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# Department of Computing

Academic Year 2019-2020

*Page created Thu Feb 20 02:15:19 GMT 2020*

CATE

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0 0 1 1 9 0 0 3 5 9 6 5

Electronic submission

**Tue - 18 Feb 2020 03:19:39**

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## Exercise Information

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| **Module:** | 499 Modal Logic for Strategic Reasoning in AI | **Issued:** | Wed - 05 Feb 2020 |
| **Exercise:** | 6 (CW) | **Due:** | Wed - 19 Feb 2020 |
| **Title:** | Coursework2 | **Assessment:** | Individual |
| **FAO:** | Belardinelli, Francesco (fbelard) | **Submission:** | Electronic |

## Student Declaration - Version 1

- I declare that this final submitted version is my unaided work.

Signed: (electronic signature) Date: 2020-02-18 03:19:21

**For Markers only:** (circle appropriate grade)

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Imperial College London

COURSEWORK


IMPERIAL COLLEGE LONDON

DEPARTMENT OF COMPUTING


# Modal Logic for Strategic Reasoning in AI


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Date: February 18, 2020

# Question 1 Exercise on LTL

## Question 1.a

$\lambda \models \phi R \psi$ iff $\lambda[i...\infty] \models \psi$ for all $i \geq 0$, or $\lambda[i...\infty] \models \phi$ for some $i \geq 0$ and $\lambda[j...\infty] \models \psi$ for all $i \geq j \geq 0$.

## Question 1.b

$(M,a) \models \phi R \psi \equiv (M,a) \models (\psi U(\psi \wedge \phi)) \vee \neg(trueU\neg\psi)$

## Question 1.c

When $\lambda[i...\infty] \models \psi$ for all $i \geq 0$ is satisfied, $\neg(trueU\neg\psi)$ is satisfied. When there exists a $i$ such that $\lambda[i...\infty] \models \phi$ for some $i \geq 0$ and $\lambda[j...\infty] \models \psi$ for all $i \geq j \geq 0$, $(\psi U(\psi \wedge \phi))$ is satisfied. So the conjunction of two formulas is satisfied when at least one of the two conditions is satisfied.

## Question 1.d

$$\begin{aligned}
\perp R\psi \equiv &(\psi U(\psi \wedge \perp)) \vee \neg(trueU\neg\psi) && (1)\\
\equiv &(\psi U \perp) \vee \neg(trueU\neg\psi) && (2)\\
\equiv &\neg(trueU\neg\psi) && (3)\\
\equiv &\neg F \neg\psi && (4)\\
\equiv &G\psi && (5)
\end{aligned}$$

# Question 2 Exercise on CTL

(a) $(M,q) \models EF\Phi$ is equivalent to $(M,q) \models E(trueU\Phi)$, whose semantic is that for some path $\lambda$ starting from $q$, $(M,\lambda[i]) \models \Phi$ for some $i \geq 0$ and $(M,\lambda[j]) \models true$ for all $i \geq j \geq 0$. This is the same as that for some path $\lambda$ starting from $q$, $(M,\lambda[i]) \models \Phi$ for some $i \geq 0$, because $(M,\lambda[j]) \models true$ always holds.

(b) $(M,q) \models AF\Phi$ is equivalent to $(M,q) \models A(trueU\Phi)$, whose semantic is that for all path $\lambda$ starting from $q$, $(M,\lambda[i]) \models \Phi$ for some $i \geq 0$ and $(M,\lambda[j]) \models true$ for all $i \geq j \geq 0$. This is the same as that for all path $\lambda$ starting from $q$, $(M,\lambda[i]) \models \Phi$ for some $i \geq 0$, because $(M,\lambda[j]) \models true$ always holds.

(c) $(M,q) \models EG\Phi$ is equivalent to $(M,q) \models E\neg(trueU\neg\Phi)$, whose semantic is that for some path $\lambda$ starting from $q$, there's no such an $i$ that $(M,\lambda[i]) \models \neg\Phi$. This is the same as that for some path $\lambda$ starting from $q$, $(M, \lambda[i]) \models \Phi$ for all $i \geq 0$.

(d) $(M, q) \models AG\Phi$ is equivalent to $(M, q) \models A\neg(trueU\neg\Phi)$, whose semantic is that for all path $\lambda$ starting from $q$, there's no such an $i$ that $(M, \lambda[i]) \models \neg\Phi$. This is the same as that for all path $\lambda$ starting from $q$, $(M, \lambda[i]) \models \Phi$ for all $i \geq 0$.

# Question 3 Exercise on CTL*

Proving (a) is trivial because all the operators and operands in the syntax of CTL are included in the syntax of CTL*.

(b) can be proved by a counterexample, such as $E\neg Xa$, where $a$ is an atom. This belongs to CTL* because $\neg Xa$ is a legal path formula in CTL*. But this doesn't belong to CTL because $\neg\psi$ is not a legal path formula in CTL.

# Question 4 Semantics of CTL*

Because $\phi$, $\neg\psi$ and $\psi \wedge \psi'$($\psi$ and $\psi'$ are path formulas and $\phi$ is state formula) are not CTL path formulas, we don't need to consider their CTL* semantics when restricting CTL* semantics to CTL formulas. We can find that the remaining semantics are exactly same as the semantics of CTL, so they have the same truth conditions.

# Question 5

(a) Because the syntax of CTL is included in CTL* and all operators have same semantics, it's obvious that, for a formula $\Phi$ in CTL, there exists an equivalent formula $\Phi'$ in CTL* if $\Phi$ and $\Phi'$ have the same form.

(b) This can be proved by an example. A formula $FGa$ in LTL is equivalent to formula $A(\top U(\neg(\top U\neg a)))$ in CTL*, where $a$ is an atom. However, it's shown in the lecture that $FGa$ in LTL is not expressible in CTL. So $A(\top U(\neg(\top U\neg a)))$ in CTL* is also not expressible in CTL.

# Question 6 Exercises on Bisimulations

First, we assume that $(M, t) \models \Phi$ iff $(M', t') \models \Phi$ if $t$ and $t'$ are bisimilar. Based on this assumption, if $\pi$ and $\pi'$ are bisimilar, for any state formula $\Phi$ and $i \geq 0$ we can show that $(M, \pi[i]) \models \Phi$ iff $(M', \pi'[i]) \models \Phi$, because $\pi[i]$ and $\pi'[i]$ are bisimilar.

We can find in the semantics of path formula operators that the truth conditions of formulas are totally dependent on the truth conditions of each state in the path. As assumed above, any $\pi[i]$ and $\pi'[i]$ have the same truth condition for any state formula $\Phi$, so for any path formula $\psi$ we have $(M, \pi) \models \psi$ iff $(M', \pi') \models \psi$.

Second, we prove the assumption that $(M, t) \models \Phi$ iff $(M', t') \models \Phi$ when $t$ and $t'$ are bisimilar.

- **When $\Phi$ doesn't include $E$ or $A$:** because for any atom $p$ we have $t \in V(p)$ iff $t' \in V'(p)$, $\Phi$ has the same truth value in $t$ and $t'$.

- **When $\Phi = E\psi$ and $\psi$ doesn't include $E$ or $A$:** if $(M, t) \models E\psi$, there exists a path $\pi = t, t[1], t[2], ..$ such that $(M, \pi) \models \psi$. Because $t$ and $t'$ are bisimilar, there exists a $t'[1]$ such that $R'(t', t'[1])$ and $t[1]$ is bisimilar as $t'[1]$. Recursively, for any $i \geq 1$, there exists a $t'[i]$ such that $R'(t'[i-1], t'[i])$ and $t[i]$ is bisimilar as $t'[i]$, i.e. there exists a path $\pi' = t', t'[1], t'[2], ..$ such that $\pi$ and $\pi'$ are bisimilar. As proved above, we know that $(M, \pi) \models \psi$ iff $(M', \pi') \models \psi$. Because such a path exists, $(M, t) \models E\psi$ iff $(M', t') \models E\psi$ holds.

- **When $\Phi = A\psi$ and $\psi$ doesn't include $E$ or $A$:** this can be proved by reduction to absurdity. If $(M, t) \models A\psi$ but there exists a path $\pi' = t', t'[1], ...$ such that $(M, \pi') \models \neg\psi$, then we can find a path $\pi = t, t[1]...$ such that $\pi$ and $\pi'$ are bisimilar, because $t$ and $t'$ are bisimilar. As proved above, $(M, \pi) \models \neg\psi$ is entailed by the bisimulation, but this cannot hold because $(M, t) \models A\psi$.

- **When $\Phi = A\psi$ or $\Phi = E\psi$, and $\psi$ can include $E$ and $A$:** because we have proved that $(M, t) \models \psi$ iff $(M', t') \models \psi$ when $\psi$ contains one $E$ or $A$, this can be recursively proved.

As we have enumerated all forms of a state formula $\Phi$, we find that $(M, t) \models \Phi$ iff $(M', t') \models \Phi$ for any state formula $\Phi$.

# Question 7

We can prove that $(M, t)$ and $(M', t')$ are bisimilar by proving that CTL-equivalence is a bisimulation, which satisfies the three conditions of being a bisimulation.

Condition 1 is trivially true, because any atom can be a CTL formula.

To prove that condition 2 is satisfied, we first assume that there exists a state $s$ such that $R(t,s)$. We try to derive a contradiction by assuming condition 2 is not satisfied, i.e., for all $s'_i \in \{s'|R'(t',s')\}$, $s'_i$ and $s$ are not modally equivalent. This results that for all $s'_i$ there exists some $\psi_i$ such that $(M,s) \models \psi_i$ and $(M',s'_i) \not\models \psi_i$.

It follows that $(M,t) \models EX(\psi_1 \wedge ... \wedge \psi_n)$ and $(M,t') \not\models EX(\psi_1 \wedge ... \wedge \psi_n)$, which is contradictive to that $(M,t)$ and $(M',t')$ are modally equivalent.

Condition 3 can be proved to be satisfied in a similar way as condition 2(exchanging $t$ and $t'$, $s_i$ and $s'_i$).

# Question 8

We need to prove that $(M,s)$ and $(M',s')$ are CTL-equivalent iff $(M,s)$ and $(M',s')$ are CTL*-equivalent.

First, we prove that $(M,s)$ and $(M',s')$ are CTL-equivalent $\implies$ $(M,s)$ and $(M',s')$ are CTL*-equivalent. In (6) we have proved that if $(M,s)$ and $(M',s')$ are bisimilar, then $(M,s)$ and $(M',s')$ are CTL*-equivalent. In (7) we have proved that if $(M,s)$ and $(M',s')$ are CTL-equivalent, then $(M,s)$ and $(M',s')$ are bisimilar. So we can conclude that if $(M,s)$ and $(M',s')$ are CTL-equivalent, then $(M,s)$ and $(M',s')$ are CTL*-equivalent.

Then, we prove that $(M,s)$ and $(M',s')$ are CTL*-equivalent $\implies$ $(M,s)$ and $(M',s')$ are CTL-equivalent. We have shown in (5)(a) that the same formula $\Phi$ in CTL is equivalent to $\Phi$ in CTL*. So for any CTL formula $\Phi$, $(M,s)$ and $(M',s')$ are equivalent because they are CTL*-equivalent.

In conclusion, we have that $(M,s)$ and $(M',s')$ are CTL-equivalent $\iff$ $(M,s)$ and $(M',s')$ are CTL*-equivalent.