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Adaptive synchronization of two coupled non-identical Hindmarsh-Rose systems by the Speed Gradient method ⋆

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Abstract: The adaptive synchronization problem between two coupled non-identical Hindmarsh-Rose systems was considered. It was shown that the usage of the developed controller, which is based on the speed gradient method, ensures to achieve synchronized behavior of the studied systems. The obtained results were mathematically proved and confirmed by the simulations.

Keywords: Synchronization, Adaptive control, Speed Gradient method, Neural dynamics, Hindmarsh-Rose system.

1. INTRODUCTION

The numerous studies of synchronization in dynamical systems have created a wide interdisciplinary area which includes a variety of scientific fields with their applications (Blekhman, 1988; Fradkov, 2007; Pikovsky et al., 2003). In particular, such fields are biology and medicine. There are many biological and medical systems which can demonstrate synchronous regimes in their behavior. The examples of such systems are the coordinated activity of cardiac pacemaker cells, a population of fireflies that flashes synchonously within its swarm and a population of birds that gathers in a flock (Peskin, 1975; Buck and Buck, 1968). The most important example of this type of systems are the neuronal populations and their dynamics in the brain of a human or an animal. Indeed, it is well known that the synchronization of a large number of neurons of the central nervous system plays a key role in the formation of the brain waves (Pikovsky et al., 2003; Strogatz and Stewart, 1993). Futhermore, it was ascertained that many pathological states and diseases of the central and peripheral nervous systems, such as essential tremor, epilepsies and Parkinson's disease, relate directly to the anomalous synchronization of the certain groups of neurons (Milton and Jung, 2013; Rosenblum et al., 2000; Uhlhaas et al., 2009). Today the methods, which are relied on suppression of pathological synchronization in the nervous system, are actively used in the therapy of these diseases. Obviously, the development of such methods requires to apply qualitative mathematical tools. Our approach is based on the applying of the tools of control theory.

Nowadays, there are many scientific works dedicated to the synchronization of mathematical models of neurons. The majority of these works is devoted to a non-adaptive synchronization (Plotnikov, 2015; Castanedo-Guerra et al., 2016; Steur et al., 2009), which requires accurate knowledge of the model parameters to design the controller. Moreover, biological neurons have different physiological characteristics, that in turn lead to non-identical parameters in the models (Plotnikov et al., 2016). Therefore, the methods of adaptive control should be used for the effective control of synchronization between biological neurons. In our study, we apply the method of adaptive control which is called the speed gradient method. This method is based on the usage of Lyapunov functions and requires to define the control goal as a objective function (Fradkov, 2007).

The rest of this paper is organized as follows. In Section 2 we describe the mathematical model. Section 3 gives brief exposition of the speed gradient method in the differential form. Section 4 deals with the adaptive synchronization problem of two interconnected Hindmarsh-Rose systems. The results of numerical simulation are given in Section 5.

2. A MATHEMATICAL MODEL

In our work, we consider a model which consists of two interconnected non-identical Hindmarsh-Rose systems

$$\begin{aligned}
\dot{x}_1 &= y_1 - ax_1^3 + bx_1^2 - z_1 + \lambda(x_2 - x_1) + u,\\
\dot{y}_1 &= c - dx_1^2 - y_1,\\
\dot{z}_1 &= \varepsilon[s(x_1 - r_1) - z_1];\\
\dot{x}_2 &= y_2 - ax_2^3 + bx_2^2 - z_2 + \lambda(x_1 - x_2),\\
\dot{y}_2 &= c - dx_2^2 - y_2,\\
\dot{z}_2 &= \varepsilon[s(x_2 - r_2) - z_2].
\end{aligned} \tag{1}$$

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Here $x_i(t)$, $y_i(t)$ $z_i(t)$ are the state variables of the i-th system; $u(t)$ is the control variable; λ is the coupling stength. The Hindmarsh-Rose system is a dynamical model of a biological neuron. In this model $x_i(t)$ describes the dynamics of the membrane potentional, and $y_i(t)$, $z_i(t)$ illustrate how to the sodium-potassium pump works. Since the rate of changing $z_i(t)$ is determined by ε, such that $0 < \varepsilon \ll 1$, then $y_i(t)$ describes the dynamics of the slow potassium current and $z_i(t)$ describes the dynamics of the fast sodium current. The Hindmarsh-Rose model is a simplified version of the Hodgkin-Huxley model. Nevertheless, this model is able to demonstrate the most of the behavioral regimes of a biological neuron such as spiking and bursting.

3. SPEED GRADIENT METHOD

In this section, we briefly describe the speed gradient method. Consider the following dynamical system

$$\dot{x} = F(x, \theta, t), \tag{2}$$

where $x \in \mathbf{R}^n$ — a state vector, $\theta \in \mathbf{R}^m$ — an input vector and $F(x, \theta, t)$ — a function, which is piecewise continuous in t and continuously differentiable in x, θ. The control goal can be defined by one of two following ways

$$Q_t \to 0, \text{ while } t \to \infty \quad (\text{or } Q_t \le \Delta \ \forall t \ge t_*), \tag{3}$$

where $Q_t \ge 0$ is a goal function and Δ, t_* are constants. In our study we consider the case when Q_t is a local functional, i.e. $Q_t = Q(x(t), t)$ and Q_t is a smooth scalar function. In case of a synchronization problem, we can define the goal function as a quadratic form $Q(x) = (x_1 - x_2)P(x_1 - x_2)$, where $P = P^T > 0$ and $x = \{x_1, x_2\}$ is extended state space of the overall system.

In order to design a control algorithm, the derivative of the goal function $\dot{Q}_t = \omega(x, \theta, t)$ is calculated, i.e. the speed (rate) of changing Q_t along trajectories of the system (2):

$$\omega(x, \theta, t) = \frac{\partial Q(x, t)}{\partial t} + \left[\nabla_x Q(x, t)\right]^T F(x, \theta, t). \tag{4}$$

Then the gradient of $\omega(x, \theta, t)$ is evaluated with respect to the input vector θ as

$$\nabla_\theta \omega(x, \theta, t) = \left[\frac{\partial \omega}{\partial \theta}\right]^T = \left[\frac{\partial F}{\partial \theta}\right]^T \nabla_x Q(x, t). \tag{5}$$

Finally, the algorithm of changing θ is determined by the following differential equation

$$\dot{\theta} = -\Gamma \nabla_\theta \omega(x, \theta, t), \tag{6}$$

where Γ is a symmetric and positive definite matrix, e.g., $\Gamma = \mathrm{diag}\{\gamma_1, \ldots, \gamma_m\}$, $\gamma_i > 0$.

The main idea of the algorithm (6) can be explained as follows. In order to achieve the control goal (3), it is necessary to change θ in the direction of decreasing Q_t. However, it may be problematic since Q_t does not depend explicitly on θ. Instead of this, we may try to decrease $\dot{Q}_t$ in order to achieve inequality $\dot{Q}_t < 0$, which implies decrease of Q_t. Now we can write the algorithm (6) since the function $\dot{Q}_t = \omega(x, \theta, t)$ depends directly on θ.

The convergence of the speed gradient method depends on a reasonable choice of θ. There are several conditions which can help to make a correct choice. This conditions can be found in (Fradkov and Pogromsky, 1996).

4. ADAPTIVE SYNCHRONIZATION OF THE NON-IDENTICAL HINDMARSH-ROSE SYSTEMS

Let us do some transformations in (1) before formulating of the adaptive synchronization problem. In the beginning, we subtract the equations of the second system from the equations of the first one, and set $\psi(t) = x_1^2(t) + x_1(t)x_2(t) + x_2^2(t)$, $\varphi(t) = x_1(t) + x_2(t)$. Then the following system holds:

$$\begin{aligned} \dot{\delta_x} &= -(a\psi - b\varphi + 2\lambda)\delta_x + \delta_y - \delta_z + u, \\ \dot{\delta_y} &= -d\varphi\delta_x - \delta_y, \\ \dot{\delta_z} &= \varepsilon\left(s\delta_x - \delta_z - sr\right), \end{aligned} \tag{7}$$

where $\delta_x(t) = x_1(t) - x_2(t)$, $\delta_y(t)f = y_1(t) - y_2(t)$, $\delta_z(t) = z_1(t) - z_2(t)$ and $r = r_1 - r_2$. Next, we perform the change of variables $e_x(t) = \delta_x(t)$, $e_y(t) = \delta_y(t)$, $e_z(t) = \delta_z(t) + sr$, and get the system (7) in the new coordinates:

$$\begin{aligned} \dot{e_x} &= -(a\psi - b\varphi + 2\lambda)e_x + e_y - e_z + sr + u, \\ \dot{e_y} &= -d\varphi e_x - e_y, \\ \dot{e_z} &= \varepsilon\left(se_x - e_z\right). \end{aligned} \tag{8}$$

Now we are ready to formulate the adaptive synchronization problem. Obviously, the stability of the system (8) indicates the presence of synchronization between the systems (1). Then, the control goal can be defined as

$$e_x(t) \to 0, \ e_y(t) \to 0, \ e_z(t) \to 0, \text{ while } t \to \infty. \tag{9}$$

In order to achieve this control goal, we propose to use a controller in the following form:

$$u(t) = -(\gamma_0 - \theta_1\varphi)e_x + \theta_2\varphi e_y + \theta_3, \tag{10}$$

where γ_0 is a controller gain; $\theta_1(t)$, $\theta_2(t)$ and $\theta_3(t)$ are tunable parameters. We will adjust these parameters by the speed gradient method. To do this, we define the objective function (11) which corresponds to the control goal (9).

$$Q(e(t)) = \frac{1}{2}\left(e_x^2(t) + e_y^2(t) + \frac{1}{\varepsilon s}e_z^2(t)\right), \tag{11}$$

where $e(t) = \mathrm{col}\left(e_x(t), e_y(t), e_z(t)\right)$. We find its derivative according to the system (8) and get the following

$$\begin{aligned} \dot{Q} = &-\left(a\psi - b\varphi + 2\lambda\right)e_x^2 + \left(1 - d\varphi\right)e_xe_y + \\ &+ e_y^2 + \frac{1}{s}e_z^2 + sre_x + ue_x. \end{aligned} \tag{12}$$

Since $\psi(t) \ge 0$ and $\varepsilon Q\left(e(t)\right) \le \frac{1}{2}\left(e_x^2(t) + e_y^2(t) + e_z^2(t)/s\right)$ hold $\forall t \ge 0$ then we can get the upper estimate of the derivative (12):

$$\begin{aligned} \dot{Q} \le &-\varepsilon Q - \left(2\lambda - b\varphi - 1/2\right)e_x^2 + sre_x + \\ &+ ue_x + \left(1 - d\varphi\right)e_xe_y + e_y^2/2 + e_z^2/2s. \end{aligned} \tag{13}$$

Now we substitute the control (10) in the last inequality and write the result in the following form

$$\begin{aligned} \dot{Q} \le &-\varepsilon Q + e^*We + (\theta_1 - b)\varphi e_x^2 + \\ &+ (\theta_2 - d)\varphi e_xe_y + (\theta_3 + sr)e_z, \end{aligned} \tag{14}$$

where

$$W = \begin{bmatrix} -(\gamma_0 + 2\lambda - 1/2) & 1/2 & 0 \\ 1/2 & -1/2 & 0 \\ 0 & 0 & -1/2s \end{bmatrix}.$$

Let us find γ_0 and λ for which the quadratic form in the inequality is negative definite. Using the Sylvester's criterion, we obtain the following condition

$$\gamma_0 + 2\lambda - 1 \ge 0. \tag{15}$$

If the condition (15) is satisfied, the inequality takes the form

$$\dot{Q} \leq -\varepsilon Q + \omega(e, \theta). \tag{16}$$

where $\theta = \mathrm{col}(\theta_1, \theta_2, \theta_3)$ and

$$\omega(e, \theta) = (\theta_1 - b)\varphi e_x^2 + (\theta_2 - d)\varphi e_x e_y + (\theta_3 + sr)e_z.$$

In order to adjust θ_1, θ_2 and θ_3, we find the gradient of $\omega(e, \theta)$ with respect to θ:

$$\nabla_\theta \omega(e, \theta) = e_x \begin{bmatrix} \varphi e_x \\ \varphi e_y \\ 1 \end{bmatrix}. \tag{17}$$

Whence we get equations for the adjusting of θ_1, θ_2 and θ_3:

$$\dot{\theta} = -\gamma \nabla_\theta \omega(e, \theta), \tag{18}$$

where γ is a gain of the speed gradient method. Finally, using the well-known theorems of the speed gradient method, we can establish an asymptotic stability of the system (8). It, in turn, indicates the achievement of the control goal (9). This fact can be formulated as a theorem.

Theorem 1. Suppose that λ is the coupling strength of the systems (1), and γ_0, γ are gains in the control algorithm. Then, the controller $u(t)$ ensures the achievement of the control goal (9) $\forall \mathrm{x}_i(0)$ $(i = 1, 2)$ and $\forall \theta(0)$ of the systems if the condition (15) is satisfied.

Thus, the theorem 1 is a sufficient condition for the synchronization of the systems (1). In addition, we have the asymptotic synchronization with respect to $x_i(t)$ and $y_i(t)$ $(i = 1, 2)$. Moreover, from the inequality (16), we can conclude that the synchronization has an exponential rate.

5. SIMULATION

We perform a numerical simulation to confirm the correctness of the theorem.

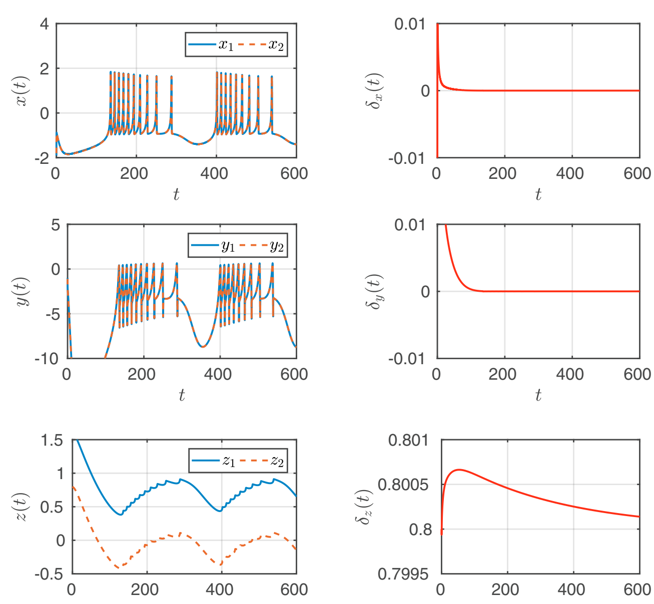

Fig. 1. Synchronization of two coupled Hindmarsh-Rose systems. (a), (c) and (e) are dynamics of the systems; (b), (c) and (b) are the synchronization errors.

For the simulation we choose the following values of the paremeters: $a = 1$, $b = 3$, $c = 1$, $d = 5$, $\varepsilon = 3 \cdot 10^{-3}$, $r_1 = -1$, $r_2 = -0.8$, $\lambda = 0.4$, $\gamma_0 = 4$. Since γ_0 and λ satisfy condition (15), according to the theorem 1, the controller (10) ensures the achievement of the goal (9), hence the system are synchronized, as shown in Fig. 1.

6. CONCLUSION

The adaptive synchronization problem of two non-identical interconnected Hindmarsh-Rose systems is considered in the article. In our study we used the speed gradient method to adjust the control parameters. It was mathematically proven that such synchronization is achievable through the controller which is given in our paper. Moreover, there is the asymptotic synchronization with respect to $x_i(t)$ and $y_i(t)$ $(i = 1, 2)$. The numerical experiments are confirmed the adequacy of obtained theoretical results.

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