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Department of Computing
Academic Year **2019-2020**
Page created Thu Feb 20 02:15:18 GMT 2020

CATE

499 fbelard 6 a5 xr119 v1 0011900355555	Electronic submission **Sun - 16 Feb 2020 12:23:52** xr119

Exercise Information

Module:	499 Modal Logic for Strategic Reasoning in AI	**Issued:**	Wed - 05 Feb 2020
Exercise:	6 (CW)	**Due:**	Wed - 19 Feb 2020
Title:	Coursework2	**Assessment:**	Individual
FAO:	Belardinelli, Francesco (fbelard)	**Submission:**	Electronic

Student Declaration - Version 1

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Signed: (electronic signature) Date: 2020-02-15 23:27:02

For Markers only: (circle appropriate grade)

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COURSEWORK

IMPERIAL COLLEGE LONDON

DEPARTMENT OF COMPUTING

Modal Logic for Strategic Reasoning in AI

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Date: February 15, 2020

1 Question 1

(a) For a path λ, $\lambda \vDash \phi R \psi$ can be defined as: if i is the smallest such that $\lambda[i..\infty] \vDash \phi$, then for all $0 \leq j \leq i$ we have $\lambda[j..\infty] \vDash \psi$. If such i does not exist, then we need that $\lambda[j..\infty] \vDash \psi$ for all $0 \leq j$.

(b) $\phi R \psi$ can be formalized as $\neg(\neg\phi U \neg\psi$ (so in some sense R and U are dual as well).

(c) Given a path λ, the LTL formula is true iff it's not the case that $(\neg\phi U \neg\psi)$ is true. There are two cases here: either $\neg\psi$ is never true, i.e. ψ is always true; or for all $0 \leq i$ such that $\lambda[i..\infty] \vDash \neg\psi$, there is already some $j < i$ such that $\lambda[j..\infty] \nvDash \neg\phi$. This would mean that the first point where $\neg\phi$ is true on the path λ takes place strictly before the first point that $\neg\psi$ is true – if we let i denote that first point where $\neg\phi$ is true, this would mean that for all $0 \leq j \leq i$ we have $\lambda[j..\infty] \vDash \psi$, which exactly corresponds to our truth condition.

(d) We have

$$\perp R\psi \iff \neg(\neg \perp U \neg\psi) \iff \neg(trueU\neg\psi) \iff \neg(F\neg\psi) \iff G\psi.$$

2 Question 2

• $(M,q) \vDash EF\Phi$ iff for some path λ starting from q we have $(M,\lambda) \vDash trueU\Phi$ iff there is some $0 \leq j$ such that $(M,\lambda[j]) \vDash \Phi$ (since true is trivially true at any state).

• $(M,q) \vDash AF\Phi$ iff for all paths λ starting from q we have $(M,\lambda) \vDash trueU\Phi$ iff for all paths λ from q there is some $0 \leq j$ such that $(M,\lambda[j]) \vDash \Phi$.

• $(M,q) \vDash EG\Phi$ iff $(M,q) \vDash \neg AF\neg\Phi$, iff there exists some path λ starting from q such that there is no $0 \leq j$ with $(M,\lambda[j]) \vDash \neg\Phi$, i.e. $(M,\lambda[j]) \vDash \Phi$ for all $0 \leq j$.

• $(M,q) \vDash AG\Phi$ iff $(M,q) \vDash \neg EF\neg\Phi$, iff for all path λ from q we do not have a $0 \leq j$ such that $(M,\lambda[j]) \vDash \neg\Phi$, i.e. for all path λ from q and all $0 \leq j$ we have $(M,\lambda[j]) \vDash \Phi$.

3 Question 3

(a) Note that the CFG for state formula for both CTL and CTL* are exactly the same. For path formulas, we only have $X\Phi$ and $\Phi U \Psi$ in CTL. Both of these appear in the grammar of CTL* as well, and by definition formuals in CTL are all and only the state formulas as well, so every formula in CTL is a formula in CTL*.

(b) This has to come from one of the first three cases in path formulas in the grammar of CTL*. One simple example is the formula AEp – this is a CTL* formula: we know from definition that every state formula in CTL* is also a path formula, so p is a path formula; so Ep is a state formula, therefore is also a path formula; so AEp is a state formula and therefore a formula in CTL*. However, it is not a formula in CTL, as Ep is not a path formula in CTL.

4 Question 4

The definition of truths are exactly the same for state formulas, so we only need to consider the definition of path formulas where the two definitions differ.
For $X\Phi$, in CTL we have $(M,\lambda) \vDash X\Phi$ iff $(M,\lambda[1]) \vDash \Phi$. Now consider the case in CTL*: we have $(M,\lambda) \vDash X\Phi$ iff $(M,\lambda[1...\infty]) \vDash \Phi$, iff $(M,\lambda[1]) \vDash \Phi$ which is exactly the same as CTL (the last iff comes from the definition of $(M,\pi) \vDash \Phi$).
For $\Phi U \Psi$ the proof is similar: in CTL*, $(M,\lambda) \vDash \Phi U \Psi$ iff $(M,\lambda[i...\infty]) \vDash \Psi$ for some $i \geq 0$, and $(M,\lambda[j...\infty]) \vDash \Phi$ for all $0 \leq j < i$; this is true iff $(M,\lambda[i]) \vDash \Psi$ for some $i \geq 0$, and $(M,\lambda[j]) \vDash \Phi$ for all $0 \leq j < i$, which is the same as the definition in CTL (similarly, the last iff comes from the definition of $(M,\pi) \vDash \Phi$).
These are the only two cases of path formulas in CTL. So we obtain the same truth conditions as in CTL by restricting the definition here.

5 Question 5

(a) By definitions, formulas in CTL are all and only the state formulas. By Question 3a, if Φ is a formula in CTL then it is also a formula in CTL*; then by question 4, we have the same truth conditions for Φ in CTL and CTL*. So just exactly the same formula Φ (in CTL*) will satisfy the requirement in this part.
(b) Consider the formula $AFGp$, where F and G are merely syntactic sugars as defined in lectures ($F\phi = trueU\phi$, $G\phi = \neg F \neg \phi$). This is a CTL* formula, but not a CTL formula. But from the lecture we know that this is not expressible in CTL.

6 Question 6

We do induction on formula complexity, and simultaneously for both path formulas and state formulas. Suppose (M,t) and (M',t') are bisimilar with bisimulation B. Write $M = (W,R,V)$ and $M' = (W',R',V')$.

• Base case is $\Phi = p$ an atom, treated as a state formula. Then $(M,t) \vDash p$ iff $t \in V(p)$. Since $B(t,t')$, the above is true iff $t' \in V'(p)$, i.e. iff $(M',t') \vDash p$.

We now deal with the other 9 inductive cases:

• $(M,t) \vDash \neg\Phi$. This is true iff it's not the case that $(M,t) \vDash \Phi$, and by inductive hypothesis (abbreviated as IH from now on) is equivalent to that $(M',t') \nvDash \Phi$, i.e. iff

$(M',t') \models \neg\Phi$.

• This case is the same as the case above, but we'll still write the proof out for the last time. $(M,t) \models \Phi \wedge \Phi'$ iff $(M,t) \models \Phi$ and $(M,t) \models \Phi'$ iff (by IH) $(M',t') \models \Phi$ and $(M',t') \models \Phi'$ iff $(M',t') \models \Phi \wedge \Phi'$.

• $(M,t) \models E\psi$ iff there exists a path starting π starting from s with $(M,\pi) \models \psi$. Now if we can prove that there is a path π' in M' starting from t' that is bisimilar with π (and vice versa), then we can quote IH (simultaneous induction) and the above is equivalent to the existence of a path π' starting from s' with $(M,\pi') \models \psi$, which is then equivalent to $(M',t') \models E\psi$.
To prove the claim, first suppose we have such a path π starting from s in M. So $\pi[0] = s$, and $s \to \pi[1]$. By the forth condition, there exists some $v' \in W'$ with $s' \to v'$ and $B(\pi[1],v')$. Let this v' be our next state in the desired path π', i.e. $\pi'[1] = v'$. We continue to run this process to generate a path in π'. Note that by our algorithm, we have $B(\pi[i],\pi'[i])$ for each $i \geq 0$, so π and π' are bisimilar.
Conversely, suppose we have a path π' starting from s' in M', so $\pi'[0] = s'$, and $s' \to \pi'[1]$. By the back condition, we have some $v \in W$ with $s \to v$ and $B(v,\pi'[1])$. Let this v be $\pi[1]$, and continue to run this algorithm to generate a path π in M. By similar reasoning as above we see that π and π' are bisimilar.

• The case $(M,s) \models A\psi$ is the same as the case above.

• $(M,\pi) \models \Phi$. This is true iff $(M,\pi[0]) \models \Phi$, and by IH (simultaneous induction) is equivalent to $(M',\pi'[0]) \models \Phi$ (note that by definition of bisimilar paths we have $B(\pi[0],\pi'[0])$, so we know $(M,\pi[0])$ and $(M',\pi'[0])$ are bisimilar – this is why we can apply IH).

• The case of $\neg\psi$, $\psi \wedge \psi'$ and $X\psi$ are the same as the corresponding case of state formulas (note that if π and π' are bisimilar, then $\pi[1...\infty]$ and $\pi'[1...\infty]$ are bisimilar – this is trivial from definition).

• $(M,\pi) \models \psi U \psi'$ iff $(M,\pi[i...\infty] \models \psi'$ for some $i \geq 0$ and $(M,\pi[j...\infty]) \models \psi$ for all $0 \leq j < i$, iff (by IH, noting that if π and π' are bisimilar then $\pi[i...\infty]$ and $\pi'[i...\infty]$ are bisimilar for all $i \geq 0$) $(M',\pi'[i...\infty] \models \psi'$ for some $i \geq 0$ and $(M',\pi'[j...\infty]) \models \psi$ for all $0 \leq j < i$, iff $(M',\pi') \models \psi U \psi'$.

This concludes the proof.

To sum up, formulas in CTL* are state formulas, and we see from above that they are preserved by bisimulations. So the truth of CTL* formulas is preserved by bisimulations.

7 Question 7

Define a relation $B \subseteq St \times St'$ such that $B(t, t')$ iff t and t' are CTL-equivalent. So firstly we have $B(t, t')$. We need to prove that B is indeed a bisimulation.

• atom: $t \in V(p)$ for some atom p iff $(M, t) \vDash p$ iff $(M', t') \vDash p$ (t and t' are CTL-equivalent) iff $t' \in V'(p)$.

• forth: suppose $t \to v$. We need to prove that there is a v' such that $t' \to v'$ and $B(v, v')$, i.e. v and v' are CTL-equivalent. First note that there is at least some v' such that $t' \to v'$: to see this, note that $t \to v$, so $(M, t) \vDash EXtrue$. So by CTL-equivalent we have $(M', t') \vDash EXtrue$; in particular there must be some v' that $t' \to v'$, else there is no path with length greater than 0 starting from t'.
Now take the set of all those v' such that $t' \to v'$; denote it by S. For a contradiction, suppose otherwise, that none of the state $v' \in S$ is CTL-equivalent to v. Then for each $v'_i \in S$, there is some formula Φ_i such that $(M, v) \vDash \Phi_i$ but $(M', v'_i) \nvDash \Phi_i$. Now consider the formula

$$EX(\Phi_1 \wedge \Phi_2 \wedge \ldots \wedge \Phi_n$$

this is true at (M, t), since it is true at v and $t \to v$. On the other hand, this formula cannot be true at (M', t'), since for each $t' \to v'_i$, v'_i does not satisfy Φ_i. So t and t' are not CTL-equivalent. Contradiction.

8 Question 8

To prove the same result for CTL* formulas, we only need to take into account the additional cases in path formulas. However, nothing is really different, as our above proof doesn't look into the specific semantics of formulas either; so the same proof in Question 7 will flow through.

For the last part, the result of question 5 only says that CTL* is strictly more *expressive* than CTL; but that doesn't necessarily mean that CTL* has stronger distinguishing power. One possibility is that the extra properties brought by CTL* do not introduce any more distinguishing power; Furthermore, it is also possible that distinguishing up to bisimulation is already the strongest distinguishing power that can be possibly achieved (only a guess, based on the name *bisimilar*) – in that case obviously we can't do any better than CTL if CTL is already able to distinguish states up to bisimulation. But without further studies it's impossible to know if this is true (and I'm guessing there is no need to prove this – it seems to be too much for a coursework question).