PeriodicActivationFunctions

October 20, 2021

Load required libraries

[1]:
```
using Random
using CairoMakie
using Turing, KernelFunctions, NNlib, SpecialFunctions
using StatsFuns: logistic, norminvcdf
using Memoization, ReverseDiff
```

0.1 Periodic Activations Function

We will first define the periodic functions described in the manuscript. We refer to the appendix for details on the correct scaling.

[2]:
```
const triangleconst =  /(2*sqrt(2))
const preluconst =  /4
_trianglewave(x) = (x -   * floor(x/  + 0.5)) * (-1)^floor(x/  + 0.5)
_trianglewave(x::ReverseDiff.TrackedReal) = ReverseDiff.
 ↪ForwardOptimize(_trianglewave)(x)

heaviside(x) = x < zero(x) ? zero(x) : x >= zero(x) ? one(x) : oftype(x,0.5)
lsmatern(x, q,  ,  ) = q*heaviside(x)*x^( -1/2)*exp(- *x)
heaviside(x::ReverseDiff.TrackedReal) = ReverseDiff.
 ↪ForwardOptimize(heaviside)(x)
```

[2]:
```
heaviside (generic function with 2 methods)
```

[3]:
```
abstract type PeriodicFunction end

hasBias(::PeriodicFunction) = true

struct SinActivation <: PeriodicFunction end
const sineconst = sqrt(2)
( ::SinActivation)(x::T) where T = T(sineconst) * sin(x)

struct SinCosActivation <: PeriodicFunction end
( ::SinCosActivation)(x) = sin(x) + cos(x)
hasBias(::SinCosActivation) = false
```

```
struct TriangleWave <: PeriodicFunction end
(::TriangleWave)(x) = triangleconst * _trianglewave(x)

struct PeriodicReLU <: PeriodicFunction end
(::PeriodicReLU)(x) = preluconst * (_trianglewave(x) + _trianglewave(x + /2))
```

[4]:
```
abstract type ActivationFunction end

struct RBF <: ActivationFunction
    ::PeriodicFunction
end
(::RBF)(x) = .(x)
prior_weights(::RBF) = Normal()
prior_bias(::RBF) = Uniform(-, )

struct Matern <: ActivationFunction
    ::PeriodicFunction
     # smoothness
end
(::Matern)(x) = .(x)
prior_weights(::Matern) = . == 0.5 ? Cauchy() : TDist(2*.)
prior_bias(::Matern) = Uniform(-, )
```

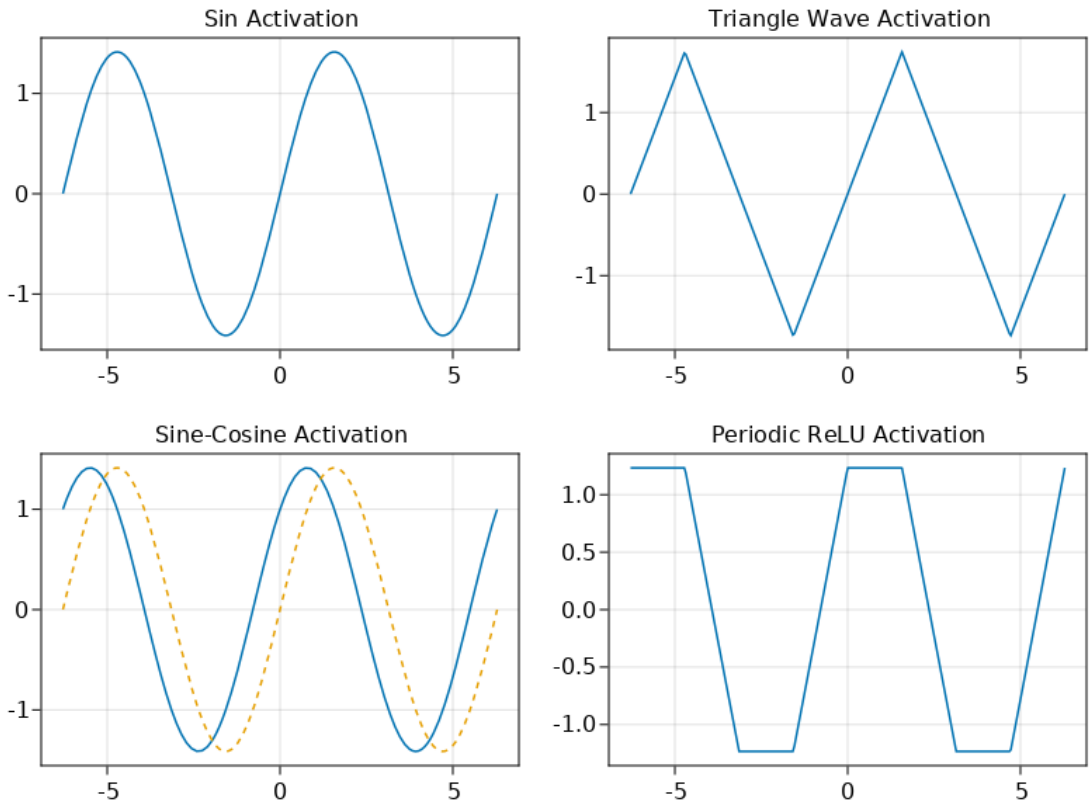
[4]:
```
prior_bias (generic function with 2 methods)
```

0.2 Visualisations

Next we will visualize the different activation functions over two periods.

[5]:
```
fig = Figure()

ax1 = fig[1,1] = Axis(fig, title = "Sin Activation")
lines!(ax1, -2*..2, x -> SinActivation()(x))

ax2 = fig[2,1] = Axis(fig, title = "Sine-Cosine Activation")
lines!(ax2, -2*..2, x -> SinCosActivation()(x))
lines!(ax2, -2*..2, x -> SinActivation()(x), linestyle = :dash)

ax3 = fig[1,2] = Axis(fig, title = "Triangle Wave Activation")
lines!(ax3, -2*..2, x -> TriangleWave()(x))

ax4 = fig[2,2] = Axis(fig, title = "Periodic ReLU Activation")
lines!(ax4, -2*..2, x -> PeriodicReLU()(x))

fig
```

[5]:



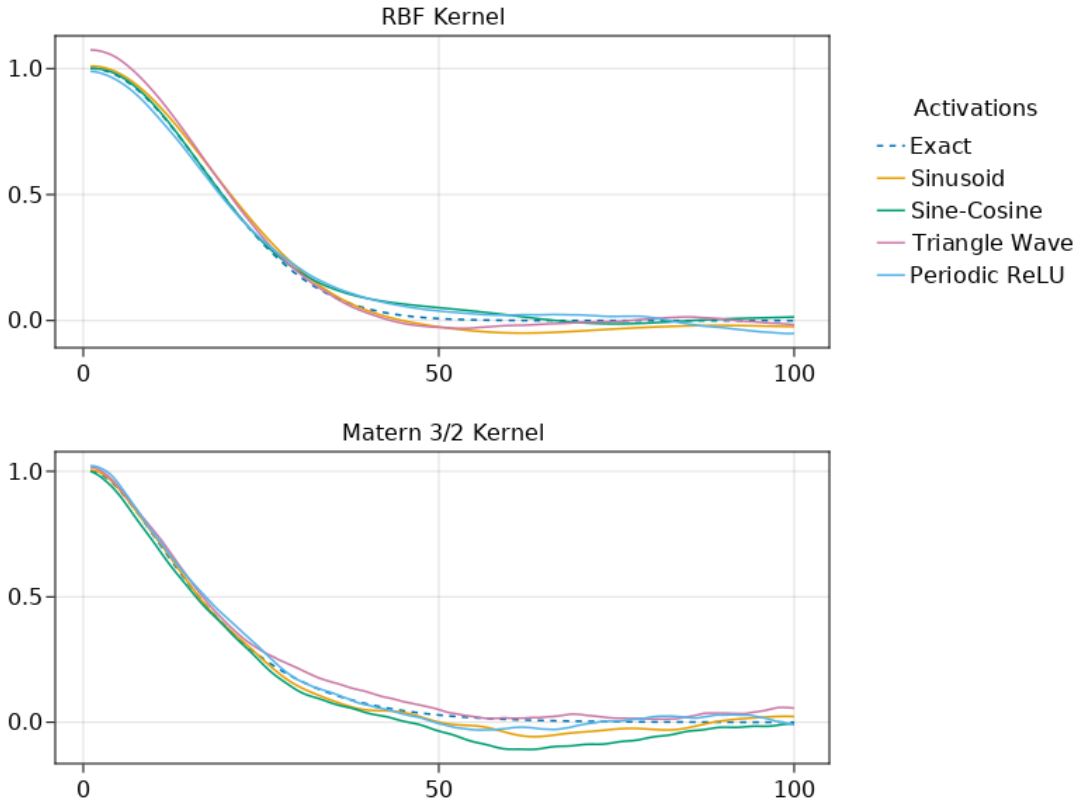
0.3 Covariance

Next we will estimate the induced covariance structure using MC integration and match it to the exact solutions.

[7]:
```
function  (x, y, afun; samples = 1000)

    b = hasBias(afun. ) ? rand(prior_bias(afun), 1, samples) : 0.0
    w = rand(prior_weights(afun), samples)


    vec(mean(afun.(x * w' .+ b) .* afun.(y * w' .+ b), dims=2))
end



x = collect(range(0, 2* , length=500))


fig = Figure()
```

```
ax1 = fig[1,1] = Axis(fig, title = "RBF Kernel")

k = SqExponentialKernel()
K = kernelmatrix(k,reshape(x, :, 1),obsdim=1)

lines!(ax1, 1..100, K[1,:], linestyle = :dash, label = "Exact")
lines!(ax1, 1..100,  (x, x[1], RBF(SinActivation())), label = "Sinusoid")
lines!(ax1, 1..100,  (x, x[1], RBF(SinCosActivation())), label = "Sine-Cosine")
lines!(ax1, 1..100,  (x, x[1], RBF(TriangleWave())), label = "Triangle Wave")
lines!(ax1, 1..100,  (x, x[1], RBF(PeriodicReLU())), label = "Periodic ReLU")

ax2 = fig[2,1] = Axis(fig, title = "Matern 3/2 Kernel")

k = Matern32Kernel()
K = kernelmatrix(k,reshape(x, :, 1),obsdim=1)

lines!(ax2, 1..100, K[1,:], linestyle = :dash, label = "Exact")
lines!(ax2, 1..100,  (x, x[1], Matern(SinActivation(), 3/2)), label =␣
↪"Sinusoid")
lines!(ax2, 1..100,  (x, x[1], Matern(SinCosActivation(), 3/2)), label =␣
↪"Sine-Cosine")
lines!(ax2, 1..100,  (x, x[1], Matern(TriangleWave(), 3/2)), label = "Triangle␣
↪Wave")
lines!(ax2, 1..100,  (x, x[1], Matern(PeriodicReLU(), 3/2)), label = "Periodic␣
↪ReLU")

fig[1,2] = Legend(fig, ax1, "Activations", framevisible = false)

fig
```

[7]:

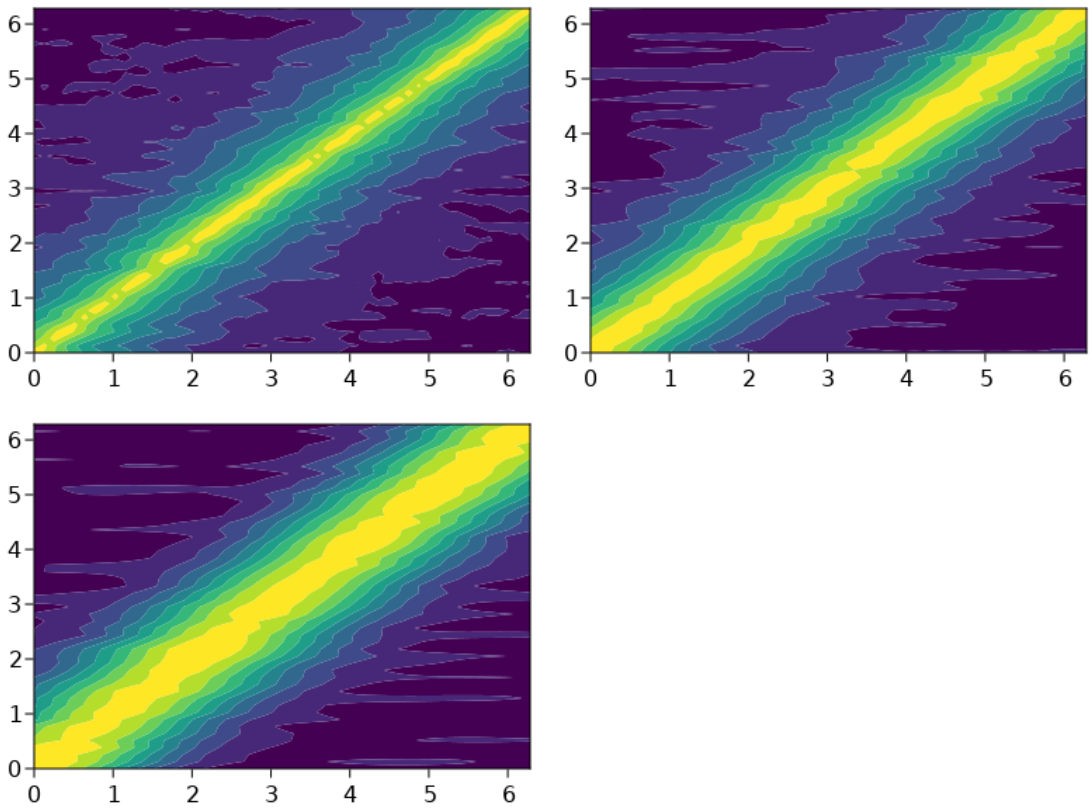


We can also directly visualize the Gram matrix.

[8]:
```
x = collect(range(0, 2* , length=50))
G  = mapreduce(xi ->  (x, xi, Matern(SinActivation(), 1/2)), hcat, x)
G  = mapreduce(xi ->  (x, xi, Matern(SinActivation(), 3/2)), hcat, x)
G  = mapreduce(xi ->  (x, xi, RBF(SinActivation())), hcat, x);
```

[9]:
```
f = Figure()
ax1 = Axis(f[1,1])
ax2 = Axis(f[1,2])
ax3 = Axis(f[2,1])

contourf!(ax1, x, x, G )
contourf!(ax2, x, x, G )
contourf!(ax3, x, x, G )

f
```

[9]:



0.4 Regression

In the next section of the notebook, we will use a stationary inducing

[10]:
```
N = 20
xs = (rand(N) * 2) .- 2
ys = exp.(abs.(xs))

ys .+= randn(N) * 0.1;
ys .-= mean(ys)
ys /= var(ys);
```

[11]:
```
@model function bnn(x, y, afun, K)

    w ~ filldist(prior_weights(afun), 1, K)
    b ~ filldist(prior_bias(afun), 1, K)

    l ~ truncated(Normal(1,5), 0, Inf)

    z = afun.(x * l * w .+ b)
```

```
    wo ~ filldist(Normal(), K)
    bo ~ Normal()

    f = (z * (wo / sqrt(K)) .+ bo)

    Turing.@addlogprob! -sum(abs2(yj-fj) for (fj,yj) in zip(f,y))
end
```

[11]: bnn (generic function with 1 method)

[12]:
```
Turing.setadbackend(:reversediff)
Turing.setrdcache(true);
```

[13]:
```
function predict(xtest, afun, K, chain; burnin = 1)

    w_names = filter(n -> contains(string(n), "w["), names(chain))
    wo_names = filter(n -> contains(string(n), "wo["), names(chain))
    b_names = filter(n -> contains(string(n), "b["), names(chain))

    w = chain[w_names]
    b = chain[b_names]
    wo = chain[wo_names]
    bo = chain["bo"]
    l = chain["l"]

    nsamples = length(w)

    latents = map(s -> begin
            w_ = Array(w[s])
            b_ = Array(b[s])

            wo_ = vec(Array(wo[s]))
            bo_ = bo[s]

            l_ = l[s]

            z = afun.(xtest * w_ * l_ .+ b_)
            f = (z * (wo_ / sqrt(K)) .+ bo_)
        end, burnin:nsamples)

    err_o = 1.96*sqrt.(var(latents)) .+ 1
    err = 1.96*std(latents)

    return mean(latents), err, err_o
end
```

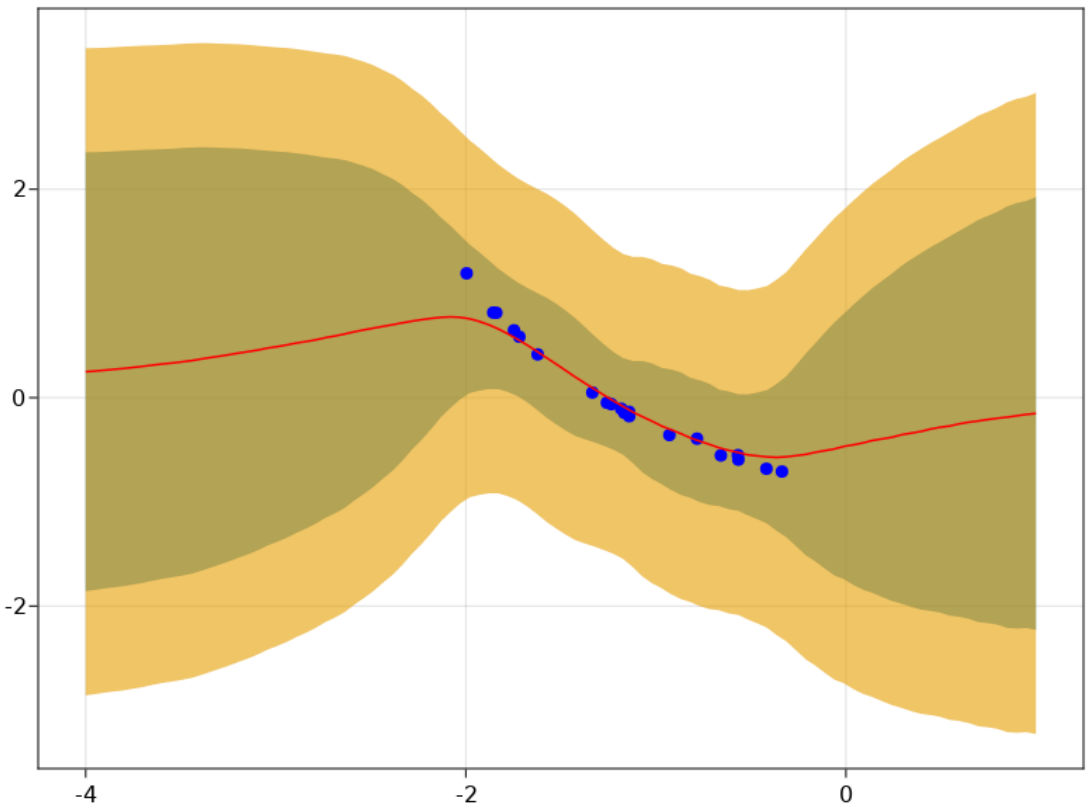
[13]: predict (generic function with 1 method)

[14]:
```
model = bnn(xs, ys, Matern(PeriodicReLU(), 3/2), 10);
chain = sample(model, NUTS(1000, 0.8), 4000);
```

```
 Info: Found initial step size
     = 0.4
 @ Turing.Inference
/u/54/meronel1/unix/.julia/packages/Turing/28kgo/src/inference/hmc.jl:188
 Warning: The current proposal will be rejected due to numerical error(s).
   isfinite.(( , r,  ,  )) = (true, false, false, false)
 @ AdvancedHMC
/u/54/meronel1/unix/.julia/packages/AdvancedHMC/MIxdK/src/hamiltonian.jl:47
Sampling:  17%|                                          |  ETA:
0:12:54  Warning: The current proposal will be rejected due to numerical
error(s).
   isfinite.(( , r,  ,  )) = (true, false, false, false)
 @ AdvancedHMC
/u/54/meronel1/unix/.julia/packages/AdvancedHMC/MIxdK/src/hamiltonian.jl:47
Sampling: 100%|                          | Time:
0:16:20
```

[15]:
```
xt = collect(range(-4, 1, length = 100))

yhat, f, o = predict(xt, Matern(PeriodicReLU(), 3/2), 10, chain);

f = Figure()
Axis(f[1, 1])

band!(xt, yhat + f, yhat - f)
band!(xt, yhat + o, yhat - o)
scatter!(xs, ys, color = :blue)
lines!(xt, yhat, color = :red)

f
```

[15]:


2
0
-2
-4
-2
0

[ ]: