WARD, Francis (frw19)

Imperial College London

Department of Computing
Academic Year **2019-2020**
Page created Thu Feb 20 02:15:22 GMT 2020

CATE

499 fbelard 6 t5 frw19 v1 0011900036167	Electronic submission **Wed - 19 Feb 2020 15:52:32** frw19

Exercise Information

Module:	499 Modal Logic for Strategic Reasoning in AI	**Issued:**	Wed - 05 Feb 2020
Exercise:	6 (CW)	**Due:**	Wed - 19 Feb 2020
Title:	Coursework2	**Assessment:**	Individual
FAO:	Belardinelli, Francesco (fbelard)	**Submission:**	Electronic

Student Declaration - Version 1

- I declare that this final submitted version is my unaided work.

Signed: (electronic signature) Date: 2020-02-19 15:51:52

For Markers only: (circle appropriate grade)

WARD, Francis (frw19)	01819196	t5	2020-02-19 15:51:52	**A***	**A**	**B**	**C**	**D**	**E**	**F**

COURSEWORK 2: TEMPORAL LOGICS

IMPERIAL COLLEGE LONDON

DEPARTMENT OF COMPUTING

Modal Logic for Strategic Reasoning

Author:
Francis Rhys Ward (CID: 01819196)

Date: February 19, 2020

1

a) $\phi R\psi$

$$(M,\lambda) \models \phi R\psi \text{ iff } \lambda[i,...,\infty] \models \psi, \forall i \geq 0, \text{ or, } (\exists j \lambda[j,...,\infty] \models \phi \text{ and } \lambda[i,...,\infty] \models \psi \forall 0 \leq i \leq j) \tag{1}$$ 2

b)

$$\phi R\psi = \neg(\neg\phi \cup \neg\psi) \tag{2}$$ 2

Solution correct and in fully simplified form

c)

We wish to show that $(M,\lambda) \models \phi R\psi \iff (M,\lambda) \models \neg(\neg\phi \cup \neg\psi)$.
$(M,\lambda) \models \neg(\neg\phi \cup \neg\psi)$ iff $\neg i \geq 0(\lambda[i,...,\infty] \models \neg\psi \geq 0$ and $\lambda[j,...,\infty] \models \neg\phi \forall 0 \leq j < i)$
equivalently $\neg\exists i \geq 0($ such that $\lambda[i,...,\infty] \not\models \psi$ and $\lambda[j,...\infty] \not\models \phi \forall 0 \leq j < i)$
This is equivalent to
$\forall i \geq 0 \neg(\lambda[i,...,\infty] \not\models \psi$ and $\lambda[j,...\infty] \not\models \phi \forall 0 \leq j < i)$
by De Morgan's law we have
$\forall i \geq 0(\neg\lambda[i,...,\infty] \not\models \psi$ or $\neg\lambda[j,...\infty] \not\models \phi \forall 0 \leq j < i)$
i.e.
$\forall i \geq 0(\lambda[i,...,\infty] \models \psi$ or $\exists 0 \leq j < i \lambda[j,...\infty] \models \phi)$
which is equivalent to
$\forall i \geq 0 \lambda[i,...,\infty] \models \psi$ or $\exists 0 \leq j \leq i(\lambda[j,...\infty] \models \phi and \lambda[j,...,\infty] \models \psi)$ and this is our definition from a) as required.

3

Solution correct and very well explained

d)

We have by definition $\lambda \models G\psi$ iff $\lambda[i,...,\infty] \models \psi \forall i \geq 0$ And $\lambda \models \bot R\psi \iff \forall i \geq 0 \lambda[i,...,\infty] \models \psi$ or $\exists 0 \leq j \leq i(\lambda[j,...\infty] \models \bot and \lambda[j,...,\infty] \models \psi)$
but clearly there is no such j that satisfies $\lambda[j,...\infty] \models \bot$. So we have: $\lambda \models \bot R\psi \iff \forall i \geq 0 \lambda[i,...,\infty] \models \psi$ as required.

3

Solution is correct and explained but could have been presented with more clarity. Please clearly separate steps in solutions

2

i. $(M,q) \models EF\phi$ iff $\exists\lambda[q]$ (a path starting at q) such that
$(M,\lambda[q]) \models F\phi$ i.e. $\exists\lambda[q]$ such that $(M,\lambda[q]) \models (\top \cup \phi)$ Well, this is true iff $\exists\lambda[q] =: \lambda$ such that ,

$$(M,\lambda[i]) \models \phi \text{ for some } i \geq 0 \text{ and } (M,\lambda[j]) \models \top \forall 0 \leq j \leq i \tag{3}$$

Clearly this is equivalent to

$$\exists\lambda[q] \text{ such that } (M,\lambda[i]) \models \phi \text{ for some } i \geq 0 \tag{4}$$ 2

as required.
ii.

$$(M,q) \models AF\phi \text{ iff } \forall\lambda[q],(M,\lambda) \models F\phi \tag{5}$$

Now just follow the same steps as in i. to show that the statement holds. 1
iii. Steps aren't shown. Even though they are similar to (i), the steps should be made clear
$EG\phi$ is equivalent to $\neg AF\neg\phi \equiv \neg A(\top \cup \neg\phi)$. Hence,

$$(M,q) \models EG\phi \tag{6}$$

iff

$$(M,q) \models \neg A(\top \cup \neg\phi) \tag{7}$$

which is true iff

$$\forall\lambda[q],(M,\lambda) \nvDash (\top \cup \neg\phi) \tag{8}$$

i.e.

$$\exists\lambda[q],(M,\lambda) \models (\top \cup \neg\phi) \tag{9}$$

This holds iff $\exists\lambda$ starting at q such that

$$\exists i \geq 0, \text{ with } (M,\lambda[i]) \models \neg\phi \text{ and } (M,\lambda[j]) \models \top\forall 0 \leq j \leq i \tag{10}$$

Now, this is clearly equivalent to

$$\exists\lambda\forall i \geq 0,(M,\lambda[i]) \models \phi \tag{11}$$

2

iv.
$(M,q) \models AG\phi$ iff $(M,q) \models \neg EF\neg\phi$ By definition this holds iff, $\neg(\exists\lambda[q].(M,\lambda) \nvDash F\phi)$ i.e. $\forall\lambda[q],(M,\lambda[q]) \models F\phi$ equivalently, $\forall\lambda[q],(M,\lambda) \models (\top \cup \phi)$. Now we can similarly follow the final two steps in iii. to show the result. 1

Again steps should be shown

3

a)

Our CTL formulas are

$$\Phi ::= p|\neg\Phi|\Phi \wedge \Phi|EX\Phi|AX\Phi|E(\Phi \cup \Phi)|A(\Phi \cup \Phi) \tag{12}$$

We trivially have that $p|\neg\Phi|\Phi \wedge \Phi|$ are formulas of CTL* by definition 1.
Do we have $EX\Phi$? We can see that $X\Phi$ is a path formula of CTL* by definition 1 (as Φ is itself a path formula). Hence, $EX\Phi$ is a state formula of CTL* (i.e. a formula). Following the same reasoning we can show that $AX\Phi$, $E(\Phi \cup \Phi)$, and $A(\Phi \cup \Phi)$ are also all state formulas of CTL* (since $\Phi \cup \Phi$ is a path formula). Thus, every formula of CTL is also a formula of CTL*. □ 3

b)

Consider, $AFGp \equiv A(\top \cup (\perp \cup p))$. This is indeed a formula of CTL* (take $\perp := p \wedge \neg p$ and $\top := \neg\perp$). By definition 1, $\perp$ and $\top$ are state formulas of CTL*, so they are also path formulas, so $\perp \cup p$ is a path formula, therefore $\top \cup (\perp \cup p)$ is a path formula. Finally $A(\top \cup (\perp \cup p))$ is a state formula of CTL*.
But, state formulas in CTL are not in general path formulas. In particular, $\perp \cup p$ is a path formula but not a state formula. So $\top \cup (\perp \cup p)$ is not a path formula of CTL and so $A(\top \cup (\perp \cup p))$ is not a (state) formula of CTL. □ 2

4

To recover CTL from CTL* we restrict the quantifiers so that each temporal quantifier is preceded directly by a path quantifier. Equivalently we restrict the formulas of CTL* to CTL. Our formula of CTL are:

$$\Phi ::= p|\neg\Phi|\Phi \wedge \Phi|EX\Phi|AX\Phi|E(\Phi \cup \Phi)|A(\Phi \cup \Phi) \tag{13}$$

In particlar, compared to CTL* we are excluding the path formulas

$$\psi = \Phi|\neg\psi|\psi \wedge \psi \tag{14}$$

and keeping only

$$\psi = X\psi|\psi \cup \psi \tag{15}$$

Clearly, satisfaction on the state formulas is completely equivalent in CTL as in CTL* (the definitions are identical). So we need only show that satisfaction of $X\psi$ and $\psi \cup \psi'$ is preserved.
By definition 2

$$(M,\pi) \models X\psi \text{ iff } (M,\pi[1,...,\infty]) \models \psi \text{ iff } (M,\pi[1]) \models \psi \tag{16}$$

so we have recovered satisfaction of $X\psi$. Now consider

$$(M,\pi) \models \psi \cup \psi' \text{ iff } (M,\pi[i,...,\infty]) \models \psi' \text{ for some } i \geq 0 \text{ and } (M,\pi[j,...,\infty]) \models \psi \forall 0 \leq j \leq i \tag{17}$$

following definition 2 this is

$$(M,\pi) \models \psi \cup \psi' \text{ iff } (M,\pi[i]) \models \psi' \text{ for some } i \geq 0 \text{ and } (M,\pi[j]) \models \psi \forall 0 \leq j \leq i \tag{18}$$

and we have recovered satisfaction of until. □ 5

5

a)

By question 3 CTL is a strict fragment of CTL* i.e. for every formula Φ of CTL, Φ is also a formula of CTL*. Furthermore, by 4, for the formulas in CTL, CTL and CTL* are semantically equivalent, i.e. they have the same truth conditions.So the Φ' we are looking for is just Φ.

2

b)

Take the example FGp from lecture 5. This is an LTL formula hence also a CTL* formula. But there is no equivalent CTL formula by the Clarke Draghicescu lemma and the example shown in the lecture slides.

2

6

Proceed by induction on the structure of Φ and ψ. Since (M,t) and (M',t') are bisimilar we have from definition 3 a) that $\forall p \in AP, t \in V(p)$ iff $t' \in V'(p)$. So we have that $(M,t) \models p$ iff $(M',t') \models p$, and hence trivially that $(M,t) \models \neg\Phi$ iff $(M',t') \models \neg\Phi$ and $(M,t) \models \Phi \wedge \Phi$ iff $(M',t') \models \Phi \wedge \Phi$.

To show that $(M,t) \models E\psi$ iff $(M',t') \models E\psi$ consider that $(M,t) \models E\psi$ iff $\exists\pi$ starting from t such that $(M,\pi) \models \psi$, well if there is such a π then, by the forth property of bisimulation we can find a corresponding bisimilar state in M' for each state in π such that the relations between states in the path are preserved, hence we can construct a π' from these states and this π' is bisimilar to π. Satisfaction is preserved between these paths since they are state-wise bisimilar and we have shown that bismulations between states preserve truth. We can similarly show that $\models A\psi$ iff $(M',t') \models A\psi$.

Now consider satisfaction on paths, $(M,\pi) \models \Phi$ iff $(M,\pi[0]) \models \Phi$ (from definition 2) and likewise for π' and $\pi'[0]$. But $\pi[0]$ and $\pi'[0]$ are bisimilar by definition 3 so we have $(M,\pi[0]) \models \Phi$ iff $(M',\pi'[0]) \models \Phi$ by above and hence $(M,\pi) \models \Phi$ iff $(M',\pi') \models \Phi$. Then we trivially have the equivalences for satisfaction of $\neg\psi$ and $\psi \wedge \psi'$.

Is satisfaction of $X\psi$ preserved by bisimulations? Well, since $\pi \approx \pi'$ we also have $\pi[1,...,\infty] \approx \pi'[1,...,\infty]$. $(M,\pi) \models X\psi$ iff $(M,\pi[1,...,\infty]) \models \psi$ iff $(M,\pi[1]) \models \psi$ by definition 2, and by above $(M,\pi[1]) \models \psi$ iff $(M',\pi'[1]) \models \psi$. So $(M,\pi) \models X\psi$ iff $(M',\pi') \models X\psi$.

Now consider the truth of $\psi \cup \psi'$. We can similarly show that this is preserved by bisimulations by reducing the definition to satisfaction on states which we have shown is preserved.

So the truth of CTL* formulas is preserved by bisimulations.

5

Induction is well carried out. It would have been better, however, to insert the proof concerning the A operator, rather than state its similarity to E

7

We wish to show that if $(M,t) \equiv (M',t')$ in CTL then they $(M,t) \approx (M',t')$. By definition of equivalence we have that for any formula Φ if $(M,t) \models \Phi$ then $(M',t') \models \Phi$, assume they are equivalent and we will show that the 3 properties of bisimulation from question 6 hold. a) is trivial as equivalent worlds satisfy the same atoms. To show b) assume that the forth condition does not hold, i.e. there is some $v \in M$ and $t \to v$ with no $v' \in M'$ such that $t' \to v'$ and $v \approx v'$.
Now let $S' = \{u' \in M' | t' \to u'\}$ this is nonempty as the relation $\to$ is serial and the sets of states in M and M' are finite by assumption.
Now, by our previous assumption $\forall u_i' \in S' \exists$ a formula ψ_i such that $(M,v) \models \psi_i$ but $(M',u_i') \not\models \psi_i$ (as u_i not bisimular to v). But then
$(M,t) \models EX(\wedge_i \psi_i)$ but $(M',t') \not\models EX(\wedge_i \psi_i)$ and we have derived a contradiction and the forth property must hold.
We can similaly prove the back property and hence that if $(M,t) \equiv (M',t')$ then they are bisimilar. □

5

Correct methodology but no actual attempt is seen to prove the back relation

8

What facts do we have? 5: CTL* ¿ CTL; 6: Truth of CTL* is preserved by bisimulations; 7: $(M,t) \equiv_{CTL} (M',t') \Rightarrow (M,t) \approx (M',t')$.
We have to show that $(M,t) \equiv_{CTL} (M',t')$ iff $(M,t) \equiv_{CTL*} (M',t')$
First $\Rightarrow$ direction: Assume (M,t) and (M',t') satisfy the same formulas in CTL, i.e. they are equivalent in CTL. Then by 7 they are bisimular. Now, since thye are bisimilar, by 6 we have that they are equivalent in CTL*.
Now, $\Leftarrow$ direction: Assume (M,t) and (M',t') satisfy the same formulas in CTL*, well by 5 CTL is a strict fragment of CTL* and we know that every CTL formula is also a formula of CTL* - so this direction is trivial. □
It is perhaps surprising thatthe satisfaction of formulas in CTL restricts which formulas can be satisfied in CTL*, even though CTL* some formulas cannot be expressed in CTL.

4

All correct but no attempt to resolve the contradiction

Out of **49**

1			
a/**2**	b/**2**	c/**3**	d/**3**
	Solution correct and in fully simplified form	Solution correct and very well explained	explained but could have been presented with more clarity. Please clearly separate steps in solutions
2	2	3	3

2			
a/**2**	b/**2**	c/**2**	d/**2**
	Steps aren't shown. Even though they are similar to (i), the steps should be made clear		Again steps should be shown
2	1	2	1

3	
a/**3**	b/**2**
3	2

4
/5
5

5	
a/**2**	b/**2**
2	2

6	7	8
/6	**/6**	**/5**
Induction is well carried out. It would have been better, however, to insert the proof concerning the A operator, rather than state its similarity to E	Correct methodology but no actual attempt is seen to prove the back relation	All correct but no attempt to resolve the contradiction
6	5	4