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### Student Declaration - Version 1

- I declare that this final submitted version is my unaided work.

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# COURSEWORK

IMPERIAL COLLEGE LONDON

DEPARTMENT OF COMPUTING

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## **499 - Modal Logic for Strategic Reasoning in AI**

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1.

a)

$(M, \pi) \models \rho R \psi$  iff either for some  $i \geq 0$   $(M, \pi[i..\infty]) \models \rho$  and for all  $0 \leq j \leq i$   $(M, \pi[0..j]) \models \psi$ ,  
or for all  $k \geq 0$   $(M, \pi[k..\infty]) \models \psi$

b)

$$a \models \rho R \psi \iff (\psi U \rho) \vee (\psi \wedge \rho) \vee (\psi \wedge X(\psi))$$

c)

$$\begin{aligned} (M, \pi) \models \rho R \psi &\iff (\psi U \rho) \vee (\psi \wedge \rho) \vee (\psi \wedge X(\psi)) && \text{(by definition)} \\ &\iff (\psi U \rho) \vee (\psi \wedge \rho) \vee G(\psi) && \text{(expansion (slides p.94) and logic*)} \\ &\iff \text{there exists } i \geq 0 \text{ such that } (M, \pi[i..\infty]) \models \rho, \text{ and for all } 0 \leq j < i, (M, \pi[j..\infty]) \models \psi \\ &\quad \text{or } (M, \pi[0..\infty]) \models \psi \text{ and } (M, \pi[0..\infty]) \models \rho \\ &\quad \text{or for all } k \geq 0 (M, \pi[k..\infty]) \models \psi \\ &\iff \text{there exists } i \geq 0 \text{ such that } (M, \pi[i..\infty]) \models \rho, \text{ and for all } 0 \leq j \leq i, (M, \pi[j..\infty]) \models \psi \\ &\quad \text{or for all } k \geq 0 (M, \pi[k..\infty]) \models \psi && (< i \text{ and } = i \text{ mean } \leq i) \end{aligned}$$

d)

$$\begin{aligned} (M, \pi) \models \perp R \psi & \\ &\iff \text{there exists } i \geq 0 \text{ such that } (M, \pi[i..\infty]) \models \perp, \text{ and for all } 0 \leq j \leq i, (M, \pi[j..\infty]) \models \psi \\ &\quad \text{or for all } k \geq 0 (M, \pi[k..\infty]) \models \psi && (< i \text{ and } = i \text{ mean } \leq i) \\ &\iff \perp \text{ and for all } 0 \leq j \leq i, (M, \pi[j..\infty]) \models \psi \\ &\quad \text{or for all } k \geq 0 (M, \pi[k..\infty]) \models \psi && \text{(first part not satisfiable )} \\ &\iff \text{for all } k \geq 0 (M, \pi[k..\infty]) \models \psi && (\perp \wedge p \equiv \perp) \\ &\iff G\psi && \text{(by definition of G)} \end{aligned}$$

\*: if  $\alpha$  is such that  $\alpha \equiv \psi \wedge X\alpha$  then  $\alpha \implies G\psi$

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## 2.

$$\begin{aligned}(M, q) \models EF\Phi &\iff E(trueU\Phi) && \text{(abbreviations in CTL)} \\ &\iff \text{for some path } \lambda \text{ starting from } q, (M, \lambda) \models trueU\Phi && \text{(definition of E)} \\ &\iff \text{for some path } \lambda \text{ starting from } q, (M, \lambda[j]) \models \Phi \text{ for some } j \geq 0, \\ &\quad \text{and } (M, \lambda[i]) \models true \text{ for all } 0 \leq i < j && \text{(definition of U)} \\ &\iff \text{for some path } \lambda \text{ starting from } q, (M, \lambda[j]) \models \Phi \text{ for some } j \geq 0 \text{ (true is true everywhere)}\blacksquare\end{aligned}$$

$$\begin{aligned}(M, q) \models AF\Phi &\iff A(trueU\Phi) && \text{(abbreviations in CTL)} \\ &\iff \text{for every path } \lambda \text{ starting from } q, (M, \lambda) \models trueU\Phi && \text{(definition of A)} \\ &\iff \text{for every path } \lambda \text{ starting from } q, (M, \lambda[j]) \models \Phi \text{ for some } j \geq 0, \\ &\quad \text{and } (M, \lambda[i]) \models true \text{ for all } 0 \leq i < j && \text{(definition of U)} \\ &\iff \text{for every path } \lambda \text{ starting from } q, (M, \lambda[j]) \models \Phi \text{ for some } j \geq 0 \text{ (true is true everywhere)}\blacksquare\end{aligned}$$

$$\begin{aligned}(M, q) \models EG\Phi & \\ &\iff \text{for some path } \lambda \text{ starting from } q, (M, \lambda) \models G\Phi && \text{(definition of E)} \\ &\iff \text{for some path } \lambda \text{ starting from } q, (M, \lambda[j]) \models \Phi \text{ for all } j \geq 0, && \text{(definition of G)}\end{aligned}$$

$$\begin{aligned}(M, q) \models AG\Phi & \\ &\iff \text{for every path } \lambda \text{ starting from } q, (M, \lambda) \models G\Phi && \text{(definition of A)} \\ &\iff \text{for every path } \lambda \text{ starting from } q, (M, \lambda[j]) \models \Phi \text{ for all } j \geq 0, && \text{(definition of G)}\end{aligned}$$

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### 3.

#### a)

Prove by induction of structure of CTL, we show each construct in CTL is defined also in CTL\* by the given definition:

atoms, are also state formulas in CTL\*.

negation of a state formulae is also a state formula in CTL\*.

conjunction of state formulae is a state formula in CTL\*.

$E\phi$  is also a state formula in CTL\*

$A\phi$  is also a state formula in CTL\*

$X\psi$  is also a path formula in CTL\*

$\phi U \psi$  is also a path formula in CTL\*

Therefore all formulas in CTL are defined in CTL\*. (CTL\* is only a superset with addition of conjunction, negation and path formulas)

#### b)

Everybody will always be safe, from some moment on:

in CTL\*:  $AFG(\bigwedge_{i \in people} \text{safe}_i)$

This is not defined in CTL because we cannot have FG (path formula followed by path formula)

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## 4.

To do this we can remove the rules for (conjunction, negation and path formulas) from our state formulas, then we would be removing the rules for  $(M, \pi) \models \Phi$ ,  $(M, \pi) \models \neg\psi$  and  $(M, \pi) \models \phi \wedge \phi'$  and are exactly left with the definition with def 1.7 and 1.8 in the slides.

---

5.

a)

This proof is similar to 3. Suppose  $(M, s) \models \Phi$  in CTL. Then  $\Phi$  must be a state formula and is either an atom, negation of a state formula, conjunction of two state formulae, E of a path formula or A of a path formula. Then  $\Phi$  is also expressed using the corresponding rules in CTL\* as the rules in CTL\* are a superset of the rules in CTL (as seen in 4.). Then we also have  $(M, s) \models \Phi'$  with  $\Phi' = \Phi$  in CTL\*.

b)

$\Psi \implies G\Phi \equiv \neg(\Psi \wedge \neg G\Phi)$

is not expressible in CTL because we cannot express a condition of a state with a path at the same time. ( $\Psi \wedge \neg G\Phi$  is not expressible in CTL).

## 6.

Assume  $(M, t)$  and  $(M, t')$  are bisimilar

Then, we assume  $(M, t) \models \Phi$  for each state formula  $\Phi$  and show  $(M, t') \models \Phi$

---

Case  $(M, t) \models p$

$\iff t \in V(p)$  (by definition 2)

$\iff t' \in V'(p)$  (by bisimilarity of  $(M, t)$  and  $(M', t')$  and condition a of definition 3)

$\iff (M', t') \models p$  (by definition 2)

---

Case  $(M, t) \models \neg\Phi$

Suppose  $(M', t') \models \Phi$

$\implies (M, t) \models \Phi$  (by bisimilarity of  $(M, t)$  and  $(M', t')$  and condition c (back) of definition 3)

$\perp$  (by  $(M, t) \models \neg\Phi$  and  $(M, t) \models \Phi$ )

Then it must be that  $(M', t') \models \neg\Phi$  by contradiction

---

Case  $(M, t) \models \Phi \wedge \Phi'$

$\iff (M, t) \models \Phi$  and  $(M, t) \models \Phi'$  (by definition 2)

$\iff (M', t') \models \Phi$  and  $(M', t') \models \Phi'$  (by bisimilarity of  $(M, t)$  and  $(M', t')$  and condition b (forth) of definition 3 for both  $(M', t') \models \Phi$  and  $(M', t') \models \Phi'$ )

$\iff (M', t') \models \Phi \wedge \Phi'$  (by definition 2)

---

Case  $(M, t) \models E\psi$

$\iff$  for some path  $\pi$  starting from  $t$ ,  $(M, \pi) \models \psi$  (by definition 2)

$\iff$  for some path  $\pi'$  starting from  $t'$ ,  $(M, \pi') \models \psi$   
 (by bisimilarity of  $(M, t)$  and  $(M', t')$  and definition 3:  
 $(M, \pi)$  and  $(M', \pi')$  are bisimilar if for every  
 $i \geq 0$   $(M, \pi[i])$  and  $(M', \pi'[i])$  are bisimilar)

$\iff (M', t') \models E\psi$  (by definition 2)

---

Case  $(M, t) \models A\psi$

$\iff$  for every path  $\pi$  starting from  $t$ ,  $(M, \pi) \models \psi$  (by definition 2)

$\iff$  for every path  $\pi'$  starting from  $t'$ ,  $(M, \pi') \models \psi$   
 (by bisimilarity of  $(M, t)$  and  $(M', t')$  and definition 3:  
 $(M, \pi)$  and  $(M', \pi')$  are bisimilar if for every  
 $i \geq 0$   $(M, \pi[i])$  and  $(M', \pi'[i])$  are bisimilar)

$\iff (M', t') \models A\psi$  (by definition 2)



---

Assume  $(M, \pi)$  and  $(M, \pi')$  are bisimilar

Now for path formulae again assume  $(M, \pi) \models \psi$  for some arbitrary state formula  $\psi$  and show  $(M, \pi') \models \psi$

---

Case  $(M, \pi) \models \Phi$

$\iff (M, \pi[0]) \models \Phi$  where  $\pi[0]$  is the initial state of path  $\pi$  (by definition 2)

$\iff (M', \pi'[0]) \models \Phi$  where  $\pi'[0]$  is the initial state of path  $\pi'$

(We have  $(M, \pi) \approx (M', \pi')$  Then by Definition 3, for every  $i \geq 0$   $(M, \pi[i])$  and  $(M', \pi'[i])$  are bisimilar. Namely, at  $i=0$   $(M, \pi[0]) \approx (M', \pi'[0])$  then we can use the previous proof on state formulae)

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$\iff (M', \pi') \models \Phi$  (by definition 2)

---

Case  $(M, \pi) \models \neg\psi$

Suppose  $(M', \pi') \models \neg\psi$

$\implies (M, \pi) \models \psi$  (by bisimilarity of  $(M, \pi)$  and  $(M', \pi')$  and definition 3c (back))

$\perp$  (by contradiction of  $(M, \pi) \models \neg\psi$  and  $(M, \pi) \models \psi$ )

---

Then it must be that  $(M', \pi') \models \neg\psi$  by contradiction (as required)

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Case  $(M, \pi) \models \psi \wedge \neg\psi$

$\iff (M, \pi) \models \psi$  and  $(M, \pi) \models \neg\psi$  (by definition 2)

$\iff (M', \pi') \models \psi$  and  $(M', \pi') \models \neg\psi$  (by bisimilarity of  $(M, \pi)$  and  $(M', \pi')$  and condition b (forth) of definition 3 for both  $(M', \pi') \models \psi$  and  $(M', \pi') \models \neg\psi$ )

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$\iff (M', \pi') \models \psi \wedge \neg\psi$  (by definition 2)

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Case  $(M, \pi) \models X\psi$

$\iff (M, \pi[1..\infty]) \models \psi$  (by definition 2)

$\iff (M', \pi'[1..\infty]) \models \psi$  (by bisimilarity of  $(M, \pi)$  and  $(M', \pi')$  and definition 3)

---

$\iff (M', \pi') \models X\psi$

---

Case  $(M, \pi) \models \psi U \psi'$

$\iff (M, \pi[1..\infty]) \models \psi'$  for some  $i \geq 0$ , and

$(M, \pi[j..\infty]) \models \psi$  for all  $0 \leq j < i$  (by definition 2)

$\iff (M', \pi'[1..\infty]) \models \psi'$  for some  $i \geq 0$ , and

$(M', \pi'[j..\infty]) \models \psi$  for all  $0 \leq j < i$  ( $(M, \pi) \approx (M', \pi')$ )

---

$\iff (M', \pi') \models \psi U \psi'$  (by definition 2)

---

we have shown that  $(M, t) \models \Phi \text{ iff } (M, t') \models \Phi$  and  $(M, t) \models \psi \text{ iff } (M, t') \models \psi$ . This means if a state formula  $\Phi$  or path formula  $\psi$  is true on one side of the bisimulation it must be true on the other side as well (by def of iff).

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## 7.

condition a)  $t, t'$  satisfy the same formulas in CTL. Therefore, they must also satisfy the same atoms.

condition b) Assume there is a state  $t$  mapped to  $t'$  ( $B(t, t')$ ) and a state  $u$  such that  $R(t, u)$ . Now assume there is no  $u'$  such that  $B(u, u')$  and  $R(t', u')$ . As  $S' = \{x' \in St' | R(t', x')\}$  cannot be empty (a single world is not defined in kripke semantics) and is finite, for all states in  $St'$  there must be a formula such that  $(M, t) \models \psi_i$  and  $(M', t') \not\models \psi_i$ . Therefore we have  $(M, t) \models E(\wedge \psi_i)$  but  $(M', t') \not\models E(\wedge \psi_i)$ . However, this cannot be true as  $t$  and  $t'$  are equivalent so we have a contradiction.

condition c) Assume there is a state  $t'$ , mapped to  $t$  ( $B(t, t')$ ) and a state  $u'$  such that  $R(t', u')$ . Now assume there is no  $u$  such that  $B(u, u')$  and  $R(t, u)$ . As  $S = \{x \in St | R(t, x)\}$  cannot be empty (a single world is not defined in kripke semantics) and is finite, for all states in  $St$  there must be a formula such that  $(M', t') \models \psi_i$  and  $(M, t) \not\models \psi_i$ . Therefore we have  $(M', t') \models E(\wedge \psi_i)$  but  $(M, t) \not\models E(\wedge \psi_i)$ . However, this cannot be true as  $t$  and  $t'$  are equivalent so we have a contradiction.

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## 8.

if  $(M, t)$  and  $(M', t')$  satisfy the same formulas in CTL, by (7.) we know that they are bisimilar. Then by (6) we know that the truth of formulas in bisimilar relations in CTL\* are preserved. Therefore, they satisfy the same formulas in CTL\*.

Now if  $(M, t)$  and  $(M', t')$  satisfy the same formulas in CTL\*, then by (6) they are bisimilar. So they satisfy the same formulae in CTL.

(5.b) shows that CTL\* is more expressive than CTL which means we can express formulae in CTL\* that we cannot express in CTL. However, here we are not saying that formulae that are expressible in CTL\* must have a corresponding formula in CTL to satisfy. Only that if  $t, t'$  satisfy formulae in CTL\* they must satisfy same formulas in CTL (if such correspondence exists). However, the opposite holds always as all formulae in CTL are expressible in CTL\*.

1			
a/2	b/2	c/3	d/3
<div>Proof is complete, but would have liked to see a stronger explanation of the removal of the false statement</div>			
<div>Solution is correct and explanation provided but no proof is given</div>			
2	0	2	3

2			
a/2	b/2	c/2	d/2
<div>Abbreviation not applied in solution</div>			
<div>Abbreviation not applied in solution</div>			
2	2	1	1

3	
a/3	b/2
3	2

4			
/5			
<div>The state formulas have been justified but no attempt is made towards showing the truth conditions for the path formulas</div>			
2			

5	
a/2	b/2
2	2

6	7	8
/6	/6	/5
<div></div>		
6	6	5