Discrete Signal Processing on Graphs: Sampling Theory Report

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1 Abstract

The overall aim of this paper is to be able to perfectly reconstruct a given graph(directed or undirected). Formulating a proper sampling theory(which works on both directed and undirected graphs) is the main objective of this paper. Generally sampling theory on graphs are not very well understood since graphs lie on complicated and irregular structures. Traversing a graph cannot be done very simply with a proper sampling theory in mind. This is much more easily done in a 2D format as we can see with the Nyquist-Shannon sampling theorem. This paper covers sampling through the usage of discrete signal processing and uses simple linear algebra. It also covers the the process in which the sampling operator is designed for a given graph. We generally search for the qualified sampling operator which is designed to guarantee perfect recovery and robustness to noise.

2 Important concepts

This paper contains many concepts mentioned in their predecessor's attempts to sample graph signals.

Here is a table of all the relevant symbols and their dimensions:

Symbol	Description	Dimension
A	graph shift(adjacency matrix)	$N \times N$
x	graph signal	N
V^{-1}	graph Fourier transform matrix	$N \times N$
Λ	diagonal eigenvalue matrix	$N \times N$
$\check{x}$	graph signal in the frequency domain	N
ψ	sampling operator	$M \times N$
ϕ	interpolation operator	$N \times M$
$\mathcal{M}$	sampled indices	
$x_{\mathcal{M}}$	sampled signal coeffcients of x	M
$\check{x}_{(K)}$	first K coeffcients of bx	K
$V_{(K)}$	first K columns of V	$N \times K$

2.1 Fourier Transform:

$$A = V\Delta V^{-1}$$

Here we take the adjacency matrix A and perform eigenvalue decomposition. V^{-1} is the fourier transform required to change the signal coefficients to the frequency domain.

$$x' = V^{-1}x$$

2.2 Bandwidth:

Generally bandwidth refers to the specific band of frquencies that are present in the signal.

In this paper we consider bandwidth as the total number of non-zero signal coefficients in the graph Fourier domain.Since each signal coefficient in the graph Fourier domain corresponds to a graph frequency, the bandwidth definition is also based on the number of graph frequencies.

A graph signal is called bandlimited when there exists a $K \epsilon \{0, 1, 2 \ldots\ldots , N-1\}$ such that its graph Fourier transform x' satisfies

$$x_k{}' = 0 \; for \; all \; k \geqslant K.$$

2.3 Sampling operator(ψ)

Next we have to construct the sampling operator from k. $\mathcal{M}$ is just a vector containing the indices of any k rows to sample. We construct the sampling operator ψ based off of $\mathcal{M}$.

$$\psi_{i,j} = \begin{cases} 1 & j = \mathcal{M}_i; \\ 0 & otherwise \end{cases}$$

For simplicities sake ,in the model $\mathcal{M}$ is taken as the first k rows but as stated before $\mathcal{M}$ can be any k rows.

$$x_{\mathcal{M}} = \psi * x$$

2.4 Interpolation Operator(ϕ)

Now that we have the sampled coefficients we require the interpolation operator.

$$\phi = V_{(k)}U$$
$$U = (\psi V_{(k)})^{-1}$$

3 Proposed methods

First and foremost to test the program I took the adjacency graph from gsp_sensor, taking 100 nodes. For the signal, I made a random vector. Since the aim of this paper was to prove that perfect reconstruction is possible I had to bandlimit the signal in some way. To do this I did the following. I set the cutoff frequency to some 'x' % of the original such that only x% of the total frequencies are passed through. From here I followed the paper in constructing the sampling operator(ψ) and then the interpolation operator(ϕ). Using the sampled set of signal coefficients and ϕ we can construct the original signal. From here I could compare the original signal coefficients and the reconstructed coefficients in order to confirm they were identical to certain error.

4 Simulations and Result

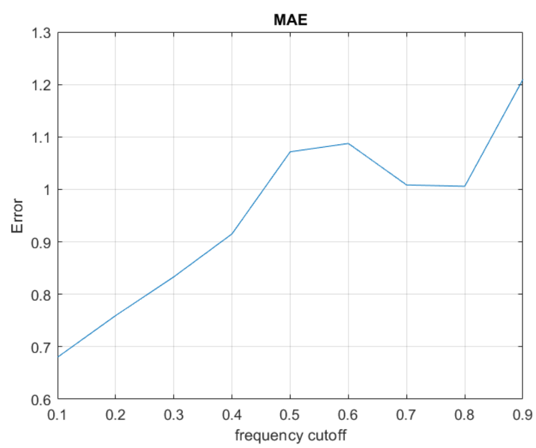

Figure 1: MAE

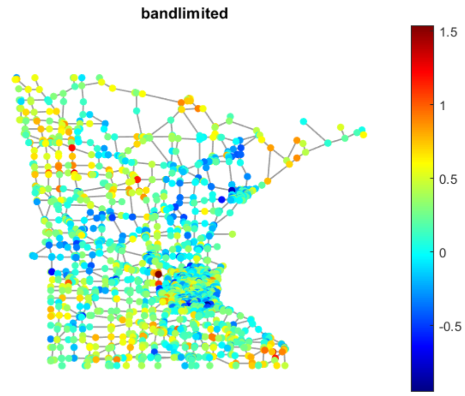

Figure 2: Minnesota:bandlimited

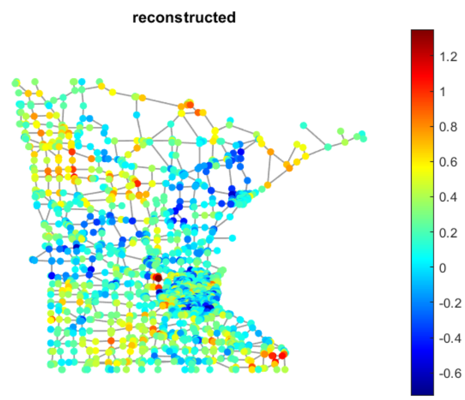

Figure 3: Minnesota: Reconstructed

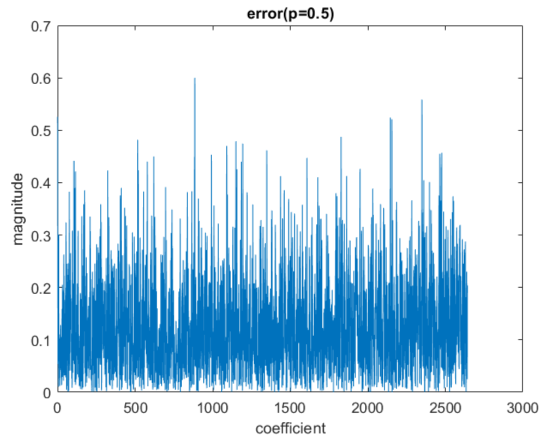

Figure 4: Minnesota: Error

As stated before in order to test the validity of the program I had used a random graph along with random signal coefficients that were later bandlimited. To test the error for every cutoff I set I ran the program 10 times and calculated the mean average error over those 10 trials. I did this for cutoff percentages ranging from 10% to 90% at 10% intervals. The results are shown in Graph 1. The next step after this was to take a more real life data set. For this I used the Minnesota road system . For the signal coefficients I used the same as before. The results of the Minnesota graph and its reconstruction are shown in Figure 2 and Figure 3 respectively. The error of the reconstructed coefficients is shown in Figure 4. The MAE of the graph was found to be 0.1247.

5 Conclusion

Any bandlimited graph can be perfectly reconstucted as long as atleast the minimum number of samples are taken(that being number of number of nonzero fourier domain coefficients). This holds for graphs with large number of nodes, as we can see with the Minnesota graph($\approx 2600\ nodes$).

6 References

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