Order **1** (1985), 317–331. 0167–8094/85.15.

Random Orders

PETER WINKLER *
Emory University, Atlanta, Georgia, U.S.A.

Communicated by P. C. Fishburn

(Received: 24 July 1984; accepted: 19 October 1984)

Abstract. Let k and n be positive integers and fix a set S of cardinality n; let $P_k(n)$ be the (partial) order on S given by the intersection of k randomly and independently chosen linear orders on S. We begin study of the basic parameters of $P_k(n)$ (e.g., height, width, number of extremal elements) for fixed k and large n. Our object is to illustrate some techniques for dealing with these 'random orders' and to lay the groundwork for future research, hoping that they will be found to have useful properties not obtainable by known constructions.

AMS (MOS) subject classifications (1980). 06A10, 60C05.

Key words. Partially ordered set, probabilistic methods, dimension.

1. Introduction

Random structures were introduced in graph theory 25 years ago, primarily by Erdös and Rényi ([4, 5]), and their beauty and utility have now been often demonstrated; see, e.g., Erdös and Spencer [6] or Bollobás's forthcoming book [2]. Unfortunately, transitivity prevents independent selection of pairs in an order relation, so the 'standard' model $G(n, p)$ for random graphs has no direct analog in the theory of ordered sets. Nonetheless, there is a natural (we think) model for random orders *of bounded dimension* which has several useful independence properties. While our model lacks the flexibility and power of random graphs and does not weight orders uniformly, it admits a variety of approaches and may yet prove useful.

By an *order* P we mean a transitive, irreflexive binary relation on some underlying set S, i.e., a partial ordering of S; we denote $(x, y) \in P$ ordinarily by $x < y$ or $y > x$. An order L is *linear* if $x < y$ or $y < x$ for every $x \neq y$ in S; a linear order on the underlying set of P which contains P is called a *linear extension* of P. It follows from Szpilrajn's Theorem [18] that P is the intersection of its linear extensions, and thus for any finite P there is a least number k such that P can be written as the intersection of just the linear orders $L_1, L_2, \ldots, L_k$. This number k is called the *dimension* of P and denoted $\dim(P)$. The concept of dimension in ordered sets was introduced by Dushnik and Miller [3] in 1941 and seems quite fundamental; see e.g., [11] and [15].

* Supported by NSF grant MCS 84-02054.

We denote by I^k the unit hypercube $[0,1]^k$ in Euclidean k-space, endowed with Lebesgue measure and the ordinary 'product' order: $x < y$ iff $x \neq y$ and $x_i \leqslant y_i$ for each i, $1 \leqslant i \leqslant k$. It is immediate that a finite ordered set P has dimension $\leqslant k$ iff it can be order-embedded in I^k.

We now proceed to the definition of a 'random order' $P_k(n)$. Fix positive integers k and n and let S be the set $\{x(1), x(2), \ldots, x(n)\}$; let $\mathscr{L}_k(n)$ be the set of all k-tuples of linear orders on S. Define $P_k(n) = L_1 \cap L_2 \cap \ldots \cap L_k$, where $(L_1, L_2, \ldots, L_k)$ is randomly chosen from among the $n!^k$ members of $\mathscr{L}_k(n)$. Then $P_k(n)$ is a random variable whose values are the orders P of dimension $\leqslant k$ on S, each taken with probability $w(P)/n!^k$ where $w(P)$ is the number of ways of representing P as the (ordered) intersection of k linear orders.

A second, equivalent definition of $P_k(n)$ is obtained by choosing n points $x(1), x(2), \ldots, x(n)$ independently from the uniform probability distribution on I^k, and letting $P_k(n)$ be the induced suborder. Since the probability that two points in S have the same ith coordinate is zero, the real numbers $x(1)_i, x(2)_i, \ldots, x(n)_i$ fall into a random linear order and the intersection of these k independent orders is, of course, $P_k(n)$. Thus, the random order $P_k(n)$ can be thought of either as the result of k independent choices (of linear orders) or of n independent choices (of points in I^k). (See Figure 1.)

In this paper we will use only elementary probability theory, primarily the following facts: (1) if $A_1, A_2, \ldots, A_m$ are independent events then $\mathrm{pr}(A_1 \text{ and } A_2 \text{ and } \ldots \text{ and } A_m) = \mathrm{pr}(A_1) \cdot \mathrm{pr}(A_2) \cdot \ldots \cdot \mathrm{pr}(A_m)$; (2) if $X_1, X_2, \ldots, X_m$ is *any* sequence of random variables then $E(X_1 + X_2 + \cdots + X_m) = E(X_1) + \cdots + E(X_m)$, where $E(X)$ is the expectation of X; and (3) if X is a random variable whose values are nonnegative integers, then $\mathrm{pr}(X > 0) \leqslant E(X)$.

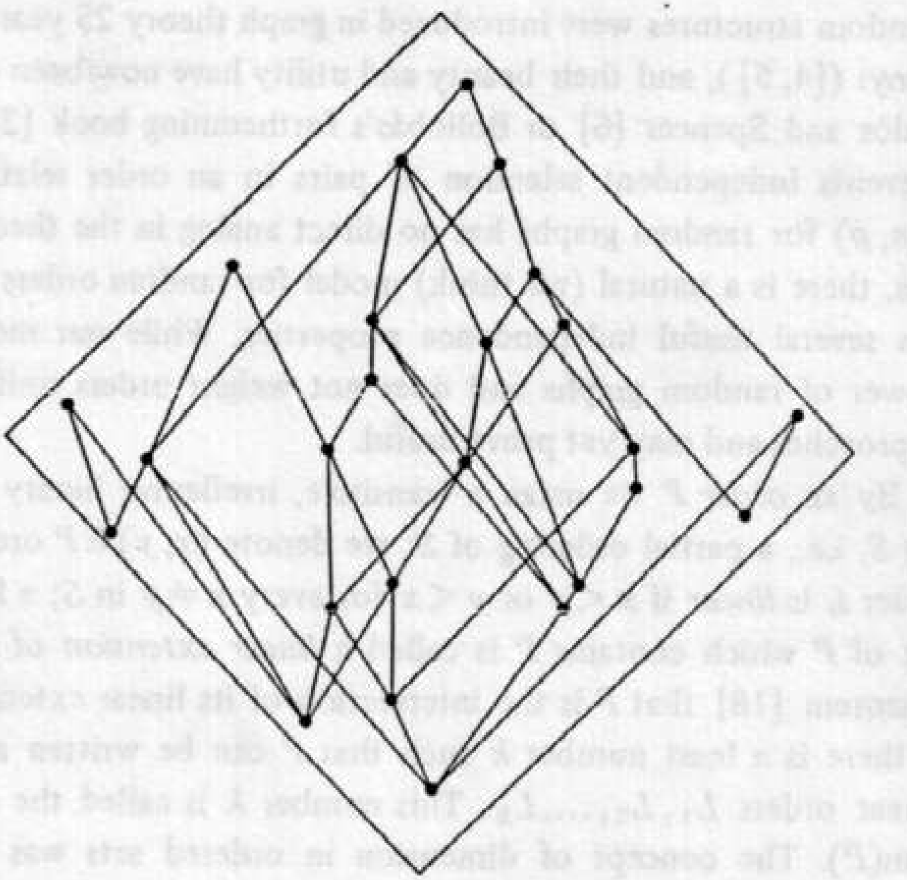

Fig. 1. Hasse diagram of a $P_2(25)$ generated by random points in a square.

2. Suborders of a Random Order

The material of this section will be self-evident to readers familiar with random structures. We have organized it into three 'observations'.

(1) Let A be a fixed subset of $S = \{x(1), \ldots, x(n)\}$ of size m; then the suborder induced on A by $P_k(n)$ has the same distribution as $P_k(m)$. Moreover, if A and B are disjoint subsets of S, the orders induced on them are independent.

This follows from the equivalent fact about random linear orders, or, even more easily, by choosing the points of A first in the random point model.

(2) For any fixed order Q of dimension $\leqslant k$, $\mathrm{pr}(P_k(n)$ contains an isomorphic copy of $Q) \to 1$ as $n \to \infty$.

To see this let m be the number of points in Q and let $A_1, \ldots, A_{\lfloor n/m \rfloor}$ be disjoint subsets of $S = \{x(1), \ldots, x(n)\}$, each of size m. If $r = \mathrm{pr}(P_k(m)$ is isomorphic to $Q)$ then $r > 0$ and $\mathrm{pr}(P_k(n)$ contains no copy of $Q) \leqslant (1-r)^{\lfloor n/m \rfloor} \to 0$.

In general if $\mathscr{P}$ is a property of ordered sets such that $\mathrm{pr}(P_k(n)$ has $\mathscr{P}) \to 1$ as $n \to \infty$, we say that 'almost every P_k has $\mathscr{P}$'. Thus, for example, almost every P_k has dimension k since we may take $\dim(Q) = k$ above. (Such orders exist for any k; see, e.g., [11].)

(3) For any order Q on m points, the expected number of suborders of $P_k(n)$ which are isomorphic to Q is

$$\binom{n}{m} \cdot \mathrm{pr}(P_k(m) \simeq Q).$$

In particular, the expected number of comparable pairs in $P_k(n)$ is

$$\binom{n}{2} \cdot 2^{1-k}.$$

The suborders above are regarded as unlabelled, i.e., there can be at most one copy of Q on a given set of m points. If A is such a set and the random variable X_A is defined to be 1 if the order induced on A by $P_k(n)$ is isomorphic to Q and 0 otherwise, the result follows by summing the expectations of the X_A's.

Two elements x, y of an ordered set are *comparable* if $x < y$ or $y < x$; of course $\mathrm{pr}(x < y) = \mathrm{pr}(y < x) = 2^{-k}$ in $P_k(n)$. Thus, the comparability graph $G(P_k(n))$, whose vertices are the points of $P_k(n)$ and edges the comparable pairs, is a random graph with edge-probability 2^{1-k} but with edges chosen in a highly dependent manner. Not surprisingly, the behavior of $G(P_k(n))$ will turn out to be very different from that of the ordinary random graph $G(n, 2^{1-k})$.

3. Extremal Elements

An element x of an ordered set Q is *minimal* (resp. *maximal*) if there is no y in Q with $y < x$ (resp. $y > x$). On account of duality the expected number of minimal elements in $P_k(n)$ is the same as the expected number of maximals; let $M_k(n)$ stand for either.

THEOREM 1. *The expected number of minimal elements in $P_k(n)$ is increasing in k and n and asymptotic to $(\ln(n))^{k-1}/(k-1)!$ for large n*, i.e., $\lim_{n\to\infty} M_k(n)\cdot(k-1)!/(\ln(n))^{k-1}=1$.

Proof. We first show, by induction on k, that

$$M_k(n)=\sum_{i=1}^{n}\frac{M_{k-1}(i)}{i}$$

for all $n \geqslant 1$, $k \geqslant 2$. (The monotonicity assertion follows.)

Let $x=(x_1,\ldots,x_k)$ be the first point chosen in the hypercube I^k during the creation of $P_k(n)$; then the probability that the next point is below x is Πx_i, the volume of the set of points below x in I^k. (Henceforth we use the unadorned symbol Π to denote the product from $i=1$ to k, and Π' to denote the product from $i=1$ to $k-1$.) It follows that $\mathrm{pr}(x \text{ is minimal in } P_k(n))=(1-\Pi x_i)^{n-1}$, and therefore that the probability that a *random* x will be minimal is

$$\int_{I^k}(1-\Pi x_i)^{n-1}\,\Pi\,dx_i$$

Summing expectations over the n points in the underlying set of $P_k(n)$ yields

$$M_k(n)=n\int_0^1\int_0^1\cdots\int_0^1(1-\Pi x_i)^{n-1}\,\Pi\,dx_i.$$

$$=\int_{I^{k-1}}\frac{1}{\Pi' x_i}\,(1-(1-\Pi' x_i)^n)\Pi'\,dx_i$$

$$=\int_{I^{k-1}}\left(\sum_{i=1}^{n}(1-\Pi' x_i)^{i-1}\right)\Pi'\,dx_i$$

$$=\sum_{i=1}^{n}\frac{M_{k-1}(i)}{i}\quad\text{as promised.}$$

We now show that

$$\frac{(\ln(n))^{k-1}}{(k-1)!}\leqslant M_k(n)\leqslant\sum_{i=0}^{k-1}\frac{(\ln(n))^i}{i!}$$

Which suffices to establish the asymptotic value.

Noting that $M_1(n)=M_k(1)=1$ for all k and n and that $M_k(n)/n$ is *decreasing* in n, we proceed again by induction on k:

$$\frac{(\ln(n))^{k-1}}{(k-1)!}=\int_1^n\frac{(\ln(u))^{k-2}}{(k-2)!\,u}\,du\leqslant\sum_{i=1}^{n}\frac{(\ln(i))^{k-2}}{(k-2)!\,i}$$

$$\leqslant \sum_{i=1}^{n} \frac{M_{k-1}(i)}{i} = M_k(n) \leqslant \sum_{i=1}^{n} \sum_{j=0}^{k-2} \frac{(\ln(i))^j}{j!\,i}$$

$$\leqslant 1 + \int_1^n \left(\sum_{j=0}^{k-2} \frac{(\ln(u))^j}{j!\,u} \right) du = \sum_{i=0}^{k-1} \frac{(\ln(n))^i}{i!}. \qquad \square$$

4. Isolated Elements

An element of an ordered set is *isolated* if no other element is comparable to it, i.e., if it is an isolated point of the comparability graph. Our principal aim in this section is to show that almost every P_k has no isolated elements. Notice that a random element of $P_k(n)$ has probability *at least* $k(k-1)/n^2$ of being isolated, for it may appear at the top of one linear order from the generating sequence $(L_1, \ldots, L_k)$ and at the bottom of another; this is already much higher than the corresponding probability for the random graph $G(n, 2^{1-k})$, namely $(1 - 2^{1-k})^{n-1}$.

It suffices, of course, to prove that the expected number $I_k(n)$ of isolated elements in $P_k(n)$ tends to zero; the method of Section 3 gives

$$I_k(n) = n \int_0^1 \int_0^1 \cdots \int_0^1 (1 - \Pi x_i - \Pi(1 - x_i))^{n-1} \Pi\, dx_i$$

which seems to be difficult to evaluate.

Instead we try to exploit the observation that the minimality and maximality of a random element of $P_k(n)$ must surely be negatively correlated, that is, the occurrence of one makes the other less likely; then the probability that x is minimal *and* maximal, i.e., isolated, would be less than $(M_k(n)/n)^2$. To do this we employ a kind of continuous version of the FKG inequality [9], which asserts the nonnegative correlation of increasing functions on a finite distributive lattice. The FKG inequality, in various forms, is fast becoming a familiar tool in the theory of ordered sets; see, e.g., [16, 8].

A function $f\colon I^k \to \mathbb{R}$ is *increasing* if whenever $x_i \leqslant y_i$ for each i, $1 \leqslant i \leqslant k$, we have $f(x_1, \ldots, x_k) \leqslant f(y_1, \ldots, y_k)$; i.e., f is order-preserving relative to our product order on I^k. Such functions can be shown to be integrable, but in fact we need the following result only for polynomials.

LEMMA. *Let f and g be increasing functions from the unit k-dimensional hupercube I^k to the reals; then*

$$\int_{I^k} f \cdot g \geqslant \int_{I^k} f \cdot \int_{I^k} g.$$

Proof. For $k = 1$ the statement is well known and easy; see, e.g., [1], problem 7.17, p. 177. Higher dimensions can be handled by applying the FKG inequality to Riemann sums, but we prefer to use the following simple fact, deducible from Kleitman's inequality [12] for ideals in a Boolean lattice: if p and q are increasing functions on $\{0, 1\}^k$ then

$$2^k \cdot \sum_{a\in\{0,1\}^k} p(a)q(a) \geqslant \sum_{a,\,b\in\{0,1\}^k} p(a)q(b).$$

Let a' denote the *complement* of an element $a = (a_1, \ldots, a_k)$ in $\{0, 1\}^k$, i.e., $a'_i = 1$ iff $a_i = 0$. Since the function q^* given by $q^*(a) = -q(a')$ is also increasing, we have

$$-2^k \cdot \sum_{a\in\{0,1\}^k} p(a)q^*(a) \leqslant - \sum_{a,\,b\in\{0,1\}^k} p(a)q^*(b)$$

$$= \sum_{a,\,b\in\{0,1\}^k} p(a)q(b).$$

hence,

$$\sum_{a\in\{0,1\}^k} p(a)q(a) \geqslant \sum_{a\in\{0,1\}^k} p(a)q(a').$$

Now let f and g satisfy the hypothesis of the Lemma and fix points $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$ in I^k. For each $a \in \{0, 1\}^k$ define the point $\hat{a}$ in I^k by

$$\hat{a}_i = \begin{cases} \max(x_i, y_i) & \text{if } a_i = 1, \\ \min(x_i, y_i) & \text{if } a_i = 0. \end{cases}$$

For example, if x happens to be below y in I^k and $a = (1, 0, 1)$ then $\hat{a} = (y_1, x_2, y_3)$, and $\hat{a}' = (x_1, y_2, x_3)$.

Then, for any $x, y \in I^k$, we have

$$\sum_{a\in\{0,1\}^k} f(\hat{a})g(\hat{a}) \geqslant \sum_{a\in\{0,1\}^k} f(\hat{a})g(\hat{a}')$$

and, hence,

$$0 \leqslant \int_{I^k}\int_{I^k}\left(\sum_{a\in\{0,1\}^k} f(\hat{a})g(\hat{a}) - \sum_{a\in\{0,1\}^k} f(\hat{a})g(\hat{a}')\right)\Pi\, \mathrm{d}x_i\, \Pi\, \mathrm{d}y_i$$

$$= 2^k \int_{I^k}\int_{I^k} f(x)g(x)\,\Pi\, \mathrm{d}x_i\, \Pi\, \mathrm{d}y_i - 2^k \int_{I^k}\int_{I^k} f(x)g(y)\,\Pi\, \mathrm{d}x_i\, \Pi\, \mathrm{d}y_i$$

(by renaming the variables in each summand)

$$= 2^k \left(\int_{I^k} f(x)g(x)\,\Pi\, \mathrm{d}x_i - \int_{I^k} f(x)\,\Pi\, \mathrm{d}x_i \int_{I^k} g(x)\,\Pi\, \mathrm{d}x_i\right)$$

and the lemma follows. Finally:

THEOREM 2. *The expected number of isolated elements in $P_k(n)$ is bounded above by $(\ln(n))^{2k-2}/((k-1)!^2 n)$ asymptotically in n; in particular, almost every P_k has no isolated elements.*

Proof.

$$I_k(n) = n \int_{I^k} (1 - \Pi x_i - \Pi(1 - x_i))^{n-1} \Pi \, dx_i$$

$$\leqslant n \int_{I^k} (1 - \Pi x_i - \Pi(1 - x_i) + \Pi x_i \cdot \Pi(1 - x_i))^{n-1} \Pi \, dx_i$$

$$= n \int_{I^k} (1 - \Pi x_i)^{n-1} (1 - \Pi(1 - x_i))^{n-1} \Pi \, dx_i$$

$$\leqslant n \int_{I^k} (1 - \Pi x_i)^{n-1} \Pi \, dx_i \int_{I^k} (1 - \Pi(1 - x_i))^{n-1} \Pi \, dx_i$$

(by the Lemma, with $f(x) = (1 - \Pi x_i)^{n-1}$ and $g(x) = -(1 - \Pi(1 - x_i))^{n-1}$)

$$= \frac{1}{n} \left(n \int_{I^k} (1 - \Pi x_i)^{n-1} \Pi \, dx_i \right) \left(n \int_{I^k} (1 - \Pi y_i)^{n-1} \Pi \, dy_i \right)$$

$$= \frac{1}{n} (M_k(n))^2$$

and the result now follows from Theorem 1.

5. Height and Width

A *chain* in an ordered set Q is a subset on which the induced order is linear, and an *antichain* is a subset whose elements are pairwise incomparable. The *height* of Q is the cardinality of a largest chain (assuming Q is finite) and the *width* of Q is the cardinality of a largest antichain; these correspond to the clique number and independence number, respectively, of the comparability graph of Q. These numbers are of order $c_k \log(n)$ in the random graph $G(n, 2^{1-k})$, but the situation in $G(P_k(n))$ must be vastly different since the product of the height and width of any ordered set of cardinality n is at least n. (If each element is ranked according to the length of the longest chain it dominates, there are height-many ranks and each is an antichain.)

THEOREM 3. *There is a constant c (depending only on k) with $0 < c < e$, such that almost every P_k has height between $cn^{1/k}$ and $en^{1/k}$.*

Proof. To obtain the upper bound, set $m = \lceil en^{1/k} \rceil$. Then

$$\text{pr}(\text{height}(P_k(n)) > en^{1/k})$$

$$\leqslant E \text{ (number of chains of length } m \text{ in } P_k(n))$$

$$= \binom{n}{m} \cdot \frac{1}{m!^{k-1}} = \frac{n!}{(n - \lceil en^{1/k} \rceil)! \lceil en^{1/k} \rceil !^k}$$

which, using standard approximations, can be shown to approach zero (barely) as $n \to \infty$.

For the lower bound we switch to geometric methods with the idea of 'constructing'

$\bar{x}) \in I^k$ be the orthogonal projection of x onto the main diagonal of the hypercube.

In what follows we make use of numbers $c_j > 0$ which depend on k but are constant relative to n. Let $\| \;\|$ be the Euclidean norm, and for any set $T \subset \mathbb{R}^k$ let $\alpha(T) = \{\alpha(x) : x \in T\}$. Let

$$U = \{x \in \mathbb{R}^k : \|x - \alpha(x)\| \leqslant n^{-1/k} \text{ and } (\sqrt{k} - \sqrt{k-1})/2 \leqslant \bar{x} \leqslant (\sqrt{k} + \sqrt{k-1})/2\};$$

then U is a cylinder which lies inside I^k for sufficiently large n, and whose axis $\alpha(U)$ is an interval of length $\sqrt{k-1}$ in $\alpha(I^k)$. Choose n random points in I^k to make up the set S which underlies $P_k(n)$; since the volume of U is

$$\sqrt{k-1} \cdot \frac{\pi^{(k-1)/2} (n^{-1/k})^{k-1}}{\Gamma((k+1)/2)} = c_1 n^{-(k-1)/k},$$

U will amost always contain at least $c_2 n^{1/k}$ points of S, for any $c_2 < c_1$. If V is a random subset of $U \cap S$ of cardinality $c_2 n^{1/k}$, then $\alpha(V)$ is simply a collection of $c_2 n^{1/k}$ points randomly and independently chosen from the uniform distribution on the line segment $\alpha(U)$. (See Figure 2.)

Partition $\alpha(U)$ into $\frac{1}{2} n^{1/k}$ consecutive intervals $A_1, A_2, \ldots$ each of length $2\sqrt{k-1} \cdot n^{-1/k}$. If $X_j = |A_j \cap \alpha(V)|$ then X_j is asymptotically Poisson distributed with mean $2c_2$. It follows that if

$$c_3 < \tfrac{1}{2} \cdot \tfrac{1}{2} \cdot (1 - e^{-2c_2})$$

then almost always at least $c_3 n^{1/k}$ of the *odd-numbered* intervals A_{2j-1} will contain points of $\alpha(V)$. Thus $\alpha(V)$ contains a set $\alpha(W)$ of cardinality $c_3 n^{1/k}$ any two elements of which are at distance at least $2\sqrt{k-1} \cdot n^{-1/k}$. We claim that its inverse image $W \subset U \cap S$ is a chain in $P_k(n)$.

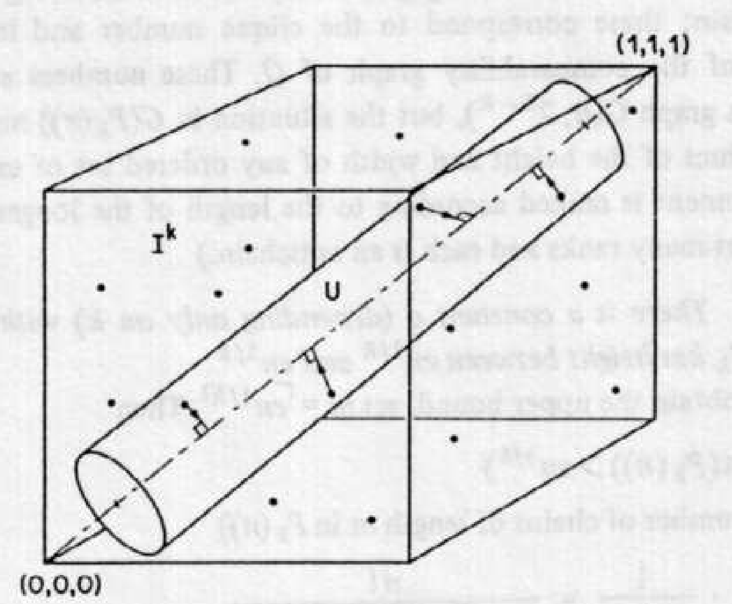

Fig. 2. The finding of a chain in $P_k(n)$; for the case $k = 3$.

To see this let x be any point in U and notice that

$$(n^{-1/k})^2 \geqslant \|x - \alpha(x)\|^2$$

$$= (\bar{x} - x_k)^2 + \sum_{i=1}^{k-1} (x_i - \bar{x})^2$$

$$\geqslant (\bar{x} - x_k)^2 + \frac{1}{k-1}\left(\sum_{i=1}^{k-1} (x_i - \bar{x})\right)^2$$

$$= (\bar{x} - x_k)^2 + ((k\bar{x} - x_k) - (k-1)\bar{x})^2/(k-1)$$

$$= \frac{k}{k-1}(\bar{x} - x_k)^2.$$

Now if x and y are incomparable points of U, we may assume $\bar{x} > \bar{y}$ but $x_k < y_k$; then $|\bar{x} - \bar{y}| < |\bar{x} - x_k| + |y_k - \bar{y}| \leqslant ((k-1)/k)^{1/2} n^{-1/k} \cdot 2$. Thus $\|\alpha(x) - \alpha(y)\| < 2\sqrt{k-1} \cdot n^{-1/k}$ and so x and y cannot be in W, and the proof is complete. □

Notice that in the case $k = 2$, the height and the width of $P_k(n)$ are identically distributed; for, rotating the square 90° (or, equivalently, reversing one of the generating linear orders) turns chains into antichains and antichains into chains. Thus, both the height and width of almost every $P_2(n)$ are between $(1/e)n^{1/2}$ and $en^{1/2}$. A conjecture of Ulam's (see [6], p. 95) would imply that the 'correct' constant is, in fact, 2.

Notice that if $X_1, \ldots, X_m$ is a sequence of nonnegative random variables whose product is always at least a certain constant c, then the product of the expectations of the X_i's is at least c; for, $\ln(c) \leqslant E(\ln(X_1)) + E(\ln(X_2)) + \cdots + E(\ln(X_m)) \leqslant \ln(E(X_1) + \cdots + E(X_m))$ by concavity of the log function. Applying this to the height and width of $P_2(n)$ tells us that the *expected* height (and expected width) must be at least $\sqrt{n}$. The argument can be generalized to $P_k(n)$, but unfortunately it yields only a lower bound for the expected height of $n^{2^{1-k}}$ instead of $cn^{1/k}$ from Theorem 3.

For $k > 2$ we were unable to get an upper bound of $cn^{(k-1)/k}$ for the width of $P_k(n)$; in the following theorem we settle for $c = \ln(n)$ although that is faster-growing than necessary.

THEOREM 4. *The width of almost every P_k lies between $e^{-1}n^{(k-1)/k}$ and $n^{(k-1)/k}$ $\ln(n)$.*

Proof. The lower bound is immediate from Theorem 3. For the upper bound we employ geometric methods similar to those above, so we will be somewhat sketchier here. As usual we choose n random points in I^k to constitute the underlying set S of $P_k(n)$; let $m = \lceil n^{1/k} \rceil$ and for each $b = (b_1, \ldots, b_k) \in \{1, 2, \ldots, m\}^k$ let

$$C_b = \{x \in I^k : (b_i - 1)/m < x_i < b_i/m \text{ for each } i, 1 \leqslant i \leqslant k\}.$$

Then the C_b's partition I^k into approximately n small hypercubes.

Let

$$S \cap C_b = \{y(b, j) : 1 \leqslant j \leqslant |S \cap C_b|\}$$

and define

$$S_j = \{y(b, j) : b \in \{1, \ldots, m\}^k\}.$$

Let

$$S' = \bigcup \{S_j : j > c_1 \ln(n)\}$$

where c_1 is a constant (relative to n) to be determined later; since the numbers $S \cap C_b$ are asymptotically Poisson distributed (with mean 1), $|S'| < n^{(k-1)/k}$ for sufficiently large n.

Now for any $a \in \{0, 1\}^k$ let

$$T_a = \bigcup \{C_b : b_i = a_i \pmod 2, 1 \leqslant i \leqslant k\};$$

then all pairs of points in $S_j \cap T_a$ are at least distance $1/m$ apart. We claim that at most $c_2 n^{(k-1)/k}$ points of $S_j \cap T_a$ can be part of an antichain; this would suffice to prove the theorem since there are only $2^k c_1 \ln(n)$ sets $S_j \cap T_a$ for $j < c_1 \ln(n)$ (we can choose $c_1 < 2^{-k} c_2$) and the remainder S' is small.

For any $x \in I^k$, let $\check{x} = \min\{x_i : 1 \leqslant i \leqslant k\}$ and set $\gamma(x) = (\check{x}, \check{x}, \ldots, \check{x})$. Then $\delta(x) = x - \gamma(x)$ is the projection of x parallel to the main diagonal onto one of the k faces of the hypercube which contain the origin. Suppose that x and y are incomparable and $\|x - y\| > 1/m$; we may then assume $\check{x} > \check{y}$ but $x_k < y_k$. If $|\check{x} - \check{y}| < \|x - y\|/(1 + \sqrt{k})$ then

$$\begin{aligned}
&\|\delta(x) - \delta(y)\| \\
&\quad = \|x - \gamma(x) - y + \gamma(y)\| \geqslant \|x - y\| - \|\gamma(x) - \gamma(y)\| \\
&\quad = \|x - y\| - \sqrt{k}\,|\check{x} - \check{y}| \geqslant \|x - y\|/(1 + \sqrt{k}) \geqslant 1/(m(1 + \sqrt{k}));
\end{aligned}$$

on the other hand if $|\check{x} - \check{y}| \geqslant \|x - y\|/(1 + \sqrt{k})$ then $|(\delta(x))_k - (\delta(y))_k| > |\check{x} - \check{y}|$ and again $\|\delta(x) - \delta(y)\| > \|x - y\|/(1 + \sqrt{k}) \geqslant 1/(m(1 + \sqrt{k}))$. Thus, the balls $B(\delta(x))$ and $B(\delta(y))$ of radius $1/(2m(1 + \sqrt{k}))$ about $\delta(x)$ and $\delta(y)$ are disjoint.

If A is an antichain in $S_j \cap T_a$ and $x \in A$ then the area (i.e., $(k-1)$-dimensional volume) of $B(\delta(x)) \cap F$, where F is one of the aforementioned faces, is at least $c_3 (n^{-1/k})^{k-1}$ for appropriate c_3; but the total area of these faces is only k. Thus, $|A| < c_2 n^{(k-1)/k}$ for suitable c_2, and the proof is complete. □

6. First-Order Properties

In 1976, Ronald Fagin [7] showed that any graph property definable by a first-order statement is either almost always true or almost always false in random graphs. Closely related to this result is the fact that a random graph on $\aleph_0$ vertices is isomorphic, with probability 1, to the unique countably infinite model of a certain complete first-order theory. Our random orders provide some interesting parallels and contrasts.

In the interest of brevity we will need to assume some background in model theory in this section; our proofs will tend to emphasize aspects peculiar to orders and order-dimension. Readers are referred to Shoenfield [17] for logic background.

There are two formal languages useful in the study of ordered sets of dimension at most k. The 'strong' language $\mathscr{L}_k$ contains binary relation symbols $<_1, <_2, \ldots, <_k$ as well as $=$ for equality; we let T_k be the theory whose axioms state that each $<_i$ is a linear order. A model of T_k will be called a *k-order*.

The 'weak' language $\mathscr{L}$ contains only equality and the binary relation symbol $<$; in it we let U_k be the theory of ordered sets of dimension $\leqslant k$. U_k is axiomatized by the usual axioms for a (partial) order $<$, together with a sentence for each minimal $(k+1)$-dimensional ordered set Q to the effect that Q cannot occur as a suborder. It follows by compactness that every model of U_k can be augmented (possibly in many ways) to a model of T_k in which $x<y$ iff $x<_i y$ for each i; and of course each model of T_k restricts uniquely to a model of U_k with $<$ defined as above.

Now fix k and $m \geqslant 1$ and let $a=(a_1, \ldots, a_k) \in \{0, 1, \ldots, m\}^k$. Let $z, y(1), \ldots, y(m)$ be variables and let $\alpha_a(z; y(1), \ldots, y(m))$ be a quantifier-free formula in $\mathscr{L}_k'$ which says the following: for each i, $1 \leqslant i \leqslant k$, there is a permutation σ_i of $\{1, \ldots, m\}$ such that $y(\sigma_i(j)) <_i y(\sigma_i(j+1))$ for $1 \leqslant j < m$ and $y(\sigma_i(j)) <_i z$ for $j \leqslant a_i$ and $y(\sigma_i) >_i z$ for $j > a_i$.

Let T_k^* be the union of T_k and the following collection of axioms, one for each $m \geqslant 1$ and $a \in \{0, 1, \ldots, m\}^k$:

$$\forall y(1)\, \forall y(2) \cdots \forall y(m)\ \exists z \Big(\bigwedge_{i \neq j} y(i) \neq y(j) \to \alpha_a(z; y(1), \ldots, y(m)) \Big).$$

THEOREM 5. *T_k^* is complete and $\aleph_0$-categorical, and is the model-completion of T_k.*

We omit the proof, which is analogous to the proof of the well known case where $k=1$; T_1^* is the theory of dense linear orderings without endpoints and T_1 is just the theory of linear orderings. Note that to show T_k^* is $\aleph_0$-categorical, which means that any two countably infinite models are isomorphic, a standard back-and-forth argument is employed; the axioms above provide exactly the property required.

Returning now to random orders, we define $P_k(\omega)$ as the order induced by I^k on a countably infinite sequence $x(1), x(2), \ldots$ of points randomly and independently chosen from I^k. (We are implicitly using as our probability density the product measure on $(I^k)^\omega$; this is defined as in Halmos [10], pp. 154–158.) $P_k(\omega)$ is naturally a k-order, with $x <_i y$ iff $x_i < y_i$ as real numbers. If $y(1), \ldots, y(m)$ are distinct points of $P_k(\omega)$ and $a \in \{0, 1, \ldots, m\}^k$, then $\{z \in I^k : \alpha_a(z; y(1), \ldots, y(m))\}$ has, with probability 1, Lebesgue measure greater than 0; hence, with probability 1 there is a $z \in P_k(\omega)$ such that $y(\sigma_i(j)) <_i y(\sigma_i(j+1))$ for $1 \leqslant j < m$ and $y(\sigma_i(j)) <_i z$ for $j \leqslant a_i$ and $y(\sigma_i(j)) >_i z$ product measure that with probability 1, $P_k(\omega)$ satisfies all the axioms of T_k^*; hence, we have

THEOREM 6. *Let P and Q be two random orders of dimension k on countably infinite sets, i.e., two values of $P_k(\omega)$. Then with probability 1 they are isomorphic.*

This isomorphism is of a strong type (as k-orders with labelled orders) which will become even stronger as a result of Theorem 7 below.

What we would like now is a theory in the weak language $\mathscr{L}$ which bears the same relationship to U_k that T_k^* does to T_k. The existence of such a theory follows from:

THEOREM 7. *There is a theory U_k^* for each $k \geqslant 1$ such that the following are equivalent for any ordered set P:*

(1) *P is a model of U_k^*;*

(2) *there is an augmentation of P (i.e., a list of linear extensions $<_1, \ldots, <_k$ whose intersection is the order on P) which is a model of T_k^*;*

(3) *every augmentation of P is a model of T_k^*.*

Proof. Of course the orders $<_1, \ldots, <_k$ cannot generally be recovered from $<$, but they can be defined up to permutation of $1, \ldots, k$ by existential $\mathscr{L}$-formulas inside models of T_k^*. To be precise, let $\delta(b(1), \ldots, b(m))$ be the conjunction of all statements of the type '$b(i) <_j b(i')$' which are true of a particular list $b(1), \ldots, b(m)$ of distinct elements of some k-order. Then there is an existential $\mathscr{L}$-formula γ such that for any $y(1), \ldots, y(m)$ in a model of T_k^*, $\gamma(y(1), \ldots, y(m))$ holds if and only if for some permutation σ of $\{1, \ldots, k\}$ the formula $\delta_\sigma(y(1), \ldots, y(m))$ holds, where δ_σ is the result of replacing each $<_i$ in δ by $<_{\sigma(i)}$. It follows then that in any k-order, $\gamma(y(1), \ldots, y(m))$ *implies* the disjunction over σ of $\delta_\sigma(y(1), \ldots, y(m))$. In particular, the formula

$$\bigwedge_{a \in \{0,1,\ldots,m\}^k} \alpha_a(z(a); y(1), \ldots, y(m)),$$

which is symmetric in $<_1, \ldots, <_k$, has a corresponding $\mathscr{L}$-formula $\gamma(z(00\cdots0), z(00\cdots1), \ldots, z(mm\cdots m); y(1), \ldots, y(m))$ which is equivalent to it in T_k^* and implies it in T_k. With these T_k^* can be axiomatized in $\mathscr{L}$.

In the case $m = 2$ we need to be able to say in $\mathscr{L}$ that '$x <_i y$ for exactly h values of i' for $0 \leqslant h \leqslant k$, which we abbreviate by '$x \,{}^h\!\!< y$'. Here $\gamma(x, y)$ will say that there are points $u(1), \ldots, u(k)$ and $v(1), \ldots, v(k)$ such that

(1) $u(i) < v(j)$ for $i \neq j$;

(2) $u(i) < x < v(j)$ and $u(j) < y < v(i)$ for $j \leqslant h < i$; and

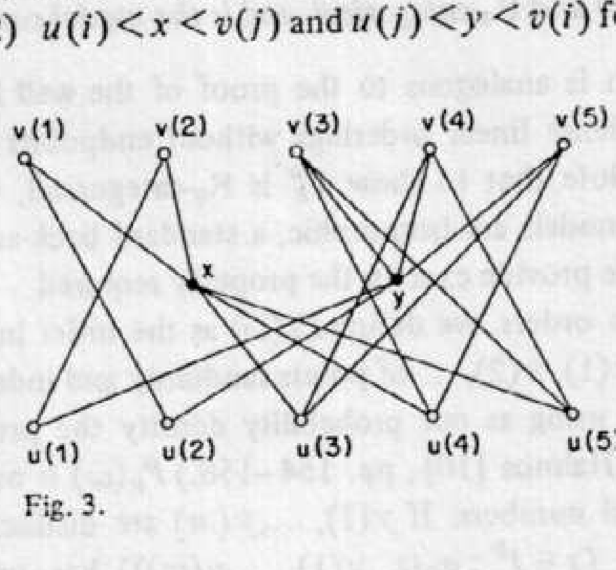

Fig. 3.

(3) no other $<$-comparabilities exist among the u's, v's, x and y except possibly between x and y.

(A diagram of this situation is shown in Figure 3, where $k = 5$ and $h = 2$.) Notice that in any augmentation, if $<_i$ is the (necessarily unique) order in which $v(i) < u(i)$, then perforce $x <_i y$ when $i \leqslant h$ and $y \leqslant_i x$ otherwise. Thus $\gamma(x, y) \to x \,{}^h\!\!< y$ in any k-order and they are equivalent in models of T_k^*, where the $u(i)$'s and $v(i)$'s will exist whenever possible.

In general it suffices to be able to say, for any $m \geqslant 1$, any permutation τ of $\{1, \ldots, m\}$, and any h with $1 \leqslant h \leqslant k$, that there are at least h values of i for which $y(\tau(1)) <_i y(\tau(2)) <_i \cdots <_i y(\tau(m))$. This will occur in a model of T_k^* just when there exist w and $z(1), \ldots, z(m)$ such that

(1) $z(i) < z(j)$ for $1 \leqslant i < j \leqslant m$;

(2) $z(i) < y(\tau(j))$ for $1 \leqslant i < j \leqslant m$;

(3) $y(\tau(i)) \,{}^h\!\!< z(j)$ for $1 \leqslant i \leqslant j \leqslant m$; and

(4) $y(i) < w \,{}^h\!\!< z(m)$ for $1 \leqslant i \leqslant m$.

Figure 4 shows an example of this situation with $k = 5$, $m = 3$, $h = 2$ and τ equal to the identity permutation. For this particular $y(1)$, $y(2)$ and $y(3)$, three more schemes, each with $h = 1$, will suffice to nail down the relationship of the $y(i)$'s up to permutation of $<_1, \ldots, <_5$.

COROLLARY. *Let P be a random k-dimensional order on a countably infinite set of points. Then with probability* 1, *P will have the following property: any two realizations of P as the intersection of a sequence of k linear orders are isomorphic as k-orders.*

In the theory of random graphs, after $\aleph_0$-categoricity and completeness are established for the theory analogous to our U_k^*, it is noted that each axiom of the theory is true in almost every *finite* random graph; Fagin's theorem follows. Alas, many single statements which are consequences of U_k^* (e.g., nonexistence of minimal elements) fail in *every* finite ordered set.

It is not known whether Fagin's theorem holds when an order is chosen at random

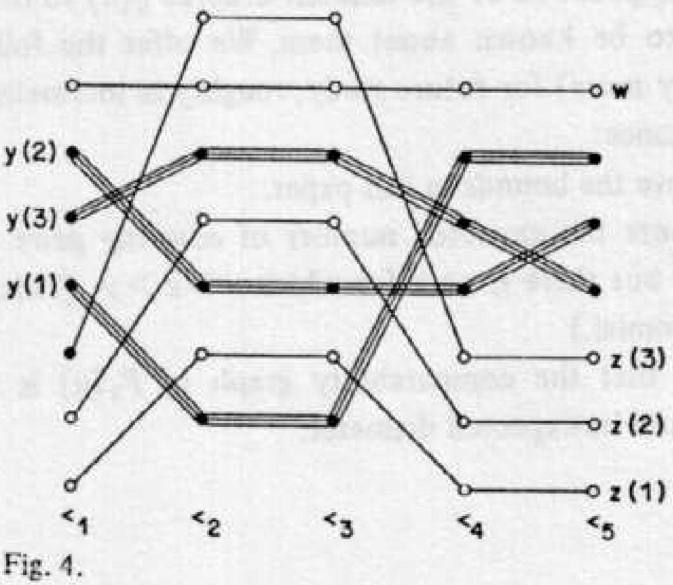

Fig. 4.

the work of Kleitman and Rothschild [13] of what most such orders look like; this problem was posed by Rolf Möhring [14]. In our case, however, the theorem fails already for $k = 2$.

THEOREM 8. *There is a first-order property β in the language of ordered sets such that*

$$0 < \lim_{n\to\infty} (\mathrm{pr}(P_2(n) \textit{ satisfies } \beta)) < 1.$$

Proof. Let β be the following $\mathscr{L}$-sentence:

$$\exists x \; \exists y \; \forall z \, [(x < y) \wedge \wedge (x < z \rightarrow (y < z \vee y = z)) \wedge \wedge (z < y \rightarrow (z < x \vee z = x))].$$

It is easily checked that β holds in $P_2(n)$ just when there is a pair of elements x, y such that y is the successor of x in both generating orders L_1 and L_2. We may assume that L_1 is the order $x(1) < x(2) < \cdots < x(n)$ and that the mapping which takes i to the height of $x(i)$ in L_2 is a random permutation σ of $\{1, \ldots, n\}$. Thus, we need only to compute the probability that for some $i < n$, $\sigma(i+1) = 1 + \sigma(i)$; a question similar to the famous *'problème des rencontres'*. Let X_r be number of r-tuples $(i_1, \ldots, i_r)$ such that $\sigma(i_j + 1) = 1 + \sigma(i_j)$ for each j, $1 \leqslant j \leqslant r$; then

$$\lim_{n\to\infty} (E(X_r)) = \lim_{n\to\infty} \frac{n!}{(n-r)!} ((n-1)(n-3) \cdots (n-2r+1))^{-1} = 1.$$

Thus X_1 is asymptotically Poisson distributed with mean 1 (see, e.g., [2] for the method of factorial moments) and hence,

$$\lim_{n\to\infty} (\mathrm{pr}(\beta \text{ holds in } P_2(n))) = \lim_{n\to\infty} (\mathrm{pr}(X_1 \geqslant 1)) = 1 - \frac{1}{e}$$

and the proof is complete. □

7. Problems

To facilitate application of the random orders $P_k(n)$ to the theory of ordered sets, much more needs to be known about them. We offer the following problems (readers will think of many more) for future study, roughly in increasing order of anticipated difficulty and importance:

(1) Improve the bounds in this paper.

(2) Compute the expected number of *covering pairs* in $P_k(n)$, where x covers y if $x > y$ but there is no z for which $x > z > y$. (This reduces to integrating a large polynomial.)

(3) Prove that the comparability graph of $P_k(n)$ is almost always connected and compute its expected diameter.

(4) Compute the expected number of linear extensions of $P_k(n)$; equivalently, compute the probability that $P_{k+1}(n)$ is an antichain.

(5) Prove that $\lim_{n\to\infty}(\mathrm{pr}(P_k(n)$ satisfies $\gamma))$ exists for any first-order sentence γ in $\mathscr{L}$ (or, even better, in $\mathscr{L}_k$) and determine what values these limits may take.

(6) Let $Q_k(n)$ be a random k-dimensional order on a labelled n-element set, each such order being weighted equally. Find a theorem relating the asymptotic probabilities that α holds in $P_k(n)$ and in $Q_k(n)$, for some suitable class of properties α.

Acknowledgements

This work was inspired by the Meeting on Graphs and Order (Banff, AB, Canada: May 1984) and the NSF-CBMS Conference on Extremal Graph Theory (Emory University, Atlanta, Georgia, U.S.A.: June 1984), for which we thank Ivan Rival and Béla Bollobás. Subsequent conversations with Dwight Duffus were indispensable.

References

1. T. Apostol (1974) *Mathematical Analysis*, 2nd edn., Addison-Wesley, Reading, Mass.
2. B. Bollobás (1984) *Random Graphs*, in preparation.
3. B. Dushnik and E. W. Miller (1941) Partially ordered sets, *Amer. J. Math.* 63, 600–610.
4. P. Erdös and A. Rényi (1959) On random graphs I, *Publ. Math. Debrecen* 6, 290–297.
5. P. Erdös and A. Rényi (1960) On evolution of random graphs, *Mat. Kutató Int. Közl.* 5, 17–60.
6. P. Erdös and J. Spencer (1974) *Probabilistic Methods in Combinatorics*, Academic Press, New York.
7. R. Fagin (1976) Probabilities on finite models, *J. Symbolic Logic* 41, 50–58.
8. P. C. Fishburn (1984) A correlational inequality for linear extensions of a poset, *Order* 1, 127–137.
9. C. M. Fortuin, P. W. Kasteleyn, and J. Ginibre (1971) Correlation inequalities on some partially ordered sets, *Commun. Math. Phys.* 22, 89–103.
10. P. R. Halmos (1950) *Measure Theory*, Van Nostrand, Princeton.
11. D. Kelly and W. T. Trotter (1982) Dimension theory for ordered sets, in *Ordered Sets* (ed. I. Rival), D. Reidel, Dordrecht.
12. D. J. Kleitman (1966) Families of non-disjoint sets, *J. Comb. Theory* 1, 153–155.
13. D. J. Kleitman and B. L. Rothschild (1975) Asymptotic enumeration of partial orders on a finite set, *Trans. Am. Math. Soc.* 205, 205–210.
14. R. Möhring (1985) Asymptotic behavior of ordered sets (Enumeration problem #4) in *Graphs and Order* (ed. I. Rival), D. Reidel, Dordrecht.
15. M. Paoli, W. T. Trotter, and J. W. Walker (1985) Graphs and orders in Ramsey theory and in dimension theory, in *Graphs and Order* (ed. I. Rival), D. Reidel, Dordrecht (to appear).
16. L. A. Shepp (1982) The XYZ conjecture and the FKG inequality, *Ann. Prob.* 10, 824–827.
17. J. R. Shoenfield (1967) *Mathematical Logic*, Addison-Wesley, Reading, Mass.
18. E. Szpilrajn (1930) Sur l'extension de l'ordre partiel, *Fund. Math.* 16, 386–389.

NOTE ADDED IN PROOF. It has been brought to the author's attention that there are results in the statistical theory of reliability that imply the lemma of Section 4 above, for example in G. A. Battle and L. Rosen, *J. Stat. Phys.* 2 (1980). The conjecture of Ulam's mentioned in Section 5 was proved by A. M. Versik and S. V. Kerov, *Dokl. Akad. Nauk SSSR* 233 (1977).