BROWANG, Matteo (mgb19)

Imperial College London

Department of Computing
Academic Year 2019-2020
Page created Thu Feb 20 02:15:19 GMT 2020

CATE

499 fbelard 6
s5 mgb19 v1

0 0 1 1 9 0 0 3 3 4 1 7

Electronic submission

Mon - 17 Feb 2020
17:33:16

mgb19

Exercise Information

Module:	499 Modal Logic for Strategic Reasoning in AI	**Issued:**	Wed - 05 Feb 2020
Exercise:	6 (CW)	**Due:**	Wed - 19 Feb 2020
Title:	Coursework2	**Assessment:**	Individual
FAO:	Belardinelli, Francesco (fbelard)	**Submission:**	Electronic

Student Declaration - Version 1

- I declare that this final submitted version is my unaided work.

Signed: (electronic signature) Date: 2020-02-06 16:36:17

For Markers only: (circle appropriate grade)

BROWANG, Matteo (mgb19)	01797461	s5	2020-02-06 16:36:17	**A* A B C D E F**

Imperial College London

COURSEWORK 2 :
TEMPORAL LOGIC

IMPERIAL COLLEGE LONDON

DEPARTMENT OF COMPUTING

Modal Logic

Author:
Matteo Browang (CID: 01797461)
matteo.browang19@imperial.ac.uk

Date: February 17, 2020

Question 1

(a) The truth condition that defines $\varphi R \psi$ are:
$\pi \models \varphi R \psi$ iff $\pi[i \dots \infty] \models \varphi$ for some $i \geq 0$, and $\pi[j \dots \infty] \models \psi$ for all $0 \leq j \leq i$

(b) A LTL formula that formalizes the meaning of $\varphi R \psi$ is:

$$(\psi \wedge X\psi) U \varphi$$

(c) We translate the formula found in **(b)** to the corresponding truth conditions according to the definition in Lecture 5:

$$\begin{aligned}
\pi \models (\psi \wedge X\psi) U \varphi & \text{ iff } \pi[i \dots \infty] \models \varphi \text{ for some } i \geq 0, \text{ and } \pi[j \dots \infty] \models (\psi \wedge X\psi) \text{ for all } 0 \leq j < i \\
& \text{ iff } \pi[i \dots \infty] \models \varphi \text{ for some } i \geq 0, \text{ and } \pi[j \dots \infty] \models \psi \text{ and} \\
& \pi[j \dots \infty] \models X\psi \text{ for all } 0 \leq j < i \\
& \text{ iff } \pi[i \dots \infty] \models \varphi \text{ for some } i \geq 0, \text{ and } \pi[j \dots \infty] \models \psi \text{ and} \\
& \pi[j+1 \dots \infty] \models \psi \text{ for all } 0 \leq j < i \\
& \text{ iff } \pi[i \dots \infty] \models \varphi \text{ for some } i \geq 0, \text{ and } \pi[j \dots \infty] \models \psi \text{ for all } 0 \leq j \leq i
\end{aligned}$$

This last truth condition exaclty corresponds to the truth condition that we found in **(a)**.

(d) We write the truth condition (that we found **(a)**) corresponding to $\perp R \psi$:

$$\pi \models \perp R \psi \text{ iff } \pi[i \dots \infty] \models \perp \text{ for some } i \geq 0, \text{ and } \pi[j \dots \infty] \models \psi \text{ for all } 0 \leq j \leq i$$

By definition $\perp$ is never true, so this truth condition becomes:

$$\pi \models \perp R \psi \text{ iff } \pi[j \dots \infty] \models \psi \text{ for all } j \geq 0$$

This is the truth condition of $G\psi$ and so $G\psi$ can be expressed as $\perp R \psi$.

Question 2

$$\begin{aligned}
(M, q) \models EF\phi & \text{ iff } (M, q) \models E(true U \phi) \\
& \text{ iff for some path } \lambda \text{ starting from } q, (M, \lambda) \models true U \phi \\
& \text{ iff for some path } \lambda \text{ starting from } q, \text{ for some } j \geq 0, (M, \lambda[j]) \models \phi
\end{aligned}$$

and $(M, \lambda[i]) \models true$ for all $0 \leq i < j$
iff for some path λ starting from q, for some $j \geq 0$, $(M, \lambda[j]) \models \phi$

$(M, q) \models EF\phi$ iff $(M, q) \models E(trueU\phi)$
iff for every path λ starting from q, $(M, \lambda) \models trueU\phi$
iff for every path λ starting from q, for some $j \geq 0$, $(M, \lambda[j]) \models \phi$
and $(M, \lambda[i]) \models true$ for all $0 \leq i < j$
iff for every path λ starting from q, for some $j \geq 0$, $(M, \lambda[j]) \models \phi$

$(M, q) \models EG\phi$ iff $(M, q) \models \neg AF\neg\phi$
iif $(M, q) \not\models AF\neg\phi$
iff for some path λ starting from q, for all $j \geq 0$, $(M, \lambda[j]) \not\models \neg\phi$
(this is the negation of the truth condition of $AF\neg\phi$ according to the previous result)
iff for some path λ starting from q, for all $j \geq 0$, $(M, \lambda[j]) \models \phi$

$(M, q) \models AG\phi$ iff $(M, q) \models \neg EF\neg\phi$
iif $(M, q) \not\models EF\neg\phi$
iff for every path λ starting from q, for all $j \geq 0$, $(M, \lambda[j]) \not\models \neg\phi$
(this is the negation of the truth condition of $EF\neg\phi$ according to the first result)
iff for every path λ starting from q, for all $j \geq 0$, $(M, \lambda[j]) \models \phi$

Question 3

(a) Let ψ be a path formula of CTL and ϕ be a formula of CTL.
We prove by mutual induction on their structure that ψ is a path formula of CTL* and ϕ is a state formula of CTL*:

- $\psi = X\phi$ with ϕ a path CTL* formula: by Def. 1., ψ is a CTL* path formula.
- $\psi = \varphi U\phi$ with φ and ϕ path CTL* formulas: by Def. 1., ψ is a CTL* path formula.
- $\phi = a \in AP$: by Def. 1., ϕ is a CTL* formula.
- $\phi = \neg\psi$ with ψ a CTL* formula: by Def. 1., ϕ is a CTL* formula.
- $\phi = \varphi \wedge \psi$ with φ and ψ CTL* formulas: by Def. 1., ϕ is a CTL* formula.
- $\phi = E\psi$ with ψ a CTL* path formula: by Def. 1., ϕ is a CTL* formula.
- $\phi = A\psi$ with ψ a CTL* path formula: by Def. 1., ϕ is a CTL* formula.

So every formula of CTL is a formula of CTL*.

(b) We consider the formula $\phi = AXXp$ with p an atom.
Then, by Def. 1., ϕ is CTL* formula but not a CTL formula (the X operator is followed by a path formula).

Question 4

The 5 state formulas of CTL* exaclty correspond to the 5 state formulas of CTL.
Concerning the path formulas, if we restrict these formulas to CTL, then we can't have path formulas of the form ϕ (with ϕ a state formula), $\neg\psi$ and $\psi \wedge \psi'$ (with ψ and ψ' path formula), so we can't apply the rules $(M,\pi) \models \phi$, $(M,\pi) \models \neg\psi$ and $(M,\pi) \models \psi \wedge \psi'$.
Then it remains the path formulas $(M,\pi) \models X\psi$ and $(M,\pi) \models \psi U \psi'$ which correspond exaclty to the path formulas of CTL.
So if we restrict these formulas to CTL, we obtain the same truth conditions as in CTL.

Question 5

(a) We show in **Question 3** that every CTL formula is also a CTL* formula. Since every formula is equivalent to itself, then every formula of CTL has an equivalent formula (itself) in CTL* so CTL* is more expressive than CTL.

(b) Let's consider the formula $\phi = F(a \wedge Xa)$ with a an atom and F is defined as in CTL ($F\psi = trueU\psi$).
By Def. 1., ϕ is a state formula of CTL* but ϕ is not a CTL formula.
We show in the lecture that ϕ is also a LTL formula and that there is no CTL formula ϕ' equivalent to ϕ.
Then CTL* is strictly more expressive than CTL.

Question 6

We prove this result by mutual induction on ϕ and ψ (p is an atom, φ and φ' are state formulas, θ and θ' are path formulas):

- $\phi = p$: $(M,t) \models \phi \Leftrightarrow t \in V(p) \Leftrightarrow t' \in V'(p) \Leftrightarrow (M',t') \models \phi$
- $\phi = \neg\varphi$: $(M,t) \models \phi \Leftrightarrow (M,t) \not\models \varphi \Leftrightarrow (M',t') \not\models \varphi \Leftrightarrow (M',t') \models \phi$
- $\phi = \varphi \wedge \varphi'$: $(M,t) \models \phi \Leftrightarrow (M,t) \models \varphi$ and $(M,t) \models \varphi' \Leftrightarrow (M',t') \models \varphi$ and $(M',t') \models \varphi' \Leftrightarrow (M',t') \models \phi$
- $\psi = \varphi$: Since, by definition $(M,\pi[0])$ and $(M',\pi'[0])$ are bisimilar, we have :

  $(M,\pi) \models \psi \Leftrightarrow (M,\pi[0]) \models \varphi \Leftrightarrow (M',\pi'[0]) \models \varphi \Leftrightarrow (M',\pi') \models \varphi$
- $\psi = \neg\theta$: $(M,\pi) \models \psi \Leftrightarrow (M,\pi) \not\models \theta \Leftrightarrow (M',\pi') \not\models \theta \Leftrightarrow (M',\pi') \models \psi$
- $\psi = \theta \wedge \theta'$: $(M,\pi) \models \psi \Leftrightarrow (M,\pi) \models \theta$ and $(M,\pi) \models \theta' \Leftrightarrow (M',\pi') \models \theta$ and $(M',\pi') \models \theta' \Leftrightarrow (M',\pi') \models \psi$

- $\psi = X\theta$: By definition of bisimularity between path, for every $i \geq 0$, $(M, \pi[i])$ and $(M', \pi'[i])$ are bisimilar, so for every $i \geq 0$, $(M, \pi[i+1])$ and $(M', \pi'[i+1])$ are bisimilar, so $(M, \pi[1\ldots\infty])$ and $(M', \pi'[1\ldots\infty])$ are also bisimilar, then:

  $(M, \pi) \models \psi \Leftrightarrow (M, \pi[1\ldots\infty]) \models \theta \Leftrightarrow (M', \pi'[1\ldots\infty]) \models \theta \Leftrightarrow (M', \pi') \models \psi$

- $\psi = \theta U \theta'$: By definition of bisimularity between path, for every $i \geq 0$, $(M, \pi[i])$ and $(M', \pi'[i])$ are bisimilar, so for every k $(M, \pi[k\ldots\infty])$ and $(M', \pi'[k\ldots\infty])$ are also bisimilar, then:

  $(M, \pi) \models \psi \Leftrightarrow (M, \pi[i\ldots\infty]) \models \theta'$ for some $i \geq 0$ and $(M, \pi[j\ldots\infty]) \models \theta$ for every $0 \leq j < i \Leftrightarrow (M', \pi'[i\ldots\infty]) \models \theta'$ and $(M, \pi[j\ldots\infty]) \models \theta$ for every $0 \leq j < i \Leftrightarrow (M', \pi') \models \psi$

- $\phi = E\theta$: By definition of the forth property of a bisimulation, for every path $\pi = (t = t_0, t_1, t_2, \ldots)$ starting from t and for every $i \geq 0$, since $t_i \to t_{i+1}$, there is t'_i and t'_{i+1} such that $B(t_i, t'_i)$, $B(t_{i+1}, t'_{i+1})$ and $t'_i \to t'_{i+1}$. So there exists a path $\pi' = (t' = t'_0, t'_1, t'_2, \ldots)$ starting from t' such that for every $i \geq 0$, (M, t_i) and (M', t'_i) are bisimilar. By definition, π and π' are bisimilar. This is also true in the other way with the back property. Then:

  $(M, t) \models \phi \Leftrightarrow$ For some path π starting from t, $(M, \pi) \models \theta \Leftrightarrow$ For some path π' (bisimilar to π) starting from t', $(M', \pi') \models \theta \Leftrightarrow (M', t') \models \theta$

- $\phi = A\theta$: With the previous result we can show that for every path π starting from t' there exists exactlty one path π' starting from t' such that π and π' are bisimilar and viceversa. Then:

  $(M, t) \models \phi \Leftrightarrow$ For every path π starting from t, $(M, \pi) \models \theta \Leftrightarrow$ For every path π' starting from t', $(M', \pi') \models \theta \Leftrightarrow (M', t') \models \phi$

Hence,

$$(M, t) \models \phi \Leftrightarrow (M', t') \models \phi$$

$$(M, \pi) \models \psi \Leftrightarrow (M', \pi') \models \psi$$

This means that CTL* formulas are invariant by bisimulation, so the truth of CTL* formulas is preserved by bisimulation.

Question 7

We consider the relation B such that for every $w, w' \in St \times St'$, $B(w, w')$ iff w and w' avec CTL-equivalent (then we have $B(t, t')$).
Let's show that B is a bisimulation. We consider two states $w, w' \in St \times St'$ such that $B(w, w')$.

- Since w and w' satisfy the same formulas, in particular they satisfy the same atomic formulas, so for all atoms p, $w \in V(p)$ iff $w' \in V'(p)$.

- To prove the Forth property, we consider $v \in St$ such that $w \to v$. Let's assume that there is no $v' \in M'$ such that $w' \to' v'$ and $B(v,v')$. We define $S' = \{u' \in M' | w' \to' u'\}$. Since we suppose that St and St' are finite, then $S' = u'_1, u'_2, \ldots, u'_k$ is finite. By assumption, for every $u'_i \in S'$ there is a CTL formula ψ_i such that $(M,v) \models \psi_i$ and $(M', u'_i) \nvDash \psi_i$.

  Then, we have $(M,w) \models AX(\psi_1 \wedge \psi_2 \wedge \cdots \wedge \psi_k)$ (we can just consider the paths of the form $(w, v, \ldots)$) but $(M', w') \nvDash AX(\psi_1 \wedge \psi_2 \wedge \cdots \wedge \psi_k)$ since none of the succesor states (items of S') of w' satisfies $\psi_1 \wedge \psi_2 \wedge \cdots \wedge \psi_k$.

  This result contracticts the equivalence of w and w', so there exists $v' \in M'$ such that $w' \to' v'$ and $B(v,v')$.

- In the same way, we can prove the Back property.

Hence, B is a bisimulation between M and M' and $B(t,t')$ so (M,t) and (M',t') are bisimilar.

Question 8

We suppose that (M,t) and (M',t') satisfy the same formulas in CTL*.
Then, since every CTL formula is equivalent to a CTL* formula according to **Question 5**, every CTL formula ψ that (M,t) satisfies is equivalent to a CTL* formula ψ', and (M,t) also satisfies ψ', so (M',t') also satisfies ψ'. Since ψ and ψ' are equivalent, (M',t') also satisfies ψ.
Then, (M,t) and (M',t') satisfy the same formulas in CTL.

We suppose that (M,t) and (M',t') satisfy the same formulas in CTL, so (M,t) and (M',t') are CTL-equivalent.
Then, according to **Question 7**, (M,t) and (M',t') are bisimilar.
According to **Question 6**, the truth of CTL* are preserved by bisimulation, which means that (M,t) and (M',t') satisfy the same formulas in CTL*.