S-520 Intro to Stats
Problem Set 4

Ans. 1 PDF:

$$f(x) = \begin{cases} 1/20, & 20 \le x < 40 \\ 0, & \text{otherwise} \end{cases}$$

a) $f(x)$ is $1/20$ for $20 \le x < 40$ & 0 otherwise.
This means that it is given that it is non negative for all values of x.
To show area under $f(x) = 1$

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

$$= \underbrace{\int_{-\infty}^{20} f(x)\, dx}_{0} + \int_{20}^{40} f(x)\, dx + \underbrace{\int_{40}^{\infty} f(x)\, dx}_{0}$$

$$= \int_{20}^{40} f(x)\, dx = \int_{20}^{40} 1/20\, dx$$

$$= \frac{1}{20} \times [x]_{20}^{40} = \frac{1}{20}[40-20]$$

$$= \boxed{1}$$

$\therefore$ f is a PDF

(b) $F(y) = P(X \leq y) = \int_{-\infty}^{y} f(x)\, dx$

$= \int_{20}^{y} f(x)\, dx = \int_{20}^{y} \frac{1}{20}\, dx$

$= \frac{1}{20} [x]_{20}^{y}$

$\boxed{F(y) = \frac{y-20}{20}}$

$$\boxed{F(y) = \begin{cases} 0 & y < 20 \\ \frac{y-20}{20} & 20 \leq y < 40 \\ 1 & y \geq 40 \end{cases}}$$

(c) $P(30 \leq X < 50)$

$= \int_{30}^{50} f(x)\, dx = \int_{30}^{40} f(x)\, dx + \int_{40}^{50} \underbrace{f(x)\, dx}_{0}$

$= \int_{30}^{40} \frac{1}{20}\, dx = \frac{1}{20} [x]_{30}^{40}$

$= \frac{10}{20} = \boxed{\frac{1}{2}}$

(d) $E(x) = \int_{20}^{40} x \cdot f(x)\, dx$

$= \frac{1}{20} \times \left[\frac{x^2}{2}\right]_{20}^{40}$

$= \frac{1}{4\cancel{0}} \times 120\cancel{0} = \boxed{30}$

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(e) Variance of a uniform random variable $= \frac{(b-a)^2}{12}$

$$= \frac{(40-20)^2}{12}$$

$$= \frac{400}{12} = \boxed{33.33}$$

Standard Deviation $(x) = \sqrt{Var(x)} = \sqrt{33.33}$

$$= \boxed{5.7732|14}$$

Ans. 2 a)

$$f(x) = \begin{cases} 1/30 & 0 \le x < 20 \\ 1/60 & 20 \le x < 40 \\ 0 & \text{otherwise} \end{cases}$$

$$F(y) = P(X \le y) = \int_{-\infty}^{y} f(y)\, dy$$

Ans.

$$\boxed{F(y) = \begin{cases} 0 & y < 0 \\ y/30 & 0 \le y < 20 \\ \frac{y+20}{60} & 20 \le y < 40 \\ 1 & y > 40 \end{cases}}$$

$$= \int_{0}^{20} \frac{1}{30}\, dy + \int_{20}^{y} \frac{1}{60}\, dy$$

$$= \frac{1}{30} \times [y]_0^{20} + \frac{1}{60} [y]_{20}^{y}$$

$$= \frac{2}{3} + \frac{y-20}{60}$$

(b) let x be a random variable that denotes length of expected waiting time

Suppose person arrives b/w 9 to 9:40, then $x \in (0, 20)$

Suppose person arrives between ~~from~~ ~~9:40~~ forty minutes past the hour to 20 mins past the hour. $x \in (20, 40)$

$\therefore x$ goes from $(0, 40)$

$$\therefore E(x) = \int_0^{40} x f(x)$$

$$= \int_0^{20} x f(x) + \int_{20}^{40} x f(x)$$

$$= \int_0^{20} x \times \frac{1}{30} + \int_{20}^{40} x \times \frac{1}{60}$$

$$= \frac{1}{30}\left[\frac{x^2}{2}\right]_0^{20} + \frac{1}{60} \times \left[\frac{x^2}{2}\right]_{20}^{40}$$

$$= \frac{1}{3\cancel{60}}[\overset{20}{\cancel{400}}] + \frac{1}{12\cancel{0}}[120\cancel{0}]$$

$$\boxed{E(x) = \frac{50}{3}} = 16 \text{ mins } 40 \text{ secs}$$

(c) $F(y) = 0.5$

$$\frac{2}{3} + \frac{y-20}{60} = 0.5$$

$$\Rightarrow \frac{40+y-20}{60} = 0.5$$

$$\Rightarrow 20+y = 30$$

$$\Rightarrow \times \boxed{y = 10}$$ ~~It is smaller than E(x)~~ → This doesn't lie b/w 20 & 40

$$\Rightarrow \frac{y}{30} = \frac{0.5}{2}$$

$$\Rightarrow \checkmark \boxed{y = 15}$$ which lies b/w 0 & 20

Ans.

$\therefore \checkmark \boxed{y = 15}$. It is smaller than $E(x)$

Ans. 3(b) $f(x) = \begin{cases} 0 & \text{if } x < 1 \\ 2(x-1) & \text{if } 1 \le x \le 2 \\ 0 & \text{if } x > 2 \end{cases}$

(b) f is a pdf if:

(i) $f(x) \ge 0$ i.e it is non negative for all values of x.
(ii) $\int f(x)\,dx = 1$.

$= \int_1^2 (2x-2)\,dx$

$= \frac{\cancel{2}}{\cancel{2}} \times [x^2]_1^2 - 2 \times [x]_1^2$

$= 3 - (2 \times 1)$

$= 1$

Hence f is non negative for all values of x
& $\int_x f(x)\,dx = 1$ for all values of x.

(c) $P(1.5 < x < 1.75)$

$= \int_{1.5}^{1.75} f(x)\,dx = \int_{1.5}^{1.75} (2x-2)\,dx$

$= \frac{\cancel{2}}{\cancel{2}} \times [x^2]_{1.5}^{1.75} - 2\,[x]_{1.5}^{1.75}$

$= [3.0625 - 2.25] - 2[0.25]$

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$= \boxed{0.3125}$

Ans.4 $X \sim N(-5, 100)$ Here $\mu = -5$ & $\sigma = 10$

a) $P(X<0)$

$= P\left[\frac{X-\mu}{\sigma} < \frac{0+5}{10}\right]$

$= P\left[Z < 0.5\right]$

# R code
pnorm(0.5, 0, 1)

$= 0.6914625$

(b) $P(X>5)$

$= 1 - P(X<5)$

$= 1 - P\left[\frac{X-\mu}{\sigma} < \frac{5+5}{10}\right]$

$= 1 - P\left[Z < 1\right]$

= # Rcode

$= 1 - \text{pnorm}(1, 0, 1)$

$= 1 - 0.8413447$

$= \boxed{0.1586553}$

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(c) $P(-3 < X < 7)$

$P\left(\frac{-3+5}{10} < \frac{x-\mu}{6} < \frac{7+5}{10}\right)$

$P(0.2 < Z < 1.2)$

Rcode

$=$ pnorm(1.2, 0, 1) − pnorm(0.2, 0, 1)

$= \boxed{0.3056706}$

(d) $P(|x+5| < 10)$

$= P(-10 < X+5 < 10)$

$= P(-15 < X < 5)$

$= P\left(\frac{-15+5}{10} < \frac{X-\mu}{6} < \frac{5+5}{10}\right)$

$= P(-1 < Z < 1)$

Rcode

$=$ pnorm(1, 0, 1) − pnorm(-1, 0, 1)

$= \boxed{0.6826895}$

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(e) $P(|X-3| > 2)$

$= 1 - P(|X-3| < 2)$

$= 1 - P(-2 < X-3 < 2)$

$= 1 - P(1 < X < 5)$

$= 1 - P\left(\frac{1+5}{10} < \frac{X-\mu}{\sigma} < \frac{5+5}{10}\right)$

$= 1 - P(0.6 < Z < 1)$

$= 1 - [\text{pnorm}(1,0,1) - \text{pnorm}(0.6,0,1)]$

$= 1 - 0.1155979$

$= \boxed{0.8844021}$

Ans. 5 $X \sim N(1, 9)$

Here $\mu_1 = 1$ $\sigma_1^2 = 9$

$X_2 \sim N(3, 16)$

Here $\mu_2 = 3$ $\sigma_2^2 = 16$

(a) $X_1 + X_2 \sim N(1+3, 9+16)$

$\boxed{X_1 + X_2 \sim N(4, 25) \quad \therefore \mu = 4 \text{ \& } \sigma^2 = 25}$

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(b) $-X_2 \sim N(-1 \times 3, (-1)^2 \times 16)$

$\sim N(-3, 16)$

$\therefore \boxed{\mu = -3 \text{ , } \sigma^2 = 16}$

(c) $X_1 - X_2 \sim N(1 \times 1 - 1 \times 3, (1)^2 \times 9 + (-1)^2 \times 16)$

$\sim N(-2, 25)$

$\therefore \boxed{\mu = -2 \text{ , } \sigma^2 = 25}$

(d) $2X_1 \sim N(2 \times 1, (2)^2 \times 9)$

$\sim N(2, 36)$

$\boxed{\mu = 2 \text{ , } \sigma^2 = 36}$

(e) $2X_1 - 2X_2 \sim N(2 \times 1 - 2 \times 3, 2^2 \times 9 + (-2)^2 \times 16)$

$\sim N(-4, 100)$

$\boxed{\mu = -4 \text{ , } \sigma^2 = 100}$

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Ans. 6 let BPM be a normal random variable X

$X \sim N(120, 400)$

a) $P(115 < X < 135)$

$P\left(\frac{115-120}{20} < \frac{X-\mu}{\sigma} < \frac{135-120}{20}\right)$

$P\left(\frac{-5}{20} < Z < \frac{15}{20}\right)$

$P(-0.25 < Z < 0.75)$

Rcode

= pnorm(0.75, 0, 1) - pnorm(-0.25, 0, 1)

[1] 0.372079

(b) $P(X > 160)$

∵ 10 discs have probability of BPM < 160

$= 1 - P(X < 160)^{10}$

$= 1 - \left[P\left(\frac{X-\mu}{\sigma} < \frac{160-120}{20}\right)\right]^{10}$

$= 1 - [P(Z < 2)]^{10}$

Rcode

= ~~1 - 0.97~~ a<-1-[pnorm(2,0,1)]^10

print(a)

[1] $\boxed{0.205569}$

c) X_1 & X_2 be the randomly distributed BPMs of the 2 songs

$P(X_1+X_2) > 320$

$= 1 - P(X_1+X_2 < 320)$

$\Rightarrow E(X_1+X_2) = 120+120 = 240$

$Var(X_1+X_2) = 1^2 Var(X_1) + 1^2 Var(X_2) = 800$

$Sd(X_1+X_2) = \sigma = \sqrt{800}$

R code

= 1 - pnorm(320, 240, 800^(1/2))

$= \boxed{0.002338867}$

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