dt}(\vec{a} \times \vec{b}) = (4t^3 - 18t)\vec{i} + (18t^2 - 12t^5)\vec{j} + (18t^2 - 4t)\vec{k}.$$

Now,

$$\vec{a} \times \frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt} \times \vec{b}$$

$$= (2t^3, t, 3t^2) \times (2, 0, 3t^2) + (6t^2, 1, 6t) \times (2t, 3, t^3)$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2t^3 & t & 3t^2 \\ 2 & 0 & 3t^2 \end{vmatrix} + \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 6t^2 & 1 & 6t \\ 2t & 3 & t^3 \end{vmatrix}$$

$$= [3t^3\vec{i} + (6t^2 - 6t^5)\vec{j} + (-2t)\vec{k}] + [(t^3 - 18t)\vec{i} + (12t^2 - 6t^5)\vec{j} + (18t^2 - 2t)\vec{k}]$$

$$= (4t^3 - 18t)\vec{i} + (18t^2 - 12t^5)\vec{j} + (18t^2 - 4t)\vec{k}$$

$$= \frac{d}{dt}(\vec{a} \times \vec{b})$$

Next,

$$\vec{a}\,.\,\frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt}\,.\,\vec{b} = (2t^3, t, 3t^2)\,.\,(2, 0, 3t^2) + (6t^2, 1, 6t)\,(2t, 3, t^3)$$

$$= (4t^3 + 0 + 9t^4) + (12t^3 + 3 + 6t^4)$$

$$= 16t^3 + 3 + 15t^4 = \frac{d}{dt}(\vec{a}\,.\,\vec{b})$$

Thus, $\frac{d}{dt}(\vec{a} \times \vec{b}) = \vec{a} \times \frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt}\,.\,\vec{b}$.

7. Find the unit tangent vector at any point on the curve $x = 3\cos t$, $y = 3\sin t$, $z = 4t$.

Solution: Let, $x = 3\cos t$, $y = 3\sin t$ and $z = 4t$.

Then,

$$\vec{r} = x_i + y\vec{j} + z\vec{k} + (x, y, z) = (3\cos t, 3\sin t, 4t)$$

So, $\frac{d\vec{r}}{dt} = (-3\sin t, 3\cos t, 4)$

Therefore,

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{(-3\sin t)^2 + (3\cos t)^2 + 4^2}$$
$$= \sqrt{9(\sin^2 t + \cos^2 t) + 16} = \sqrt{9+16} = 5.$$

Now, the unit tangent vector be,

$$\left(\frac{d\hat{r}}{dt}\right) = \frac{\frac{d\vec{r}}{dt}}{\left|\frac{d\vec{r}}{dt}\right|} = \frac{1}{5}(-3\sin t, 3\cos t, 4).$$

8. Find the angle between the tangents to the curve $x = t, y = t^2, z = t^3$ at $t = \pm 1$.

Solution: Let, $x = t, y = t^2, z = t^3$

Then, $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = (x, y, z) = (t, t^2, t^3)$.

So, $T = \frac{d\vec{r}}{dt} = (1, 2t, 3t^2)$.

At $t = 1$, $\frac{d\vec{r}}{dt} = T_1 = (1, 2, 3)$ and at $t = -1$, $\frac{d\vec{r}}{dt} = T_2 = (1, -2, 3)$.

Let, θ be the angle between T_1 and T_2 then,

$$\cos\theta = \frac{T_1.T_2}{|T_1||T_2|} = \frac{(1, 2, 3).(1, -2, 3)}{|(1, 2, 3)||(1, -2, 3)|} = \frac{1-4+9}{\sqrt{1+4+9}\sqrt{1+4+9}} = \frac{6}{14} = \frac{3}{7}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{3}{7}\right)$$

Thus, the required angle be $\theta = \cos^{-1}\left(\frac{3}{7}\right)$.

9. A particle moves along the curve $x = e^{-t}$, $y = 2\cos 3t$, $z = 2\sin 3t$, where t is the time. Determine its velocity and acceleration vectors and also the magnitude of velocity and acceleration at $t = 0$. **[2010 Fall Q.No. 3(a)]**

Solution: Given curve is

$x = e^{-t}$, $y = 2\cos 3t$ and $z = 2\sin 3t$

Then, $\vec{r} = (x, y, z) = (e^{-t}, 2\cos 3t, 2\sin 3t)$.

So that,

$$\frac{d\vec{r}}{dt} = (-e^{-t}, -6\sin 3t, 6\cos 3t) \quad \text{and} \quad \frac{d^2\vec{r}}{dt^2} = (e^{-t}, -18\cos 3t, -18\sin 3t).$$

We know that, velocity of a curve is, $\vec{v} = \frac{d\vec{r}}{dt}$ and acceleration is $\vec{a} = \frac{d^2\vec{r}}{dt^2}$.

Therefore, velocity vector be

$$\vec{v} = \frac{d\vec{r}}{dt} = (-e^{-t}, -6\sin 3t, 6\cos 3t)$$

and acceleration vector be

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = (e^{-t}, -18\cos 3t, -18\sin 3t)$$

Also, the velocity vector at $t = 0$ is

$$\vec{v}_{at\ t=0} = (-e^{-0}, -6\sin 0, 6\cos 0) = (-1, 0, 6)$$

and acceleration vector at $t = 0$ is

$$\vec{a}_{at\ t=0} = (1, -18, 0)$$

Therefore, magnitude of velocity at $t = 0$ is

$$|\vec{v}_{at\ t=0}| = \sqrt{(-1)^2 + 6^2} = \sqrt{1+36} = \sqrt{37}$$

and magnitude of acceleration at $t = 0$ is

$$|\vec{a}_{at\ t=0}| = \sqrt{1^2 + (-18)^2} = \sqrt{1+324} = \sqrt{325} = 5\sqrt{13}$$

10. A particle moves along the curve $x = t^3 + 1, y = t^2, z = 2t + 5$. Find the component of its velocity and acceleration at $t = 1$ in the direction $\vec{i} + \vec{j} + 3\vec{k}$. **[2012 Fall Q.No. 3(a)] [2009 Fall Q.No. 3(a)]**

Solution: Given curve be

$$\vec{x} = t^3 + 1, \quad \vec{y} = t^2 \quad \text{and} \quad \vec{z} = 2t + 5.$$

Then the position vector of any point of the curve be,

$$\vec{r} = (x, y, z) = (t^3 + 1, t^2, 2t + 5)$$

So that $\frac{d\vec{r}}{dt} = (3t^2, 2t, 2)$ and $\frac{d^2\vec{r}}{dt^2} = (6t, 2, 0)$

at $t = 1$, $\frac{d\vec{r}}{dt}(3, 2, 2)$ and $\frac{d^2\vec{r}}{dt^2} = (6, 2, 0)$

We know that the velocity vector of $\vec{r}$ at $t = 1$ is

$$\vec{v} = \frac{d\vec{r}}{dt} \text{ at } t = 1 \quad \text{i.e. } \vec{v} = (3, 2, 2)$$

and the acceleration vector of $\vec{r}$ at $t = 1$ is

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} \text{ at } t = 1 \quad \text{i.e. } \vec{a} = (6, 2, 0)$$

Also, given that a vector $\vec{i} + \vec{j} + 3\vec{k} = (1, 1, 3) = \vec{n}$ (say)

So, the unit vector along $(1, 1, 3)$ is

$$\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{(1, 1, 3)}{\sqrt{1+1+9}} = \frac{(1, 1, 3)}{\sqrt{11}}$$

Thus, the velocity component of $\vec{r}$ along $\vec{n}$ is

$$= \vec{v}.\hat{n}$$
$$= (3, 2, 2)\frac{(1, 1, 3)}{\sqrt{11}} = \frac{3+2+6}{\sqrt{11}} = \frac{11}{\sqrt{11}} = \sqrt{11}$$

and the acceleration component of $\vec{r}$ along $\vec{n}$ is

$$= \vec{a}.\hat{n}$$
$$= (6, 2, 0)\frac{(1, 1, 3)}{\sqrt{11}} = \frac{6+2+0}{\sqrt{11}} = \frac{8}{\sqrt{11}}$$

11. A particle moves so that its position vector is given by $\vec{r} = \cos wt\,\vec{i} + \sin wt\,\vec{j}$ show that the velocity $\vec{v}$ of the particle is perpendicular to $\vec{r}$ and show that $\vec{r} \times \vec{v}$ is a constant vector.

Solution: Given position vector is

$$\vec{r} = \cos wt\,\vec{i} + \sin wt\,\vec{j} = (\cos wt, \sin wt, 0)$$

Then. $$\frac{d\vec{r}}{dt} = (-w \sin wt, w \cos wt, 0)$$

We know that the velocity vector to $\vec{r}$ is $\vec{v} = \frac{d\vec{r}}{dt}$

Now,

$$\vec{r}.\vec{v} = \vec{r}.\frac{d\vec{r}}{dt} = (\cos wt, \sin wt, 0).(-w\sin wt, w\cos wt, 0)$$
$$= -w \sin wt \cos wt + w \sin wt \cos wt + 0$$
$$= 0$$

This shows that $\vec{r}$ and $\vec{v}$ are perpendicular to each other.

Next,

$$\vec{r}\times\vec{v} = \vec{r}\times\frac{d\vec{r}}{dt} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos wt & \sin wt & 0 \\ -w\sin wt & w\cos wt & 0 \end{vmatrix}$$
$$= 0\,\vec{i} + 0\,\vec{j} + (w\cos^2 wt + w\sin^2 wt)\,\vec{k}$$
$$= w\,\vec{k} \qquad [\cos^2 wt + \sin^2 wt = 1.]$$

This shows that $\vec{r}\times\vec{v}$ is constant vector.

12. A particle moves along the curves $x = 4\cos t$, $y = 4\sin t$, $z = 6t$. Find the velocity and acceleration at time $t = 0$ and $t = \pi/2$. **[2011 Spring Q. No. 6(c)]**

Solution: Given curve is

$$x = 4\cos t,\ y = 4\sin t \text{ and } z = 6t$$

Then the position vector of any point of the curve is,

$$\vec{r} = (x, y, z) = (4\cos t, 4\sin t, 6t)$$

Then, $\frac{d\vec{r}}{dt} = (-4\sin t, 4\cos t, 6)$ and $\frac{d^2\vec{r}}{dt^2} = (-4\cos t, -4\sin t, 0)$

At $t = 0$,

$$\frac{d\vec{r}}{dt} = (0, 4, 6) \quad \text{and} \quad \frac{d^2\vec{r}}{dt^2} = (-4, 0, 0)$$

and at $t = \frac{\pi}{2}$, $$\frac{d\vec{r}}{dt} = (-4, 0, 6) \quad \text{and} \quad \frac{d^2\vec{r}}{dt^2} = (0, -4, 0)$$

We know that, velocity along a curve is $\vec{v} = \frac{d\vec{r}}{dt}$ and acceleration is, $\vec{a} = \frac{d^2\vec{r}}{dt^2}$

Therefore, velocity at $t = 0$ is, $\vec{v} = (0, 4, 6) = 4\vec{j} + 6\vec{k}$

and velocity at $t = \frac{\pi}{2}$ is, $\vec{v} = (-4, 0, 6) = -4\vec{i} + 6\vec{k}$

Also, acceleration at $t = 0$ is, $\vec{a} = (-4, 0, 0) = -4\vec{i}$

and acceleration at $t = \frac{\pi}{2}$ is, $\vec{a} = (0, -4, 0) = -4\vec{j}$.

[2001 Q.No. 3(a)]

A particle moves along the curve, $x = a\cos t$, $y = a\sin t$ and z bt. Find the velocity and acceleration at $t = 0$ and $t = \pi/2$.

Note: See the above solution with replacing 4 by a.

13. A particle moves along the curve $x = t^3 + 1$, $y = t^2$, $z = 2t + 5$. Find the velocity and acceleration at $t = 1$.

Solution: Part of solution of Q. 10.

14. If $\vec{a} = x^2\vec{i} - y\vec{j} + xz\vec{k}$ and $\vec{b} = y\vec{i} + x\vec{j} - xyz\vec{k}$ verify that $\frac{\partial^2}{\partial x\partial y}(\vec{a}\times\vec{b}) = \frac{\partial^2}{\partial x\partial y}(\vec{a}\times\vec{b})$

Solution: Let,

$$\vec{a} = x^2\vec{i} - y\vec{j} + xz\vec{k} = (x^2, -y, xz)$$

and $$\vec{b} = y\vec{i} + x\vec{j} - xyz\vec{k} = (y, x, -xyz)$$

Then,

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x^2 & -y & xz \\ y & x & -xyz \end{vmatrix}$$
$$= (xy^2z - x^2z)\,\vec{i} + (xyz + x^3yz)\,\vec{j} + (x^3 + y^2)\,\vec{k}$$

So that,

$$\frac{\partial(\vec{a}\times\vec{b})}{\partial x} = (y^2z - 2xz)\,\vec{i} + (yz + 3x^2yz)\,\vec{j} + 3x^2\vec{k}$$

and $$\frac{\partial^2(\vec{a}\times\vec{b})}{\partial x\partial y} = 2yz\,\vec{i} + (z + 3x^2z)\,\vec{j} + 0 \quad \text{..........(1)}$$

Also,

$$\frac{\partial(\vec{a}\times\vec{b})}{\partial y} = 2xyz\,\vec{i} + (xz + x^3z)\,\vec{j} + 2y\,\vec{k}$$

and $$\frac{\partial^2(\vec{a}\times\vec{b})}{\partial x\partial y} = 2yz\,\vec{i} + (z + 3x^2z)\,\vec{j} + 0 \quad \text{..........(2)}$$

From (1) and (2), we have

$$\frac{\partial^2(\vec{a}\times\vec{b})}{\partial x\partial y} = \frac{\partial^2(\vec{a}\times\vec{b})}{\partial x\partial y}$$

15. If $\vec{r} = x^2y\vec{i} - 2y^2z\vec{j} + xy^2z^2\vec{k}$. Show that $\left|\frac{\partial^2\vec{r}}{\partial x^2}\times\frac{\partial^2\vec{r}}{\partial y^2}\right|$ at the point (2, 1, –1) is $8\sqrt{2}$

Solution: Let, $\vec{r} = x^2y\vec{i} - 2y^2z\vec{j}, xy^2z^2\vec{k} = (x^2y, -2y^2z, xy^2z^2)$

Then,

$$\frac{\partial\vec{r}}{\partial x} = (2x, 0, y^2z^2), \qquad \frac{\partial\vec{r}}{\partial y} = (x^2, -4yz, 2xyz)$$

and,

$$\frac{\partial^2\vec{r}}{\partial x^2} = (2, 0, 0) \qquad \frac{\partial^2\vec{r}}{\partial y^2} = (0, -4z, 2xz^2)$$

So that,

$$\frac{\partial^2\vec{r}}{\partial x^2}\times\frac{\partial^2\vec{r}}{\partial y^2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & 0 \\ 0 & -4z & 2z^2 \end{vmatrix} = 0.\vec{i} - 4xz^2\vec{j} - 8z\vec{k}$$

Therefore,

$$\left|\frac{\partial^2\vec{r}}{\partial x^2}\times\frac{\partial^2\vec{r}}{\partial y^2}\right| = \sqrt{0 + (-4xz^2)^2 + (-8z)^2}$$

At the point (2, 1, –1)

$$\left|\frac{\partial^2\vec{r}}{\partial x^2}\times\frac{\partial^2\vec{r}}{\partial y^2}\right| = \sqrt{0 + (4(2)(-1)^2)^2 + (8(-1)^2)}$$

$$= \sqrt{0 + 64 + 864} = \sqrt{2\times 64} = 8\sqrt{2}.$$

16. Show that the unit tangent to the curve $\vec{r} = t\vec{i} + t^2\vec{j} + t^3\vec{k}$ at t = 1 is $\frac{1}{\sqrt{14}}(\vec{i} + 2\vec{j} + 3\vec{k})$

Solution: Given curve is

$$\vec{r} = t\vec{i} + t^2\vec{j} + t^3\vec{k} = (t, t^2, t^3)$$

So, the tangent vector of $\vec{r}$ is, $\frac{d\vec{r}}{dt} = (1, 2t, 3t^2)$

Therefore, the unit tangent vector of $\vec{r}$ is

$$\left(\frac{d\vec{r}}{dt}\right) = \frac{\frac{d\vec{r}}{dt}}{\left|\frac{d\vec{r}}{dt}\right|} = \frac{(1, 2t, 3t^2)}{\sqrt{1 + 4t^2 + 9t^4}}$$

At t = 1, the unit tangent vector of $\vec{r}$ is

$$\left(\frac{d\vec{r}}{dt}\right) = \frac{(1, 2, 3)}{\sqrt{1 + 4 + 9}}$$

$$= \frac{(1, 2, 3)}{\sqrt{14}} = \frac{1}{\sqrt{14}}(\vec{i} + 2\vec{j} + 3\vec{k}).$$

17. A particle P is moving on a circle of radius a with constant angular velocity $w = \frac{d\theta}{dt}$, then show that the acceleration of the particle is $-w^2\vec{r}$.

Solution: Since the particle is moving on a circle of radius a and with constant angular velocity $w = \frac{d\theta}{dt}$.

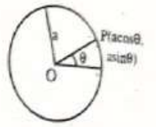
So, $\vec{r} = a\cos\theta\,\vec{i} + a\sin\theta\,\vec{j}$. Then, $\frac{d\vec{r}}{dt} = -a\sin\theta\frac{d\theta}{dt}\vec{i} + a\cos\theta\frac{d\theta}{dt}\vec{j}$

And,

$$\frac{d^2\vec{r}}{dt^2} = -a\cos\theta\left(\frac{d\theta}{dt}\right)^2\vec{i} - a\sin\theta\left(\frac{d\theta}{dt}\right)^2\vec{j}$$

$$= -[a\cos\theta\, w^2\vec{i} + a\sin\theta.\, w^2\vec{j}]$$

$$= -w^2(a\cos\theta\,\vec{i} + a\sin\theta\,\vec{j}) = -w^2\vec{r}$$



Since the acceleration of the particle is $\frac{d^2\vec{r}}{dt^2}$ i.e. $-w^2\vec{r}$.

EXERCISE 4.2

1. Find grad f, where

(i) $f = x^2 + yz$ **(ii)** $f = x^3 + y^3 + 3xyz$ (iii) $f = \log(x^2 + y^2 + z^2)$

Solution: (i) Given that, $f = x^2 + yz$

Then,

$$\text{Grad}(f) = \nabla f = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(x^2 + yz)$$

$$= \vec{i}\frac{\partial}{\partial x}(x^2 + yz) + \vec{j}\frac{\partial}{\partial y}(x^2 + yz) + \vec{k}\frac{\partial}{\partial z}(x^2 + yz)$$

$$= \vec{i}.2x + \vec{j}.z + \vec{k}.y$$

$$= 2x\vec{i} + z\vec{j} + y\vec{k}$$

(ii) Given that, $f = x^3 + y^3 + 3xyz$

Then,

$$\text{Grad}(f) = \nabla f = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(x^3 + y^3 + 3xyz)$$

$$= (3x^2 + 3yz)\vec{i} + (3y^2 + 3zx)\vec{j} + 3xy\vec{k}.$$

(iii) Given that, $f = \log(x^2 + y^2 + z^2)$

Then,

$$\text{Grad}(f) = \nabla f = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(\log(x^2 + y^2 + z^2))$$

$$= \vec{i}\left(\frac{2x}{x^2 + y^2 + z^2}\right) + \vec{j}\left(\frac{2y}{x^2 + y^2 + z^2}\right) + \vec{k}\left(\frac{2z}{x^2 + y^2 + z^2}\right)$$

$$= \frac{2}{x^2 + y^2 + z^2}(x\vec{i} + y\vec{j} + z\vec{k}).$$

2. Find a unit normal to the surface

(i) $xy^3z^2 = 4$ at $(-1, -1, 2)$ (ii) $x^2y + 2xz = 4$ at $(2, -2, 3)$

Solution: (i) Let given surface is,

$$f = xy^3z^2 - 4$$

We have, grad (f) is the normal to the given surface.

Then,

$$\text{grad}(f) = \nabla f = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(xy^3z^2 - 4),$$

$$= y^3z^2\,\vec{i} + 3xy^2z^2\,\vec{j} + 2xy^3z\,\vec{k}$$

At point $(-1, -1, 2)$, grad (f) $= -4\vec{i} - 12\vec{j} + 4\vec{k}$

Thus, $(-4, -12, 4)$ is normal to f at $(-1, -1, 2)$.

And, the unit vector of grad (f) is

$$\hat{n} = \frac{\text{grad}(f)}{|\text{grad}(f)|} = \frac{(-4, -12, 4)}{\sqrt{16 + 144 + 16}} = \frac{4(-1, -3, 1)}{4\sqrt{1 + 9 + 1}} = \frac{(-1, -3, 1)}{\sqrt{11}}.$$

(ii) Let the given surface be,

$$f = x^2y + 2xz - 4$$

We know, the normal vector to the surface f is ∇f.

Here,

$$\nabla f = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(x^2y + 2xz - 4)$$

$$= (2xy + 2z)\vec{i} + x^2\,\vec{j} + 2x\,\vec{k}$$

At point $(2, -2, 3)$, $\nabla f = (-8 + 6)\vec{i} + 4\vec{j} + 4\vec{k}$

$$= 2(-\vec{i} + 2\vec{j} + 2\vec{k})$$

Now, unit vector of ∇f is

$$\hat{n} = \frac{\nabla f}{|\nabla f|} = \frac{2(\vec{i} + 2\vec{j} + 2\vec{k})}{2\sqrt{1 + 4 + 4}} = \frac{\vec{i} + 2\vec{j} + 2\vec{k}}{3}$$

Thus, the unit vector normal to f at $(2, -2, 3)$ is $\frac{1}{3}(-\vec{i} + 2\vec{j} + 2\vec{k})$.

3. Find the directional derivatives of f at P in the direction $\vec{a}$, where

(i) $f = x^2 + y^2$, $P(1, 1)$, $\vec{a} = 2\vec{i} - 4\vec{j}$

Solution: Given surface be, $f = x^2 + y^2$

Then,

$$\text{grad}(f) = \nabla f = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(x^2 + y^2)$$

$$= 2(x\,\vec{i} + y\,\vec{j})$$

At point P(1, 1), grad (f) $= 2(\vec{i} + \vec{j})$

Also, given that $\vec{a} = 2\vec{i} - 4\vec{j}$. So, unit vector of $\vec{a}$ is

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{2\vec{i} - 4\vec{j}}{\sqrt{4 + 16}} = \frac{\vec{i} - 2\vec{j}}{\sqrt{5}}$$

Now, the direction derivative of f at P alone $\vec{a}$ is

$$\nabla f.\,\hat{a} = \frac{2}{\sqrt{5}}(\vec{i} + \vec{j}).(\vec{i} - 2\vec{j}) = \frac{2}{\sqrt{5}}(1 - 2) = -\frac{2}{\sqrt{5}}$$

(ii) $f = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$, $P(3, 0, 4)$; $\vec{a} = \vec{i} + \vec{j} + \vec{k}$. **[2006 Spring Q.No. 3(a)]**

Solution: Given surface is,

$$f = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$$

We know the directional derivative of f at P in the direction of $\vec{a}$ is $\nabla f.\,\hat{a}$ at P.

Here,

$$\nabla f = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\frac{1}{\sqrt{x^2 + y^2 + z^2}}$$

$$= -2(x^2 + y^2 + z^2)^{-3/2}\,(x\,\vec{i} + y\,\vec{j} + z\,\vec{k})$$

At point P(3, 0, 4), $\nabla f = -2(9 + 0 + 16)^{-3/2}(3\vec{i} + 4\vec{k})$

$$= -\frac{2}{125}(3\vec{i} + 4\vec{k})$$

Also, given that, $\vec{a} = \vec{i} + \vec{j} + \vec{k}$. So, unit vector of $\vec{a}$ is

$$\hat{a} = \frac{\vec{i} + \vec{j} + \vec{k}}{\sqrt{1 + 1 + 1}} = \frac{\vec{i} + \vec{j} + \vec{k}}{\sqrt{3}}$$

Now,

$$\nabla f.\,\hat{a} = -\frac{2}{125}(3\vec{i} + 4\vec{k}).\frac{1}{\sqrt{3}}(\vec{i} + \vec{j} + \vec{k})$$

$$= -\frac{2}{125\sqrt{3}}(3 + 0 + 4) = -\frac{14}{125\sqrt{3}}$$

Thus, the directional derivative of f at P in the direction of $\vec{a}$ is, $-\frac{14}{125\sqrt{3}}$.

(iii) $f = xyz$, $P(-1, 1, 3)$, $\vec{a} = \vec{i} - 2\vec{j} + 2\vec{k}$

(iv) $f = e^x \cos y$, $P(2, \pi, 0)$, $\vec{a} = 2\vec{i} + 3\vec{k}$

(v) $f = xy^2 + yz^3$, $P(2, -1, 3)$, $\vec{a} = \vec{i} + 2\vec{i} + 2\vec{k}$ **[2009 Fall – Short]**

(vi) $f = 2xy + z^2$, $P(1, -1, 3)$, $\vec{a} = \vec{i} + 2\vec{j} + 2\vec{k}$

Solution: (iii) – (vi) – process as (ii).

(vii) $f = 4xz^3 - 3x^2yz^2$ **at (2, −1, 2) along z-axis.**

Solution: Given that, $f = 4xz^3 - 3x^2yz^2$

Then the directional derivative of f at (2, −1, 2) along z-axis is,

$\nabla f \cdot \hat{a}$ at $(2, -1, 2)$ and where $\vec{a} = \vec{k}$.

Here,

$$\nabla f = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(4xz^3 - 3x^2yz^2)$$

$$= (4z^3 - 6xyz^2)\vec{i} + (-3x^2z^2)\vec{j} + (12xz^2 - 6x^2yz)\vec{k}$$

So that,

$$\nabla f.\hat{a} = ((4z^3 - 6xyz^2)\vec{i} - 3x^2z^2\vec{j} + (12xz^2 - 6x^2yz)\vec{k}) \cdot \vec{k}$$

$$[\because |\vec{a}| = 1 \text{ and } \hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{\vec{k}}{1} = \vec{k}]$$

$$= 12xz^2 - 6x^2yz$$

at point $(2, -1, 2)$, $\nabla f \cdot \hat{a} = 12(2)(2)^2 - 6(2)^2(-1)(2) = 96 + 48 = 144$

Thus, the directional derivative of f at $(2, -1, 2)$ along z–axis is 144.

(viii) $f = xy^2 + yz^3$ at $(2, -1, 1)$ along the direction of the normal to the surface $x \log z$, $y^2 + 4 = 0$ at $(-1, 2, 1)$.

Solution: Given that, $f = xy^2 + yz^3$

And the surface is, $\phi = x\log(z) - y^2 + 4$

Then,

$$\nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{j}\frac{\partial}{\partial z}\right)(x\log(z) - y^2 + 4) = \log(z)\,\vec{i} - 2y\,\vec{j} + \frac{x}{z}\vec{k}$$

at $(-1, 2, 1)$,

$$\nabla\phi = \log(1)\,\vec{i} - 4\vec{j} - \vec{k} = 0\vec{i} - 4\vec{j} - \vec{k}$$

Also,

$$\text{grad}(f) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{j}\frac{\partial}{\partial z}\right)(xy^2 + yz^3)$$

$$= y^2\vec{i} + (2xy + z^3)\vec{j} + 3yz^2\vec{k}$$

at $(2, -1, 1)$,

$$\text{grad}(f) = \vec{i} + (-4 + 1)\vec{j} - 3\vec{k} = \vec{i} - 3\vec{j} - 3\vec{k}$$

Now, directional derivative of f is

$$= (\text{grad}(f)|_{\text{at }(2,-1,1)}) \cdot \left(\frac{\nabla\phi}{|\nabla\phi|}_{\text{at }(-1,2,1)}\right)$$

$$= (\vec{i} - 3\vec{j} - 3\vec{k}) \cdot \left(\frac{0\vec{i} - 4\vec{j} - \vec{k}}{\sqrt{0 + 16 + 1}}\right) = \frac{1}{\sqrt{17}}(0 + 12 + 3) = \frac{15}{\sqrt{17}}$$

4. Find the angle between the tangent planes to the surface $x \log z = y^2 - 1$ and $x^2y = 2 - z$ at $(1, 1, 1)$.

Solution: Let the given surface is, $f = x\log(z) - y^2 + 1$

Then,

$$\nabla f = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) = (\log(z)\,\vec{i} - 2y\,\vec{j} + \frac{x}{z}\vec{k}) \quad \text{.........(i)}$$

And the given surface is, $F = x^2y - 2 + z$

Then,

$$\nabla F = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) = 2xy\,\vec{i} + x^2\vec{j} + \vec{k}$$

Let θ be the angle between the tangent planes of surface $x\log(z) = y^2 - 1$ and $x^2y = 2 - z$. Then θ be the angle between ∇f and ∇F.

Therefore,

$$\cos\theta = \frac{\nabla f \cdot \nabla F}{|\nabla f||\nabla F|} = \frac{2xy\log(z) - 2x^2y + x/z}{\sqrt{(\log(z))^2 + 4y^2 + x^2/z^2}\sqrt{4x^2y^2 + x^4 + 1}}$$

at point $(1, 1, 1)$,

$$\cos\theta = \frac{2(0) - 2 + 1}{\sqrt{0 + 4 + 1}\sqrt{4 + 1 + 1}} \quad [\because \log(1) = 0]$$

$$= \frac{-1}{\sqrt{5}\sqrt{6}} = -\frac{1}{\sqrt{30}}$$

Thus, angle between the tangent planes to the given surfaces $x\log(z) = y^2 - 1$ and $x^2y = 2 - z$ at $(1, 1, 1)$ is $\cos^{-1}\left(-\frac{1}{\sqrt{30}}\right)$.

5. Find the angle between the tangent planes to the surface $xy = z^2$ at the point $(4, 1, 2)$ and $(3, 3, -3)$.

Solution: Given surface is, $f = xy - z^2$

Then the normal to the surface f is ∇f.

Here,

$$\nabla f = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(xy - z^2) = y\vec{i} + x\vec{j} - 2z\vec{k}.$$

At point, $(4, 1, 2)$, $\nabla f = \vec{i} + 4\vec{j} - 4\vec{k}$.

and at point $(3, 3, -3)$, $\nabla f = 3\vec{i} + 3\vec{j} + 6\vec{k}$.

Let θ be the angle between ∇f at $(4, 1, 2)$ and at $(3, 3, -3)$

Then,

$$\cos\theta = \frac{(\nabla f \text{ at } (4, 1, 2)) \cdot (\nabla f \text{ at } (3, 3, -3))}{|\nabla f \text{ at } (4, 1, 2)||\nabla f \text{ at } (3, 3, -3)|}$$

$$= \frac{(\vec{i} + 4\vec{j} - 4\vec{k}) \cdot (3\vec{i} + 3\vec{j} + 6\vec{k})}{|\vec{i} + 4\vec{j} - 4\vec{k}||3\vec{i} + 3\vec{j} + 6\vec{k}|}$$

$$= \frac{3 + 12 - 24}{\sqrt{1 + 16 + 16}\sqrt{9 + 9 + 36}} = \frac{-9}{\sqrt{33}\sqrt{54}} = \frac{-9}{9\sqrt{11}\sqrt{2}} = \frac{-1}{\sqrt{22}}$$

$$\Rightarrow \theta = \cos^{-1}\left(-\frac{1}{\sqrt{22}}\right).$$

Thus, the angle between the normal to $xy = z^2$ at $(4, 1, 2)$ and $(3, 3, -3)$ is, $\cos^{-1}\left(-\frac{1}{\sqrt{22}}\right)$.

6. If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, show that

(i) grad r = grad (r) = $\frac{\vec{r}}{r}$ (ii) grad $\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}$

(iii) grad $(r^n) = nr^{n-2}\,\vec{r}$ (iv) **grad $(\vec{a}\,.\,\vec{r}) = \vec{a}$, where $\vec{a}$ is a constant vector.**

Solution: Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = (x, y, z)$

Then, $|\vec{r}| = \sqrt{x^2+y^2+z^2}$ ....(i)

So that,

$$\frac{\vec{r}}{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{(x, y, z)}{\sqrt{x^2+y^2+z^2}} \quad ......(ii)$$

and, $$-\frac{\vec{r}}{r^3} = -\frac{\vec{r}}{|\vec{r}|^3} = \frac{-(x, y, z)}{(x^2+y^2+z^2)^{3/2}} \quad ....(iii)$$

Also, $r^{(n-2)}\,\vec{r} = |\vec{r}|^{(n-2)}\,\vec{r} = (x^2+y^2+z^2)^{(n-2)/2}\,(x, y, z)$ ....(iv)

Now,

(i) $$\text{grad (r)} = \nabla r = \nabla|\vec{r}| = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\sqrt{x^2+y^2+z^2}$$

$$= \frac{1}{2\sqrt{x^2+y^2+z^2}}(2x\vec{i} + 2y\vec{j} + 2z\vec{k})$$

$$= \frac{x\vec{i} + y\vec{j} + z\vec{k}}{\sqrt{x^2+y^2+z^2}} = \frac{(x, y, z)}{\sqrt{x^2+y^2+z^2}} = \frac{\vec{r}}{r}$$

Thus, grad (r) = $\frac{\vec{r}}{r}$

(ii) $$\text{grad}\left(\frac{1}{r}\right) = \nabla\left(\frac{1}{r}\right) = \nabla\frac{1}{|\vec{r}|} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).\frac{1}{\sqrt{x^2+y^2+z^2}}$$

$$= -\frac{1}{2(x^2+y^2+z^2)^{3/2}}(2x\vec{i} + 2y\vec{j} + 2z\vec{k})$$

$$= -\frac{(x, y, z)}{(x^2+y^2+z^2)^{3/2}} = -\frac{\vec{r}}{r^3}.$$

Thus, grad $\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}$.

(iii) $$\text{grad}(r^n) = \nabla(r^n) = \nabla|\vec{r}|^n = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)^{n/2}$$

$$= \frac{n}{2}(x^2+y^2+z^2)^{n/2-1}(2x\vec{i} + 2y\vec{j} + 2z\vec{k})$$

$$= n(x, y, z).(x^2+y^2+z^2)^{(n-2)/2}$$

$$= nr^{n-2}\,\vec{r}$$

Thus, grad $(r^n) = nr^{n-2}\,\vec{r}$

(iv) $$\text{grad}(\vec{a}\,.\,\vec{r}) = \nabla((a_1\vec{i} + a_2\vec{j} + a_3\vec{k})\,(x\vec{i} + y\vec{j} + z\vec{k}))$$

$$= \nabla(a_1x + a_2y + a_3z)$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(a_1x + a_2y + a_3z)$$

$$= a_1\vec{i} + a_2\vec{j} + a_3\vec{k} = \vec{a}.$$

Thus, grad $(\vec{a}\,.\,\vec{r}) = \vec{a}$.

7. **In what direction from (3, 1, –2) is the directional derivative of $f = x^2y^2z^4$ maximum and what is its magnitude?**

Solution: Given that, $f = x^2y^2z^4$

Since we have the directional derivative of f is maximum in the direction of grad(f).

Here,

$$\text{grad } f = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(x^2y^2z^4)$$

$$= 2xy^2z^4\,\vec{i} + 2x^2yz^4\,\vec{j} + 4x^2y^2z^3\,\vec{k}$$

at (3, 1, –2), $\text{grad } f = 2(3)(1)^2(-2)^4\,\vec{i} + 2(3)^2(1)(-2)^4\,\vec{j} + 4(3)^2(1)^2(-2)^3\,\vec{k}$

$$= 96\vec{i} + 288\vec{j} - 288\vec{k}.$$

And its magnitude is

$$|\text{grad } f| = \sqrt{(96)^2 + (288)^2 + (-288)^2}$$

$$= \sqrt{9216 + 82944 + 82944} = \sqrt{175104} = 96\sqrt{19}.$$

Thus in the direction of $96\vec{i} + 288\vec{j} - 288\vec{k}$ from (3, 1, –2) is the maximum directional derivative of $f = x^2y^2z^4$ and its magnitude is $96\sqrt{19}$.

8. **What is the greatest rate of increase of $u = x^2 + yz^2$ at the point (1, –1, 3)?**

Solution: Given that, $u = x^2 + yz^2$.

Since we have the greatest rate of increase of u at the point (α, β, γ) is the maximum value of the directional derivative at (α, β, γ). So,

greatest rate of increase of u at (α, β, γ)

= $|\nabla u|$ at (α, β, γ)

Here,

$$\nabla u = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(x^2 + yz^2) = 2x\vec{i} + z^2\vec{j} + 2yz\vec{k}$$

At point (1, –1, 3), $\nabla u = 2\vec{i} + 9\vec{j} - 6\vec{k}$

Then, value of $|\nabla u|$ at (1, –1, 3) is,

$$|\nabla u| = |(2, 9, -6)| \quad \text{at } (1, -1, 3)$$

$$= \sqrt{4 + 81 + 26} = \sqrt{121} = 11.$$

Thus, the greatest rate of increase of $u = x^2 + yz^2$ at (1, –1, 3) is 11.

9. **The temperature at a point (x, y, z) in space is given by $T = x^2 + y^2 - z$. A mosquito located at (1, 1, 2) desire to fly in such a direction that it wing get warm as soon as possible. In what direction should it fly?**

Solution: Given that, $T = x^2 + y^2 - z$

Then,

$$\nabla T = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 - z) = 2x\vec{i} + 2y\vec{j} - \vec{k}$$

at point (1, 1, 2), $\nabla T = 2\vec{i} + 2\vec{j} - \vec{k}$.

Given that a mosquito desires to fly in a direction so that its wing gets warm as soon as possible.

That means, the mosquito wants to fly in the direction where it get maximum temperature.

Thus, the mosquito should fly in the direction of $2\vec{i} + 2\vec{j} - \vec{k}$.

10. If q is the acute angle between the surfaces $xy^2z = 3x + z^2$ and $3x^2 - y^2 + 2z = 1$ at the point (1, –2, 1), show that $\cos\theta = \frac{3}{7\sqrt{6}}$.

Solution: Given surfaces are

$$f = xy^2z - 3x - z^2 \quad \text{and} \quad F = 3x^2 - y^2 + 2z - 1$$

Then,

$$\text{grad } f = (y^2z - 3)\vec{i} + 2xyz\vec{j} + (xy^2 - 2z)\vec{k}$$

and $$\text{grad } F = 6x\vec{i} - 2y\vec{j} + 2\vec{k}$$

At point, (1, –2, 1), $\text{grad } f = \vec{i} - 4\vec{j} + 2\vec{k}$

and $\text{grad } F = 6\vec{i} + 4\vec{j} + 2\vec{k}$

Let θ be the angle between f and F at (1, –2, 1). Then,

$$\cos\theta = \frac{\text{grad } f \,.\, \text{grad } F}{|\text{grad } f|\,|\text{grad } F|} \quad \text{at point } (1, -2, 1)$$

$$= \frac{(\vec{i} - 4\vec{j} + 2\vec{k}).(6\vec{i} + 4\vec{j} + 2\vec{k})}{|(\vec{i} - 4\vec{j} + 2\vec{k})|\,|(6\vec{i} + 4\vec{j} + 2\vec{k})|}$$

$$= \frac{6 - 16 + 4}{\sqrt{1 + 16 + 4}\sqrt{36 + 16 + 4}} = \frac{-6}{\sqrt{21}\sqrt{56}} = \frac{-6}{14\sqrt{6}} = \frac{-3}{7\sqrt{6}}.$$

11. Find the value of constants l and u so that the surface $\lambda x^2 - \mu yz = (\lambda + 2)x$ and $4x^2y + z^3 = 4$ may intersect orthogonally at the point (1, –1, 2).

Solution: Given surfaces are

$$\lambda x^2 - \mu yz = (\lambda + 2)x \quad \text{.........(i)}$$
$$4x^2y + z^3 = 4 \quad \text{.......(ii)}$$

Given that the surfaces intersect orthogonally at (1, –1, 2). So, the point lies on both surfaces.

Then at (1, –1, 2), (i) becomes

$$\lambda + 2\mu = \lambda + 2 \Rightarrow \mu = 1.$$

Set, $$\phi_1 = \lambda x^2 - (\lambda + 2)x - yz \quad [\because \mu = 1]$$
$$\phi_2 = 4x^2y + z^3 - 4$$

So, $$\vec{r}_1 = \text{grad}(\phi_1) = (2\lambda x - \lambda - 2)\vec{i} - z\vec{j} - y\vec{k}$$
$$\vec{r}_2 = \text{grad}(\phi_2) = 8xy\vec{i} + 4x^2\vec{j} + 3z^2\vec{k}$$

at (1, –1, 2)

$$\vec{r}_1 = (\lambda - 2)\vec{i} - 2\vec{j} + \vec{k} \quad \text{and} \quad \vec{r}_2 = -8\vec{i} + 4\vec{j} + 12\vec{k}$$

Given that ϕ_1 and ϕ_2 are orthogonal to each other at (1, –1, 2). So, at (1, –1, 2), we should have,

$$\vec{r}_1.\vec{r}_2 = 0 \Rightarrow -8(\lambda - 2) - 8 + 12 = 0$$
$$\Rightarrow -8(\lambda - 2) + 4 = 0 \Rightarrow \lambda - 2 = -1/2 \Rightarrow \lambda = \frac{3}{2} = 2.5$$

Thus, λ = 2.5 and μ = 1.

EXERCISE 4.3

1. Find divergence of

(i) $x\vec{i} + y\vec{j} + z\vec{k}$ **(ii)** $e^x(\cos y\,\vec{i} + \sin y\,\vec{j})$

(iii) $\left(-\frac{y\vec{i} + x\vec{j}}{x^2 + y^2}\right)$ **(iv)** $e^x\vec{i} + ye^{-x}\vec{j} + 2z\sinh x\,\vec{k}$

Solution: (i) Let, $\vec{v} = x\vec{i} + y\vec{j} + z\vec{k}$

Then,

$$\text{Div}(\vec{v}) = \nabla.\vec{v} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).(x\vec{i} + y\vec{j} + z\vec{k})$$
$$= 1 + 1 + 1 = 3.$$

Thus, divergence of $x\vec{i} + y\vec{j} + z\vec{k}$ is 3.

(ii) Let $\vec{v} = e^x(\cos y\,\vec{i} + \sin y\,\vec{j})$.

Then the divergence of $\vec{v}$ is,

$$\text{div.}\,\vec{v} = \nabla.\vec{v} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(e^x\cos y\,\vec{i} + e^x\sin y\,\vec{j})$$
$$= e^x\cos y + e^x\cos y = 2e^x\cos y.$$

Thus, the divergence of $e^x(\cos y\,\vec{i} + \sin y\,\vec{j})$ is $2e^x\cos y$.

(iii) Let $\vec{v} = \left(-\frac{y\vec{i} + x\vec{j}}{x^2 + y^2}\right)$. Then the divergence of $\vec{v}$ is;

$$\text{div.}\,\vec{v} = \nabla.\vec{v} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y}\right)\left(-\frac{y\vec{i} + x\vec{j}}{x^2 + y^2}\right)$$
$$= \frac{1}{x^2 + y^2}\left[-\frac{\partial y}{\partial x} + \frac{\partial x}{\partial y}\right] = \frac{1}{x^2 + y^2}(0 + 0) = 0.$$

Thus, the divergence of $\vec{v}$ is 0.

(iv) Let $\vec{v} = e^x\,\vec{i} + ye^{-x}\,\vec{j} + 2z\sinh x\,\vec{k}$. Then the divergence of $\vec{v}$ is,

$$\text{div}\,\vec{v} = \nabla.\vec{v} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).\left(e^x\,\vec{i} + ye^{-x}\,\vec{j} + 2z\sinh x\,\vec{k}\right)$$
$$= e^x + e^{-x} + 2\sinh x$$
$$= e^x + e^{-x} + 2\sinh x$$
$$= e^x + e^{-x} + 2\left(\frac{e^x - e^{-x}}{2}\right) = e^x + e^{-x} + e^x - e^{-x} = 2e^x.$$

Thus, divergence of $e^x\,\vec{i} + ye^{-x}\,\vec{j} + 2z\sinh x\,\vec{k}$ is $2e^x$.

2. Find curl of

(i) $\frac{1}{2}(x^2 + y^2 + z^2)(\vec{i} + \vec{j} + \vec{k})$ **(ii)** $(x^2 + y^2 + z^2)^{-3/2}(x\vec{i} + y\vec{j} + z\vec{k})$

(iii) $xyz\,(x\vec{i} + y\vec{j} + z\vec{k})$

Solution: (i) Let $\vec{v} = \frac{1}{2}(x^2 + y^2 + z^2)(\vec{i} + \vec{j} + \vec{k})$.

Then the curl of $\vec{v}$ is,

$$\text{curl}\,\vec{v} = \nabla\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 1/2\,(x^2+y^2+z^2) & 1/2(x^2+y^2+z^2) & 1/2\,(x^2+y^2+z^2) \end{vmatrix}$$
$$= (y - z)\,\vec{i} + (z - x)\,\vec{j} + (x - y)\,\vec{k}$$

Thus, curl of $\vec{v}$ is $(y - z)\,\vec{i} + (z - x)\,\vec{j} + (x - y)\,\vec{k}$

(ii) – (iii) Similar to (i).

3. Evaluate: (i) div. $(3x^2\vec{i} + 5xy^2\vec{j} + xyz^3\vec{k})$ **at (1, 2, 3).**

(ii) div $(xy\sin z\,\vec{i} + y^2\sin x\,\vec{j} + z^2\sin xy\,\vec{k})$ **at** $\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right)$.

Solution: (i) Here,

$$\text{div.}\,(3x^2\vec{i} + 5xy^2\vec{j} + xyz^3\vec{k})$$
$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).(3x^2\vec{i} + 5xy^2\vec{j} + xyz^3\vec{k})$$
$$= 6x + 10xy + 3xyz^2$$

At point (1, 2, 3), $\text{div.}\,(3x^2\vec{i} + 5xy^2\vec{j} + xyz^3\vec{k}) = 6 + 20 + 54 = 80.$

(ii) Here,

$$\text{div}\,(xy\sin z\,\vec{i} + y^2\sin x\,\vec{j} + z^2\sin xy\,\vec{k})$$
$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).(xy\sin z\,\vec{i} + y^2\sin x\,\vec{j} + z^2\sin xy\,\vec{k})$$
$$= y\sin z + 2y\sin x + 2z\sin xy$$

At point $\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right)$, $\text{div}\,(xy\sin z\,\vec{i} + y^2\sin x\,\vec{j} + z^2\sin xy\,\vec{k})$
$$= \frac{\pi}{2}\sin\frac{\pi}{2} + 2\frac{\pi}{2}\sin 0 + 2\frac{\pi}{2}\sin 0$$
$$= \frac{\pi}{2}.1 + \pi.0 + \pi.0 = \frac{\pi}{2}.$$

4. Find the divergence and curl of vectors

(i) $\vec{v} = xyz\,\vec{i} + 3x^2y\,\vec{j} + (xz^2 - y^2z)\,\vec{k}$

(ii) $\vec{v} = (x^2 + yz)\,\vec{i} + (y^2 + zx)\,\vec{j} + (z^2 + xy)\,\vec{k}$

Solution: (i) Let $\vec{v} = xyz\,\vec{i} + 3x^2y\,\vec{j} + (xz^2 - y^2z)\,\vec{k}$

Then divergence of $\vec{v}$ is,

$$\text{div.}\,\vec{v} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).(xyz\,\vec{i} + 3x^2y\,\vec{j} + (xz^2 - y^2z)\,\vec{k})$$
$$= yz + 3x^2 + 2xz - y^2$$

And, curl of $\vec{v}$ is, $\text{curl}\,\vec{v} = \nabla\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xyz & 3x^2y & xy^2 - y^2z \end{vmatrix}$

$$= -2yz\,\vec{i} - (z^2 - xy)\,\vec{j} + (6xy - xz)\,\vec{k}$$

Thus, divergence of $\vec{v}$ is $yz + 3x^2 + 2xz - y^2$ and curl of $\vec{v}$ is $-2yz\,\vec{i} + (xy - z^2)\,\vec{j} + x(6y - z)\,\vec{k}$.

(ii) Similar to (i)

5. If $\vec{v} = \dfrac{x\vec{i} + y\vec{j} + z\vec{k}}{\sqrt{x^2 + y^2 + z^2}}$. **Show that:** $\nabla.\vec{v} = \dfrac{2}{\sqrt{x^2 + y^2 + z^2}}$ **and** $\nabla\times\vec{v} = \vec{0}$.

[2011 Fall Q.No. 6(a)]

Solution: Let,

$$\vec{v} = \frac{x\vec{i} + y\vec{j} + z\vec{k}}{\sqrt{x^2 + y^2 + z^2}}$$

Then,

$$\nabla.\vec{v} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).\left(\frac{x\vec{i} + y\vec{j} + z\vec{k}}{\sqrt{x^2 + y^2 + z^2}}\right)$$
$$= (x^2 + y^2 + z^2)^{-1/2} - \frac{x}{2}(x^2 + y^2 + z^2)^{-1/2}.2x + (x^2 + y^2 + z^2)^{-1/2} -$$
$$\frac{y}{2}(x^2 + y^2 + z^2)^{-3/2}.2y + (x^2 + y^2 + z^2)^{-1/2} - \frac{z}{2}(x^2 + y^2 + z^2)^{-3/2}.2z$$
$$= \frac{3}{\sqrt{x^2 + y^2 + z^2}} - \frac{1}{(x^2 + y^2 + z^2)^{3/2}}(x^2 + y^2 + z^2)$$
$$= \frac{3}{\sqrt{x^2 + y^2 + z^2}} - \frac{1}{(x^2 + y^2 + z^2)^{1/2}}$$
$$= \frac{2}{\sqrt{x^2 + y^2 + z^2}}$$

And,

$$\nabla\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x.(x^2+y^2+z^2)^{-1/2} & y(x^2+y^2+z^2)^{-1/2} & z(x^2+y^2+z^2) \end{vmatrix}$$

$$= \left[-\frac{z}{2}(x^2+y^2+z^2)^{-3/2}.\,2y + \frac{y}{2}(x^2+y^2+z^2)^{-3/2}.\,2z\right]\vec{i} -$$

$$\left[-\frac{z}{2}(x^2+y^2+z^2)^{-3/2}.\,2x + \frac{x}{2}(x^2+y^2+z^2)^{-3/2}.\,2z\right]\vec{j} +$$

$$\left[-\frac{y}{2}(x^2+y^2+z^2)^{-3/2}.\,2x + \frac{x}{2}(x^2+y^2+z^2)^{-3/2}.\,2y\right]\vec{k}$$

$$= \frac{1}{(x^2+y^2+z^2)^{3/2}}[(-yz+yz)\vec{i}\ (-xz+xz)\vec{j} + (-xy+xy)\vec{k}]$$

$$= \frac{1}{(x^2+y^2+z^2)^{3/2}}(0\vec{i} - 0\vec{j} + 0\vec{k})$$

$$= \vec{0}$$

Thus, $\nabla.\vec{v} = \frac{2}{\sqrt{x^2+y^2+z^2}}$ and $\nabla\times\vec{v} = \vec{0}$.

6. If $\vec{A} = 3xz^2\vec{i} - yz\vec{j} + (x+2z)\vec{k}$, Find curl (curl $\vec{A}$)

Solution: Let $\vec{A} = 3xz^2\vec{i} - yz\vec{j} + (x+2z)\vec{k}$

Then,

$$\text{curl}(\vec{A}) = \nabla\times\vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3xz^2 & -yz & x+2z \end{vmatrix} = y\vec{i} + (6xz-1)\vec{j} + 0\vec{k}$$

So,

$$\text{curl}(\text{curl}(\vec{A})) = \nabla\times\text{curl}\,\vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & 6xz-1 & 0 \end{vmatrix}$$

$$= -6x\vec{i} + 0\vec{j} + (6z-1)\vec{k}.$$

Thus, curl (curl $\vec{A}$) $= -6x\vec{i} + (6z-1)\vec{k}$.

7. Show that the vector $\vec{v} = (x+3y)\vec{i} + (y-3z)\vec{j} + (x-2z)\vec{k}$ is solenoidal.

Solution:

Note: If divergence of $\vec{v}$ is zero i.e. div $\vec{v} = 0$ then $\vec{v}$ is called solenoidal.

Let, $\vec{v} = (x+3y)\vec{i} + (y-3z)\vec{j} + (x-2z)\vec{k}$

Then,

$$\text{div}.\,\vec{v} = \nabla.\vec{v} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)((x+3y)\vec{i} + (y-3z)\vec{j} + (x-2z)\vec{k}))$$

$$= 1 + 1 - 2 = 0.$$

This shows that $\vec{v}$ is solenoidal.

8. If $u = x^2+y^2+z^2$ and $\vec{v} = x\vec{i} + y\vec{j} + z\vec{k}$. Show that div. $(u\vec{v}) = 5u$

Solution: Let, $u = x^2+y^2+z^2$ and $\vec{v} = x\vec{i} + y\vec{j} + z\vec{k}$

Then,

$$\text{div}(u\vec{v}) = \nabla.(u\vec{v}) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).\,((x^2+y^2+z^2)(x\vec{i} + y\vec{j} + z\vec{k}))$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)[(x^3+xy^2+xz^2)\vec{i} +$$

$$(x^2y+y^3+yz^2)\vec{j} + (x^2z+y^2z+z^3)\vec{k}]$$

$$= 3x^2+y^2+z^2+x^2+3y^2+z^2+x^2+y^2+3z^2$$

$$= 5(x^2+y^2+z^2) = 5u$$

Thus, div. $(u\vec{v}) = 5u$.

9. Show that the vector $\vec{v} = (\sin y+z)\vec{i} + (x\cos y - z)\vec{j} + (x-y)\vec{k}$ is irrotational.

Solution:

Note: If curl of $\vec{v}$ is zero i.e. curl $\vec{v} = 0$ then $\vec{v}$ is called irrotational.

Let, $\vec{v} = (\sin y + z)\vec{i} + (x\cos y - z)\vec{j} + (x-y)\vec{k}$

Then $\vec{v}$ is irretational if curl $\vec{v} = \vec{0}$.

Here,

$$\text{curl}\,\vec{v} = \nabla\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \sin y + z & x\cos y - z & x-y \end{vmatrix}.$$

$$= (-1+1)\vec{i} + (1-1)\vec{j} + (\cos y - \cos y)\vec{k} = \vec{0}.$$

This shows that $\vec{v}$ is irrotational.

10. Show that $\vec{v} = 2xyz^3\vec{i} + x^2z^3\vec{j} + 3x^2yz^2\vec{k}$ is irrotational.

Solution: Let, $\vec{v} = 2xyz^3\vec{i} + x^2z^3\vec{j} + 3x^2yz^2\vec{k}$

Then $\vec{v}$ is irratational if curl $\vec{v} = \vec{0}$.

Here,

$$\text{curl}\,\vec{v} = \nabla\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xyz^3 & x^2z^3 & 3x^2yz^2 \end{vmatrix}$$

$$= (3x^2z^2 - 3x^2z^2)\vec{i} - (6xyz^2 - 6xyz^2)\vec{j} + (2xz^3 - 2xz^3)\vec{k}$$

$$= \vec{0}.$$

This shows that $\vec{v}$ is irratational.

11. If $\phi = \log(x^2+y^2+z^2)$, find div (grad ϕ) and curl (grad ϕ).

[2010 Fall: If $\mu = \log(x^2+y^2+z^2)$, find div (grad μ).]

Solution: Let, $\phi = \log(x^2+y^2+z^2)$

Then,