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Modal Logic CW 2

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1. (a) For a path π, then $\pi \models \phi \; R \; \psi$ iff either

   - $\pi[i..\infty] \models \phi$ for some $i \geq 0$ and $\pi[j..\infty] \models \psi$ for all $0 \leq j \leq i$, or
   - $\pi[i..\infty] \models \psi$ for all $i \geq 0$

   (b)
   $$(\psi \; U \; (\psi \wedge \phi)) \vee \neg(\top \; U \; \neg\psi)$$

   (c) $\pi \models (\psi \; U \; (\psi \wedge \phi)) \vee \neg(\top \; U \; \neg\psi)$ if and only if $\pi \models \psi \; U \; (\psi \wedge \phi)$ or $\pi \models \neg(\top \; U \; \neg\psi)$

   $\pi \models \psi \; U \; (\psi \wedge \phi)$, iff (from lecture) $\pi[i..\infty] \models \psi \wedge \phi$ for some $i \geq 0$, and $\pi[j..\infty] \models \psi$ for all $0 \leq j < i$. Then $\pi[i..\infty] \models \psi \wedge \phi$ iff $\pi[i..\infty] \models \psi$ and $\pi[i..\infty] \models \phi$. Clearly, this is true iff $\pi[i..\infty] \models \phi$, and $\pi[j..\infty] \models \psi$ for $0 \leq j \leq i$ - which is precisely the first condition.

   $\pi \models \neg(\top \; U \; \neg\psi)$ iff $\pi \not\models \top \; U \; \neg\psi$. $\pi \not\models \top \; U \; \neg\psi$ if, for all $i \geq 0$ such that $\pi[i..\infty] \models \neg\psi$, then there exists $0 \leq j < i$ such that $\pi[j..\infty] \not\models \top$. But by the semantics of $\top$, this cannot be the case - so in fact there exists no such $i \geq 0$ such that $\pi[i..\infty] \models \neg\psi$ - or equivalently $\pi[i..\infty] \not\models \psi$. So this is true iff $\forall i \geq 0$, then $\pi[i..\infty] \models \psi$ - which is precisely the second condition.

   (d) We already know that $G\psi \equiv \neg(\top \; U \; \neg\psi)$ (from lecture) - so it would suffice to show that $\psi \; U \; (\psi \wedge \bot) \equiv \bot$. Using $\psi \wedge \bot \equiv \bot$, this can be simplified to showing that $\psi \; U \; \bot \equiv \bot$.

   Let λ be any path such that $\lambda \models \psi \; U \; \bot$. Then for some $i \geq 0$, $\lambda[i..\infty] \models \bot$ and for all $0 \leq j < i$, $\lambda[j..\infty] \models \psi$ - but this is a contradiction, so no such λ exists. Therefore, the paths satisfying $\psi \; U \; \bot$ and $\bot$ in any model are the same - so $\psi \; U \; \bot \equiv \bot$.

2. We first show that $(M, \pi) \models (\top \; U \; \Phi)$ iff for some $j \geq 0$, $(M, \pi[j]) \models \Phi$.

   By definition of U, then $(M, \pi) \models (\top \; U \; \Phi)$ iff for some $j \geq 0$, $(M, \pi[j]) \models \Phi$ and $\forall 0 \leq i < j$, $(M, \pi[i]) \models \top$. But by the definition of $\top$, then this second condition always holds - so $(M,\pi) \models (\top\ U\ \Phi)$ iff for some $j \geq 0$, $(M,\pi[j]) \models \Phi$.

- $(M,q) \models EF\Phi$ iff for some path λ from q, then $(M,\lambda) \models \top\ U\ \Phi$ - but this is iff for some $j \geq 0$, $M(,\lambda[j]) \models \Phi$.
  So $(M,q) \models EF\Phi$ iff for some path λ from q, for some $j \geq 0$, then $(M,\lambda[j]) \models \Phi$.
- $(M,q) \models AF\Phi$ iff for every path λ from q, then $(M,\lambda) \models \top\ U\ \Phi$ - but this is iff for some $j \geq 0$, $M(,\lambda[j]) \models \Phi$.
  So $(M,q) \models AF\Phi$ iff for every path λ from q, for some $j \geq 0$, then $(M,\lambda[j]) \models \Phi$.
- $(M,q) \models EG\Phi$ iff $(M,q) \not\models AF\neg\Phi$ - so iff it is not true that for every path λ from q, for some $j \geq 0$, $(M,\lambda[j]) \models \neg\Phi$.
  Then this is not true iff for some path λ from q, every $j \geq 0$, $(M,\lambda[j]) \not\models \neg\Phi$ - so $(M,\lambda[j]) \models \Phi$.
  So $(M,q) \models EG\Phi$ iff for some path λ from q, for every $j \geq 0$, then $(M,\lambda[j]) \models \Phi$.
- $(M,q) \models AG\Phi$ iff $(M,q) \not\models EF\neg\Phi$ - so iff it is not true that for some path λ from q, for some $j \geq 0$, $(M,\lambda[j]) \models \neg\Phi$.
  Then this is not true iff for every path λ from q, every $j \geq 0$, $(M,\lambda[j]) \not\models \neg\Phi$ - so $(M,\lambda[j]) \models \Phi$.
  So $(M,q) \models EG\Phi$ iff for every path λ from q, for every $j \geq 0$, then $(M,\lambda[j]) \models \Phi$.

3. (a) We show this by structural induction on the sufficient connectives $\neg, \wedge, A, E, X, U$, over both path and state formulae.

   If $p \in \text{AP}$ is a state formula of CTL, then by construction p is a state formula of CTL*.

   Assume $\neg\Phi$ is a state formula of CTL. Then Φ is a state formula of CTL, so Φ is a state formula of CTL*. So by construction $\neg\Phi$ is a state formula of CTL*.

   Assume $\Phi \wedge \Psi$ is a state formula of CTL. Then Φ, Ψ are state formulae of CTL - so they are state formulae of CTL*. So by construction, $\Phi \wedge \Psi$ is a state formula of CTL*.

   Assume $E\phi$ is a state formula of CTL. Then ϕ is a path formula of CTL - so it is a path formula of CTL*. So by construction $E\psi$ is a path formula of CTL*.

   Assume $A\phi$ is a state formula of CTL. Then ϕ is a path formula of CTL - so it is a path formula of CTL*. So by construction $A\psi$ is a path formula of CTL*.

Assume $X\Phi$ is a path formula of CTL. Then Φ is a state formula of CTL, and so Φ is a state formula of CTL*. So by construction $X\Phi$ is a path formula of CTL*.

Assume $\Phi\ U\ \Psi$ is a path formula of CTL. Then Φ, Ψ are state formulae of CTL - so they are state formula of CTL*. So by construction $\Phi\ U\ \Psi$ is a path formula of CTL*.

So by induction, all state and path formulae of CTL are state and path formulae of CTL*, respectively.

Since the formulae of CTL are all and exactly the state formulae, and all state formulae of CTL are state formulae of CTL*, and the formulae of CTL* are all and exactly the state formulae - then CTL is a syntactic fragment of CTL*.

(b) $E((\neg Xp) \wedge q)$ - this does not belong to CTL since the contents of the E is a path formula containing conjunction and propositional atoms, both not allowed in CTL.

4. Denote the semantic judgement of CTL as $\models_{\text{CTL}}$. We prove by induction that if $\Phi, \psi \in$ CTL, then $(M,s) \models \Phi \iff (M,s) \models_{\text{CTL}} \Phi$, and $(M,\pi) \models \psi \iff (M,\pi) \models_{\text{CTL}} \psi$.

   For the state formula p, then the result is trivial, since $(M,s) \models p \iff s \in V(p) \iff (M,s) \models_{\text{CTL}} p$.

   For the state formula $\neg\Phi$, then $(M,s) \models \neg\Phi \iff (M,s) \not\models \Phi$ - and by the inductive hypothesis - $\iff (M,s) \not\models_{\text{CTL}} \Phi \iff (M,s) \models_{\text{CTL}} \neg\Phi$.

   For the state formula $\Phi \wedge \Phi'$, then $(M,s) \models \Phi \wedge \Phi'$ iff $(M,s) \models \Phi$ and $(M,s) \models \Phi'$. By the inductive hypothesis, this is iff $(M,s) \models_{\text{CTL}} \Phi$ and $(M,s) \models_{\text{CTL}} \Phi'$, so iff $(M,s) \models_{\text{CTL}} \Phi \wedge \Phi'$.

   For the state formula $E\psi$, then $(M,s) \models E\psi$ iff for some path π starting from s, $(M,\pi) \models \psi$. By the inductive hypothesis, this is iff for some path π starting from s, $(M,\pi) \models_{\text{CTL}} \psi$, and so iff $(M,s) \models_{\text{CTL}} E\psi$.

   For the state formula $A\psi$, then $(M,s) \models A\psi$ iff for every path π starting from s, $(M,\pi) \models \psi$. By the inductive hypothesis, this is iff for every path π starting from s, $(M,\pi) \models_{\text{CTL}} \psi$, and so iff $(M,s) \models_{\text{CTL}} A\psi$.

   For the path formula $X\psi$ then for $X\psi$ to be a formula of the CTL fragment, then $\psi = \Phi$. Then $(M,\pi) \models X\Phi$ iff $(M,\pi[1..]) \models \Phi$ - and by definition, this is iff $(M,\pi[1]) \models \Phi$. By the inductive hypothesis, then this is iff $(M,\pi[1]) \models_{\text{CTL}} \Phi$ - which is iff $(M,\pi) \models_{\text{CTL}} X\Phi$.

For the path formula $\psi\ U\ \psi'$, the for this to be a formula of the CTL fragment, then $\psi = \Phi, \psi' = \Phi'$. Then $(M, \pi) \models \Phi\ U\ \Phi'$ iff $(M, \pi[i..\infty]) \models \Phi'$ for some $i \geq 0$, and for all $0 \leq j < i$, then $(M, \pi[j..\infty]) \models \Phi$. By definition this is iff $(M, \pi[i]) \models \Phi'$ for some $i \geq 0$, and for all $0 \leq j < i$, then $(M, \pi[j]) \models \Phi$. By the inductive definition, then this is iff $(M, \pi[i]) \models_{\text{CTL}} \Phi'$ for some $i \geq 0$, and for all $0 \leq j < i$, then $(M, \pi[j]) \models_{\text{CTL}} \Phi$ - which is iff $(M, \pi) \models_{\text{CTL}} \Phi\ U\ \Phi'$.

5. (a) Take $\Phi' = \Phi$. By Question 4, the semantics of formulae of CTL* which lie inside the CTL fragment are the same as for the corresponding CTL formula. Therefore,

$$(M, s) \models \Phi \text{ (in CTL*)} \iff (M, s) \models \Phi \text{ (in CTL)}$$

   (b) We consider the LTL formula $F(p \wedge Xp)$, which from lectures is not expressible in CTL.

   This formula is equivalent to the state formula $AF(p \wedge Xp)$ in CTL*.

6. We use the fact that if there exists a bisimulation between M and M', then for every path $\pi \in M$, there exists a bisimilar path $\pi' \in M'$ such that $\pi[0]$ is bisimilar to $\pi'[0]$.

   We can construct a finite prefix of such a path of length n for any choice of n, simply taking $\pi'[0]$ to be any world bisimilar to $\pi[0]$ and repeatedly applying the forth condition.

   Using this fact, we can perform the proof by mutual induction.

   For the state formula p, as (M, t) and (M', t') are bisimilar, then

$$(M, t) \models p \iff t \in V(p) \iff t' \in V'(p) \iff (M', t') \models p$$

   For the state formula $\neg\Phi$, then $(M, t) \models \neg\Phi$ iff $(M, t) \not\models \Phi$. By the inductive hypothesis, this is iff $(M', t') \not\models \Phi$, so iff $(M', t') \models \neg\Phi$.

   For the state formula $\Phi \wedge \Phi'$, then $(M, t) \models \Phi \wedge \Phi'$ iff $(M, t) \models \Phi$ and $(M, t) \models \Phi'$. By the inductive hypothesis, this is iff $(M', t') \models \Phi$ and $(M', t') \models \Phi'$ - so iff $(M', t') \models \Phi \wedge \Phi'$.

   For the state formula $E\psi$, then $(M, t) \models E\psi$ only if for some π starting from t, then $(M, \pi) \models \psi$. Then by the above result, there exists $\pi' \in M'$ starting from t' such that π and π' are bisimilar - so by the inductive hypothesis $(M', \pi') \models \psi$ - and as $\pi'[0] = t'$, then $(M', t') \models E\psi$. Since this argument is symmetric, the result is an iff.

For the state formula $A\psi$, then $(M,t) \models A\psi$ only if for every π starting from t, then $(M,\pi) \models \psi$. Let $\pi' \in M'$ be a path starting from t'- then by the above result there exists $\pi \in M$ starting from t such that π is bisimilar to π'. But as π starts from t, then $(M,\pi) \models \psi$ - so by the inductive hypothesis, $(M',\pi') \models \psi$. Since π' was arbitrary, then $(M',t') \models A\psi$. Since this argument is symmetric, the result is an iff.

For the path formula Φ, then $(M,\pi) \models \Phi$ iff $(M,\pi[0]) \models \Phi$. As π' is bisimilar to π, then $\pi'[0]$ is bisimilar to $\pi[0]$ - so by the inductive hypothesis, this is iff $(M',\pi'[0]) \models \Phi$, and so iff $(M',\pi') \models \Phi$.

For the path formulae $\neg\psi, \psi \wedge \psi'$, the proof is the same as for the state formulae version.

For the path formula $X\psi$, then $(M,\pi) \models X\psi$ iff $(M,\pi[1..\infty]) \models \psi$. As π is bisimilar to π', then $\pi[1..\infty]$ is bisimilar to $\pi'[1..\infty]$ - so by the inductive hypothesis, this is iff $(M',\pi'[1..\infty]) \models \psi$, and so iff $(M',\pi') \models \Phi$.

For the path formula $\psi \; U \; \psi'$, then $(M,\pi) \models \psi \; U \; \psi'$ only if for some $i \geq 0$, $(M,\pi[i..\infty]) \models \psi'$ and for all $0 \leq j < i$, $(M,\pi[j..\infty]) \models \psi$. As π is bisimilar to π', then all of these subpaths are bisimilar to their corresponding subpaths in π' - so by the inductive hypothesis, this is only if $(M',\pi'[i..\infty]) \models \psi'$ and for all $0 \leq j < i$, $(M',\pi'[j..\infty]) \models \psi$. So $(M',\pi') \models \psi \; U \; \psi'$. Since this argument is symmetric, the result is an iff.

7. $(\implies)$ - as in the proof of Theorem 35, it suffices to show that $\leftrightsquigarrow$ is a bisimulation. Assume $u \in M, u' \in M'$ such that $u \leftrightsquigarrow u'$.

   Then

$$u \in V(p) \iff (M,u) \models p \iff (M',u') \models p \iff u' \in V'(p)$$

   Assume $v \in M$ such that $u \to v$. Assume for a contradiction that there exists no $v' \in M'$ such that $u' \to v'$ and $v \leftrightsquigarrow v'$. By the assumption that the set of states of M' is finite, then the set

$$S = \{v' \in M \mid u' \to v'\}$$

   is finite. As we only consider infinite paths, it must also be nonempty. Therefore, for each $w \in S$, we can take Φ_w some state formula in CTL such that $(M,v) \models \Phi_w$ and $(M',w) \not\models \Phi_w$. Then the assertion $\bigwedge_{w \in S} \Phi_w$ has the property that $(M,v) \models \bigwedge_{w \in S} \Phi_w$.

   Considering some path starting at u for which the next state is v, we then have that $(M,u) \models EX \bigwedge_{w \in S} \Phi_w$. As this is a formula of CTL, then by the assumption that $u \leftrightsquigarrow u'$, then $(M',u') \models EX \bigwedge_{w \in S} \Phi_w$ - so for some path π starting from u', then $(M',\pi[1]) \models \bigwedge_{w \in S} \Phi_w$. But then $\pi[1] \in S$, so in particular $(M', \pi[1]) \not\models \Phi_{\pi[1]}$, so $(M', \pi[1]) \not\models \bigwedge_{w \in S} \Phi_w$ - so we have a contradiction. So in fact there exists some $v' \in M'$ such that $u' \to v'$ and $v \leftrightsquigarrow v'$. So the forth condition is satisfied.

Since the above argument is symmetric, the back condition is also satisfied.

So $\leftrightsquigarrow$ is a bisimulation, and by assumption $t \leftrightsquigarrow t'$, so (M, t) and (M', t') are bisimilar.

$(\Longleftarrow)$ - follows from 4 and 6. If (M, t) and (M', t') are bisimilar, then they satisfy the same CTL* formulae - and so they satisfy the same CTL formulae in the CTL* fragment, and so they satisfy the same CTL formulae.

8. $(\Longrightarrow)$ - Assume (M, t) and (M', t') satisfy the same formulae of CTL. Then by 7, they are bisimilar. Then by 6, $(M, t) \models \Phi \iff (M', t') \models \Phi$, for $\Phi \in$ CTL* - so they satisfy the same formulae of CTL*.

   $(\Longleftarrow)$ - Assume (M, t) and (M', t') satisfy the same formulae of CTL*.

   Then $(M, t) \models_{\text{CTL}} \Phi$, for $\Phi \in$ CTL if and only $(M, t) \models \Phi$ as a formula in the syntactic fragment of CTL* - if and only if $(M', t') \models \Phi$ as a formula in the syntactic fragment, if and only if $(M', t') \models_{\text{CTL}} \Phi$.

   While more formulae could be expressible in CTL* than in CTL, those formulae are not more expressive in a way that can distinguish between bisimilar states - and can only distinguish between states which are not bisimilar, perhaps for the reason that they are already not bisimilar.