2) $A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -1 \end{bmatrix}$

$$A = U \Sigma V^T$$

$$A A^T = U \Sigma V^T V \Sigma^T \cancel{A} U^T$$

$$= U \Sigma \Sigma^T U^T$$

$$= U (\Sigma^2) U^T$$

not exactly Σ^2 but diagonals elements are squreos of σ_i^2

$$A A^T = U (\Sigma^2) U^T$$

$\Rightarrow$ entries of Σ^2 = eigen values of $A A^T$

$\Rightarrow$ entries of Σ (singular values of A) $= \sqrt{\text{eigen values of } A A^T}$

→ Columns of U = eigen vectors of $A A^T$.

→ Columns of V = " " " $A^T A$.

$$AA^T = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -1 \end{bmatrix} \times \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 17 & 10 \\ 10 & 14 \end{bmatrix}$$

$$|AA^T - \lambda I| = 0$$

$$\Rightarrow \quad (17-\lambda)(14-\lambda) - 100 = 0$$

$$\lambda^2 - 31\lambda + 138 = 0$$

$$\lambda = \frac{31 \pm \sqrt{823}}{2}$$

eigen vectors of AA^T are given by

$$AA^T v_i = \lambda_i v_i$$

$$(AA^T \lambda_i I) v_i = 0$$

$$\begin{bmatrix} 17-\lambda_i & 10 \\ 10 & 14-\lambda_i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$(17-\lambda_i)x + 10y = 0$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ \lambda_i - 17 \end{bmatrix}$$

→ Columns of U = eigen vectors of AA^T of unit magnitude.

→ eigen values of A^TA are $\lambda_1, \lambda_2, 0$

⇒ eigen vectors of A^TA

$$A^TA\, v_i = \lambda_i v_i$$

$$\Rightarrow (A^TA - \lambda_i I)\, v_i = 0$$

↳ $0, \lambda_1, \lambda_2$

→ rows of V^T = eigen vectors of A^TA of unit magnitude.

$$A = \begin{bmatrix} | & | \\ u_1 & u_2 \\ | & | \end{bmatrix} \begin{bmatrix} \sqrt{\lambda_1} & 0 & 0 \\ 0 & \sqrt{\lambda_2} & 0 \end{bmatrix} \times \begin{bmatrix} - & v_1^T & - \\ - & v_2^T & - \\ - & v_3 & - \end{bmatrix}$$

$$A = \begin{bmatrix} | & | \\ u_1 & u_2 \\ | & | \end{bmatrix} \left(\begin{bmatrix} \sqrt{\lambda_1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \sqrt{\lambda_2} & 0 \end{bmatrix} \right) \begin{bmatrix} - & v_1^T & - \\ - & v_2^T & - \\ - & v_3^T & - \end{bmatrix}$$

$$= \sqrt{\lambda_1}\, \underbrace{u_1 v_1^T}_{\text{rank-1}} + \sqrt{\lambda_2}\, \underbrace{u_2 . v_2^T}_{\text{rank-1}}$$