Benchmark Specifications

1 Duffing's oscillator

1.1 System

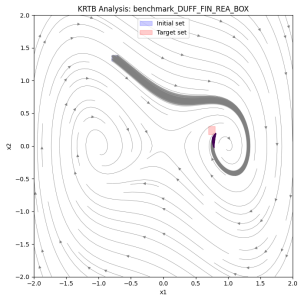
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -0.5x_2 - x_1(x_1^2 - 1) \end{bmatrix},$$

1.2 Benchmark DUFF_REA_BOX

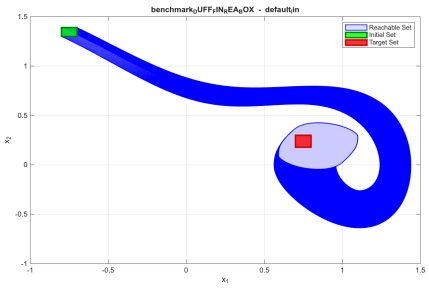
1.2.1 Specification

- Initial Set: $X_0 = [-0.8, -0.7] \times [1.3, 1.39]$
- Target Set: $X_F = [0.7, 0.8] \times [0.178, 0.3]$
- Time Horizon: $[0, 6]$
- Expected Result: Reachable

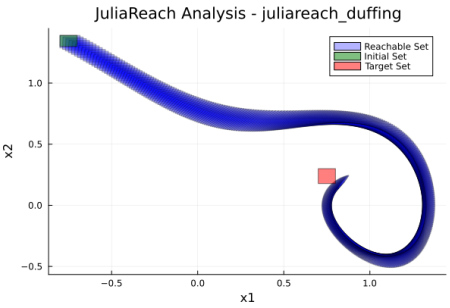
1.2.2 Results



(a) Result from Reach-time Bounds method (Ours) for benchmark DUFF_REA_BOX.



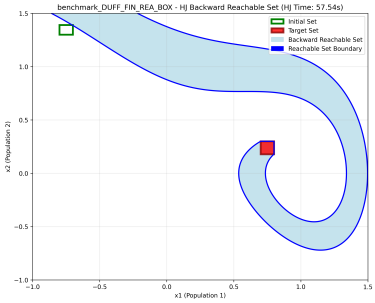
(b) Result from Set-Propagation method (CORA) for benchmark DUFF_REA_BOX.



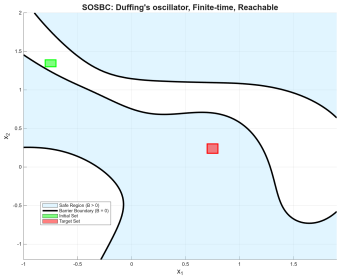
(c) Result from Set-Propagation method (JuliaReach) for benchmark DUFF_REA_BOX.


```
{
  "timestamp": "2025-07-14T17:38:34.084357"
  ,
  "dreach_execution_time_seconds": 93.118,
  "return_code": 0,
  "stdout": "dReal Options:\nSMT:
    duffing_fin_rea_box_0_0.smt2, PATH :
    [1]\ndelta-sat with delta =
    0.00100000000000000\nFor k = 0, [1] --
    SAT\n",
  "stderr": "\nreal\t1m33.118s\nuser\t1m32
    .990s\nsys\t0m0.054s\n",
  "success": true,
  "tool": "dreach",
  "setting": "setting_0",
  "benchmark": "DUFF_FIN_REA_BOX",
  "drh_file": "duffing_fin_rea_box.drh",
  "container_runtime": "podman"
}
```


(d) Result from Counterexample Searching method (dreach) for benchmark DUFF_REA_BOX.



(e) Result from Hamilton-Jacobian PDE method (hj-reachability) for benchmark DUFF_REA_BOX.



(f) Result from Barrier Certificate method (SOSTOOLS) for benchmark DUFF_REA_BOX (Infeasible Solution).

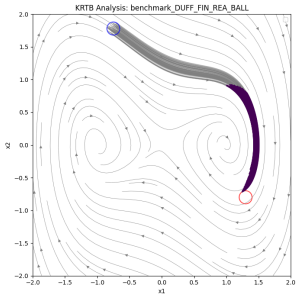
Figure 1: Results for the benchmark DUFF_REA_BOX.

1.3 Benchmark DUFF_REA_CVXLS

1.3.1 Specification

- Initial Set: $X_0 = \{x \in \mathbb{R}^2 | (x_1 + 0.75)^2 + (x_2 - 1.78)^2 - 0.01 \leq 0\}$
- Target Set: $X_F = \{x \in \mathbb{R}^2 | (x1 - 1.3)^2 + (x2 + 0.8)^2 - 0.01 \leq 0\}$
- Time Horizon: $[0, 6]$
- Expected Result: Reachable

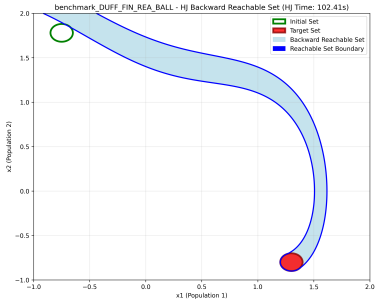
1.3.2 Results



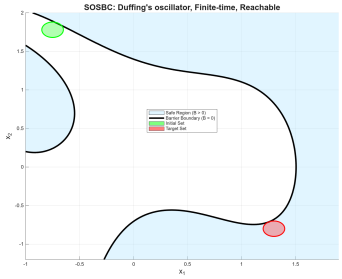
(a) Result from Reach-time Bounds method (Ours) for benchmark DUFF_REA_CVX_LS.


```
{
  "timestamp": "2025-07-14T18:05:35.173642",
  "dreach_execution_time_seconds": 3.126,
  "return_code": 0,
  "stdout": "dReal Options:\nSMT: duffing_fin_rea_ball_0_0.smt2,
    PATH : [1]\ndelta-sat with delta = 0.00100000000000000\nFor
    k = 0, [1] -- SAT\n",
  "stderr": "\nreal\t0m3.126s\nuser\t0m3.111s\nsys\t0m0.017s\n",
  "success": true,
  "tool": "dreach",
  "setting": "setting_0",
  "benchmark": "DUFF_FIN_REA_BALL",
  "drh_file": "duffing_fin_rea_ball.drh",
  "container_runtime": "podman"
}
```


(b) Result from Counterexample Searching method (dreach) for benchmark DUFF_REA_CVX_LS.



(c) Result from Hamilton-Jacobian PDE method (SOSTOOLS) for benchmark DUFF_REA_CVXLS.



(d) Result from Barrier Certificate method (SOSTOOLS) for benchmark DUFF_REA_CVXLS (Infeasible Solution).

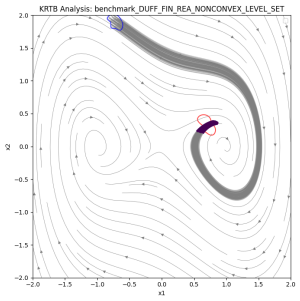
Figure 2: Results for the benchmark DUFF_REA_CVX_LS.

1.4 Benchmark DUFF_REA_NCVX_LS

1.4.1 Specification

- Initial Set: $X_0 = \{x \in \mathbb{R}^2 | -\left(1 - \frac{x_1+0.75}{3} + \left(\frac{x_2-1.85}{0.05}\right)^5 + 2\left(\frac{x_1+0.75}{0.05}\right)^3\right)\exp\left(-\left[\left(\frac{x_1+0.75}{0.05}\right)^2 + \left(\frac{x_2-1.85}{0.05}\right)^2\right]\right) + 0.01 \leq 0\}$
- Target Set: $X_F = \{x \in \mathbb{R}^2 | -\left(1 - \frac{x_1-0.65}{3} + 3\left(\frac{x_2-0.25}{0.1}\right)^5 + 4\left(\frac{x_1-0.65}{0.1}\right)^3\right)\exp\left(-\left[\left(\frac{x_1-0.65}{0.1}\right)^2 + \left(\frac{x_2-0.25}{0.1}\right)^2\right]\right) + 0.9 \leq 0\}$
- Time Horizon: $[0, 6]$
- Expected Result: Reachable

1.4.2 Results



(a) Result from Reach-time Bounds method (Ours) for benchmark DUFF_REA_NCVX_LS.


```
{
  "timestamp": "2025-07-13T17:53:23.294714",
  "dreach_execution_time_seconds": 71.053,
  "return_code": 0,
  "stdout": "dReal Options:\nSMT: duffing_fin_rea_0_0.smt2, PATH
    : [1]\ndelta-sat with delta = 0.00100000000000000\nFor k =
    0, [1] -- SAT\n",
  "stderr": "\nreal\t1m11.053s\nuser\t1m10.830s\nsys\t0m0.106s\n
    ",
  "success": true,
  "tool": "dreach",
  "setting": "setting_0",
  "benchmark": "DUFF_FIN_REA",
  "drh_file": "duffing_fin_rea.drh",
  "container_runtime": "podman"
}
```


(b) Result from Counterexample Searching method (dreach) for benchmark DUFF_REA_CVX_LS.

Figure 3: Results for the benchmark DUFF_REA_NCVX_LS.

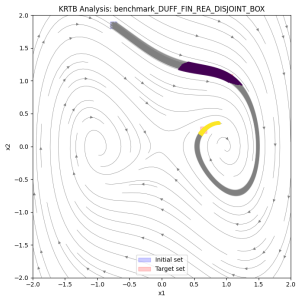
1.5 Benchmark DUFF_REA_DSJT_BOX

1.5.1 Specification

- Initial Set: $X_0 = [-0.8, -0.7] \times [1.8, 1.9]$

- Target Set: $X_F = [0.6, 0.7] \times [0.2, 0.3] \cup [1.0, 1.1] \times [1.0, 1.1]$
- Time Horizon: $[0, 6]$
- Expected Result: Reachable

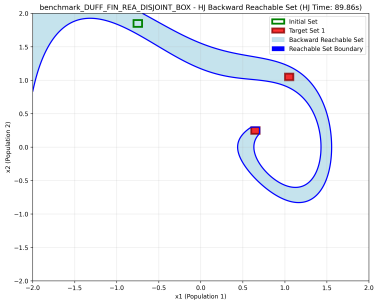
1.5.2 Results



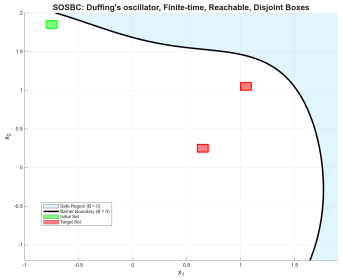
(a) Result from Reach-time Bounds method (Ours) for benchmark DUFF_REA_DSJT_BOX.


```
{
  "timestamp": "2025-07-15T14:14:45.368702",
  "dreach_execution_time_seconds": 0.119,
  "return_code": 0,
  "stdout": "dReal Options:\nSMT: duffing_fin_rea_disjoint_box_0_0.smt2,
   PATH : [1]\nunsat\nFor k = 0, dReach tried 1 feasible paths, all of
   them are unsat.\nSMT: duffing_fin_rea_disjoint_box_1_0.smt2, PATH :
   [1,1]\nunsat\nFor k = 1, dReach tried 1 feasible paths, all of them are
    unsat.\nSMT: duffing_fin_rea_disjoint_box_2_0.smt2, PATH : [1,1,1]\
   nunsat\nFor k = 2, dReach tried 1 feasible paths, all of them are unsat
   .\nSMT: duffing_fin_rea_disjoint_box_3_0.smt2, PATH : [1,1,1,1]\nunsat\
   nFor k = 3, dReach tried 1 feasible paths, all of them are unsat.\n",
  "stderr": "\nreal\t0m0.119s\nuser\t0m0.034s\nsys\t0m0.041s\n",
  "success": true,
  "tool": "dreach",
  "setting": "setting_0",
  "benchmark": "DUFF_FIN_REA_DISJOINT_BOX",
  "drh_file": "duffing_fin_rea_disjoint_box.drh",
  "container_runtime": "podman"
}
```


(b) Result from Counterexample Searching method (dreach) for benchmark DUFF_REA_CVX_LS.



(c) Result from Hamilton-Jacobian PDE method (hj-reachability) for benchmark DUFF_REA_DSJT_BOX.



(d) Result from Barrier Certificate method (SOSTOOLS) for benchmark DUFF_REA_DSJT_BOX (Infeasible Solution).

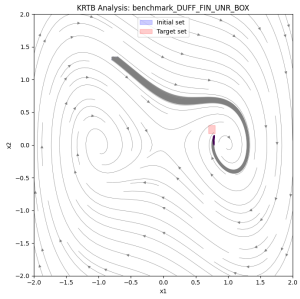
Figure 4: Results for the benchmark DUFF_REA_DSJT_BOX

1.6 Benchmark DUFF_UNR_BOX

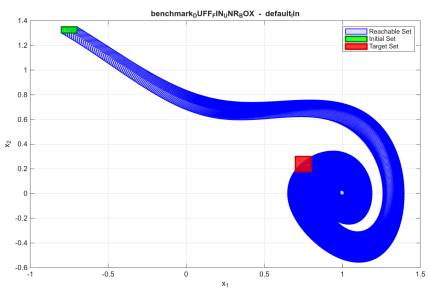
1.6.1 Specification

- Initial Set: $X_0 = [-0.8, -0.7] \times [1.3, 1.35]$
- Target Set: $X_F = [0.7, 0.8] \times [0.178, 0.3]$
- Time Horizon: $[0, 20]$
- Expected Result: Unreachable

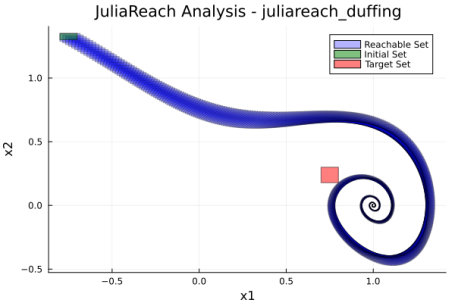
1.6.2 Results



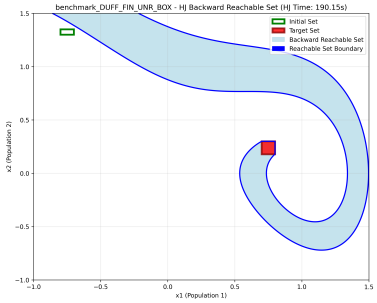
(a) Result from Reach-time Bounds method (Ours) for benchmark DUFF_UNR_BOX.



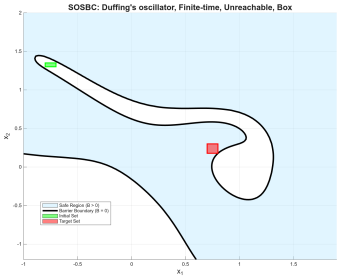
(b) Result from Set-Propagation method (CORA) for benchmark DUFF_UNR_BOX.



(c) Result from Set-Propagation method (JuliaReach) for benchmark DUFF_UNR_BOX.



(d) Result from Hamilton-Jacobian PDE method (hj-reachability) for benchmark DUFF_UNR_BOX.



(e) Result from Barrier Certificate method (SOSTOOLS) for benchmark DUFF_UNR_BOX.

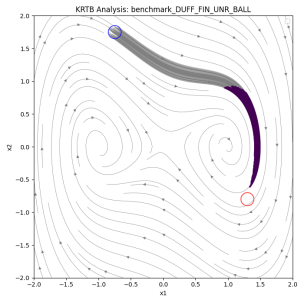
Figure 5: Results for the benchmark DUFF_UNR_BOX

1.7 Benchmark DUFF_UNR_CVX_LS

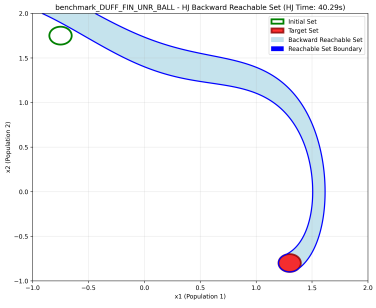
1.7.1 Specification

- Initial Set: $X_0 = \{x \in \mathbb{R}^2 | (x1 + 0.75)^2 + (x2 - 1.75)^2 - 0.01 \leq 0\}$
- Target Set: $X_F = \{x \in \mathbb{R}^2 | (x1 - 1.3)^2 + (x2 + 0.8)^2 - 0.01 \leq 0\}$
- Time Horizon: $[0, 6]$
- Expected Result: Unreachable

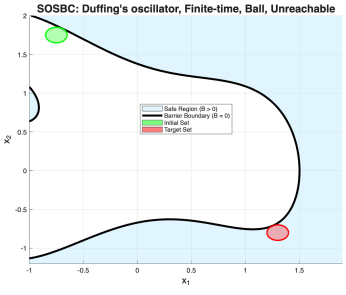
1.7.2 Results



(a) Result from Reach-time Bounds method (Ours) for benchmark DUFF_UNR_CVX_LS.



(b) Result from Hamilton-Jacobian PDE method (hj-reachability) for benchmark DUFF_UNR_CVX_LS.



(c) Result from Barrier Certificate method (SOSTOOLS) for benchmark DUFF_UNR_CVX_LS.

Figure 6: Results for the benchmark DUFF_UNR_CVX_LS

2 Nonlinear System with Stable Limit Cycle

2.1 System

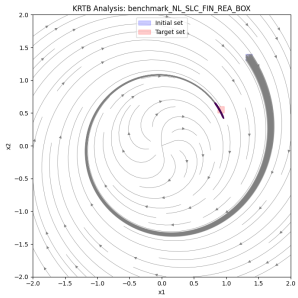
$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}\frac{-x_1^3+4x_1^2x_2-x_1x_2^2+x_1+4x_2^3}{4(x_1^2+x_2^2)}\\ \frac{-4x_1^3-x_1^2x_2-4x_1x_2^2-x_2^3+x_2}{4(x_1^2+x_2^2)}\end{bmatrix}$$

2.2 Benchmark NL-SLC_REA_BOX

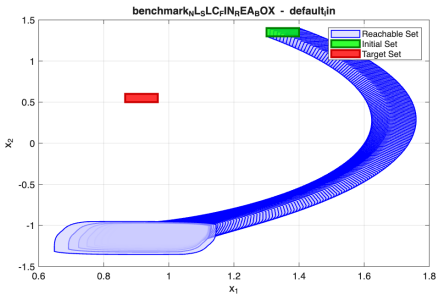
2.2.1 Specification

- Initial Set: $X_0 = [1.3, 1.4] \times [1.3, 1.4]$
- Target Set: $X_F = [\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}] \times [0.5, 0.6]$
- Time Horizon: $[0, 15]$
- Expected Result: Reachable

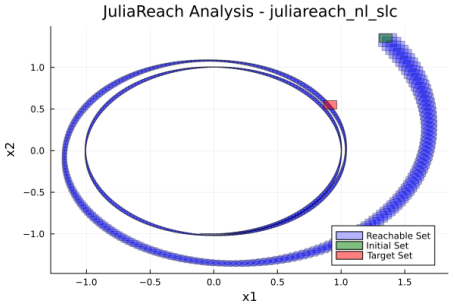
2.2.2 Results



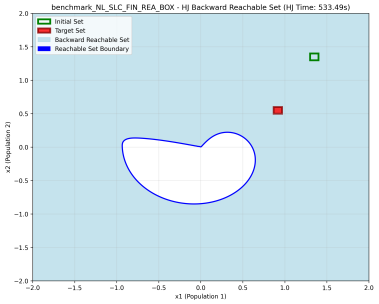
(a) Result from Reach-time Bounds method (Ours) for benchmark NL-SLC_REA_BOX.



(b) Result from Set-Propagation method (CORA) for benchmark NL-SLC_REA_BOX (Set Explosion).



(c) Result from Set-Propagation method (JuliaReach) for benchmark NL-SLC_REA_BOX.



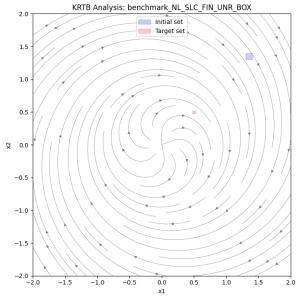
(d) Result from Hamilton-Jacobian PDE method (hj-reachability) for benchmark NL-SLC_REA_BOX.

Figure 7: Results for the benchmark NL-SLC_REA_BOX

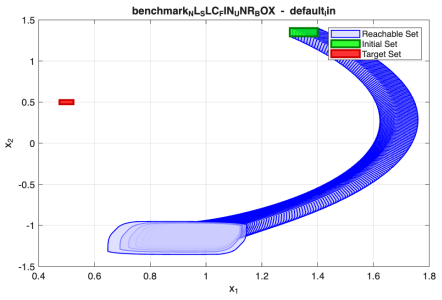
2.3 Benchmark NL-SLC_UNR_BOX

2.3.1 Specification

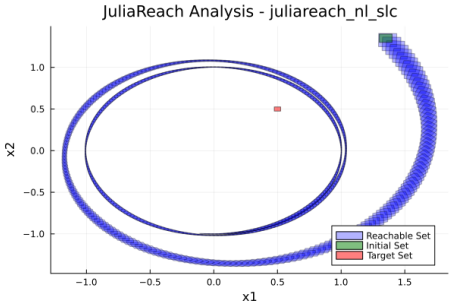
- Initial Set: $X_0 = [1.3, 1.4] \times [1.3, 1.4]$
- Target Set: $X_F = [0.475, 0.525] \times [0.475, 0.525]$
- Time Horizon: $[0, 15]$
- Expected Result: Unreachable



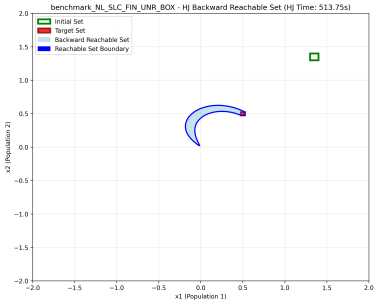
(a) Result from Reach-time Bounds method (Ours) for benchmark NL-SLC_UNR_BOX.



(b) Result from Set-Propagation method (CORA) for benchmark NL-SLC_REA_BOX (Set Explosion).



(c) Result from Set-Propagation method (JuliaReach) for benchmark NL-SLC_UNR_BOX.



(d) Result from Hamilton-Jacobian PDE method (hj-reachability) for benchmark NL-SLC_REA_BOX.

Figure 8: Results for the benchmark NL-SLC_UNR_BOX

3 Nonlinear System with Known Eigenfunctions

3.1 System

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = [\nabla \Psi(x)]^{-1} \begin{bmatrix} -1 & 0 \\ 0 & 2.5 \end{bmatrix} \Psi(x)$$

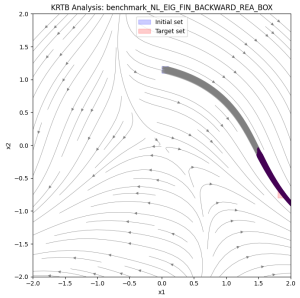
where the principal eigenfunctions are denoted as $\Psi(x) = [\psi_1(x), \psi_2(x)]^T$ with $\psi_1(x) = x_1^2 + 2x_2 + x_2^3$ and $\psi_2(x) = x_1 + \sin(x_2) + x_1^3$, and the associated eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 2.5$ at the unstable equilibrium $(0, 0)$.

3.2 Benchmark NL-EIG_BWD_REA_BOX

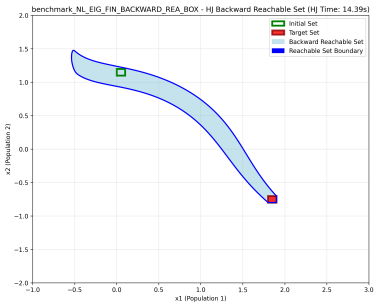
3.2.1 Specification

- Initial Set: $X_0 = [0, 0.1] \times [1.1, 1.2]$
- Target Set: $X_F = [1.8, 1.9] \times [-0.8, -0.7]$
- Time Horizon: $[0, 1.3]$
- Expected Result: Reachable

3.2.2 Results



(a) Result from Reach-time Bounds method (Ours) for benchmark NL-EIG_BWD_REA_BOX.



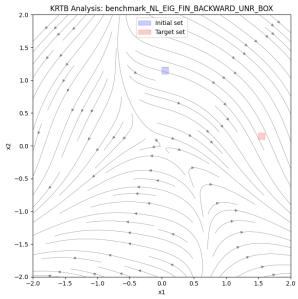
(b) Result from Hamilton-Jacobian PDE method (hj-reachability) for benchmark NL-EIG_BWD_REA_BOX.

Figure 9: Results for the benchmark NL-EIG_BWD_UNR_BOX

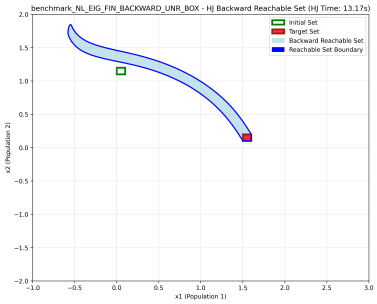
3.3 Benchmark NL-EIG_BWD_UNR_BOX

- Initial Set: $X_0 = [0, 0.1] \times [1.1, 1.2]$
- Target Set: $X_F = [1.5, 1.6] \times [0.1, 0.2]$
- Time Horizon: $[0, 1.3]$
- Expected Result: Reachable

3.3.1 Results



(a) Result from Reach-time Bounds method (Ours) for benchmark NL-EIG_BWD_UNR_BOX.



(b) Result from Hamilton-Jacobian PDE method (hj-reachability) for benchmark NL-EIG_BWD_UNR_BOX.

Figure 10: Results for the benchmark NL-EIG_BWD_UNR_BOX

4 Cart-pole System

4.1 System

$$\dot{x} = \begin{bmatrix} \dot{p} \\ \dot{v} \\ \dot{\theta} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} v \\ \frac{[F+m_p \sin(\theta)(l\omega^2+g\cos(\theta))]}{m_c+m_p \sin(\theta)^2} \\ \omega \\ \frac{[-F\cos(\theta)-m_p l\omega^2 \cos(\theta)\sin(\theta)-(m_c+m_p)g\sin(\theta)]}{l(m_c+m_p \sin(\theta)^2)} \end{bmatrix}$$

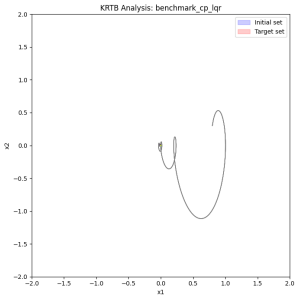
where $m_c = 2, m_p = 1, l = 1$, and $g = 9.8065$. The system equipped with the LQR controller $u(x) = [-3.162x_0 - 6.698x_1 - 5.027x_2 - 1.617x_3]$

4.2 Benchmark CP-LQR_REA_CVG

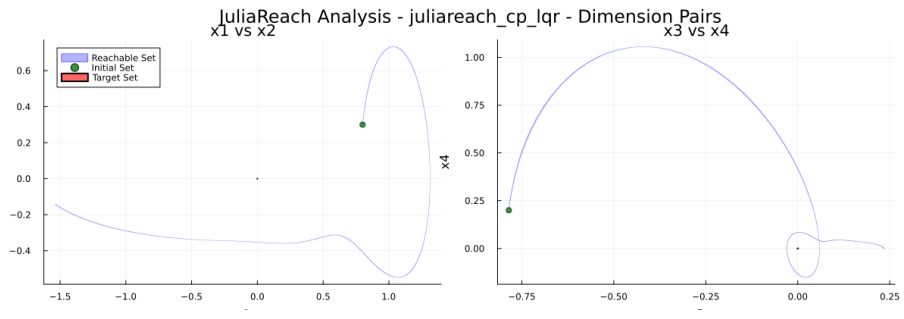
4.2.1 Specification

- Initial Set: $X_0 = [0.8, 0.3, -\frac{\pi}{4}, 0.2]$
- Target Set: $X_F = \{x \,|\, 10^{-4} \leq \|x\|_\infty \leq 10^{-2}, x \in X\}$
- Time Horizon: $[0, 10]$
- Expected Result: Reachable

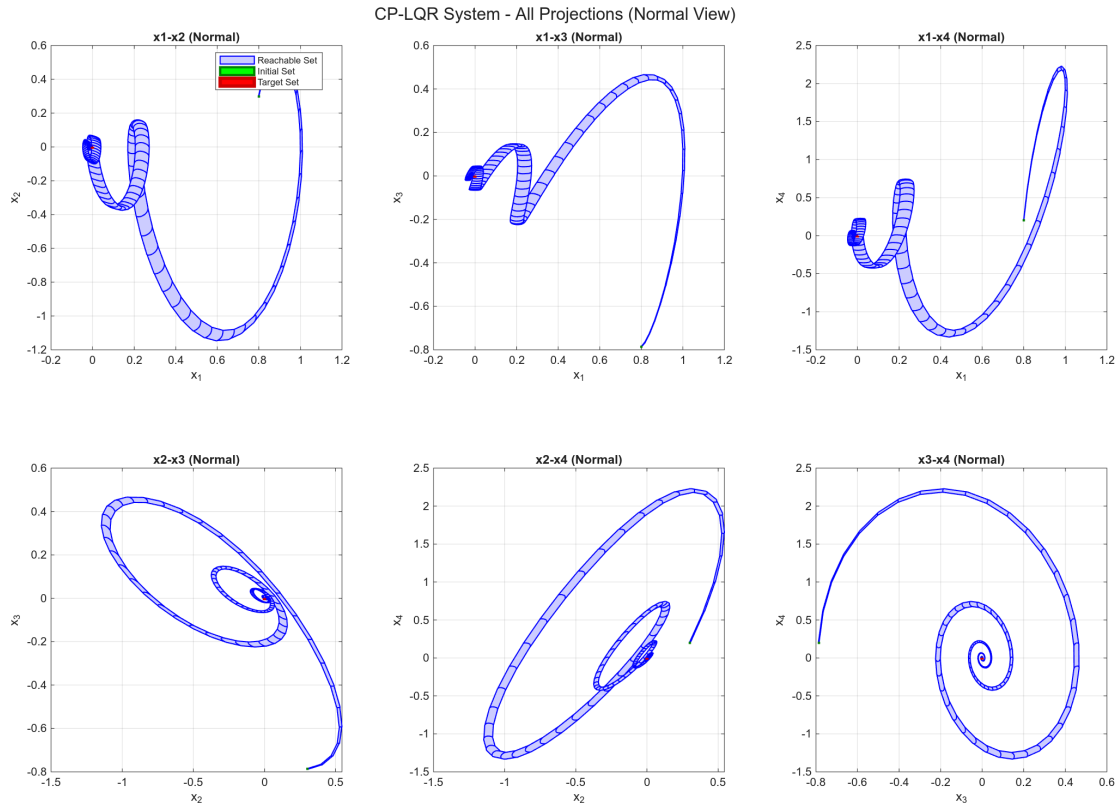
4.2.2 Results



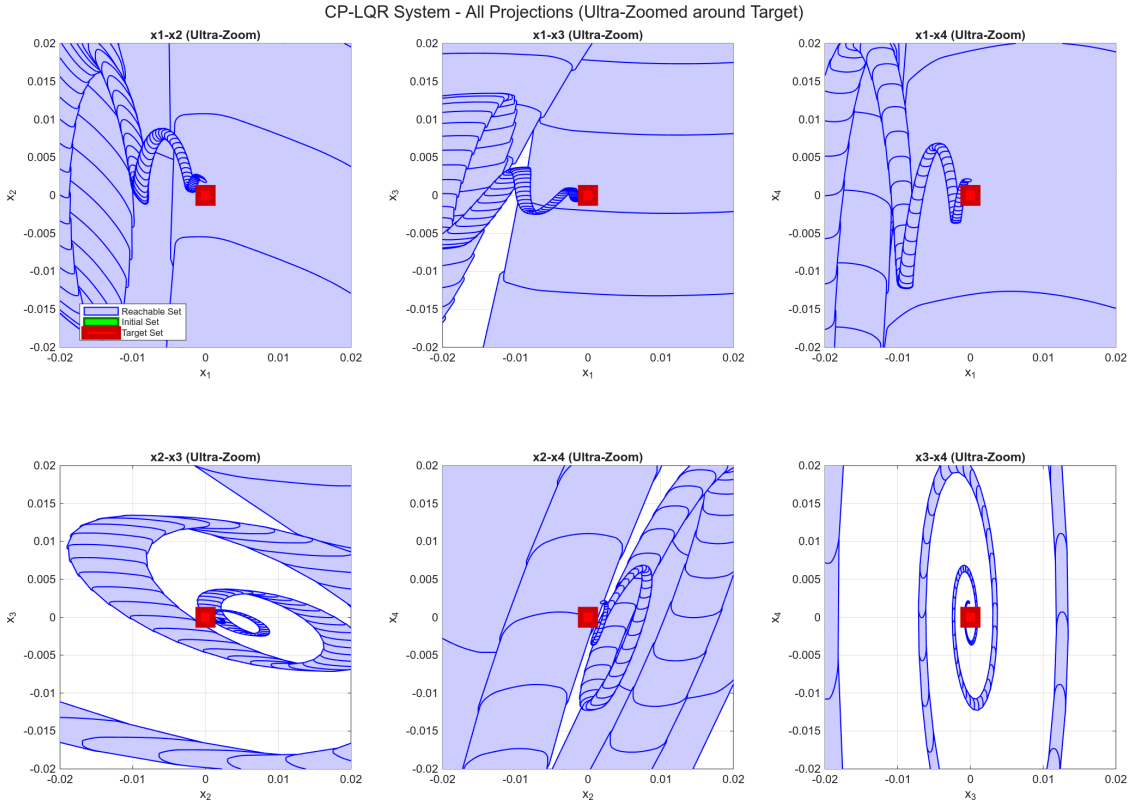
(a) Result from Reach-time Bounds method (Ours) for benchmark CP-LQR_REA_CVG.



(b) Result from Set-Propagation method (JuliaReach) for benchmark CP-LQR_REA_CVG. (Unsound Result)



(a) Result from Set-Propagation method (CORA) for benchmark CP-LQR_REA_CVG (normal scale).



(b) Result from Set-Propagation method (CORA) for benchmark CP-LQR_REA_CVG (zoomed scale). (Unsound Result)

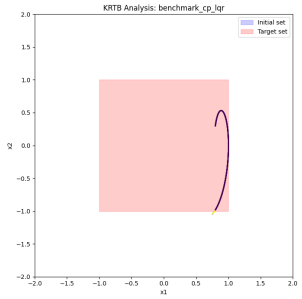
Figure 12: Results for the benchmark CP-LQR_REA_CVG

4.3 Benchmark CP-LQR_REA_USF

4.3.1 Specification

- Initial Set: $X_0 = [0.8, 0.3, -\frac{\pi}{4}, 0.2]$
- Target Set: $X_F = \{x \,|\, p + l \sin(\theta) > 1.3\}$
- Time Horizon: $[0, 10]$
- Expected Result: Reachable

4.3.2 Results



(a) Result from Reach-time Bounds method (Ours) for benchmark CP-LQR_REA_USF.


```
{
  "timestamp": "2025-07-14T12:46:56.358411",
  "dreach_execution_time_seconds": 0.425,
  "return_code": 0,
  "stdout": "dReal Options:\nSMT: cp_lqr_fin_rea_unsafe_0_0.smt2, PATH :
   [1]\ndelta-sat with delta = 0.00100000000000000\nFor k = 0, [1] -- SAT\
   n",
  "stderr": "\nreal\t0m0.425s\nuser\t0m0.348s\nsys\t0m0.019s\n",
  "success": true,
  "tool": "dreach",
  "setting": "setting_0",
  "benchmark": "CP_LQR_FIN_REA_UNSAFE",
  "drh_file": "cp_lqr_fin_rea_unsafe.drh",
  "container_runtime": "podman"
}
```


(b) Result from Counterexample Searching method (dreach) for benchmark CP-LQR_REA_USF.

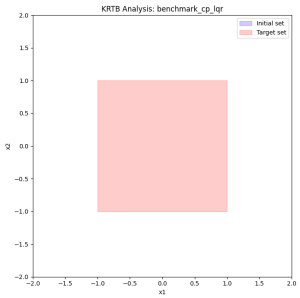
Figure 13: Results for the benchmark CP-LQR_REA_USF

4.4 Benchmark CP-LQR_UNR_SAF

4.4.1 Specification

- Initial Set: $X_0 = [0.8, 0.3, -\frac{\pi}{4}, 0.2]$
- Target Set: $X_F = \{x \,|\, p + l \sin(\theta) > 1.7\}$
- Time Horizon: $[0, 10]$
- Expected Result: Reachable

4.4.2 Results



(a) Result from Reach-time Bounds method (Ours) for benchmark CP-LQR_REA_USF (No Feasible Trajectory).

Figure 14: Results for the benchmark CP-LQR_REA_USF

5 Multi-agent Consensus System

5.1 System

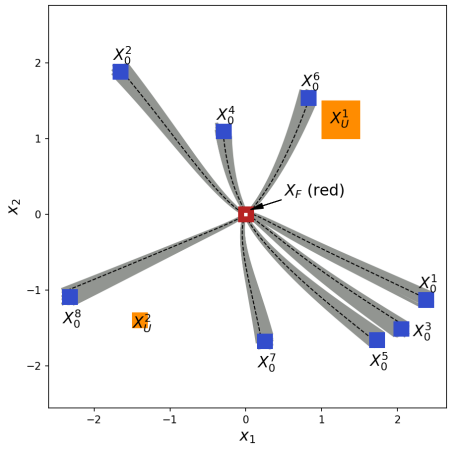
As defined in Fig. 15.

5.2 Benchmark MAS-CON_REA_BOX

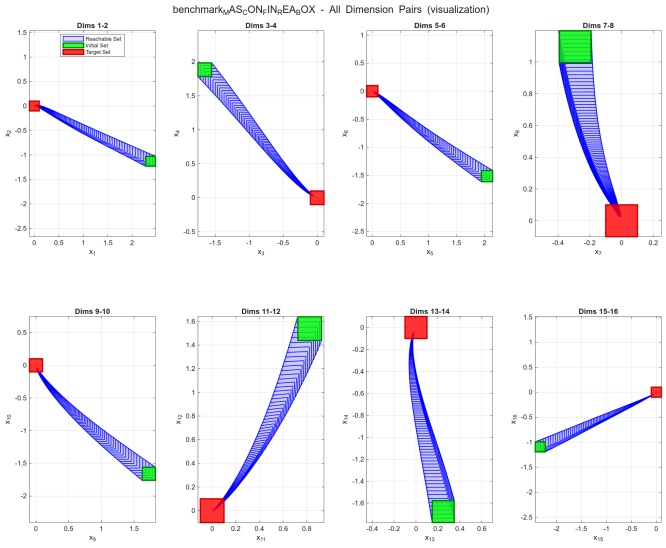
5.2.1 Specification

- Initial Set: $X_0 = \{x \in \mathbb{R}^{16} | \|x - c\|_\infty \leq 0.1\}$,
  where $c = [2.38, -1.13, -1.65, 1.89, 2.05, -1.51, -0.29, 1.10, 1.73, -1.66, 0.83, 1.54, 0.25, -1.68, -2.32, -1.09]^\top$
- Target Set: $X_F = \{x \in \mathbb{R}^{16} \,|\, 0.02 \leq \|x\|_\infty \leq 0.1, x \in X\}$
- Time Horizon: $[0, 7]$
- Expected Result: Reachable

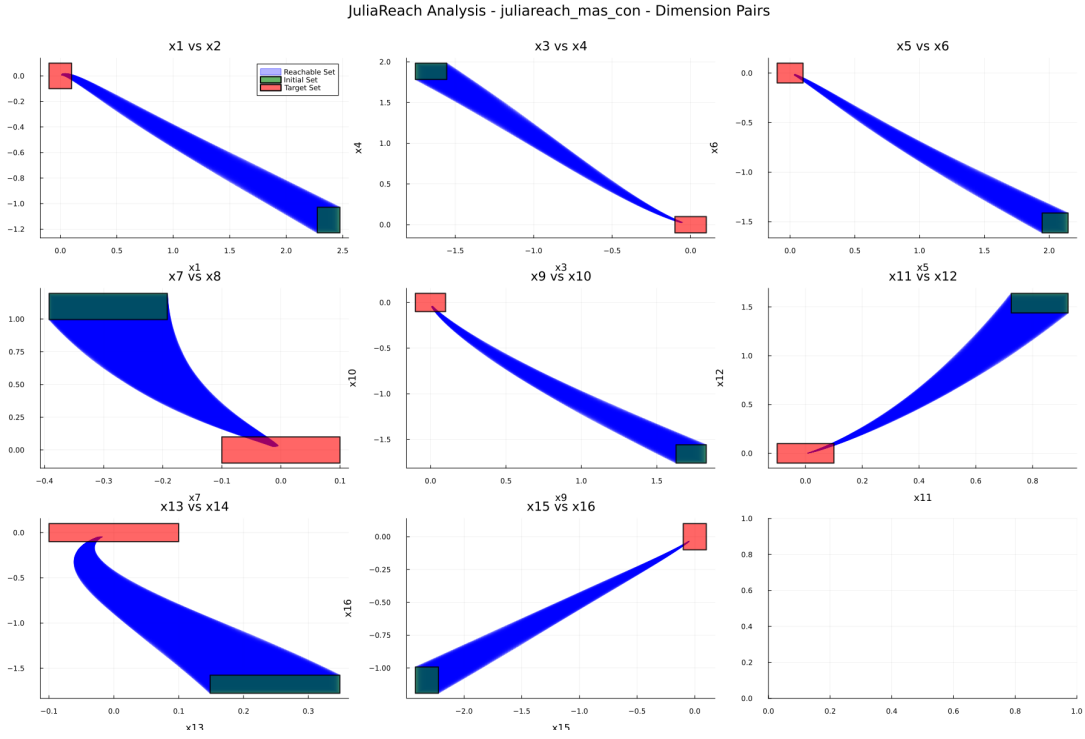
5.2.2 Results



(a) Result from Reach-time Bounds method (Ours) for benchmark MAS-CON_REA_BOX.



(b) Result from Set-Propagation method (CORA) for benchmark MAS-CON_REA_BOX.



(c) Result from Set-Propagation method (JuliaReach) for benchmark MAS-CON_REA_BOX.

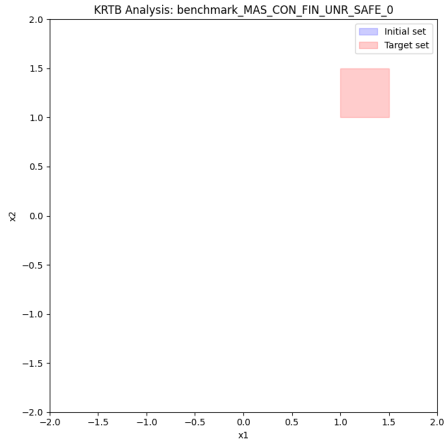
Figure 16: Results for the benchmark MAS-CON_REA_BOX

5.3 Benchmark MAS-CON_UNR_SAF_0

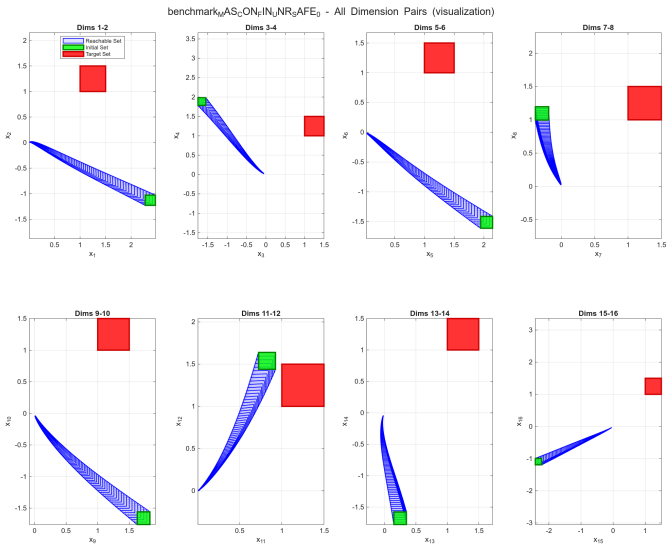
5.3.1 Specification

- Initial Set: $X_0 = \{x \in \mathbb{R}^{16} | \|x - c_0\|_\infty \leq 0.1\}$,
  where $c_0 = [2.38, -1.13, -1.65, 1.89, 2.05, -1.51, -0.29, 1.10, 1.73, -1.66, 0.83, 1.54, 0.25, -1.68, -2.32, -1.09]^\top$
- Target Set: $X_F = \{x \in \mathbb{R}^{16} \,|\, \|x - c_F\|_\infty \leq 0.2, x \in X\}$,
  where $c_F = [1.3, 1.3, 1.3, 1.3, 1.3, 1.3, 1.3, 1.3, 1.3, 1.3, 1.3, 1.3, 1.3, 1.3, 1.3, 1.3]^\top$
- Time Horizon: $[0, 7]$
- Expected Result: Unreachable

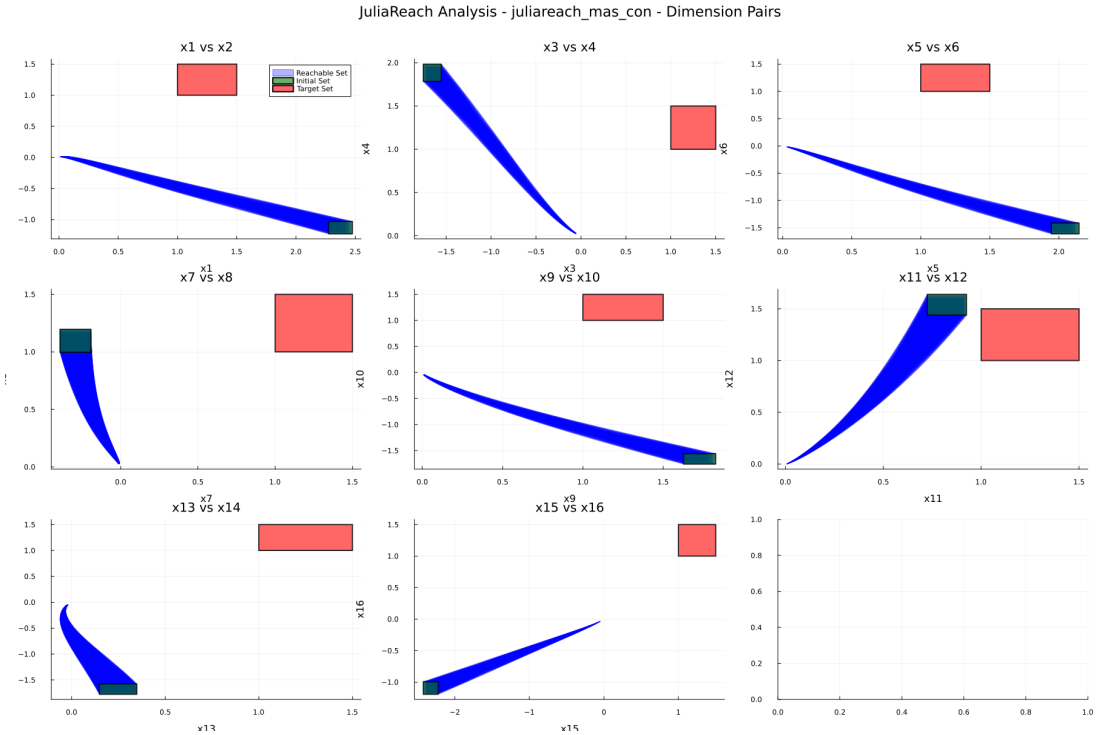
5.3.2 Results



(a) Result from Reach-time Bounds method (Ours) for benchmark MAS-CON_UNR_SAF_0. (No Feasible Trajectory)



(b) Result from Set-Propagation method (CORA) for benchmark MAS-CON_UNR_SAF_0.



(c) Result from Set-Propagation method (JuliaReach) for benchmark MAS-CON_UNR_SAR_0.

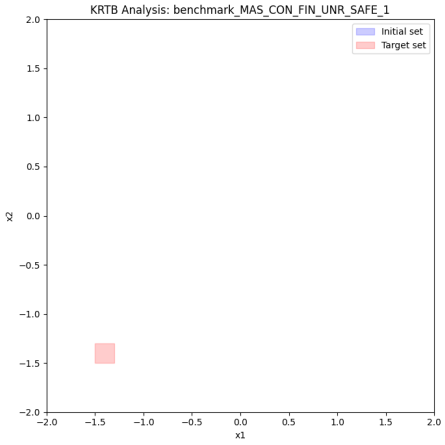
Figure 17: Results for the benchmark MAS-CON_UNR_SAF_0

5.4 Benchmark MAS-CON_UNR_SAF_1

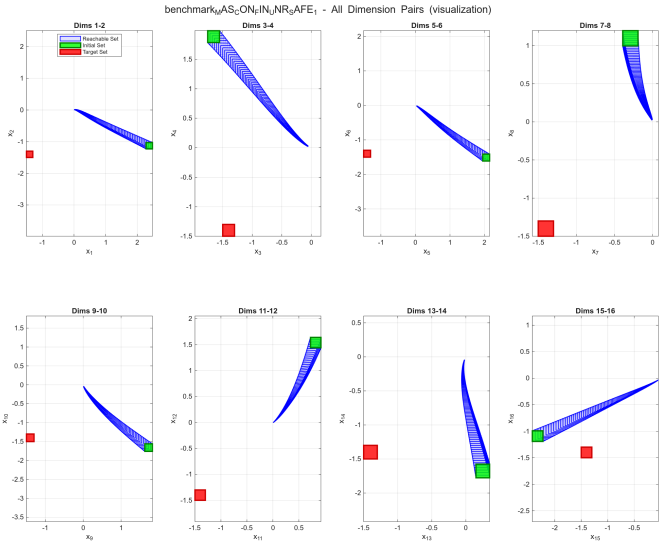
5.4.1 Specification

- Initial Set: $X_0 = \{x \in \mathbb{R}^{16} | \|x - c_0\|_\infty \leq 0.1\}$,
  where $c_0 = [2.38, -1.13, -1.65, 1.89, 2.05, -1.51, -0.29, 1.10, 1.73, -1.66, 0.83, 1.54, 0.25, -1.68, -2.32, -1.09]^\top$
- Target Set: $X_F = \{x \in \mathbb{R}^{16} \,|\, \|x - c_F\|_\infty \leq 0.2, x \in X\}$,
  where $c_F = [1.3, 1.3, 1.3, 1.3, 1.3, 1.3, 1.3, 1.3, 1.3, 1.3, 1.3, 1.3, 1.3, 1.3, 1.3, 1.3]^\top$
- Time Horizon: $[0, 7]$
- Expected Result: Unreachable

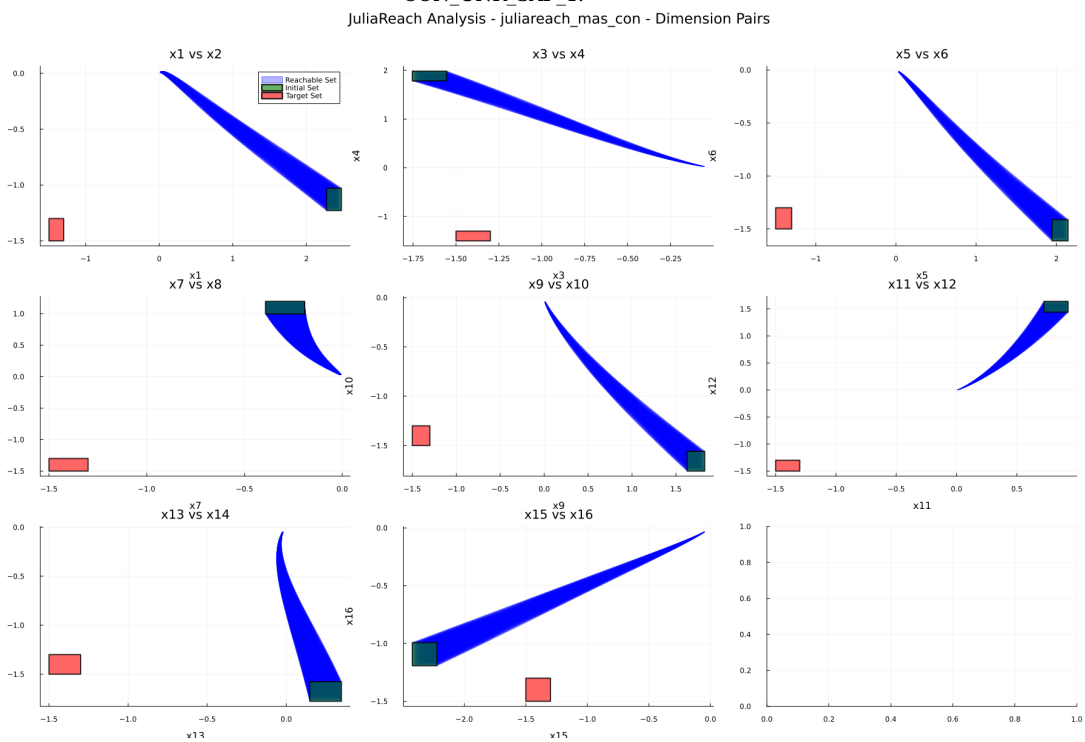
5.4.2 Results



(a) Result from Reach-time Bounds method (Ours) for benchmark MAS-CON_UNR_SAF_1. (No Feasible Trajectory)



(b) Result from Set-Propagation method (CORA) for benchmark MAS-CON_UNR_SAF_1.



(c) Result from Set-Propagation method (JuliaReach) for benchmark MAS-CON_UNR_SAR_1.

Figure 18: Results for the benchmark MAS-CON_UNR_SAF_1

The dynamics of the i-th agent in the 16-agent system, for $i \in \{1, \ldots, 16\}$, are governed by the general form:

$$\dot{x}_i = -\alpha x_i - \beta \cdot \frac{F_i(\mathbf{x})}{1 + e^{-x_i}} \tag{1}$$

where $\mathbf{x} = [x_1, \ldots, x_{16}]^T$ is the joint state vector, $\alpha = 0.4419$ is the uniform self-damping rate, and $\beta = 0.1$ is the coupling strength. The nonlinear coupling functions $F_i(\mathbf{x})$, which define the network's topology and interaction laws, are explicitly defined below.

$$\begin{aligned}
F_1(\mathbf{x}) = {} & 0.7799\sin(x_1 - x_2) + 0.4384\sin(x_1 - x_3) \\
& + 0.7235\sin(x_1 - x_4) + 0.5385\sin(x_1 - x_6) \\
& + 0.9359\tanh(x_1 - x_2) + 0.5261\tanh(x_1 - x_3) \\
& + 0.8682\tanh(x_1 - x_4) + 0.6462\tanh(x_1 - x_6)
\end{aligned}$$

$$\begin{aligned}
F_2(\mathbf{x}) = {} & -0.2684\sin(x_1 - x_2) + 0.8037\sin(x_2 - x_4) \\
& + 0.0659\sin(x_2 - x_6) + 0.2881\sin(x_2 - x_7) \\
& + 0.9096\sin(x_2 - x_8) - 0.3221\tanh(x_1 - x_2) \\
& + 0.9645\tanh(x_2 - x_4) + 0.0791\tanh(x_2 - x_6) \\
& + 0.3458\tanh(x_2 - x_7) + 1.0915\tanh(x_2 - x_8)
\end{aligned}$$

$$\begin{aligned}
F_3(\mathbf{x}) = {} & -0.2134\sin(x_1 - x_3) + 0.0249\sin(x_3 - x_4) \\
& + 0.2303\sin(x_3 - x_7) - 0.2561\tanh(x_1 - x_3) \\
& + 0.0299\tanh(x_3 - x_4) + 0.2764\tanh(x_3 - x_7)
\end{aligned}$$

$$\begin{aligned}
F_4(\mathbf{x}) = {} & -0.9091\sin(x_1 - x_4) - 0.1332\sin(x_2 - x_4) \\
& - 0.5234\sin(x_3 - x_4) + 0.6690\sin(x_4 - x_5) \\
& + 0.4908\sin(x_4 - x_8) - 1.0910\tanh(x_1 - x_4) \\
& - 0.1598\tanh(x_2 - x_4) - 0.6281\tanh(x_3 - x_4) \\
& + 0.8028\tanh(x_4 - x_5) + 0.5889\tanh(x_4 - x_8)
\end{aligned}$$

$$\begin{aligned}
F_5(\mathbf{x}) = {} & -0.8379\sin(x_4 - x_5) + 0.3140\sin(x_5 - x_6) \\
& + 0.5726\sin(x_5 - x_7) - 1.0055\tanh(x_4 - x_5) \\
& + 0.3768\tanh(x_5 - x_6) + 0.6872\tanh(x_5 - x_7)
\end{aligned}$$

$$\begin{aligned}
F_6(\mathbf{x}) = {} & -0.4528\sin(x_1 - x_6) - 0.3530\sin(x_2 - x_6) \\
& - 0.4591\sin(x_5 - x_6) + 0.4130\sin(x_6 - x_7) \\
& - 0.5434\tanh(x_1 - x_6) - 0.4236\tanh(x_2 - x_6) \\
& - 0.5509\tanh(x_5 - x_6) + 0.4956\tanh(x_6 - x_7)
\end{aligned}$$

$$\begin{aligned}
F_7(\mathbf{x}) = {} & -0.7411\sin(x_2 - x_7) - 0.4224\sin(x_3 - x_7) \\
& - 0.6344\sin(x_5 - x_7) - 0.5229\sin(x_6 - x_7) \\
& + 0.0014\sin(x_7 - x_8) - 0.8893\tanh(x_2 - x_7) \\
& - 0.5068\tanh(x_3 - x_7) - 0.7613\tanh(x_5 - x_7) \\
& - 0.6275\tanh(x_6 - x_7) + 0.0017\tanh(x_7 - x_8)
\end{aligned}$$

$$\begin{aligned}
F_8(\mathbf{x}) = {} & -0.7094\sin(x_2 - x_8) - 0.6962\sin(x_4 - x_8) \\
& - 0.0531\sin(x_7 - x_8) - 0.8513\tanh(x_2 - x_8) \\
& - 0.8354\tanh(x_4 - x_8) - 0.0638\tanh(x_7 - x_8)
\end{aligned}$$

$$\begin{aligned}
F_9(\mathbf{x}) = {} & -0.9650\sin(x_{11} - x_9) - 0.9450\sin(x_{12} - x_9) \\
& - 0.4723\sin(x_{14} - x_9) + 0.2351\sin(x_9 - x_{10}) \\
& - 1.1580\tanh(x_{11} - x_9) - 1.1341\tanh(x_{12} - x_9) \\
& - 0.5668\tanh(x_{14} - x_9) + 0.2821\tanh(x_9 - x_{10})
\end{aligned}$$

$$\begin{aligned}
F_{10}(\mathbf{x}) = {} & -0.4858\sin(x_{12} - x_{10}) - 0.3435\sin(x_{14} - x_{10}) \\
& - 0.3244\sin(x_{15} - x_{10}) - 0.3004\sin(x_{16} - x_{10}) \\
& - 0.3087\sin(x_9 - x_{10}) - 0.5830\tanh(x_{12} - x_{10}) \\
& - 0.4122\tanh(x_{14} - x_{10}) - 0.3893\tanh(x_{15} - x_{10}) \\
& - 0.3605\tanh(x_{16} - x_{10}) - 0.3705\tanh(x_9 - x_{10})
\end{aligned}$$

$$\begin{aligned}
F_{11}(\mathbf{x}) = {} & 0.7749\sin(x_{11} - x_{12}) + 0.4606\sin(x_{11} - x_{15}) \\
& + 0.1655\sin(x_{11} - x_9) + 0.9299\tanh(x_{11} - x_{12}) \\
& + 0.5528\tanh(x_{11} - x_{15}) + 0.1986\tanh(x_{11} - x_9)
\end{aligned}$$

$$\begin{aligned}
F_{12}(\mathbf{x}) = {} & -0.8005\sin(x_{11} - x_{12}) + 0.0407\sin(x_{12} - x_{13}) \\
& + 0.4758\sin(x_{12} - x_{16}) + 0.8873\sin(x_{12} - x_9) \\
& + 0.6749\sin(x_{12} - x_{10}) - 0.9606\tanh(x_{11} - x_{12}) \\
& + 0.0488\tanh(x_{12} - x_{13}) + 0.5709\tanh(x_{12} - x_{16}) \\
& + 1.0647\tanh(x_{12} - x_9) + 0.8099\tanh(x_{12} - x_{10})
\end{aligned}$$

$$\begin{aligned}
F_{13}(\mathbf{x}) = {} & -0.6587\sin(x_{12} - x_{13}) + 0.3571\sin(x_{13} - x_{14}) \\
& + 0.8128\sin(x_{13} - x_{15}) - 0.7905\tanh(x_{12} - x_{13}) \\
& + 0.4285\tanh(x_{13} - x_{14}) + 0.9754\tanh(x_{13} - x_{15})
\end{aligned}$$

$$\begin{aligned}
F_{14}(\mathbf{x}) = {} & -0.0071\sin(x_{13} - x_{14}) + 0.4631\sin(x_{14} - x_{15}) \\
& + 0.5999\sin(x_{14} - x_9) + 0.7282\sin(x_{14} - x_{10}) \\
& - 0.0086\tanh(x_{13} - x_{14}) + 0.5558\tanh(x_{14} - x_{15}) \\
& + 0.7198\tanh(x_{14} - x_9) + 0.8738\tanh(x_{14} - x_{10})
\end{aligned}$$

$$\begin{aligned}
F_{15}(\mathbf{x}) = {} & -0.8285\sin(x_{11} - x_{15}) - 0.1281\sin(x_{13} - x_{15}) \\
& - 0.2304\sin(x_{14} - x_{15}) + 0.1325\sin(x_{15} - x_{16}) \\
& + 0.6078\sin(x_{15} - x_{10}) - 0.9941\tanh(x_{11} - x_{15}) \\
& - 0.1538\tanh(x_{13} - x_{15}) - 0.2765\tanh(x_{14} - x_{15}) \\
& + 0.1590\tanh(x_{15} - x_{16}) + 0.7293\tanh(x_{15} - x_{10})
\end{aligned}$$

$$\begin{aligned}
F_{16}(\mathbf{x}) = {} & -0.7822\sin(x_{12} - x_{16}) - 0.5326\sin(x_{15} - x_{16}) \\
& + 0.5749\sin(x_{16} - x_{10}) - 0.9387\tanh(x_{12} - x_{16}) \\
& - 0.6392\tanh(x_{15} - x_{16}) + 0.6900\tanh(x_{16} - x_{10})
\end{aligned}$$

Figure 15: The complete definition of the coupling functions $F_i(\mathbf{x})$ for the 16-agent system, as referenced in Eq. (1). The parameters are provided with full precision (rounded to four decimal places for display purposes).