FU, Ziyan (zf119)

Imperial College London

Department of Computing

Academic Year **2019-2020**

Page created Thu Feb 20 02:15:18 GMT 2020

CATE

499 fbelard 6 a5 zf119 v1 0 0 1 1 9 0 0 3 6 0 9 4	Electronic submission **Wed - 19 Feb 2020 21:07:00** zf119

Exercise Information

Module:	499 Modal Logic for Strategic Reasoning in AI	**Issued:**	Wed - 05 Feb 2020
Exercise:	6 (CW)	**Due:**	Wed - 19 Feb 2020
Title:	Coursework2	**Assessment:**	Individual
FAO:	Belardinelli, Francesco (fbelard)	**Submission:**	Electronic

Student Declaration - Version 1

- I declare that this final submitted version is my unaided work.

Signed: (electronic signature) Date: 2020-02-19 08:25:57

For Markers only: (circle appropriate grade)

FU, Ziyan (zf119)	01742528	a5	2020-02-19 08:25:57	**A* A B C D E F**

C499 Modal Logic for Strategic Reasoning in AI Coursework 2

Ziyan Fu

01742528 zf119

ziyan.fu19@imperial.ac.uk

1.

(a)

$\pi \vDash \varphi R \psi$ iff

$\forall j \geq 0. \pi[j..\infty] \vDash \psi$ or $\exists i \geq 0 \left((\pi[i..\infty] \vDash \varphi) \wedge \forall\, 0 \leq k \leq i\, (\pi[k..\infty] \vDash \psi)\right)$

(b)

$\varphi R \psi \equiv \neg\left((\neg\varphi) U (\neg\psi)\right)$

(c)

Truth conditions of the LTL formula in (b), which is $\neg\left((\neg\varphi) U (\neg\psi)\right)$

iff (definition of R)

$\neg\exists j \geq 0. \left((\pi[j..\infty] \vDash \neg\psi) \wedge \forall 0 \leq i < j. (\pi[i..\infty] \vDash \neg\varphi)\right)$

iff (semantics of negation)

$\neg\exists j \geq 0. \left((\pi[j..\infty] \nvDash \psi) \wedge \forall 0 \leq i < j. (\pi[i..\infty] \nvDash \varphi)\right)$

iff (duality of $\exists$ and $\forall$)

$\forall j \geq 0. \neg\left((\pi[j..\infty] \nvDash \psi) \wedge \forall 0 \leq i < j. (\pi[i..\infty] \nvDash \varphi)\right)$

iff (de Morgan’s law)

$\forall j \geq 0. \left(\neg(\pi[j..\infty] \nvDash \psi) \vee \neg\forall 0 \leq i < j. (\pi[i..\infty] \nvDash \varphi)\right)$

iff (semantics of negation)

$\forall j \geq 0. (\pi[j..\infty] \vDash \psi \vee \exists 0 \leq i < j. \pi[i..\infty] \vDash \varphi)$

iff $\forall j \geq 0. \pi[j..\infty] \vDash \psi$ or $\exists i \geq 0. \left((\pi[i..\infty] \vDash \varphi) \wedge \forall\, 0 \leq k \leq i. (\pi[k..\infty] \vDash \psi)\right)$

iff Truth conditions provided in (a)

(d)

$\perp R \psi$

iff $(\forall i \geq 0, \pi[i..\infty] \vDash \psi)\ \vee\ \exists i \geq 0, \left(\pi[i..\infty] \vDash \perp \wedge (\forall 0 \leq j \leq i, \pi[j..\infty] \vDash \psi)\right)$

iff $(\forall i \geq 0, \pi[i..\infty] \vDash \psi)\ \vee\ \exists i \geq 0, \left(\perp \wedge (\forall 0 \leq j \leq i, \pi[j..\infty] \vDash \psi)\right)$

iff $(\forall i \geq 0, \pi[i..\infty] \vDash \psi)\ \vee\ \perp$

iff $(\forall i \geq 0, \pi[i..\infty] \vDash \psi)$

iff $G\psi$

2.

$(M, q) \vDash EF\Phi$

iff $(M, q) \vDash E(trueU\Phi)$

iff for some path λ from q, $(M, \lambda) \vDash (trueU\Phi)$

iff for some path λ from q, for some $j \geq 0$, $(M, \lambda[j]) \vDash \Phi$ and $(M, \lambda[k]) \vDash \top$ for all $0 \leq k < j$

iff for some path λ from q, for some $j \geq 0$, $(M, \lambda[j]) \vDash \Phi$

$(M, q) \vDash AF\Phi$

iff $(M, q) \vDash A(trueU\Phi)$

iff for every path λ from q, $(M, \lambda) \vDash (trueU\Phi)$

iff for every path λ from q, for some $j \geq 0$, $(M, \lambda[j]) \vDash \Phi$ and $(M, \lambda[k]) \vDash \top$ for all $0 \leq k < j$

iff for every path λ from q, for some $j \geq 0$, $(M, \lambda[j]) \vDash \Phi$

$(M, q) \vDash EG\Phi$

iff $(M, q) \vDash \neg AF\neg\Phi$

iff It's not the case that, for all path λ from q, $(M, \lambda) \vDash F\neg\Phi$

iff for some path λ from q, it's not the case that, $(trueU\neg\Phi)$

iff for some path λ from q, it's not the case that, for some $j \geq 0$, $(M, \lambda[j]) \vDash \neg\Phi$ and $(M, \lambda[k]) \vDash \top$, for all $0 \leq k < j$

iff for some path λ from q, for all $j \geq 0$, $(M, \lambda[j]) \vDash \Phi$

$(M, q) \vDash AG\Phi$

iff $(M, q) \vDash \neg EF\neg\Phi$

iff It's not the case that, for some path λ from q, $(M, \lambda) \vDash F\neg\Phi$

iff for every path λ from q, it's not the case that, $(trueU\neg\Phi)$

iff for every path λ from q, it's not the case that, for some $j \geq 0$, $(M, \lambda[j]) \vDash \neg\Phi$ and $(M, \lambda[k]) \vDash \top$, for all $0 \leq k < j$

iff for every path λ from q, for all $j \geq 0$, $(M, \lambda[j]) \vDash \Phi$

3.

(a)

Temporal logic on infinite trees [Clarke & Emerson 1981]

- State formulas Φ, Ψ:

(1)	- $a \in AP$	atoms
(2)	- $\neg\Phi$	negation
(3)	- $\Phi \wedge \Psi$	conjunction
(4)	- $\mathsf{E}\phi$	for some path ϕ is true
(5)	- $\mathsf{A}\phi$	for every path ϕ is true

- Path formulas ϕ:

(6)	- $\mathsf{X}\Phi$	*neXt* Φ
(7)	- $\Phi \mathsf{U} \Psi$	Φ *Until* Ψ

Definition 1 (Syntax of CTL*) *State (Φ) and path (ψ) formulas in CTL* are defined in Backus-Naur form as follows, where p is an atom:*

$$\begin{aligned} \Phi \quad &::= \quad \overset{(1)}{p} \mid \overset{(2)}{\neg\Phi} \mid \overset{(3)}{\Phi \wedge \Phi} \mid \overset{(4)}{E\psi} \mid \overset{(5)}{A\psi} \\ \psi \quad &::= \quad \underset{(6)}{\Phi} \mid \underset{(7)}{\neg\psi} \mid \underset{(8)}{\psi \wedge \psi} \mid \underset{(9)}{X\psi} \mid \underset{(10)}{\psi U \psi} \end{aligned}$$

The formulas of CTL are all and only the* state *formulas.*

We need to show that

For every formula Φ, if Φ is a formula of CTL, then Φ is also a formula in CTL*

Proof by induction

Base case:

For a formula $\Phi = p, p \in AP$,

if Φ is a formula of CTL as condition (1) of CTL syntax,

then Φ is also a formula of CTL* since condition (1) of CTL* syntax.

Inductive cases:

Let state formula Ψ is a CTL and CLT* formula.

For a formula $\Phi = \neg\Psi$,

if Φ is a formula of CTL as condition (2) of CTL syntax,

then Φ is also a formula of CTL* since condition (2) of CTL* syntax.

Let both state formulas $\Psi1$ and $\Psi2$ are CTL and CLT* formulas.

For a formula $\Phi = \Psi1 \wedge \Psi2$,

if Φ is a formula of CTL as condition (3) of CTL syntax,

then Φ is also a formula of CTL* since condition (3) of CTL* syntax.

Let γ be a path formula, for a formula $\Phi = E\gamma$,

if Φ is a formula of CTL as condition (4) of CTL syntax,

it means that γ can only be $X\,\Psi1$ or $\Psi1\,U\,\Psi2$ where both $\Psi1$ and $\Psi2$ are state formulas as conditions (6) and (7) of CTL syntax,

then Φ is also a formula of CTL* since conditions (4), (9) and (10) of CTL* syntax.

Let γ be a path formula, for a formula $\Phi = A\gamma$,

if Φ is a formula of CTL as condition (5) of CTL syntax,

it means that γ can only be $X\,\Psi1$ or $\Psi1\,U\,\Psi2$ where both $\Psi1$ and $\Psi2$ are state formulas as conditions (6) and (7) of CTL syntax,

then Φ is also a formula of CTL* since conditions (5), (9) and (10) of CTL* syntax.

In conclusion, for every formula Φ, if Φ is a formula of CTL, then Φ is also a formula in CTL*

(b)

Let $\Phi = A(Xp \wedge Xq)$

Φ is a CTL* formula, since $\Phi = A\psi$, $\psi = Xp \wedge Xq$ and condition (5) of CTL* syntax,
since $\psi = Xp \wedge Xq$, $\varphi_1 = Xp$, $\varphi_2 = Xp$ and condition (9) of CTL* syntax,
since $\varphi_1 = \Psi_1$, $\Psi_1 = p$ and conditions (6) and (1) of CTL* syntax, and
$\varphi_2 = \Psi_2$, $\Psi_2 = q$ and conditions (6) and (1) of CTL* syntax.

Φ is NOT a CTL formula, since $\Phi = A\psi$, $\psi = Xp \wedge Xq$ and condition (5) of CTL syntax,
since $(Xp \wedge Xq)$ is NOT a path formula of CTL syntax.

NOTE:

There exists some formula Φ' in CTL* such that $\Phi = A(Xp \wedge Xq)$ and $\Phi' = AX(p \wedge q)$ are equivalent, BUT $\Phi = A(Xp \wedge Xq)$ itself is NOT a CTL formula.

4.

Definition 2 (Semantics of CTL*) *Let M be a model, s a state, π a path, Φ and Φ' state formulas, and ψ' path formulas. Then,*

(1)	$(M,s) \models p$	*iff*	$s \in V(p)$
(2)	$(M,s) \models \neg\Phi$	*iff*	$(M,s) \not\models \Phi$
(3)	$(M,s) \models \Phi \wedge \Phi'$	*iff*	$(M,s) \models \Phi$ *and* $(M,s) \models \Phi'$
(4)	$(M,s) \models E\psi$	*iff*	*for some path* π *starting from* s, $(M,\pi) \models \psi$
(5)	$(M,s) \models A\psi$	*iff*	*for all paths* π *starting from* s, $(M,\pi) \models \psi$
(6)	$(M,\pi) \models \Phi$	*iff*	$(M,\pi[0]) \models \Phi$, *where* $\pi[0]$ *is the initial state in path* π.
(7)	$(M,\pi) \models \neg\psi$	*iff*	$(M,\pi) \not\models \psi$
(8)	$(M,\pi) \models \psi \wedge \psi'$	*iff*	$(M,\pi) \models \psi$ *and* $(M,\pi) \models \psi'$
(9)	$(M,\pi) \models X\psi$	*iff*	$(M,\pi[1\ldots\infty]) \models \psi$
(10)	$(M,\pi) \models \psi U \psi'$	*iff*	$(M,\pi[i\ldots\infty]) \models \psi'$ *for some* $i \geq 0$, *and* $(M,\pi[j\ldots\infty]) \models \psi$ *for all* $0 \leq j < i$

Slide from Lecture 5:

Let

- $M = \langle St, \longrightarrow, V \rangle$ be a model defined on a transition system
- Φ, Ψ be state formulas
- γ be a path formula.

Definition 1.7 (Semantics of CTL: state formulas)

(1)	$(M,q) \models a$	iff	$q \in V(a)$
(2)	$(M,q) \models \neg\Phi$	iff	$(M,q) \not\models \Phi$
(3)	$(M,q) \models \Phi \wedge \Psi$	iff	$(M,q) \models \Phi$ and $(M,q) \models \Psi$
(4)	$(M,q) \models \mathrm{E}\gamma$	iff	for *some* path λ starting from q, $(M,\lambda) \models \gamma$
(5)	$(M,q) \models \mathrm{A}\gamma$	iff	for *all* paths λ starting from q, $(M,\lambda) \models \gamma$

Definition 1.8 (Semantics of CTL: path formulas)

(6)	$(M,\lambda) \models \mathrm{X}\Phi$	iff	$(M,\lambda[1]) \models \Phi$
(7)	$(M,\lambda) \models \Phi \mathrm{U} \Psi$	iff	$(M,\lambda[i]) \models \Psi$ for some $i \geq 0$, and $(M,\lambda[j]) \models \Phi$ for all $0 \leq j < i$

By restricting Def. 2 to formulas in CTL, we **freeze** the conditions (6), (7) and (8). We need to show that the Def. 2 after freezing will obtain the same truth conditions as the semantics of CTL.

After restricting, a user can only call the conditions (1)-(5) directly to build a CTL formula, which are the entailment from a **state** s, but NOT allowed to call the conditions (6)-(10) directly, which are the entailment from a **path** π, hence CTL only can be applied on a state.

For **conditions (1), (2) and (3)** in both CTL and CLT* semantics, these pairs are exactly same. Hence, after restricting, the conditions (1), (2) and (3) of CTL* semantics will obtain the same truth conditions as conditions (1), (2) and (3) semantics of CTL.

For **condition** (4) in CTL* semantics:

$(M, s) \vDash E\varphi$ in CTL* semantics

iff for some path π starting from s, $(M, \pi) \vDash \varphi$ in CTL* semantics

iff for some path π starting from s, $(M, \pi) \vDash X\psi$ in CTL* semantics or
$(M, \pi) \vDash \psi U \psi'$ in CTL* semantics

iff for some path π starting from s, $(M, \pi[1 \cdots \infty]) \vDash \psi$ or
$(M, \pi[i \cdots \infty]) \vDash \psi'$ for some $i \geq 0$, and
$(M, \pi[j \cdots \infty]) \vDash \psi$ for all $0 \leq j < i$

(at following step, we applied the condition (6) of CTL semantics)*

iff for some path π starting from s,
$(M, \pi[1]) \vDash \Phi$, where $\pi[1]$ is the initial state of path $\pi[1 \cdots \infty]$ and $\Phi = \psi$ or
$(M, \pi[i]) \vDash \Phi'$ and $\Phi' = \psi'$ for some $i \geq 0$, and $(M, \pi[j]) \vDash \Phi$ and
$\Phi = \psi$ for all $0 \leq j < i$

iff for some path π starting from s,
$(M, \pi[1]) \vDash X\Phi$ in CTL semantics such that state formula $\Phi = \psi$ or
$(M, \pi[i]) \vDash \Phi U \Phi'$ in CTL semantics such that state formula $\Phi = \psi$ and $\Phi' = \psi'$.

iff for some path π starting from s, $(M, \pi) \vDash \varphi$ in CTL semantics.

iff $(M, \pi) \vDash E\varphi$ in CTL semantics.

For **condition** (5) in CTL* semantics:

$(M, s) \vDash A\varphi$ in CTL* semantics

iff for all path π starting from s, $(M, \pi) \vDash \varphi$ in CTL* semantics

iff for all path π starting from s, $(M, \pi) \vDash X\psi$ in CTL* semantics or
$(M, \pi) \vDash \psi U \psi'$ in CTL* semantics

iff for all path π starting from s, $(M, \pi[1 \cdots \infty]) \vDash \psi$ or
$(M, \pi[i \cdots \infty]) \vDash \psi'$ for some $i \geq 0$, and
$(M, \pi[j \cdots \infty]) \vDash \psi$ for all $0 \leq j < i$

(at following step, we applied the condition (6) of CTL semantics)*

iff for all path π starting from s,
$(M, \pi[1]) \vDash \Phi$, where $\pi[1]$ is the initial state of path $\pi[1 \cdots \infty]$ and $\Phi = \psi$ or
$(M, \pi[i]) \vDash \Phi'$ and $\Phi' = \psi'$ for some $i \geq 0$, and $(M, \pi[j]) \vDash \Phi$ and
$\Phi = \psi$ for all $0 \leq j < i$

iff for all path π starting from s,
$(M, \pi[1]) \vDash \Phi\psi$ in CTL semantics such that state formula $\Phi = \psi$ or
$(M, \pi[i]) \vDash \Phi U \Phi'$ in CTL semantics such that state formula $\Phi = \psi$ and $\Phi' = \psi'$.

iff for all path π starting from s, $(M, \pi) \vDash \varphi$ in CTL semantics.

iff $(M, \pi) \vDash A\varphi$ in CTL semantics.

In conclusion, the Def. 2 after freezing will obtain the same truth conditions as in Def. 1.7 and 1.8 in Lecture 5 (semantics of CTL).

5.

(a)

Proof by induction

Base case:

For condition (1) in CTL semantics:

$(M, s) \vDash p,\ p \in AP$, w.r.t. CTL semantics

iff $s \in N(p)$

$(M, s) \vDash p,\ p \in AP$, w.r.t. CTL* semantics

Inductive cases:

For condition (2) in CTL semantics:

Let Φ be a state formula of CTL, by the conclusion of question (3), Φ is also a CTL* state formula.

$(M, s) \vDash \neg\Phi$, w.r.t. CTL semantics

iff $(M, s) \nvDash \Phi$

$(M, s) \vDash \neg\Phi$, w.r.t. CTL* semantics

For condition (3) in CTL semantics:

Let Φ and Φ' be state formulas of CTL, by the conclusion of question (3), Φ and Φ' are also CTL* state formulas.

$(M, s) \vDash \Phi \wedge \Phi'$, w.r.t. CTL semantics

iff $(M, s) \vDash \Phi$ and $(M, s) \vDash \Phi'$

$(M, s) \vDash \Phi \wedge \Phi'$, w.r.t. CTL* semantics

For condition (4) in CTL semantics:

$(M, \pi) \vDash E\varphi$ w.r.t. CTL semantics.

iff for some path π starting from s, $(M, \pi) \vDash \varphi$ w.r.t. CTL semantics.

iff for some path π starting from s,
$(M, \pi[1]) \vDash X\Phi$ w.r.t. CTL semantics such that state formula $\Phi = \psi$ or
$(M, \pi[i]) \vDash \Phi U \Phi'$ w.r.t. CTL semantics such that state formula $\Phi = \psi$ and $\Phi' = \psi'$.

iff for some path π starting from s,
$(M, \pi[1]) \vDash \Phi$, where $\pi[1]$ is the initial state of path $\pi[1 \cdots \infty]$ and $\Phi = \psi$ or
$(M, \pi[i]) \vDash \Phi'$ and $\Phi' = \psi'$ for some $i \geq 0$, and $(M, \pi[j]) \vDash \Phi$ and

iff for some path π starting from s, $(M, \pi[1 \cdots \infty]) \vDash \psi$ or
$(M, \pi[i \cdots \infty]) \vDash \psi'$ for some $i \geq 0$, and
$(M, \pi[j \cdots \infty]) \vDash \psi$ for all $0 \leq j < i$

iff for some path π starting from s, $(M, \pi) \vDash X\psi$ w.r.t. CTL* semantics or
$(M, \pi) \vDash \psi U \psi'$ w.r.t. CTL* semantics

iff for some path π starting from s, $(M, \pi) \vDash \varphi$ w.r.t. CTL* semantics
$\Phi = \psi$ for all $0 \leq j < i$

iff $(M, s) \vDash E\varphi$, w.r.t CTL semantics

For **condition** (5) in CTL semantics:

$(M,\pi) \vDash A\varphi$ w.r.t. CTL semantics.

iff for all path π starting from s, $(M,\pi) \vDash \varphi$ w.r.t. CTL semantics.

iff for all path π starting from s,

$(M,\pi[1]) \vDash X\Phi$ w.r.t. CTL semantics such that state formula $\Phi = \psi$ or

$(M,\pi[i]) \vDash \Phi U \Phi'$ w.r.t. CTL semantics such that state formula $\Phi = \psi$ and $\Phi' = \psi'$.

iff for all path π starting from s,

$(M,\pi[1]) \vDash \Phi$, where $\pi[1]$ is the initial state of path $\pi[1 \cdots \infty]$ and $\Phi = \psi$ or

$(M,\pi[i]) \vDash \Phi'$ and $\Phi' = \psi'$ for some $i \geq 0$, and $(M,\pi[j]) \vDash \Phi$ and

iff for all path π starting from s, $(M,\pi[1 \cdots \infty]) \vDash \psi$ or

$(M,\pi[i \cdots \infty]) \vDash \psi'$ for some $i \geq 0$, and

$(M,\pi[j \cdots \infty]) \vDash \psi$ for all $0 \leq j < i$

iff for all path π starting from s, $(M,\pi) \vDash X\psi$ w.r.t. CTL* semantics or

$(M,\pi) \vDash \psi U \psi'$ w.r.t. CTL* semantics

iff for all path π starting from s, $(M,\pi) \vDash \varphi$ w.r.t. CTL* semantics

$\Phi = \psi$ for all $0 \leq j < i$

iff $(M,s) \vDash A\varphi$, w.r.t CTL semantics

In conclusion, for every formula Φ of CTL, there exists some formula Φ' in CTL* such that Φ and Φ' are equivalent.

(b)

We can embed LTL in CTL* by making the implicit universal path quantifiers of LTL explicit. Given a state s we have:

$$s \vDash_{LTL} \varphi \Leftrightarrow s \vDash_{CTL*} \boldsymbol{A}\varphi$$

Suppose φ is LTL formula $\boldsymbol{FG}p$, the equivalent CTL* formula is $\boldsymbol{AFG}p$.

As we proved above, CTL is a fragment of CTL*. However, there is no way to express LTL formula $\boldsymbol{FG}p$ in CTL, hence, CTL* formula $\boldsymbol{AFG}p$ cannot be expressed in CTL.

In conclusion, there exists some formula $\Phi = \boldsymbol{AFG}p$ in CTL* for which there exists no equivalent formula Φ' in CTL