DZHANIBEKOV EFFECT

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Introduction

We try to understand the Dzhanibekov effect,which at first seems very non-intuitive.So, we try and provide a mathematical model to help us understand this better. We do this by first finding out the angular velocities in the **BFCS** in terms of elliptical integrals.Then,we obtain the Euler Angles by integrating the angular velocities. Further,using the initial arbitrary conditions we make a quantitative model of the Dzhanibekov Effect which will be accessible to the students of the course ES0209.

1 BackGround

- We have a body of mass m, and a Cartesian Coordinate System $\{\mathcal{E}(t), G, \hat{e_i}(t)\}$ is attached to the rigid body which is known as the **BFCS**.
- There is another Frame Of Reference known as **Observer Frame** which is fixed to the ground $\{\mathcal{E}_o, O, \hat{E_i}\}$.
- The body is such that $I_1 \neq I_2 \neq I_3$, where:
  - I_1:Moment Of Inertia of the body wrt e_1 axis.
  - I_2:Moment Of Inertia of the body wrt e_2 axis.
  - I_3:Moment Of Inertia of the body wrt e_3 axis.
- $\mathcal{I}^G = I_1\hat{e_1} \otimes \hat{e_1} + I_2\hat{e_2} \otimes \hat{e_2} + I_3\hat{e_3} \otimes \hat{e_3}$.

2 Equations Of Motion Of A Rigid Body

2.1 Newton's Equation of Motion of the Center of Mass

- The position vector of the center of mass of the body wrt O is given as:
  $\underline{r}^{\mathrm{G/O}}(t) = r_{1\mathrm{G}}(t)\hat{E}_1 + r_{2\mathrm{G}}(t)\hat{E}_2 + r_{3\mathrm{G}}(t)\hat{E}_3$, where $r_{\mathrm{iG}}(t)$ is the displacement of G wrt. O in the i-th direction respectively.
- On differentiating wrt time, we obtain the velocity of the centre of mass as:
  $\underline{v}^{\mathrm{G/O}}(t) = \dot{r}_{1\mathrm{G}}(t)\hat{E}_1 + \dot{r}_{2\mathrm{G}}(t)\hat{E}_2 + \dot{r}_{3\mathrm{G}}(t)\hat{E}_3$, where $\dot{r}_{\mathrm{iG}}(t)$ are the velocities in the i-th directions respectively.
- Considering the effect of gravity, we have:

$$\mathrm{m}\dot{\underline{v}^{\mathrm{G/O}}}(t) = \text{-mg}\hat{E}_3 \tag{1}$$

- So,we get :
  - $\ddot{\underline{r}_{1\mathrm{G}}}(t) = 0$
  - $\ddot{\underline{r}_{2\mathrm{G}}}(t) = 0$
  - $\ddot{\underline{r}_{3\mathrm{G}}}(t) = -g$

2.2 Euler's Equation For a torque-free Rigid Body

2.2.1 Calculation Of $\Omega(t)$ and $\underline{w}^B(t)$:

- The two frames are connected as: $\mathcal{E}_o \xrightarrow{\mathcal{R}_o(t)} \mathcal{E}(t)$
- So,we have $\hat{e_i}(t) = \mathcal{R}_o(t).\hat{E_i}$
- On rearranging, taking the time-derivative and using the property of rotation tensors (i.e, $\mathcal{R}_o(t).\mathcal{R}_o^T(t) = \mathcal{I}_d$) we get $\dot{\hat{e_i}}(t) = \Omega(t).\hat{e_i}(t)$, where $\dot{\mathcal{R}_o}(t).\mathcal{R}_o^T(t) = \Omega(t)$.
- We get the angular velocity corresponding to $\Omega(t)$ as $\underline{w}^B(t) = ax(\Omega(t))$.
- So, we have $\dot{\hat{e_i}}(t) = \underline{w}^B(t) \times \hat{e_i}(t)$.

2.2.2 Calculations Of $\underline{h}^G$ and $\underline{M}^G$:

- The Angular Momentum Vector about the **COM** is given as:
  $\underline{h}^G = \mathcal{I}^G.\underline{w}^B$.
- So, we have $\underline{h}^G = \mathcal{I}_1 w_1 \hat{\mathbf{e_1}} + \mathcal{I}_2 w_2 \hat{\mathbf{e_2}} + \mathcal{I}_3 w_3 \hat{\mathbf{e_3}}$.
- $\underline{M}^G = \underline{w}^B \times (\mathcal{I}^G.\underline{w}^B) + \mathcal{I}^G.\dot{\underline{w}}^B$.
- Here, in absence of any external torque, i.e, $\underline{M}^G = 0$.
- So, on substituting the values in the total moment equation, we get the **Euler's Equation Of Motion** as:
  - $\mathcal{I}_1 \mathbf{\dot{w}_1} - (\mathcal{I}_2 - \mathcal{I}_3)\mathbf{w_2 w_3} = \mathbf{0}$
  - $\mathcal{I}_2 \mathbf{\dot{w}_2} - (\mathcal{I}_3 - \mathcal{I}_1)\mathbf{w_3 w_1} = \mathbf{0}$
  - $\mathcal{I}_3 \mathbf{\dot{w}_3} - (\mathcal{I}_1 - \mathcal{I}_2)\mathbf{w_1 w_2} = \mathbf{0}$

2.3 Euler Angle Representation Of The Rotation Tensor $\mathcal{R}_o$

- We consider the 3-1-3 Euler Angle Representation of the entire Rotation Tensor, as shown below:
- $\mathcal{E}_o \xrightarrow{\mathcal{R}_\varphi, \hat{E_3}} \mathcal{E}' \xrightarrow{\mathcal{R}_\theta, \hat{e_1'}} \mathcal{E}" \xrightarrow{\mathcal{R}_\phi, \hat{e_3"}} \mathcal{E}$.
- Considering the above, w_i's are related to the Euler Angles and their rate of change as follows:
-
  - $\mathbf{w_1} = \dot{\psi}\sin\theta\sin\phi + \dot{\theta}\cos\phi$
  - $\mathbf{w_2} = \dot{\psi}\sin\theta\cos\phi - \dot{\theta}\sin\phi$
  - $\mathbf{w_3} = \dot{\psi}\cos\theta + \dot{\phi}$

2.4 Energy And Angular Momentum Ellipsoids

- We have, $\mathcal{K}_{rot} = \frac{1}{2} \times \underline{w}^B.(\mathcal{I}^G.\underline{w}^B)$.We can see that $\dot{K_{rot}} = 0$.
- So, we have that the total rotational Kinetic Energy ($\mathcal{K}_{rot}$) remains constant at it's initial value, let that value be given as $\mathcal{K}$.
- So, we have: $\mathcal{K} = \frac{1}{2}\mathcal{I}_1 w_1^2 + \frac{1}{2}\mathcal{I}_2 w_2^2 + \frac{1}{2}\mathcal{I}_3 w_3^2$.
- In absence of any external torque, i.e. $\underline{M}^G = 0$, we have that $\underline{h}^G$ remains constant at say it's initial value(magnitude) $\mathcal{H}$.
- So, we have: $\mathcal{H}^2 = \mathcal{I}^2{}_1 w_1^2 + \mathcal{I}^2{}_2 w_2^2 + \mathcal{I}^2{}_3 w_3^2$.

2.5 Non-dimensional Equations

- Non-dimensional angular momentum: $\bar{H}_{\mathrm{ic}} = \frac{H_{\mathrm{ic}}}{\sqrt{2\mathcal{K}\mathcal{I}_2}}$
- let $D = \frac{\mathcal{H}^2}{2\mathcal{K}}$, then non-dimensional D: $\bar{d} = \frac{D}{\mathcal{I}_2}$
- Non-dimensional time: $\bar{t} = \frac{t}{t_{\mathrm{r}}}$, where $t_{\mathrm{r}} = \sqrt{\frac{\mathcal{I}_2}{2\mathcal{K}}}$
- Non-dimensional Euler's Equations:
  * $\frac{d}{dt}\bar{H}_{1\mathrm{c}} + (1 - \frac{\mathcal{I}_2}{\mathcal{I}_3})\bar{H}_{2\mathrm{c}}\bar{H}_{3\mathrm{c}} = 0$
  * $\frac{d}{dt}\bar{H}_{2\mathrm{c}} + (\frac{\mathcal{I}_2}{\mathcal{I}_3} - \frac{\mathcal{I}_2}{\mathcal{I}_1})\bar{H}_{3\mathrm{c}}\bar{H}_{1\mathrm{c}} = 0$
  * $\frac{d}{dt}\bar{H}_{3\mathrm{c}} + (\frac{\mathcal{I}_2}{\mathcal{I}_1} - 1)\bar{H}_{1\mathrm{c}}\bar{H}_{2\mathrm{c}} = 0$
- Non-dimensional constant energy ellipsoid: $\frac{\mathcal{I}_2}{\mathcal{I}_1}\bar{H_{1\mathrm{c}}}^2 + \bar{H_{2\mathrm{c}}}^2 + \frac{\mathcal{I}_2}{\mathcal{I}_3}\bar{H_{3\mathrm{c}}}^2 = 0$
- Non-dimensional angular momentum sphere: $\bar{H_{1\mathrm{c}}}^2 + \bar{H_{2\mathrm{c}}}^2 + \bar{H_{3\mathrm{c}}}^2 = \bar{d}$,
  the radius $\sqrt{\bar{d}}$ satisfies $\frac{\mathcal{I}_3}{\mathcal{I}_2} \leq \bar{d} \leq \frac{\mathcal{I}_1}{\mathcal{I}_2}$

2.6 Geometrical Solution

- To get the geometrical solutions, we can find the intersection between the energy and the angular momentum ellipsoid.
- We initially eliminate $\bar{\mathcal{H}}_{1\mathrm{c}}$ from the two ellipsoids to get the projection of the trajectory on the $\bar{\mathcal{H}}_{2\mathrm{c}}$ $\bar{\mathcal{H}}_{3\mathrm{c}}$ plane to get the equation:

  $\bar{\mathcal{H}_{2\mathrm{c}}}^2(1 - \frac{\mathcal{I}_1}{\mathcal{I}_2}) + \bar{\mathcal{H}_{3\mathrm{c}}}^2(\frac{\mathcal{I}_\in}{\mathcal{I}_\ni} - \frac{\mathcal{I}_2}{\mathcal{I}_1}) = (1 - \frac{\mathcal{I}_2}{\mathcal{I}_1}\bar{d})$

- The coefficients of the angular momentum terms on the left hand side are both positive.Thus,this represents an ellipse if the right hand side does not vanish.If, the RHS vanishes then it depicts a state of permanent rotation corresponding to $\bar{\mathcal{H}}_{2\mathrm{c}} = 0, \bar{\mathcal{H}}_{3\mathrm{c}} = 0, \bar{\mathcal{H}}_{1\mathrm{c}} = \sqrt{\bar{d}}$.
- Similarly if projection on the $\bar{\mathcal{H}}_{1\mathrm{c}}$ $\bar{\mathcal{H}}_{2\mathrm{c}}$ plane to get the equation:
  We get an ellipse if $\bar{d} > \frac{\mathcal{I}_3}{\mathcal{I}_2}$. Else if there is an equality , we get a state of permanent rotation corresponding to $\bar{\mathcal{H}}_{1\mathrm{c}} = 0, \bar{\mathcal{H}}_{2\mathrm{c}} = 0, \bar{\mathcal{H}}_{3\mathrm{c}} = \sqrt{\bar{d}}$.
- If projection on the $\bar{\mathcal{H}}_{1\mathrm{c}}$ $\bar{\mathcal{H}}_{3\mathrm{c}}$ plane is considered,we get:

  $-\bar{\mathcal{H}_{1\mathrm{c}}}^2(1 - \frac{\mathcal{I}_2}{\mathcal{I}_1}) + \bar{\mathcal{H}_{2\mathrm{c}}}^2(\frac{\mathcal{I}_\in}{\mathcal{I}_\ni} - 1) = (1 - \bar{d})$

- Here, the coefficients on the LHS in the parantheses are positive.Hence, this is an equation of a Hyperbola, with equation of it's assymptotes as:
  $\mathcal{H}_{3\mathrm{c}} = \pm\sqrt{\frac{\mathcal{I}_{3\mathrm{c}}(\mathcal{I}_{1\mathrm{c}} - \mathcal{I}_{2\mathrm{c}})}{\mathcal{I}_{1\mathrm{c}}(\mathcal{I}_{2\mathrm{c}} - \mathcal{I}_{3\mathrm{c}})}}\mathcal{H}_{1\mathrm{c}}$
- If $\bar{d} > 1$ the vertices and focal points of the hyperbola are on the $\mathcal{H}_{1\mathrm{c}}$ axis.Therefore, the energy ellipsoid is closed and circulates about the $\mathcal{H}_{1\mathrm{c}}$ axis.
- If $\bar{d} < 1$ the vertices and focal points of the hyperbola are on the $\mathcal{H}_{3\mathrm{c}}$ axis.Therefore, the energy ellipsoid is closed and circulates about the $\mathcal{H}_{3\mathrm{c}}$ axis.
- If $\bar{d} = 1$ the trajectory becomes the assymptotes eqn. becoming sepatrices.

2.7 Solution of the Non-dimensional Euler's Equation

- $\bar{H_{1c}}$ and $\bar{H_{3c}}$ can be written as functions of $\bar{H_{2c}}$ as follows:
  - $\bar{H_{1c}}^2 = \frac{\bar{\mathcal{I}_1}(\bar{\mathcal{I}_2}-\bar{\mathcal{I}_3})}{\bar{\mathcal{I}_2}(\bar{\mathcal{I}_1}-\bar{\mathcal{I}_3})}(\bar{a}^2 - \bar{H_{2c}}^2)$
  - $\bar{H_{3c}}^2 = \frac{\bar{\mathcal{I}_3}(\bar{\mathcal{I}_1}-\bar{\mathcal{I}_2})}{\bar{\mathcal{I}_2}(\bar{\mathcal{I}_1}-\bar{\mathcal{I}_3})}(\bar{c}^2 - \bar{H_{2c}}^2)$

  where $\bar{a}^2 = \frac{D-\mathcal{I}_3}{\mathcal{I}_2-\mathcal{I}_3}$, $\bar{b}^2 = \frac{\mathcal{I}_1-D}{\mathcal{I}_1-\mathcal{I}_2}$ and
  $\bar{a}^2 - \bar{b}^2 = \frac{(D-\mathcal{I}_2)(\mathcal{I}_1-\mathcal{I}_3)}{(\mathcal{I}_2-\mathcal{I}_3)(\mathcal{I}_1-\mathcal{I}_2)}$.
- Substituting these values in Euler's second equation:
  $\frac{d\bar{H_{2c}}}{\sqrt{(\bar{a}^2-\bar{H_{2c}}^2)(\bar{b}^2-\bar{H_{2c}}^2)}} = -\sqrt{\frac{(\mathcal{I}_1-\mathcal{I}_2)(\mathcal{I}_2-\mathcal{I}_3)}{\mathcal{I}_1\mathcal{I}_3}}d\bar{t}$
- To solve this equation, 3 cases are considered as follows:
  - Considering $\bar{a}^2 < \bar{b}^2$ which leads to $D < \mathcal{I}_2$:

    Modifying the above equation gives us:
    $\int \frac{d(\bar{H_{2c}}/a)}{\sqrt{(1-(\bar{H_{2c}}/\bar{a})^2)(1-\frac{\bar{a}^2}{\bar{b}^2}(\bar{H_{2c}}/\bar{a})^2)}} = -\int \bar{b}\sqrt{\frac{(\mathcal{I}_1-\mathcal{I}_2)(\mathcal{I}_2-\mathcal{I}_3)}{\mathcal{I}_1\mathcal{I}_3}}d\bar{t}$
    The LHS is an Elliptic Integral of the 1st kind.Therefore, we get: $\bar{\mathcal{H}}_{2c} = -\bar{a}sn(\bar{\xi}(\bar{t}))$, where $\bar{\xi}(\bar{t}) = -\int \bar{b}\sqrt{\frac{(\mathcal{I}_1-\mathcal{I}_2)(\mathcal{I}_2-\mathcal{I}_3)}{\mathcal{I}_1\mathcal{I}_3}}(\bar{t} - \bar{t}_o)$.
    On using the equations:
    $sn^2\tau + cn^2\tau = 1$ and $dn^2\tau + k^2n^2\tau = 1$ where $k^2 = \bar{a}^2/\bar{b}^2$.
    We get:
    $\bar{\mathcal{H}_{1c}} = \pm\bar{a}\sqrt{\frac{\mathcal{I}_1(\mathcal{I}_2-\mathcal{I}_3)}{\mathcal{I}_2(\mathcal{I}_1-\mathcal{I}_3)}}cn(\xi(\bar{t}))$ and $\bar{\mathcal{H}_{3c}} = \pm\bar{b}\sqrt{\frac{\mathcal{I}_3(\mathcal{I}_1-\mathcal{I}_2)}{\mathcal{I}_2(\mathcal{I}_1-\mathcal{I}_3)}}dn(\xi(\bar{t}))$.

    The sign of each term is determined using the initial conditions and Euler's eqn of motion.So,if $\bar{\mathcal{H}_{3c}} > 0$,
    $\bar{\mathcal{H}_{1c}} = +\bar{a}\sqrt{\frac{\mathcal{I}_1(\mathcal{I}_2-\mathcal{I}_3)}{\mathcal{I}_2(\mathcal{I}_1-\mathcal{I}_3)}}cn(\xi(\bar{t}))$, $\bar{\mathcal{H}_{3c}} = +\bar{b}\sqrt{\frac{\mathcal{I}_3(\mathcal{I}_1-\mathcal{I}_2)}{\mathcal{I}_2(\mathcal{I}_1-\mathcal{I}_3)}}dn(\xi(\bar{t}))$ and $\bar{\mathcal{H}_{2c}} = -\bar{a}sn(\xi(\bar{t}))$
    and the opposite signs for $\bar{\mathcal{H}_{1c}}$ and $\bar{\mathcal{H}_{3c}}$ provided $\bar{\mathcal{H}_{3c}} < 0$.

    where, $\bar{a}, \bar{b}$ are given as:
    $\bar{a} = \pm\sqrt{\frac{D-\mathcal{I}_3}{\mathcal{I}_2-\mathcal{I}_3}}$ and $\bar{b} = \sqrt{\frac{\mathcal{I}_1-D}{\mathcal{I}_1-\mathcal{I}_2}}$.
    The non-dimensional period of the Jacobi elliptic functions is obtained from the complete elliptic integral of the first kind where sn and cn are 4K periodic and dn is 2K periodic.
  - Considering $\bar{a}^2 > \bar{b}^2$ which leads to $D > \mathcal{I}_2$:

    Modifying the above equation gives us:
    $\int \frac{d(\bar{H_{2c}}/\bar{b})}{\sqrt{(1-(\bar{H_{2c}}/\bar{b})^2)(1-\frac{\bar{b}^2}{\bar{a}^2}(\bar{H_{2c}}/\bar{b})^2)}} = -\int \bar{a}\sqrt{\frac{(\mathcal{I}_1-\mathcal{I}_2)(\mathcal{I}_2-\mathcal{I}_3)}{\mathcal{I}_1\mathcal{I}_3}}d\bar{t}$
    In a similar way,
    We get:
    If $\bar{\mathcal{H}_{1c}} > 0$,
    $\bar{\mathcal{H}_{1c}} = +\bar{a}\sqrt{\frac{\mathcal{I}_1(\mathcal{I}_2-\mathcal{I}_3)}{\mathcal{I}_2(\mathcal{I}_1-\mathcal{I}_3)}}dn(\eta(\bar{t}))$, $\bar{\mathcal{H}_{3c}} = +\bar{b}\sqrt{\frac{\mathcal{I}_3(\mathcal{I}_1-\mathcal{I}_2)}{\mathcal{I}_2(\mathcal{I}_1-\mathcal{I}_3)}}cn(\eta(\bar{t}))$ and $\bar{\mathcal{H}_{2c}} = -\bar{a}sn(\eta(\bar{t}))$
    and the opposite signs for $\bar{\mathcal{H}_{1c}}$ and $\bar{\mathcal{H}_{3c}}$ provided $\bar{\mathcal{H}_{3c}} < 0$.
    where, $\eta(\bar{t}), \bar{a}, \bar{b}$ are given as:
    $\eta(\bar{t}) = -\int \bar{a}\sqrt{\frac{(\mathcal{I}_1-\mathcal{I}_2)(\mathcal{I}_2-\mathcal{I}_3)}{\mathcal{I}_1\mathcal{I}_3}}(\bar{t} - \bar{t}_o)$, $\bar{a} = \sqrt{\frac{D-\mathcal{I}_3}{\mathcal{I}_2-\mathcal{I}_3}}$ and $\bar{b} = \pm\sqrt{\frac{\mathcal{I}_1-D}{\mathcal{I}_1-\mathcal{I}_2}}$.
  - Considering $\bar{a}^2 = \bar{b}^2$ which leads to $D = \mathcal{I}_2$:

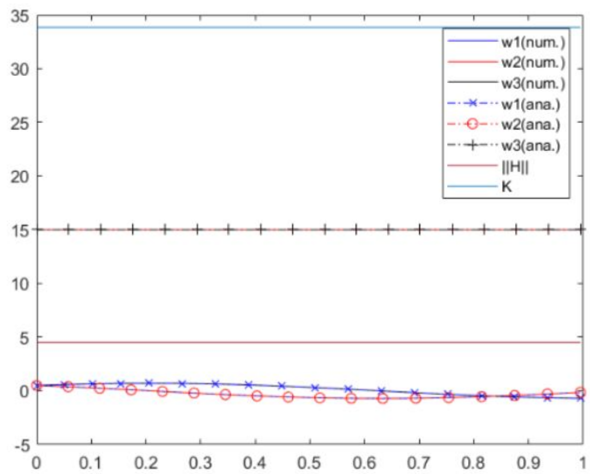
    Modifying the above equation gives us:
    $\int \frac{d(\bar{H_{2c}})}{\sqrt{(\bar{a}^2-(\bar{H_{2c}})^2)}} = -\int \sqrt{\frac{(\mathcal{I}_1-\mathcal{I}_2)(\mathcal{I}_2-\mathcal{I}_3)}{\mathcal{I}_1\mathcal{I}_3}}d\bar{t}$

In a similar way,
We get: $tanh^{-1}(\bar{\mathcal{H}}_{2c}/\bar{a}) = -\bar{a}\sqrt{\frac{(\mathcal{I}_1-\mathcal{I}_2)(\mathcal{I}_2-\mathcal{I}_3)}{\mathcal{I}_1\mathcal{I}_3}}(\bar{t}-\bar{t}_o)$,
So, $\bar{\mathcal{H}}_{2c} = -\bar{a}tanh(\eta(\bar{t}))$ where, $\eta(\bar{t})$ is given as:
$\eta(\bar{t}) = \bar{a}\sqrt{\frac{(\mathcal{I}_1-\mathcal{I}_2)(\mathcal{I}_2-\mathcal{I}_3)}{\mathcal{I}_1\mathcal{I}_3}}(\bar{t}-\bar{t}_o)$.
So,we finally have: $\bar{\mathcal{H}}_{1c} = \pm\bar{a}\sqrt{\frac{\mathcal{I}_1(\mathcal{I}_2-\mathcal{I}_3)}{\mathcal{I}_2(\mathcal{I}_1-\mathcal{I}_3)}}\frac{1}{cosh(\eta(\bar{t}))}$ and $\bar{\mathcal{H}}_{3c} = \pm\bar{a}\sqrt{\frac{\mathcal{I}_3(\mathcal{I}_1-\mathcal{I}_2)}{\mathcal{I}_2(\mathcal{I}_1-\mathcal{I}_3)}}\frac{1}{cosh(\eta(\bar{t}))}$.
In this case, the solutions are not periodic whereas the solutions of case 1 and case 2 are periodic.
Therefore, as $\eta(\bar{t})$ approaches ∞,$\bar{\mathcal{H}}_{1c} = \bar{\mathcal{H}}_{3c} = 0$ and $\bar{\mathcal{H}}_{2c}$ approaches $-\bar{a}$.
As,$\bar{d}$ approaches 1,the solutions of **Case 1** and **Case 2** converge.

3 Axisymmetric Cases :

- The analytical solutions for the cases involving equal moments of inertia are obtained by considering the limiting cases of the above solutions obtained.
- Initially, the case where the intermediate mass moment of inertia approaches the largest moment of inertia is considered.
  - So, we have $\mathbf{I_2} = \mathbf{I_1} - \epsilon$, where $|\epsilon| << 1$.
  - On substituting the above eqn. in **CASE 1** ,and taking limit as ϵ goes to **0** we get, for $\mathbf{H_{3c}} > 0$:
    $\bar{H}_{1c} = a_s\cos\xi_s(\bar{t})$
    $\bar{H}_{2c} = -a_s\sin\xi_s(\bar{t})$
    $\bar{H}_{3c} = \sqrt{\frac{I_3(I_1-D)}{I_1(I_1-I_3)}}$ = constant,where
    $\bar{a}_s = \pm\sqrt{\frac{D-I_3}{I_1-I_3}}$ and $\xi_s(\bar{t}) = \sqrt{\frac{(I_1-D)(I_1-I_3)}{I_1I_3}}(\bar{t}-\bar{t}_o)$.
  - The signs of $\bar{H}_{1c}$ and $\bar{H}_{3c}$ are reversed for $\bar{H}_{3c} < 0$.

```
I1 = 0.4000
I2 = 0.4000
I3 = 0.3000
w1_0 = 0.5000
w2_0 = 0.5000
w3_0 = 15
```

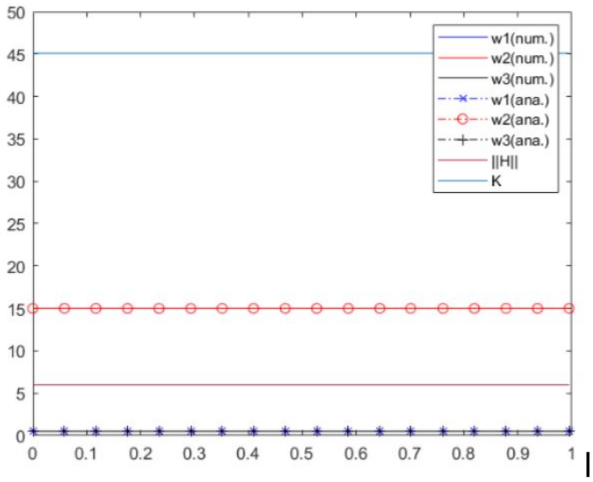


- Now, considering the case where the Intermediate moment of inertia approaches the smallest moment of inertia.
  - So, we have $\mathbf{I_2} = \mathbf{I_3} + \epsilon$, where $|\epsilon| << 1$.

- On substituting the above eqn. in **CASE 2** ,and taking limit as ϵ goes to **0** we get, for $\mathbf{H_{1c}} > 0$:
  $\bar{H}_{3c} = \bar{b}_s \cos \xi_s(\bar{t})$
  $\bar{H}_{2c} = -\bar{b}_s \sin \xi_s(\bar{t})$
  $\bar{H}_{1c} = \sqrt{\frac{I_1(D-I_3)}{I_3(I_1-I_3)}}$ = constant,where
  $\bar{b}_s = \pm\sqrt{\frac{I_1-D}{I_1-I_3}}$ and $\eta_s(\bar{t}) = \sqrt{\frac{(I_1-I_3)(D-I_3)}{I_1 I_3}}(\bar{t} - \bar{t}_o)$.
- The signs of $\bar{H}_{1c}$ and $\bar{H}_{3c}$ are reversed for $\bar{H}_{1c} < 0$.

- Now, considering the case wherein all the three moment's of inertia are equal :
  On substituting $\mathbf{I_2} = \mathbf{I_3} = \mathbf{I_1}$,in the Non-Dimensional Euler Eqns, we get :
  $\bar{H}_{1c} = \bar{H}_{1c}(0)$, $\bar{H}_{2c} = \bar{H}_{2c}(0)$, $\bar{H}_{3c} = \bar{H}_{3c}(0)$,
  that is all of them remain constant at their initial values.

```
I1 = 0.4000
I2 = 0.4000
I3 = 0.4000
w1_0 = 0.5000
w2_0 = 15
w3_0 = 0.5000
```



4 Pure Spin State Cases :

- Considering 3 different Moments Of Inertia and considering pure spin state about **2nd axis**.

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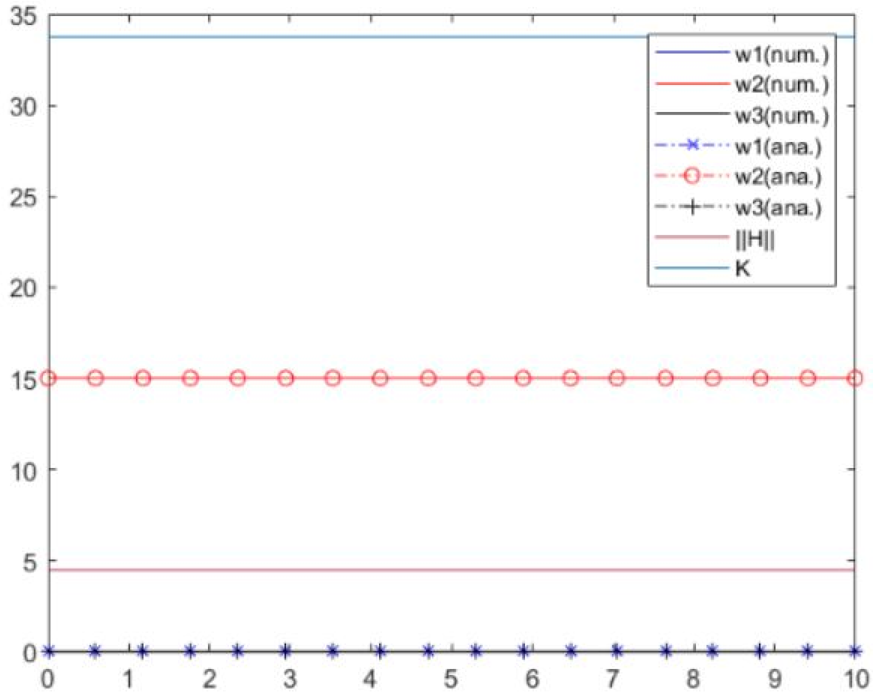
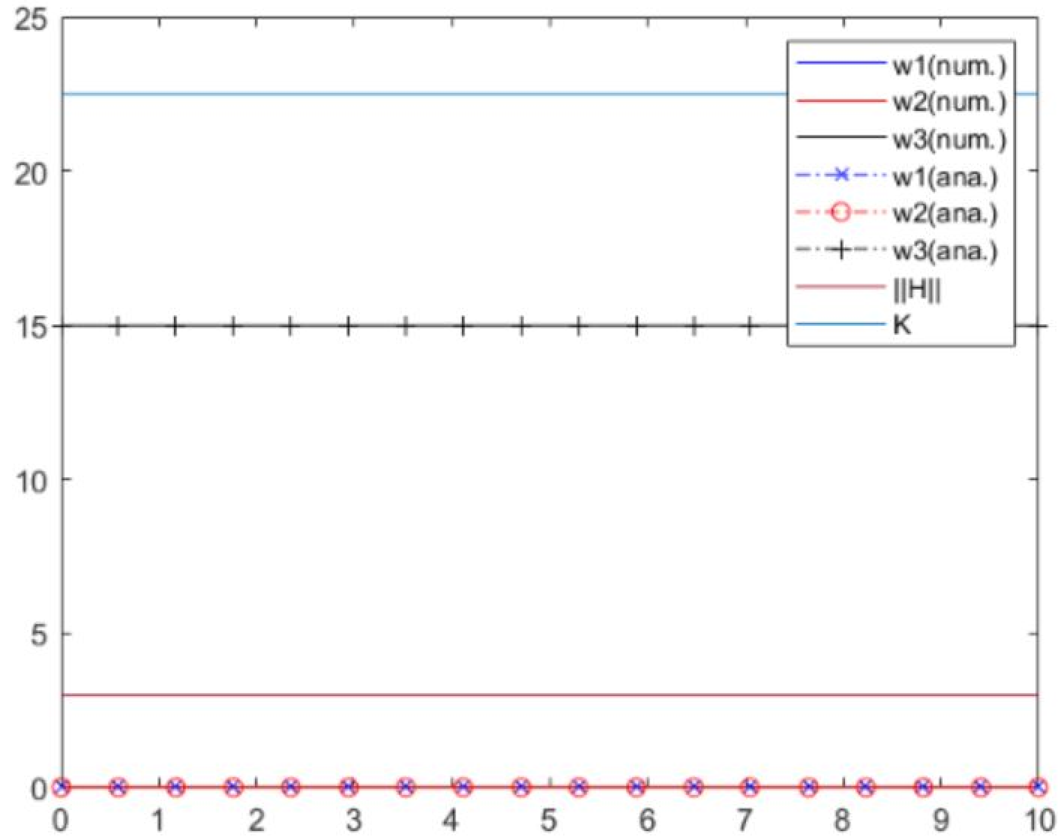
I1 = 0.4000
I2 = 0.3000
I3 = 0.2000
w1_0 = 0
w2_0 = 15
w3_0 = 0
```



- Considering 3 different Moments Of Inertia and considering pure spin state about **3rd axis**.

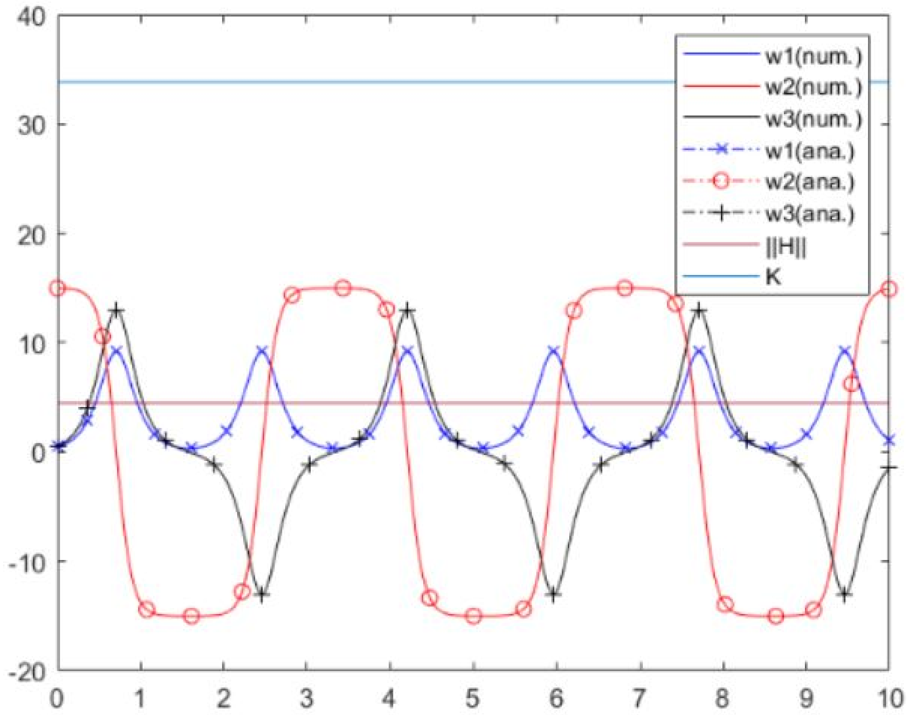
```
I1 = 0.4000
I2 = 0.3000
I3 = 0.2000
w1_0 = 0
w2_0 = 0
w3_0 = 15
```



5 Validation :

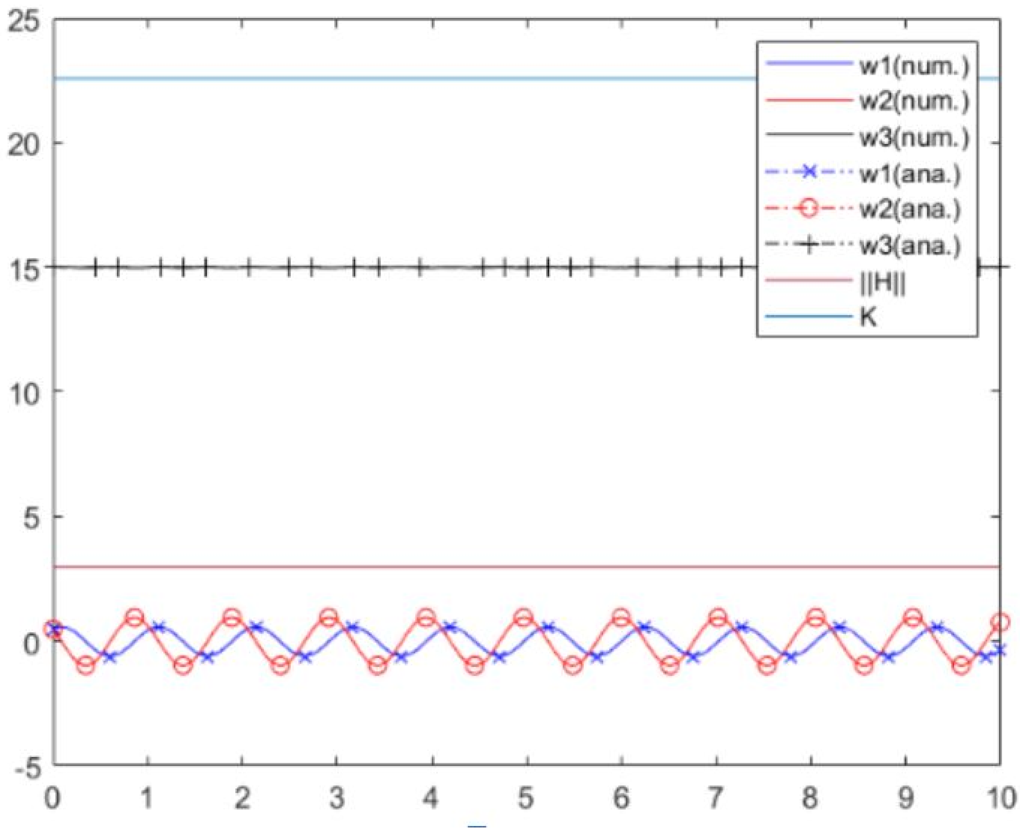
- Considering 3 different Moments Of Inertia and considering spin about **2nd axis** and slight perturbations about the other two.

```
I1 = 0.4000
I2 = 0.3000
I3 = 0.2000
w1_0 = 0.5000
w2_0 = 15
w3_0 = 0.5000
```



- Considering 3 different Moments Of Inertia and considering spin about **3rd axis** and slight perturbations about the other two.

```
I1 = 0.4000
I2 = 0.3000
I3 = 0.2000
w1_0 = 0.5000
w2_0 = 0.5000
w3_0 = 15
```



6 Finding Euler's Angles

- We have made a system of First-Order Differential Equations that solve for the motion of the body over time.

- These equations are represented in the form :
$M(t, y)\bar{y} = f(t, y)$.

- We used this equation for Numerical Integration in MATLAB.

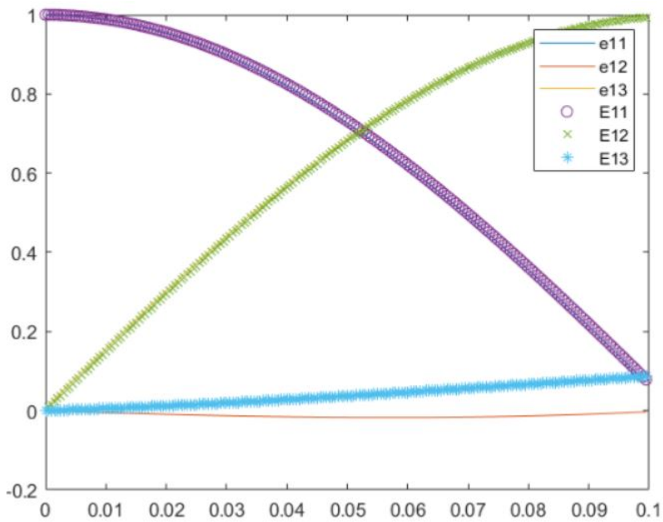
- M(t,y) = $\begin{bmatrix} I_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sin\theta\sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & \sin\theta\cos\phi & -\sin\phi & 0 \\ 0 & 0 & 0 & \cos\theta & 0 & 1 \end{bmatrix}$ and f(t,y) = $\begin{bmatrix} -(I_3 - I_2)w_2w_3 \\ -(I_1 - I_3)w_1w_3 \\ -(I_2 - I_1)w_2w_1 \\ w_1 \\ w_2 \\ w_3 \end{bmatrix}$ and y = $\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ \psi \\ \theta \\ \phi \end{bmatrix}$

7 Another Method For Finding Euler Angles:

- Since the components of the angular velocity and angular momentum vectors have been obtained with respect to the moving frame, we must express the vector components in the inertial frame.
- Thus, even the rotation matrix R(t) must be updated every time with the angular velocity.
- We have : $R\dot{(}t) = R(t)w(t)$,
- The eqn is integrated from t to t + δt considering the angular velocity w(t) to be constant at w_o.
- Considering the angular velocity constant and the initial value of R(t) as equal to R(0), we get : $R(t) = R(0)exp(tw_o)$.
- The matrix exponential is defined as : $exp(t) = I_d + tw_o + \frac{1}{2}t^2w_o^2$ + ... =
$$\sum_{k=0}^{\infty} \frac{t^k}{k!}(w_o)^k$$
- This eqn. on solving using Cayley-Hamiltonian Theorem: $exp(tw_o) = I_d + \frac{w_o}{||w_o||} \sin t||w_o|| + (\frac{w_o}{||w_o||})^2(1 - \cos t||w_o||)$.
- While the constant angular velocity is not met under the Dzhanibekov phenomena, it can be applied by approximation to each time step of the Runge-Kutta integration. From this perspective, the mid-point integration method is adopted by using the mean value of the angular velocity,as : $w(t + \delta t) = (w(t) + w(t + \delta t))/2$.
- Assuming that the angular velocity is constant during the time step from t to $t + \delta t$ and the initial eqn. is integrated analytically for the initial value of R(t) to find R(t + δt): $R(t + \delta t) = R(t)exp(\delta t * w((t + \delta t)/2)$.

8 Comparing Euler Angles Found By Both Methods :

- **E11,E12,E13** are from direct integration of ODEs.
e11,e12,e13 are from approximation that **w** is constant between t , $t + \delta t$.



The link to the model is : here.

9 Understanding Jacobi Elliptic Integrals:

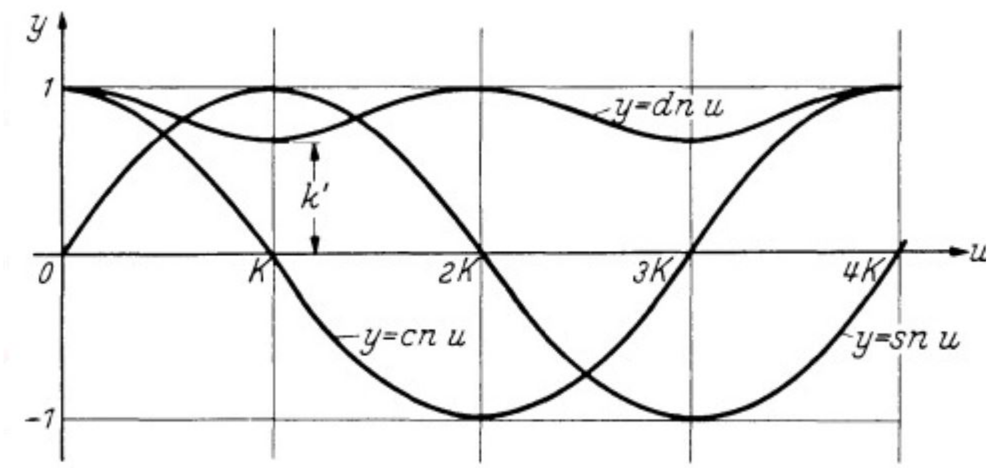
- For studying Elliptic Functions,we have to study the differential equation:
  $(\frac{dy}{dx})^2 = (1-y^2)*(1-k^2y^2)$, where k is a constant such that 0<k<1.
- We define the Jacobian Elliptic Function snx to be that solution of the above differential equation which satisfies
  $y = 0$ and $\frac{dy}{dx} > 0$ at $x = 0$.
- Here, k is known as the modulus of the function.
  We can write sn(x) as sn(x,k) but we drop the k for convenience.
- The general solution for the differential eqn. is given as $y = sn(x+c)$ for some constant c since even on translating the curve along the x-direction gives a solution as the relation between the slope and ordinate remains unchanged.
- Due to certain properties we get to know that $sn(x)$ is a periodic function, i.e. we have $sn(x+4K) = sn(x)$,where $K = \int_0^1 \frac{dy}{\sqrt{(1-y^2)(1-k^2y^2)}}$.
- Thus,all in all we have :
  $(\frac{d(snx)}{dx})^2 = (1-sn^2(x))(1-k^2sn^2(x))$.
  $sn(0) = 0$
  $sn(x+4K) = sn(x)$.
- We define other elliptic functions as :
  $cn^2x = 1 - sn^2x, cn(0) = 1$
  $dn^2x = 1 - k^2sn^2x, dn(0) = 1$.
- sn and cn have a period of 4K while dn has a period of 2K.
- 

Fig. 10.

10 Motion Of A Simple Pendulum In Terms Of Elliptic Functions:

- The energy equation for a simple pendulum with a bob of mass m and length a is given as:
  $E = \frac{1}{2}ma^2\dot{\theta}^2 - mgacos\theta$.
- Let the amplitude of the motion be α so that at the extreme positions i.e. $\theta = \pm\alpha$, we have $\dot{\theta} = 0$.
  We get $E = -mgacos\alpha$ at this position.

- Equating the above 2 equations for E ,we get:
$\dot{\theta}^2 = 4p^2(\sin^2 \alpha/2 - \sin^2 \theta/2) - - - - - - - - - - - (*)$
where $p^2 = \frac{g}{a}$.

- Let us define ϕ as:
$\sin(\theta/2) = \sin(\alpha/2)\sin\phi$.On differentiating wrt. time(t) :
$\frac{1}{2}\cos(\theta/2)\dot{\theta} = \sin(\alpha/2)\cos\phi(\dot{\phi})$.

- Multiplying (*) by $\frac{1}{4}(\cos^2\theta/2)\dot{\theta}^2$ on both sides, we get:
$\dot{\phi}^2 = p^2(1 - (\sin^2\alpha/2)(\sin^2\phi))$.

- On multiplying this by $\cos^2\phi$ on both the sides and substituting $y = \sin\phi$ and $k = \sin\alpha/2$,we get:
$\dot{y}^2 = p^2(1 - y^2)(1 - k^2y^2)$.

- This seems similar to the differential equation corresponding to the Elliptic Functions except the factor **p**.

- Hence, we make a substitution $x = pt$,thus getting:
$(\frac{dy}{dx})^2 = (1 - y^2) * (1 - k^2y^2)$, the general solution of which is $y = sn(x + c)$.

- Thus, we get : $\sin(\theta/2) = \sin(\alpha/2)sn[p(t - t_o)]$ and $\cos(\theta/2) = dn[p(t - t_o)]$, where t_o is a constant of integration.

- On differentiating, we get : $\dot{\theta} = 2\sin(\alpha/2)cn[p(t - t_o)]$.

- Thus, we can see that the motion will repeat itself after a period of
$4K/p = \frac{4}{p}\int_0^1 \frac{dy}{\sqrt{(1-y^2)(1-k^2y^2)}}$.

- Putting $y = \sin\phi$, we get: $T = \frac{4}{p}\int_0^{\pi/2} \frac{d\phi}{1-k^2\sin^2\phi}$.

- Using the Taylor's Expansion Series, we get:$T = \frac{4}{p}\int_0^{\pi/2}(1 + \frac{1}{2}k^2\sin^2\phi + \frac{1.3}{2.4}k^4\sin^4\phi + ...)d\phi$.

- We get: $T = \frac{2\pi}{p}[1 + \frac{1}{2}^2k^2 + \frac{1.3}{2.4}^2k^4 + ...]$.

- For very small amplitude : $T = 2\pi\sqrt{\frac{a}{g}}$.

- Considering the next approximation, we have : $T = 2\pi\sqrt{\frac{a}{g}}(1 + \frac{\alpha^2}{16})$. Thus,we conclude that the Time Period (T) increases with the amplitude (α).