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Network aggregative games: Distributed convergence to Nash equilibria

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***Abstract*— We consider quasi-aggregative games for large populations of heterogeneous agents, whose interaction is determined by an underlying communication network. Specifically, each agent minimizes a quadratic cost function, which depends on its own strategy and on a convex combination of the strategies of its neighbors, and is subject to heterogeneous convex constraints. We suggest two distributed algorithms that can be implemented to steer the best responses of the rational agents to a Nash equilibrium configuration. The convergence of these schemes is guaranteed under different sufficient conditions depending on the matrices defining the agents' cost functions and on the communication network.**

I. INTRODUCTION

In recent years, there has been an increasing interest in the modeling and control of populations of agents that interact through a network. If the agents are noncooperative and profit maximizing, these systems can be studied combining ideas of game theory and network analysis [1], [2], [3]. Traditionally, the literature on network game theory has focused on games in which each agent aims at maximizing a payoff that depends on its *one-to-one* interactions with the neighbors. When the size of the population becomes very large, however, the analysis of these models may become computationally intractable. Moreover, in many applications involving large populations, as demand side management in smart grids [4], [5], [6], charging coordination of plug-in electric vehicles [7], [8], congestion control [9] and economic markets [10], the agents are not influenced by the other players individually, but only by their *aggregate* effect. This feature can be exploited by using the framework of aggregative games [11]. When the population size is very large, aggregative games can be analyzed more efficiently by assuming a continuum of players, that are influenced only by the population distribution, and by exploiting the so-called *mean field* approximation [12], [13]. Recently, a similar framework has been used to describe the dynamics of deterministic systems characterized by countably many players, that are affected by the average population behavior and are subject to heterogenous convex constraints [14].

The aggregate models described above are based, in their classic formulation, on the assumption that all the agents have the same relevance and that they are affected by the population in the same way. Consequently, their aim is not to describe applications in which the interaction among the agents, even though not one-to-one as in typical network games, still possesses a well determined structure. This is the case, for example, of leader-follower games [15] where some agents have more control authority and knowledge than others, games with local/agent dependent cost functions [16], games where the agents have limited communication capabilities [17], [18], and games where the agents have different stubbornness [19], [20]. A very important subclass of these problems is the one of *quasi-aggregative* games [21]. Therein, each agent is affected by the population according to a different agent-dependent aggregation function.

In this paper we consider a new class of quasi-aggregative games in which the agents communicate through a network and consequently use as aggregate function the convex combination of the strategies of their neighbors. Motivated by engineering applications, we consider quadratic cost functions and, in contrast to classical aggregative games, we assume that the agents' strategies are multidimensional vectors belonging to personalized convex constraint sets. For this class of systems, we are interested in characterizing the asymptotic behavior of a population where each agent updates synchronously its strategy in response to the strategies of the others. The simplest strategy update rule that we investigate is the myopic Best Response (BR), obtained when each agent selects the strategy that minimizes its cost given the current neighbors state. As first technical contribution, we derive conditions on the cost function and on the network structure that guarantee convergence of the BR dynamics to a Nash Equilibrium, that is, to a set of strategies where no agent has interest in unilaterally deviating from its behavior. For cases where the BR dynamics are not guaranteed to converge, we propose a different strategy update scheme that exploits memory in order to ensure distributed convergence to a Nash equilibrium under less stringent assumptions. Finally, we investigate by simulations the effect of the network structure and population size on the convergence speed for a resource allocation problem. Further case studies and theoretical results can be found in [22], where the proposed algorithms are applied to multidimensional opinion dynamics and distributed demand-response models.

The paper is organized as follows. Section II introduces the new class of network aggregative games. Section III provides a sufficient condition for a set of strategies to be a Nash equilibrium. Section IV illustrates the main convergence results. Section V provides an illustrative example and a comparison between this framework, when the network is fully connected, and classic aggregative games. Section VI concludes the paper. The proofs are given in the Appendix.

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Notation

$\mathbb{R}, \mathbb{R}_{>0}, \mathbb{R}_{\geq 0}, \mathbb{Z}$ respectively denote the set of real, positive real, non-negative real and integer numbers; for $a, b \in \mathbb{Z}$, $a \leq b$, $\mathbb{Z}[a,b]$ denotes the integer interval $\{a, a+1, \ldots, b\}$. For a given $Q \in \mathbb{R}^{n \times n}$, $Q \succ 0$, we denote by $\mathcal{H}_Q$ the Hilbert space $\mathbb{R}^n$ with inner product $\langle \cdot, \cdot \rangle_Q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined as $\langle x, y \rangle_Q := x^\top Q y$, and induced norm $\|\cdot\|_Q : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ defined as $\|x\|_Q := \sqrt{x^\top Q x}$. I_n denotes the n-dimensional identity matrix. $A \otimes B$ denotes the Kronecker product between matrices A and B. Given $\mathcal{S} \subseteq \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$, $A\mathcal{S} + b$ denotes the set $\{Ax + b \in \mathbb{R}^n \mid x \in \mathcal{S}\}$. With $\mathcal{X}^{1 \times N}$ we indicate $\mathcal{X}^1 \times \cdots \times \mathcal{X}^N$. $\mathbb{1}_n \in \mathbb{R}^n$ denotes the vector of all ones. Let $x^i \in \mathbb{R}^n$, for $i \in \mathbb{Z}[1, N]$, we denote by $\boldsymbol{x} := [x^1; \ldots; x^N] := [x^{1\top}, \ldots, x^{N\top}]^\top \in \mathbb{R}^{Nn}$.

II. Network aggregative games

We consider a game played among a population of N heterogeneous agents whose interactions are specified by an underlying communication network. Specifically, we assume that the agents communicate according to a *row stochastic* matrix $P \in \mathbb{R}^{N \times N}$, whose element $P_{ij} \in [0,1]$ indicates the strength (or relevance) of the communication from agent j to agent i, where $P_{ij} = 0$ denotes no communication, and the diagonal elements are set to zero, that is $P_{ii} = 0$ for all $i \in \mathbb{Z}[1, N]$. In the following, we denote by $\mathcal{N}^i$ the set of neighbors of agent i, that is $\mathcal{N}^i := \{j \in \mathbb{Z}[1, N] \mid P_{ij} > 0\}$. Note that we consider j to be a neighbor of agent i if agent i *receives* communications from j. Moreover, we assume that the interaction between neighbors is not one-to-one, but each agent i is influenced only by the aggregate strategies of its neighbors $\mathcal{N}^i$. More in detail, each agent i tries to minimize a cost function $J^i(x^i, \sigma^i)$ that depends on its own deterministic state $x^i \in \mathcal{X}^i \subseteq \mathbb{R}^n$ and on the *neighbors aggregate state*

$$\sigma^i := \textstyle\sum_{j \neq i} P_{ij} x^j \in \mathbb{R}^n. \tag{1}$$

Equivalently, each agent $i \in \mathbb{Z}[1, N]$ aims at solving the optimization problem

$$x^{i\,\star}(\sigma^i) := \arg\min_{x \in \mathcal{X}^i} J^i(x, \sigma^i). \tag{2}$$

In the following, we focus on games with quadratic cost

$$J^i(x, \sigma^i) := q_i x^\top Q x + 2\left(C_i \sigma^i + c_i\right)^\top x, \tag{3}$$

where $q_i > 0$, Q, $C_i \in \mathbb{R}^{n \times n}$, $Q \succ 0$, $c_i \in \mathbb{R}^n$ and we suppose that each agent has a different constraint set $\mathcal{X}^i$ satisfiying the following standing assumption.

Standing assumption 1 (Constraint sets): The set $\mathcal{X}^i$ is convex and compact for all $i \in \mathbb{Z}[1, N]$. □

A very important notion in noncooperative game theory is that of Nash Equilibrium. Specifically, a set of strategies $\{\bar{x}^i\}_{i=1}^N$ is said to be a Nash equilibrium if no agent has interest in deviating from its own strategy, given the strategies of the others. In the context of network aggregative games, this concept can be defined mathematically as follows.

Definition 1 (Nash equilibrium): Given a set of cost functions $\{J^i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}\}_{i=1}^N$ and a communication matrix $P \in \mathbb{R}^{N \times N}$, a set of strategies $\{\bar{x}^i \in \mathcal{X}^i\}_{i=1}^N$ is a Nash equilibrium for the game (2) if, for all $i \in \mathbb{Z}[1, N]$, it holds

$$J^i\left(\bar{x}^i, \textstyle\sum_{j \neq i} P_{ij} \bar{x}^j\right) = \min_{y \in \mathcal{X}^i} J^i\left(y, \textstyle\sum_{j \neq i} P_{ij} \bar{x}^j\right). \quad \square$$

III. Characterization of the Nash equilibria

For a given initial set of strategies $\{x^i\}_{i=1}^N$, let $\boldsymbol{\sigma} := [\sigma^1; \ldots; \sigma^N] \in \mathbb{R}^{Nn}$ be a vector composed by all the neighbors aggregate states, computed *simultaneously* by the agents according to (1), and define the extended mapping $\boldsymbol{x}^\star : \mathbb{R}^{Nn} \to \mathcal{X}^{1 \times N}$ as

$$\boldsymbol{x}^\star(\boldsymbol{\sigma}) := \left[x^{1\,\star}(\sigma^1); \ldots; x^{N\,\star}(\sigma^N)\right] \in \mathbb{R}^{Nn}, \tag{4}$$

whose elements are the optimal strategies computed *simultaneously* by each agent i, in response to the measured neighbors aggregate state σ^i. The mapping $\boldsymbol{x}^\star$ in (4) can be used to define an aggregation mapping $\boldsymbol{\mathcal{A}}$ that, given the vector of aggregate quantities $\boldsymbol{\sigma}$, returns the updated aggregate quantities, after one optimization and one communication step. Formally, $\boldsymbol{\mathcal{A}} : \mathbb{R}^{Nn} \to (P \otimes I_n)\mathcal{X}^{1 \times N}$ is defined as

$$\begin{aligned}\boldsymbol{\mathcal{A}}(\boldsymbol{\sigma}) &:= \begin{bmatrix} \mathcal{A}^1(\boldsymbol{\sigma}) \\ \vdots \\ \mathcal{A}^N(\boldsymbol{\sigma}) \end{bmatrix} := \begin{bmatrix} \sum_{j \neq 1} P_{1j} x^{j\,\star}(\sigma^j) \\ \vdots \\ \sum_{j \neq N} P_{Nj} x^{j\,\star}(\sigma^j) \end{bmatrix} \\ &= (P \otimes I_n)\boldsymbol{x}^\star(\boldsymbol{\sigma}) =: \boldsymbol{\mathcal{P}}\boldsymbol{x}^\star(\boldsymbol{\sigma}).\end{aligned} \tag{5}$$

In other words, the aggregation mapping $\boldsymbol{\mathcal{A}}$ is the composition of the best response extended mapping $\boldsymbol{x}^\star$ with the mapping $\boldsymbol{\mathcal{P}}$, which corresponds to one round of communications through the network. In the following proposition we show that any fixed point $\bar{\boldsymbol{\sigma}}$ of the aggregation mapping $\boldsymbol{\mathcal{A}}(\cdot)$ in (5) can be directly used to compute a Nash equilibrium.

Proposition 1: The mapping $\boldsymbol{\mathcal{A}}$ in (5) admits at least one fixed point. Given any fixed point $\bar{\boldsymbol{\sigma}} = \boldsymbol{\mathcal{A}}(\bar{\boldsymbol{\sigma}})$, the set of strategies $\{x^{i\,\star}(\bar{\sigma}^i)\}_{i=1}^N$ is a Nash equilibrium for the game in (2). □

IV. Iterative schemes and convergence properties

The aggregation mapping $\boldsymbol{\mathcal{A}}$ mathematically characterizes what happens if, at a given instant of time, all the agents synchronously compute their neighbors aggregate state and consequently update their strategies. In the following, we are interested in characterizing the asymptotic properties of the strategies evolution when these two steps (communication and optimization) are repeated indefinitely. In the next subsection we formalize this scenario, also known as myopic best response [23], as an iterative scheme and we derive conditions on the problem data under which the set of updated strategies converges to a Nash equilibrium.

A. Iterative scheme without memory

Given an initial set of states $\{x^i_{(0)}\}_{i=1}^N$ suppose that each agent simultaneously repeats the procedure in Algorithm 1. That is, at every step k it computes

1) its optimal strategy $x^{i\,\star}(\sigma^i_{(k)})$;
2) the updated neighbor aggregate state $\sigma^i_{(k+1)} = \sum_{j \neq i} P_{ij} x^{j\,\star}(\sigma^j_{(k)})$.

Algorithm 1: Distributed iteration with no memory

Initialization: set $k \leftarrow 0$. Each agent i computes the initial neighbors aggregate state $\sigma^i_{(0)} = \sum_{j\neq i} P_{ij} x^j_{(0)}$.

Iterate:

Optimization step: each agent i computes its optimal strategy w.r.t. the current neighbor aggregate state $\sigma^i_{(k)}$

$$x^{i\star}_{(k+1)} \leftarrow \arg\min_{x\in\mathcal{X}^i} J^i(x, \sigma^i_{(k)}); \tag{6}$$

Communication step: each agent i updates its neighbors aggregate state

$$\sigma^i_{(k+1)} \leftarrow \sum_{j\neq i} P_{ij} x^{j\star}_{(k+1)}; \tag{7}$$

$k \leftarrow k+1$.

The stage update vector is therefore $\boldsymbol{\sigma}_{(k+1)} = \boldsymbol{\mathcal{A}}(\boldsymbol{\sigma}_{(k)})$.

Theorem 1: Suppose that $\|P\| \leq 1$ and for all $i \in \mathbb{Z}[1, N]$

$$\begin{bmatrix} q_i Q & -C_i \\ -C_i^\top & q_i Q \end{bmatrix} \succ 0. \tag{8}$$

Then for any set of initial states $\{x^i_{(0)}\}_{i=1}^N$, the sequence $(\boldsymbol{\sigma}_{(k)})_{k=0}^\infty$ in (7) converges to a fixed point $\bar{\boldsymbol{\sigma}}$ of $\boldsymbol{\mathcal{A}}$ in (5). Consequently, for each $i \in \mathbb{Z}[1, N]$, the sequence $(x^{i\star}_{(k)})_{k=0}^\infty$ defined in (6) converges to $x^{i\star}(\bar{\sigma}^i)$. The set of strategies $\{x^{i\star}(\bar{\sigma}^i)\}_{i=1}^N$ is a Nash equilibrium of the game in (2). □

Remark 1: The condition $\|P\| \leq 1$ is satisfied, for example, if the matrix P is *doubly-stochastic*, that is, if all its rows and columns sum up to one. This is always the case for symmetric networks.

B. Iterative scheme with memory

The conditions under which Algorithm 1 converges may be in some cases too restrictive. In this scenario, one may consider some variants of Algorithm 1, where the agents do not update their strategy according to the current neighbors aggregate state only, but they use some of the information gained in the previous iterations. As a particular case, we assume here that the agents compute their best response with respect to a reference signal, $z^i_{(k)}$, that is a convex combination between the current neighbor aggregate state $\sigma^i_{(k)}$ and the reference signal used at the previous iteration $z^i_{(k-1)}$, as illustrated in Algorithm 2. Similar iterations have been suggested, for example, in [14], [17], [18].

Theorem 2: Suppose that $\|P\| \leq 1$ and for all $i \in \mathbb{Z}[1, N]$

$$\begin{bmatrix} q_i Q & -C_i \\ -C_i^\top & q_i Q \end{bmatrix} \succcurlyeq 0. \tag{11}$$

Then for any set of initial states $\{x^i_{(0)}\}_{i=1}^N$, the sequence $(\boldsymbol{z}_{(k)})_{k=0}^\infty$ in (10) converges to a fixed point $\bar{\boldsymbol{\sigma}}$ of $\boldsymbol{\mathcal{A}}$ in (5). Consequently, for each agent $i \in \mathbb{Z}[1, N]$, the sequence $(x^{i\star}_{(k)})_{k=0}^\infty$ defined in (9) converges to $x^{i\star}(\bar{\sigma}^i)$. The set of strategies $\{x^{i\star}(\bar{\sigma}^i)\}_{i=1}^N$ is a Nash equilibrium of the game in (2). □

Algorithm 2: Distributed iteration with memory

Initialization: set $k \leftarrow 0$. Fix $\lambda \in (0, 1)$. Each agent i computes the initial neighbors aggregate state $\sigma^i_{(0)} = \sum_{j\neq i} P_{ij} x^j_{(0)}$ and sets $z^i_{(0)} = \sigma^i_{(0)}$.

Iterate:

Optimization step: each agent i computes its optimal strategy w.r.t. the current reference $z^i_{(k)}$

$$x^{i\star}_{(k+1)} \leftarrow \arg\min_{x\in\mathcal{X}^i} J^i(x, z^i_{(k)}); \tag{9}$$

Communication step: each agent i updates the neighbors aggregate state

$$\sigma^i_{(k+1)} \leftarrow \sum_{j\neq i} P_{ij} x^{j\star}_{(k+1)};$$

and updates the reference

$$z^i_{(k+1)} \leftarrow \lambda z^i_{(k)} + (1-\lambda)\sigma^i_{(k+1)}; \tag{10}$$

$k \leftarrow k+1$.

Remark 2: Note that, even though in the iterative scheme with memory each agent computes its best response with respect to the signal $z^i_{(k)}$ instead of the neighbor aggregate state $\sigma^i_{(k)}$, the sequence $(\boldsymbol{z}_{(k)})_{k=0}^\infty$ in (10) converges to a fixed point $\bar{\boldsymbol{\sigma}}$ of the same mapping $\boldsymbol{\mathcal{A}}$ as in Algorithm 1. As a consequence, in both cases the strategies of the agents converge to a Nash equilibrium of the game in (2). □

V. Illustrative examples

A. Fully connected networks and deterministic mean field games

The theory presented above can be used to analyze games in which each agent plays against the average behavior of all the others, itself excluded. In other terms, these are cases where the communication network is fully connected (FC). Mathematically, this is obtained by imposing $\sigma^i_{\text{fc}} = \frac{1}{N-1}\sum_{j\neq i} x^j$ for all $i \in \mathbb{Z}[1, N]$, which results in the communication matrix

$$P_{\text{fc}} := \tfrac{1}{N-1}\left(\mathbb{1}_N \mathbb{1}_N^\top - I_N\right).$$

Note that this matrix is doubly stochastic and therefore satisfies the condition $\|P\| \leq 1$. Consequently, Theorem 1 and Theorem 2 guarantee the convergence of the iterative scheme, without or with memory, to a Nash equilibrium, if condition (8) or (11) is satisfied, respectively. This class of games has a very strong connection with *deterministic mean field* games (MF) and *aggregative* games. These two categories of games can in fact be rewritten, in terms of the setting considered in this paper, by imposing that the neighbors aggregate state is the same for all agents and coincides with the average population state, that is, $\sigma^i_{\text{mf}} = \frac{1}{N}\sum_{j=1}^N x^j$, for all $i \in \mathbb{Z}[1, N]$. Note that the only difference between σ^i_{mf} and σ^i_{fc} is whether each agent weights its own contribution in the neighbors aggregate state or not. While this seems a mild difference, adopting the mean field setting would actually complicate the game since in this case each agent could, by selecting its strategy, influence also the value of σ^i_{mf}. The convergence properties of algorithms similar to Algorithm 1 and 2 for this latter scenario have been studied in [14]. Therein, it was shown that, under the same conditions as in (8) and (11), a central coordinator can steer the population to an ε-Nash equilibrium, in which each agent can improve its cost of at most a quantity $\varepsilon > 0$, with $\varepsilon \sim \mathcal{O}(1/N)$. As a consequence, while using σ^i_{fc} the algorithms converge to an exact Nash equilibrium for any population size, the same holds for σ^i_{mf} only in the limit of infinite population. Note that the algorithms in [14] require the presence of a central coordinator. In [22] we extend the theory presented in this paper to allow convergence to a ε-Nash equilibrium of the game with σ^i_{mf} using only local communications.

B. Illustrative example

As illustrative example we consider a simplified resource allocation game in which each agent has to complete a given task in n time slots. Let us denote by $x^i_h \in [0,1]$ the percentage of the task that agent i allocates at time slot h and by $M^i \in [0,1]^n$ a personalized vector of upper constraints on the maximum allowed allocation for each time slot. The set of possible allocation vectors is therefore given by

$$\mathcal{X}^i := \{x \in \mathbb{R}^n \mid 0 \le x \le M^i, \mathbb{1}_n^\top x = 1\}, \tag{12}$$

where the inequalities are understood component-wise. Among these possible choices, each agent selects the strategy satisfying (12) that minimizes

$$J^i(x, \sigma^i) := q_i \|x - \hat{x}^i\|^2 + 2 \cdot {\sigma^i}^\top x. \tag{13}$$

This cost is composed by two terms: the first one represents an individual cost that agent i encounters for deviating from a prefixed allocation schedule $\hat{x}^i$, the second one models an additional price that is proportional to the neighbors aggregate allocation. That is, each agent has an incentive to allocate its task in time slots that are not congested by the neighbors. The cost function (13) can be rewritten, up to constant terms, as $J^i(x, \sigma^i) := q_i \|x\|^2 + 2(\sigma^i - q_i \hat{x}^i)^\top x$, which coincides with (3) for $Q = C_i := I_n, c_i := -q_i \hat{x}^i$. Accordingly, the following equivalence holds

$$\begin{bmatrix} q_i Q & -C_i \\ -C_i^\top & q_i Q \end{bmatrix} = \begin{bmatrix} q_i I_n & -I_n \\ -I_n & q_i I_n \end{bmatrix} = \begin{bmatrix} q_i & -1 \\ -1 & q_i \end{bmatrix} \otimes I_n.$$

The eigenvalues of the two dimensional matrix above are $q_i \pm 1$. Since the eigenvalues of the Kronecker product are the product of the eigenvalues, condition (8) is satisfied if $q_i > 1$ and condition (11) is satisfied if $q_i \ge 1$.

Numerical study: Algorithm 1

For the numerical study we fixed $n = 5$ and we considered populations with bounds M^i randomly generated and such that $\mathbb{1}^\top M^i = 2$, so that the optimization problems are guaranteed to be feasible. The initial condition and the desired allocation vector are set for each agent to the projection of the vector $e_1 := [1, 0, \ldots, 0]^\top$ into the constraint set $\mathcal{X}^i$, that is $x^i_{(0)} = \hat{x}^i := \text{Proj}^{I_n}_{\mathcal{X}^i}(e_1)$. The interpretation is that each agent would like to complete its task as soon as possible, given its constraints. The weights $\{q_i\}_{i=1}^N$ are randomly generated in the interval $(1, 2)$, so that condition (8) is satisfied. To compare different Nash equilibria we define the index

$$d(\{\bar{x}^i\}_{i=1}^N) := \left\| \tfrac{1}{n} \mathbb{1}_n - \tfrac{1}{N} \textstyle\sum_{i=1}^N \bar{x}^i \right\|,$$

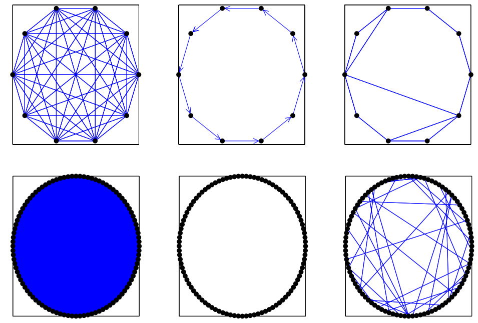

Fig. 1. Three different network topologies: fully connected, directed ring and undirected small world. The first line is for a population of $N = 10$ agents, the second for $N = 100$. All these matrices verify $\|P\| = 1$.

that is the distance of the average allocation demand of the whole population from the uniform distribution (that is achieved, for example, if everybody allocates $1/n$ of its task in each time slot). In Figure 2, we show the average distance and average number of iterations required by Algorithm 1 to converge, over 40 different randomly generated populations, as a function of the population size N, for three different types of networks, that are illustrated in Figure 1. In all plots and for each topology, the solid line denotes the average number of iterations and the shaded region encloses the average ± 1 standard deviation. While fully connected and ring topologies are the same for each randomly generated population, we used a different small world network for each replicate. These have been generated adding undirected shortcut links to the undirected ring topology, each with probability 0.3 [24]. The weights have been assigned so that the resulting P matrix was doubly stochastic. From the figures, it emerges that the network topology does not influence significantly the distance of the Nash equilibrium from the uniform distribution, but has an influence on the average number of iterations required to reach it. Specifically, the convergence is faster for the fully connected network. Finally, it is interesting to note that the number of iterations is almost independent from the population size. Therefore, the proposed algorithm can be used for large-size populations.

Numerical study: Algorithm 2

As a second case study, we used populations generated as in the previous case except for the fact that we imposed $q_i = 1$ for all i, so that only condition (11) is satisfied and Algorithm 1 is not guaranteed to converge. Figure 3 shows the results obtained using Algorithm 2 with parameter $\lambda = 0.8$. From the comparison of the two algorithms, it emerges that the average number of iterations is approximately the same under the fully connected and small world topology, while

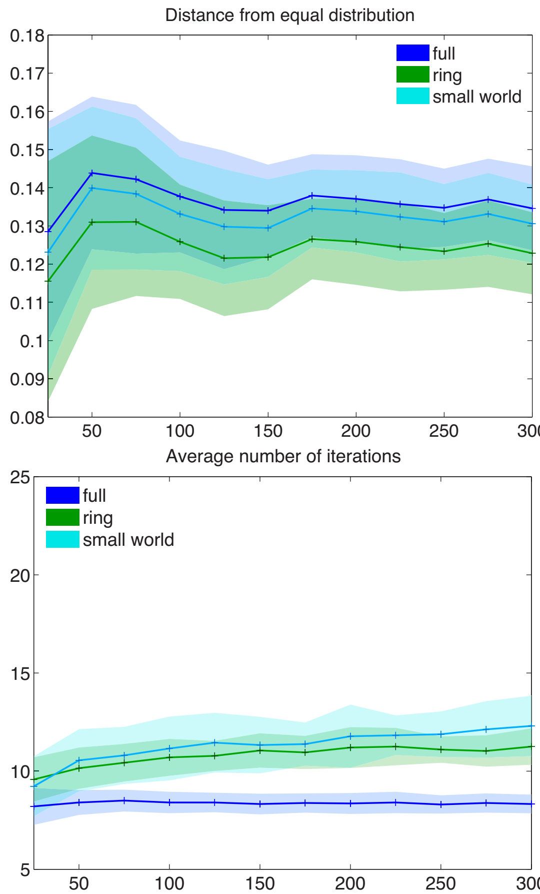

Fig. 2. Distance from the uniform task allocation (top) and average number of iterations (bottom), using Algorithm 1, as a function of the population size N.

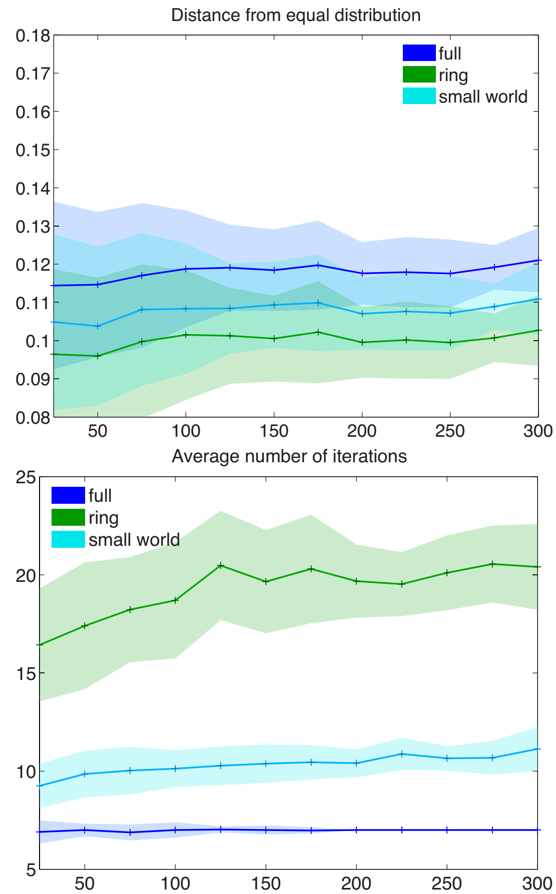

Fig. 3. Distance from the uniform task allocation (top) and average number of iterations (bottom), using Algorithm 2, as a function of the population size N.

is higher in the case of the ring topology. This observation may suggest that small world networks are preferable over ring topologies in terms of convergence properties.

VI. Conclusion

Computing the Nash equilibria of games played among large populations of agents over a network is a challenging problem. In this paper we proposed two iterative schemes that can be used to coordinate the agents best responses in the case of network aggregative games with quadratic cost and heterogeneous convex constraints. A remarkable feature of the proposed algorithms is that they are totally distributed and therefore scale well with the population size. We verified by simulations that, for a simple test case, the number of iterations needed to reach convergence is typically small and indeed does not depend drastically on the population size. We note as a drawback that the proposed approach requires synchronous updates of all the agents. As future work, it would be interesting to derive conditions under which similar algorithms can work asynchronously.

VII. Appendix

Proof of Proposition 1

Since the mapping $\boldsymbol{\mathcal{A}} : \mathbb{R}^{Nn} \to (P \otimes I_n)\mathcal{X}^{1\times N}$ is continuous and compact valued, it has at least one fixed point [25, Theorem 4.1.5 (b)]. The second statement follows from the definition of Nash equilibrium given in Definition 1. Specifically, let $\bar{x}^i := x^{i\star}\left(\bar{\sigma}^i\right)$, then it follows from the definition of fixed point that for every agent $i \in [1, N]$ $\bar{x}^i := \arg\min_{y\in\mathcal{X}^i} J^i\left(y, \bar{\sigma}^i\right) = \arg\min_{y\in\mathcal{X}^i} J^i(y, \sum_{j\neq i} P_{ij}\bar{x}^j)$ and $J^i\left(\bar{x}^i, \sum_{j\neq i} P_{ij}\bar{x}^j\right) = \min_{y\in\mathcal{X}^i} J^i\left(y, \sum_{j\neq i} P_{ij}\bar{x}^j\right)$. Consequently, the set of strategies $\left\{\bar{x}^i := x^{i\star}\left(\bar{\sigma}^i\right)\right\}_{i=1}^N$ is a Nash equilibrium for the game in (2). ■

Notions from operator theory

The main idea behind the proof of Theorem 1 is to derive sufficient conditions to guarantee that the aggregation mapping $\boldsymbol{\mathcal{A}}(\cdot)$ in (5) possesses one of the regularity properties listed in the following definition.

Definition 2 (Regularity properties): Consider the Hilbert space $\mathcal{H}_S$ defined by the matrix $S \in \mathbb{R}^{n\times n}$, $S = S^\top \succ 0$. A mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ is

1) a Contraction (CON) [26, Definition 1.6] in $\mathcal{H}_S$ if there exists $\delta \in (0, 1]$ such that $\|f(x) - f(y)\|_S \leq (1-\delta)\|x - y\|_S, \quad \forall x, y \in \mathbb{R}^n$.
2) Non-Expansive (NE) [27, Definition 4.1 (ii)] in $\mathcal{H}_S$ if $\|f(x) - f(y)\|_S \leq \|x - y\|_S, \quad \forall x, y \in \mathbb{R}^n$. □

If a mapping f is a CON, then the Picard–Banach iteration, $z_{k+1} = f(z_k)$, converges, for any initial point $z_0 \in \mathbb{R}^n$, to its unique fixed point [26, Theorem 2.1]. If a mapping $f : \mathcal{C} \to \mathcal{C}$ is NE, with $\mathcal{C} \subset \mathbb{R}^n$ compact and convex, then the Krasnoselskij iteration $z_{k+1} = \lambda z_k + (1-\lambda) f(z_k)$, where $\lambda \in (0,1)$, converges, for any initial point $z_0 \in \mathcal{C}$, to a fixed point of f [26, Theorem 3.2].

Lemma 1: Consider a linear mapping $f : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto f(x) := Fx$, $F \in \mathbb{R}^{n \times n}$. The following statements are equivalent: 1) f is NE in $\mathcal{H}_S$; 2) $\|F\|_S \leq 1$; 3) $F^\top S F - S \preccurlyeq 0$.

Proof: $(1) \Leftrightarrow \|Fr - Fs\|_S \leq \|r-s\|_S \ \forall r,s \Leftrightarrow \|F(r-s)\|_S \leq \|r-s\|_S \ \forall r,s \Leftrightarrow \|Fx\|_S \leq \|x\|_S \ \forall x \Leftrightarrow (2) \Leftrightarrow \|Fx\|_S \leq \|x\|_S \ \forall x \Leftrightarrow \|Fx\|_S^2 \leq \|x\|_S^2 \ \forall x \Leftrightarrow x^\top F^\top S F x \leq x^\top S x \ \forall x \Leftrightarrow x^\top (F^\top S F - S) x \leq 0 \ \forall x \Leftrightarrow (3)$ ■

Proof of Theorem 1

We notice that a single iteration k of Algorithm 1 is described by $\boldsymbol{\sigma}_{(k+1)} = \boldsymbol{\mathcal{A}}(\boldsymbol{\sigma}_{(k)})$, which is a Picard-Banach iteration relative to the aggregation mapping. Therefore we prove that $(\boldsymbol{\sigma}_{(k)})_{k=0}^{\infty}$ converges to a fixed point of $\boldsymbol{\mathcal{A}}$ by showing that the mapping $\boldsymbol{\mathcal{A}}(\cdot)$ is a CON in $\mathcal{H}_{\boldsymbol{\mathcal{Q}}}$, where $\boldsymbol{\mathcal{Q}} := I_N \otimes Q$. To this end, since $\boldsymbol{\mathcal{A}}(\cdot) = \boldsymbol{\mathcal{P}} \boldsymbol{x}^\star(\cdot)$, we first prove that $\boldsymbol{x}^\star(\cdot)$ is a CON in $\mathcal{H}_{\boldsymbol{\mathcal{Q}}}$, and then that $\boldsymbol{\mathcal{P}}$ is NE in $\mathcal{H}_{\boldsymbol{\mathcal{Q}}}$. If (8) holds, by [14, Theorem 2], the mapping $x^{i\,\star}$ is a CON in $\mathcal{H}_{q_i Q}$ (which is equivalent to be a CON in $\mathcal{H}_Q$) for some rate $\delta^i \in (0,1]$. Therefore for any $\boldsymbol{r}, \boldsymbol{s} \in \mathbb{R}^{Nn}$ $\|\boldsymbol{x}^\star(\boldsymbol{r}) - \boldsymbol{x}^\star(\boldsymbol{s})\|_{\boldsymbol{\mathcal{Q}}}^2 = \| \left[x^{1\,\star}(r^1) - x^{1\,\star}(s^1);\ \ldots; x^{N\,\star}(r^N) - x^{N\,\star}(s^N) \right] \|_{\boldsymbol{\mathcal{Q}}}^2 = \|x^{1\,\star}(r^1) - x^{1\,\star}(s^1)\|_Q^2 + \ldots + \|x^{N\,\star}(r^N) - x^{N\,\star}(s^N)\|_Q^2 \leq (1-\delta^1)^2 \|r^1 - s^1\|_Q^2 + \ldots + (1-\delta^N)^2 \|r^N - s^N\|_Q^2 \leq (1-\delta)^2 \|\boldsymbol{r} - \boldsymbol{s}\|_{\boldsymbol{\mathcal{Q}}}^2$, where $\delta := \min\{\delta^1, \ldots, \delta^N\} \in (0,1]$. This proves that $\boldsymbol{x}^\star(\cdot)$ is a CON in $\mathcal{H}_{\boldsymbol{\mathcal{Q}}}$. Let us now show that $\boldsymbol{\mathcal{P}}$ is NE in $\mathcal{H}_{\boldsymbol{\mathcal{Q}}}$; by Lemma 1, the condition $\|P\| \leq 1$ is equivalent to $P^\top P - I_N \preccurlyeq 0$. Moreover $\|\boldsymbol{\mathcal{P}}\|_{\boldsymbol{\mathcal{Q}}} \leq 1 \Leftrightarrow \boldsymbol{\mathcal{P}}^\top \boldsymbol{\mathcal{Q}} \boldsymbol{\mathcal{P}} - \boldsymbol{\mathcal{Q}} \preccurlyeq 0 \Leftrightarrow (P \otimes I_n)^\top (I_N \otimes Q)(P \otimes I_n) - I_N \otimes Q \preccurlyeq 0 \Leftrightarrow \left(P^\top \otimes I_n^\top\right)(I_N \otimes Q)(P \otimes I_n) - I_N \otimes Q \preccurlyeq 0 \Leftrightarrow \left(P^\top I_N P \otimes I_n^\top Q I_n\right) - I_N \otimes Q \preccurlyeq 0 \Leftrightarrow \left(P^\top P\right) \otimes Q - I_N \otimes Q \preccurlyeq 0 \Leftrightarrow \left(P^\top P - I_N\right) \otimes Q \preccurlyeq 0$. Since $Q \succ 0$ and $\left(P^\top P - I_N\right) \preccurlyeq 0$, by the properties of the Kronecker product $\left(P^\top P - I_N\right) \otimes Q \preccurlyeq 0$. Finally, by the previous equivalence $\|\boldsymbol{\mathcal{P}}\|_{\boldsymbol{\mathcal{Q}}} \leq 1$ and $\boldsymbol{\mathcal{P}}$ is NE in $\mathcal{H}_{\boldsymbol{\mathcal{Q}}}$, by Lemma 1. Therefore, $\boldsymbol{\mathcal{A}}$ is a composition of a NE mapping with a CON and is therefore a CON. Consequently the Picard-Bannach iteration, used in Algorithm 1, leads to a fixed point of $\boldsymbol{\mathcal{A}}$. The result then follows from the fact that the mappings $x^{i\,\star}$ are continuous and from Proposition 1.

Proof of Theorem 2

The proof of this theorem follows the same lines of the one of Theorem 1. Condition (11) guarantees that the mappings $x^{i\,\star}$ are NE in $\mathcal{H}_{q_i Q}$ (and consequently in $\mathcal{H}_Q$) [14, Theorem 2]. As a result, $\boldsymbol{\mathcal{A}}$ is a composition of two NE mappings and hence it is NE. A single iteration of Algorithm 2 can be described by $\boldsymbol{z}_{(k+1)} \leftarrow \lambda \boldsymbol{z}_{(k)} + (1-\lambda) \boldsymbol{\mathcal{A}}(\boldsymbol{z}_{(k)})$, which is the Krasnoselskij iteration relative to the aggregation mapping in (5). The fact that this is NE is sufficient to guarantee that the scheme converges to one of its fixed points.

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