Distributed projected-reflected-gradient algorithms for stochastic generalized Nash equilibrium problems

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Abstract—We consider the stochastic generalized Nash equilibrium problem (SGNEP) with joint feasibility constraints and expected-value cost functions. We propose a distributed stochastic projected reflected gradient algorithm and show its almost sure convergence when the pseudogradient mapping is monotone and the solution is unique. The algorithm is based on monotone operator splitting methods for SGNEPs when the expected-value pseudogradient mapping is approximated at each iteration via an increasing number of samples of the random variable, an approach known as stochastic approximation with variance reduction. Finally, we show that a preconditioned variant of our proposed algorithm has convergence guarantees when the pseudogradient mapping is cocoercive.

I. Introduction

Stochastic generalized Nash equilibrium problems (SGNEPs) have received some attention from the system and control community [1], [2], [3]. In a SGNEP, a set of agents interacts with the aim of minimizing their expected-value cost functions while subject to some joint feasibility constraints. The main feature is that both the cost function and the constraints depend on the strategies chosen by the other agents. Stochastic equilibrium problems arise when there is some uncertainty, modelled via a random variable with an unknown distribution. One main reason for the interest is related to their possible applications. For instance, any networked Cournot game with market capacity constraints and uncertainty in the demand can be modelled as a SGNEP [4], [5]. Other applications can be found in transportation systems, where the drivers perception of travel time is a possible source of uncertainty [6], and in electricity markets where companies schedule their energy production without fully knowing the demand [7].

When the random variable is known, the expected-value formulation can be solved via a standard technique for deterministic GNEPs [8], [9]. Therefore, similarly to the deterministic case, one possible approach for SGNEPs is to recast the problem as a stochastic variational inequality (SVI) through the use of the Karush-Kuhn-Tucker conditions. Then, the problem can be rewritten as a monotone inclusion and solved via operator splitting techniques. Besides the fact that we use the algorithm to find a SGNE, the difficulty in the stochastic case is that the pseudogradient is usually not directly accessible, for instance because the expected value is hard or expensive to compute. For this reason, in many

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situations, the search for a solution of a SVI relies on samples of the random variable.

Two are the main approximation schemes used in the literature: the stochastic approximation (SA) and the sample average approximation (SAA). In the first case, the approximation is done by using only one (or a finite number of) realization of the random variable. SA was first presented in [10], it is computationally less expensive than SAA but usually it requires stronger assumptions on the pseudogradient mapping and on the parameters [2], [11], [12]. The second approximation scheme takes instead a huge number of samples and therefore it has the disadvantage of being computationally costly. However, it usually requires weaker assumptions to ensure convergence [13].

Depending on the monotonicity assumptions on the pseudogradient mapping or the affordable computational complexity, there are different algorithms that can be used to find a SGNE. Among others, one can consider the stochastic preconditioned forward-backward (SpFB) algorithm [14] which is guaranteed to converge to a Nash equilibrium under cocoercivity of the pseudogradient and by demanding one projection step per iteration. However, cocoercivity is not the weakest possible assumption, therefore one would like to have an algorithm that converges under mere monotonicity. Under this assumption, the stochastic forward-backwardforward (SFBF) algorithm involves only one projection step per iteration but two costly evaluation of the pseudogradient mapping [15]. Another alternative is the stochastic extragradient (SEG) algorithm whose iterates are characterized by two projection steps and two evaluation of the pseudogradient mapping which may be expensive [16], [17]. To summarize, having weaker assumptions comes at the price of implementing computationally expensive algorithms.

In this paper, we propose two special instances of the stochastic projected reflected gradient (SPRG) algorithm for SGNEPs. The SPRG involves only one projection step and one evaluation of the pseudogradient, computed in the reflection of second-last iterate on the last one, i.e., it uses the previous two iterates for the updates. Specifically, our contributions are the following.

- We exploit two splitting techniques to obtain two distributed algorithms, with and without preconditioning, that are instances of the SPRG scheme.
- We prove convergence of the first algorithm to a SGNE if the pseudogradient mapping is monotone and the solution is unique (Section V), differently from [17] where it is assumed to be monotone and "weak sharp".
- The weak sharpness property, discussed in Section IV, is

hard to check on the problem data but it is guaranteed to hold on a cocoercive operator. Therefore, we prove convergence of the second, preconditioned algorithm when the pseudogradient mapping is cocoercive (Section VI).

We note that he projected reflected gradient (PRG) was first presented for deterministic VI by Malitsky in [18] and for SVI by Cui and Shanbhag in [11], [17]. Nevertheless, this is the first time that the PRG algorithm is designed for SGNEPs, namely, SVI with special structure, including the coupling constraints between the agents. Remarkably, under uniqueness of the solution, our algorithm has convergence guaranteed also for merely monotone pseudogradient mappings. This is a significant advantage compared to the SpFB which may not converge in that case, see Section VII for an example of non-convergence.

Notation: $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ denotes the standard inner product and $\|\cdot\|$ is the associated euclidean norm. We indicate that a matrix A is positive definite, i.e., $x^{T}Ax > 0$, with $A \succ 0$. $A \otimes B$ indicates the Kronecker product between A and B. Given a symmetric $\Phi \succ 0$, the Φ -induced inner product is $\langle x,y\rangle_{\Phi}=\underline{\langle \Phi x,y\rangle}$ and the associated norm is defined as $||x||_{\Phi} = \sqrt{\langle \Phi x, x \rangle}$. $\mathbf{0}_m$ indicates the vector with m entries all equal to 0. Given $\underline{x}_1, \ldots, x_N \in \mathbb{R}^n, \boldsymbol{x} :=$ $\operatorname{col}(x_1, \dots, x_N) = [x_1^\top, \dots, x_N^\top]^\top \cdot J_F = (\operatorname{Id} + F)^{-1} \text{ is}$ the resolvent of the operator F where Id is the identity operator. For a closed set $C \subseteq \mathbb{R}^n$, the mapping proj_C : $\mathbb{R}^n \to C$ denotes the projection onto C, i.e., $\operatorname{proj}_C(x) =$ $\operatorname{argmin}_{y \in C} \|y - x\|$. ι_C is the indicator function of the set C, that is, $\iota_C(x) = 1$ if $x \in C$ and $\iota_C(x) = 0$ otherwise. The set-valued mapping $N_C: \mathbb{R}^n \to \mathbb{R}^n$ denotes the normal cone operator of the set C, i.e., $N_C(x) = \emptyset$ if $x \notin C, \{v \in \mathbb{R}^n | \sup_{z \in C} v^{\top}(z - x) \le 0\}$ otherwise.

For all $x,y \in C$, a mapping $F: C \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is: pseudomonotone if $(x-y)^T F(y) \geq 0 \Rightarrow (x-y)^T F(x) \geq 0$; monotone if $(F(x)-F(y))^T (x-y) \geq 0$ [19, Def. 2.3.1]; maximally monotone if there exists no monotone operator $G: C \to \mathbb{R}^n$ such that the graph of G properly contains the graph of F [20, Def. 20.20]; β -cocoercive if for $\beta > 0$ $(F(x)-F(y))^T (x-y) \geq \beta \|F(x)-F(y)\|^2$ [19, Def. 2.3.9]; L-Lipschitz continuous if for L > 0 $\|F(x)-F(y)\| \leq L \|x-y\|$ [20, Def. 1.47].

II. STOCHASTIC GENERALIZED NASH EQUILIBRIUM PROBLEM

A. Problem Setup

We consider a set of agents $\mathcal{I}=\{1,\ldots,N\}$, each of them choosing its decision variable $x_i\in\mathbb{R}^{n_i}$ from its local decision set $\Omega_i\subseteq\mathbb{R}^{n_i}$. The aim of each agent is to minimize its local cost function $\mathbb{J}_i:\mathbb{R}^n\to\mathbb{R}$ defined as

$$\mathbb{J}_i(x_i, \boldsymbol{x}_{-i}) := \mathbb{E}_{\xi}[J_i(x_i, \boldsymbol{x}_{-i}, \xi(\omega))]$$

for some measurable function $J_i: \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}$ and $n = \sum_{i=1}^N n_i$. We note that the cost function depends on the local variable x_i , on the decisions of the other agents $\boldsymbol{x}_{-i} = \operatorname{col}((x_j)_{j \neq i})$ and on the random variable $\xi(\omega)$. Specifically, the random variable $\xi: \Xi \to \mathbb{R}^d$ expresses the

fact that there is some uncertainty in the cost function, given the associated probability space $(\Xi, \mathcal{F}, \mathbb{P})$. \mathbb{E}_{ξ} represents the mathematical expectation with respect to the distribution of ξ^1 . We assume that $\mathbb{E}[J_i(\boldsymbol{x},\xi)]$ is well defined for all the feasible $\boldsymbol{x}=\operatorname{col}(x_1,\ldots,x_N)$.

The cost function should satisfy some assumptions, postulated next, to make our analysis possible. Such assumptions are standard for (stochastic) Nash equilibrium problems [1].

Assumption 1: For each $i \in \mathcal{I}$ and $x_{-i} \in \mathcal{X}_{-i}$ the function $\mathbb{J}_i(\cdot,x_{-i})$ is convex and continuously differentiable. For each $i \in \mathcal{I}$ and $\xi \in \Xi$, the function $J_i(\cdot,x_{-i},\xi)$ is convex, Lipschitz continuous and continuously differentiable. For each x_{-i} , the function $J_i(x_i,x_{-i},\cdot)$ is measurable and Lipschitz continuous with constant $\ell_i(x_{-i},\xi)$, integrable in ξ .

Since we consider a SGNEP, we introduce affine shared constraints of the form $Ax \leq b$, where $A = [A_1, \ldots, A_N] \in \mathbb{R}^{m \times n}$, $A_i \in \mathbb{R}^{m \times n_i}$ indicates how agent i is involved in the coupling constraints and $b \in \mathbb{R}^m$. Thus, we denote each agent $i \in \mathcal{I}$ feasible decision set with the set-valued mapping

$$\mathcal{X}_i(\boldsymbol{x}_{-i}) := \left\{ y_i \in \Omega_i \mid A_i y_i \le b - \sum_{j \ne i}^N A_j x_j \right\}, \quad (1)$$

and the collective feasible set as

$$\mathcal{X} = \mathbf{\Omega} \cap \{ \mathbf{y} \in \mathbb{R}^n | A\mathbf{y} - b \le \mathbf{0}_m \}$$
 (2)

where $\Omega = \prod_{i=1}^{N} \Omega_i$. We suppose that there is no uncertainty in the constraints.

Standard assumptions for the constraints sets are postulated next [21].

Assumption 2: For each $i \in \mathcal{I}$, the set Ω_i is nonempty, compact and convex. The set \mathcal{X} satisfies Slater's constraint qualification.

The aim of each agent i, given the decision variables of the other agents x_{-i} , is to choose a decision x_i , that solves its local optimization problem, i.e.,

$$\forall i \in \mathcal{I}: \begin{cases} \min_{x_i \in \Omega_i} & \mathbb{J}_i(x_i, \boldsymbol{x}_{-i}) \\ \text{s.t.} & A_i x_i \leq b - \sum_{j \neq i}^N A_j x_j. \end{cases}$$
(3)

From a game-theoretic perspective, we aim at computing a stochastic generalized Nash equilibrium (SGNE) [1].

Definition 1: A collective variable $x^* \in \mathcal{X}$ is a stochastic generalized Nash equilibrium if, for all $i \in \mathcal{I}$:

$$\mathbb{J}_i(x_i^*, \boldsymbol{x}_{-i}^*) \le \inf\{\mathbb{J}_i(y, \boldsymbol{x}_{-i}^*) \mid y \in \mathcal{X}_i(\boldsymbol{x}_{-i}^*)\}.$$

In other words, a SGNE is a vector of strategies where none of the agents can decrease its cost function by unilaterally deviating from its decision variable.

When Assumptions 1-2 hold, existence of a SGNE of the game in (3) is guaranteed by [1, §3.1] but uniqueness does not hold in general [1, §3.2].

Among all the possible Nash equilibria, we focus on those that are also solutions of an associated (stochastic) variational inequality. First let us define the pseudogradient mapping as

$$\mathbb{F}(\boldsymbol{x}) = \operatorname{col}\left(\mathbb{E}[\nabla_{x_i} J_i(x_i, \boldsymbol{x}_{-i}, \xi_i)]_{i \in \mathcal{I}}\right),\tag{4}$$

where the exchange between the expected value and the gradient is possible because of Assumption 1 [1, Lem. 3.4].

¹For brevity, we use ξ instead of $\xi(\omega)$, $\omega \in \Xi$, and \mathbb{E} instead of \mathbb{E}_{ξ} .

Then, the stochastic variational inequality $\mathrm{SVI}(\mathcal{X},\mathbb{F})$ is the problem of finding $x^* \in \mathcal{X}$ such that

$$\langle \mathbb{F}(\boldsymbol{x}^*), \boldsymbol{x} - \boldsymbol{x}^* \rangle \ge 0$$
, for all $\boldsymbol{x} \in \boldsymbol{\mathcal{X}}$. (5)

with $\mathbb{F}(x)$ as in (4) and \mathcal{X} as in (1). We also note that any solution of $SVI(\mathcal{X}, \mathbb{F})$ is a SGNE of the game in (3) while the opposite does not hold in general. In fact, a game may have a Nash equilibrium while the associated VI may have no solution [22, Prop. 12.7].

Remark 1: Under Assumptions 1-2, the solution set of $SVI(\mathcal{X}, \mathbb{F})$ is non empty and compact, i.e. $SOL(\mathcal{X}, \mathbb{F}) \neq \emptyset$ [19, Corollary 2.2.5].

B. Operator-theoretic characterization

In this section, we rewrite the SGNEP as a monotone inclusion, i.e., the problem of finding a zero of a set-valued monotone operator.

To this aim, we characterize the SGNE of the game in terms of the Karush–Kuhn–Tucker (KKT) conditions for the coupled optimization problems in (3). For each agent $i \in \mathcal{I}$, let us denote with $\lambda_i \in \mathbb{R}^m_{\geq 0}$ the dual variable associated with the coupling constraints. Then, each Lagrangian function is given by $\mathcal{L}_i(\boldsymbol{x}, \lambda_i) := \mathbb{J}_i(x_i, \boldsymbol{x}_{-i}) + \iota_{\Omega_i}(x_i) + \lambda_i^{\top}(A\boldsymbol{x} - b)$. It holds that the set of strategies \boldsymbol{x}^* is a SGNE if and only if the following KKT conditions are satisfied [23, Th. 4.6]:

$$\forall i \in \mathcal{I} : \begin{cases} 0 \in \mathbb{E}[\nabla_{x_i} J_i(x_i^*, \boldsymbol{x}_{-i}^*, \xi)] + \mathcal{N}_{\Omega_i}(x_i^*) + A_i^{\top} \lambda_i \\ 0 \in -(A\boldsymbol{x}^* - b) + \mathcal{N}_{\mathbb{R}_{\geq 0}^m}(\lambda^*). \end{cases}$$

$$(6)$$

Similarly, a variational SGNE (v-SGNE) can be characterized by using the KKT conditions associated to the SVI in (5) [19, Proposition 1.2.1]. The associated KKT optimality conditions reads as

$$\forall i \in \mathcal{I} : \begin{cases} 0 \in \mathbb{E}[\nabla_{x_i} J_i(x_i^*, \boldsymbol{x}_{-i}^*, \xi)] + \mathcal{N}_{\Omega_i}(x_i^*) + A_i^\top \lambda^*, \\ 0 \in -(A\boldsymbol{x}^* - b) + \mathcal{N}_{\mathbb{R}^m_{\geq 0}}(\lambda^*). \end{cases}$$
(7)

The connection between the KKT conditions in (7) and a v-SGNE is summarized next.

From [21, Th. 3.1], it follows that if \boldsymbol{x}^* is a solution of $SVI(\boldsymbol{\mathcal{X}}, \mathbb{F})$ at which the KKT conditions (7) hold, then \boldsymbol{x}^* is a solution of the SGNEP at which the KKT conditions (6) hold with $\lambda_1 = \lambda_2 = \cdots = \lambda_N = \lambda^*$. Viceversa, if \boldsymbol{x}^* is a solution of the SGNEP at which KKT conditions (6) hold with $\lambda_1 = \lambda_2 = \cdots = \lambda_N = \lambda^*$, \boldsymbol{x}^* is a solution of $SVI(\boldsymbol{\mathcal{X}}, \mathbb{F})$ in (5). In words, [21, Th. 3.1] says that variational equilibria are those such that the shared constraints have the same dual variable λ^* for all the agents.

We conclude this section rewriting the KKT conditions in (7) as

$$0 \in \mathcal{T}(\boldsymbol{x}, \boldsymbol{\lambda}) := \begin{bmatrix} N_{\Omega}(\boldsymbol{x}) + \mathbb{F}(\boldsymbol{x}) + A^{\top} \boldsymbol{\lambda} \\ N_{\mathbb{R}_{>0}^{m}}(\boldsymbol{\lambda}) - (A\boldsymbol{x} - b) \end{bmatrix}, \quad (8)$$

where $\mathcal{T}: \mathcal{X} \times \mathbb{R}^m_{\geq 0} \rightrightarrows \mathbb{R}^n \times \mathbb{R}^m$ is a set-valued mapping. This notation in helpful to obtain the algorithms and for the convergence analysis.

C. Approximation scheme

Since the distribution of the random variable is unknown, in the algorithms we replace the expected value with an approximation. For the convergence analysis, we use the stochastic approximation (SA) scheme. We assume to have access to a pool of i.i.d. samples of the random variable collected, for all $k \in \mathbb{N}$ and for each agent $i \in \mathcal{I}$, in the vectors $\bar{\xi}_i^k = \operatorname{col}(\xi_i^{(1)}, \dots, \xi_i^{(N_k)})$. At each time k, the approximation is

$$F_i^{\text{SA}}(x_i^k, \boldsymbol{x}_{-i}^k, \bar{\xi}_i^k) = \frac{1}{N_k} \sum_{i=1}^{N_k} \nabla_{x_i} J_i(x_i^k, \boldsymbol{x}_{-i}^k, \xi_i^{(s)})$$
 (9)

where N_k is the batch size, i.e., the number of sample to be taken. Since the number of samples varies at each iteration, algorithms using (9) are also called variable sample-size schemes. We define the distance between the expected value and its approximation as

$$\epsilon^k = F^{\text{SA}}(\boldsymbol{x}^k, \boldsymbol{\xi}^k) - \mathbb{F}(\boldsymbol{x}^k),$$

where $F^{\rm SA}(\boldsymbol{x},\boldsymbol{\xi})=\operatorname{col}(F_i^{\rm SA}(\boldsymbol{x},\bar{\xi}_i))$ and $\boldsymbol{\xi}=\operatorname{col}(\bar{\xi}_1,\ldots,\bar{\xi}_n)$. Since there is no uncertainty in the constraints, we have

$$\mathcal{A}^{\text{SA}}(\boldsymbol{\omega}^k, \boldsymbol{\xi}^k) - \mathcal{A}(\boldsymbol{\omega}^k) = \varepsilon_k = \text{col}(\epsilon_k, 0, 0),$$

where, from now on, the superscript SA indicates that we use F^{SA} as in (9). Let us introduce the filtration $\mathcal{F} = \{\mathcal{F}_k\}$, i.e., a family of σ -algebras such that $\mathcal{F}_0 = \sigma(X_0)$, for all $k \geq 1$, $\mathcal{F}_k = \sigma(X_0, \xi_1, \xi_2, \dots, \xi_k)$ and $\mathcal{F}_k \subseteq \mathcal{F}_{k+1}$ for all $k \geq 0$. The filtration \mathcal{F} collects the informations that each agent has at the beginning of each iteration k. We note that the process ϵ_k is adapted to \mathcal{F}_k and it satisfies the following assumption [15], [16].

Assumption 3: For al $k \ge 0$, $\mathbb{E}\left[\epsilon_k | \mathcal{F}_k\right] = 0$ a.s.. \blacksquare Moreover, the stochastic error has a vanishing second moment that depends on the increasing number of samples N_k taken at each iteration.

Assumption 4: There exist $c, k_0, a > 0$ such that, for all $k \in \mathbb{N}$,

$$N_k \ge c(k+k_0)^{a+1}. (10)$$

For all k and C > 0, the stochastic error is such that

$$\mathbb{E}[\|\epsilon^k\||\mathcal{F}_k] \le \frac{C\sigma^2}{N_k} \text{ a.s..} \tag{11}$$

The bound for the stochastic error in (11) can be obtained as a consequence of some milder assumptions and it is called variance reduction; we refer to [16, Lem. 3.12], [14, Lem. 6] for more details. Concerning the batch size, the law in (10) is standard in the SA literature [16, Eq. 11], [17, Eq. v-SPRG] since it implies that the sequence $(1/N_k)_{k\in\mathbb{N}}$ is summable.

III. DISTRIBUTED STOCHASTIC PROJECTED REFLECTED GRADIENT ALGORITHMS

In this section, we propose two distributed instances of a stochastic projected reflected gradient (SPRG) algorithm for finding a v-SGNE of the game in (3). The iterations are presented in Algorithms 1 and 2 and are inspired by [17], Algorithm 1 Distributed stochastic projected reflected gradient (SPRG)

Initialization: $x_i^0 \in \Omega_i, \lambda_i^0 \in \mathbb{R}^m_{\geq 0}$, and $z_i^0 \in \mathbb{R}^m$. Iteration k: Agent i

(1): Updates

$$\begin{split} \tilde{x}_{i}^{k} &= 2x_{i}^{k} - x_{i}^{k-1} \\ \tilde{z}_{i}^{k} &= 2z_{i}^{k} - z_{i}^{k-1} \\ \tilde{\lambda}_{i}^{k} &= 2\lambda_{i}^{k} - \lambda_{i}^{k-1} \end{split}$$

(2): Receives x_j^k for all $j\in\mathcal{N}_i^J$ and z_j^k,λ_j^k for $j\in\mathcal{N}_i^\lambda$, then updates:

$$\begin{split} x_i^{k+1} &= \operatorname{proj}_{\Omega_i}[x_i^k - \alpha_i(F_i^{\text{SA}}(\tilde{x}_i^k, \tilde{\boldsymbol{x}}_{-i}^k, \boldsymbol{\xi}_i^k) + A_i^\top \tilde{\lambda}_i^k)] \\ z_i^{k+1} &= z_i^k - \nu_i \sum_{j \in \mathcal{N}_i^\lambda} w_{i,j} (\tilde{\lambda}_i^k - \tilde{\lambda}_j^k) \\ \lambda_i^{k+1} &= \operatorname{proj}_{\mathbb{R}^m_{\geq 0}} \{\lambda_i^k + \tau_i (A_i \tilde{x}_i^k - b_i) \\ &- \tau \sum_{j \in \mathcal{N}_i^\lambda} w_{i,j} [(\tilde{z}_i^k - \tilde{z}_j^k) - (\tilde{\lambda}_i^k - \tilde{\lambda}_j^k)]\} \end{split}$$

[18]. For each agent i, the variables x_i^k , z_i^k and λ_i^k denote the local variables x_i , z_i and λ_i at the iteration time k while α_i , ν_i and σ_i are the step sizes. Since we want the algorithm to be distributed, we assume that each agent i only knows its local data, i.e., Ω_i , A_i and b_i . Moreover, each player is able to compute the approximation $F^{\text{SA}}(\boldsymbol{x},\boldsymbol{\xi})$ in (9) of $\mathbb{F}(\boldsymbol{x})$ in (4), given the collective decision \boldsymbol{x} . We assume therefore that each agent has access to all the decision variables that affect its pseudogradient (full decision information setup). These information are collected, for each agent i, in the set \mathcal{N}_i^J , that is, the set of agents j whose decision x_j explicitly influences J_i .

Since the v-SGNE requires consensus of the dual variables, we introduce an auxiliary variable $z_i \in \mathbb{R}^m$ for all $i \in \mathcal{I}$. The role of $z = \operatorname{col}(z_1, \ldots, z_N)$ is to help reaching consensus and it is further discussed later in this section. The auxiliary variable z_i and a local copy of the dual variable λ_i are shared through the graph $\mathcal{G}^{\lambda} = (\mathcal{I}, \mathcal{E}^{\lambda})$. The set of edges \mathcal{E}^{λ} represents the exchange of the private information on the dual variables: $(i,j) \in \mathcal{E}^{\lambda}$ if agent i can receive $\{\lambda_j, z_j\}$ from agent j. The set of neighbours of i in \mathcal{G}^{λ} is indicated with $\mathcal{N}_i^{\lambda} = \{j|(j,i) \in \mathcal{E}_{\lambda}\}$ [8], [3]. Since each agent feasible set implicitly depends on all the other agents decisions (through the shared constraints), to reach consensus of the dual variables, all agents must coordinate and therefore, \mathcal{G}^{λ} must be connected.

Assumption 5: The dual-variable communication graph \mathcal{G}^{λ} is undirected and connected.

The weighted adjacency matrix of the dual variables graph is indicated with $W \in \mathbb{R}^{N \times N}$. Let $L = D - W \in \mathbb{R}^{N \times N}$ be the Laplacian matrix associated to W, where $D = \operatorname{diag}(d_1,\ldots,d_N)$ is the diagonal matrix of the agents degrees $d_i = \sum_{j=1}^N w_{i,j}$. It follows from Assumption 5 that the adjacency matrix W and the Laplacian L are both symmetric, i.e., $W = W^{\top}$ and $L = L^{\top}$.

Algorithm 2 Distributed stochastic preconditioned projected reflected gradient (SpPRG)

Initialization: $x_i^0 \in \Omega_i, \lambda_i^0 \in \mathbb{R}^m_{\geq 0}$, and $z_i^0 \in \mathbb{R}^m$. Iteration k: Agent i

(1) Updates

$$\tilde{x}_i^k = 2x_i^k - x_i^{k-1}$$
$$\tilde{\lambda}_i^k = 2\lambda_i^k - \lambda_i^{k-1}$$

(2): Receives \tilde{x}_j^k for $j\in\mathcal{N}_i^J$ and λ_j^k for $j\in\mathcal{N}_i^\lambda$ then updates:

$$\begin{aligned} x_i^{k+1} &= \operatorname{proj}_{\Omega_i}[x_i^k - \alpha_i(F_i^{\text{SA}}(\tilde{x}_i^k, \tilde{x}_{-i}^k, \xi_i^k) - A_i^\top \lambda_i^k)] \\ z_i^{k+1} &= z_i^k + v_i \sum\nolimits_{j \in \mathcal{N}_i^\lambda} w_{i,j}(\lambda_i^k - \lambda_j^k) \end{aligned}$$

(3): Receives \tilde{x}_j^{k+1} for $j\in\mathcal{N}_i^J$ and $z_j^{k+1},\,\tilde{\lambda}_j^k$ for $j\in\mathcal{N}_i^\lambda$ then updates:

$$\lambda_i^{k+1} = \operatorname{proj}_{\mathbb{R}_+^m} \left[\lambda_i^k + \sigma_i \left(A_i (2x_i^{k+1} - x_i^k) - b_i \right) + \sigma_i \sum_{j \in \mathcal{N}_i^{\lambda}} w_{i,j} \left(2(z_i^{k+1} - z_j^{k+1}) - (z_i^k - z_j^k) \right) - \sigma_i \sum_{j \in \mathcal{N}_i^{\lambda}} w_{i,j} (\tilde{\lambda}_i^k - \tilde{\lambda}_j^k) \right]$$

To obtain the distributed iterations presented in Algorithm 1 and 2, we exploit a splitting technique starting from the operator \mathcal{T} in (8). First, let us note that operator \mathcal{T} can be written as $\mathcal{T} = \mathcal{Q} + \mathcal{P} + \mathcal{R}$ where

$$Q: \begin{bmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{bmatrix} \mapsto \begin{bmatrix} \mathbb{F}(\mathbf{x}) \\ b \end{bmatrix},$$

$$\mathcal{R}: \begin{bmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{bmatrix} \mapsto \begin{bmatrix} 0 & A^{\top} \\ -A & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{bmatrix}, \qquad (12)$$

$$\mathcal{P}: \begin{bmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{bmatrix} \mapsto \begin{bmatrix} N_{\Omega}(\mathbf{x}) \\ N_{\mathbb{R}^{m}_{\geq 0}}(\lambda) \end{bmatrix},$$

and $\lambda = \operatorname{col}(\lambda_1, \dots, \lambda_N) \in \mathbb{R}^{Nm}$. Moreover, let L be the Laplacian matrix of \mathcal{G}^{λ} and set $\mathbf{L} = L \otimes \mathbf{I}_m \in \mathbb{R}^{Nm \times Nm}$. Let $\mathbf{A} = \operatorname{diag}\{A_1, \dots, A_N\} \in \mathbb{R}^{Nm \times n}$ and and let us define the column vector \boldsymbol{b} of suitable dimensions.

Following [8], to force consensus on the dual variables, we impose the Laplacian constraint $\mathbf{L}\boldsymbol{\lambda}=0$. Then, to preserve monotonicity, we augment the operators introducing the auxiliary variable \boldsymbol{z} . Exploiting the splitting $\mathcal{T}=(\mathcal{Q}+\mathcal{R})+\mathcal{P}$, we consider the following extended operators:

$$\mathcal{A}: \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{z} \\ \boldsymbol{\lambda} \end{bmatrix} \mapsto \begin{bmatrix} \mathbb{F}(\boldsymbol{x}) \\ 0 \\ \mathbf{L}\boldsymbol{\lambda} + \boldsymbol{b} \end{bmatrix} + \begin{bmatrix} 0 & 0 & \mathbf{A}^{\top} \\ 0 & 0 & \mathbf{L} \\ -\mathbf{A} & -\mathbf{L} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{z} \\ \boldsymbol{\lambda} \end{bmatrix} \\
\mathcal{B}: \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{z} \\ \boldsymbol{\lambda} \end{bmatrix} \mapsto \begin{bmatrix} N_{\Omega}(\boldsymbol{x}) \\ \mathbf{0} \\ N_{\mathbb{R}_{>0}^{m}}(\boldsymbol{\lambda}) \end{bmatrix}.$$
(13)

Since the distribution of the random variable is unknown, we replace $\ensuremath{\mathcal{A}}$ with

$$\hat{\mathcal{A}}: egin{bmatrix} (oldsymbol{x}, \xi) \ oldsymbol{z} \ oldsymbol{\lambda} \end{bmatrix} \mapsto egin{bmatrix} F^{\mathrm{SA}}(oldsymbol{x}, \xi) \ 0 \ \mathbf{L} \ + \mathbf{b} \end{bmatrix} + egin{bmatrix} 0 & 0 & \mathbf{A}^{ op} \ 0 & 0 & \mathbf{L} \ -\mathbf{A} & -\mathbf{L} & 0 \end{bmatrix} egin{bmatrix} oldsymbol{x} \ oldsymbol{z} \ oldsymbol{\lambda} \ \end{bmatrix}.$$

where F^{SA} is the approximation in (9) of the expected value mapping \mathbb{F} in (4). Then, given $\omega = \operatorname{col}(x, z, \lambda)$, Algorithm 1 in compact form reads as the SPRG iteration

$$\boldsymbol{\omega}^{k+1} = (\operatorname{Id} + \Phi^{-1}\bar{\mathcal{B}})^{-1} (\boldsymbol{\omega}^k - \Phi^{-1}\hat{\mathcal{A}}(2\boldsymbol{\omega}^k - \boldsymbol{\omega}^{k-1})), (15)$$

where $\Phi \succ 0$ contains the inverse of step size sequences

$$\Phi = \text{diag}(\alpha^{-1}, \nu^{-1}, \sigma^{-1}), \tag{16}$$

and α^{-1} , ν^{-1} , σ^{-1} are diagonal matrices.

Another possible splitting of the operator \mathcal{T} in (8) can be considered, namely, $\mathcal{T} = \mathcal{Q} + (\mathcal{P} + \mathcal{R})$. Therefore, we can write a different couple of extended operators as

$$C: \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{z} \\ \boldsymbol{\lambda} \end{bmatrix} \mapsto \begin{bmatrix} \mathbb{F}(\boldsymbol{x}) \\ 0 \\ \boldsymbol{b} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \mathbf{L}\boldsymbol{\lambda} \end{bmatrix}$$

$$D: \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{z} \\ \boldsymbol{\lambda} \end{bmatrix} \mapsto \begin{bmatrix} N_{\Omega}(\boldsymbol{x}) \\ \mathbf{0}_{mN} \\ N_{\mathbb{R}_{>0}^{m}}(\boldsymbol{\lambda}) \end{bmatrix} + \begin{bmatrix} 0 & 0 & \mathbf{A}^{\top} \\ 0 & 0 & \mathbf{L} \\ -\mathbf{A} & -\mathbf{L} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{z} \\ \boldsymbol{\lambda} \end{bmatrix}.$$
(17)

Since the expected value can be hard to compute, we take an approximation. Analogously to (14), we replace \mathbb{F} in (4) with the approximation F^{SA} in (9) and \mathcal{C} with

$$\hat{C}: \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{z} \\ \boldsymbol{\lambda} \end{bmatrix} \mapsto \begin{bmatrix} F^{SA}(\boldsymbol{x}, \xi) \\ 0 \\ \boldsymbol{b} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \mathbf{L}\boldsymbol{\lambda} \end{bmatrix}$$
(18)

Then, given \hat{C} in (18) and D in (17), we can write Algorithm 2 in compact form as

$$\omega^{k+1} = (\operatorname{Id} + \Psi^{-1} \mathcal{D})^{-1} (\omega^k - \Psi^{-1} \hat{\mathcal{C}} (2\omega^k - \omega^{k-1}))$$
 (19)

where Ψ is the preconditioning matrix. Specifically, let $\alpha^{-1}=\operatorname{diag}\{\alpha_1^{-1}\operatorname{I}_{n_1},\ldots,\alpha_N^{-1}\operatorname{I}_{n_N}\}\in\mathbb{R}^{n\times n}$ and similarly σ^{-1} and ν^{-1} of suitable dimensions. Then, we have

$$\Psi = \begin{bmatrix} \alpha^{-1} & 0 & -\mathbf{A}^{\top} \\ 0 & \nu^{-1} & -\mathbf{L} \\ -\mathbf{A} & -\mathbf{L} & \sigma^{-1} \end{bmatrix}.$$
 (20)

By expanding (19) with \hat{C} as in (18), \mathcal{D} as in (17) and Ψ as in (20), we obtain the iterations in Algorithm 2.

We note that in general the extended operators in (13) and (17) have different monotonicity properties. Which one specifically is discussed for the convergence analysis (Section V and VI, respectively) that follows from the considerations in the next section.

IV. TECHNICAL DISCUSSION ON WEAK SHARPNESS AND COCOERCIVITY

The original proof of the SPRG presented in [17] for SVI shows convergence under the assumption of monotonicity and weak sharpness. The weak sharpness property was first introduced to characterize the minima of

$$\min_{x \in \mathcal{X}} f(x) \tag{21}$$

with $f: \mathcal{X} \to \overline{\mathbb{R}}$ [24]. It was presented as en extension of the concept of strong (or sharp) solution, i.e., for all $x^* \in \mathcal{X}^* = \mathrm{SOL}(f, \mathcal{X})$

$$f(x) > f(x^*) + \rho ||x - x^*||,$$

which holds if there is only one minimum. For generalizing non-unique solutions, the following definition was proposed in [24]: a set \mathcal{X}^* is a set of weak sharp minima for the function f if, for all $x \in \mathcal{X}$ and $x^* \in \mathcal{X}^*$,

$$f(x) \ge f(x^*) + \rho \operatorname{dist}(x, \mathcal{X}^*) \tag{22}$$

where $\operatorname{dist}(x, \mathcal{X}^*) = \inf_{x^* \in \mathcal{X}^*} \|x - x^*\|$. We note that a strong solution is also a weak sharp minimum while the contrary holds only if the solution is unique [24].

The concept was later extended to variational inequalities in [25], using the formal definition

$$-\mathbb{F}(\boldsymbol{x}^*) \in \operatorname{int}\left(\bigcap_{\boldsymbol{x} \in \boldsymbol{\mathcal{X}}^*} [\mathrm{T}_{\boldsymbol{\mathcal{X}}}(\boldsymbol{x}) \cap \mathrm{N}_{\boldsymbol{\mathcal{X}}^*}(\boldsymbol{x})]^{\circ}\right), \quad (23)$$

which was already proved to be equivalent to (22) for the problem in (21) when $\mathbb{F}(x^*) = \nabla f(x^*)$.

Unfortunately, the characterization in (23) is hard to use in a convergence proof. Therefore, more practical conditions have been proposed. The first one [25] relies on the gap function G and it reads as

$$G(x) = \max_{y \in \mathcal{X}} \langle \mathbb{F}(y), x - y \rangle \ge \rho \operatorname{dist}(x, \mathcal{X}^*). \tag{24}$$

Another condition, used in the convergence proof of the SPRG [17], was proposed in [12]:

$$\langle \mathbb{F}(\boldsymbol{x}^*), \boldsymbol{x} - \boldsymbol{x}^* \rangle \ge \rho \operatorname{dist}(\boldsymbol{x}, \boldsymbol{\mathcal{X}}^*),$$
 (25)

for all $x^* \in \mathcal{X}^*$ and $x \in \mathcal{X}$. For the weak sharpness definition in (23) to be equivalent to (24) and (25), the (pseudogradient) mapping should have the *F-unique* property, i.e., $\mathbb{F}(\mathrm{SOL}(\mathbb{F},\mathcal{X}))$ should be at most a singleton [19, Section 2.3.1]. The class of operators that certainly have this property is that of $monotone^+$ operators, namely, a monotone mapping F such that for all $x, y \in \mathcal{X}$

$$\langle F(\boldsymbol{y}) - F(\boldsymbol{x}), \boldsymbol{y} - \boldsymbol{x} \rangle = 0 \quad \Rightarrow \quad F(\boldsymbol{y}) = F(\boldsymbol{x}).$$

If a mapping is monotone⁺, then (23) is equivalent to (24) and (25) [25, Thm. 4.1],[26, Prop. 2].

The monotone⁺ property does not necessarily hold for the extended operator C in (17), even if it holds for \mathbb{F} . However, it holds if the operator is cocoercive [19, Def. 2.3.9]. For more details on monotone⁺ operators and the weak sharpness property, we refer to [27], [28], [25], [29].

We conclude this section with some examples showing that the condition in (25) may hold also if the mapping is not monotone⁺ and that the domains are relevant for the validity of the assumption.

Example 1: [25] Consider the variational inequality in (5) where $F(x) = \operatorname{col}(-x_2, 2x_1)$ and $\mathcal{X} = [0, 1]^2$. The mapping F is pseudomonotone but not monotone⁺ on \mathcal{X} . The solution set is $\mathcal{X}^* = \{x \in \mathcal{X} : x_2 = 0\}$ and it holds that

$$G(\boldsymbol{x}) = \max_{\boldsymbol{y} \in \boldsymbol{\mathcal{X}}} \langle F(\boldsymbol{y}), \boldsymbol{x} - \boldsymbol{y} \rangle$$

$$= \max_{\boldsymbol{y} \in \boldsymbol{\mathcal{X}}} -x_1 y_2 - y_1 y_2 + 2x_2 y_1$$

$$= 2x_2 = 2 \operatorname{dist}(\boldsymbol{x}, \boldsymbol{\mathcal{X}}^*).$$
(26)

Therefore, \mathcal{X}^* satisfies (24) with $\rho=2$ but, for any $\boldsymbol{x}^*\in\mathcal{X}^*$, $[T_{\mathcal{X}}(\boldsymbol{x}^*)\cap N_{\mathcal{X}^*}(\boldsymbol{x}^*)]^\circ=\{x_2^*\leq 0\}$ and $-F(\boldsymbol{x}^*)\notin\mathcal{X}^*$

 $\bigcap_{\boldsymbol{x}^*\in\boldsymbol{\mathcal{X}}}[T_{\boldsymbol{\mathcal{X}}}(\boldsymbol{x}^*)\cap N_{\boldsymbol{\mathcal{X}}^*}(\boldsymbol{x}^*)]^\circ. \text{ Thus, the solution set }\boldsymbol{\mathcal{X}}^* \text{ is not weakly sharp.}$

Example 2: Consider the mapping $F(\mathbf{x}) = \operatorname{col}(-x_2, x_1)$ and the associate variational inequality in (5) with $\mathcal{X} = [0,1]^2$. Then the mapping F is monotone but not monotone⁺ on \mathcal{X} . The solution set is $\mathcal{X}^* = \{\mathbf{x} \in \mathcal{X} : x_2 = 0\}$ and, similarly to (26), the conditions (24) and (25) hold.

Now, let $\mathcal{X} = \mathbb{R}^2$. In this case, there is only one solution and $\mathcal{X}^* = \{\mathbf{0}_2\}$. However, (25) reads as

$$\langle F(\mathbf{0}), \boldsymbol{x} \rangle = 0 \ge \rho \operatorname{dist}(\boldsymbol{x}, \mathcal{X}^*) = ||\boldsymbol{x}||,$$

which is false.

V. Convergence under uniqueness of solution

In light of the considerations in Section IV, we know that a unique solution is also a weak solution and that (25) may hold even if the mapping is not monotone⁺. Therefore, here we consider the case of merely monotone operators but with unique solution and prove that the proposed (non-preconditioned) Algorithm 1 converges to a v-SGNE.

First, the following lemma ensure that the zeros of A + B are v-SGNEs.

Lemma 1: Let Assumptions 1-5 hold. Consider the operators \mathcal{T} in (8) and \mathcal{A} and \mathcal{B} in (13). Then, the following statements hold.

- (i) If $\omega^* \in \operatorname{zer}(\mathcal{A} + \mathcal{B})$, then x^* is a v-SGNE of game in (3), i.e., x^* solves the $\operatorname{SVI}(\mathcal{X}, \mathbb{F})$ in (5). Moreover $\lambda^* = \mathbf{1}_N \otimes \lambda^*$, and (x^*, λ^*) satisfy the KKT condition in (7) i.e., $\operatorname{col}(x^*, \lambda^*) \in \operatorname{zer}(\mathcal{T})$
- (ii) $\operatorname{zer}(\mathcal{T}) \neq \emptyset$ and $\operatorname{zer}(\mathcal{A} + \mathcal{B}) \neq \emptyset$

Proof: It follows from [8, Th. 2].

Now, to ensure that \mathcal{A} and \mathcal{B} have the properties that we use for the convergence result, we make the following assumption.

Assumption 6: \mathbb{F} as in (4) is monotone and $\ell_{\mathbb{F}}$ -Lipschitz continuous for some $\ell_{\mathbb{F}} > 0$.

Then, the two operators \mathcal{A} and \mathcal{B} in (13) have the following properties.

Lemma 2: Let Assumptions 1 and 6 hold and let $\Phi \succ 0$. Then, \mathcal{A} and \mathcal{B} in (13) have the following properties.

- (i) A is monotone and ℓ_A -Lipschitz continuous.
- (ii) The operator \mathcal{B} is maximally monotone.
- (iii) $\Phi^{-1}\mathcal{A}$ is monotone and ℓ_{Φ} -Lipschitz continuous.
- (iv) $\Phi^{-1}\mathcal{B}$ is maximally monotone.

Proof: It follows from [8] and [14].

To guarantee that the weak sharpness property holds, we assume to have a strong solution.

Assumption 7: The ŠVI in (5) has a unique solution.
We can now state the convergence result.

Theorem 1: Let Assumptions 1-6 hold. Then, the sequence $(x_k)_{k\in\mathbb{N}}$ generated by Algorithm 1 with F^{SA} as in (9) converges a.s. to a v-SGNE of the game in (3).

Proof: The iterations of Algorithm 1 are obtained by expanding (15) and solving for x_k , z_k and λ_k . Therefore, Algorithm 1 is a SPRG iteration as in (15). The convergence of the sequence (x^k, λ^k) to a v-GNE of the game in (3) then follows by [17, Prop. 10] and Lemma 1 since $\Phi^{-1}A$ is monotone by Lemma 2 and has a unique solution.

VI. CONVERGENCE UNDER COCOERCIVITY

We know from Section IV that the weak sharpness property holds for cocoercive operators. Here we consider this case. Since the properties of the extend operators operators depends on the properties of the mapping \mathbb{F} in (4), we postulate the following assumption.

Assumption 8: \mathbb{F} is β -cocoercive for some $\beta > 0$.

Remark 2: If a function is β -cocoercive, it is also $1/\beta$ -Lipschitz continuous [20, Remark 4.15].

We note that the operator \mathcal{A} in (13) contains a skew symmetric matrix that is not cocoercive. For this reason we consider the splitting in (17). While Lemma 1 guarantees that the zeros of $\mathcal{C} + \mathcal{D}$ are the same as the zeros of \mathcal{T} in (8), we now show the necessary monotonicity properties of the extended operators \mathcal{C} and \mathcal{D} in (17).

Lemma 3: Let Assumptions 1 and 8 hold and let $\Psi \succ 0$. Then, \mathcal{C} and \mathcal{D} in (17) have the following properties.

- (i) C is θ -cocoercive where $0 < \theta \le \min\left\{\frac{1}{2d^*}, \beta\right\}$ and d^* is the maximum weighted degree of \mathcal{G}^{λ} ;
- (ii) \mathcal{D} is maximally monotone;
- (iii) $\Psi^{-1}\mathcal{C}$ is $\theta\tau$ -cocoercive, with $\tau = \frac{1}{|\Psi^{-1}|}$;
- (iv) $\Psi^{-1}\mathcal{D}$ is maximally monotone.

Proof: It follows from [8, Lem. 5] and [8, Lem. 7]. ■ Furthermore, since the preconditioning matrix must be positive definite, we postulate the following assumption on the step sizes.

Assumption 9: Let θ be the cocoercivity constant as in Lemma 3, $\tau=\frac{1}{|\Psi^{-1}|}\in(0,\frac{\theta}{8})$ and the step sizes α , ν and σ satisfy, for all $i\in\mathcal{I}$,

$$0 < \alpha_{i} \le \left(\max_{j=1,\dots,n_{i}} \left\{ \sum_{k=1}^{m} |[A_{i}^{T}]_{jk}| \right\} + \tau \right)^{-1}$$

$$0 < \nu_{i} \le \left(2d_{i} + \tau \right)^{-1}$$

$$0 < \sigma_{i} \le \left(\max_{j=1,\dots,m} \left\{ \sum_{k=1}^{k_{i}} |[A_{i}]_{jk}| \right\} + 2d_{i} + \tau \right)^{-1}$$

$$(27)$$

where $[A_i^{\top}]_{jk}$ indicates the entry (j,k) of the matrix A_i^{\top} . \blacksquare For example, we can obtain conditions (27) imposing that the preconditioning matrix Ψ to be diagonally dominant which, since it is symmetric, implies that Ψ is positive definite [8].

We are now ready to state our convergence result.

Theorem 2: Let Assumptions 1-5 and 8-9 hold. Then, the sequence $(x_k)_{k\in\mathbb{N}}$ generated by Algorithm 2 with F^{SA} as in (9) converges a.s. to a v-SGNE of the game in (3).

Proof: The iterations of Algorithm 2 are obtained by expanding (19) and solving for x_k , z_k and λ_k . Therefore, Algorithm 2 is a SPRG iteration as in (19). The convergence of the sequence (x^k, λ^k) to a v-GNE of the game in (3) then follows by [17, Prop. 10] and Lemma 1 since $\Psi^{-1}\mathcal{C}$ is cocoercive by Lemma 3.

Remark 3: We note that adopting a SA scheme is not possible in this case because a vanishing step should be taken to control the stochastic error [11]. However, having a time-varying step implies using a variable metric, induced by the preconditioning matrix Φ_k which depends on α^k , σ^k and ν^k , for the convergence analysis. Although analysing a variable metric is possible, the matrix Φ_k should satisfy additional assumptions that typically do not hold if the step size is vanishing [30, Prop. 3.4].

VII. NUMERICAL SIMULATIONS

Let us propose some numerical evaluations to validate the analysis: an illustrative example and a Nash-Cournot game. While the first comes from Example 2, the second is a realistic application to an electricity market with capacity constraints [8], [17].

All the simulations are performed on Matlab R2019b with a 2,3 GHz Intel Core i5 and 8 GB LPDDR3 RAM.

A. Illustrative example

We start with the stochastic counterpart of Example 2, that is, a monotone (non-cocoercive) stochastic Nash equilibrium problem with two players with strategies x_1 and x_2 respectively, and pseudogradient mapping

$$\mathbb{F}(\boldsymbol{x}) = \left[\begin{array}{cc} 0 & R_1(\xi) \\ -R_2(\xi) & 0 \end{array} \right] \left[\begin{array}{c} x_1 \\ x_2 \end{array} \right].$$

The random variables are sampled from a normal distribution with mean 1 and finite variance, following Assumption 4. The problem is unconstrained and the optimal solution is (0,0). The step sizes are taken to be the highest possible and we compare our SpPRG with the stochastic distributed preconditioned forward–backward (SpFB) which is guaranteed to converge under the same cocoercivity assumption with the SA scheme [14].

Figure 1 shows that the SpFB does not converge while, due to the uniqueness of the solution, the SpPRG does.

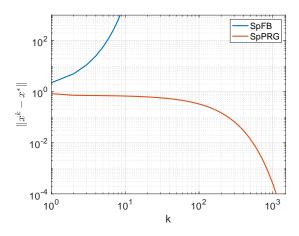


Fig. 1. Relative distance of the primal variable from the solution.

B. Nash-Cournot game with market capacity constraints

Now, we consider an electricity market problem that can be casted as a network Cournot game with markets capacity constraints [8], [3], [17]. We consider a set of N=20 companies selling their product to a set of m=7 markets. Each generator decides the quantity of energy x_i to deliver to the n_i markets it is connected with. Each company i has a local constraint, i.e., a production limit, of the form $0 < x_i < \gamma_i$ where each component of γ_i is randomly drawn from [1,1.5]. Each company has a cost of production $c_i(x_i) = 1.5x_i + q_i$, where q_i is a given constant, that is not

uncertain. For simplicity, we assume the transportation costs are zero.

Each market j has a bounded capacity b_j , randomly drawn from [0.5, 1]. The collective constraints are then given by $Ax \leq b$ where $A = [A_1, \ldots, A_N]$ and each A_i specifies in which market each company participates.

The prices of the markets are collected in $P: \mathbb{R}^m \times \Xi \to \mathbb{R}^m$. The uncertainty variable, ξ which represents the demand uncertainty, appears in this functional. P is supposed to be a linear function and reads as $P(\xi) = \bar{P}(\xi) - DAx$. Each component of $\bar{P} = \operatorname{col}(\bar{P}_1(\xi), \dots, \bar{P}_7(\xi))$ is taken with a normal distribution with mean 3 and finite variance. The entries of D are randomly taken in [0.5, 1].

The cost function of each agent is then given by

$$\mathbb{J}_i(x_i, x_{-i}, \xi) = c_i(x_i) - \mathbb{E}[P(\xi)^\top (A\boldsymbol{x}) A_i x_i].$$

and it is influenced by the variables of the companies selling in the same market as in [8, Fig. 1]. The dual variables graph is a cycle graph with the addiction of the edges (2,15) and (6,13) [8].

We simulate the SpFB, the forward-backward-forward (SFBF) and the extragradient (SEG) algorithms to make a comparison with our SPRG and SpPRG, using the SA scheme. The parameters α , ν and σ are taken to be the highest possible that guarantee convergence.

As a measure of the distance from the solution, we consider the residual, $\operatorname{res}(x^k) = \|x^k - \operatorname{proj}_C(x^k - F(x^k))\|$, which is equal zero if and only if x is a solution. The plots in Fig. 2 shows how the residual varies in the number of iterations while the plot in Fig. 3 shows the number of times that the pseudogradient mapping is computed. As one can see from the plots, the performances of SpPRG and SPRG are very similar. The difference in the trajectory is related to the different step sizes which depend on the Lipschitz constant of $\mathcal C$ in (17) and $\mathcal A$ in (13) respectively.

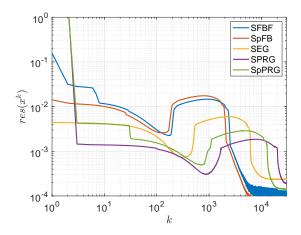


Fig. 2. Relative distance of the primal variable from the solution.

VIII. CONCLUSION

The stochastic projected reflected gradient algorithm is applicable to distributed stochastic generalized Nash equilibrium seeking. To guarantee convergence to a solution and

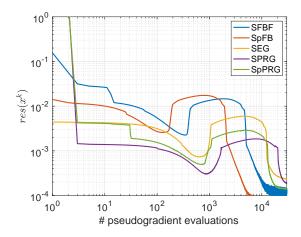


Fig. 3. Relative distance of the primal variable from the solution.

to obtain a distributed algorithm, preconditioning may be used and the pseudogradient mapping should be cocoercive. However, should the equilibrium be unique, the cocoercivity assumption can be reduced to mere monotonicity.

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