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STOCHASTIC MODEL PREDICTIVE CONTROL WITH APPLICATION TO DISTRIBUTED CONTROL SYSTEMS

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A chi crede in me da sempre.

Abstract

The goal of this Thesis is twofold. Firstly, we deal with the analysis and the development of Stochastic Model Predictive Control algorithms (SMPC) for linear discrete-time systems with additive stochastic disturbances and probabilistic constraints on the states and the inputs. Secondly, we consider the development of distributed Model Predictive Control algorithms for uncertain linear discrete-time systems and we extend the techniques described in the first part to the distributed framework.

SMPC techniques are based on the idea of taking advantage of the information available on the probabilistic characterization of the uncertainty affecting the system to relax the problem constraints with respect to classical worst-case approaches. In this setup, a novel SMPC algorithm, named probabilistic-SMPC or p-SMPC, is proposed both in the state-feedback and output-feedback framework and its application is discussed in several examples. The main advantages of p-SMPC rely on the guaranteed recursive feasibility and convergence properties, even in the case of disturbances with possibly unbounded support, and the reduced computational load, similar to the one required by standard nominal MPC techniques.

The idea behind distributed MPC algorithms is based on the assumption that the coupling terms among the subsystems can be interpreted as disturbances to be rejected. Initially, a regulation problem for dynamically coupled subsystems with local probabilistic constraints is considered and a novel algorithm, based on p-SMPC, extends the nice recursive feasibility and convergence properties of the centralized approach to the distributed case. Secondly, the problem of tracking a reference output signal is discussed and a multilevel scheme is proposed, that relies on robust MPC techniques to handle the couplings between subsystems. Particular attention is devoted to the application of the proposed approach to a real mobile robot coordination problem. Finally a stochastic distributed MPC algorithm for tracking reference signals is presented for dynamically decoupled subsystems subject to local and collective probabilistic constraints. The mobile robot coordination problem is extended also to this framework to show the viability of the approach.

Contents

CHAPTER 1

Introduction

Model Predictive Control (MPC) is nowadays a standard in many industrial contexts, see e.g. [131], due to its ability to cope with complex control problems and to the availability of theoretical results guaranteeing feasibility and stability properties, see [100]. These reasons have motivated the many efforts devoted to develop MPC algorithms robust with respect to unknown, but bounded, disturbances or model uncertainties, see for example [94,102]. The problem of designing robust deterministic MPC schemes has many solutions, see for example [83, 139], however, these algorithms can suffer from some issues. Firstly, feasibility, convergence, and stability properties are usually achieved by resorting, implicitly or explicitly, to a worst-case analysis, which may turn out to be very conservative or even impossible in the case of unbounded uncertainties and it may require the solution to difficult on-line min-max optimization problems, that are computationally very demanding, see [94]. Secondly, the tuning of robust algorithms, for instance the off-line procedures required to compute robust positive invariant sets, can be computationally demanding or even impracticable for systems of medium-high order, see [101]. Thirdly, and perhaps most importantly, they do not consider the possible a-priori knowledge of the statistical properties of the disturbances, i.e. their distribution function,

which can be assumed to be available in many problems. In this setup, if the uncertainties or the state and control disturbances are characterized as stochastic processes, constraints should be reformulated in a probabilistic framework [68, 156]. Starting from the pioneering papers [82, 145], these reasons have motivated the development of MPC algorithms for systems affected by stochastic noise and subject to probabilistic state and/or input constraints. Stochastic MPC (SMPC) has already been considered in several application fields, such as temperature and HVAC control in buildings [39, 86, 89–91, 113, 114, 123, 159, 162], process control [152], power production, management, and dispatch in systems with renewable energy sources [3, 67, 98, 109, 119, 122, 124, 135, 141, 165], cellular networks management [155], driver steering, scheduling, and energy management in vehicles [13, 37, 49, 53, 57, 85, 132, 133, 140], path planning and formation control [14, 17, 48, 158], air traffic control [80], inventory control and supply chain management [69, 72, 166], resource allocation [30], portfolio optimization and finance [38, 61, 62, 111, 129, 146].
In spite of this large number of applications of SMPC and the already available theoretical results, many tough challenges emerge in this setup, related to the development of methods with guaranteed stability and feasibility properties. Indeed, while for bounded disturbances recursive feasibility and convergence can be established both in a deterministic or a stochastic setup, the more general case of unbounded noise poses more difficulties and some specific solutions and reformulations of these properties have been adopted. For example in [28] the concept of invariance with probability p is used, while in [117] the definition of probabilistic resolvability is introduced. Due to the large variety of available SMPC techniques, the literature on the topic is quite vast and sometimes not very consistent. Thus, the first part of this Thesis will be devoted to discuss the main key points and the many possible formulations of SMPC algorithms with the aim of clarifying the most important aspect about the topic. Moreover, an algorithm, denoted as probabilistic-SMPC or p-SMPC, will be proposed and its properties and applications thoroughly discussed.

In recent years, another very significant topic is represented by the development of distributed control algorithms, that allow to handle large-scale systems whose complexity is related to the the presence of a high number of small or medium-scale subsystems interacting via inputs, states, outputs or constraints. As discussed before, due to its flexibility, robustness, and the vast literature on the topic, among all the possible solutions, particularly interesting appear to be those based on Model Predictive Control. The most important aspect, when controlling large-scale systems in a distributed framework, is that using local MPC controllers, the predicted trajectories of inputs, states and outputs are directly available and can be used as information to be transmitted to other local controllers to coordinate their actions (see for example [93]). This data exchange can greatly simplify the design of a distributed control system and can allow one to obtain performances close to those of a centralized controller. The goal of the second part of this Thesis is to extend the SMPC technique proposed in the first part to the distributed framework. In particular, the first problem to be addressed will be the control of dynamically coupled linear systems with additive disturbances and subject to local probabilistic constraints. Then, the problem of designing a distributed controller for tracking reference signals will be discussed and a solution based on the well-established DPC algorithm, see e.g. [43, 47, 93], is presented. The main idea of DPC is that, at every sampling time, each subsystem transmits to its neighbors the reference trajectories of its inputs and states over the prediction horizon and guarantees that the real values of their inputs and states lie in a specified invariant neighbor of the corresponding reference trajectories. In this way, each subsystem has to solve an MPC problem where the reference trajectories received from the other controllers represent a disturbance known over all the prediction horizon, while the differences between the reference and the real values of its neighbors inputs and states can be treated as unknown bounded disturbances to be rejected. To this end, a robust tube-based MPC formulation is adopted and implemented using the theory of polytopic invariant sets. Finally the problem of tracking a reference signals in presence of probabilistic constraints will be discussed for dynamically decoupled subsystems and the presented algorithm applied to a mobile robot coordination problem.

Structure of the Thesis

The Thesis is organized as follows. In Chapter 2 the stochastic constrained control problem is introduced and several possible modeling assumptions on the constraints, control laws and cost functions are discussed with the goal of presenting the main ingredients of a stochastic MPC (SMPC) problem. After this quick overview of the topic, a detailed analysis of the state of the art of stochastic MPC approaches already available in the literature is presented and results are summarized.

In Chapter 3 the analytic p-SMPC approach is presented for linear discrete-time systems with additive possibly unbounded disturbances, measurable state and subject to one or more individual chance constraints on the state and the input. The constraints are reformulated into deterministic ones and

the resulting algorithm has similar computational complexity to standard MPC approaches. Moreover, an efficient LMI formulation of the problem to be solved is presented. Finally, resorting to proper terminal constraints and a switching initialization strategy, recursive feasibility and convergence are guaranteed even in the case of unbounded uncertainties.
In Chapter 4 the state-feedback p-SMPC approach described in the Chapter 3 is applied to some academic and realistic benchmark examples, together with three other notable approaches selected from the literature, namely the tube-based SMPC [77], the average SMPC [75] and a scenario MPC [144]. These algorithms are firstly presented and some hints on their implementation are given with the aim of showing the difficulties that may arise in enforcing feasibility and convergence properties to the MPC scheme. After this quick overview, the first example analyzes the behavior of the controlled system in terms of actual violation of one single chance constraint at a single time instant, with the goal of showing the effectiveness of the various approaches in reaching the desired level of relaxation. The second example is a temperature control problem for a small four-room apartment and the goal is to show the behavior of the selected SMPC controllers in terms of violation of the constraints over time. In both the examples the algorithms are compared in terms of conservatism (actual violation frequency) and computation time. Finally the third example is the more complex problem of controlling the temperature inside a realistic model of a building. In particular, the model of a real building is exploited together with real measurements of the disturbance (the external temperature) collected from a meteorological company. In this example together with a classical nominal MPC algorithm and a robust worst-case approach, only the p-SMPC approach proposed in Chapter 3 and the av-SMPC in [75] are selected to be implemented thanks to their efficiency and ease of implementation. Results are presented with the main goal of showing the advantages of such stochastic MPC approaches.
In Chapter 5 the state-feedback p-SMPC approach is extended to the case of discrete-time linear systems with additive disturbances, both on the state and the output, and non measurable states and the output-feedback p-SMPC algorithm is presented. In particular a state observer is introduced in the stochastic control scheme whose gain is selected, together with the feedback controller gain, by solving a proper MPC problem. Individual chance constraints inside such problem are handled by means of a second order description of the system variables and through the use of the Cantelli-Chebyshev's inequality in case of disturbances with unknown distribution. The choice of proper terminal constraints for both the expected value and

the variance of the state of the system and an intialization strategy similar to the one adopted in Chapter 3 allow to guarantee recursive feasibility and convergence properties even in the case of unbounded uncertainties. Moreover, the main implementation issues are discussed and a way of reformulating constraints as LMI is proposed. Finally two simulation examples are shown to prove the efficacy of the algorithm.
In Chapter 6 the problem of designing a distributed stochastic MPC controller for dynamically coupled systems and local probabilistic constraints is discussed. With respect to classical robust distributed predictive control techniques the aim is to cope with disturbances with possibly unbounded support. To this end, the p-SMPC approach introduced in Chapter 3 is adapted to allow for a distributed implementation and, in particular, the recursive feasibility and convergence guarantees are extended by means of similar conditions on the initialization of the algorithm. A simple example shows the efficacy of the proposed approach.
In Chapter 7 a distributed control scheme for tracking reference signals is presented that is based on well-known robust MPC techniques to ensure the coordination of the agents. The control scheme is composed by three different levels, namely an output feedback trajectory generation level, a state and input trajectory generation level and a tracking MPC level that take advantage of the exchange of information between subsystems to achieve local goals while satisfying collective constraints and compensating for the coupled dynamics. Feasibility and convergence properties are proven by means of worst-case considerations. In the end, a simulation example and a real experiment are presented to show the efficacy and the flexibility of the approach. In particular, the real example is a mobile robot coordination problem in which the single robots are required to reach their own final goals while avoiding fixed obstacle and collisions.
In Chapter 8 the distributed tracking problem is extended to the stochastic case with the aim of using it to solve the mobile robots coordination problem introduced in Chapter 7. To this end, the analysis is limited only to dynamically decoupled subsystems with local and coupling probabilistic constraints. Firstly, an approach inspired from [127] is quickly described for solving the general problem. Secondly, all the previous results are formulated for the mobile agents coordination problem and the collision constraints are handled by extending the approach described in [2] to the probabilistic case. Some simulation results are shown to analyze the stochastic behavior of the controlled system in terms of number of detected collisions. Finally some results from a real coordination example are shown to prove the efficacy of the proposed approach.

In Chapter 9 some conclusion on the overall work presented in this Thesis are drawn and possible future directions are discussed.

CHAPTER 2

State of the art of Stochastic MPC

The problem of designing robust deterministic MPC schemes has nowadays many solutions, see for example [83, 139]. However, the available approaches are in general computationally very demanding, since they either require the solution to difficult on-line min-max optimization problems, see [94], or the off-line computation of polytopic robust positive invariant sets, see [101]. In addition they are conservative, mainly because they implicitly or explicitly rely on worst-case approaches. Indeed, even if the uncertainties are characterized as stochastic processes, worst-case deterministic methods do not take advantage of the available knowledge on the characteristics of the process noise, such as their probability density function, and cannot even guarantee recursive feasibility in case of possibly unbounded disturbances. To overcome these limitations, an emerging field of research concerns the design of innovative Stochastic Model Predictive Control (SMPC) algorithms, aimed at exploiting the stochastic nature of the uncertainty and, when available, its statistical description. In this framework, hard constraints on the system variables have to be reformulated as stochastic ones, allowing the controlled system to violate them in prescribed probabilistic terms. In this scenario, it is possible to consider also unbounded disturbances and/or uncertainties, for example in case they are characterized by a Gaussian distribution.

In spite of the large number of applications of SMPC (see Chapter 1), the classification of the many available SMPC algorithms can be quite difficult due to the large variety of problem formulations and solutions. For example, design methods have been developed for linear or nonlinear, discrete-time or continuous-time systems, with additive, multiplicative or parametric uncertainties, finite or infinite horizon cost functions, polytopic, quadratic or more complex probabilistic constraints. Also, linear systems with known state have generally been considered, with the notable exceptions of [24, 65, 157], where output feedback methods have been proposed. For these reasons, the goal of this chapter is to present the most widely used problem formulations, with particular emphasis on the definition of state and control constraints in probabilistic terms, on the cost function to be minimized, and on the structure of the adopted control law. Then, we propose a classification of the available methods based on the system's assumptions, the adopted MPC formulation, and the feasibility and convergence properties. A final section of conclusions closes the analysis and gives the motivation to the algorithms proposed in the next chapters.
Before continuing, a note on the terminology is due. Here we denote with the expression "hard constraint" the ideal constraint that one wants to impose in a deterministic framework while we use the term "stochastic constraint" to denote in general its relaxed version obtained allowing a partial violation, in a sense that will be clarified later, of the original requirement. However, the work will be focused on a subset of these stochastic constraints denoted as "probabilistic", or "chance", constraints with the meaning that the relaxation is expressed in terms of maximum allowed probability of violating the requirement.

2.1 Stochastic systems and constraints

In the most general case assume that the system under control is described by the following discrete-time, nonlinear model

$$x_{t+1} = f(x_t, u_t, w_t) \tag{2.1}$$

In (2.1) $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $w \in \mathcal{W} \subseteq \mathbb{R}^{n_w}$ are the state, input, and stochastic noise vectors, respectively, which must satisfy, at least ideally, a set of constraints described in very general terms by the inequalities

$$g(x_t, u_t, w_t) \leq 0 \tag{2.2}$$

where the function $g(\cdot) : \mathbb{R}^{n \times m \times n_w} \rightarrow \mathbb{R}^r$, can take different forms, as specified in the following. In particular, depending on the noise characteristics, these constraints can be satisfied deterministically, or, in order to consider that their deterministic fulfillment can be too tight or even impossible due to the presence of the stochastic noise, a partial violation is allowed.

Many SMPC algorithms have been developed with specific reference to linear systems with additive or multiplicative uncertainty. In the case of additive uncertainty, the adopted model is

$$x_{t+1} = Ax_t + B_u u_t + B_w w_t \tag{2.3}$$

where the noise term has the role of a real disturbance acting on the system or can be used to represent an unmodeled dynamics.
Systems with multiplicative uncertainty are described by the model

$$x_{t+1} = Ax_t + B_u u_t + \sum_{j=1}^{q} [A_j x_t + B_j u_t] w_{jt} \tag{2.4}$$

and their use is widely popular in specific application fields, such as in financial applications, see e.g., [38,129], where stock prices and the portfolio wealth dynamics are represented as in (2.4).

Before going into details, it is easy to see that, in view of the stochastic nature of the noise w, the inclusion of hard constraints (2.2) in the problem formulation can lead to infeasibility. For instance, when the uncertainty acting on the system has an unbounded support, i.e. the set $\mathcal{W}$ is unbounded as in the case of a Gaussian noise, there is no way to ensure hard constraints on the state variable. Moreover, even if the uncertainty is bounded, the worst-case scenario that one has to consider to ensure the satisfaction of the hard constraints might be too conservative, and performances of the obtained solution could be increased by resorting to a stochastic reformulation of (2.2).

In particular, three types of stochastic constraints are usually encountered in the literature:

- Expectation constraints $\quad \mathbb{E}[g(x, u, w)] \leq 0$
- Probabilistic or chance constraints $\quad \mathcal{P}\left\{g(x, u, w) \leq 0\right\} \geq 1 - p$
- Integrated chance constraints $\quad \int_0^\infty \mathcal{P}\left\{g(x, u, w) \geq s\right\} ds \leq p$

where $\mathcal{P}\{\varphi\}$ is the probability of φ and p is a design parameter to be tuned in order to obtain a trade-off between performances and constraint violation.

The use of expectation constraints at the place of deterministic ones represents the simplest solution and amounts to ensure that the constraints are satisfied on average for the considered problem. A notable example is [130], where it is required that the expected value of quadratic functions of state and input variables respects given bounds. In this way, however, the number of occurred violations or the amount of the single violation are not controlled directly. To this end, one can think about extending the formulation considering a tuning parameter, for example requiring that $\mathbb{E}[g(x,u,w)] \leq \alpha$ for some parameter α whose choice is however difficult and somehow artificial.

On the other hand, probabilistic constraints are commonly adopted in view of the fact that many problems can be naturally formulated using this framework, for example every time the violation of certain constraints up to a specified frequency is allowed. In this formulation, however, the constraint is not always convex and further approximations are needed to use it within an MPC scheme, that will be discussed in detail in the sequel. With respect to the previous solution, notice that the probabilistic constraints can be interpreted as the expected value of the indicator function of the event "constraints satisfied".

Finally, integrated chance constraints [58] [59], are a useful tool to express the idea of constraint violation in a more quantitative way. Roughly speaking, in this formulation the constraint violation is allowed with high probability if the amount by which it is violated is small enough.

For a more detailed explanation of the constraint models and a clear analysis of their effects on the problem, the reader is referred to, e.g., [15, 33].

To simplify the setup, in this Thesis focus will be placed on the second class of constraints, the so-called probabilistic, or chance, constraints. However, based on the form of $g(\cdot)$, another distinction is due, that is widely used in the literature. In particular, when $g(x,u,w)$ is a vector, for example when the goal is to express the probability that the state and/or the control are inside a certain set, the constraint is called "joint chance constraint". On the other hand, when $g(x,u,w)$ is a scalar function, the constraint is addressed as "individual chance constraint". Even if the joint representation seems to be more natural, its exact tractable representation usually does not exist, despite the convexity of the constraint itself. This is because the mere evaluation of the constraint requires the computation of a multivariate integral, which is known to become prohibitive in high dimensions.

For this reason, from a practical point of view, joint chance constraints need to be approximated to obtain a tractable expression. A clear overview of the problem can be found for example in [15], [110] and in the references therein. Besides the use of confidence ellipsoids or sampling techniques that will not be considered in this work, the simplest way to work with a joint chance constraint is to approximate it by splitting the overall set into a sequence of individual chance constraints, whose single probabilities sum up to the original one, as described in [14].

In a more formal way, if $g(x,u,w) = [g_1(x,u,w),\dots,g_r(x,u,w)]^T$, the joint chance constraint can be rewritten as follows

$$\mathcal{P}\left\{g(x,u,w) \leq 0\right\} = \mathcal{P}\left\{\bigwedge_{i=1}^{r} g_i(x,u,w) \leq 0\right\} \geq 1-p$$

or, in other words, the probability that at least one of the single violation occurs needs to be less than p, i.e.,

$$\mathcal{P}\left\{\bigvee_{i=1}^{r} g_i(x,u,w) > 0\right\} \leq p$$

Applying the Boole's inequality to the last expression we obtain the following bound

$$\mathcal{P}\left\{\bigvee_{i=1}^{r} g_i(x,u,w) > 0\right\} \leq \sum_{i=1}^{r} \mathcal{P}\left\{g_i(x,u,w) > 0\right\}$$

and thus one can choose a set of r parameters p_i such that

$$\mathcal{P}\left\{g_i(x,u,w) \leq 0\right\} \geq 1-p_i \quad \forall i = 1\dots r\,, \quad \sum_{i=1}^{r} p_i = p$$

Due to the fact that $\sum_{i=1}^{r} p_i = p$, the above equations clearly give a conservative approximation of the original constraint. Based on that, an easy choice can be to equally subdivide the overall risk p setting $p_i = p/r, i =$

$1, \ldots, r$ or to choose the single risks offline based on same heuristics. However, if a less conservative solution is required, the approximation can be reduced including the values p_i as free variables in the optimization problem. This iterative risk allocation technique is discussed in [120] and [15] for the case of Gaussian uncertainty. In the rest of this work we will assume that constraints are already given in the form of multiple individual chance constraints to ease the setup.

Before continuing in the analysis, a remark is due. Note that, once the model for the constraints has been defined, for example chosen from the ones presented above, we are still dealing with a stochastic programming problem that is generally difficult to solve in its original version and thus proper reformulations and/or approximations based on the available description of the uncertainty are needed to make it easier to implement. Together with this, we need to stress the fact that, when dealing with MPC schemes, the simple application of the stochastic constraints to the future predictions of the system variables is in general not sufficient to guarantee the recursive feasibility of the algorithm. Both these facts will be clarified in the sequel and an overview of the techniques used in the literature to enforce feasibility will be discussed in a simple case.

2.2 Stochastic MPC, formulations and properties.

Once the model used to represent the stochastic system has been chosen and the state and control constraints have been properly reformulated as discussed in the previous section, the MPC optimization problem can be stated by defining a suitable cost function together with additional constraints which can be added to achieve recursive feasibility and stability properties. Then, specific algorithms can be developed according to different approaches.

It is possible to roughly cluster the different stochastic MPC methods nowadays available in two main classes: the first one, i.e. the so-called *analytic methods* (referred in [163] as probabilistic approximation methods), is based on the reformulation of probabilistic-type constraints and of the cost function in terms of variables whose behavior can be characterized in deterministic terms (e.g., mean values and variances), to be included in the MPC formulation. The second class of approaches relies on the *randomized*, or *scenario-based methods*, i.e., on the on-line random generation of a sufficient number of noise realizations, and on the solution to a suitable constrained optimization problem based on these scenarios. The main features of these methods will be now discussed.

2.2.1 Cost function

In a deterministic framework, the MPC cost function J_t^N is usually selected to weight, over a finite number of steps defined by the prediction horizon N, a stage cost $l(x,u)$, plus a cost related to the state at the end of the horizon, $l_f(x)$. Since in a stochastic framework, the state and possibly the control variables are random processes, J_t^N is itself a random variable that depends on the uncertainty affecting the system. This makes the derivation of the cost function to be minimized in a probabilistic setup arbitrary, to some extent. Some of the possible choices taken in the literature are listed below.

- The most commonly used cost function in a stochastic framework is the following

$$\bar{J}_t = \mathbb{E}[\sum_{i=t}^{t+N-1} l(x_i,u_i) + l_f(x_{t+N})] \tag{2.5}$$

  where the expectation is taken over the distribution of the disturbance. In the analytic framework, e.g., [96, 130] , the cost function above can be reformulated as a function of the mean value and the variance of the system variables by selecting $l(x,u) = \|x\|_Q^2 + \|u\|_R^2$ and $l_f(x) = \|x\|_P^2$. Then, defining $\mathbb{E}[x_i] = \bar{x}_i$, $\mathbb{E}[u_i] = \bar{u}_i$, X_i=var(x_i), and U_i=var(u_i), one can write

$$\mathbb{E}[l(x_i,u_i)] = \mathbb{E}[\|x_i\|_Q^2 + \|u_i\|_R^2] = \|\bar{x}_i\|_Q^2 + \|\bar{u}_i\|_R^2 + \text{tr}(QX_i + RU_i) \tag{2.6a}$$

$$\mathbb{E}[l_f(x_{t+N})] = \mathbb{E}[\|x_{t+N}\|_P^2] = \|\bar{x}_{t+N}\|_P^2 + \text{tr}(PX_{t+N}) \tag{2.6b}$$

- An alternative to (2.5) can be found by resorting to the certainty equivalence principle (see, e.g., [5, 143, 144, 160]). More specifically, we can define the cost function as the deterministic one

$$\hat{J}_t = \sum_{i=t}^{t+N-1} l(\hat{x}_i,u_i) + l_f(\hat{x}_{t+N}) \tag{2.7}$$

  where the nominal system trajectory $\hat{x}_i$ $i = t,\dots,t+N$ is obtained using the update equation $\hat{x}_{i+1} = f(\hat{x}_i,u_i,\hat{w}_i)$ with initial condition

$\hat{x}_t = x_t$ and where $\hat{w}_i$, $i = t, \ldots, t+N-1$, is a nominal disturbance trajectory, e.g., defined as the expected value or the optimal predictor of w_i.

- In a scenario-based framework (see, e.g. [4, 5, 128, 144, 160]), a sampled average over N_s noise realizations can be considered at the place of (2.5), i.e.,

$$\bar{J}_t \simeq \frac{1}{N_s} \sum_{k=1}^{N_s} J_t^{[k]} \tag{2.8}$$

where

$$J_t^{[k]} = \sum_{i=t}^{t+N-1} l(x_i^{[k]}, u_i) + l_f(x_{t+N}^{[k]}) \tag{2.9}$$

and where, denoting by $w_i^{[k]}$ $i = t, \ldots, t+N-1$ the k-th noise realization (with $k = 1, \ldots, N_s$), the trajectory $x_i^{[k]}$ $i = t+1, \ldots, t+N$ is computed using the update equation $x_{i+1}^{[k]} = f(x_i^{[k]}, u_i, w_i^{[k]})$ with initial condition $x_t^{[k]} = x_t$, i.e., the measured state.

- In the context of scenario-based SMPC, a worst-case optimization procedure can be also employed. For example, in [20, 144] the following cost function is minimized

$$J_t^{max} = \max_{k=1,\ldots,N_s} (J_t^{[k]}) \tag{2.10}$$

in which the worst-case cost is taken from all the extracted realization.

2.2.2 Control strategies

In the following, to simplify the setup as much as possible, focus will be placed on linear systems of the type (2.3) with additive zero-mean white noise with bounded support.

In view of the superposition principle, it is always possible, at time t, to write the future evolution of the state variable as the sum of two components, $x_{t+i|t} = \bar{x}_{t+i} + e_{t+i}$, with $\bar{x}_t = x_t$, $e_t = 0$, and where $\bar{x}_{t+i}$ evolves independently of the noise w_{t+i}, while e_{t+i} depends (linearly) just on the evolution of exogenous variable w_{t+i}. Defining the sequence of possible disturbances along the horizon, $\mathbf{w}_t = \begin{bmatrix} w_t^T & \ldots & w_{t+N-1}^T \end{bmatrix}^T \in \mathcal{W}^N$, it is possible to write

$$e_{t+i} = E_i \mathbf{w}_t \tag{2.11}$$

where E_i is a suitable matrix representing the effect of the noise on the uncertainty of the evolution of the state variable, and in turn on the reliability of the prediction given by $\bar{x}_{t+i}$. In the following we will highlight that E_i can assume different values depending on the adopted control strategy. More specifically, we will describe how E_i depends on the chosen control law and its degrees of freedom, similarly to the discussion given in [107].

Open loop control

Some approaches (e.g., [17, 60, 120]) require that, at time t, the candidate control sequence $u_t, \ldots, u_{t+N-1}$ to be applied to the system (2.3) is computed directly as a result of the optimization problem. This means that $u_t, \ldots, u_{t+N-1}$ are independent of $\mathbf{w}_t$ and are deterministically defined as a function of the current state x_t or, in other words, we can write $u_{t+i} = \bar{u}_{t+i|t}$. Therefore, the input sequence is defined as $\mathbf{u}_t = \begin{bmatrix} \bar{u}_{t|t}^T & \ldots & \bar{u}_{t+N-1|t}^T \end{bmatrix}^T$. With this choice, the evolution of the deterministic state $\bar{x}_{t+i}$ is described by

$$\bar{x}_{k+1} = A\bar{x}_k + B_u \bar{u}_{k|t} \tag{2.12}$$

for $k = t, \ldots, t+N-1$, while the "open loop" evolution of perturbed component of the state variable is

$$e_{k+1} = Ae_k + B_w w_k$$

for $k = t, \ldots, t+N-1$. In this case, it follows that the matrix E_i in (2.11) corresponds to the i-th row-block of the matrix

$$\begin{bmatrix} B_w & 0 & \ldots & 0 \\ AB_w & B_w & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1}B_w & A^{N-2}B_w & \ldots & B_w \end{bmatrix}$$

It is clear that, in this case, the variance of e_{t+i} (i.e., the uncertainty on the evolution of the state variable) evolves in an uncontrolled fashion. Especially in case the system is unstable, this approach has significant drawbacks, since it may induce serious feasibility problems.

Disturbance feedback control

The disturbance-feedback approach is employed in different works, see, e.g., [36, 76, 128, 160]. In this case, exploiting the fact that our prediction are made in a closed-loop fashion, the input sequence $\mathbf{u}_t$ is defined as a function of the disturbance sequence $\mathbf{w}_t$. In particular, information on the state along the horizon are captured indirectly by the disturbance since, once the input is known, it is always possible to recover w_{t+i} from x_{t+i+1} and x_{t+i}.
The most common choice corresponds to the affine feedback case, where it is set

$$\mathbf{u}_t = \mathbf{c}_{t|t} + \Theta_{t|t}\mathbf{w}_t \tag{2.13}$$

Here both $\mathbf{c}_{t|t}$ and $\Theta_{t|t}$ are optimal values, computed at time t, of the degrees of freedom $\mathbf{c}_t = \begin{bmatrix} c_t^T & \dots & c_{t+N-1}^T \end{bmatrix}^T$ and

$$\Theta_t = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ \theta_{t+1,t} & 0 & 0 & \dots & 0 \\ \theta_{t+2,t} & \theta_{t+2,t+1} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \theta_{t+N-1,t} & \theta_{t+N-1,t+1} & \theta_{t+N-1,t+2} & \dots & 0 \end{bmatrix}$$

Note that, for causality reasons, Θ_t is set to be a lower-block triangular matrix with zero diagonal blocks. Moreover, since the number of free variables grows quadratically with the horizon N, it is common to see reduced parametrizations for Θ_t.

Also in this case, the evolution of the deterministic state $\bar{x}_{t+i}$ is described by (2.12) for $k = t, \dots, t+N-1$, where the deterministic input is equivalent to $\bar{u}_{k|t} = H_{t-k+1}\mathbf{c}_{t|t}$ and $H_i \in \mathbb{R}^{n \times nN}$ is the matrix selecting the i-th vector element from $\mathbf{c}_{t|t}$. The evolution of the perturbed component of the state variable is

$$e_{k+1} = Ae_k + B_u H_i \Theta_{t|t}\mathbf{w}_t + B_w w_k$$

for $k = t, \dots, t+N-1$. In this case, it follows that the matrix E_i corresponds to the i-th block-row of the matrix

$$\begin{bmatrix} B_w & 0 & \dots & 0 \\ AB_w & B_w & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1}B_w & A^{N-2}B_w & \dots & B_w \end{bmatrix} + \begin{bmatrix} B_u & 0 & \dots & 0 \\ AB_u & B_u & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{N-1}B_u & A^{N-2}B_u & \dots & B_u \end{bmatrix} \Theta_{t|t}$$

From the latter it is clear that, at the optimization level, the choice of $\Theta_{t|t}$ can greatly reduce the effect of the noise sequence $\mathbf{w}_t$ on e_{t+i}, and eventually its variance.

State feedback control

State feedback approaches include, e.g., [25,27–29,77]. In this case, the input variable u_{t+i} is defined as a function of x_{t+i}. Slightly different versions of state feedback control laws have been proposed in the literature. More specifically

- in [96, 130] it is set

$$u_{t+i} = \bar{u}_{t+i|t} + K_{t+i|t}(x_{t+i} - \bar{x}_{t+i}) \tag{2.14}$$

  where $\bar{x}_k$ evolves according to (2.12) for $k = t, \dots, t+N-1$. In this case, the results of the optimization problem are the sequence of gains $K_{t+i|t}$, and the sequence of open-loop terms $\bar{u}_{t+i|t}$, for $i = 0, \dots, N-1$. The evolution of perturbed component of the state variable is

$$e_{k+1} = (A + B_u K_{k|t})e_k + B_w w_k$$

  for $k = t, \dots, t+N-1$. Defining $\Phi_k = A + B_u K_{k|t}$, it follows that the matrix E_i corresponds to the i-th block-row of the matrix

$$\begin{bmatrix} B_w & 0 & \dots & 0 \\ \Phi_{t+1}B_w & B_w & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{t+N}\dots\Phi_{t+1}B_w & \Phi_{t+N}\dots\Phi_{t+2}B_w & \dots & B_w \end{bmatrix} \tag{2.15}$$

  As it is discussed in [55], it is possible to obtain an equivalent disturbance feedback formulation, with fewer degrees of freedom with respect to the ones in matrix Θ_k. It is worth mentioning, however, that both formulations result in convex problems, see e.g., [130].

- in, e.g., [24,27,28,77], the control law is

$$u_{t+i} = \bar{u}_{t+i|t} + Kx_{t+i} \tag{2.16}$$

  Here the result of the optimization problem is only the sequence of terms $\bar{u}_{t+i|t}$, for $i = 0, \ldots, N-1$, while the gain K is fixed offline. The evolution of the deterministic state $\bar{x}_{t+i}$ is described by

$$\bar{x}_{k+1} = \Phi\bar{x}_k + B_u\bar{u}_{k|t} \tag{2.17}$$

  for $k = t, \ldots, t+N-1$, where $\Phi = A + B_uK$ and the matrix E_i corresponds to the i-th block-row of the matrix (2.15), where we set $\Phi_k = \Phi$ for all k.

In both cases, it is clear that a proper choice of the control gain (which can be an optimization variable or a design parameter as in [27]) can reduce the effect of the noise sequence $\mathbf{w}_t$ on e_{t+i}, and eventually its variance. This results in a larger feasibility region with respect to the case of open loop solutions (especially when the system is unstable).

2.2.3 Reformulation of state constraints

In this section, to simplify the setup as much as possible, individual linear chance constraints on the state will be considered, while the possible presence of deterministic or stochastic constraints on the control variable u will be thoroughly discussed in the sequel.
Assume that, at time t, the goal is to impose the following state constraint in the next $i \geq 1$ prediction steps

$$\mathcal{P}\left\{g^T x_{t+i|t} \leq h\right\} \geq 1-p \tag{2.18}$$

where $g \in \mathbb{R}^n, h \in \mathbb{R}$ are fixed and $x_{t|t} = x_t$. As discussed in Section 2.2.2, in view of (2.11) we can write (2.18) as

$$\mathcal{P}\left\{g^T(\bar{x}_{t+i} + E_i\mathbf{w}_t) \leq h\right\} \geq 1-p \tag{2.19}$$

and two main approaches are available for enforcing (2.19). First we discuss how this issue is approached by analytic methods, then the scenario-based ones will be considered.

Analytic reformulation of state constraints

In general, in an analytic framework, we can guarantee (2.19) by verifying the following constraint

$$g^T \bar{x}_{t+i} \leq h - q_i(1-p) \tag{2.20}$$

where the constraint tightening level, $q_i(1-p)$, can be characterized as discussed below, based on the available information regarding the noise sequence $\mathbf{w}_t$, its bounds, and its properties. In particular, consider the following cases.

- Requiring no constraint violation is equivalent to impose (2.19) with $p = 0$. This corresponds to the worst-case tightening adopted in the deterministic framework, and amounts to setting

  $$q_i(1) = \max_{\mathbf{w}_t \in \mathcal{W}^N} g^T E_i \mathbf{w}_t \tag{2.21}$$

  In this case, it is easy to understand that (2.20) may admit a solution only under the assumption that $\mathcal{W}$ is bounded.

- In case a non-zero probability of violation is allowed, it is possible to use the knowledge on the distribution of the noise, if available. Following, for example, the approach proposed in [77, 87], the term $q_i(1-p)$ can be computed as

  $$q_i(1-p) = \arg\min_q q, \ s.t. \ \mathcal{P}\left\{g^T E_i \mathbf{w}_t \leq q\right\} = 1-p \tag{2.22}$$

  In general, the previous expression cannot be computed analytically since it involves the evaluation of a multivariate convolution integral. It is then necessary to approximate it numerically, for example by discretizing the distributions of w or using sample-based approaches. However, if the value of p and the shape of the constraints are fixed in the problem, this computation can be performed off-line only once and with an arbitrary precision.

- Under the assumption that the noise is unbounded, Gaussian with expected value $\bar{w} = 0$ and covariance matrix W, also the variable e_{t+i} is Gaussian and the exact constraint tightening can be computed analytically (e.g., in [17, 120]). Specifically,

  $$q_i(1-p) = \sqrt{g^T E_i \mathbf{W} E_i^T g}\, \mathcal{N}^{-1}(1-p) \tag{2.23}$$

  where $\mathbf{W}$ =diag$(W, \ldots, W)$ and $\mathcal{N}$ is the cumulative probability function of a Gaussian variable with zero mean and unitary variance.

- Finally, if the distribution of the noise is not specified, but its expected value $\bar{w}$ and covariance matrix W are known, it is possible to resort to the Cantelli-Chebyshev inequality [97] that is reported in the following lemma

  Lemma 1. ***(Cantelli's inequality)*** *Let y be a (scalar) random variable with mean $\bar{y}$ and variance Y. Then for every $\mathbb{R} \ni \alpha \geq 0$ it holds that*

  $$\mathcal{P}(y \geq \bar{y} + \alpha) \leq \frac{Y}{Y + \alpha^2} \tag{2.24}$$

  Recalling that (without loss of generality) we are now considering $\bar{w} = 0$, it is possible to use the approach proposed in [96]. In particular, consider the constraint in (2.18) and assume that the following expression holds for some $\delta \geq 0$

  $$g^T \bar{x}_{t+i} \leq h - \delta \tag{2.25}$$

  Thus, by the use of (2.25) and (2.24) we have that

  $$\mathcal{P}\{g^T x_{t+i|t} \geq h\} \leq \mathcal{P}\{g^T x_{t+i|t} \geq g^T \bar{x}_{t+i} + \delta\} \leq \frac{g^T E_i \mathbf{W} E_i^T g}{g^T E_i \mathbf{W} E_i^T g + \delta^2}$$

  where, again, $\mathbf{W} = \text{diag}(W, \dots, W)$. Therefore, the original inequality (2.18) is satisfied by imposing

  $$\frac{g^T E_i \mathbf{W} E_i^T g}{g^T E_i \mathbf{W} E_i^T g + \delta^2} \leq p \tag{2.26}$$

  that in turn can be rearranged as

  $$g^T E_i \mathbf{W} E_i^T g (\frac{1-p}{p}) \leq \delta^2 \tag{2.27}$$

  Now, combining (2.25) and (2.27), the probabilistic constraint (2.18) is approximated by (2.20) where a bound for $q_i(1-p)$ is provided as

  $$q_i(1-p) \leq \sqrt{g^T E_i \mathbf{W} E_i^T g} \sqrt{\frac{1-p}{p}} \tag{2.28}$$

  Note that in (2.23) and (2.28) the term $q_i(1-p)$ has the same shape if we define a function $f(1-p)$ that is

  $$f(1-p) = \mathcal{N}^{-1}(1-p)$$

whenever the uncertainty is Gaussian and is bounded by

$$f(1-p) = \sqrt{\frac{1-p}{p}}$$

when the distribution of the uncertainty is not known. Of course, the choice (2.28) is in general conservative. For example, in Figure 2.1 we show the comparison between the values of $f(1-p)$ in (2.23) and (2.28).

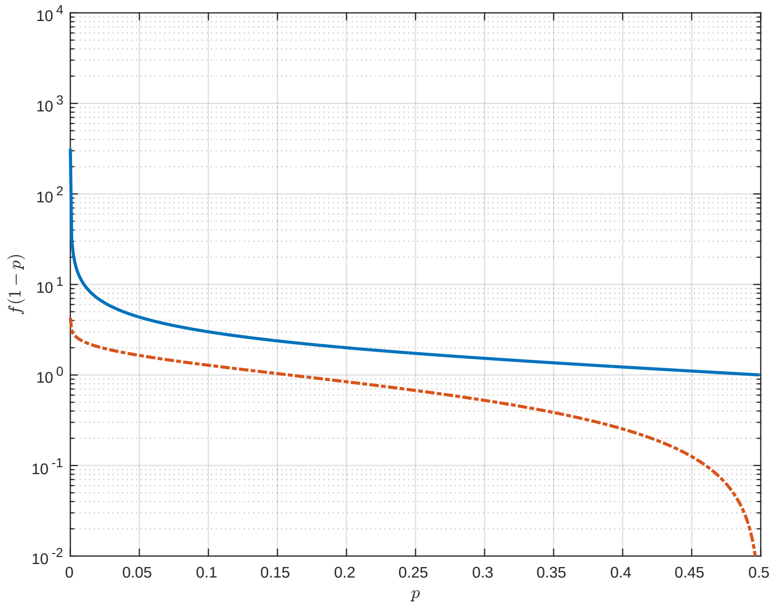


Figure 2.1: *Comparison between the values of* $f(1-p) = \sqrt{\frac{1-p}{p}}$ *(solid line) and* $f(1-p) = \mathcal{N}^{-1}(1-p)$ *(dotted line).*

From this discussion is clear that the choice of the matrix E_i (which in turn depends on the adopted control strategy, see Section 2.2.2) greatly affects the size of $q_i(1-p)$, which can be minimized to eventually enhance the feasibility properties of the SMPC-based control scheme.

Reformulation of state constraints in a scenario framework

In scenario-based stochastic MPC methods, constraints are verified for a finite number of sampled deterministic prediction sequences, computed on the basis of proper extractions of the uncertainty. The number of the extracted samples is carefully selected to ensure a prescribed level of constraint violation.

Specifically, recalling (2.11), the probabilistic constraint (2.18) is guaranteed by enforcing, at any time instant i along the prediction horizon, the following collection of deterministic constraints

$$g^T(\bar{x}_{t+i} + E_i \mathbf{w}_t^{[i,k_i]}) \leq h \tag{2.29}$$

for all $k_i = 1, \dots, N_{s,i}$, where $\mathbf{w}_t^{[i,k_i]}$ is the k_i-th noise realization relative to the i-th constraint, and where it is assumed that $\mathbf{w}_t^{[i,k]}$ is independent of $\mathbf{w}_t^{[j,h]}$ for all $i \neq j$ or $h \neq k$.
The number $N_{s,i}$ of noise realizations must be carefully chosen in order to have precise stochastic guarantees on the solution. More in detail, denote by d_i the number of variables involved in the optimization problem referred to the i-th prediction. The probability that $g^T x_{t+i} > h$ is smaller than p with a given confidence level β is (see e.g. [20, 144])

$$\sum_{k=1}^{d_i-1} \binom{N_{s,i}}{k} p^k (1-p)^{N_{s,i}-k} \geq \beta \tag{2.30}$$

The value of $N_{s,i}$ can thus be chosen so as to satisfy

$$N_{s,i} \geq \frac{d_i + 1 + \ln(1/\beta) + \sqrt{2(d_i+1)\ln(1/\beta)}}{p} \tag{2.31}$$

Note that this number is mainly affected by the dimension of the problem (d_i) and by the choice of the parameter p while it is only marginally increased by the confidence level β, due to the logarithmic dependence, that thus can be chosen to be very small. Moreover, it may happen, especially when the required number of samples, $N_{s,i}$ is low, that the solution to the problem is too conservative due to "unlucky" extractions of the uncertainty used to reformulate the constraint. To avoid this situation, it is possible to implement iterative sample-removal algorithms, which are able to detect the more stringent constraints to be ignored. In this case, the expression in (2.30) is not valid anymore and a different procedure, described for example in [23, 144] is required to compute both the number of samples $N_{s,i}$ and the number of constraints $N_{r,i}$ to be removed given the probability p and the confidence β. Of course, due to the removed samples, the number of extractions that are needed to approximate the chance constraint correctly is increased. An advantage of using this removal technique is that it is easier to reach the level of relaxation required in the problem, while the main drawback is related to the increase of the computational complexity due to both the increased number of samples and to the removal procedure itself.

Finally note that the equation (2.29) applies when open loop predictions are performed at time t. In this case, in principle, to enforce (2.19) for each considered future instant $i = 1, \dots, N$, one should extract a number $N_{s,i}$ of independent realizations, resulting in a a total number of realizations required at time t of $\sum_{i=1}^{N} N_{s,i}$. Recall that the number of independent optimization variables d_i (and hence $N_{s,i}$) for the single constraint increases with i. This, however, results unnecessary in a closed-loop receding-horizon control setting, as thoroughly discussed and proved in [144], for more details see Section 4.1.3.

2.2.4 Constraints on input variables

Contrarily to the case of state constraints, there is no general consensus in the SMPC literature on input constraints and saturations. More specifically, the issue is wether to reformulate input constraints as probabilistic ones (as it is generally done in the case of state constraints) or not. In a practical context, indeed, it is desirable that input variables lie in specified ranges; however, this requirement is not always compatible with the adopted control strategy and with recursive feasibility requirements.
Considering open loop control strategies, since the input sequence $\bar{u}_{t+i}$ is deterministically defined at time t, hard bounds on it could be enforced regardless of the entity of the disturbance (both in the bounded and in the unbounded case). On the other hand, when state feedback and disturbance feedback policies are adopted, the values taken by $u_{t+1}, \dots, u_{t+N-1}$ depend on $w_t, \dots, w_{t+N-2}$, and therefore saturation constraints can be imposed at time t at most only on u_t in case the noise is unbounded. Of course, if the noise support is bounded, hard constraints can be enforced on the whole input trajectory at the price of applying robust worst-case arguments. Consistently with this discussion, hard constraints are assumed on the input variables in two cases: (i) when the noise is bounded, e.g., [27, 76]; (ii) when recursive feasibility requirements are relaxed, e.g., [128, 144]. A notable exception is discussed in [65], where a nonlinear (output) feedback policy is adopted; in this case, the use of bounded nonlinear functions of $y_{t+i} - \hat{y}_{t+i}$ (where y is the system output and $\hat{y}$ is the system output nominal prediction), together with the assumptions that the system is stable and that state constraints are absent, allows hard bounds on inputs to be enforced at any time instant and to guarantee recursive feasibility.
In the strategy presented here, constraints on input variables must be formulated in general as probabilistic ones. Some exceptions will be later discussed, e.g., when the system is stable and the degrees of freedom on

$K_t, \ldots, K_{t+N-1}$ are reduced.

2.2.5 Feasibility and convergence properties

In the context of stochastic MPC the problem of guaranteeing recursive feasibility is still largely open. Some notable solutions are available (for both state feedback and disturbance feedback approaches, see e.g., [25, 27, 74, 76]), under the assumption that the disturbances are bounded. They are based on imposing suitable mixed probabilistic/worst-case constraint tightening to equation (2.20) (or, in other words, by amplifying the term $q_i(1-p)$) for $i > 1$ by accounting for bounded sets where the state can evolve due to the bounded noise affecting the state equation.
The more general case of unbounded noise poses more difficulties and some specific solutions and reformulations of these properties have been adopted; for example in [28, 29] the concept of invariance with probability p is used, while in [117] the definition of probabilistic resolvability is introduced. In some notable works (e.g., [20,60,144]), including the scenario-based methods, the feasibility of the optimization problem is assumed at each time step, and possibly enforced, e.g., by reformulating the problem constraints in a soft fashion.

Concerning stability and convergence results, while in case of bounded deterministic disturbances practical stability (i.e., convergence in a neighborhood of the origin) can be established, in the stochastic framework mean square stability results are generally addressed, with the notable exception of [60], where convergence of the mean value of $x(k)$ to zero is proven. Indeed, in [24, 27, 65, 87], and in general in case of additive noise, it can be proven that

$$\lim_{k\to\infty} \mathbb{E}[\|x_k\|^2] = \lim_{k\to\infty} (\|\mathbb{E}[x_k]\|^2 + \text{var}(x_k)) \leq \text{const} \tag{2.32}$$

This means that the state of the system is driven to a neighborhood of the steady state condition (whose dimension depends on the amplitude of the input noise and on the adopted control policy).
On the other hand, when modeling uncertainties are described (e.g., for linear systems with multiplicative uncertainties [28, 41, 130]), point-wise convergence can be obtained, e.g., that $\lim_{k\to\infty} \mathbb{E}[\|x_k\|^2] = 0$.

Note that, for simplicity in the following summary the property (2.32) will be denoted *mean square convergence* and the constant value will be possibly equal to zero.

		UNCERTAINTY	
		BOUNDED	UNBOUNDED
	NONLINEAR	[85], [56]	[16], [104] [116], [42], [16], [154], [18]
LINEAR	ADDITIVE NOISE	[24], [87], [76], [7], [115], [143], [75], [24], [107], [74], [27], [78], [79], [26], [96], [112]	[159], [161], [113], [163], [161], [60], [45], [36], [148], [128], [117], [99], [90], [65], [164], [33], [31], [147], [118], [64], [63], [66], [15], [120], [10], [14], [157], [152], [6], [151], [150], [82], [145] [126]
	MULTIPLICATIVE NOISE	[28], [41], [9], [25], [29], [8]	[81], [70], [130], [28], [34]
	PARAMETRIC UNCERTAINTY	[20], [21]	[144], [22], [108] [125]

Table 2.1: *Classification of available SMPC techniques based on system assumptions.*

2.2.6 Summarizing tables

In this section we give a schematic overview of the many different approaches proposed in the recent literature on SMPC. First, in Table 2.1 we focus on the system assumptions required by the different algorithms. Secondly, in Table 2.2 we stress, for the available methods, how constraints are handled, and the information available/assumed on the disturbance acting on the system. Finally, in Table 2.3 we review the main theoretical properties guaranteed in the different algorithms.
In addition to the information summarized in Tables 2.1, 2.2, 2.3, the following comments are due to highlight the main characteristics of the approaches proposed so far.

- The main advantage of scenario approaches is that they are applicable to wide classes of systems (linear, nonlinear) affected by general disturbances (additive, multiplicative, parametric, bounded or unbounded) with constraints of general type on the inputs, states, and outputs, provided that the problem is convex in the optimization variables. Their drawbacks come from the computational complexity that arises from the high number of samples usually required to reach the desired probabilistic guarantees and eventually to the need of sample-removal strategies. In addition, stability and recursive feasibility properties are not available at present, as it can be deduced from a joint analysis of Tables 2.2, 2.3.

- In the wide class of analytical methods, algorithms with recursive

			STATE CONSTRAINTS TYPE		
			EXPECTED VALUE	JOINT CC	INDEPENDENT CC
ANALYTICAL	KNOWN PDF	ALL	[42], [64], [130]	×	[104], [125] [87], [76], [75], [24], [107], [74], [27], [147], [78], [79], [26], [25], [29], [10]
		GAUSSIAN	×	[81], [126] [117], [116], [118], [15], [120], [150], [82]	[113], [148], [14], [157], [152], [151], [145]
	SECOND ORDER		×	[163], [164]	[46], [85], [45], [60], [96]
RANDOMIZED (KNOWN PDF)			×	[159], [161], [144], [115], [20], [36], [143], [22], [128], [99], [41], [16]	×

Table 2.2: *Classification of available SMPC techniques based on algorithms structure.*

		CONVERGENCE			
		IN PROBABILITY	EXPECTED VALUE	MEAN SQUARE	NO
FEASIBILITY	YES	[20], [22], [34]	[9]	[87], [45], [41], [24], [65], [107], [27], [31], [78], [79], [26], [63], [96], [25], [29]	[85], [76], [36], [75], [74]
	NO	×	×	[81], [42], [130], [126]	[159], [161], [144], [113], [104], [125] [163], [115], [148], [143], [128], [99], [164], [147], [118], [64], [16], [15], [120], [10], [14], [157], [152], [151], [150], [82], [145]
	PROBABILISTIC	×	[60]	[28]	[117], [116]

Table 2.3: *Classification of available SMPC techniques based on feasibility and convergence properties.*

feasibility and convergence are already available both for bounded and unbounded noise. However, in order to rigorously reformulate costs and constraints in an analytical fashion, the most popular assumption is that the model is linear and affected by additive or multiplicative white noise. A notable exception is represented by the approaches that rely on polynomial chaos expansions, e.g., [104, 125, 126]. Thanks to the possibility of approximating general analytic functions using series of basis functions, these methods handle systems affected by parametric uncertainties with known distribution; for example, continuous-time and discrete-time linear systems are considered in [125] and [126], respectively, while nonlinear discrete-time systems are addressed in [104].

- Very few algorithms, see [24, 65], have been extended to deal with the output feedback case, which still represents a largely open issue.

2.3 Numerical examples

In this section we show, through some simple examples, the effect of the modeling choices discussed in this chapter. In particular, the first example compares a robust and a probabilistic tightening procedure for different distributions of the noise. To this end, the probabilistic bounds are obtained based on the Chebyshev inequality as described in Section 2.2.3.
The second example, similarly to the first one, describes the effect of the several control policies discussed in Section 2.2.2.
Finally, the third example shows the approximation we introduce by splitting a joint chance constraint into several individual chance constraints using risk allocation procedures.

2.3.1 Comparison between stochastic and robust tightenings

Consider the following scalar system

$$x_{t+1} = ax_t + u_t + w_t \tag{2.33}$$

where $0 < a < 1$, $w \in [-w_{max}, w_{max}]$, $w_{max} > 0$, and the measurable state is constrained as follows

$$x_t \leq x_{max} \tag{2.34}$$

The limitations imposed by a deterministic robust approach (for example in [102]) and a probabilistic method, are now compared. For both the

algorithms an open-loop control law of the form $u_t = \bar{u}_t$ is considered, where, similarly to equation (2.12), $\bar{u}$ is the input of the deterministic system $\bar{x}_{t+1} = a\bar{x}_t + b\bar{u}_t$.
In a probabilistic framework, we allow the constraint (2.34) to be violated with probability p, i.e.,

$$\mathcal{P}\{x \geq x_{max}\} \leq p \tag{2.35}$$

To verify (2.34) and (2.35) the tightened constraint $\bar{x}_k \leq x_{max} - \Delta x$ must be fulfilled in both the approaches where, in case of the deterministic robust approach [102]

$$\Delta x = \Delta x_{RPI} = \sum_{i=0}^{+\infty} a^i w_{max} = \frac{1}{1-a} w_{max}$$

while, having defined w as a stochastic process with zero mean and variance W, in the probabilistic framework

$$\Delta x = \Delta x_S(p) = \sqrt{X(1-p)/p}$$

and X is the steady state variance satisfying the algebraic equation $X = a^2 X + W$, i.e. $X = W/(1-a^2)$. Notably, W takes different values depending upon the noise distribution.
It results that the deterministic tightened constraints are more conservative provided that $\Delta x_S(p) < \Delta x_{RPI}$, i.e.

$$p > \frac{(1-a)^2}{b(1-a^2) + (1-a)^2} \tag{2.36}$$

Consider now the three distributions depicted in Figure 2.2 setting $W = w_{max}^2/b$, with

- $b = 3$ for uniform distribution (A)
- $b = 18$ for triangular distribution (B)
- $b = 25$ for truncated Gaussian distribution (C)

Setting, for example, $a = 0.9$, condition (2.36) is verified for $p > 0.0172$ in case (A), $p > 0.0029$ in case (B), and $p > 0.0021$ in case (C).
Note that, although formally truncated, the distribution in case (C) can be well approximated with a non-truncated Gaussian distribution: if this information were available, one could use $\Delta x_S(p) = \sqrt{X}\,\mathcal{N}^{-1}(1-p)$ for

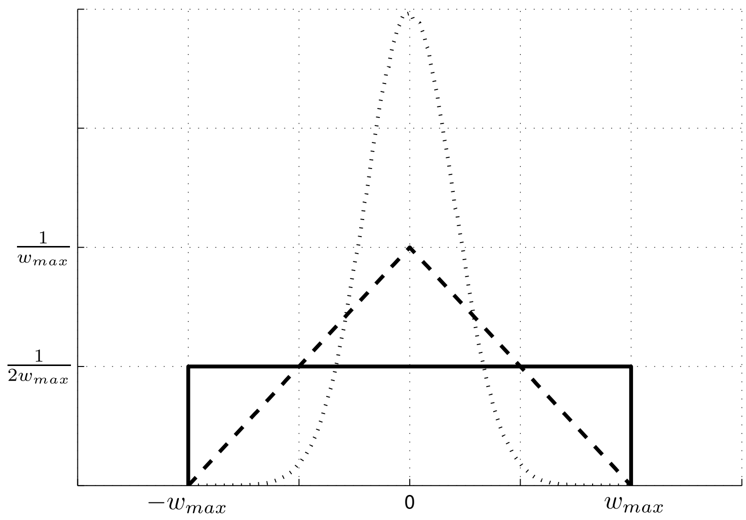


Figure 2.2: *Distributions: uniform (case A, solid line), triangular (case B, dashed line), truncated Gaussian (case C, dotted line).*

constraint tightening, and in this case $\Delta x_S(p) < \Delta x_{RPI}$ would be verified with

$$p > 1 - \mathcal{N}\left(\frac{(1-a^2)b}{(1-a)^2}\right) \simeq 0$$

2.3.2 Choice of the control strategy and tightening approaches

In this simple example we show the effect of the choice of the control strategies discussed in Section 2.2.2 and of the analytic tightening approaches presented in Section 2.2.3.
In particular, consider the following discrete-time linear scalar system

$$x_{t+1} = ax_t + u_t + w_t$$

where $a = 0.6$ and the disturbance w_t is a zero mean white noise with variance $W = 10^{-2}$ and such that $\|w_t\| \leq 0.5$.
Again, to simplify as much as possible the setup, we do not consider input constraints while the focus is on a single state chance-constraint

$$\mathcal{P}\left\{gx_t \leq h\right\} \geq 1 - p, \quad \forall t \geq 0$$

with $g = 1$, $h = 3$ and $p = 0.2$. Predictions are computed within an horizon of length $N = 10$.

Results are shown in Figure. 2.3 and 2.4 for three different tightening approaches, namely the worst-case in (2.21), the direct quantile estimation in (2.22) and the use of the Chebyshev inequality in (2.28) and applying respectively an open-loop policy, a fixed gain state-feedback policy and a fixed gain affine disturbance-feedback policy.

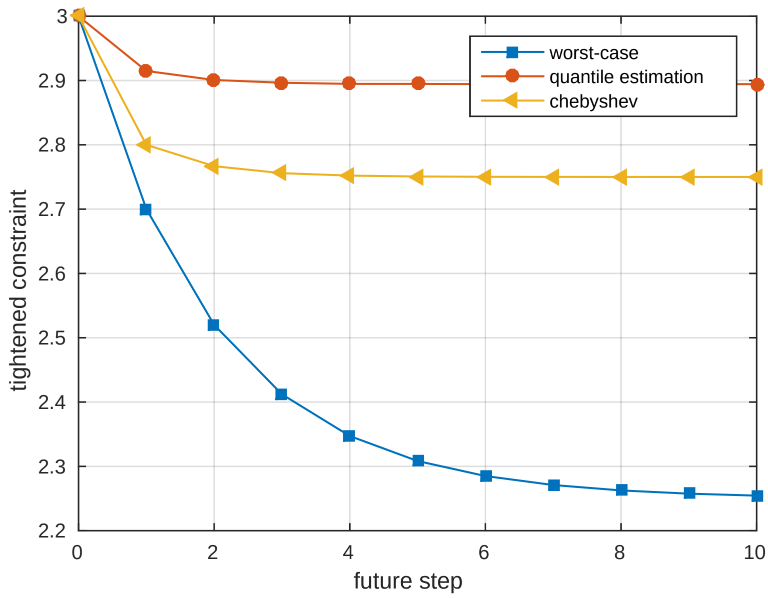


Figure 2.3: *Constraint tightening on a scalar example over $N = 10$ prediction steps. Computed using an open-loop control strategy.*

It is easy to show that, besides the choice of the control strategy, the use of a stochastic constraint in place of the deterministic worst-case one allows one to reduce the tightening level both in the case of known (quantile) and unknown (Chebyshev) distribution. This greatly motivates the use of such approaches inside a Model Predictive Control scheme. Moreover, it is clear that the use of a closed-loop control strategy such as the fixed gain state-feedback or the fixed gain disturbance-feedback easily overcomes the open-loop solution, and this happens without adding any further degree of freedom to the policy at the price of possibly relaxing any input constraint along the horizon. A more general comparison of the different control policies, obtained using (2.22) to compute the tightenings, is shown in Figure. 2.5. Obviously, the time-varying state-feedback and disturbance-

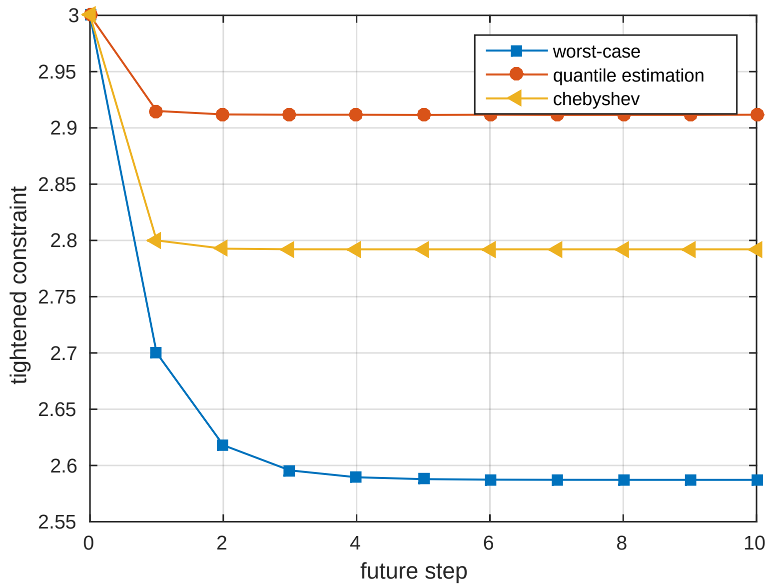


Figure 2.4: *Constraint tightening on a scalar example over $N = 10$ prediction steps. Computed using a state-feedback or an affine disturbance-feedback control strategy (in this case the same result is obtained).*

feedback strategies are both able to achieve the lowest possible tightening level, due to the fact that we are explicitly compensating for the effect of the disturbance. However, in this simple example, the difference between such time-varying policies and their simplified version with fixed gains, is not so crucial and thus one can prefer the constant solutions that reduces the number of degrees of freedom. This fact will be discussed in the sequel with respect to the proposed algorithm.

2.3.3 Joint chance constraint approximation

In this example we consider the problem of approximating a single joint chance constraint into multiple individual chance constraints as described in Section 2.1. In particular, consider the following two independent Gaussian stochastic variables

$$x \sim \mathcal{N}(\bar{x}, 1) \qquad y \sim \mathcal{N}(\bar{y}, 1)$$

The problem is to choose their expected values, respectively $\bar{x}$ and $\bar{y}$, so that the following constraint is satisfied

$$\mathcal{P}\left\{\begin{bmatrix} x \\ y \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\} \geq p$$

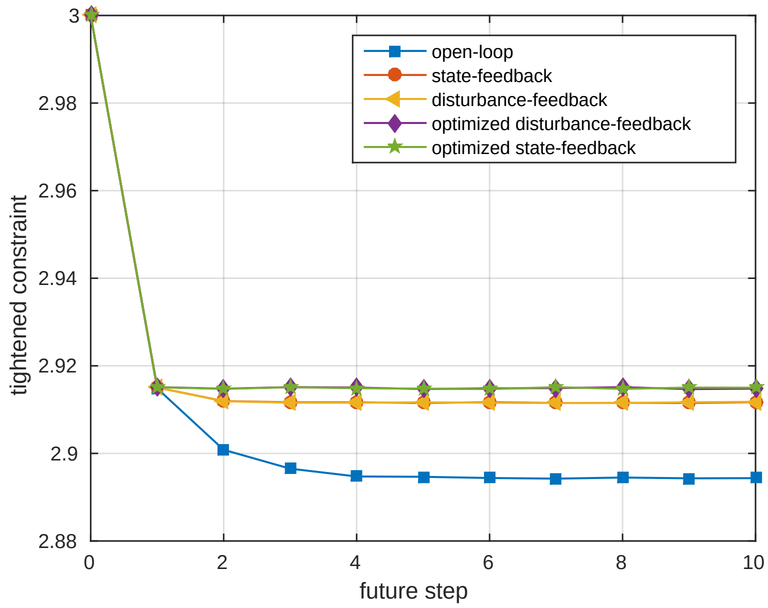


Figure 2.5: *Constraint tightening on a scalar example over $N = 10$ prediction steps. Comparison between different control strategies.*

with $p = 0.5$ (of course, due to the unbounded support, we cannot have $p = 1$, i.e., an hard constraint).

Following the procedure described in Section 2.1 the joint chance constraint can be approximated with two individual constraints as follows

$$\mathcal{P}\{x \geq 0 \wedge y \geq 0\} \geq p \;\rightarrow\; \begin{cases} \mathcal{P}\{x \geq 0\} \leq p_x \\ \mathcal{P}\{y \geq 0\} \leq p_y \end{cases},\; p_x + p_y = 0.5$$

where the terms p_x and p_y are now design parameters to be chosen such that $p_x + p_y = p$. To this end, we can select offline two possible values or we can consider the two parameters as extra degrees of freedom to be selected through a proper algorithm (risk allocation).

Focusing on the feasible regions of both the original joint chance constrained problem and the approximated one, results are shown in Figure 2.6. In particular, it is easy to show that the region corresponding to a single choice of p_x and p_u has the shape of a rectangle and it results to be much smaller than the feasible region of the original problem. Moreover, even using a dynamic risk allocation procedure, i.e. keeping the parameters p_x and p_y as free, we cannot reach the same feasibility region of the joint chance constraint, as shown by the dotted line.

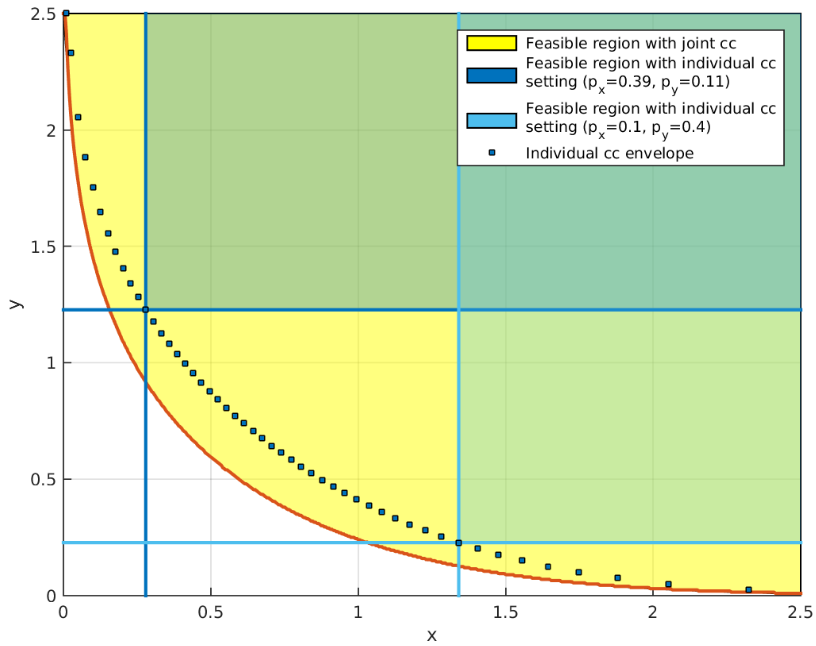


Figure 2.6: *Feasible region for the joint chance constraint and for the pair of individual chance constraints corresponding to two different choices of p_x and p_y. The points represent the envelope that can be obtained using risk allocation.*

2.4 Comments

In this chapter we discussed the importance of considering a stochastic Model Predictive Control framework to be able to deal with stochastic, possibly unbounded, disturbances and to reduce the conservativeness with respect to classical robust approaches in the bounded case. This is particularly significant when information on the probabilistic distribution of the disturbances acting on the system are available and the problem constraints can be relaxed in a stochastic way, meaning that a partial violation can be accepted. A very simple example showed the effect of this change in perspective.

After an overview on the commonly used models for uncertain discrete-time systems we focused on linear systems with additive disturbances. Similarly, among the possible ways to express stochastic constraints, we chose to adopt the so-called individual chance constraints, whose formulation appears to be more natural in several contexts like for example the temperature and energy consumption control in buildings or the power production planning in presence of renewable energy sources , but can be applied also to different problems like robots path planning and obstacle avoidance.

With respect to this setup, the main ingredients of an MPC control problem have been analyzed in a stochastic framework. The most common ways adopted to handle stochastic constraints have been presented and their application has been discussed under several possible control strategies. Moreover, we have shown how to specify cost functions and we have summarized the main results available in the literature for convergence and feasibility guarantees.

Based on the previous discussion, it is apparent that there is a need for stochastic MPC algorithms that are able to provide convergence and recursive feasibility guarantees without exploiting worst-case arguments. In particular, this would allow handling disturbances with a-priori unbounded support directly inside an MPC scheme, without specifying any additional recovery strategy (of course at the price of some relaxation). Together with this, there is also need for output-feedback control strategies that share the same guarantees in terms of convergence and feasibility. Motivated by this analysis, in the sequel an analytic SMPC algorithm (denoted as p-SMPC) will be presented, both for the state-feedback and the output-feedback case, that is able to give the desired guarantees at the price of relaxing hard input bounds into probabilistic requirements. With respect to sample-based solutions, the proposed algorithm is much less flexible, in the sense that it can be used only in the case of linear systems with additive disturbance and individual chance constraints, but the computational load is similar to the one of a standard deterministic algorithm.

CHAPTER 3

State-feedback probabilistic SMPC

In this chapter a novel stochastic MPC technique is presented for discrete-time linear systems with additive disturbances characterized by a possibly unbounded support.
As thoroughly discussed in Chapter 2, the knowledge of the stochastic distribution of the uncertainty is exploited to reduce the conservativeness with respect to well known robust approaches and to consider also possibly unbounded disturbances. The constraints acting on the system are multiple individual linear chance constraints both on the state and the input, but extensions to joint chance constraints can be obtained using, for example, risk allocation techniques. The perfect knowledge of the system state is firstly assumed and a state-feedback control algorithm is derived and compared to other paradigmatic examples from the literature. Results are given in terms of mean square convergence and recursive feasibility.

For linear systems with additive noise, a simple, yet effective, way to handle probabilistic constraints and to reformulate a probabilistic MPC problem in terms of a deterministic one has been proposed in [96]. In order to handle uncertainties with unknown distribution, this algorithm is based on the use of the Cantelli-Chebyshev's inequality, as discussed in Section 2.2.3. The

main weakness of the method in [96] is due to the assumption that the noise is bounded and to the inclusion in the MPC problem of some quite artificial constraints to guarantee ISS convergence.
In this chapter, the algorithm described in [96] is deeply revisited and extended to derive a computationally efficient MPC method for systems subject to possibly unbounded disturbances. Specifically, the algorithm here proposed is characterized by

a) A computational burden only slightly heavier that the one required by stabilizing MPC methods for undisturbed linear systems

b) The possibility to consider unbounded noises

c) Guaranteed recursive feasibility and convergence under mild conditions. These properties are obtained by considering at any time instant the current expected value and the covariance of the state as optimization variables, to be properly selected according to two alternative strategies, and by imposing some constraints to their value at the end of the prediction horizon.

3.1 Problem statement

Consider, as discussed in the previous chapter, the following discrete-time linear system

$$x_{t+1} = Ax_t + B_u u_t + B_w w_t \quad t \geq 0 \tag{3.1}$$

where $x_t \in \mathbb{R}^n$ is the state, $u_t \in \mathbb{R}^m$ is the input and $w_t \in \mathbb{R}^{n_w}$ is a zero-mean white noise with variance W and a-priori unbounded support. At this stage no extra assumptions on the noise distribution are made. Perfect state information is assumed, together with the reachability of the pairs (A, B_u) and $(A, \tilde{B}_w)$, where $\tilde{B}_w \tilde{B}_w^T = B_w$.

In line with the discussion in Chapter 2 and due to the possibly unbounded support of the disturbance, constraints on the state and input variables of system (3.1) are imposed in a probabilistic sense, i.e., at time t the following multiple individual chance constraints are considered

$$\mathcal{P}\{b_r^T x_{t+k} \leq x_r^{max}\} \geq 1 - p_r^x \,, \quad \forall k > 0 \,, \quad r = 1, \ldots, n_r \tag{3.2}$$

$$\mathcal{P}\{c_s^T u_{t+k} \leq u_s^{max}\} \geq 1 - p_s^u \,, \quad \forall k \geq 0 \,, \quad s = 1, \ldots, n_s \tag{3.3}$$

where $\mathcal{P}(\phi)$ denotes the probability of ϕ, b_r, c_s are constant vectors, x_r^{max}, u_s^{max} are bounds for the state and control variables and p_s^x, p_s^u are design parameters that can be chosen independently for each constraint. It is also assumed that the set of relations $b_r^T x \leq x_r^{max}$, $r = 1, \ldots, n_r$ ($c_s^T u \leq u_s^{max}$, $s = 1, \ldots, n_s$), defines a compact set $\mathbb{X}$ ($\mathbb{U}$) containing the origin in its interior.

3.1.1 Regulator structure

For system (3.1), the goal is to design with MPC a feedback control law of the form

$$u_t = \bar{u}_t + K_t(x_t - \bar{x}_t) \tag{3.4}$$

where the open-loop term $\bar{u}_t$ and the time-varying gain K_t are free parameters that must be computed by solving at any time instant a suitable optimization problem. Moreover, in (3.4), $\bar{x}_t$ denotes the (possibly conditional) expected value of the state x_t at time t, commonly denoted as $\bar{x}_t = \mathbb{E}\{x_t\}$.

Remark 1. *Note that, due to the choice of the state-feedback control law* (3.4)*, we are forced to use probabilistic constraints also on the inputs of the system, as discussed in Section 2.2.2, but we can still choose the parameters p_s^u to be sufficiently small to avoid frequent violations.*

In order to handle the chance constraints (3.2) and (3.3), the system variables, and later their future predictions, are firstly described in terms of their expected value and variance. In particular, we define the nominal model that evolves as follows

$$\bar{x}_{t+1} = A\bar{x}_t + B_u\bar{u}_t \quad k \geq 0 \tag{3.5}$$

where the term $\bar{u}_t$ is the same as in (3.4) and is again a degree of freedom of our optimization problem. Of course, since $\mathbb{E}\{w_t\} = 0$ for all t, it is easy to verify that $\bar{x}_{t+1} = \mathbb{E}[x_{t+1}]$ and thus the nominal model represents the dynamics of the expected value of the state. Starting from this, define the error variable

$$\delta x_t = x_t - \bar{x}_t \tag{3.6}$$

that has the meaning of the difference between the real state and its expected value. Moreover, according to (3.1)-(3.5), the prediction of the state error evolves as

$$\delta x_{t+1} = (A + B_u K_t)\delta x_t + B_w w_t \tag{3.7}$$

where again the gains K_t are degrees of freedom of the optimization problem. Due to the fact this error is zero mean, if we denote the predicted covariance matrix of the state as X_t, its evolution can be computed as

$$\begin{aligned} X_{t+1} &= \mathbb{E}\{\delta x_{t+1}\delta x_{t+1}^T\} \\ &= (A + B_u K_t)X_t(A + B_u K_t)^T + B_w W B_w^T \end{aligned} \tag{3.8}$$

Note that, in general, expression (3.8) is nonlinear when the gains are considered as free variables, however, resorting to standard manipulations in matrix computation, it is possible to rewrite it as an LMI. This fact will be discussed in the sequel.

3.1.2 Reformulation of the probabilistic constraints

In order to state an MPC problem efficiently solvable on-line, the probabilistic constraints (3.2) and (3.3) are now transformed in deterministic, although tighter, ones using the expressions in (3.5) and (3.8) and resorting to the results presented in Paragraph 2.2.3. In particular, constraint (3.2) is satisfied if the following inequality holds

$$b_r^T \bar{x}_{t+k} \leq x_r^{max} - \sqrt{b_r^T X_{t+k} b_r}\, f(1 - p_r^x), \quad \forall k \geq 0,\ r = 1, \ldots, n_r \tag{3.9}$$

where the variance of the state, X_{t+k} plays the role of the variance of the uncertainty, namely e_{t+k} in (2.11) that appears in (2.28). Moreover, similarly to the discussion in Paragraph 2.2.3, the constant term $f(1 - p_r^x)$ can be computed as $\mathcal{N}^{-1}(1 - p_r^x)$, where $\mathcal{N}(\cdot)$ is the cumulative density function of a normal distribution, in case the uncertainty (and due to linearity also the state and the input) is Gaussian, while it can always be bounded by the term $\sqrt{(1 - p_r^x)/p_r^x}$, resorting to the Cantelli-Chebyshev's lemma [97], in case of unknown distributions. For further details on the procedure or a comparison with the other available methods the reader is referred to Chapter 2.

Inequality (3.9), together with (3.5) and (3.8), clearly shows the different effects of the terms appearing in control law (3.4) and motivates its structure. In fact, the open-loop term $\bar{u}$ influences the evolution of the mean $\bar{x}$ (recall (3.5)), while the gain K can be chosen to limit the variance X (recall (3.8)) which evolution is otherwise fixed by the initial condition, as thoroughly discussed in Section 2.2.2. However, when K is considered as a degree of freedom, the practical application of the deterministic constraint (3.9) is still hampered by its nonlinear dependence on X, so that a further

linearization step is useful to finally derive a linear constraint to be included in the MPC optimization problem.
To this end, letting $\bar{\delta} = \varepsilon x^{max}$, where $\varepsilon \in [0, 1]$ is an additional design parameter, and at the price of an additional slight tightening of the constraint, a standard linearization procedure allows one to reformulate (3.9) as follows

$$b_r^T \bar{x}_{t+k} \leq (1 - 0.5\varepsilon) x_r^{max} - \frac{f(p_r^x)^2}{2\varepsilon x_r^{max}} \, b_r^T X_{t+k} b_r, \quad \forall k \geq 0, \; r = 1, \ldots, n_r \tag{3.10}$$

The same procedure used for the state constraints can be used to reformulate the input constraints (3.3). In particular, denoting by

$$\begin{aligned} U_{t+k} &= \mathbb{E}\{(\bar{u}_{t+k} + K_{t+k}\delta x_{t+k})(\bar{u}_{t+k} + K_{t+k}\delta x_{t+k})^T\} \\ &= K_{t+k} X_{t+k} K_{t+k}^T \end{aligned} \tag{3.11}$$

the covariance matrix of the input variable, the corresponding deterministic constraint becomes

$$c_s^T \bar{u}_{t+k} \leq u_s^{max} - \sqrt{c_s^T U_{t+k} c_s} \, f(p_s^u), \quad \forall k \geq 0, \; s = 1, \ldots, n_s \tag{3.12}$$

and its linear (tighter) counterpart is

$$c_s^T \bar{u}_{t+k} \leq (1 - 0.5\varepsilon) u_s^{max} - \frac{f(p_s^u)^2}{2\varepsilon u_s^{max}} \, c_s^T U_{t+k} c_s, \quad \forall k \geq 0, \; s = 1, \ldots, n_s \tag{3.13}$$

While the use of Cantelli's inequality allows one to reformulate the problem without making any assumption on the distribution of the disturbance w, we stress again the fact that the values of $f(1-p)$ in case of unknown distribution are greater than in the Gaussian case (e.g., about an order of magnitude in the range $(0.1, 0.4)$) as thoroughly discussed in the previous chapter.

Remark 2. *Note that the linearization procedure adopted in* (5.31a) *and* (5.31b) *is not necessary if the gain that appears in* (3.4) *is fixed and not considered as a degree of freedom of the MPC algorithm. From one hand, this choice simplifies the online usage of the algorithm and reduces the conservatism of the constraint approximation, while on the other hand it doesn't allow to explicitly control the variances of the state and the input, that depend only on their initial values, thus representing a sort of trade-off in the design phase.*

3.2 MPC algorithm: formulation and properties

To formally state the MPC algorithm for the computation of the regulator parameters $\bar{u}_t$, and K_t, the following notation will be adopted: given a variable z or a matrix Z, at any time step t we will denote by z_{t+k} and Z_{t+k}, $k \geq 0$, their generic values in the future, while $z_{t+k|t}$ and $Z_{t+k|t}$ will represent their specific values computed based on the knowledge (e.g., measurements) available at time t. According to the standard procedure of MPC, at any time instant t a future prediction horizon of length N is considered and a suitable optimization problem is solved. The main ingredients of the optimization problem are now introduced.

3.2.1 Cost function

Assume to be at time t and denote by $\bar{u}_{t,\ldots,t+N-1} = \{\bar{u}_t, \ldots, \bar{u}_{t+N-1}\}$ the nominal input sequence over a future prediction horizon of length N. Moreover, define by $K_{t,\ldots,t+N-1} = \{K_t, \ldots, K_{t+N-1}\}$ the sequence of the future control gains, and recall that the covariance $X_{t+k} = \mathbb{E}\left\{\delta x_{t+k} \delta x_{t+k}^T\right\}$ evolves, starting from X_t, according to (3.8).

In line with the discussion in Section 2.2.1, the cost function we consider here is the following

$$J = \mathbb{E}\{ \sum_{i=t}^{t+N-1} (\|x_i\|_Q^2 + \|u_i\|_R^2) + \|x_{t+N}\|_S^2 \} \tag{3.14}$$

where Q and R are positive definite, symmetric matrices of appropriate size, S is the solution of the algebraic Lyapunov equation

$$(A + B_u\bar{K})^T S (A + B_u\bar{K}) - S = -Q - \bar{K}^T R \bar{K} \tag{3.15}$$

and $\bar{K}$ can be any stabilizing gain for the error system (3.7). Without loss of generality we choose here to set $\bar{K}$ as the solution of an LQ control problem, with state and control weights Q, R, for the nominal model (3.5).

To exploit the structure of the cost function, define, in general, the expected value along the horizon, $\bar{x}_{t+k} = \mathbb{E}\{x_{t+k}\}$, which are predictions computed according to

$$\bar{x}_{t+k+1} = A\bar{x}_{t+k} + B_u\bar{u}_{t+k} \tag{3.16a}$$

Also, let $u_{t+k} = \bar{u}_{t+k} + K_k(x_{t+k} - \bar{x}_{t+k})$, and $X_{t+k} = \mathbb{E}\{(x_{t+k} - \bar{x}_{t+k})(x_{t+k} - \bar{x}_{t+k}^T)\}$, which evolves according to

$$X_{t+k+1} = (A + B_u K_{t+k}) X_{t+k} (A + B_u K_{t+k})^T + B_w W B_w^T \tag{3.16b}$$

Accordingly, as described in Section 2.2.1, the cost function (3.14) can be represented as a sum of two components accounting for the expected value and the variance of the state variable, respectively

$$J = J_m(\bar{x}_t, \bar{u}_{t \dots t+N-1}) + J_v(X_t, K_{t \dots t+N-1}) \tag{3.17}$$

where

$$J_m = \sum_{i=t}^{t+N-1} (\|\bar{x}_i\|_Q^2 + \|\bar{u}_i\|_R^2) + \|\bar{x}_{t+N}\|_S^2 \tag{3.18}$$

$$J_v = \sum_{i=t}^{t+N-1} tr\{(Q + K_i^T R K_i) X_i\} + tr\{S X_{t+N}\} \tag{3.19}$$

Note that J_m depends on the nominal inputs $\bar{u}_{t \dots t+N-1} = \{\bar{u}_t, \dots, \bar{u}_{t+N-1}\}$ and on the mean value initial condition $\bar{x}_t$. On the other hand, J_v depends on the gains $K_{t \dots t+N-1} = \{K_t, \dots, K_{t+N-1}\}$ and on the initial covariance X_t. Therefore, in the control law (3.4) $\bar{u}$ can be used to drive the mean value of the state, while K can be selected to reduce its variance as much as possible. Furthermore, the pair $(\bar{x}_t, X_t)$ will be also accounted for as an argument of the MPC optimization, as later discussed.

3.2.2 Terminal constraints

As usual in MPC with guaranteed stability, see e.g. [100], also in the algorithm proposed here some constraints must be imposed at the end of the prediction horizon for both the mean value $\bar{x}_{t+N}$ and the variance X_{t+N}. Specifically, the terminal constraints we consider are the following

$$\bar{x}_{t+N} \in \bar{\mathbb{X}}_f \tag{3.20}$$

$$X_{t+N} \leq \bar{X} \tag{3.21}$$

In (3.20), the set $\bar{\mathbb{X}}_f$ is a classic positively invariant set (see [73]) for the system under the auxiliary LQ control law, satisfying

$$(A + B_u \bar{K})\bar{x} \in \bar{\mathbb{X}}_f \quad \forall x \in \bar{\mathbb{X}}_f \tag{3.22}$$

As for the terminal condition on the variance, $\bar{X}$, we can obtain something with the same invariant property by deriving the steady state solution of the

Lyapunov equation (3.8) computed by considering a noise variance $\bar{W} \geq W$ and assuming the constant gain $\bar{K}$, i.e.

$$\bar{X} = (A + B_u\bar{K})\bar{X}(A + B_u\bar{K})^T + B_w\bar{W}B_w^T \tag{3.23}$$

Indeed, recalling equation (3.8) and using the constant gain $\bar{K}$, from (3.21) we can write

$$\begin{aligned} X_{t+N+1} &= (A + B_u\bar{K})X_{t+N}(A + B_u\bar{K})^T + B_wWB_w^T \\ &\preceq (A + B_u\bar{K})\bar{X}(A + B_u\bar{K})^T + B_w\bar{W}B_w^T \\ &\preceq \bar{X} \end{aligned} \tag{3.24}$$

so that $\bar{X}$ plays the role of an invariant set for the variance.

Together with the previous definitions, we need to ensure that inside the terminal set the probabilistic requirements on the state and input are met and thus the following conditions, involving both $\bar{\mathbb{X}}_f$ and $\bar{X}$ at the same time, must hold

$$b^T\bar{x} \leq (1 - 0.5\varepsilon)x^{max} - \frac{f(p^x)^2}{2\varepsilon x^{max}}\, b^T\bar{X}b \tag{3.25a}$$

$$c^T\bar{K}\bar{x} \leq (1 - 0.5\varepsilon)u^{max} - \frac{f(p^u)}{2\varepsilon u^{max}}\, c^T\bar{U}c \tag{3.25b}$$

for all $\bar{x} \in \bar{\mathbb{X}}_f$, where we denoted the steady state input covariance matrix as $\bar{U} = \bar{K}\bar{X}\bar{K}^T$.

Remark 3. *In the computation of the terminal conditions through* (3.25a) *and* (3.25b)*, we used the linearized version of the deterministic constraints obtained from the original probabilistic ones. Note that if we are able to use the nonlinear form that appears in* (3.9) *and* (3.12) *the previous expression can be slightly modified to obtain a less tight approximation of the original constraint. This is useful when the gain sequence $K_{t+k|t}$ is fixed and not considered as a degree of freedom as already discussed in Remark 2.*

3.2.3 Initial conditions for the mean and the covariance

The proposed stochastic MPC problem aims at minimizing the cost function (3.14) also with respect to the initial conditions $(\bar{x}_t, X_t)$. Specifically, the selected "optimal" values of $\bar{x}_t$ and X_t are $\bar{x}_{t|t}$ and $X_{t|t}$, respectively. In order to use the most recent information available on the state, at any time instant it would be natural to set the current value of the nominal state $\bar{x}_{t|t}$ to x_t (i.e., letting $\bar{x}_{t|t}$ be the optimal conditional expectation value, using the *a posteriori* data x_t), corresponding to setting $X_{t|t}$ to zero. However, since we do not exclude the possibility of unbounded disturbances, this choice would lead in such cases to an infeasible optimization problem, and the fundamental property of recursive feasibility would be lost. On the other hand, and in view of the terminal constraints (3.20), (3.21), it is quite simple to see that recursive feasibility is guaranteed provided that the considered mean is updated according to the prediction equation (3.5), which corresponds to a variance update given by update (3.8). These considerations lead to define the following two alternative strategies.

Strategy 1 - Reset of the initial state: in the MPC optimization problem set $\bar{x}_{t|t} = x_t$, $X_{t|t} = 0$. This corresponds to using all the information available at time t from the measures.
Strategy 2 - Prediction: in the MPC optimization problem set $\bar{x}_{t|t} = \bar{x}_{t|t-1}$, $X_{t|t} = X_{t|t-1}$,i.e., use the nominal prediction from the past optimal solution

$$\bar{x}_{t|t-1} = A\bar{x}_{t-1|t-1} + B\bar{u}_{t-1|t-1}$$

and

$$X_{t|t-1} = (A + B_u K_{t-1|t-1})X_{t-1|t-1}(A + B_u K_{t-1|t-1})^T + B_w W B_w^T$$

In conclusion, also the initial conditions $(\bar{x}_t, X_t)$ are optimization parameters which must be chosen according to either Strategy 1 or Strategy 2, i.e.

$$(\bar{x}_{t|t}, X_{t|t}) \in \{(x_t, 0), (\bar{x}_{t|t-1}, X_{t|t-1})\} \tag{3.26}$$

3.2.4 Statement of the state feedback p-SMPC problem

The probabilistic state-feedback SMPC (p-SMPC) problem can now be stated as follows.

p-SMPC problem: *at any time instant t minimize, with respect to the sequences $\bar{u}_{t\ldots t+N-1}$, $K_{t\ldots t+N-1}$, and to the pair $(\bar{x}_t, X_t)$, the performance index* (3.17) *subject to the dynamics* (3.16a) *and* (3.16b)*, to the linear deterministic reformulations* (3.10)*,* (3.13) *of the probabilistic constraints* (3.2)*,* (3.3) *for all $k = 0, \ldots, N-1$, to the initial constraint* (3.26)*, and to the* *terminal constraints* (3.20), (3.21).

In particular, the problem we solve at time t is the following

$$
\begin{aligned}
&\min_{\substack{\bar{u}_{t\dots t+N-1},\, K_{t\dots t+N-1} \\ (\bar{x}_t, X_t)}} \mathbb{E}[\sum_{k=0}^{N-1}(\|x_{t+k}\|_Q^2 + \|u_{t+k}\|_R^2) + \|x_{t+N}\|_S^2] \\
&s.t. \\
&\quad \bar{x}_{t+k+1} = A\bar{x}_{t+k} + B\bar{u}_{t+k} \\
&\quad X_{t+k+1} = (A + BK_{t+k})X_{t+k} + (A + BK_{t+k})^T + B_w W B_w^T \\
&\quad b_r^T \bar{x}_{t+k} \leq (1 - 0.5\varepsilon)x^{max} - \frac{f(p^x)^2}{2\varepsilon x^{max}} b_r^T X_{t+k} b_r,\ \forall r, k \\
&\quad c_s^T \bar{u}_{t+k} \leq (1 - 0.5\varepsilon)u^{max} - \frac{f(p^u)^2}{2\varepsilon u^{max}} c_s^T K_{t+k} X_{t+k} K_{t+k}^T c_s,\ \forall s, k \\
&\quad \bar{x}_{t+N} \in \bar{\mathbb{X}}_f \\
&\quad X_{t+N} \preceq \bar{X} \\
&\quad (\bar{x}_t, X_t) \in \{(x_t, 0), (\bar{x}_{t|t-1}, X_{t|t-1})\}
\end{aligned}
\tag{3.27}
$$

Denoting by $\{\bar{u}_{t|t}, \dots, \bar{u}_{t+N-1|t}\}$, $\{K_{t|t}, \dots, K_{t+N-1|t}\}$, and $(\bar{x}_{t|t}, X_{t|t})$ the optimal solution of the p-SMPC problem, and according to the receding horizon principle, the feedback control law actually used is then given by (3.4) as

$$
u_t = \bar{u}_{t|t} + K_{t|t}(x_t - \bar{x}_{t|t}) \tag{3.28}
$$

where obviously, when the reset strategy is used, the feedback term vanishes.

In order to state the properties of the proposed algorithm, we define the p-SMPC problem feasibility set as the set

$$
\Xi := \left\{ (\bar{x}_0, X_0) : \begin{array}{l} \exists \bar{u}_{0,\dots,N-1}, K_{0,\dots,N-1} \text{ such that (3.10), (3.13) hold for} \\ \text{all } k = 0, \dots, N-1 \text{ and (3.20), (3.21) are verified} \end{array} \right\}
$$

Note that, in view of the compactness of $\mathbb{X}$, see (3.2), the set Ξ results to be compact. The recursive feasibility and convergence properties of the resulting control system are discussed in detail in the sequel. However, some preliminary comments are in order.

- At the initial time $t = 0$, the algorithm must be initialized by setting $\bar{x}_{0|0} = x_0$ and $X_{0|0} = 0$. In view of this, feasibility at time $t = 0$ amounts to $(x_0, 0) \in \Xi$.
- According to the problem statement, feasibility of p-SMPC at time $t > 0$ amounts to $\{(x_t, 0), (\bar{x}_{t|t-1}, X_{t|t-1})\} \bigcap \Xi \neq \emptyset$.
- The binary choice between Strategies 1 and 2 for the initialization of $\bar{x}_{t|t}$, $X_{t|t}$, see constraint (3.26), requires to solve at any time instant two optimization problems. However, notice that in Strategy 1 (reset of the initial nominal state and covariance), the on-line minimization of the term J_v in the cost function J is not needed. In fact, it is possible to compute off-line the optimal sequence of control gains $\{K_{0|0}, \dots, K_{N-1|0}\}$ minimizing J_v with $X_0 = 0$ and to use them in on-line operations. It follows that the corresponding MPC problem is a pretty standard one, where only J_m is to be minimized with respect to the sequence $\{\bar{u}_t, \dots, \bar{u}_{t+N-1}\}$.
- A sequential procedure can be adopted to reduce the average overall computational burden. The optimization problem corresponding to Strategy 1 is first solved, then, if it is infeasible, Strategy 2 must be used. On the contrary, if it is feasible, it is possible to compare the resulting value of the optimal cost function with the value of the cost that corresponds to the use of the sequences $\{\bar{u}_{t|t-1}, \dots, \bar{u}_{t+N-2|t-1}, \bar{K}\bar{x}_{t+N-1|t}\}$, $\{K_{t|t-1}, \dots, K_{t+N-2|t-1}, \bar{K}\}$. If the optimal cost with Strategy 1 is lower, Strategy 1 can be used without solving the MPC problem for Strategy 2. This does not guarantee optimality, but the convergence properties of the method stated in the result below are recovered and the computational effort is reduced.

Now we are in the position to state the main result concerning the convergence properties of the algorithm.

Theorem 1. *Assume that there exists $\rho \in (0, 1)$ such that the noise variance W verifies*

$$\frac{(N + \frac{\lambda_{max}(S)}{\lambda_{min}(Q)})}{\lambda_{min}(Q)} tr(SB_wWB_w^T) < \min(\rho\bar{\sigma}^2, \rho\lambda_{min}(\bar{X})) \tag{3.29}$$

where $\bar{\sigma}$ is the maximum radius of a ball, centered at the origin, included in $\bar{\mathbb{X}}_F$. Then, if at time $t = 0$ the S-MPC problem admits a solution, the optimization problem is recursively feasible and, as $t \to +\infty$,

$$\mathbb{E}\{\|x_t\|^2\} \to \frac{1}{\rho\lambda_{min}(Q)}(N + \frac{\lambda_{max}(S)}{\lambda_{min}(Q)})\, tr(SB_wWB_w^T)$$

while satisfying the state and input probabilistic constraints (3.3) *and* (3.2) *for all $t \geq 0$.*

3.3 Implementation issues

In this section the proposed algorithm is reformulated into a SDP problem to obtain an efficient solution procedure, based on specific solvers, that guarantees a low computational cost.

To this end, note that both the expected value and the variance of the state and input along the prediction horizon $t, \dots, t+N$ depend on the optimization variables $\bar{u}_{t \dots t+N-1|t}$ and $K_{t \dots t+N-1|t}$. In particular, the expected value is related only to the nominal input $\bar{u}$ and this dependence is linear and governed by equation (3.5). On the other hand, the variance of the state evolves as in equation (3.8) and therefore shows a nonlinear dependence with respect to the gain K.
To transform equation (3.8) into a linear one it is possible to resort to Schur complements, following, for example, the ideas in [96, 130]. Recall first Schur's lemma for matrices (see for example Appendix 5.5 of [19])

Lemma 2. Schur complement. *Suppose that D is a non singular matrix and A, B, C are matrices of proper dimensions, than the following two expressions are equivalent:*

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \succeq 0, \qquad A - BD^{-1}C \succeq 0$$

in which $\succeq 0$ stands for the positive semidefiniteness of the matrix equation.

Now consider the evolution of the state variance (3.8) and define a new set of optimization variables, G_k, that are computed at each time step as

$G_k = K_k X_k,\ \forall k = t, \ldots, t+N-1$. Then we can write

$$\begin{aligned} X_{k+1} &= (A + B_u K_k) X_k (A + B_u K_k)^T + B_w W B_w^T \\ &= (AX_k + B_u G_k) X_k^{-1} (AX_k + B_u G_k)^T + B_w W W^{-1} (B_w W)^T \end{aligned}$$

which can be reformulated as

$$X_{k+1} - \begin{bmatrix} (AX_k + B_u G_k) & B_w W \end{bmatrix} \begin{bmatrix} X_k & 0 \\ 0 & W \end{bmatrix} \begin{bmatrix} (AX_k + B_u G_k)^T \\ (B_w W)^T \end{bmatrix} = 0$$

Finally, recalling that we are minimizing the variance through the term J_v, it is possible to relax the equality into an inequality and use the Schur's lemma to obtain the LMI reformulation of the constraint

$$\begin{bmatrix} X_{i+1} & (AX_i + B_u G_i) & (B_w W) \\ (AX_i + B_u G_i)^T & X_i & 0 \\ (B_w W)^T & 0 & W \end{bmatrix} \succeq 0 \tag{3.30}$$

that can take the place of (3.8) in the optimization problem. Notice that in (3.30) both X_{k+1} and X_k are considered as optimization variables at each step of the prediction horizon while the gain K_k is replaced by G_k.

With the same procedure we need to rewrite the covariance matrix U_k of the input variable to replace the gain K_k. In particular we obtain

$$U_k = K_k^T X_k K_k = G_k^T X_k^{-1} G_k$$

and again the previous expression can be written as the following LMI

$$\begin{bmatrix} U_k & G_k^T \\ G_k & X_k \end{bmatrix} \succeq 0 \tag{3.31}$$

in which the sequence of $U_k,\ \forall k = t, \ldots, t+N-1$ must be considered as extra optimization variables for the problem.

The cost function can be treated as follows. Starting directly from (3.14), it can be written as

$$
\begin{aligned}
J(\cdot) &= \mathbb{E}\{\sum_{i=t}^{t+N-1}(\|x_i\|_Q^2 + \|u_i\|_R^2) + \|x_{t+N}\|_S^2\} \\
&= \mathbb{E}\{\sum_{i=t}^{t+N-1}\| \begin{bmatrix} x_i \\ u_i \end{bmatrix} \|_M^2 + \|x_{t+N}\|_S^2\} \\
&= \sum_{i=t}^{t+N-1}\mathbb{E}\{\begin{bmatrix} x_i \\ u_i \end{bmatrix}^T M \begin{bmatrix} x_i \\ u_i \end{bmatrix}\} + \mathbb{E}\{\|x_{t+N}\|_S^2\} \\
&= \sum_{i=t}^{t+N-1} tr\{M\mathbb{E}\{\begin{bmatrix} x_i \\ u_i \end{bmatrix}\begin{bmatrix} x_i \\ u_i \end{bmatrix}^T\}\} + tr\{S\mathbb{E}\{x_{t+N}x_{t+N}^T\}\}
\end{aligned}
$$

where $M = diag(Q, R)$. Now defining a new optimization variable $P_i \in \mathbb{R}^{n+m\times n+m}$ as

$$
P_i = \mathbb{E}\{\begin{bmatrix} x_i \\ u_i \end{bmatrix}\begin{bmatrix} x_i \\ u_i \end{bmatrix}^T\} \tag{3.32}
$$

and a partition P_i^x of the first n rows and n columns, the cost function becomes

$$
J(\cdot) = \sum_{i=t}^{t+N-1} tr\{MP_i\} + tr\{SP_{t+N}^x\} \tag{3.33}
$$

that is linear in the sequence of new optimization variables $P_{t,\dots,t+N}$.
Taking the expression in (3.32) it is possible to compute the following expression

$$
\begin{aligned}
P(i) &= \mathbb{E}\{\begin{bmatrix} \bar{x}_i + \delta x_i \\ \bar{u}_i + K_i\delta x_i \end{bmatrix}\begin{bmatrix} \bar{x}_i + \delta x_i \\ \bar{u}_i + K_i\delta x_i \end{bmatrix}^T\} \\
&= \mathbb{E}\{(\begin{bmatrix} \bar{x}_i \\ \bar{u}_i \end{bmatrix} + \begin{bmatrix} I \\ K_i \end{bmatrix}\delta x_i)(\begin{bmatrix} \bar{x}_i \\ \bar{u}_i \end{bmatrix} + \begin{bmatrix} I \\ K_i \end{bmatrix}\delta x_i)^T\} \\
&= \mathbb{E}\{\begin{bmatrix} \bar{x}_i \\ \bar{u}_i \end{bmatrix}\begin{bmatrix} \bar{x}_i \\ \bar{u}_i \end{bmatrix}^T\} + \begin{bmatrix} X_i \\ G_i \end{bmatrix} X_i^{-1}\mathbb{E}\{\delta x_i \delta x_i^T\}X_i^{-1}\begin{bmatrix} X_i \\ G_i \end{bmatrix}^T \\
&= \begin{bmatrix} \bar{x}_i \\ \bar{u}_i \end{bmatrix}\begin{bmatrix} \bar{x}_i \\ \bar{u}_i \end{bmatrix}^T + \begin{bmatrix} X_i \\ G_i \end{bmatrix} X_i^{-1}\begin{bmatrix} X_i \\ G_i \end{bmatrix}^T
\end{aligned}
$$

rearranging the above terms the definition of P_i can be imposed through the following LMI

$$\begin{bmatrix} P_i & \begin{bmatrix} X_i \\ G_i \end{bmatrix} & \begin{bmatrix} \bar{x}_i \\ \bar{u}_i \end{bmatrix} \\ \begin{bmatrix} X_i & G_i^T \end{bmatrix} & X_i & 0 \\ \begin{bmatrix} \bar{x}_i^T & \bar{u}_i^T \end{bmatrix} & 0 & 1 \end{bmatrix} \succeq 0 \tag{3.34}$$

that must be added to the problem to make use of the linear cost function (3.33). Note that this constraint holds along the prediction horizon, for $i = t+1, \ldots, t+N-1$, while particular attention must be given to the initial step, $i = t$, and to the last step, $i = t+N$. If at time t we have $X_t = 0$, since we are using all the available information on the state, (3.34) becomes

$$\begin{bmatrix} P_i & \begin{bmatrix} \bar{x}_i \\ \bar{u}_i \end{bmatrix} \\ \begin{bmatrix} \bar{x}_i^T & \bar{u}_i^T \end{bmatrix} & 1 \end{bmatrix} \succeq 0 \tag{3.35}$$

while if one makes use of the predicted state of the system, as described by Strategy II, and hence $X_t \succ 0$, it is possible to apply (3.34) also for $i = t$. For $t = N$ in both the cases it results

$$\begin{bmatrix} P_{t+N} - X_{t+N} & \bar{x}_{t+N} \\ \bar{x}_{t+N}^T & 1 \end{bmatrix} \succeq 0 \tag{3.36}$$

3.4 Proof of the main Theorem

3.4.1 Proof of recursive feasibility

Assume that, at time instant t, a feasible solution of p-SMPC is available, i.e., $(\bar{x}_{t|t}, X_{t|t}) \in \Xi$ with optimal sequences $\{\bar{u}_{t|t}, \ldots, \bar{u}_{t+N-1|t}\}$ and $\{K_{t|t}, \ldots, K_{t+N-1|t}\}$.
We prove that, at time $t+1$, at least a feasible solution to p-SMPC exists, i.e., $(\bar{x}_{t+1|t}, X_{t+1|t}) \in \Xi$ with feasible, possibly suboptimal, sequences $\{\bar{u}_{t+1|t}, \ldots, \bar{u}_{t+N-1|t}, \bar{K}\bar{x}_{t+N|t}\}$ and $\{K_{t+1|t}, \ldots, K_{t+N-1|t}, \bar{K}\}$.

Constraint (3.10) is verified for all pairs $(\bar{x}_{t+1+k|t}, X_{t+1+k|t})$, $k = 0, \ldots, N-2$, in view of the feasibility of the p-SMPC at time t. Furthermore, in view of (3.20), (3.21), and the condition (3.25a), we have that

$$\begin{aligned} b_r^T \bar{x}_{t+N|t} &\leq (1-0.5\varepsilon)x_r^{max} - \frac{f(p_r^x)^2}{2\varepsilon x^{max}}\, b_r^T \bar{X} b_r \\ &\leq (1-0.5\varepsilon)x_r^{max} - \frac{f(p_r^x)^2}{2\varepsilon x_r^{max}}\, b_r^T X_{t+N|t} b_r \end{aligned}$$

for all $r = 1, \ldots, n_r$, i.e., constraint (3.10) is verified.
Analogously, constraint (3.13) is verified for all pairs $(\bar{u}_{t+1+k|t}, U_{t+1+k|t})$, $k = 0, \ldots, N-2$, in view of the feasibility of S-MPC at time t. Furthermore, in view of (3.20), (3.21), and the condition (3.25b), we have that

$$\begin{aligned} c_s^T \bar{K} \bar{x}_{t+N|t} &\leq (1-0.5\varepsilon)u_s^{max} - \frac{f(p_s^u)^2}{2\varepsilon u_s^{max}}\, c_s^T \bar{U} c_s \\ &\leq (1-0.5\varepsilon)u_s^{max} - \frac{f(p_s^u)}{2\varepsilon u_s^{max}}\, c_s^T U_{t+N|t} c_s \end{aligned}$$

for all $s = 1, \ldots, n_s$ i.e., constraint (3.13) is verified.
In view of (3.20) and of the invariance property (3.22) it follows that $\bar{x}_{t+N+1|t} = (A + B\bar{K})\bar{x}_{t+N|t} \in \bar{\mathbb{X}}_f$ and, in view of (3.21)

$$\begin{aligned} X_{t+N+1|t} &= (A + B_u\bar{K})X_{t+N|t}(A + B_u\bar{K})^T + B_w W B_w^T \\ &\preceq (A + B_u\bar{K})\bar{X}(A + B_u\bar{K})^T + B_w \bar{W} B_w^T = \bar{X} \end{aligned}$$

hence verifying both (3.20) and (3.21) at time $t+1$.

3.4.2 Proof of convergence

In view of the feasibility, at time $t+1$ of the possibly suboptimal solution given by $\{\bar{u}_{t+1|t}, \ldots, \bar{u}_{t+N-1|t}, \bar{K}\bar{x}_{t+N|t}\}$, $\{K_{t+1|t}, \ldots, K_{t+N-1|t}, \bar{K}\}$, and $(\bar{x}_{t+1|t}, X_{t+1|t})$, we have that the optimal cost function computed at time $t+1$ is $J^*(t+1) = J_m^*(t+1) + J_v^*(t+1)$[1].
In view of the optimality of $J^*(t+1)$ we have

$$\begin{aligned} J^*(t+1) \leq\; & J_m(\bar{x}_{t+1|t}, u_{t+1|t}, \ldots, u_{t+N-1|t}, \bar{K}\bar{x}_{t+N|t}) \\ & + J_v(X_{t+1|t}, K_{t+1|t}, \ldots, K_{t+N-1|t}, \bar{K}) \end{aligned} \tag{3.37}$$

[1]For brevity, we denote $J^*(x_t, \bar{x}_{t|t-1}, X_{t|t-1})$ with $J^*(t)$, $J_m^*(x_t, \bar{x}_{t|t-1}, X_{t|t-1})$ with J_m^*(t), and $J_v^*(x_t, \bar{x}_{t|t-1}, X_{t|t-1})$ with $J_v^*(t)$

Note that, for the expected value component

$$J_m(\bar{x}_{t+1|t}, u_{t+1|t}, \ldots, u_{t+N-1|t}, \bar{K}\bar{x}_{t+N|t}) =$$
$$J_m(\bar{x}_{t|t}, u_{t|t}, \ldots, u_{t+N-1|t}) - \|\bar{x}_{t|t}\|_Q^2 - \|\bar{u}_{t|t}\|_R^2 +$$
$$\|\bar{x}_{t+N|t}\|_Q^2 + \|\bar{K}\bar{x}_{t+N|t}\|_R^2 - \|\bar{x}_{t+N|t}\|_S^2 +$$
$$\|(A + B_u\bar{K})\bar{x}_{t+N|t}\|_S^2$$

and in view of the definition of S, given by (3.15), we have also

$$\begin{aligned} &\|\bar{x}_{t+N|t}\|_Q^2 + \|\bar{K}\bar{x}_{t+N|t}\|_R^2 - \|\bar{x}_{t+N|t}\|_S^2 + \\ &\|(A + B_u\bar{K})\bar{x}_{t+N|t}\|_S^2 = 0 \end{aligned} \tag{3.38}$$

Furthermore, note that

$$J_m(\bar{x}_{t|t}, u_{t|t}, \ldots, u_{t+N-1|t}) = J_m^*(t) \tag{3.39}$$

and thus it is possible to write

$$\begin{aligned} &J_m(\bar{x}_{t+1|t}, u_{t+1|t}, \ldots, u_{t+N-1|t}, \bar{K}\bar{x}_{t+N|t}) \leq \\ &J_m^*(t) - \|\bar{x}_{t|t}\|_Q^2 - \|\bar{u}_{t|t}\|_R^2 \end{aligned} \tag{3.40}$$

Consider now the variance component, J_v. We compute that

$$\begin{aligned} &J_v(X_{t+1|t}, K_{t+1|t}, \ldots, K_{t+N-1|t}, \bar{K}) = \\ &J_v(X_{t|t}, K_{t|t}, \ldots, K_{t+N-1|t}) - tr\{(Q + K_{t|t}^T R K_{t|t})X_{t|t}\} \\ &+ tr\{(Q + \bar{K}^T R\bar{K})X_{t+N|t} + \\ &S(A + B_u\bar{K})X_{t+N|t}(A + B_u\bar{K})^T + SB_wWB_w^T - SX_{t+N|t}\} \end{aligned} \tag{3.41}$$

Recall the following properties of the trace: $tr(A + B) = tr(A) + tr(B)$, $tr(AB) = tr(BA)$, being A and B matrices of suitable dimensions. In view of this, and recalling (3.15)

$$\begin{aligned} &tr\{(Q + \bar{K}^T R\bar{K})X_{t+N|t} + S(A + B_u\bar{K})X_{t+N|t}(A + B_u\bar{K})^T\} = \\ &tr\{(Q + \bar{K}^T R\bar{K} + (A + B_u\bar{K})^T S(A + B_u\bar{K}))X_{t+N|t}\} = \\ &tr\{SX_{t+N|t}\} \end{aligned} \tag{3.42}$$

and therefore we have that

$$J_v^*(t + 1) \leq J_v^*(t) - tr\{(Q + K_{t|t}^T R K_{t|t})X_{t|t}\} + tr(B_wWB_w^T) \tag{3.43}$$

From (3.37)-(3.42) we obtain

$$\begin{aligned} J^*(t + 1) &\leq J^*(t) - \mathbb{E}\{\|x_t\|_Q^2 + \|u_t\|_R^2\} + tr(SB_wWB_w^T) \\ &\leq J^*(t) - \lambda_{min}(Q)\mathbb{E}\{\|x_t\|^2\} + tr(SB_wWB_w^T) \end{aligned} \tag{3.44}$$

where $\lambda_{min}(Q)$ denotes the smallest eigenvalue of the matrix Q.
Furthermore, from the definition of $J^*(t)$ we also have that

$$\begin{array}{rcl} J^*(t) & \geq & \mathbb{E}\{\|x_t\|_Q^2 + \|u_t\|_R^2\} \\ & \geq & \lambda_{min}(Q)\mathbb{E}\{\|x_t\|^2\} \end{array} \tag{3.45}$$

Define now the terminal set $\Omega_F = \{(\bar{x}, X) : \bar{x} \in \bar{\mathbb{X}}_F, X \preceq \bar{X}\}$. Assuming that $(\bar{x}_{t|t}, X_{t|t}) \in \Omega_F$ we have that $J^*(t) \leq J_m^{aux}(t) + J_v^{aux}(t)$, where

$$\begin{array}{rl} J_m^{aux}(t) & = \sum_{k=t}^{t+N-1} \|(A + B_u\bar{K})^{t-k}\bar{x}_{t|t}\|_Q^2 \\ & +\|\bar{K}(A + B_u\bar{K})^{t-k}\bar{x}_{t|t}\|_R^2 + \|(A + B_u\bar{K})^N\bar{x}_{t|t}\|_S^2 \end{array}$$

since $\{\bar{K}\bar{x}_{t|t}, \ldots, \bar{K}(A+B_u\bar{K})^{N-1}\bar{x}_{t|t}\}$ is a feasible input sequence. Therefore, recalling (3.15),

$$J_m^{aux}(t) = \|x_{t|t}\|_S^2 \tag{3.46}$$

Similarly, and recalling the properties of the trace and (3.15), we obtain that $J_v^{aux}(t)$ is equal to

$$\begin{array}{l} \sum_{k=0}^{N-1} tr\{(Q + \bar{K}^T R\bar{K})[(A + B_u\bar{K})^k X_{t|t}(A + B_u\bar{K})^{T(k)} \\ + \sum_{i=0}^{k-1}(A + B_u\bar{K})^i B_w W B_w^T(A + B_u\bar{K})^{T(i)}]\} \\ +tr\{S[(A + B_u\bar{K})^N X_{t|t}(A + B_u\bar{K})^{T(N)} \\ + \sum_{i=0}^{N-1}(A + B_u\bar{K})^i B_w W B_w^T(A + B_u\bar{K})^{T(i)}]\} \\ = tr\{[\sum_{k=0}^{N-1}(A + B_u\bar{K})^{T(k)}(Q + \bar{K}^T R\bar{K})(A + B_u\bar{K})^k \\ +(A + B_u\bar{K})^{T(N)}S(A + B_u\bar{K})^N]X_{t|t}\} \\ +tr\{[\sum_{k=1}^{N-1}\sum_{i=0}^{k-1}(A + B_u\bar{K})^{T(i)}(Q + \bar{K}^T R\bar{K})(A + B_u\bar{K})^i \\ + \sum_{i=0}^{N-1}(A + B_u\bar{K})^{T(i)}S(A + B_u\bar{K})^i]B_w W B_w^T\} \\ = tr\{SX_{t|t}\} + tr\{[S + \sum_{k=1}^{N-1}((A + B_u\bar{K})^{T(k)}S(A + B_u\bar{K})^k) \\ + \sum_{i=1}^{N-1}(A + B_u\bar{K})^{T(i)}(Q + \bar{K}^T R\bar{K})(A + B_u\bar{K})^i]B_w W B_w^T\} \\ = tr\{SX_{t|t}\} + tr\{[S + \sum_{i=1}^{N-1} S]B_w W B_w^T\} \end{array}$$

Therefore

$$J_v^{aux}(t) = tr\{SX_{t|t}\} + N tr\{SB_w W B_w^T\} \tag{3.47}$$

Combining (3.46) and (3.47) we obtain that

$$\begin{array}{rcl} J^*(t) & \leq & \mathbb{E}\{\|x_t\|_S^2\} + N\,tr\{SB_w W B_w^T\} \\ & \leq & \lambda_{max}(S)\mathbb{E}\{\|x_t\|^2\} + N\,tr\{SB_w W B_w^T\} \end{array} \tag{3.48}$$

Remark that assumptions (3.44), (3.45), and (3.48) are similar to the ones needed in the framework of input-to-state stability.

From (3.44), (3.45) and (3.48) it is possible to derive robust stability-related results. If $(\bar{x}_{t|t}, X_{t|t}) \in \Omega_F$ then, in view of (3.48), (3.44)

$$\begin{aligned} J^*(t+1) \leq & J^*(t)(1 - \frac{\lambda_{min}(Q)}{\lambda_{max}(S)}) \\ & + (\frac{\lambda_{min}(Q)}{\lambda_{max}(S)} N + 1) tr(SB_w W B_w^T) \end{aligned} \tag{3.49}$$

Denote $b = \frac{1}{\rho}(N + \frac{\lambda_{max}(S)}{\lambda_{min}(Q)})$. If $J^*(t) \leq b\, tr(SB_w W B_w^T)$, and provided that inequalities (3.29) are verified, then one can prove that $(\bar{x}_{t|t}, X_{t|t}) \in \mathcal{I}\Omega_F$, where $\mathcal{I}\Omega_F$ denotes the interior of Ω_F. In fact, $J^*(t) \leq b\, tr(SB_w W B_w^T)$ implies that, in view of (3.45)

$$\mathbb{E}\{\|x_t\|^2\} = \|\bar{x}_{t|t}\|^2 + tr(X_{t|t}) \leq \frac{b}{\lambda_{min}(Q)} tr(SB_w W B_w^T)$$

This, considering (3.29), implies that

$$\|\bar{x}_{t|t}\|^2 < \bar{\sigma}^2 \tag{3.50a}$$

$$tr(X_{t|t}) < \lambda_{min}(\bar{X}) \tag{3.50b}$$

In view of (3.50a), then $\bar{x}_{t|t} \in \bar{\mathbb{X}}_F$. Furthermore, (3.50b) implies that $\lambda_{max}(X_{t|t}) < \lambda_{min}(\bar{X})$, which in turn implies that $X_{t|t} \preceq \bar{X}$. Therefore, recalling (3.49), if $J^*(t) \leq b\, tr(SB_w W B_w^T)$, then $J^*(t+1) \leq b\, tr(SB_w W B_w^T)$. and the positive invariance of the set

$$D = \{(\bar{x}, X) : J^*(t) \leq b\, tr(SB_w W B_w^T)\} \tag{3.51}$$

is guaranteed. From this point on, the proof follows similarly to [95, 136].

For $(\bar{x}_{t|t}, X_{t|t}) \in \Omega_F \backslash D$, it holds that

$$J^*(t) > b\, tr(SB_w W B_w^T) \tag{3.52}$$

which, in view of (3.48), implies that

$$\mathbb{E}\{\|x_t\|^2\} > \frac{1}{\lambda_{min}(Q)} tr\{SB_w W B_w^T\}$$

Therefore, considering (3.44)

$$J^*(t+1) - J^*(t) < 0 \tag{3.53}$$

On the other hand, there exists constant $\bar{c} > 0$ such that, for all x_t with $(\bar{x}_{t|t}, X_{t|t}) \in \Xi \backslash \Omega_F$ there exists x_Ω with $(\bar{x}_\Omega, X_\Omega) \in \Omega_F \backslash D$ such that

$$-\lambda_{min}(Q)\mathbb{E}\{\|x_t\|^2\} \leq -\lambda_{min}(Q)\mathbb{E}\{\|x_\Omega\|^2\} - \bar{c}$$

This, in view of (3.44) and (3.53), implies that, for all x_t with $(\bar{x}_{t|t}, X_{t|t}) \in \Xi \backslash \Omega_F$

$$J^*(t+1) - J^*(t) < -\bar{c} \tag{3.54}$$

and, in turn, this implies that there exists a time instant $T_1 > 0$ such that x_{t+T_1} is such that $(\bar{x}_{t+T_1|t+T_1}, X_{t+T_1|t+T_1}) \in \Omega_F$.
If, on the one hand $(\bar{x}_{t+T_1|t+T_1}, X_{t+T_1|t+T_1}) \in D$, in view of the positive invariance of D, $(\bar{x}_{t+k|t+k}, X_{t+k|t+k}) \in D$ for all $k \geq T_1$. If, on the other hand, $(\bar{x}_{t+T_1|t+T_1}, X_{t+T_1|t+T_1}) \in \Omega_F \backslash D$, recalling (3.49), (3.52), and (3.45)

$$\begin{aligned} J^*(t+T_1+1) - & J^*(t+T_1) \\ & \leq -(1-\rho)\frac{\lambda_{min}(Q)}{\lambda_{max}(S)} J^*(t+T_1) \\ & \leq -(1-\rho)\frac{\lambda_{min}(Q)^2}{\lambda_{max}(S)} \mathbb{E}\{\|x_{t+T_1}\|^2\} \end{aligned} \tag{3.55}$$

In view of (3.54)-(3.55) there exists $T_2 \geq T_1$ such that, for all $\varepsilon > 0$

$$J(t+k) \leq \varepsilon + b\, tr(SB_wWB_w^T)$$

for all $k \geq T_2$, which proves that $\mathbb{E}\{\|x_t\|^2\} \to \frac{1}{\lambda_{min}(Q)} b\, tr(SB_wWB_w^T)$ asymptotically, as $t \to +\infty$.

3.5 Simulation example

In this section, a very simple simulation example is proposed to show the efficacy of the p-SMPC scheme presented in this chapter. In particular, the example we consider is inspired by the one that appears in [117] where the system is a double integrator representing a point-mass moving in a two dimensional space subject to uncertainty in the position. The control goal is to drive the state of the system to the origin while remaining with a desired probability inside a certain area. The system can be described as in (3.1) with matrices

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} B = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} F = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and the disturbance w_k is a white noise with zero-mean, covariance matrix $W = I_2$ and unknown distribution. The constrained area is a rectangle with vertices $(-5, -3.5)$, $(5, -3.5)$, $(5, 3.5)$ and $(-5, 3.5)$, while the control action is subject to the constraint $\|u_k\|_\infty \leq 1$.
In the cost function (3.14), the weighting matrices have been chosen as $Q = 10^{-3} I_4$, $R = I_2$. The gain $\bar{K}$ has been obtained as the solution of the corresponding LQ control problem and has been used in (3.23) to compute the steady state covariance matrix $\bar{X}$ with $\bar{W} = W$. Setting $\varepsilon = 0.2$ and $p^u = 0.2$ the results in Figure 3.1 have been obtained for different values of the violation p^x applied to each of the constraints.

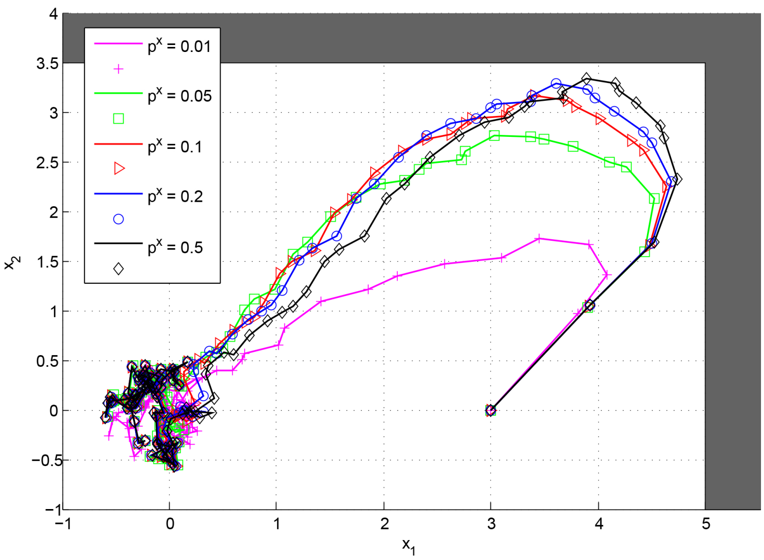


Figure 3.1: *Simulation results of the proposed S-MPC with different violation probabilities.*

From these results, it is apparent that the algorithm successfully brings the uncertain state to a neighborhood of the origin without violating the constraints on the position for each selected value of p^x. As expected, higher

probability of violation corresponds to a less conservative algorithm and to higher values of the cost function associated with the real state.
Focusing on a single simulation with $p^x = 0.1, p^u = 0.2, \varepsilon = 0.2$, it is possible to see in Figure3.2 how the reset strategy works. Clearly, Strategy 2 (Prediction) is more frequently used when the state approaches the boundary region.

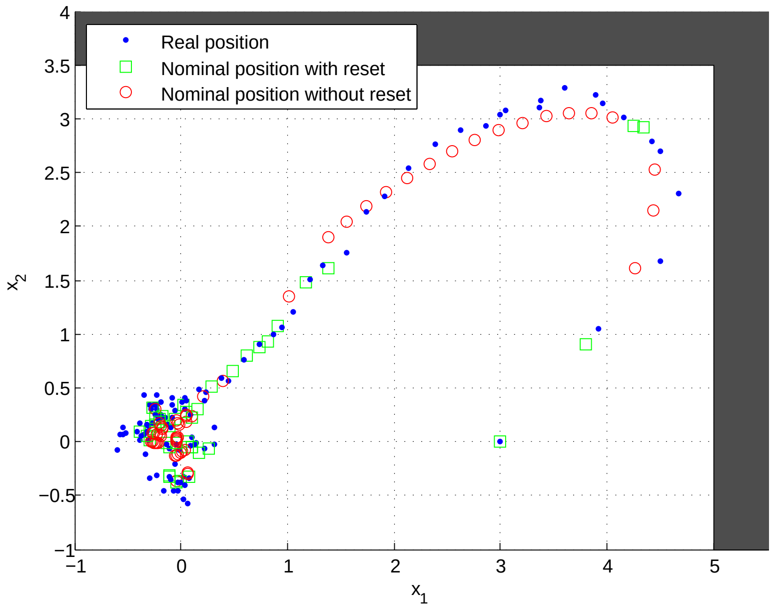


Figure 3.2: *Simulation results with $p^x = 0.1$ and $p^u = 0.2$.*

Setting $p^u = 0.2$ and $\varepsilon = 0.2$ and for different values of p^x, $T = 100$ simulations are made to verify the theoretical probability of constraint violation presented and the results are summarized in Table.3.1.

	$\mathbf{p^x = 0.01}$	$\mathbf{p^x = 0.2}$	$\mathbf{p^x = 0.5}$
Measured violation	0	0.034	0.097

Table 3.1: *Constraint violation in* 100 *simulations.*

Due to the use of Cantelli's inequality to tighten the probabilistic constraints and the linearization procedure applied to their deterministic version, the algorithm results to be very conservative in terms of actual violation. More detailed examples will be presented later to compare some of the more interesting techniques in the SMPC literature and particular attention will be devoted to the results in terms of violation probability of the constraints.

3.6 Comments

In this chapter we proposed a novel algorithm, denoted as p-SMPC, for stochastic Model Predictive Control of discrete-time linear system with additive uncertainty under a set of individual chance constraints. In line with the analysis presented in Chapter 2, the p-SMPC approach is classified as an analytic method, in which the probabilistic constraints are firstly reformulated as deterministic, although tighter, ones and then implemented into a quite standard MPC setup. With respect to similar algorithms already discussed in the literature, in the p-SMPC technique proper terminal constraints, together with a strategy for choosing the initial condition of the optimization problem, allow to obtain recursive feasibility and convergence properties even in the case of unbounded uncertainties. The main advantages and disadvantages of the proposed algorithm, are now summarized. In particular, the p-SMPC algorithm is very useful due to

1) its ability to deal with possibly unbounded disturbances acting on the system,
2) the guaranteed recursive feasibility and convergence properties both in the case of bounded and unbounded uncertainties,
3) a reduced online computational load, similar to the one of standard nominal MPC algorithms,
4) a low design complexity.

However, some drawbacks are present, mainly related to

1) the need for probabilistic constraints on the input (due to the choice of a feedback control law in presence of unbounded disturbances, we are not able to enforce them in a worst-case fashion),
2) the ability to cope only with additive disturbances
3) the quite conservative Cantelli-Chebyschev bound in the case of unknown uncertainties distribution
4) the choice of individual chance constraints (in case the original problem is specified with joint chance constraints we need to reformulate them by the use, for example, of offline risk allocation techniques).

In the next chapter, various examples of the application of the proposed approach in comparison with other notable stochastic MPC techniques are presented. The aim is to further prove the efficacy of the p-SMPC algorithm and to show the main difficulties that arise in the implementation of such SMPC schemes.

CHAPTER 4

Application examples

In this chapter we present some application examples of four stochastic MPC algorithms for linear systems with additive disturbances and measurable state. Three of these techniques are selected from the literature for the interesting idea on which they are based and the last one is the probabilistic SMPC (p-SMPC) presented in the previous chapter.
After a quick overview on the selected algorithms, that has the only goal of showing the main rationale behind them and giving a sketch on the implementation issues, several examples are presented to show the efficacy of these techniques and to make comparisons in terms of probabilistic guarantees, design complexity and online computational load. In particular the following examples are used. Firstly a toy system is presented, in which the effect of the probabilistic constraint at a fixed time step is analyzed in terms of the actual violation over multiple simulations. Then a simple multi-room model is used to show the efficacy of the selected techniques in terms of reaching the desired constraint violation level over time. In the end the problem of controlling the temperature inside a realistic building is considered. In particular results of the application of a nominal standard MPC algorithm, a robust worst-case algorithm and two of the four selected stochastic techniques are compared with the aim of showing the real benefit

of probabilistic approaches.

4.1 A quick overview on some paradigmatic SMPC algorithms

Before proceeding with the examples, we summarize here the main guidelines of three SMPC algorithms taken from the literature which will be tested in simulation, together with the algorithm proposed here (denoted in the sequel as p-SMPC). The algorithms have been selected to represent interesting classes of methods; specifically and with reference to the classification in Chapter 2, we consider two analytic methods for dealing with disturbances with bounded support, labeled as t-SMPC and av-SPMC, and a general scenario-based approach, denoted by s-SPMC. The aim is to highlight, in a simple setup, the main difficulties that arise when trying to ensure properties like stability and recursive feasibility while reducing as much as possible the level of conservatism and to further prove the efficacy of the p-SMPC developed in this thesis.

The selected methods are used to control linear models with additive uncertainty described, adopting the same notation presented in Chapter 2, as in equation (2.3). In particular the system is described by the following expression

$$x_{t+1} = Ax_t + B_u u_t + B_w w_t \tag{4.1}$$

where the state is assumed to be measurable. In the following description, a single probabilistic state constraint of the type (2.18) is considered, while the input is left unconstrained for the sake of simplicity.

4.1.1 Stochastic tube MPC (t-SMPC)

The Stochastic tube MPC method (t-SMPC) is an analytic scheme, proposed for linear systems affected by additive and/or multiplicative uncertainties, see e.g., [24,25,27–29,77]. In [28,29] no boundedness assumption is made on the disturbance affecting the system and recursive feasibility results are established in probability. In later works (e.g., [24, 25, 27, 77]), under the assumption of bounded disturbances, recursive feasibility results are given.

Here we consider a simplified version of the approach presented in [77]. In this work the authors define an offline constraint tightening procedure that, thanks to the assumption that $w_t \in \mathcal{W} := \{w : |w| \leq \alpha\}$ with

$\alpha = [\alpha_1 \dots \alpha_{n_w}]^T$, ensures the recursive feasibility of the algorithm while taking advantage of the probabilistic nature of the constraint. The resulting online computational load is comparable with the one of a deterministic MPC algorithm, at the price of reducing the flexibility with respect to online tightening procedures.
The control scheme is based on a state feedback strategy of the type (2.16), i.e., we have

$$u_{t+i} = \bar{u}_{t+i|t} + Kx_{t+i} \tag{4.2}$$

where K is a constant gain that stabilizes the matrix $A + B_u K$ and $\bar{u}$ is computed as a solution of a proper MPC problem.
In principle, transient probabilistic constraints are guaranteed as in (2.20), where $q_i(1-p)$ can be computed, as in (2.22), using the expression

$$q_i(1-p) = \arg\min_q q,\ s.t.\ \mathcal{P}\left\{g^T E_i \mathbf{w}_t \le q\right\} = 1-p \tag{4.3}$$

or obtained, in a conservative way, using the Chebyshev inequality (2.28).
To ensure recursive feasibility, the solution presented in [77] takes advantage of the known bounds on the disturbance to implement a mixed stochastic/worst-case tightening procedure, which could include some conservativeness. More specifically, (2.20) is replaced by

$$g^T \bar{x}_{t+i} \le h - \beta_i, \qquad i = 1, 2, \dots \tag{4.4}$$

where $\beta_1 = q_1(1-p)$ and $\beta_i \ge q_i(1-p)$ for all $i \ge 2$. For details on the computation of the terms β_i see [77]. The algorithm is implemented using the common dual mode prediction paradigm, where the above constraints are accounted for explicitly along the horizon defined by N and implicitly by means of a proper terminal constraint $\bar{x}_{t+N} \in \mathcal{S}_{\hat{N}}$

$$\mathcal{S}_{\hat{N}} = \left\{ \bar{x}_N : \begin{array}{c} g^T \Phi^l \bar{x}_N \le h - \beta_{N+l},\ l = 1 \dots \hat{N} \\ g^T \Phi^l \bar{x}_N \le h - \bar{\beta},\ l > \hat{N} \end{array} \right\} \tag{4.5}$$

where it is assumed that $\bar{u}_{t+i} = 0$ for all $i \ge N$ and $\hat{N}$ defines an additional prediction horizon. This allows to define an infinite-time expectation cost function of the type

$$J_t = \sum_{i=0}^{\infty} \mathbb{E}[x_{t+i}^T Q x_{t+i} + u_{t+i}^T R u_{t+i}] \tag{4.6}$$

which is minimized at each time step t. Note that, once all the terms required up to a desired numerical precision are computed offline, the remaining problem is not more complex than a classical deterministic MPC, and the algorithm shares the same feasibility properties on the closed-loop operations that we have in the robust case (for the detailed proof see [77]).

4.1.2 Stochastic MPC for controlling the average number of constraint violations (av-SMPC)

The analytic approach presented in this section has been first proposed in [74], and has been later developed in [75,76]. It applies to linear systems with bounded uncertainties, i.e., where $w_t \in \mathcal{W}$, with $\mathcal{W}$ bounded. Recursive feasibility properties can be established and hard constraints on the input variable $u_t \in \mathbb{U}$ are allowed. The approach proposed by the authors relies on a disturbance feedback control strategy with average cost function of type (2.5), although in [76] it is shown that more general cases can be encompassed. The general approach is similar to the one discussed in Section 4.1.1; however, the mixed probabilistic/worst-case constraint tightenings are relaxed thanks to the idea that the sampled average of constraint violations must be limited, rather than its probabilistic counterpart. In this way, the approach results to be in general less conservative at the price of a slight increase in the computational burden.

In this section we describe only the simplified version of the control scheme, inspired by [75] where, for simplicity, a single probabilistic constraint of the type (2.19) is considered and input constraints are neglected.
In this approach, denoting by v_t the number of constraint violations occurred up to time t (with $v_t \leq t$), the constraint (2.20) for x_{t+1} is replaced by the following one

$$g^T \bar{x}_{t+1} \leq h - q_1(1 - \beta_t) \tag{4.7}$$

where $\beta_t = \max(p(t+1) - v_t, p)$. Note that $\beta_t > p$ is equivalent to $v_t/t < p$, so that, if the average rate of violations v_t/t occurred up to time t is smaller than the prescribed limit p, we can allow for a greater probability of constraint violation at the next time step. Also, at time $t + 1$, the state must be included in a set $\mathcal{S}_{r_t}$, defined in such a way that, if $x_{t+1} \in \mathcal{S}_{r_t}$ then, for all possible realizations of the bounded noise sequence $\mathbf{w}_t$, $x_{t+r_t} \in \mathcal{S}$, where the set $\mathcal{S}$ is the so-called stochastic control invariant set. $\mathcal{S}$ is defined in such a way that if $x \in \mathcal{S}$ then there exists u such that $Ax + B_u u + B_w w \in \mathcal{S}$ for all $w \in \mathcal{W}$ and $g^T(Ax + B_u u) \leq h - q_1(1-p)$. Intuitively, the number r_t is computed as the number of steps ahead in which a constraint violation would allow to satisfy $v_{t+i}/(t+i) < p$ for all $i = 1, \ldots, r_t$. As discussed in [76], in our simplified setting $x_{t+1} \in \mathcal{S}_{r_t}$ is ensured if, for all

$i = 1, \ldots, N - r_t$

$$g^T \bar{x}_{t+i} + \max_{\mathbf{w}_t \in \mathcal{W}^N} E_i e_{t+i} \leq h - q_1(1-p)$$

It is worth noting that in [76] a more general case is considered, where a forgetting factor is allowed in the computation of v_t, as well as the use of a penalty function to quantify the distance from constraint violation (e.g., rather than just defining v_t as the number of violations, v_t can represent also how far the state has been from violating the bounds). Also, as already discussed, hard constraints on inputs are allowed.

4.1.3 Scenario MPC (s-SMPC)

As previously discussed, the main idea of all the scenario-based techniques, see for example [20], [144] and [36] and the references therein, is to take advantage of the possibility to draw samples of the uncertainty, or to use its past records, to formulate a sample-based version of the control problem. In the following simulations we have used a general scenario-based algorithm inspired by [144].

The adopted control policy is the disturbance-feedback one, i.e., u_{t+i} is computed following (2.13) as

$$u_{t+i} = c_{t+i} + \sum_{j=0}^{i-1} \theta_{ij} w_{t+j} \tag{4.8}$$

where both the open loop term c and the parameters θ_{ij} are free variables of a suitable MPC problem whose goal is to minimize the expected cost function (2.5). The latter can be done, as discussed in Section 2.2.1, by minimizing the sampled cost (4.9), that is

$$\bar{J}_t \simeq \frac{1}{N_s} \sum_{k=1}^{N_s} J_t^{[k]} \tag{4.9}$$

where $J_t^{[k]}$ is the standard average cost function (2.5) computed correspondingly to the k-th extraction of the disturbance sequence. As highlighted in Section 2.2.1, other possible cost functions can be used in this framework.

The probabilistic constraints on state variables, similar to (2.29), are derived along the lines of the general ideas presented in Section 2.2.3. However, as thoroughly discussed in [144], in a receding-horizon control framework, it is sufficient to extract a total number N_s of realizations, appearing in all future constraints at the same time

$$g^T(\bar{x}_{t+i} + E_i \mathbf{w}_t^{[k]}) \leq h \tag{4.10}$$

for all $k = 1, \ldots, N_s$. In particular, N_s is the number of realizations required for enforcing the probabilistic constraint (2.18) for $i = 1$. For example, $N_s = N_{s,1}$ as in (2.31) in case no constraint removal is performed. In addition, for further reducing the number of required realizations, in [144] it is shown that, in order to guarantee that the *average* (instead of the *pointwise*) probability of constraint violation is equal to p, $N_{s,1}$ in (2.31) can be replaced by

$$N_s \geq \frac{d_1 - p}{p}$$

Also in this case, the removal strategy can be adopted for reducing the possible conservativity of the results at the price of an high increase of the online complexity of the algorithm. For details, see again [144].

4.1.4 The proposed approach: probabilistic MPC (p-SMPC)

The approach proposed in the previous chapter falls in the category of analytic methods, and has been developed specifically for linear systems of type (2.3) with additive and possibly unbounded uncertainty w_t, assumed to be a zero mean white noise with variance W. It encompasses the case where state constraints are in the form (2.19). In view of the unboundedness of the noise affecting the system, input constraints must also be formulated, in general, as probabilistic ones.
For a detailed explanation the reader is referred to Chapter 3, however a quick summary is given below with the aim of stress the differences with the other approaches.

Here, for the sake of exposition, input constraints are neglected. The approach lies on a state feedback policy of type (2.14) and, to reduce the computational complexity but at the price of obtaining suboptimal solutions, we discard the control gains as optimization variables and set $K_k = \bar{K}$ for all $k \geq 0$. Differently from the case considered in Section 2.2.2, here it is not necessarily set $\bar{x}_t = x_t$, but it is required, more in general, that

$$\bar{x}_t = \mathbb{E}[x_t] \tag{4.11}$$

Due to the choice of fixing the gains, the probabilistic constraint (2.19) is formulated as in (2.20) to be enforced for all the prediction horizon, i.e., for $i = t, \ldots, t + N - 1$ where, in line with (2.28) $q_i(1 - p) =$

$\sqrt{g^T X_{t+i} g}\sqrt{(1-p)/p}$ In case w_t is a (non-truncated) Gaussian variable, similarly to (2.23), $q_i(1-p) = \sqrt{g^T X_{t+i} g}\mathcal{N}^{-1}(1-p)$. The cost function to be minimized is the average cost (2.5).
Recursive feasibility is established in view of two ingredients:

- *Terminal constraints*, both in the mean value and in the variance, i.e.

$$\bar{x}_{t+N} \in \bar{\mathbb{X}}_f \tag{4.12}$$
$$X_{t+N} \leq \bar{X} \tag{4.13}$$

- *Initialization*. The initial condition for the pair $(\bar{x}_t, X_t)$ is free to take two different values, i.e., $(\bar{x}_t, X_t) = (x_t, 0)$ or $(\bar{x}_t, X_t) = (\bar{x}_{t|t}, X_{t|t-1})$, where $(\bar{x}_{t|t}, X_{t|t-1})$ is computed starting from the optimal solution obtained by the p-SMPC problem solved at time $t-1$.

For further details see Chapter 3.

4.2 Simulation examples

In this section we apply the methods described in Section 4.1 to different case studies. The first two examples make reference to linear models with additive noise (4.1), with measurable state and only one probabilistic state constraint of type (2.18), while the input is unconstrained. The last example, inspired by a realistic problem, is more complex and several linear individual chance constraints on both the states and the inputs are considered. In this case, however, only the av-SMPC and p-SMPC algorithms are implemented. For better comparisons, in the tuning of all analytic methods (i.e., t-SMPC, av-SMPC, and p-SMPC), the functions q_i are computed offline as in (2.22) (with particular reference to av-SMPC, please see details in [76]).

4.2.1 Academic benchmark example

In this section we consider the problem presented, e.g., in [27]. In particular, the focus is on obtaining the desired constraint violation rate at the specific time instant $t = 1$ s. The system with additive uncertainty, see (4.1), is characterized by the matrices

$$A = \begin{bmatrix} 1.6 & 1.1 \\ -0.7 & 1.2 \end{bmatrix}, \qquad B_u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad B_w = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The disturbance w_t is a truncated Gaussian noise with zero mean, variance $\sigma^2 = 1/144$ and bounds $|w_t|_\infty \leq 0.2236$. The system is subject to a single probabilistic state constraint of the type (2.19), i.e.

$$\mathcal{P}\left\{g^T x_t \leq h\right\} \geq 1 - p, \quad \forall t \geq 0 \tag{4.14}$$

where $g^T = \begin{bmatrix}1 & 0.1364\end{bmatrix}$, $h = 0.5$, and the violation level is set to $p = 0.2$. The feasible area is shown in Figure 4.1, together with the initial condition $x_0 = \begin{bmatrix}-5 & 60\end{bmatrix}^T$. Note that, in view of the fact that x_0 does not lie in the feasible area, probabilistic constraints are enforced, for all methods, from instant $t = 1$s on.

As in [27] an LQ controller has been first used to show the unconstrained solution and compare it with the constrained case. In particular choosing the weights $Q = I_2$ and $R = I_2$, the LQ optimal gain is $K = \begin{bmatrix}1.041 & 1.047\end{bmatrix}^T$, and the transient of the closed-loop system with LQ control has the behavior shown in Figure 4.1, where all the simulation correspond to a violation of the constraint at time $t = 1$.

The four SMPC algorithms previously described have been implemented by using quadratic cost functions with the same weighting matrices Q and R considered in the design of the LQ controller and with a prediction horizon $N = 5$.
The results achieved with the t-SMPC controller described in section 4.1.1 are shown in Figure 4.2. The algorithm is able to guarantee an actual violation rate of 0.1, more conservative than the required value 0.2, at time $t = 1$ s, with the computational time of a single iteration of $t_{solve} = 0.058$ s.

The av-SMPC strategy described in Section 4.1.2 produced the results shown in Figure 4.3. Thanks to the online tightening procedure, the overall violation rate at the time instant $t = 1$ s goes up to 0.19, close to the desired level 0.2, even if the required average computational time for a single iteration increases to $t_{solve} = 0.12$ s.

The p-SMPC controller described in Section 4.1.4 has been implemented by setting $\bar{x}_0 = \mathbb{E}[x_0] = x_0$, $X_0 = 0$. Note that, since the initial state does not lie inside the feasibility region, at time $t = 0$ the constraints (4.14) have been neglected. The obtained results are reported in Figure 4.4. The constraint violation rate at $t = 1$ s is 0.18, close to the desired one. The computational time for a single iteration is approximately $t_{solve} = 0.076$ s.

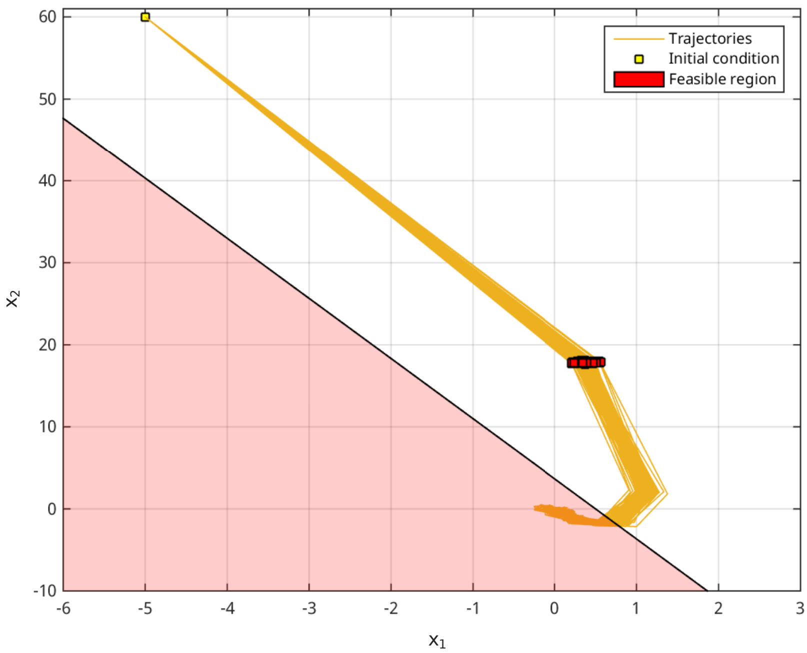


Figure 4.1: *Feasible region (pink area) and system trajectories for 100 runs with the LQ controller.*

Finally, the s-SMPC scheme described in Section 4.1.3, has been formulated in such a way that point-wise probabilistic constraint satisfaction is required with confidence level $\beta = 10^{-6}$. This results in the number of required samples and the number of removable constraints $N_s = 498$ and $N_r = 59$, respectively. The results are shown in Figure 4.5. The constraint violation rate at $t = 1$ s is 0.13 with an average computational time of $t_{solve} = 1.8$ s.

The overall results, in terms of constraint violation frequency at $t = 1$ s, and computational time are summarized in Table 4.1. This table highlights a relative conservativeness of t-SMPC and s-SMPC, besides the high computational burden of s-SMPC. However all the presented algorithms are able to solve the problem obtaining increased performances with respect to deterministic strategies without affecting too much the computation time.

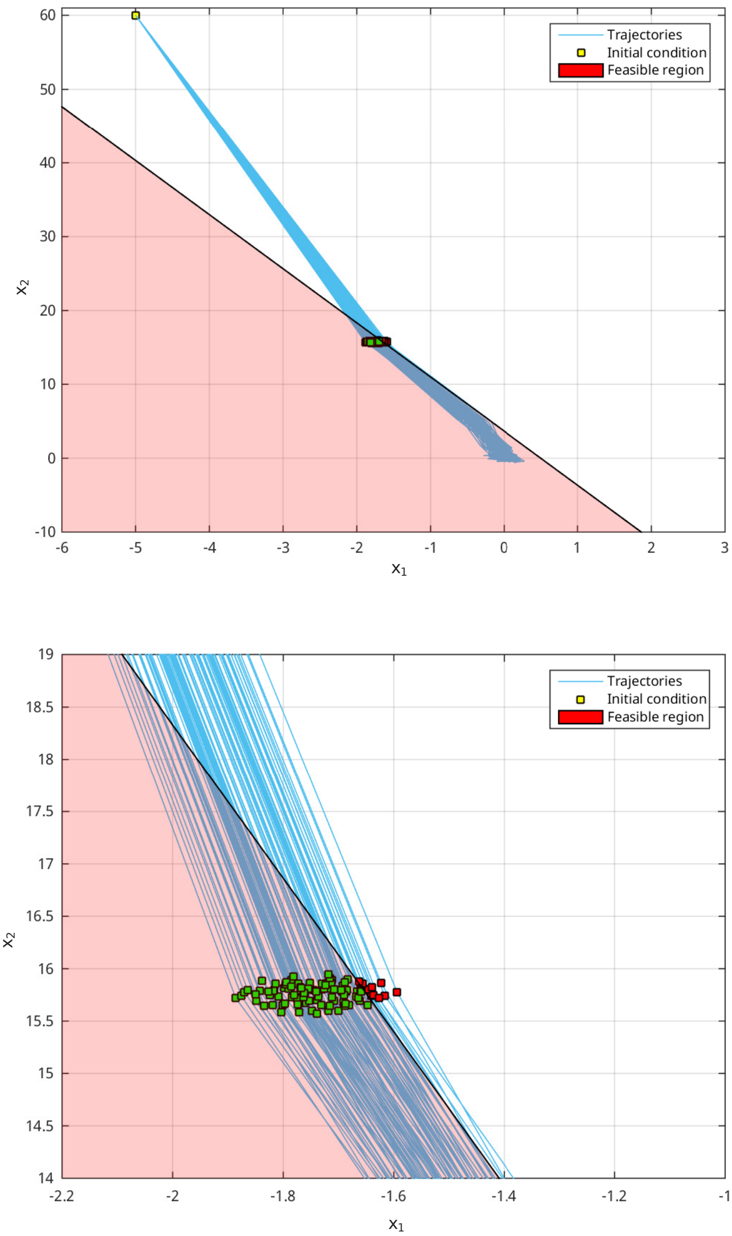
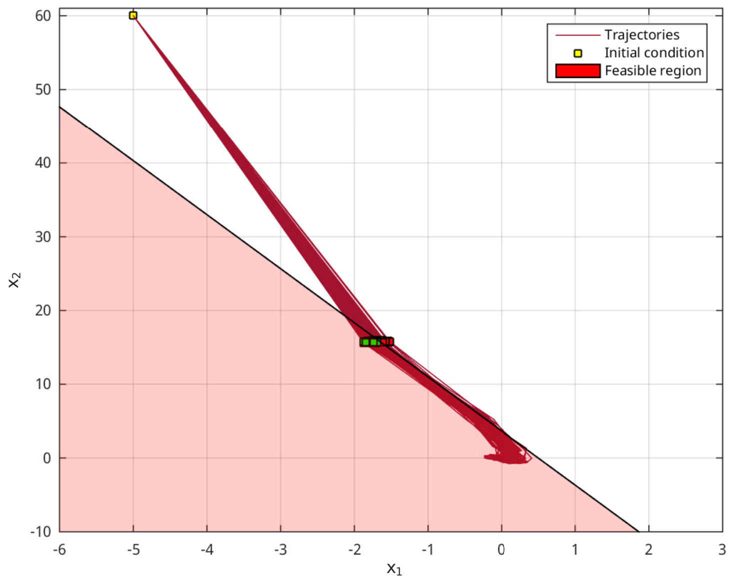


Figure 4.2: *Simulations using t-SMPC. On the right the overall system trajectories for the 100 runs. On the left the violated constraint at time $t = 1$ s.*



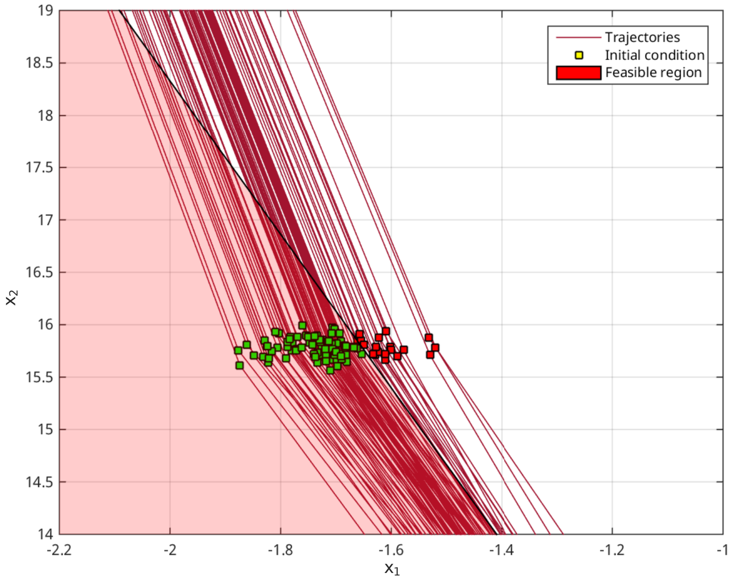


Figure 4.3: *Simulations using av-SMPC controller. On the right the overall system trajectories for the 100 runs. On the left the violated constraint at time $t = 1$ s.*

4.2.2 Multi-room temperature control

In this section we consider an example taken from [12]. The problem consists in controlling the temperatures, T_i, $i = 1, \ldots 4$ inside a simple building

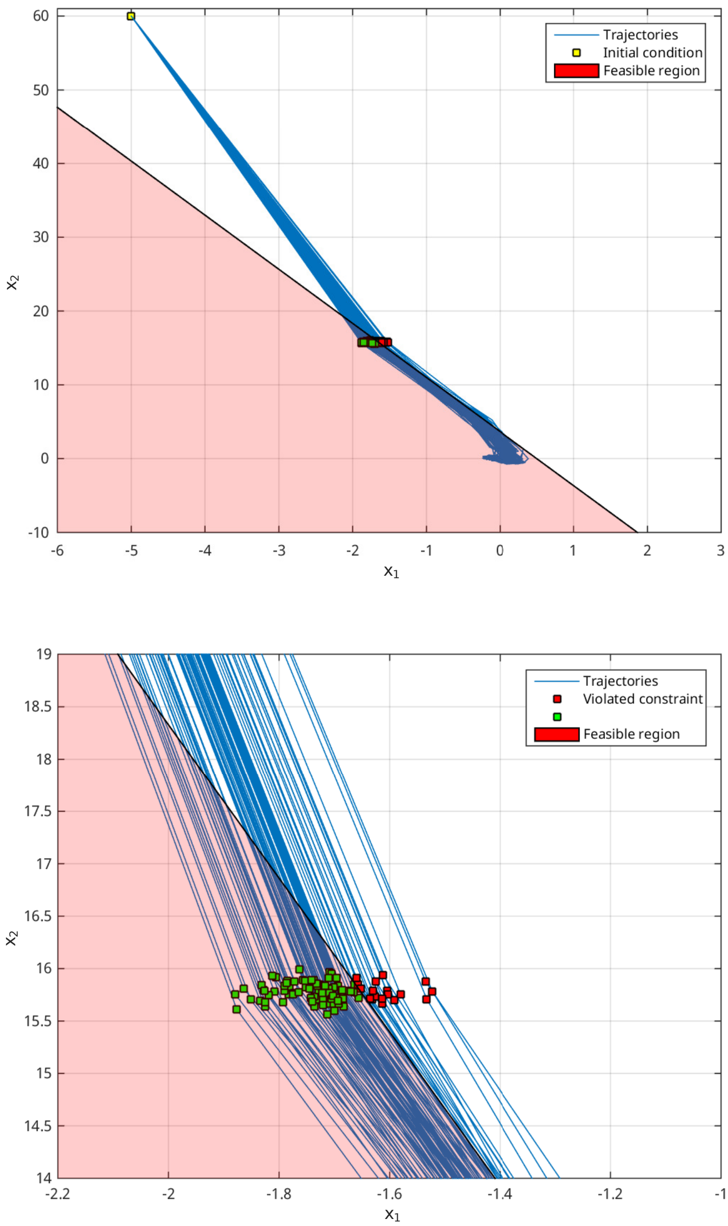
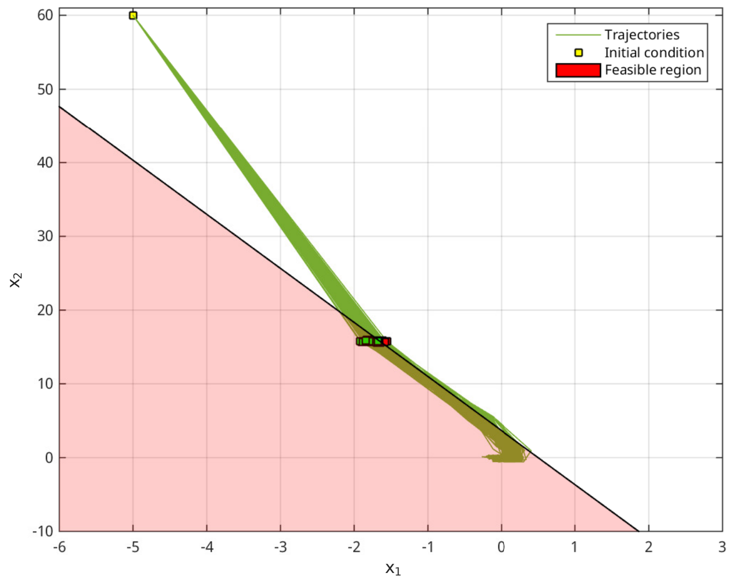


Figure 4.4: *Simulations using p-SMPC. On the right the overall system trajectories for the 100 runs. On the left the violated constraint at time $t = 1$ s.*

composed by four rooms, each one equipped with a radiator supplying heat



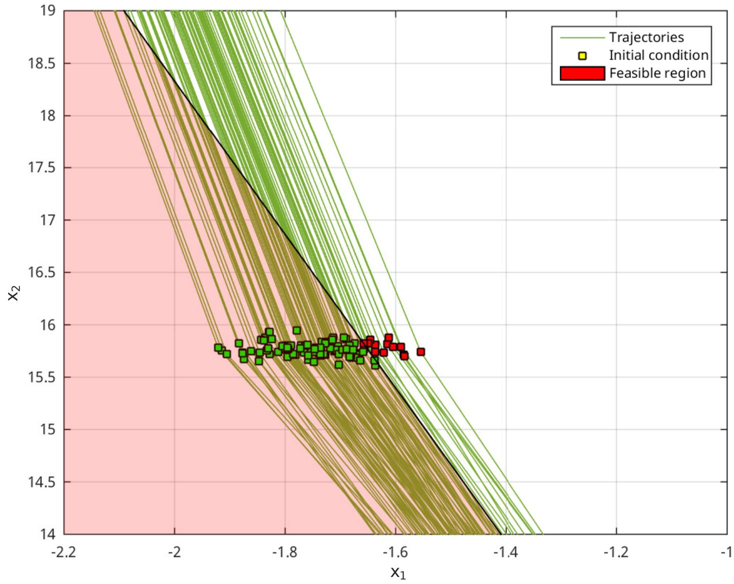


Figure 4.5: *Simulations using s-SMPC. On the right the overall system trajectories for the 100 runs. On the left the violated constraint at time $t = 1$ s.*

q_i, $i = 1 \ldots 4$. A schematic representation of the building is given in Figure

Approach	Violation frequency	Computation time
t-SMPC	0.10	0.058 sec
av-SMPC	0.19	0.12 sec
p-SMPC	0.18	0.076 sec
s-SMPC	0.13	1.8 sec

Table 4.1: *Stochastic Model Predictive Control approaches comparison - results for the academic benchmark example.*

4.6.

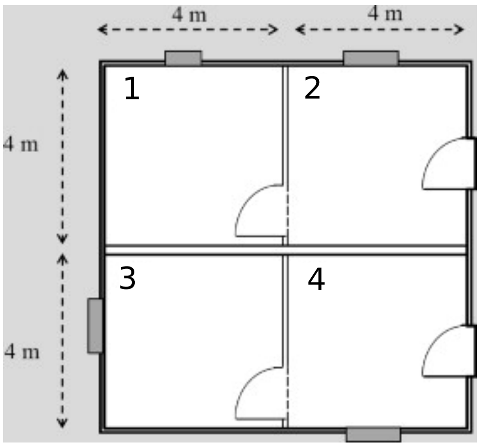


Figure 4.6: *Schematic representation of a building with two apartments.*

The nominal external temperature is fixed as $\bar{T}_E = 0$ °C and, for the sake of simplicity, solar radiation is not considered. The model of the system is linear, where the four state variables represent the temperature variations around the equilibrium point $\bar{T} = 20^oC$, for more details see [12]. The discrete-time system is obtained by ZOH discretization with sampling time 10 s, i.e., we have

$$A = \begin{bmatrix} 0.9909 & 0.005 & 0.002 & 0 \\ 0.005 & 0.9909 & 0 & 0.002 \\ 0.002 & 0 & 0.9909 & 0.005 \\ 0 & 0.002 & 0.005 & 0.9909 \end{bmatrix}$$
$$B_u = \begin{bmatrix} 9.9545 & 0.0252 & 0.0101 & 0 \\ 0.0252 & 9.9545 & 0 & 0.0101 \\ 0.0101 & 0 & 9.9545 & 0.0252 \\ 0 & 0.0101 & 0.0252 & 9.9545 \end{bmatrix}$$

and $B_w = I_4$. The state and control constraints are

$$x_{t,i} \in [-5, 5], \qquad u_{t,i} \in [-0.038, 0.030], \qquad i = 1, \ldots, 4 \qquad (4.15)$$

while the initial conditions are $x_0 = [-3.2, -2.58, -1.12, 3.55]^T$. The state constraints are relaxed allowing a $p = 0.2$ rate of violation for the temperature of each room (independent chance constraint). The uncertainty acting on the system is a truncated Gaussian additive noise with variance $W = 0.05I_4$ and bounds $\|w\|_\infty \leq 2$. Results of the application of the described algorithms are evaluated over $M = 100$ different runs. Differently from the previous example, the goal here is to estimate the level of constraint violation of the solution on the long run. For all the different control schemes, the cost function is selected with $Q = I$ and $R = 100I$, while the prediction horizon $N = 5$ has always been used.

The t-SMPC algorithm, described in 4.1.1, has been tuned by computing the feedback gain and the terminal weighting matrix with LQ. Results, referred to the temperature of room A, are shown in Figure 4.7. The average constraint violation is 0.04, while the average computational time for each iteration is $t_{solve} = 1.56$s.

The av-SMPC method described in Section 4.1.2 lead to the results reported in Figure 4.8, again referred to the temperature of room A. The controlled system violates the constraint with an average frequency of 0.19, and the average computation time is $t_{solve} = 2.2$s.

The p-SMPC controller described in Section 4.1.4 has been tuned by setting $p^u = 0.1$ and computing the input gain and the terminal weighting

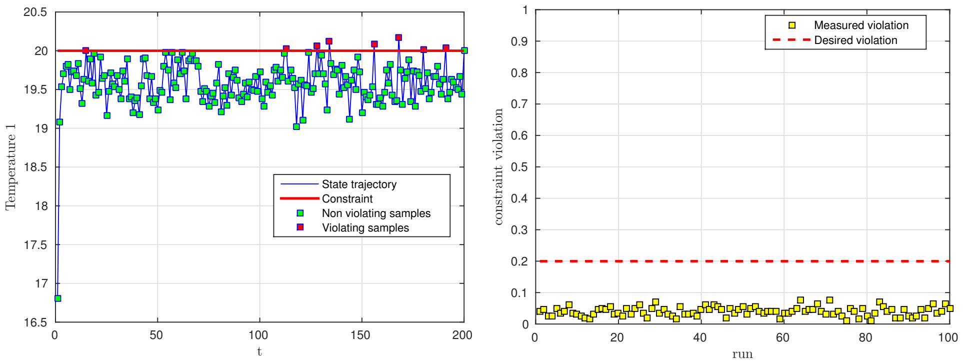


Figure 4.7: *Simulations using t-SMPC. On the left a sample trajectory T_1. On the right the violation frequency over 100 runs.*

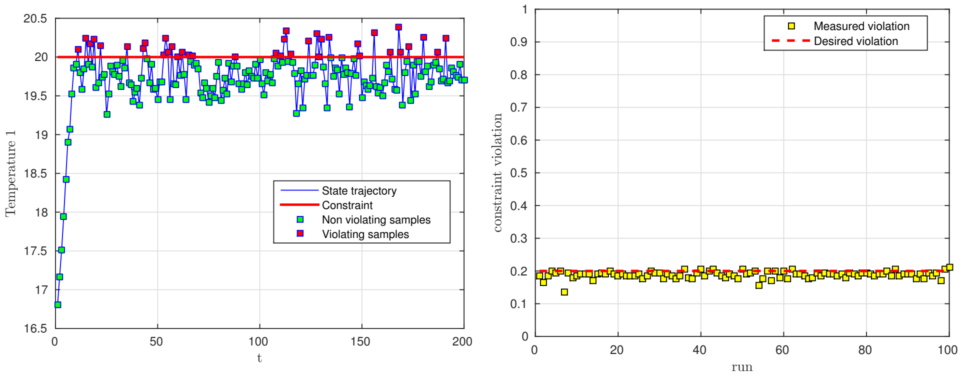


Figure 4.8: *Simulations using av-MPC controller. On the left a sample trajectory T_1. On the right the violation frequency over 100 runs.*

matrix with LQ. The behavior of the temperature of room A is shown in Figure 4.9. The constraint is violated with an average rate of 0.19, close to the desired value $p = 0.2$. The average computational time is $t_{solve} = 1.79$s.

Finally, the s-SMPC algorithm described in Section 4.1.3 has been formulated in order to guarantee time-average probabilistic constraint satisfaction. Considering $p_1 = 4$, the number of sample extractions is $N_s = 465$ with $N_r = 55$ to be removed. The simulation results are shown in Figure 4.10. The violation frequency of the constraint is 0.17 and the required average computational time is $t_{solve} = 17.6$s.

The results achieved with the four methods are summarized in Table 4.2. Even in this case, all the algorithms can provide better performances (clearly

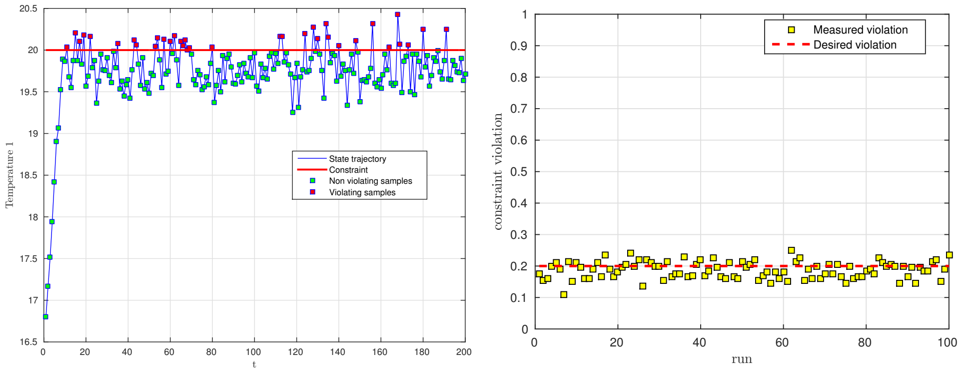


Figure 4.9: *Simulations using p-SMPC. On the left a sample trajectory T_1. On the right the violation frequency over* 100 *runs.*

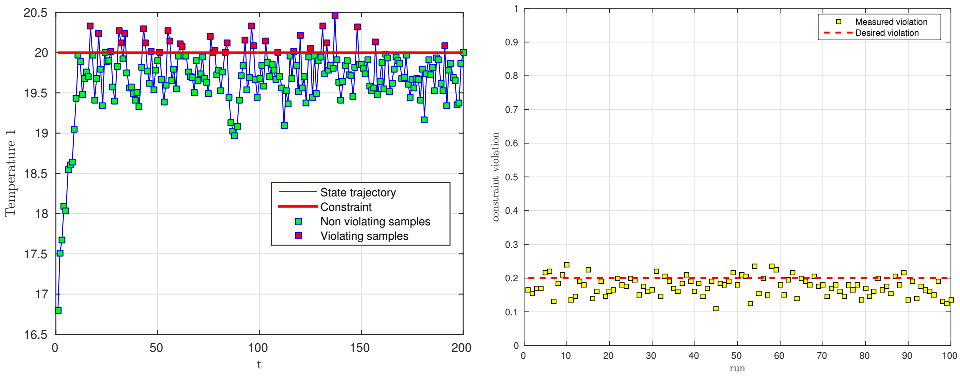


Figure 4.10: *Simulations using s-SMPC. On the left a sample trajectory T_1. On the right the violation frequency over 100 runs.*

in terms of power consumption) than a robust deterministic version, thanks to the probabilistic relaxation. However, the t-SMPC seems to be the most conservative one due to the offline tightening procedure coupled with the worst-case considerations along the horizon. On the other hand, the s-SMPC, even if it is very flexible in its usage, requires an high computation time. The av-SMPC and the p-SMPC are good compromises and thus are the ones that will be tested in the next example.

4.3 Temperature control on a realistic building

In this example, the av-SMPC and the p-SMPC algorithms previously discussed, are used to control the temperature inside a realistic medium-size commercial building with the aim of minimizing the power consumption

Approach	Violation frequency	Computation time
t-SMPC	0.04	1.56 sec
av-SMPC	0.19	2.2 sec
p-SMPC	0.19	1.79 sec
s-SMPC	0.17	17.6 sec

Table 4.2: *Stochastic Model Predictive Control approaches comparison - results for example 3*

while maintaining some comfort constraints in presence of uncertain weather conditions and building occupation. Note that, in this framework, the use of stochastic control techniques can be crucial, due to the explicit way of handling the risk associated to an uncertain environment and stochastic Model Predictive Control approaches are very useful due to the possibility of exploiting the knowledge of weather forecasts or scheduled modifications of the comfort constraints (for example desired temperatures can change during the night) anticipating the reaction of the system. In the sequel, a realistic 3 zones office building is considered, together with real data and forecasts for the city of Lausanne (Switzerland) during the first week of December 2013, collected from the web service `http://www.wunderground.com`. For people occupation a realistic profile that match the dimension of the building is considered.

The model used in the design of the control algorithms is a description of the thermal dynamics of the building obtained as discussed in [54,134]. The entire modeling process is handled in *MATLAB* thanks to the use of the toolbox *OpenBuild* (`http://la.epfl.ch/openBuild`) that integrates in an easy to use package standard tools for energy efficiency analysis like for example the well-known*Energy Plus*.

4.3.1 Building model

Here the model of the proposed 3-zones building, resulting from the toolbox *OpenBuild* (see [54]), is presented. In particular, in line with equation (4.1), we consider the following discrete-time linear time-invariant model with additive disturbances

$$\begin{aligned} x_{t+1} &= Ax_t + B_u u_t + B_d d_t \\ y_t &= Cx_t \end{aligned} \tag{4.16}$$

where the input vector u represents the power used by the building (in the case considered here the heating power), the output vector y represents the temperatures inside the three zones of the building and the vector d collects the disturbances coming respectively from the outside temperature, the weather condition, and the number of people inside the building. The matrix B_d describes the effect of the three sources of uncertainty on the states of the system. The time step used for computing the model is $T_s = 20$ minutes. For the building under study the dimension of the state is $n_x = 71$, the dimension of the input is $n_u = 3$ and the dimension of the output is $n_y = 3$. If needed it is possible to derive a reduced order model exploiting the structure of the building and approximating some of the relationships between its surfaces. In this case the state vector boils down to $n_x = 10$ elements, at the price of a slightly increased modeling error and at the price of loosing the physical meaning of each of its components. The reduced model will be used in the sequel.
As for the disturbance vector d, we assume to know in advance the weather situation (solar radiation profile) and the flow of people inside the building, so that the attention is focused only on the uncertainty coming from the unknown outside temperature. In this case, to predict its profile, we have access to real time forecasts, made by a weather company, that last for the following ten days and are updated every hour. Defining the vector of known disturbances as $\bar{d}$, the dynamics of the system become

$$x_{t+1} = Ax_t + B_u u_t + B_d \bar{d}_t + B_w w_t \tag{4.17}$$

where the matrix B_w is $B_w = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T B_d$ due to the fact that the forecast error for the outside temperature, w, is just a scalar.

The goal of the temperature controller we want to design is to maintain the temperature of each zone of the building, namely the outputs of the system, inside a given comfort region, while minimizing the power required. The bounds on the temperature can be time-varying, for example relaxing the requirements during the night, but are supposed to be known in advance, accordingly to a certain schedule. The constraints acting on the output are thus the following

$$y_t^{min} \leq y_t \leq y_t^{max} \qquad \forall t \geq 0 \tag{4.18}$$

while the bounds on the input are

$$u^{min} \leq u_t \leq u^{max} \qquad \forall t \geq 0 \tag{4.19}$$

where, without loss of generality, we can consider $u^{min} = 0$.
Due to the availability of the forecasts for the following hours and the explicit presence of constraints on the temperatures and the input powers, the use of an MPC controller can be very effective. Furthermore, in practice, violations of the proposed bounds on the temperature are usually allowed in terms of a maximum accumulated violation during the year, thus motivating the use of a stochastic approach to explicitly account for this relaxation.

4.3.2 Disturbance model

Consider now the description of the forecast error for the outside temperature, denoted with w_t, in (4.17). In particular at every time step we have access to a new forecast that we can use in the MPC controller to guess the future evolution of the temperatures along the prediction horizon, and improve our decision. Past forecasts (one hour ahead) and real measurements, can be analyzed to derive a stochastic description of the disturbance acting on the system and to improve the forecasts made by the weather company. An example of the forecast error for the period 01/12/2013 to 15/12/2013 is shown in Figure 4.11.

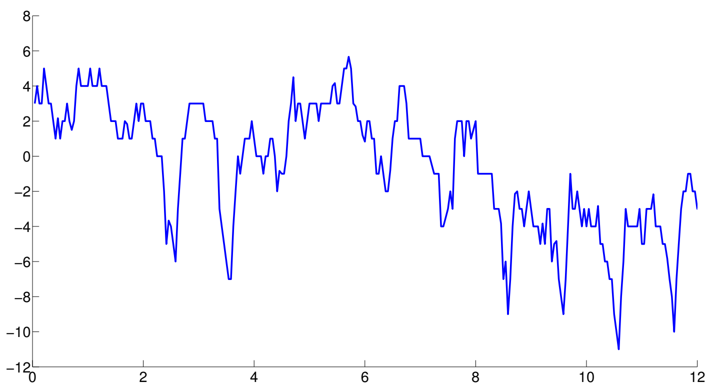


Figure 4.11: *Forecast error on the outside temperature example.*

Past data are used to identify the following model

$$e_{t+1} = 0.066 + 0.89e_t + 0.14w_{t-1} + w_t \qquad w \sim \mathcal{N}(0, 1.2) \tag{4.20}$$

Thanks to the estimated model, the residual disturbance acting on the system has zero mean and a nearly Gaussian distribution. Note that, in general, the support of the disturbance is naturally unbounded, so that, to apply robust policies, we have to put some artificial bounds and allow the possibility to violate the constraints.

4.3.3 Standard nominal MPC

With the aim of highlight the advantages of probabilistic approaches, firstly we consider the application of a standard nominal MPC scheme to the problem described so far. In this case we consider the forecasts we have as correct, just neglecting any further uncertainty. In particular, we consider the following dynamics for the system to be controlled

$$\begin{aligned} \bar{x}_{t+1} &= A\bar{x}_t + B_u\bar{u}_t + B_d\bar{d}_t \\ \bar{y}_t &= C\bar{x}_t \end{aligned} \tag{4.21}$$

In line with the discussion in Chapter 2, the chosen cost function is the standard quadratic cost on the output and the input

$$J = \sum_{k=t}^{N-1} \|\bar{x}_k\|_Q^2 + \|\bar{u}_k\|_R^2 + \|\bar{x}_{t+N}\|_S^2 \tag{4.22}$$

For the simulation we considered $y^{min} = 22\,°C$ and $y^{max} = 26\,°C$, with nightly relaxations of $4\,°C$, $u^{min} = 0\,kW$ for all the actuators and respectively $u^{max} = [10, 3, 10]\,kW$, $c_i = 1$. The state and input weighting matrices are chosen as $Q = I$, $R = 10 * I$ and the prediction horizon is set to $N = 72$ steps. As a result, we obtain an uncontrolled frequency of violation.

4.3.4 Robust MPC strategy

Consider now the application of a robust MPC strategy to avoid constraint violations due to the partially unknown outside temperature. Note that, to compute the optimal solution with respect to the worst-case disturbance sequence along the prediction horizon, we need to put artificial bounds on the disturbance that will act as a compromise between the achieved robustness and the conservativity.

In this case we take the worst case-forecast error analyzing the past data and define the set $\mathcal{W}$ such that $w \in \mathcal{W}$. The error is just a scalar, so that the set $\mathcal{W}$ is an interval

$$\mathcal{W} = \{w : \; |w| \leq w^{max}\} = \left\{w : \; \begin{bmatrix} 1 \\ -1 \end{bmatrix} w \leq w^{max} \right\} \tag{4.23}$$

Between the many available control strategies for formulating a robust MPC we choose to adopt an affine disturbance feedback policy as described in 2.2.2.
The simulations are run considering the same setup used in the nominal case and bounding the disturbance to $|w| \leq 2\,°C$. As a result, we have the behavior shown in Figure 4.12 for the zone temperatures. In this case the effect of the worst case controller is to keep the system far from the constraints so that the result is quite conservative.

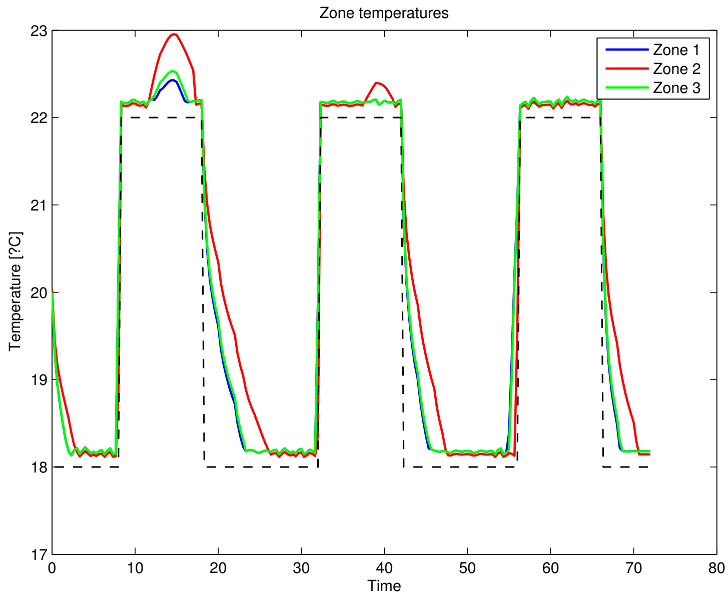


Figure 4.12: *Zone temperatures and desired constraints with robust MPC*

4.3.5 Stochastic MPC strategies

To overcome the limitations shown in the previous sections, and in particular to avoid the uncontrolled number of constraints violations of the nominal controller, or the definition of too conservative bounds of the robust case, we consider here two of the stochastic MPC techniques described in Section 4.1, namely the av-SMPC and the p-SMPC.
In the av-SMPC the constraint tightening is done accordingly to the closed-loop behavior of the system itself, i.e., taking into account the history of the occurred violations. As in the robust case, a bound on the disturbance is required for feasibility purposes and thus we assume that $w \in \mathcal{W}$. In the

p-SMPC the constraint tightening is performed in an open-loop perspective but the assumption on bounded disturbance is relaxed, at the price of introducing stochastic constraints also on the control inputs (for further details see Chapter 3).

In both the algorithms the cost function is the average of the classical quadratic cost in (4.24), i.e., we consider

$$J = \mathbb{E}\left\{ \sum_{i=t}^{t+N-1} x_i^T Q x_i + u_i^T R u_i + x_{t+N}^T S x_{t+N} \right\} \tag{4.24}$$

where $Q = C^T C$, $R = 100 * I_3$, and S is computed as in the nominal and robust case. The disturbance is approximated with a Gaussian distribution as described in Section 4.3.2 and its covariance (W in the p-SMPC) is estimated from the covariance of the available samples. The state constraints are relaxed allowing for a probability of violating the constraints of $p^x = 0.2$.
For the av-SMPC the parameters are chosen as $\bar{\xi} = 0.4$, $\xi = 0.2$, $\alpha = 0.05$, $\gamma = 0.85$ and $n_s = 3$. As for the p-SMPC the controller gain, in order to simplify the algorithm and to use the non-linear version of the constraints (3.9), is fixed to $\bar{K} = K_{LQ}$ and the input violation probability is set to $p^u = 0.1$.

The temperature inside the zones is shown in Figure 4.13 for the av-SMPC and in Figure 4.14 for the p-SMPC case. As expected in both the cases we are able to carry the system to the desired position with respect to the constraint and to obtain an actual violation frequency of 0.16 and 0.10, respectively. These values are much lower than the desired violation frequency, probably due to the not exact modeling of the disturbance, however the resulting behavior is still better than the one obtained in the robust case, thus motivating the use of the proposed approaches. Note that, in general, the av-SMPC is more flexible since a proper choice of the so-called loss function(i.e., the way the violations are measured [76]) allows to model different kind of stochastic constraint relaxations (see Chapter 2. For example choosing such function as $\mathcal{L}(y) = max\{y, 0\}$, instead of counting the number of violations, allows to model expected value constraints, exploiting specification on the maximum possible error averaged over time common in the field of temperature control.

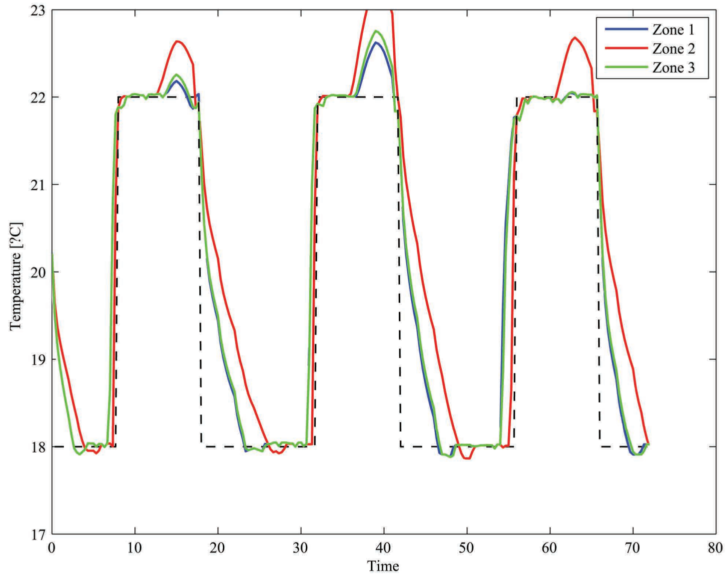


Figure 4.13: *Zone temperatures and desired constraints with av-SMPC.*

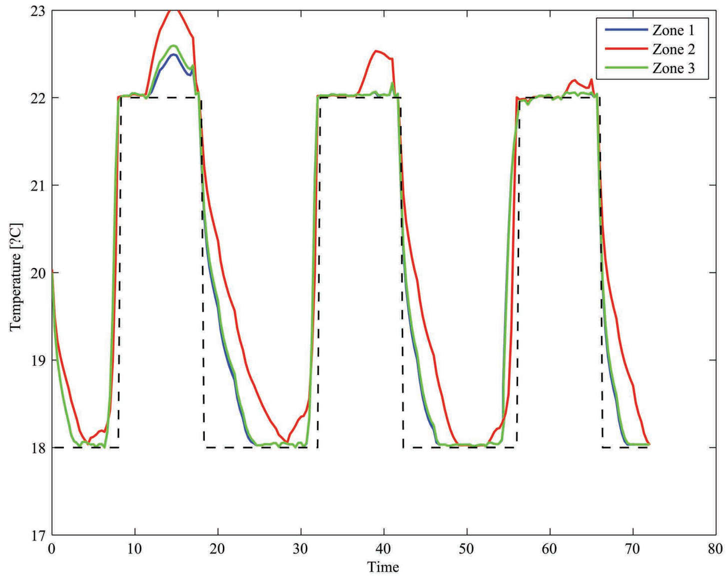


Figure 4.14: *Zone temperatures and desired constraints with p-SMPC.*

4.4 Comments

In this chapter we briefly presented four stochastic MPC algorithms for linear discrete-time systems with additive bounded disturbances and linear individual chance constraints. Following the terminology adopted in Chapter 2, three of them are analytic methods and the last one is a sample-based (scenario) method. These algorithms have been applied firstly to two different simulation examples with the aim of showing the behavior of the controlled system over multiple simulations with respect to, respectively, the number of violations of a single constraint point-wise in time and the long-run number of violations of a certain constraint in a receding horizon fashion. Comparisons are made in term of conservativeness, i.e. the measured frequency of violation with respect to the desired level, and online complexity. Of course, due to the simplicity of the examples adopted, we are not able to completely show the possibilities of each algorithm, however some general comments can be drawn. In particular we noticed that

- the **t-SMPC** algorithm has proven to be very fast and easy to implement, it gives better results than standard deterministic approaches but it is still too conservative due to the offline use of mixed probabilistic worst-case tightenings. Thanks to the assumption of bounded disturbances it guarantees feasibility and convergence.
- the **av-SMPC**, with the idea of adjusting the constraints based on actual violation measurements, represents a good compromise between tube based strategy, in which the tightenings are completely offline and sample-based approaches (scenario) that are completely online. The main limitation is due to the bounded disturbance assumption that is used to guarantee recursive feasibility of the scheme.
- the **p-SMPC** has proven to be effective and easy to design. Its main limitation is related to the need for probabilistic input constraint that is compensated, however, by the guaranteed convergence and feasibility properties even in the case of unbounded disturbances.
- the **s-SMPC** is really flexible and capable to solve a wide range of problems, that the other selected approaches are not able to manage, with a reduced offline complexity. The only assumption is basically the convexity of the optimization problem once a disturbance sample is extracted. Even joint chance constraints can be considered directly without resorting to risk allocation procedures. However, the algorithm can be very demanding especially if used together with sample

removal techniques. In general it not suited for small linear systems, like the ones tested here.

The third example is the problem of controlling the temperature inside a realistic building based on real uncertainty (external temperature) measures. The goal was to show the difference between nominal, robust and stochastic MPC strategies and to this end the p-SMPC and the av-SMPC algorithms have been implemented due to their good compromise between complexity and conservatism. From the results it is easy to understand the importance of such stochastic Model Predictive Control techniques to overcome the limitations of standard robust MPC algorithms.

CHAPTER 5

Output-feedback probabilistic SMPC

In this chapter we propose, with the aim of extending the p-SMPC technique presented in Chapter 3, an output-feedback stochastic Model Predictive Control algorithm for linear discrete-time systems affected by a possibly unbounded additive noise with not measurable state and subject to probabilistic constraints. Similarly to the p-SMPC case, if the noise distribution is unknown, the probabilistic constraints on the state and input variables are reformulated by means of the Cantelli-Chebyshev's inequality. Again the recursive feasibility is guaranteed and the convergence of the state to a suitable neighborhood of the origin is proved under mild assumptions. In addition, implementation issues are thoroughly addressed and two examples are discussed in details, with the aim of providing an insight into the performance achievable by the proposed control scheme. As in the state feedback case, the algorithm computational load can be made similar to the one of a standard stabilizing MPC algorithm with a proper choice of the design parameters (e.g., converting all constraints to linear matrix inequalities, LMIs), so that the application of the proposed approach to medium/large-scale problems is allowed.

5.1 Problem statement

In this chapter we consider the following discrete-time linear system

$$\begin{cases} x_{t+1} = Ax_t + B_u u_t + B_w w_t \quad t \geq 0 \\ y_t = Cx_t + v_t \end{cases} \tag{5.1}$$

where $x_t \in \mathbb{R}^n$ is the state, $u_t \in \mathbb{R}^m$ is the input, $y_t \in \mathbb{R}^p$ is the measured output and $w_t \in \mathbb{R}^{n_w}, v_t \in \mathbb{R}^p$ are two independent, zero-mean, white noise processes with covariance matrices $W \succeq 0$ and $V \succ 0$, respectively, and a-priori unbounded support. With respect to the previous chapter, here we remove the assumption that the state is fully accessible and we focus on the use of the uncertain measures y_t into the control scheme. To this end, the pair (A, C) is assumed to be observable, together with the reachability of the pairs (A, B_u) and $(A, \tilde{B_w})$, where matrix $\tilde{B_w}$ satisfies $\tilde{B_w}\tilde{B_w}^T = B_w W B_w^T$.

As in the state-feedback case, polytopic constraints on the state and input variables of system (5.1) are imposed as multiple independent chance constraints, i.e., it is required that, for all $t \geq 0$

$$\mathbb{P}\{b_r^T x_t \leq x_r^{max}\} \leq 1 - p_r^x \,, \quad r = 1, \ldots, n_r \tag{5.2}$$

$$\mathbb{P}\{c_s^T u_t \leq u_s^{max}\} \leq 1 - p_s^u \,, \quad s = 1, \ldots, n_s \tag{5.3}$$

where p_r^x, p_s^u are again design parameters and it is assumed that the set of relations $b_r^T x \leq x_r^{max}$, $r = 1, \ldots, n_r$ (respectively, $c_s^T u \leq u_s^{max}$, $s = 1, \ldots, n_s$), define a convex set $\mathbb{X}$ (respectively, $\mathbb{U}$) containing the origin in its interior.

5.1.1 Regulator structure

For system (5.1), we want to design a standard regulation scheme composed by the state observer

$$\hat{x}_{t+1} = A\hat{x}_t + B_u u_t + L_t(y_t - C\hat{x}_t) \tag{5.4}$$

coupled with the feedback control law

$$u_t = \bar{u}_t + K_t(\hat{x}_t - \bar{x}_t) \tag{5.5}$$

where, as in Section 3.1, $\bar{x}$ is the state of the nominal model

$$\bar{x}_{t+1} = A\bar{x}_t + B_u \bar{u}_t \tag{5.6}$$

In (5.4), (5.5), the feedforward term $\bar{u}_t$ and the time-varying gains L_t, K_t are design parameters to be selected through a proper MPC scheme to guarantee convergence properties and the fulfillment of the probabilistic constraints (5.2), (5.3).

With respect to the previous case, the distance of the state from its expected value, namely δx_t, and thus its variance, depends now on the presence of both the controller and the observer. To model this situation, define the two errors

$$e_t = x_t - \hat{x}_t \tag{5.7a}$$
$$\varepsilon_t = \hat{x}_t - \bar{x}_t \tag{5.7b}$$

Starting from (5.7) one can write the difference between the state and its expected value as the sum of the two errors e_t and ε_t

$$\delta x_t = x_t - \bar{x}_t = e_t + \varepsilon_t \tag{5.8}$$

In order to ease the notation, we can collect them into the vector $\sigma_t = \begin{bmatrix} e_t^T & \varepsilon_t^T \end{bmatrix}^T$ whose dynamics, according to (5.1)-(5.7), is described by

$$\sigma_{t+1} = \quad \Phi_t \sigma_t + \Psi_t \begin{bmatrix} w_t \\ v_t \end{bmatrix} \tag{5.9}$$

where

$$\Phi_t = \begin{bmatrix} A - L_t C & 0 \\ L_t C & A + B_u K_t \end{bmatrix}, \; \Psi_t = \begin{bmatrix} B_w & -L_t \\ 0 & L_t \end{bmatrix}$$

Note that both these matrices depend on the free variables L_t and K_t. In the following, similarly to Chapter 4 it is assumed that, by a proper initialization, i.e. $\mathbb{E}\{\sigma_0\} = 0$, and thus recalling that the processes v and w are zero mean, the enlarged state σ_t of system (5.9) is zero-mean, so that $\bar{x}_t = \mathbb{E}\{x_t\}$. Due to this assumption and denoting by $\Sigma_t = \mathbb{E}\left\{\sigma_t \sigma_t^T\right\}$ and by $\Omega = \mathrm{diag}(W, V)$ the covariance matrices of σ_t and $[w_t^T \, v_t^T]^T$ respectively, the evolution of Σ_t is governed by

$$\Sigma_{t+1} = \Phi_t \Sigma_t \Phi_t^T + \Psi_t \Omega \Psi_t^T \tag{5.10}$$

As in the state-feedback algorithm, we are interested in using the state and input covariance matrices to handle probabilistic constraints, however we are forced to consider in the computations the overall error vector, and its covariance Σ_t, due to the presence of the observer. In particular, by definition the variable δx_t defined by (5.8) is zero mean and its covariance matrix X_t can be derived from Σ_t as

$$X_t = \mathbb{E}\left\{\delta x_t \delta x_t^T\right\} = \begin{bmatrix} I & I \end{bmatrix} \Sigma_t \begin{bmatrix} I & I \end{bmatrix}^T \tag{5.11}$$

Finally, letting also $\delta u_t = u_t - \bar{u}_t = K_t(\hat{x}_t - \bar{x}_t)$, one has $\mathbb{E}\left\{\delta u_t\right\} = 0$ and also the covariance matrix $U_t = \mathbb{E}\left\{\delta u_t \delta u_t^T\right\}$ can be obtained from Σ_t as

$$U_t = \mathbb{E}\left\{K_t \varepsilon_t \varepsilon_t^T K_t^T\right\} = \begin{bmatrix} 0 & K_t \end{bmatrix} \Sigma_t \begin{bmatrix} 0 & K_t \end{bmatrix}^T \tag{5.12}$$

This second order description of the system variables can now be used, as in the state-feedback case, to reformulate the constraints.

5.1.2 Reformulation of the probabilistic constraints

To incorporate the probabilistic constraints (5.2) and (5.3) into the MPC problem, we rely on the probabilistic approximation method described in detail in Chapter 3, involving the stochastic variables second-order description, at the price of suitable tightening. This, save for the case of Gaussian disturbances, induces some conservativeness. As in the state-feedback case, we denote here with $f(p)$ a function that is $f(p) = \sqrt{(1-p)/p}$ in the case the distribution of the stochastic variable is unknown or $f(p) = \mathcal{N}^{-1}(1-p)$, where $\mathcal{N}$ is the cumulative probability function of a Gaussian variable with zero mean and unitary variance, in the case the variables are Gaussian.

The probabilistic constraints (5.2)-(5.3), at time i, are verified provided that the following (deterministic) inequalities are satisfied

$$b_r^T \bar{x}_i \leq x_r^{max} - \sqrt{b_r^T X_i b_r} f(p_r^x) \tag{5.13a}$$

$$c_s^T \bar{u}_i \leq u_s^{max} - \sqrt{c_s^T U_i c_s} f(p_s^u) \tag{5.13b}$$

where the terms X_i and U_i depend now on the overall covariance matrix Σ_t as in (5.11) and (5.12).

Note again that, if the support of the noise terms w_k and v_k is unbounded, the definition of the state and control constraints in probabilistic terms is the only way to state feasible control problems. In case of bounded noise the comparison, in terms of conservativeness, between the probabilistic framework and the deterministic one has been discussed in the example of Section 2.3.

5.2 MPC algorithm: formulation and properties

To formally state the MPC algorithm for the computation of the regulator parameters $\bar{u}_t$, L_t, K_t, the following notation will be adopted: given a variable z or a matrix Z, at any time step t we will denote by z_{t+k} and Z_{t+k}, $k \geq 0$, their generic values in the future, while $z_{t+k|t}$ and $Z_{t+k|t}$ will represent their specific values computed based on the knowledge (e.g., measurements) available at time t. The main ingredients of the optimization problem are now introduced.

5.2.1 Cost function

Assume to be at time t and denote by $\bar{u}_{t,\ldots,t+N-1} = \{\bar{u}_t, \ldots, \bar{u}_{t+N-1}\}$ the nominal input sequence over a future prediction horizon of length N. Moreover, define by $K_{t,\ldots,t+N-1} = \{K_t, \ldots, K_{t+N-1}\}$, and $L_{t,\ldots,t+N-1} = \{L_t, \ldots, L_{t+N-1}\}$ the sequences respectively of the future control and observer gains, and recall that the covariance $\Sigma_{t+k} = \mathbb{E}\left\{\sigma_{t+k}\sigma_{t+k}^T\right\}$ evolves, starting from Σ_t, according to (5.10).
In line with the discussion in Section 2.2.1 and with the choice in Chapter 3, the cost function to be minimized is the sum of two components. The first one (J_m) accounts for the expected values of the future nominal inputs and states, while the second one (J_v) is related to the variances of the future errors e, ε, and of the future inputs. Specifically, the overall performance index is

$$J = J_m(\bar{x}_t, \bar{u}_{t,\ldots,t+N-1}) + J_v(\Sigma_t, K_{t,\ldots,t+N-1}, L_{t,\ldots,t+N-1}) \tag{5.14}$$

where

$$J_m = \sum_{i=t}^{t+N-1} \|\bar{x}_i\|_Q^2 + \|\bar{u}_i\|_R^2 + \|\bar{x}_{t+N}\|_S^2 \tag{5.15}$$

$$\begin{aligned} J_v = {} & \mathbb{E}\left\{ \sum_{i=t}^{t+N-1} \|x_i - \hat{x}_i\|_{Q_L}^2 + \|x_{t+N} - \hat{x}_{t+N}\|_{S_L}^2 \right\} + \\ & \mathbb{E}\left\{ \sum_{i=t}^{t+N-1} \|\hat{x}_i - \bar{x}_i\|_Q^2 + \|u_i - \bar{u}_i\|_R^2 + \|\hat{x}_{t+N} - \bar{x}_{t+N}\|_S^2 \right\} \end{aligned} \tag{5.16}$$

where the positive definite and symmetric weights Q, Q_L, S, and S_L must satisfy the following inequality

$$\bar{Q}_T - S_T + \Phi^T S_T \Phi \preceq 0 \tag{5.17}$$

with

$$\Phi = \begin{bmatrix} A - \bar{L}C & 0 \\ \bar{L}C & A + B_u \bar{K} \end{bmatrix},$$

$\bar{Q}_T = \text{diag}(Q_L, Q + \bar{K}^T R \bar{K})$, $S_T = \text{diag}(S_L, S)$, and the fixed gains $\bar{K}$, $\bar{L}$ chosen so as to guarantee that Φ,i.e., the error system in (5.9), is asymptotically stable.

By means of standard computations, it is possible to write the cost (5.16) as follows

$$J_v = \sum_{i=t}^{t+N-1} \text{tr}(Q_{T,i}\Sigma_i) + \text{tr}(S_T \Sigma_{t+N}) \tag{5.18}$$

where $Q_{T,i} = \text{diag}(Q_L, Q + \bar{K}_i^T R \bar{K}_i)$. Similarly to the state-feedback case, from (5.14)-(5.16), it is apparent that the goal is twofold: to drive the mean $\bar{x}$ to zero by acting on the nominal input component $\bar{u}_{t,\ldots,t+N-1}$ and to minimize the variance of σ by acting on the gains $K_{t,\ldots,t+N-1}$ and $L_{t,\ldots,t+N-1}$. In addition, also the pair $(\bar{x}_t, \Sigma_t)$ must be considered as an additional argument of the MPC optimization, as later discussed, to guarantee recursive feasibility.

5.2.2 Terminal constraints

As usual in stabilizing MPC, see e.g. [100], some terminal constraints must be considered. Based on the same rationale used in the Chapter 3, also the terminal condition is applied through the second order description of the state variables. In the new setup, the mean $\bar{x}_{t+N}$ and the variance Σ_{t+N} at the end of the prediction horizon must satisfy

$$\bar{x}_{t+N} \in \bar{\mathbb{X}}_F \tag{5.19}$$

$$\Sigma_{t+N} \preceq \bar{\Sigma} \tag{5.20}$$

where $\bar{\mathbb{X}}_F$ is again the positively invariant set such that

$$(A + B_u \bar{K})\bar{x} \in \bar{\mathbb{X}}_F, \qquad \forall \bar{x} \in \bar{\mathbb{X}}_F \tag{5.21}$$

while $\bar{\Sigma}$ is obtained as the steady-state solution of the Lyapunov equation (5.10), i.e.,

$$\bar{\Sigma} = \Phi\bar{\Sigma}\Phi^T + \Psi\bar{\Omega}\Psi^T \tag{5.22}$$

where the matrix Φ has been already defined, $\Psi = \begin{bmatrix} B_w & -\bar{L} \\ 0 & \bar{L} \end{bmatrix}$ and the collective variance $\bar{\Omega} = \mathrm{diag}(\bar{W}, \bar{V})$ is built by considering (arbitrary) noise variances $\bar{W} \succeq W$ and $\bar{V} \succeq V$.
In addition, and consistently with (5.13), the following coupling conditions must be verified.

$$b_r^T \bar{x} \le x_r^{max} - \sqrt{b_r^T \bar{X} b_r} f(p_r^x), r = 1, \dots, n_r \tag{5.23a}$$

$$c_s^T \bar{K} \bar{x} \le u_s^{max} - \sqrt{c_s^T \bar{U} c_s} f(p_s^u), s = 1, \dots, n_s \tag{5.23b}$$

for all $\bar{x} \in \bar{\mathbb{X}}_F$, where

$$\bar{X} = \begin{bmatrix} I & I \end{bmatrix} \bar{\Sigma} \begin{bmatrix} I & I \end{bmatrix}^T, \bar{U} = \begin{bmatrix} 0 & \bar{K} \end{bmatrix} \bar{\Sigma} \begin{bmatrix} 0 & \bar{K} \end{bmatrix}^T \tag{5.24}$$

As it will be shown (see Theorem 4), (5.19) and (5.20) allow for recursive feasibility of the control scheme and enforce mean square convergence properties.
Note that in (5.22), the choice of $\bar{\Omega}$ is subject to a tradeoff. In fact, large variances $\bar{W}$ and $\bar{V}$ result in large $\bar{\Sigma}$ and, in view of (5.24), large $\bar{X}$ and $\bar{U}$. This enlarges the terminal constraint (5.20) but, on the other hand, reduces the size of the terminal set $\mathbb{X}_F$ compatible with (5.23).

5.3 Statement of the output feedback p-SMPC problem

The formulation of the main output-feedback p-SMPC problem requires a preliminary discussion concerning the initialization, similar to the one in Chapter 3 but with the main difference that here we have to deal with observed data.
In principle, and in order to use the most recent information available on the state, at each time instant it would be natural to set the current value of the nominal state $\bar{x}_{t|t}$ to the estimate $\hat{x}_t$ and the covariance matrix $\Sigma_{t|t}$ to $\mathrm{diag}(\Sigma_{11,t|t-1}, 0)$, where $\Sigma_{11,t|t-1}$ is the covariance of state prediction error e obtained using the observer (5.4) and cannot be reset to zero. However, as already discussed, since we do not exclude the possibility of unbounded

disturbances, in some cases this choice could lead to infeasible optimization problems. On the other hand, and in view of the terminal constraints (5.19), (5.20), it is quite easy to see that recursive feasibility is guaranteed provided that $\bar{x}$ is updated according to the prediction equation (5.6), which corresponds to the variance update given by (5.10). These considerations motivate the choice of accounting for the initial conditions $(\bar{x}_t, \Sigma_t)$ as free variables, which will be selected by the control algorithm (based on feasibility and optimality of the MPC optimization problem defined below) according to the following alternative strategies

Strategy 1 - Reset of the initial state: in the MPC optimization problem set $\bar{x}_{t|t} = \hat{x}_t$, $\Sigma_{t|t} = \mathrm{diag}(\Sigma_{11,t|t-1}, 0)$. This corresponds to using all the information available through the observer at time t.

Strategy 2 - Prediction: in the MPC optimization problem set $\bar{x}_{t|t} = \bar{x}_{t|t-1}$, $\Sigma_{t|t} = \Sigma_{t|t-1}$. This corresponds to using the optimal prediction computed from the past optimal solution for both the expected value and the variance of the state.

The output-feedback p-SMPC problem can now be stated.

S-MPC problem: at any time instant t solve

$$\min_{\substack{\bar{x}_t, \Sigma_t, \bar{u}_{t,\ldots,t+N-1},\\ K_{t,\ldots,t+N-1}, L_{t,\ldots,t+N-1}}} J$$

where J is defined in (5.14), (5.15), (5.16), subject to

- the dynamics (5.6) and (5.10);
- the constraints (5.13) for all $i = t, \ldots, t+N-1$, $r = 1, \ldots, n_r$, $s = 1, \ldots, n_s$;
- the initialization constraint (corresponding to the choice between strategies S1 and S2)

$$(\bar{x}_t, \Sigma_t) \in \{(\hat{x}_t, \mathrm{diag}(\Sigma_{11,t|t-1}, 0)), (\bar{x}_{t|t-1}, \Sigma_{t|t-1})\} \tag{5.25}$$

- the terminal constraints (5.19), (5.20). $\square$

Denoting by $\bar{u}_{t,\ldots,t+N-1|t} = \{\bar{u}_{t|t}, \ldots, \bar{u}_{t+N-1|t}\}$, $K_{t,\ldots,t+N-1|t} = \{K_{t|t}, \ldots, K_{t+N-1|t}\}$, $L_{t,\ldots,t+N-1|t} = \{L_{t|t}, \ldots, L_{t+N-1|t}\}$, and $(\bar{x}_{t|t}, \Sigma_{t|t})$ the optimal solution to the p-SMPC problem, the feedback control law actually used is then given by (5.5) with $\bar{u}_t = \bar{u}_{t|t}$, $K_t = K_{t|t}$, and the state observation evolves as in (5.4) with $L_t = L_{t|t}$.

Now, define the p-SMPC problem feasibility set as

$$\Xi := \left\{ (\bar{x}_0, \Sigma_0) : \begin{array}{l} \exists \bar{u}_{0,\dots,N-1}, K_{0,\dots,N-1}, L_{0,\dots,N-1} \text{ such that (5.6), (5.10)} \\ \text{and (5.13) hold for all } k = 0, \dots, N-1 \text{ and (5.19),} \\ \text{(5.20) are verified.} \end{array} \right\}$$

Some comments are in order.
1) At the initial time $t = 0$, the algorithm must be initialized by setting $\bar{x}_{0|0} = \hat{x}_0$ and $\Sigma_{0|0} = \mathrm{diag}(\Sigma_{11,0}, 0)$. In view of this, feasibility at time $t = 0$ amounts to $(\hat{x}_0, \Sigma_{0|0}) \in \Xi$.
2) The binary choice between strategies S1 and S2 requires to solve at any time instant two optimization problems. However, similarly to the state-feedback case, the following sequential procedure can be adopted to reduce the average overall computational burden: the optimization problem corresponding to strategy S1 is first solved and, if it is infeasible, strategy S2 must be solved and adopted. On the contrary, if the problem with strategy S1 is feasible, it is possible to compare the resulting value of the optimal cost function with the value of the cost that corresponds to the feasible sequences $\{\bar{u}_{t|t-1}, \dots, \bar{u}_{t+N-2|t-1}, \bar{K}\bar{x}_{t+N-1|t}\}$, $\{K_{t|t-1}, \dots, K_{t+N-2|t-1}, \bar{K}\}$ and $\{L_{t|t-1}, \dots, L_{t+N-2|t-1}, \bar{L}\}$. If the optimal cost with strategy S1 is lower, strategy S1 can be used without solving the MPC problem for strategy S2. This does not guarantee optimality, but the convergence properties of the method, stated in the result below, are recovered while the computational effort is reduced.

Now we are in the position to state the main result concerning the convergence properties of the algorithm.

Theorem 2. *If, at $t = 0$, the p-SMPC problem admits a solution, it is recursively feasible and the state and the input probabilistic constraints* (5.2) *and* (5.3) *are satisfied for all $t \geq 0$. Furthermore, if there exists $\rho \in (0, 1)$ such that the noise variance Ω verifies*

$$\frac{(N + \frac{\beta}{\alpha})}{\alpha} \mathrm{tr}(S_T \Psi \Omega \Psi^T) < \min(\rho \bar{\sigma}^2, \rho \lambda_{min}(\bar{\Sigma})) \tag{5.26}$$

where $\bar{\sigma}$ is the maximum radius of a ball, centered at the origin, included *in* $\bar{\mathbb{X}}_F$*, and*

$$\alpha = \min\{\lambda_{min}(Q), \mathrm{tr}\{Q^{-1} + Q_L^{-1}\}^{-1}\} \tag{5.27a}$$
$$\beta = \max\{\lambda_{max}(S), \mathrm{tr}\{S_T\}\} \tag{5.27b}$$

then, as $t \to +\infty$

$$\mathrm{dist}(\|\bar{x}_t\|^2 + \mathrm{tr}\{\Sigma_{t|t}\}, [0, \frac{1}{\alpha}(N + \frac{\beta}{\alpha})\,\mathrm{tr}(S_T \Psi \Omega \Psi^T)]) \to 0 \tag{5.28}$$

Note that, as expected, for smaller and smaller values of Ω, also $\|\bar{x}_t\|$ and $\mathrm{tr}\{\Sigma_{t|t}\}$ tend to zero.

5.4 Implementation issues

In this section two issues are addressed. First, the non linear constraints (5.13) and the non linear dependence of the covariance evolution, see (5.10), on $K_{t,\dots,t+N-1}, L_{t,\dots,t+N-1}$ make the numerical solution of S-MPC impractical. In sections 5.4.1 and 5.4.2 two possible solutions, inspired by the one adopted in Chapter 3, are described, allowing to cast the p-SMPC problem as a quadratic one, with linear constraints.
The second issue concerns the fact that, in our framework, deterministic constraints on input variable (e.g., saturations) are not accounted for. In Section 5.4.3 we propose some possible solutions to this problem.

5.4.1 Approximation of the output-feedback p-SMPC for allowing a solution with LMIs

A solution, based on an approximation of S-MPC characterized by linear constraints solely, is now presented. First define $A^D = \sqrt{2}A, B_u^D = \sqrt{2}B_u, C^D = \sqrt{2}C$, and $V^D = 2V$ and let the auxiliary gain matrices $\bar{K}$ and $\bar{L}$ be selected according to the following assumption.

Assumption 1. *The gains* $\bar{K}$ *and* $\bar{L}$ *are computed as the steady-state gains of the LQG regulator for the system* (A^D, B^D, C^D)*, with state and control weights* Q *and* R*, and noise covariances* $\bar{W} \succeq W$ *and* $\bar{V} \succeq V^D$*.*

Note that, if a gain matrix $\bar{K}$ (respectively $\bar{L}$) is stabilizing for $(A^D + B_u^D \bar{K}) = \sqrt{2}(A + B_u\bar{K})$ (respectively $(A^D - \bar{L}C^D) = \sqrt{2}(A - \bar{L}C)$), it is also stabilizing for $(A + B_u\bar{K})$ (respectively $(A - \bar{L}C)$), i.e., for the original system. The following preliminary result can be stated.

Lemma 3. *Define* $A_{L_t}^D = A^D - L_t C^D$, $A_{K_t}^D = A^D + B_u^D K_t$, *the block diagonal matrix* $\Sigma_t^D = \mathrm{diag}(\Sigma_{11,t}^D, \Sigma_{22,t}^D)$, $\Sigma_{11,t}^D \in \mathbb{R}^{n\times n}$, $\Sigma_{22,t}^D \in \mathbb{R}^{n\times n}$ *and the update equations*

$$\Sigma_{11,t+1}^D = A_{L_t}^D \Sigma_{11,t}^D (A_{L_t}^D)^T + B_w W B_w^T + L_t V^D L_t^T \tag{5.29a}$$

$$\Sigma_{22,t+1}^D = A_{K_t}^D \Sigma_{22,t}^D (A_{K_t}^D)^T + L_t C^D \Sigma_{11,t}^D C^{D\,T} L_t^T + L_t V^D L_t^T \tag{5.29b}$$

Then
I) $\Sigma_t^D \succeq \Sigma_t$ *implies that* $\Sigma_{t+1}^D = \mathrm{diag}(\Sigma_{11,t+1}^D, \Sigma_{22,t+1}^D) \succeq \Sigma_{t+1}$.
II) We can rewrite as LMIs the following inequalities

$$\Sigma_{11,t+1}^D \succeq A_{L_t}^D \Sigma_{11,t}^D (A_{L_t}^D)^T + B_w W B_w^T + L_t V^D L_t^T \tag{5.30a}$$

$$\Sigma_{22,t+1}^D \succeq A_{K_t}^D \Sigma_{22,t}^D (A_{K_t}^D)^T + L_t C^D \Sigma_{11,t}^D C^{D\,T} L_t^T + L_t V^D L_t^T \tag{5.30b}$$

Based on Lemma 3-II, we can reformulate the original problem so that the covariance matrix Σ^D is used instead of Σ. Accordingly, the update equation (5.10) is replaced by (5.29) and the output feedback p-SMPC problem is recast as an LMI one (see Appendix 5.6), greatly reducing the required computational effort at the price of a slight extra tightening.
As for the nonlinear dependence of the inequalities (5.13) on the covariance matrices X_t and U_t, it is again possible to prove that they are satisfied if

$$b_r^T \bar{x}_i \leq (1 - 0.5\alpha_x) x_r^{max} - \frac{b_r^T X_i b_r}{2\alpha_x x_r^{max}} f(p_r^x)^2 \tag{5.31a}$$

$$c_s^T \bar{u}_i \leq (1 - 0.5\alpha_u) u_s^{max} - \frac{c_s^T U_i c_s}{2\alpha_u u_s^{max}} f(p_s^u)^2 \tag{5.31b}$$

where $\alpha_x \in (0, 1]$ and $\alpha_u \in (0, 1]$ are free design parameters (for a more detailed explanation see Chapter 3). Also, note that $X_t \preceq \begin{bmatrix} I & I \end{bmatrix} \Sigma_t^D \begin{bmatrix} I & I \end{bmatrix}^T = \Sigma_{11,t}^D + \Sigma_{22,t}^D$ and that $U_t \preceq \begin{bmatrix} 0 & K_t \end{bmatrix} \Sigma_t^D \begin{bmatrix} 0 & K_t \end{bmatrix}^T = K_t \Sigma_{22,t}^D K_t^T$ so that, defining $X_t^D = \Sigma_{11,t}^D + \Sigma_{22,t}^D$ and $U_t^D = K_t \Sigma_{22,t}^D K_t^T$, (5.31) can be written as follows

$$b_r^T \bar{x}_i \leq (1 - 0.5\alpha_x) x_r^{max} - \frac{b_r^T X_i^D b_r}{2\alpha_x x_r^{max}} f(p_r^x)^2 \tag{5.32a}$$

$$c_s^T \bar{u}_i \leq (1 - 0.5\alpha_u) u_s^{max} - \frac{c_s^T U_i^D c_s}{2\alpha_u u_s^{max}} f(p_s^u)^2 \tag{5.32b}$$

Note that the reformulation of (5.13) into (5.32) has been performed at the price of additional constraint tightening. For example, on the right hand side of (5.32a), x_r^{max} is replaced by $(1 - 0.5\alpha^x)x_r^{max}$, which significantly reduces the size of the constraint set. Parameter α^x cannot be reduced at will, since it also appears at the denominator in the second additive term.
In view of Assumption 1 and resorting to the separation principle, it is possible to show [52] that the solution $\bar{\Sigma}^D$ to the steady-state equation

$$\bar{\Sigma}^D = \Phi^D \bar{\Sigma}^D (\Phi^D)^T + \Psi \bar{\Omega} \Psi^T \tag{5.33}$$

is block-diagonal, i.e., $\bar{\Sigma}^D = \mathrm{diag}(\bar{\Sigma}_{11}^D, \bar{\Sigma}_{22}^D)$, where

$$\Phi^D = \begin{bmatrix} A^D - \bar{L}C^D & 0 \\ \bar{L}C^D & A^D + B_u^D \bar{K} \end{bmatrix}$$

Based on the same rational, the terminal constraint (5.20) must be transformed into $\Sigma_{t+N}^D \preceq \bar{\Sigma}^D$ which corresponds to setting

$$\Sigma_{11,t+N}^D \preceq \bar{\Sigma}_{11}^D, \qquad \Sigma_{22,t+N}^D \preceq \bar{\Sigma}_{22}^D \tag{5.34}$$

taking advantage of the block-diagonal form.
Defining $\bar{X}^D = \bar{\Sigma}_{11}^D + \bar{\Sigma}_{22}^D$ and $\bar{U}^D = \bar{K}\bar{\Sigma}_{22}^D \bar{K}^T$, the terminal set condition (5.23) must now be reformulated as

$$b_r^T \bar{x} \leq (1 - 0.5\alpha^x)x_r^{max} - \frac{b_r^T \bar{X}^D b_r}{2\alpha^x x_r^{max}} f(p_r^x)^2 \tag{5.35a}$$

$$c_s^T \bar{K}\bar{x} \leq (1 - 0.5\alpha^u)u_s^{max} - \frac{c_s^T \bar{U}^D c_s}{2\alpha^u u_s^{max}} f(p_s^u)^2 \tag{5.35b}$$

for all $\bar{x} \in \bar{\mathbb{X}}_F$.
Also J_v must be reformulated. Indeed

$$\begin{aligned} J_v \leq J_v^D = \sum_{i=t}^{t+N-1} \mathrm{tr}\left\{Q_L \Sigma_{11,i}^D + Q\Sigma_{22,i}^D + RK_i \Sigma_{22,i}^D K_i^T\right\} \\ + \mathrm{tr}\left\{S_L \Sigma_{11,t+N}^D + S\Sigma_{22,t+N}^D\right\} \end{aligned} \tag{5.36}$$

where the terminal weights S and S_L must now satisfy the following Lyapunov-like inequalities

$$\begin{aligned} &(\bar{A}_K^D)^T S \bar{A}_K^D - S + Q + \bar{K}^T R \bar{K} \preceq 0 \\ &(\bar{A}_L^D)^T S_L \bar{A}_L^D - S_L + Q_L + (C^D)^T \bar{L}^T S \bar{L} C^D \preceq 0 \end{aligned} \tag{5.37}$$

where $\bar{A}_K^D = A^D + B_u^D \bar{K}$ and $\bar{A}_L^D = A^D - \bar{L}C^D$.

It is now possible to formally state what we call the p-SMPCl problem (to denote that here the gains are considered as free variables in opposition to what we state in the next section).

S-MPCl problem: at any time instant t solve

$$\min_{\substack{\bar{x}_t,\, \Sigma_{11,t}^D,\, \Sigma_{22,t}^D,\, \bar{u}_{t,\ldots,t+N-1},\\ K_{t,\ldots,t+N-1},\, L_{t,\ldots,t+N-1}}} J$$

where J is defined in (5.14), (5.15), (5.36), subject to

- the dynamics (5.6) and (5.29);
- the linear constraints (5.32) for all $i = t, \ldots, t+N-1$, $r = 1, \ldots, n_r$, $s = 1, \ldots, n_s$;
- the initialization constraint, corresponding to the choice between strategies S1 and S2, i.e., $(\bar{x}_t, \Sigma_{11,t}^D, \Sigma_{22,t}^D) \in \{(\hat{x}_t, \Sigma_{11,t|t-1}^D, 0), (\bar{x}_{t|t-1}, \Sigma_{11,t|t-1}^D, \Sigma_{22,t|t-1}^D)\}$
- the terminal constraints (5.19), (5.34).

□

The following corollary follows from Theorem 4.

Corollary 1. *If, at time $t = 0$, the p-SMPCl problem admits a solution, it is recursively feasible and the state and input probabilistic constraints* (5.2) *and* (5.3) *are satisfied for all $t \geq 0$. Furthermore, if there exists $\rho \in (0,1)$ such that the noise variance $\Omega^D = \mathrm{diag}(W, V^D)$ verifies*

$$\frac{(N + \frac{\beta}{\alpha})}{\alpha} \mathrm{tr}(S_T \Psi \Omega^D \Psi^T) < \min(\rho\bar{\sigma}^2, \rho\lambda_{min}(\bar{\Sigma}^D)) \tag{5.38}$$

then, as $t \to +\infty$

$$\mathrm{dist}(\|\bar{x}_t\|^2 + \mathrm{tr}\{\Sigma_{t|t}^D\}, [0, \frac{1}{\alpha}(N + \frac{\beta}{\alpha})\, \mathrm{tr}(S_T \Psi \Omega^D \Psi^T)]) \to 0$$

The proof of Corollary 1 readily follows from the proof of Theorem 4. For details please see [46].

5.4.2 Approximation of p-SMPC with constant gains

The solution presented in the following is characterized by a great simplicity and consists in setting $L_t = \bar{L}$ and $K_t = \bar{K}$ for all $t \geq 0$. In this case, the value of Σ_{t+k} (and therefore of X_{t+k} and U_{t+k}) can be directly computed for any $k > 0$ by means of (5.10) as soon as Σ_t is given. As a byproduct, the nonlinearity in the constraints (5.13) does not carry about implementation problems. Therefore, this solution has a twofold advantage: first, it is simple and requires an extremely lightweight implementation; secondly, it allows for the use of nonlinear less conservative constraint formulations. In this simplified framework, the following problem, denoted in the rest as p-SMPCc in contrast to the p-SMPCl, can be stated.

p-SMPCc problem: at any time instant t solve

$$\min_{\bar{x}_t, \Sigma_t, \bar{u}_{t,\ldots,t+N-1}} J$$

where J is defined in (5.14), (5.15), (5.16), subject to

- the dynamics (5.6) , with $K_t = \bar{K}$, and

$$\Sigma_{t+1} = \Phi \Sigma_t \Phi^T + \Psi \Omega \Psi^T \tag{5.39}$$

- the constraints (5.13) for all $i = t, \ldots, t+N-1$, $r = 1, \ldots, n_r$, $s = 1, \ldots, n_s$;
- the initialization constraint (5.25);
- the terminal constraints (5.19), (5.20).

□

An additional remark is due. The term J_v in (5.18) does not depend only on the control and observer gain sequences $K_{t,\ldots,t+N-1}$, $L_{t,\ldots,t+N-1}$, but also on the initial condition Σ_t. Therefore, it is not possible to discard it in this simplified formulation.
The following corollary can be derived from Theorem 4.

Corollary 2. *If, at time $t = 0$, the p-SMPCc problem admits a solution, it is recursively feasible and the state and input probabilistic constraints* (5.2) *and* (5.3) *are satisfied for all $t \geq 0$. Furthermore, if there exists $\rho \in (0, 1)$ such that the noise variance Ω verifies* (5.26)*, then, as $t \to +\infty$,* (5.28) *holds.*

The proof of Corollary 2 readily follows from the proof of Theorem 4. For details please see [46].

5.4.3 Boundedness of the input variables

Both the state-feedback p-SMPC presented in Chapter 3 and the output-feedback p-SMPC scheme described in the previous sections cannot handle hard constraints on the input variables. However, in general input variables are bounded in practice, and may be subject to

$$Hu_t \leq \mathbf{1} \tag{5.40}$$

where $H \in \mathbb{R}^{n_H \times m}$ is a design matrix and $\mathbf{1}$ is a vector of dimension n_H whose entries are equal to 1. Three possible approaches are proposed to account for this case.

- Inequalities (5.40) can be stated as additive probabilistic constraints (5.3) with small violation probabilities p_s^u. This solution, although not guaranteeing satisfaction of (5.40) with probability 1, is simple and easy.

- In the p-SMPCc scheme, define the gain matrix $\bar{K}$ in such a way that $A + B_u\bar{K}$ is asymptotically stable and, at the same time, $H\bar{K} = 0$. From (5.5), it follows that $Hu_t = H\bar{u}_t + H\bar{K}(\hat{x}_t - \bar{x}_t) = H\bar{u}_t$. Therefore, to verify (5.40) it is sufficient to include in the problem formulation the deterministic constraint $H\bar{u}_t \leq \mathbf{1}$.

- In the p-SMPCc scheme, if probabilistic constraints on u are absent, replace (5.5) with $u_t = \bar{u}_t$ and set $H\bar{u}_t \leq \mathbf{1}$ in the p-SMPC optimization problem to verify (5.40). If we also define $\hat{u}_t = \bar{u}_t + \bar{K}(\hat{x}_t - \bar{x}_t)$ as the input to equation (5.4), the dynamics of variable σ_t is given by (5.9) with

$$\Phi_t = \begin{bmatrix} A - \bar{L}C & B_u\bar{K} \\ \bar{L}C & A + B_u\bar{K} \end{bmatrix}$$

  and the arguments follow similarly to those proposed in the paper. It is worth mentioning, however, that matrix Φ_t must be asymptotically stable, which requires asymptotic stability of A.

5.5 Proof of the main Theorem

Recursive feasibility is first proved. Assume that, at time instant t, a feasible solution of S-MPC is available, i.e., $(\bar{x}_{t|t}, \Sigma_{t|t}) \in \Xi$ with optimal sequences $\bar{u}_{t,\ldots,t+N-1|t}$, $K_{t,\ldots,t+N-1|t}$, and $L_{t,\ldots,t+N-1|t}$. We prove that at time $t+1$ a feasible solution exists, i.e., in view of the initialization strategy S2, $(\bar{x}_{t+1|t}, \Sigma_{t+1|t}) \in \Xi$ with admissible sequences $\bar{u}^f_{t+1,\ldots,t+N|t} = \{\bar{u}_{t+1|t}, \ldots, \bar{u}_{t+N-1|t}, \bar{K}\bar{x}_{t+N|t}\}$, $K^f_{t+1,\ldots,t+N|t} = \{K_{t+1|t}, \ldots, K_{t+N-1|t}, \bar{K}\}$, and $L^f_{t+1,\ldots,t+N|t} = \{L_{t+1|t}, \ldots, L_{t+N-1|t}, \bar{L}\}$.

Constraint (5.13a) is verified for all pairs $(\bar{x}_{t+1+k|t}, X_{t+1+k|t})$, $k = 0, \ldots, N-2$, in view of the feasibility of p-SMPC at time t. Furthermore, in view of (5.19), (5.20), (5.24), and the condition (5.23a),we have that $b^T\bar{x}_{t+N|t} \leq x^{max} - \sqrt{b^T X_{t+N|t} b} f(p^x_r)$, i.e., constraint (5.13a) is verified.
Analogously, constraint (5.13b) is verified for all pairs $(\bar{u}_{t+1+k|t}, U_{t+1+k|t})$, $k = 0, \ldots, N-2$, in view of the feasibility of p-SMPC at time t. Furthermore, in view of (5.19), (5.20), (5.24), and the condition (5.23b), we have that $c^T\bar{K}\bar{x}_{t+N|t} \leq u^{max} - \sqrt{c^T U_{t+N|t} c} f(p^u_s)$, i.e., constraint (5.13b) is verified.
In view of (5.19) and of the invariance property (5.21) it follows that $\bar{x}_{t+N+1|t} = (A + B_u\bar{K})\bar{x}_{t+N|t} \in \bar{\mathbb{X}}_F$ and, in view of (5.20), (5.22) $\Sigma_{t+N+1|t} \preceq \Phi\bar{\Sigma}\Phi^T + \Psi\bar{\Omega}\Psi^T = \bar{\Sigma}$, hence verifying both (5.19) and (5.20) at time $t+1$.

The proof of convergence is partially inspired by [136]. In view of the feasibility, at time $t+1$ of the possibly suboptimal solution $\bar{u}^f_{t+1,\ldots,t+N|t}$, $K^f_{t+1,\ldots,t+N|t}$, $L^f_{t+1,\ldots,t+N|t}$, and $(\bar{x}_{t+1|t}, \Sigma_{t+1|t})$, we have that the optimal cost function computed at time $t+1$ is $J^*(t+1) = J^*_m(t+1) + J^*_v(t+1)$ [1].
In view of the optimality of $J^*(t+1)$

$$J^*(t+1) \leq J_m(\bar{x}_{t+1|t}, \bar{u}^f_{t+1,\ldots,t+N|t}) + J_v(\Sigma_{t+1|t}, K^f_{t+1,\ldots,t+N|t}, L^f_{t+1,\ldots,t+N|t}) \tag{5.41}$$

Note that, in view of (5.17)

$$J_m(\bar{x}_{t+1|t}, \bar{u}^f_{t+1,\ldots,t+N|t}) \leq J_m(\bar{x}_{t|t}, \bar{u}_{t,\ldots,t+N-1|t}) - \|\bar{x}_{t|t}\|^2_Q - \|\bar{u}_{t|t}\|^2_R \tag{5.42}$$

Furthermore

$$J_m(\bar{x}_{t|t}, \bar{u}_{t,\ldots,t+N-1|t}) = J^*_m(t) \tag{5.43}$$

[1] For brevity, we denote $J^*(x_t, \bar{x}_{t|t-1}, \Sigma_{t|t-1})$ with $J^*(t)$, $J^*_m(x_t, \bar{x}_{t|t-1}, \Sigma_{t|t-1})$ with $J^*_m(t)$, and $J^*_v(x_t, \bar{x}_{t|t-1}, \Sigma_{t|t-1})$ with $J^*_v(t)$

Now consider J_v in (5.18) and note that, in view of the properties of the trace and (5.17)

$$\begin{aligned} & J_v(X_{t+1|t}, K^f_{t+1,\dots,t+N|t}, L^f_{t+1,\dots,t+N|t}) \\ & \leq J_v(X_{t|t}, K_{t,\dots,t+N-1|t}, L_{t,\dots,t+N-1|t}) \\ & - \operatorname{tr}\{ \begin{bmatrix} Q_L & 0 \\ 0 & Q + K^T_{t|t} R K_{t|t} \end{bmatrix} \Sigma_{t|t} \} + \operatorname{tr}(S_T \Psi \Omega \Psi^T) \end{aligned} \tag{5.44}$$

From (6.32)-(5.44) we obtain

$$\begin{aligned} & J^*(t+1) \leq J^*(t) - (\|\bar{x}_{t|t}\|^2_Q + \|\bar{u}_{t|t}\|^2_R) \\ & -\operatorname{tr}\{ \begin{bmatrix} Q_L & 0 \\ 0 & Q + K^T_{t|t} R K_{t|t} \end{bmatrix} \Sigma_{t|t} \} + \operatorname{tr}(S_T \Psi \Omega \Psi^T) \end{aligned} \tag{5.45}$$

Furthermore, from the definition of $J^*(t)$

$$\begin{aligned} J^*(t) \geq & \|\bar{x}_{t|t}\|^2_Q + \|\bar{u}_{t|t}\|^2_R \\ & + \operatorname{tr} \left\{ \begin{bmatrix} Q_L & 0 \\ 0 & Q + K^T_{t|t} R K_{t|t} \end{bmatrix} \Sigma_{t|t} \right\} \end{aligned} \tag{5.46}$$

Now, denote $\Omega_F = \{(\bar{x}, \Sigma) : \bar{x} \in \bar{\mathbb{X}}_F, \Sigma \preceq \bar{\Sigma}\}$. Assuming that $(\bar{x}_{t|t}, \Sigma_{t|t}) \in \Omega_F$ we have that $J^*(t) \leq J^{aux}_m(t) + J^{aux}_v(t)$, where

$$\begin{aligned} J^{aux}_m(t) &= \textstyle\sum_{k=0}^{N-1} \|(A + B_u \bar{K})^k \bar{x}_{t|t}\|^2_Q \\ & + \|\bar{K}(A + B_u \bar{K})^k \bar{x}_{t|t}\|^2_R + \|(A + B_u \bar{K})^N \bar{x}_{t|t}\|^2_S \end{aligned}$$

since $\{\bar{K}\bar{x}_{t|t}, \dots, \bar{K}(A+B_u\bar{K})^{N-1}\bar{x}_{t|t}\}$ is a feasible input sequence. Therefore, from (5.17),

$$J^{aux}_m(t) \leq \|\bar{x}_{t|t}\|^2_S \tag{5.47}$$

Similarly, recalling (5.17), we obtain that

$$J^{aux}_v(t) \leq \operatorname{tr}\{S_T \Sigma_{t|t}\} + Ntr\{S_T \Psi \Omega \Psi^T\} \tag{5.48}$$

Combining (5.47) and (5.48) we obtain that, for all $(\bar{x}_{t|t}, \Sigma_{t|t}) \in \Omega_F$

$$J^*(t) \leq \|\bar{x}_{t|t}\|^2_S + \operatorname{tr}\{S_T \Sigma_{t|t}\} + N \operatorname{tr}\{S_T \Psi \Omega \Psi^T\} \tag{5.49}$$

From (5.45), (5.46) and (5.49) it is possible to derive robust stability-related results. Before to proceed, recall that $\mathrm{tr}\{S_T\Sigma_{t|t}\} = \mathrm{tr}\{S_T^{\frac{1}{2}T}\Sigma_{t|t}S_T^{\frac{1}{2}}\}$ where $S_T^{\frac{1}{2}}$ is a matrix that verifies $S_T^{\frac{1}{2}T}S_T^{\frac{1}{2}} = S_T$. Therefore

$$\mathrm{tr}\{S_T\Sigma_{t|t}\} = \mathrm{tr}\{S_T^{\frac{1}{2}T}\Sigma_{t|t}S_T^{\frac{1}{2}}\} = \|\Sigma_{t|t}^{\frac{1}{2}}S_T^{\frac{1}{2}}\|_F^2$$

On the other hand, denoting $Q_{T|t} = \mathrm{diag}(Q_L, Q + K_{t|t}^T R K_{t|t})$, it follows that $\mathrm{tr}\{Q_{T|t}\Sigma_{t|t}\} = \|\Sigma_{t|t}^{\frac{1}{2}}Q_{T|t}^{\frac{1}{2}}\|_F^2$. Moreover,

$$\mathrm{tr}\{S_T\Sigma_{t|t}\} \leq \|\Sigma_{t|t}^{\frac{1}{2}}\|_F^2\|S_T^{\frac{1}{2}}\|_F^2 = \mathrm{tr}\{S_T\}\mathrm{tr}\{\Sigma_{t|t}\}$$

(where $\|\cdot\|_F$ is the Frobenius norm) and, in view of the matrix inversion Lemma,

$$\begin{aligned}\mathrm{tr}\{Q_{T|t}\Sigma_{t|t}\} &\geq (\|Q_{T|t}^{-\frac{1}{2}}\|_F^2)^{-1}\|\Sigma_{t|t}^{\frac{1}{2}}\|_F^2\\ &\geq \mathrm{tr}\{(\mathrm{diag}(Q_L, Q))^{-1}\}^{-1}\mathrm{tr}\{\Sigma_{t|t}\}\\ &= \mathrm{tr}\{Q^{-1} + Q_L^{-1}\}^{-1}\mathrm{tr}\{\Sigma_{t|t}\}.\end{aligned}$$

Define $V(\bar{x}_{t|t}, \Sigma_{t|t}) = \|\bar{x}_{t|t}\|^2 + \mathrm{tr}\{\Sigma_{t|t}\}$ and $\omega = \mathrm{tr}\{S_T\Psi\Omega\Psi^T\}$. In view of this, we can reformulate (5.45), (5.46) and (5.49) as follows.

$$J^*(t+1) \leq J^*(t) - \alpha V(\bar{x}_{t|t}, \Sigma_{t|t}) + \omega \tag{5.50a}$$

$$J^*(t) \geq \alpha V(\bar{x}_{t|t}, \Sigma_{t|t}) \tag{5.50b}$$

$$J^*(t) \leq \beta V(\bar{x}_{t|t}, \Sigma_{t|t}) + N\omega \tag{5.50c}$$

If $(\bar{x}_{t|t}, \Sigma_{t|t}) \in \Omega_F$ then, in view of (5.50c), (5.50a)

$$J^*(t+1) \leq J^*(t)(1 - \frac{\alpha}{\beta}) + (\frac{\alpha}{\beta}N + 1)\omega \tag{5.51}$$

Let $\eta \in (\rho, 1)$ and denote $b = \frac{1}{\eta}(N + \frac{\beta}{\alpha})$. In view of (5.50b), if $J^*(t) \leq b\,\omega$ then $V(\bar{x}_{t|t}, \Sigma_{t|t}) \leq \frac{b}{\alpha}\omega$. This, considering (5.26), implies that

$$\|\bar{x}_{t|t}\|^2 \leq \frac{\rho}{\eta}\bar{\sigma}^2,\ \mathrm{tr}(\Sigma_{t|t}) \leq \frac{\rho}{\eta}\lambda_{min}(\bar{\Sigma}) \tag{5.52}$$

In view of (5.52), then $\bar{x}_{t|t} \in \bar{\mathbb{X}}_F$ and $\lambda_{max}(\Sigma_{t|t}) < \lambda_{min}(\bar{\Sigma})$, which in turn implies that $\Sigma_{t|t} < \bar{\Sigma}$. Therefore, recalling (5.51), if $J^*(t) \leq b\,\omega$, then $J^*(t+1) \leq b\,\omega$ and the positive invariance of the set $D = \{(\bar{x}, \Sigma) : J^*(t) \leq b\,\omega\}$ is guaranteed.
If $(\bar{x}_{t|t}, \Sigma_{t|t}) \in \Omega_F \backslash D$, it holds that $J^*(t) > b\,\omega$ which, in view of (5.50c), implies that

$$V(\bar{x}_{t|t}, \Sigma_{t|t}) > \frac{1}{\alpha}\omega \tag{5.53}$$

Since $(\bar{x}_{t|t}, \Sigma_{t|t}) \in \Omega_F \backslash D$, recalling (5.51), (5.53), and (5.50b), there exists $\bar{c}_1 > 0$ (function of η) such that

$$J^*(t+1) - J^*(t) \leq -(1-\eta)\frac{\alpha^2}{\beta} V(\bar{x}_{t|t}, \Sigma_{t|t}) \leq -\bar{c}_1 \tag{5.54}$$

On the other hand, for all x_t with $(\bar{x}_{t|t}, \Sigma_{t|t}) \in \Xi \backslash \Omega_F$, there exists constant $\bar{c}_2 > 0$ such that there exists x_Ω with $(\bar{x}_\Omega, \Sigma_\Omega) \in \Omega_F \backslash D$ such that $-\alpha V(\bar{x}_{t|t}, \Sigma_{t|t}) \leq -\alpha V(\bar{x}_\Omega, \Sigma_\Omega) - \bar{c}_2$. This, in view of (5.50a) and (5.54), implies that

$$J^*(t+1) - J^*(t) < -\bar{c}_2 \tag{5.55}$$

In view of (5.54)-(5.55), for all x_t with $(\bar{x}_{t|t}, \Sigma_{t|t}) \in \Xi \backslash D$ there exists $\bar{c}$ (function of η)

$$J^*(t+1) - J^*(t) < -\bar{c} \tag{5.56}$$

This implies that, for each $\eta \in (\rho, 1)$, there exists $T > 0$ such that x_{t+T} is such that $(\bar{x}_{t+T|t+T}, \Sigma_{t+T|t+T}) \in D$, i.e., that $\alpha V(\bar{x}_{t+k|t+k}, \Sigma_{t+k|t+k}) \leq b\omega$ for all $k \geq T$. This, for $\eta \to 1$, implies (5.28).

5.6 The p-SMPCI problem

5.6.1 Proof of Lemma 3

For the proof of Part I., the following result is used.

Lemma 4. *Given a positive semi-definite, symmetric matrix M, then*

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix} \preceq \begin{bmatrix} 2M_{11} & 0 \\ 0 & 2M_{22} \end{bmatrix}$$

Proof of Lemma 4 Since $M \succeq 0$, then

$$\begin{bmatrix}-x_1^T & x_2^T\end{bmatrix} M \begin{bmatrix}-x_1 \\ x_2\end{bmatrix} = x_1^T M_{11} x_1 + x_2^T M_{22} x_2 - x_1^T M_{12} x_2 - x_2^T M_{12}^T x_1 \succeq 0$$

for all x_1, x_2 such that $\begin{bmatrix}x_1^T & x_2^T\end{bmatrix} \neq 0$. From this, we obtain that

$$\begin{aligned}\begin{bmatrix}x_1^T & x_2^T\end{bmatrix} M \begin{bmatrix}x_1 \\ x_2\end{bmatrix} &= x_1^T M_{11} x_1 + x_2^T M_{22} x_2 + x_1^T M_{12} x_2 + x_2^T M_{12}^T x_1 \\ &\leq 2x_1^T M_{11} x_1 + 2x_2^T M_{22} x_2 = \begin{bmatrix}x_1^T & x_2^T\end{bmatrix} \begin{bmatrix}2M_{11} & 0 \\ 0 & 2M_{22}\end{bmatrix} \begin{bmatrix}x_1 \\ x_2\end{bmatrix}\end{aligned}$$

for all x_1, x_2 such that $\begin{bmatrix}x_1^T & x_2^T\end{bmatrix} \neq 0$. This concludes the proof of Lemma 4. □

Consider now matrix Σ_t and its block decomposition

$$\Sigma_t = \begin{bmatrix}\Sigma_{11,t} & \Sigma_{12,t} \\ \Sigma_{12,t}^T & \Sigma_{22,t}\end{bmatrix}$$

where $\Sigma_{ij,t} \in \mathbb{R}^{n\times n}$ for all $i, j = 1, 2$. A bound for the time evolution of the covariance matrix Σ_t is computed, iteratively, considering that

$$\Sigma_{t+1} \preceq \Phi_t \Sigma_t^D \Phi_t^T + \Psi_t \Omega \Psi_t^T \tag{5.57}$$

If we define $\Sigma_{t+1}^D = \mathrm{diag}(\Sigma_{11,t+1}^D, \Sigma_{22,t+1}^D)$, where

$$\begin{aligned}\Sigma_{11,t+1}^D = &\, 2(A - L_t C)\Sigma_{11,t}^D (A - L_t C)^T + \\ &\, B_w W B_w^T + 2L_t V L_t^T \end{aligned} \tag{5.58}$$

$$\begin{aligned}\Sigma_{22,t+1}^D = &\, 2(A + B_u K_t)\Sigma_{22,t}^D (A + B_u K_t)^T + \\ &\, 2L_t C \Sigma_{11,t}^D C^T L_t^T + 2L_t V L_t^T \end{aligned} \tag{5.59}$$

then we obtain that $\Sigma_{t+1}^D \succeq \Sigma_{t+1}$, in view of Lemma 4. The latter corresponds with (5.29).

Part IIa. LMI reformulation of the update of $\Sigma_{11,k}^D$.

We define $\Gamma_k = \mathrm{diag}(\Sigma_{11,k}^D, W^{-1}, (V^D)^{-1})$, $\tilde{\Theta}_k = \begin{bmatrix}A_{L_k}^D & FW & L_k V^D\end{bmatrix}$ and we rewrite constraint (5.30a) as $\Sigma_{11,k+1}^D - \tilde{\Theta}_k \Gamma_k \tilde{\Theta}_k^T \succeq 0$. Resorting to the Schur complement it is possible to derive the equivalent form $\Gamma_k^{-1} -$

$\tilde{\Theta}_k^T(\Sigma_{11,k+1}^D)^{-1}\tilde{\Theta}_k \succeq 0$. To obtain a linear inequality from the previous expression we define

$$Z_k = (\Sigma_{11,k+1}^D)^{-1}L_k \tag{5.60}$$

and $\tilde{\Sigma}_{11,i}^D = (\Sigma_{11,i}^D)^{-1}$, i.e., $\mathrm{diag}(\tilde{\Sigma}_{11,k}^D, W, V^D) - \Phi_k^T(\tilde{\Sigma}_{11,k+1}^D)^{-1}\Phi_k \succeq 0$, where $\Phi_k = [(\tilde{\Sigma}_{11,k+1}^D A^D - Z_k C^D)\,,\, \tilde{\Sigma}_{11,k+1}^D FW\,,\, Z_k V^D]$. The latter expression can be written as a compact LMI as follows

$$\begin{bmatrix} \begin{bmatrix} \tilde{\Sigma}_{11,k}^D & 0 & 0 \\ 0 & W & 0 \\ 0 & 0 & V^D \end{bmatrix} & \Phi_k^T \\ \Phi_k & \tilde{\Sigma}_{11,k+1}^D \end{bmatrix} \succeq 0 \tag{5.61}$$

Notice that, however, in the constraints (5.32) and in the cost function (5.36), the term $\Sigma_{11,i}^D$ appears, rather than its inverse $\tilde{\Sigma}_{11,i}^D$. To solve this issue, we define matrix Δ_k as an upper bound to $\Sigma_{11,k}^D$ (i.e., $\Delta_k \succeq \Sigma_{11,k}^D$), which can be recovered from $\tilde{\Sigma}_{11,k+1}^D$ through the following linear inequality

$$\begin{bmatrix} \Delta_k & I \\ I & \tilde{\Sigma}_{11,k}^D \end{bmatrix} \succeq 0 \tag{5.62}$$

Then, one should replace $\Sigma_{11,k}^D$ with Δ_k in (5.32) and (5.36).

Part IIb. Reformulation of the update of $\Sigma_{22,k}^D$.
Consider now the inequality (5.30b), i.e.,

$$\begin{aligned} &\Sigma_{22,k+1}^D - (A^D + B_u^D K_k)\Sigma_{22,k}^D(A^D + B_u^D K_k)^T \\ &- L_k(C^D\Sigma_{11,k}^D C^{D\,T} + V^D)L_k^T \succeq 0 \end{aligned} \tag{5.63}$$

Recalling (5.60), (5.63) can be rewritten as $\Sigma_{22,k+1}^D - (A^D + B_u^D K_k)\Sigma_{22,k}^D(A^D + B_u^D K_k)^T - \Sigma_{11,k+1}^D M_k \Sigma_{11,k+1}^D \succeq 0$, where

$$M_k = Z_k(C^D\Sigma_{11,k}^D C^{D\,T} + V^D)Z_k^T \tag{5.64}$$

By defining $\Xi_k = K_k\Sigma_{22,k}^D$, and using the matrix Δ_{k+1} in place of $\Sigma_{11,k+1}^D$, the inequality (5.63) can be recast as a suitable LMI. In fact, in view of the Schur complement Lemma and letting $\tilde{M}_k = M_k^{-1}$, we obtain

$$\begin{bmatrix} \Sigma_{22,k+1}^D & [(A^D\Sigma_{22,k}^D + B_u^D\Xi_k) \quad \Delta_{k+1}] \\ \begin{bmatrix} (A^D\Sigma_{22,k}^D + B_u^D\Xi_k)^T \\ \Delta_{k+1} \end{bmatrix} & \begin{bmatrix} \Sigma_{22,k}^D & 0 \\ 0 & \tilde{M}_k \end{bmatrix} \end{bmatrix} \succeq 0 \tag{5.65}$$

The equation (5.64) can be recast as the inequality $M_k \succeq Z_k(C^D\Sigma^D_{11,k}C^{D\,T} + V^D)Z_k^T$, which can be reformulated as

$$\begin{bmatrix} M_k & \begin{bmatrix} Z_kV^D & Z_kC^D \end{bmatrix} \\ \begin{bmatrix} (Z_kV^D)^T \\ (Z_kC^D)^T \end{bmatrix} & \begin{bmatrix} V^D & 0 \\ 0 & \tilde{\Sigma}^D_{11,k} \end{bmatrix} \end{bmatrix} \succeq 0 \tag{5.66}$$

Finally, concerning the equality $\tilde{M}_k = M_k^{-1}$, it can be solved using the approach proposed in [40]. Indeed, we solve the following LMI

$$\begin{bmatrix} M_k & I \\ I & \tilde{M}_k \end{bmatrix} \succeq 0 \tag{5.67}$$

and, at the same time, we minimize the additional cost function

$$\mathrm{tr}\{M_k\tilde{M}_k\} \tag{5.68}$$

The problem (5.67)-(5.68) can be managed using the recursive cone complementarity linearization algorithm proposed in [40] with a suitable initialization.

5.6.2 LMI reformulation of the constraints

While the constraint (5.31a) is a linear inequality (and therefore it does not need to be further reformulated), the inequality (5.31b) needs special attention. As already remarked, in (5.31b), U_k must be replaced by $\bar{U}_k$. In turn, the equality $\bar{U}_k = K_k\Sigma^D_{22,k}K_k^T = \Xi_k(\Sigma^D_{22,k})^{-1}\Xi_k^T$ must be recast as an LMI as follows:

$$\begin{bmatrix} \bar{U}_k & \Xi_k \\ \Xi_k^T & \Sigma^D_{22,k} \end{bmatrix} \succeq 0 \tag{5.69}$$

5.7 Simulation example

In this section the efficiency of the proposed approach is shown by means of a simple example inspired by [102]. In particular we consider the system in (5.1) where we set

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, F = I_2, C = I_2$$

and we assume that noise is Gaussian, with $W = 0.01I_2$ and $V = 10^{-4}I_2$. The probabilistic constraints are $\mathbb{P}\{x_2 \geq 2\} \leq 0.1$, $\mathbb{P}\{u \geq 1\} \leq 0.1$, and

$\mathbb{P}\{-u \geq 1\} \leq 0.1$. In (5.14), (5.15), and (5.18) we set $Q_L = Q = I_2$, $R = 0.01$, and $N = 9$.

In Figure 5.1 we compare the feasible sets obtained with the methods presented in Section 5.4, and under different assumptions concerning the noise (namely p-SMPCc (1), p-SMPCc (2), p-SMPCl (1), p-SMPCl (2), where (1) denotes the case of Gaussian distribution and (2) denotes the case when the distribution is unknown). Apparently, in view of the linearization of the constraints (see the discussion after (5.32)), the p-SMPCl algorithm is more conservative than p-SMPCc. On the other hand, concerning the dimension of the obtained feasibility set, in this case the use of the Chebyshev - Cantelli inequality does not carry about a dramatic performance degradation in terms of conservativeness.

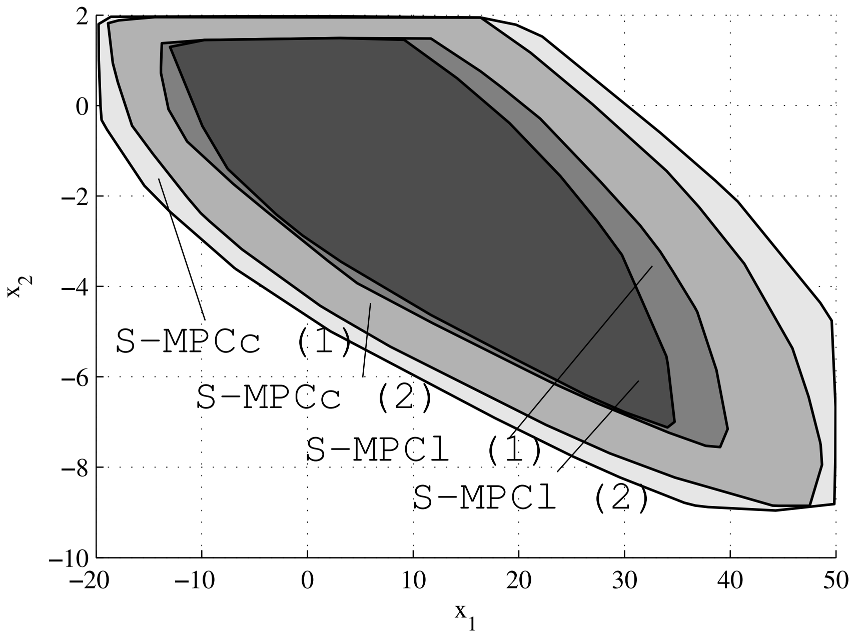


Figure 5.1: *Plots of the feasibility sets for S-MPCc (1), S-MPCc (2), S-MPCl (1), S-MPCl (2)*

In Figure 5.2 we show the evolution of the state variables x_1 and x_2, respectively, using the different control approaches, for 200 Montecarlo runs, starting from initial condition $(5, -1.5)$. Also, in Figure 5.3 we show the corresponding inputs. Apparently, the fact that the control and estimation gains are free variables makes the transient behaviour of the state responses in case of p-SMPCl more performing, with respect to the case when the p-SMPCc is used. For a more detailed analysis, please see Table 5.1, where it is witnessed that the overshoot and the variance of the dynamic state response are reduced in case of p-SMPCl, at the price of a more reactive input response.

The fact that the control and estimation gains are free variables makes the transient behaviour of the state responses in case of p-SMPCl more performing and reduces the variance of the dynamic state response (at the price of a more reactive input response), with respect to the case when p-SMPCc is used. For example, the maximum variance of $x_1(k)$ (resp. of $x_1(k)$) is about 0.33 (resp. 0.036) in case of p-SMPCc (1) and (2), while it results about 0.25 (resp. 0.035) in case of p-SMPCl (1) and (2). On the other hand, the maximum variance of $u(k)$ is about 0.006 in case of p-SMPCc, while it is 0.008 in case of p-SMPCl.

	S-MPCc (1)	S-MPCc (2)	S-MPCl (1)	S-MPCl (2)
Overshoot on $\mathbb{E}\{x_1(k)\}$	-0.3395	-0.3395	-0.2615	-0.2649
Overshoot on $\mathbb{E}\{x_2(k)\}$	0.0608	0.0608	0.0505	0.0535
Overshoot on $\mathbb{E}\{u(k)\}$	-0.0125	-0.0125	-0.0102	-0.0106
Max variance of $x_1(k)$	0.3298	0.3298	0.2521	0.2498
Max variance of $x_2(k)$	0.0358	0.0358	0.0355	0.0354
Max variance of $u(k)$	0.0058	0.0058	0.0080	0.0080

Table 5.1: *Comparison of the dynamic behaviour of the trajectories using the different approaches.*

5.8 Comments

With the aim of filling a gap in the literature of stochastic Model Predictive Control, in this chapter an extension to the output-feedback case of the so-called p-SMPC approach presented in Chapter 3 has been proposed. With respect to the state-feedback case the algorithm is complicated by the need for a state observer whose gain is considered, similarly to the feedback controller gain, as a free parameter of the optimization problem. Feasibility and convergence properties are still guaranteed in the case of unbounded disturbances. The main features of the proposed probabilistic MPC algorithm lie in its simplicity and in its light-weight computational load, both in the off-line design phase and in the online implementation. This allows for the application of the p-SMPC scheme to medium/large-scale problems, for systems affected by general disturbances.

Similarly to the state feedback p-SMPC, the main limitations with respect to existing approaches are due to the difficulty to cope with saturations in the input variables and in the suboptimality of the solution when the Cantelli-Chebyshev's inequality is used instead of the full noise characterization.

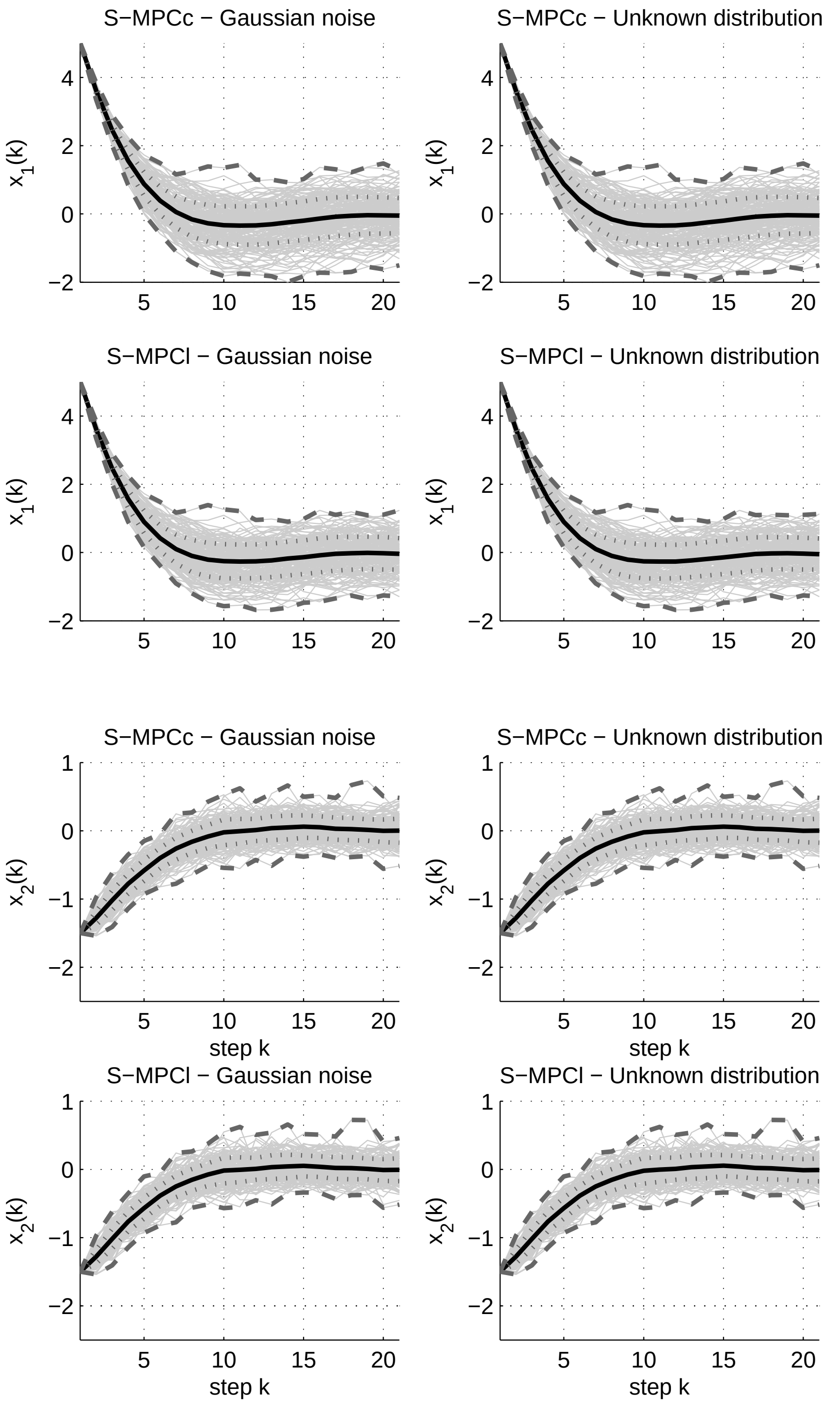


Figure 5.2: *Trajectories x_1 (left) and $x_2(k)$ (right) for 200 runs (light grey lines), sampled mean value (black solid line line), mean value $\pm$ sampled standard deviation (dotted dark grey lines), maximum and minimum values (dashed dark grey lines).*

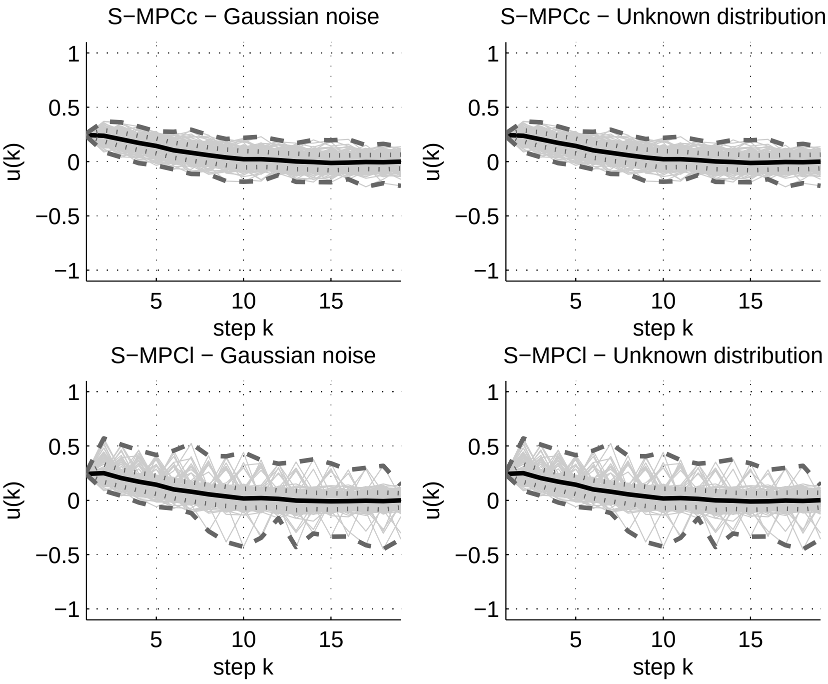


Figure 5.3: *Trajectories $u(k)$ for 200 runs (light grey lines), sampled mean value (black solid line line), mean value $\pm$ sampled standard deviation (dotted dark grey lines), maximum and minimum values (dashed dark grey lines).*

CHAPTER 6

Distributed Predictive Control for regulation: a stochastic approach

In this chapter, taking advantage of the stochastic perspective discussed in the first part of the Thesis, we propose a novel Distributed Predictive Control approach for linear dinamically interconnected subsystems affected by additive, possibly unbounded, stochastic noise and subject to probabilistic constraints on the state and the input. The aim is to obtain an algorithm that is able to deal with disturbances with possibly unbounded support. In particular, the p-SMPC algorithm described in Chapter 3 is adopted, with some modifications, inside a distributed control scheme and recursive feasibility and convergence properties are proven. Even if the results can be apparently very conservative, this method allows to move the first steps towards the development of efficient stochastic distributed predictive control solutions that can overcome the limitations of standard robust approaches. In this chapter, in order to simplify the control problem, we assume that the subsystems are coupled only through their states while each of them is subject only to local probabilistic constraints on the state and the input. Extensions to the case of coupling probabilistic constraints are possible using, for example, the approach proposed in [127].

6.1 Problem statement

Consider a set of M dynamically coupled subsystems, each one described by the following linear dynamics

$$x_{t+1}^{[i]} = A_{ii}x_t^{[i]} + B_{u,i}u_t^{[i]} + \sum_{j\neq i} A_{ij}x_t^{[j]} + B_{w,i}w_t^{[i]}, \quad \forall i = 1,\dots,M \tag{6.1}$$

where $x_t^{[i]} \in \mathbb{R}^{n_i}$, $u_t^{[i]} \in \mathbb{R}^{m_i}$ are the state and the input of the i-th subsystem, respectively, while $w_t^{[i]} \in \mathbb{R}^{p_i}$ is a zero-mean independent white noise with known variance W_i, possibly unbounded support and such that $w_t^{[i]}$ and $w_t^{[j]}$ are uncorrelated for all $i \neq j$. Perfect state information is available and all the pairs $(A_{ii}, B_{u,i})$ are assumed to be stabilizable. The i-th subsystem is denoted in the sequel with $\mathcal{S}_i$, $i = 1,\dots,M$, and any subsystem $\mathcal{S}_j$ is said to be a *neighbor* of subsystem $\mathcal{S}_i$ if and only if $A_{ij} \neq 0$, i.e., if and only if the states $x^{[j]}$ of $\mathcal{S}_j$ influence the dynamics of $\mathcal{S}_i$. The set of neighbors of $\mathcal{S}_i$ is denoted here by $\mathcal{N}_i$.

In addition, in line with the discussion in the previous chapters, we assume that each subsystem $\mathcal{S}_i$ is subject to a set of local probabilistic constraints on the state and the input of the form (2.18), i.e., for each subsystem $\mathcal{S}_i$ we consider the following set of individual chance constraints

$$\mathcal{P}\left\{b_{r,i}^T x_t^{[i]} \leq x_{r,i}^{max}\right\} \geq 1 - p_{r,i}^x,\ \forall r = 1,\dots,n_{r_i},\ t \geq 0 \tag{6.2a}$$

$$\mathcal{P}\left\{c_{s,i}^T u_t^{[i]} \leq u_{s,i}^{max}\right\} \geq 1 - p_{s,i}^u,\ \forall s = 1,\dots,n_{s_i},\ t \geq 0 \tag{6.2b}$$

where, as discussed in detail in Chapter 2, $\mathcal{P}\{\phi\}$ denotes the probability of ϕ and the values $p_{r,i}^x$ and $p_{s,i}^u$ are considered as design parameters, with the meaning that we are allowing the single constraint to be violated up to the specified probability pointwise in time. Coupling constraints are not considered here to simplify the setup, however extensions to this case are possible following the approach in [127].
Concerning the constraint set for the state

$$\mathbb{X}^{[i]} : \{x \mid b_{r,i}^T x \leq x_{r,i}^{max},\ \forall r = 1\dots n_{r_i}\}$$

and for the input

$$\mathbb{U}^{[i]} : \{u \mid c_{s,i}^T u \leq u_{s,i}^{max},\ \forall s = 1\dots n_{s_i}\}$$

we assume that they are nonempty and containing the origin in their interior.

For the development of the distributed controller, it is worth defining the dynamics of the collective system. To this end, denoting by $\mathbf{v} = (v^{[1]}, \dots, v^{[s]})$ the short-hand of a vector with $v^{[1]}, \dots, v^{[s]}$ components, let the collective state, input and disturbance vector be, respectively, $\mathbf{x}_t = (x_t^{[1]}, \dots, x_t^{[M]}) \in \mathbb{R}^n$, $n = \sum_{i=1}^M n_i$, $\mathbf{u}_t = (u_t^{[1]}, \dots, u_t^{[M]}) \in \mathbb{R}^m$, $m = \sum_{i=1}^M m_i$, and $\mathbf{w}_t = (w_t^{[1]}, \dots, w_t^{[M]}) \in \mathbb{R}^p$, $p = \sum_{i=1}^M p_i$, and define by $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{B_w}$ the collective system matrices, i.e.

$$\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1M} \\ A_{21} & A_{22} & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ A_{M1} & \dots & \dots & A_{MM} \end{bmatrix}, \quad \begin{array}{l} \mathbf{B} = \text{diag}(B_{u,1}, \dots, B_{u,M}) \\ \mathbf{B_w} = \text{diag}(B_{w,1}, \dots, B_{w,M}) \end{array}$$

Thus, the overall dynamical model is

$$\mathbf{x}_{t+1} = \mathbf{A}\mathbf{x}_t + \mathbf{B}\mathbf{u}_t + \mathbf{B_w}\mathbf{w}_t \tag{6.3}$$

We also define $\mathbf{W} = var(\mathbf{w}_t) = diag(W_1, \dots, W_M)$. With reference to the collective system in (6.3), some structural requirements are introduced by the following standing assumption

Assumption 2. *(i) The pair $(\mathbf{A}, \mathbf{G}_w)$ is stabilizable, where $\mathbf{G}_w$ is such that $\mathbf{G}_w\mathbf{G}_w^T = \mathbf{B_w}\mathbf{W}\mathbf{B_w}^T$. (ii) There exists a decentralized gain $\mathbf{K} = diag(K_1, \dots, K_M)$ such that the matrices $(\mathbf{A} + \mathbf{B}\mathbf{K})$ and $(A_{ii} + B_i K_i)$, $i = 1, \dots, M$ are Schur stable.*

6.1.1 Regulator structure

Inspired by the p-SMPC approach presented in Chapter 3 , we define the control law for the i-th subsystem, $i = 1, \dots, M$, as

$$u_t^{[i]} = \hat{u}_t^{[i]} + K_i(x_t^{[i]} - \hat{x}_t^{[i]}) \tag{6.4}$$

where the term $\hat{u}_t^{[i]}$ is obtained as a solution of a proper local MPC problem, while the parameter K_i is selected as a constant gain whose computation will be discussed in the sequel.
Moreover, the variables $\hat{x}_t^{[i]}$ and $\hat{u}_t^{[i]}$ are the state and the input, respectively, of the i-th nominal system, denoted as $\hat{\mathcal{S}}_i$, and given for all $i = 1, \dots, M$ by

$$\hat{x}_{t+1}^{[i]} = A_{ii}\hat{x}_t^{[i]} + B_{u,i}\hat{u}_t^{[i]} \tag{6.5}$$

Note that (6.5) is obtained neglecting both the effect of the uncertainty and the presence of the couplings with the neighboring subsystems, and thus the M nominal systems are completely deterministic and decentralized.

In parallel to the real and the nominal decentralized models, (6.1) and (6.5), we need to account for the behavior of the expected values of the state and input of each agent, namely $\bar{x}_t^{[i]} = \mathbb{E}[x_t^{[i]}]$ and $\bar{u}_t^{[i]} = \mathbb{E}[u_t^{[i]}]$, that will be used, in line with the approach proposed in Chapter 3, to reformulate the chance constraints (6.2). In particular, starting from (6.1) and recalling that $\mathbb{E}[w_t^{[i]}]$, $\forall t \geq 0$, the evolution of the expected value of the state of the i-th subsystem can be described as follows for all $i = 1, \dots, M$, i.e,

$$\bar{x}_{t+1}^{[i]} = A_{ii}\bar{x}_t^{[i]} + B_{u,i}\bar{u}_t^{[i]} + \sum_{j \in \mathcal{N}_i} A_{ij}\bar{x}_t^{[j]} \tag{6.6}$$

where the coupling terms, $\bar{x}_t^{[j]},\ j \in \mathcal{N}_i$, are the expected values of the state of all the neighboring subsystems of $\mathcal{S}_i$. Note that, contrarily to the nominal subsystems in (6.5), the dynamics of the expected values in (6.6) is not decoupled and thus cannot be used directly inside each local controller. Concerning the expected value of the i-th input, $\bar{u}_t^{[i]}$, recall that $\hat{x}_t^{[i]}$ and $\hat{u}_t^{[i]}$ are deterministic variables, thus, for all $i = 1, \dots, M$, we have from equation (6.4) that

$$\bar{u}_t^{[i]} = \hat{u}_t^{[i]} + K_i(\bar{x}_t^{[i]} - \hat{x}_t^{[i]}) \tag{6.7}$$

Denoting by $\bar{\mathbf{x}}_t = (\bar{x}_t^{[1]}, \dots, \bar{x}_t^{[M]}) \in \mathbb{R}^n$ and by $\bar{\mathbf{u}}_t = (\bar{u}_t^{[1]}, \dots, \bar{u}_t^{[M]}) \in \mathbb{R}^m$ the collective expected values of the states and of the inputs, respectively, the collective average dynamics of the system is given, in view of equation 6.6, by

$$\bar{\mathbf{x}}_{t+1} = \mathbf{A}\bar{\mathbf{x}}_t + \mathbf{B}\bar{\mathbf{u}}_t \tag{6.8}$$

Similarly to the procedure adopted in Chapter 3, we define for the i-th subsystem the state error variable, $\delta x_t^{[i]} = x_t^{[i]} - \bar{x}_t^{[i]}$ and, from (6.1) and (6.6), we obtain the dynamics

$$\delta x_{t+1}^{[i]} = (A_{ii} + B_{u,i}K_i)\delta x_t^{[i]} + \sum_{j \in \mathcal{N}_i} A_{ij}\delta x_t^{[j]} + B_{w,i}w_t^{[i]} \tag{6.9}$$

In (6.9) the terms $\delta x^{[j]},\ j \in \mathcal{N}_i$ that appear in the sum are the deviations of the states of the neighboring agents from their own expected values and, because of that, their expected value is zero. In particular, assuming that each subsystem is properly initialized so that $\mathbb{E}[\delta x_t^{[i]}] = 0,\ \forall i =$

$1, \dots, M$, it follows that $\mathbb{E}[\delta x_{t+k}^{[i]}] = 0, \forall k \geq 0$. Furthermore, letting $\boldsymbol{\delta x}_t = (\delta x_t^{[1]}, \dots, \delta x_t^{[M]}) \in \mathbb{R}^n$ be the collective error, from (6.9) the dynamics of the overall error is

$$\boldsymbol{\delta x}_{t+1} = (\mathbf{A} + \mathbf{B}\mathbf{K})\boldsymbol{\delta x}_t + \mathbf{B_w}\mathbf{w}_t \tag{6.10}$$

with $\mathbb{E}[\boldsymbol{\delta x}_t] = 0$, $\forall t \geq 0$. Now define the covariance matrix of the overall system in (6.3) as $\mathbf{X}_t = \text{var}(\mathbf{x}_t) = \mathbb{E}[\boldsymbol{\delta x}_t \boldsymbol{\delta x}_t^T]$. In view of equation (6.10) its evolution is described by

$$\mathbf{X}_{t+1} = (\mathbf{A} + \mathbf{B}\mathbf{K})\mathbf{X}_t(\mathbf{A} + \mathbf{B}\mathbf{K})^T + \mathbf{B_w}\mathbf{W}\mathbf{B_w}^T \tag{6.11}$$

where $\mathbf{W} = \text{var}(\mathbf{w}_t)$ is the block diagonal variance of the overall disturbance due to the independence assumption.
Finally, note the variance of variable $\delta x_t^{[i]}$ is $X_t^{[i]}$, i.e., the i-th diagonal block of the possibly full matrix $\mathbf{X}_t$.

A block-diagonal upper bound for the variance

Note that, in general, even if we assume that, at a given time t, the matrix $\mathbf{X}_t$ is block-diagonal, its evolution $\mathbf{X}_{t+k}$, $k > 0$ is full due to the couplings between the subsystems, which represents a tough bottleneck in the design of a distributed controller and for the reformulation of probabilistic constraints in deterministic terms. In order to overcome this aspect, we define a symmetric and positive semi-definite block-diagonal matrix, $\tilde{\mathbf{X}}_t$, in such a way that

$$\tilde{\mathbf{X}}_t = \begin{bmatrix} \tilde{X}_t^{[1]} & 0 & \dots & 0 \\ 0 & \tilde{X}_t^{[2]} & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & \tilde{X}_t^{[M]} \end{bmatrix} \succeq \mathbf{X}_t \tag{6.12}$$

and whose evolution $\tilde{\mathbf{X}}_{t+1} = diag(\tilde{X}_{t+1}^{[1]}, \dots, \tilde{X}_{t+1}^{[1]})$ is governed by the following set of M update equations

$$\begin{aligned} \tilde{X}_{t+1}^{[i]} = & (\tilde{A}_{ii} + \tilde{B}_{u,i}K_i)\tilde{X}_t^{[i]}(\tilde{A}_{ii} + \tilde{B}_{u,i}K_i)^T \\ & + \sum_{j \in \mathcal{N}_i} \tilde{A}_{ij}\tilde{X}_t^{[j]}\tilde{A}_{ij}^T + B_{w,i}W_iB_{w,i}^T \end{aligned} \tag{6.13}$$

where $\tilde{A}_{ii} = \sqrt{\nu_i}A_{ii}$, $\tilde{A}_{ij} = \sqrt{\nu_i}A_{ij}$ and $\tilde{B}_{u,i} = \sqrt{\nu_i}B_{u,i}$, and ν_i is the cardinality of $\tilde{\mathcal{N}}_i = \{j | A_{ji} \neq 0, j = 1, \dots, M\}$. The following lemma holds

Lemma 5. *Assume that, at time t, $\tilde{\mathbf{X}}_t \succeq \mathbf{X}_t$. Then if $\tilde{\mathbf{X}}_{t+1}$ is defined according to equation* (6.13) *we have that*

$$\tilde{\mathbf{X}}_{t+1} \succeq \mathbf{X}_{t+1}$$

Moreover, since $\tilde{\mathbf{X}}_t \succeq \mathbf{X}_t$, then also $\tilde{X}_t^{[i]} \succeq X_t^{[i]}$, $\forall t \geq 0$.

The proof of Lemma 5 is postponed to Section 6.4.1. Finally, in order to tighten the probabilistic constraints as described in Chapter 3, we are interested in characterizing the evolution of the covariance matrix of the i-th subsystem $X_t^{[i]}$. However, due to the coupled structure of the system, and in particular to the off-diagonal terms of $\mathbf{X}_t$, those constraints can only be solved in a centralized fashion. On the other hand, based on the result in Lemma 5 and the definition of $\mathbf{X}_t$ and $\tilde{\mathbf{X}}_t$, the new matrices $\tilde{X}_t^{[i]}$, $i = 1, \ldots, M$ can be used in place of the original ones to construct the deterministic, although tighter, constraints. Note, however, that in view of equation (6.13) the matrices $\tilde{X}_t^{[i]}$, $i = 1, \ldots, M$ are still coupled, and thus the evolution (6.13) needs to be computed in a distributed fashion. This fact will be discussed in the sequel.

Concerning the covariance of the input, from the structure of the controller in (6.4) and (6.7) we can define, for all $i = 1, \ldots, M$

$$U_t^{[i]} = K_i X_t^{[i]} K_i^T \tag{6.14}$$

or, collectively, define $\mathbf{U}_t$ as

$$\mathbf{U}_t = \mathbf{K}\mathbf{X}_t\mathbf{K}^T$$

Moreover, using the bounding matrices $\tilde{X}_t^{[i]}$, $i = 1, \ldots, M$, we define a new marix $\tilde{U}_t^{[i]}$ as $\tilde{U}_t^{[i]} = K_i \tilde{X}_t^{[i]} K_i^T$ which, by construction, is such that $\tilde{U}_t^{[i]} \succeq U_t^{[i]}$, $\forall t \geq 0$ and thus $\tilde{U}_t^{[i]}$, $i = 1, \ldots, M$, can be used to tighten the constraint in place of the real variance of the input.

6.1.2 Reformulation of the probabilistic constraints

Consider the set of local individual chance constraints that appears in equation (6.2). The approach described in Chapter 3 is now used to reformulate them in deterministic terms, based on the expected value and variance of the state and the input of the i-th subsystem. Moreover, in order to maintain a distributed structure of the controller, the bounds $\tilde{X}_t^{[i]}$ and $\tilde{U}_t^{[i]}$, $i = 1, \dots, M$ in place of the original state and control variances, are considered inside the definition of the constraints. This, of course, introduces a source of conservatism but greatly simplifies the setup, allowing for a distributed implementation.

In particular, starting from the state and input constraints sets, $\mathbb{X}^{[i]}$ and $\mathbb{U}^{[i]}$, define M tightened sets, $\bar{\mathbb{X}}_t^{[i]}$ and $\bar{\mathbb{U}}_t^{[i]}$, for all $i = 1, \dots, M,\ \forall t \geq 0$ as

$$\bar{\mathbb{X}}_t^{[i]} = \left\{ x \mid b_{r,i}^T x \leq x_{r,i}^{max} - \sqrt{b_{r,i}^T \tilde{X}_t^{[i]} b_{r,i}}\, f(p_{i,r}^x) \quad \forall r = 1 \dots n_{r_i} \right\} \tag{6.15}$$

and

$$\bar{\mathbb{U}}_t^{[i]} = \left\{ u \mid c_{s,i}^T u \leq u_{s,i}^{max} - \sqrt{c_{s,i}^T \tilde{U}_t^{[i]} c_{s,i}}\, f(p_{s,i}^u) \quad \forall s = 1 \dots n_{s_i} \right\} \tag{6.16}$$

As in the previous chapters, the value of the function $f(p)$ depends on the stochastic nature of the disturbance and in particular, if the distribution of the uncertainty is unknown, it is bounded by $f(p) = \sqrt{(1-p)/p}$, while if the uncertainty is Gaussian (and this, due to linearity, is true also for the state and the input) the exact value is computed as $f(p) = \mathcal{N}^{-1}(1-p)$. For further details see Chapter 2.

Finally, as discussed in Chapter 3, the original constraints in (6.2) are satisfied provided that

$$\bar{x}_t^{[i]} \in \bar{\mathbb{X}}_t^{[i]},\ \forall t \geq 0 \tag{6.17a}$$

$$\bar{u}_t^{[i]} \in \bar{\mathbb{U}}_t^{[i]},\ \forall t \geq 0 \tag{6.17b}$$

However, note that, the constraints in (6.17) are still not suitable for the implementation of a distributed MPC algorithm, where at any time t the evolution of the expected value and of the variance of the states and inputs of each subsystem must be computed in a distributed way from t to $t+N$, N being the length of the adopted prediction horizon. In fact, the following two main problems arise

P-A in view of (6.6), also the expected values $\bar{x}_{t+k}^{[i]}$, $k = 0, \dots, N$, of the states of the subsystems $\mathcal{S}_i$, $i = 1, \dots, M$, depend on the expected values $\bar{x}_{t+k}^{[j]}$ of the states of the neighbors $\mathcal{S}_j$, $j \in \mathcal{N}_i$;

P-B in view of (6.13), the sets $\bar{\mathbb{X}}^{[i]}_{t+k}$ and $\bar{\mathbb{U}}^{[i]}_{t+k}$, $k = 0, ..., N$, are affected by all the covariance bounds $\tilde{X}^{[j]}_{t+k}$ of the neighboring subsystems $\mathcal{S}_j$, $j \in \mathcal{N}_i$, so that a distributed computation is required.

In view of this and in order to derive the desired controller, we need to further reformulate the constraints.

Problem P-A

Firstly, in order to characterize the evolution of the expected value in a decentralized way, we recall that the nominal systems $\hat{\mathcal{S}}_i$ defined in (6.5) are decentralized and we resort to an approach based on the robust "tube-based solution" developed in [102] for robust centralized MPC. Specifically, for the i-th subsystem we define the difference $z^{[i]}_{t+k} = \bar{x}^{[i]}_{t+k} - \hat{x}^{[i]}_{t+k}$ and note that, in view of (6.6), (6.5), (6.7), the dynamics of $z^{[i]}_{t+k}$, for $k = 0, \ldots, N-1$, is given by

$$\begin{aligned} z^{[i]}_{t+k+1} &= (A_{ii} + B_{u,i}K_i)z^{[i]}_{t+k} + \sum_{j\in\mathcal{N}_i} A_{ij}\bar{x}^{[j]}_{t+k} \\ &= (A_{ii} + B_{u,i}K_i)z^{[i]}_{t+k} + \sum_{j\in\mathcal{N}_i} A_{ij}z^{[j]}_{t+k} + \sum_{j\in\mathcal{N}_i} A_{ij}\hat{x}^{[j]}_{t+k} \end{aligned} \tag{6.18}$$

Correspondingly, the collective vector $\mathbf{z}_{t+k} = (z^{[1]}_{t+k}, \ldots, z^{[M]}_{t+k})$ evolves according to the following dynamics

$$\mathbf{z}_{t+k+1} = (\mathbf{A} + \mathbf{BK})\mathbf{z}_{t+k} + \mathbf{d}_{t+k} \tag{6.19}$$

where $\mathbf{d}_{t+k} = (d^{[1]}_{t+k}, \ldots, d^{[M]}_{t+k})$, with $d^{[i]}_{t+k} = \sum_{j\in\mathcal{N}_i} A_{ij}\hat{x}^{[j]}_{t+k}$, can be interpreted as a vector of disturbances to be rejected.
Now assume that in the formulation of the M local MPC problems we can enforce the constraint $\hat{x}^{[i]}_{t+k} \in \hat{\mathbb{X}}^{[i]}_0$, $\forall i = 1, \ldots, M$, where $\hat{\mathbb{X}}^{[i]}_0$, $i = 1, \ldots, M$ are closed and compact sets containing the origin in their interior. Then, we can also guarantee that, for all $i = 1, \ldots, M$

$$d^{[i]}_{t+k} = \sum_{j\in\mathcal{N}_i} A_{ij}\hat{x}^{[j]}_{t+k} \in \mathbb{D}^{[i]}_{t+k} = \bigoplus_{j\in\mathcal{N}_i} A_{ij}\hat{\mathbb{X}}^{[i]}_o \tag{6.20}$$

The following standing assumption is now in order

Assumption 3. *There exists a rectangular robust positive invariant (RPI) set for system* (6.19)*, i.e.,*

$$\mathbb{Z} = \mathbb{Z}^{[1]} \times \mathbb{Z}^{[2]} \times \ldots \times \mathbb{Z}^{[M]}$$

such that $\mathbf{z}_t \in \mathbb{Z}$ *implies* $\mathbf{z}_{t+k} \in \mathbb{Z}$, $\forall \mathbf{d}_{t+k} \in \mathbb{D}^{[1]}_{t+k} \times \ldots \times \mathbb{D}^{[M]}_{t+k}$, $\forall k \geq 0$. *Similarly, the sets* $\mathbb{Z}^{[i]}$, $i = 1, \ldots, M$ *are RPI for the system in* (6.18), *meaning that* $z^{[i]}_t \in \mathbb{Z}^{[i]}$, *for all* $i = 1, \ldots, M$ *implies* $z^{[i]}_{t+k} \in \mathbb{Z}^{[i]}$, $\forall d^{[i]}_{t+k} \in \mathbb{D}^{[i]}_{t+k}$, $\forall z^{[j]}_{t+k} \in \mathbb{Z}^{[j]}$, $j \in \mathcal{N}_i$ *and* $\forall k \geq 0$.

Problem P-B

With reference to the problem **P-B**, we assume that, at time instant t, the sequence $\tilde{X}^{[i]}_k$, $k = t, \ldots, t+N-1$ of future upper bounds of the state variances is available to subsystem i. As it will be clarified in the following, we will guarantee that $\tilde{X}^{[i]}_k$ is, at all time instants $k > 0$, an upper bound of the state variance by properly defining the corresponding MPC problems.
Note that the recursive update of $\tilde{X}^{[i]}_k$ can be made in a distributed fashion. For example, thanks to a lightweight neighbor-to-neighbor communication network, the terms $\tilde{X}^{[j]}_k$, $j \in \mathcal{N}_i$ can be made available to subsystem i, and matrix $\tilde{X}^{[i]}_{k+1}$ can be computed using (6.13) and made available for the definition of the MPC problem when needed.
Concerning this, the following standing assumption is required to guarantee that such sequence does not grow unbounded.

Assumption 4. *For all* $i = 1, \ldots, M$ *there exist matrices* $\bar{X}_i$, *which are solutions to the set of algebraic equations*

$$\begin{aligned}(\tilde{A}_{ii} + \tilde{B}_{u,i}K_i)\bar{X}_i(\tilde{A}_{ii} + \tilde{B}_{u,i}K_i)^T = \\ \bar{X}_i - \sum_{j \in \mathcal{N}_i} \tilde{A}_{ij}\bar{X}_j\tilde{A}^T_{ij} - B_{w,i}W_iB^T_{w,i}\end{aligned} \tag{6.21}$$

It is important to remark that the existence of matrices $\bar{X}_i$, $i = 1, \ldots, M$, guarantee that, if $\tilde{X}^{[i]}_1 \preceq \bar{X}_i$ for all $i = 1, \ldots, M$, then $\tilde{X}^{[i]}_k \preceq \bar{X}_i$ and for all $k > 0$. This can be proved by induction computing that if, at time k, $\tilde{X}^{[i]}_k \preceq \bar{X}_i$, then $\tilde{X}^{[i]}_{k+1} = (\tilde{A}_{ii} + \tilde{B}_{u,i}K_i)\tilde{X}^{[i]}_k(\tilde{A}_{ii} + \tilde{B}_{u,i}K_i)^T + \sum_{j \in \mathcal{N}_i} \tilde{A}_{ij}\tilde{X}^{[j]}_k\tilde{A}^T_{ij} + B_{w,i}W_iB^T_{w,i}$ satisfies $\tilde{X}^{[i]}_{k+1} \preceq (\tilde{A}_{ii} + \tilde{B}_{u,i}K_i)\bar{X}_i(\tilde{A}_{ii} + \tilde{B}_{u,i}K_i)^T + \sum_{j \in \mathcal{N}_i} \tilde{A}_{ij}\bar{X}_j\tilde{A}^T_{ij} + B_{w,i}W_iB^T_{w,i} = \bar{X}_i$.

Now, in order to reformulate the constraints (6.17), it is possible to define two convex sets $\hat{\mathbb{X}}^{[i]}_t$ and $\hat{\mathbb{U}}^{[i]}_t$, satisfying

$$\hat{\mathbb{X}}^{[i]}_t = \bar{\mathbb{X}}^{[i]}_t \ominus \mathbb{Z}^{[i]}, \ \hat{\mathbb{U}}^{[i]}_t = \bar{\mathbb{U}}^{[i]}_t \ominus K_i\mathbb{Z}^{[i]} \tag{6.22}$$

respectively. The following implications hold

$$\left\{\begin{array}{l} \hat{x}_t^{[i]} \in \hat{\mathbb{X}}_t^{[i]} \\ z_t^{[i]} \in \mathbb{Z}^{[i]} \end{array}\right. \implies \bar{x}_t^{[i]} = \hat{x}_t^{[i]} + z_t^{[i]} \in \bar{\mathbb{X}}_t^{[i]} \tag{6.23a}$$

$$\left\{\begin{array}{l} \hat{u}_t^{[i]} \in \hat{\mathbb{U}}_t^{[i]} \\ z_t^{[i]} \in \mathbb{Z}^{[i]} \end{array}\right. \implies \bar{u}_t^{[i]} = \hat{u}_t^{[i]} + K_i z_t^{[i]} \in \bar{\mathbb{U}}_t^{[i]} \tag{6.23b}$$

so that the state and control constraints can be enforced on the nominal decentralized subsystems. Note however that, from (6.23a) and (6.23b), we need to correctly initialize the problem in order to guarantee $z_t^{[i]} \in \mathbb{Z}^{[i]}$, for all $i = 1, \dots, M$. This problem will be addressed in the sequel.

Finally we define the sets $\bar{\mathbb{X}}_\infty^{[i]}$ and $\bar{\mathbb{U}}_\infty^{[i]}$ as the tightened constraint sets in (6.15) and (6.16) computed in correspondence of the matrices $\bar{X}_i$, $\bar{U}_i = K_i \bar{X}_i K_i^T$, i.e.,

$$\bar{\mathbb{X}}_\infty^{[i]} = \left\{ x \mid b_{r,i}^T x \le x_{r,i}^{max} - \sqrt{b_{i,r}^T \bar{X}_i b_{r,i}}\, f(p_{i,r}^x) \right\} \tag{6.24a}$$

$$\bar{\mathbb{U}}_\infty^{[i]} = \left\{ u \mid c_{s,i}^T u \le u_{s,i}^{max} - \sqrt{c_{s,i}^T \bar{U}_i c_{s,i}}\, f(p_{s,i}^u) \right\} \tag{6.24b}$$

The following standing assumption allows us to correctly define the sets used in the proposed control scheme.

Assumption 5. *With reference to the definition in* (6.24) *the sets* $\bar{\mathbb{X}}_\infty^{[i]} \ominus \mathbb{Z}^{[i]}$ *and* $\bar{\mathbb{U}}_\infty^{[i]} \ominus K_i \mathbb{Z}^{[i]}$ *,* $i = 1, \dots, M$ *exist and contain the origin in their interior.*

The previous assumption ensure that all the sets $\hat{\mathbb{X}}_t^{[i]}$ and $\hat{\mathbb{U}}_t^{[i]}$, for $i = 1, \dots, M$ and $t \ge 0$, exist and contain the origin in their interior.

6.2 The distributed SMPC algorithm: formulation and properties

In this section the main features of the proposed algorithm are described and the i-th local MPC problem, to be solved by each subsystem at each time step, is stated. Similarly to the p-SMPC approach, in order to give feasibility guarantees, the choice of the initial conditions for the nominal state (expected value) and the variance is considered as an extra degree of freedom of the problem. The cost function, the terminal constraints, and the initializations of the M local MPC problems are now specified to guarantee feasibility and convergence of the control scheme.

6.2.1 Cost function

The local cost function to be minimized, $J^{[i]}$, $i = 1, \dots, M$, depends on the nominal values of the state and input, i.e.

$$J^{[i]} = \sum_{k=t}^{t+N-1} (\|\hat{x}_k^{[i]}\|_{Q_i}^2 + \|\hat{u}_k^{[i]}\|_{R_i}^2) + \|\hat{x}_{t+N}^{[i]}\|_{S_i}^2 \tag{6.25}$$

where, for all $i = 1, \dots, M$, $Q_i > 0$ and $R_i > 0$ are tuning knobs and the terminal weights S_i are defined such that

$$\begin{aligned} Q_i + &K_i^T R_i K_i + (\tilde{A}_{ii} + \tilde{B}_{u,i} K_i)^T S_i (\tilde{A}_{ii} + \tilde{B}_{u,i} K_i) \\ &- S_i \preceq 0 \end{aligned} \tag{6.26}$$

6.2.2 Terminal constraints

As standard in MPC with stability guaranteed, terminal constraints must be included into the problem formulation, i.e.,

$$\hat{x}_{t+N}^{[i]} \in \mathbb{X}_F^{[i]} \tag{6.27}$$

where the terminal set $\mathbb{X}_F^{[i]}$ is a positive invariant set for the nominal subsystem (6.5) under the auxiliary control law $\hat{u}_t^{[i]} = K_i \hat{x}_t^{[i]}$ where the original probabilistic state and control constraints (6.2a) and (6.2b) are verified. In our framework, this means that also the tightened constraints (6.15) and (6.16) must be satisfied inside $\mathbb{X}_F^{[i]}$, i.e.,

$$\mathbb{X}_F^{[i]} \subseteq \left\{ x^{[i]} \mid \begin{array}{cl} x^{[i]} & \in \bar{\mathbb{X}}_\infty^{[i]} \ominus \mathbb{Z}^{[i]} \\ K_i x^{[i]} & \in \bar{\mathbb{U}}_\infty^{[i]} \ominus K_i \mathbb{Z}^{[i]} \\ x^{[i]} & \in \hat{\mathbb{X}}_o^{[i]} \end{array} \right\} \tag{6.28}$$

6.2.3 Initial conditions

The initial conditions of each local optimization problem are free variables selected to guarantee recursive feasibility properties. The basic idea is to take advantage of the information available at time t whenever possible and, otherwise, to use the optimal nominal solution computed at time $t-1$ that guarantees feasibility. Specifically, two possible initialization strategies for the choice of $\hat{x}_{t|t}^{[i]}$ are defined

Strategy S1 - nominal evolution: at time t choose $\hat{x}_{t|t}^{[i]} = \hat{x}_{t|t-1}^{[i]}$. In this way we are using the optimal solution computed at time $t-1$ in order to counteract the effect of possibly unbounded disturbances that affected the state. Moreover, $\bar{x}_t^{[i]}$ is defined according to (6.6) with (6.7), which implies, from Lemma 5, that $\text{var}(x^{[i]}(t)) = X_t^{[i]} \preceq \tilde{X}_t^{[i]}$. Also, $z_t^{[i]} \in \mathbb{Z}^{[i]}$ by construction.

Strategy S2 - robust initialization: at time t set $x_t^{[i]} - \hat{x}_t^{[i]} \in \mathbb{Z}^{[i]}$. We implicitly assume that $\bar{x}_t^{[i]} = x_t^{[i]}$ and thus $\text{var}(x^{[i]}(t)) = X_t^{[i]} = 0 \preceq \tilde{X}_t^{[i]}$ and $z_t^{[i]} \in \mathbb{Z}^{[i]}$.

6.2.4 Statement of the local MPC problems and main result

We are now ready to formulate the i-th local optimization problem.
i-MPC problem: at each time step t measure the state $x_t^{[i]}$ and solve the optimization problem

$$\min_{\hat{x}_t^{[i]}, \hat{u}_{t\dots t+N-1}^{[i]}} \quad J^{[i]}(\hat{x}_t^{[i]}, \hat{u}_{t,\dots,t+N-1}^{[i]}) \tag{6.29a}$$

subject to

$$\hat{x}_t^{[i]} \in \{\hat{x}_{t|t-1}^{[i]}\} \cup (x_t^{[i]} \oplus -\mathbb{Z}^{[i]}) \tag{6.29b}$$

$$\hat{x}_{k+1}^{[i]} = A_{ii}\hat{x}_k^{[i]} + B_{u,i}\hat{u}_k^{[i]} \tag{6.29c}$$

$$\hat{x}_k^{[i]} \in \hat{\mathbb{X}}_0^{[i]} \bigcap \hat{\mathbb{X}}_k^{[i]}, k = t\dots t+N-1 \tag{6.29d}$$

$$\hat{u}_k^{[i]} \in \hat{\mathbb{U}}_k^{[i]}, k = t\dots t+N-1 \tag{6.29e}$$

$$\hat{x}_{t+N}^{[i]} \in \mathbb{X}_F^{[i]} \tag{6.29f}$$

As a solution we obtain the optimal sequence $\hat{u}_{t\dots t+N-1}^{[i]}$ and the optimal value $\hat{x}_{t|t}^{[i]}$. Thus, consistently with (6.4), at time step t we apply the following input

$$u_t^{[i]} = \hat{u}_{t|t}^{[i]} + K_i^{[i]}(x_t^{[i]} - \hat{x}_{t|t}^{[i]}) \tag{6.30}$$

Moreover we compute the optimal sequence at time t, i.e., $\hat{x}_{t\dots t+N-1|t}^{[i]}$, thanks to (6.5).

We are now in the position for state the main result.

Theorem 3. *If, at time $t=0$, all i-MPC problems, $i = 1,\dots,M$, admit a solution, then the optimization problem is recursively feasible and*

$\mathbb{E}\{\|\mathbf{x}_t\|^2\} \to tr\{\mathbf{X}_{ss}\}$ *as* $t \to +\infty$*, where* $\mathbf{X}_{ss}$ *is the unique positive semi-definite solution (recall Assumption 2) to the Lyapunov equation*

$$\mathbf{X}_{ss} - (\mathbf{A}+\mathbf{BK})\mathbf{X}_{ss}(\mathbf{A}+\mathbf{BK})^T = \mathbf{B_w}\mathbf{W}\mathbf{B_w}^T$$

Furthermore, the state and input probabilistic constraints (6.2) *are verified for all* $t \geq 0$*.*

6.3 Implementation issues

The independent and local MPC problems previously defined require the a-priori off-line computation of a number of parameters and sets which must be performed in a centralized way. In particular, the gains K_i and $\mathbf{K}$ satisfying Assumption 2 can be computed with the procedure based on the solution of LMI's reported in [12].
Additional parameters of the algorithm are the sets $\hat{\mathbb{X}}_0^{[i]}$, see (6.20), and the rectangular sets $\mathbb{Z}^{[i]}$ satisfying Assumption 3. Their choice is fundamental, since directly limits the feasibility set of the local MPC problems and thus may represent the main source of conservatism. However one can notice that, in general, the bigger are the sets $\hat{\mathbb{X}}_o^{[i]}$ the bigger will be the sets $\mathbb{Z}^{[i]}$, meaning that the two terms $\hat{x}_t^{[i]}$ and $\bar{x}_t^{[i]}$ are allowed to be far from each other. In practice, this will affect the dimension of the tightened original constraint sets $\hat{\mathbb{X}}_t^{[i]} = \bar{\mathbb{X}}_t^{[i]} \ominus \mathbb{Z}^{[i]}$ and $\hat{\mathbb{U}}_t^{[i]} = \bar{\mathbb{U}}_t^{[i]} \ominus K_i\mathbb{Z}^{[i]}$, $t \geq 0$ and again reduce the overall feasibility set for the i-th MPC problem. Thus, the choice of $\hat{\mathbb{X}}_0^{[i]}$ and $\mathbb{Z}^{[i]}$ represents a sort of tradeoff that needs to be accounted for in the design. However, it is always possible to choose a sufficiently small $\hat{\mathbb{X}}_o^{[i]}$ such that $\mathbb{Z}^{[i]}$ is not empty. Procedures for the practical computation of these sets are also described in [12].
Finally, note that conditions (e.g., LMIs or based on small-gain conditions) for guaranteeing the existence of solutions to (6.21) can be studied along the lines traced in [44].

6.4 Proofs

6.4.1 Proof of Lemma5

In this section we prove Lemma 5. In particular, consider the overall covariance matrix $\mathbf{X}_t$ and a symmetric positive semi-definite matrix $\tilde{\mathbf{X}}_t = diag(\tilde{X}_t^{[1]}, \dots, \tilde{X}_t^{[M]})$ such that $\tilde{\mathbf{X}}_t \succeq \mathbf{X}_t$. From equation (6.11) we have that

$$\begin{aligned}\mathbf{X}_{t+1} &= (\mathbf{A}+\mathbf{BK})\mathbf{X}_t(\mathbf{A}+\mathbf{BK})^T + \mathbf{B_w}\mathbf{W}\mathbf{B_w}^T \\ &\preceq (\mathbf{A}+\mathbf{BK})\tilde{\mathbf{X}}_t(\mathbf{A}+\mathbf{BK})^T + \mathbf{B_w}\mathbf{W}\mathbf{B_w}^T \end{aligned} \tag{6.31}$$

Focus now on the first term of equation (6.31), i.e.,

$$\mathbf{\Psi} = (\mathbf{A}+\mathbf{BK})\mathbf{X}_t(\mathbf{A}+\mathbf{BK})^T$$

and, in order to simplify the notation, define $\mathbf{F} = (\mathbf{A}+\mathbf{BK})$. Since $\tilde{\mathbf{X}}_t$ is positive semidefinite by assumption, then also $\mathbf{\Psi}$ is positive semidefinite and thus for every vector of proper dimension $\boldsymbol{\eta} = [\eta_1^T, \dots, \eta_M^T]^T$ we have that

$$\boldsymbol{\eta}^T\mathbf{\Psi}\boldsymbol{\eta} = \begin{bmatrix}\eta_1^T & \dots & \eta_M^T\end{bmatrix} \begin{bmatrix} F_{11} & F_{12} & \dots & F_{1M} \\ F_{21} & F_{22} & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ F_{M1} & \dots & \dots & F_{MM}\end{bmatrix} \tilde{\mathbf{X}}_t \begin{bmatrix} F_{11} & F_{12} & \dots & F_{1M} \\ F_{21} & F_{22} & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ F_{M1} & \dots & \dots & F_{MM}\end{bmatrix}^T \begin{bmatrix}\eta_1 \\ \vdots \\ \eta_M\end{bmatrix}$$

Then, by means of simple computations, the latter expression can be rewritten as follows

$$\boldsymbol{\eta}^T\mathbf{\Psi}\boldsymbol{\eta} = \begin{bmatrix}\sum_i \eta_i^T F_{i1} & \sum_i \eta_i^T F_{i2} & \dots & \sum_i \eta_i^T F_{iM}\end{bmatrix} \begin{bmatrix} \tilde{X}_t^{[1]} & 0 & \dots & 0 \\ 0 & \tilde{X}_t^{[2]} & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & \tilde{X}_t^{[M]}\end{bmatrix} \begin{bmatrix}\sum_i F_{i1}^T\eta_i \\ \sum_i F_{i2}^T\eta_i \\ \vdots \\ \sum_i F_{iM}^T\eta_i\end{bmatrix}$$

from which we have

$$\boldsymbol{\eta}^T\mathbf{\Psi}\boldsymbol{\eta} = \sum_j \sum_i \eta_i^T F_{ij}\tilde{X}_t^{[j]} \sum_h F_{hj}^T\eta_h = \sum_j \sum_{i,h} \eta_i^T F_{ij}\tilde{X}_t^{[j]} F_{hj}\eta_h$$

Now we apply the following result, that can be derived from Lemma 4 in Section 5.6, to the previous expression. If $P \succeq 0$ then for every v_i and v_j of appropriate dimensions we have $v_i^T P v_i + v_j^T P v_j \geq 2v_i^T P v_j$ and thus $v_i^T P v_j \leq 1/2(\|v_i\|_P^2 + \|v_j\|_P^2)$. In particular, given that the terms $\tilde{X}_t^{[j]}$, for all $i = 1, \dots, M$ are positive semidefinite by assumption, we have

$$\sum_j \sum_{i,h} \eta_i^T F_{ij}\tilde{X}_t^{[j]} F_{hj}\eta_h \leq \sum_j \sum_{i,h} \frac{1}{2}(\eta_i^T F_{ij}\tilde{X}_t^{[j]} F_{ij}^T\eta_i + \eta_h^T F_{hj}\tilde{X}_t^{[j]} F_{hj}^T\eta_h)$$

now the two terms in the brackets are completely equivalent and moreover we can define as ν_j the number of elements $F_{ij} \neq 0$ so that we have

$$\sum_j \sum_{i,h} \eta_i^T F_{ij} \tilde{X}_t^{[j]} F_{hj} \eta_h \leq \sum_j \nu_j \sum_i \eta_i^T F_{ij} \tilde{X}_t^{[j]} F_{ij}^T \eta_i = \sum_i \|\eta_i\|^2_{\sum_j \nu_j F_{ij}^T \tilde{X}_t^{[j]} F_{ij}}$$

and finally, defining $\tilde{F}_{ij} = \sqrt{\nu_j} F_{ij}$, this corresponds to

$$\begin{bmatrix} \eta_1^T & \dots & \eta_M^T \end{bmatrix} \begin{bmatrix} \sum_j \tilde{F}_{1j} \tilde{X}_t^{[j]} \tilde{F}_{1j}^T & & \\ & \ddots & \\ & & \sum_j \tilde{F}_{Mj} \tilde{X}_t^{[j]} \tilde{F}_{Mj}^T \end{bmatrix} \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_M \end{bmatrix}$$

Summarizing this proves that

$$\begin{bmatrix} \sum_j \tilde{F}_{1j} \tilde{X}_t^{[j]} \tilde{F}_{1j}^T & & \\ & \ddots & \\ & & \sum_j \tilde{F}_{Mj} \tilde{X}_t^{[j]} \tilde{F}_{Mj}^T \end{bmatrix} \succeq \mathbf{F} \tilde{\mathbf{X}}_t \mathbf{F}^T$$

and finally from this, and recalling that $\mathbf{B_w} \mathbf{W} \mathbf{B_w}^T$ is block diagonal, the proof is completed.

6.4.2 Proof of Theorem3

Assume that, at time instant t and for each subsystem $i = 1, \dots, M$, a feasible solution to i-MPC is available, i.e., $\hat{x}_{t|t}^{[i]}$ with optimal input sequence $\hat{u}_{t \dots t+N-1|t}^{[i]}$. We prove that, at time $t+1$, a feasible solution to each i-MPC problem exists, i.e., $\hat{x}_{t+1|t}^{[i]}$ (i.e., with strategy $S1$ in constraint (6.29b)), with input sequence $\hat{u}_{t+1 \dots t+N|t}^{[i],f} = \{\hat{u}_{t+1|t}^{[i]}, \dots, \hat{u}_{t+N-1|t}^{[i]}, K_i \hat{x}_{t+N|t}^{[i]}\}$.
Since (6.29d) and (6.29e) hold for the i-MPC problem solved at time t, i.e., for all $k = t, \dots, t+N-1$, it holds that (6.29d) and (6.29e) are verified also for the candidate feasible solution to the i-MPC problem at time $t+1$ for $k = t+1, \dots, t+N-1$.
Also, in view of the fact that $\tilde{X}_{t+N}^{[i]} \preceq \bar{X}_i$, of (6.28), and (6.29f), then $\hat{x}_{t+N|t}^{[i],f} = \hat{x}_{t+N|t}^{[i]} \in \hat{\mathbb{X}}_0^{[i]} \cap \hat{\mathbb{X}}_{t+N}^{[i]}$, as required. Finally, since $\hat{x}_{t+N|t}^{[i],f} = \hat{x}_{t+N|t}^{[i]} \in \mathbb{X}_F^{[i]}$ and since $\mathbb{X}_F^{[i]}$ is a positively invariant set for the nominal

subsystem (6.5) under the auxiliary control law, then $\hat{x}^{[i],f}_{t+N+1|t} = (A_{ii} + B_{u,i}K_i)\hat{x}^{[i]}_{t+N|t} \in \mathbb{X}^{[i]}_F$. This concludes the proof of recursive feasibility.

In view of the feasibility, at time $t+1$ of the possibly suboptimal solution $\hat{u}^{[i],f}_{t+1\dots t+N|t}$, $\bar{x}^{[i]}_{t+1|t}$, and denoting with $J^{[i],*}(t+1)$ the optimal cost function computed at time $t+1$ by subsystem i, then

$$J^{[i],*}(t+1) \leq J^{[i],f}(t+1|t) \tag{6.32}$$

where, by standard arguments, $J^{[i]}(t+1|t) = J^{[i],*}(t) - \|\hat{x}^{[i]}_{t|t}\|^2_{Q_i} - \|\hat{u}^{[i]}_{t|t}\|^2_{R_i} + \|\hat{x}^{[i]}_{t+N|t}\|^2_{Q_i} + \|K_i\hat{x}^{[i]}_{t+N|t}\|^2_{R_i} - \|\hat{x}^{[i]}_{t+N|t}\|^2_{S_i} + \|(A_{ii} + B_{u,i}K_i)\hat{x}^{[i]}_{t+N|t}\|^2_{S_i}$. In view of (6.26) and (6.32), then $J^{[i],*}(t+1) \leq J^{[i],*}(t) - (\|\hat{x}^{[i]}_{t|t}\|^2_{Q_i} + \|\hat{u}^{[i]}_{t|t}\|^2_{R_i})$. Using standard arguments in MPC, this proves that $\|\hat{x}^{[i]}_{t|t}\|^2_{Q_i} + \|\hat{u}^{[i]}_{t|t}\|^2_{R_i} \to 0$ as $t \to +\infty$ which, since $Q_i > 0$ and $R_i > 0$, implies that $\hat{x}^{[i]}_{t|t} \to 0$ and $\hat{u}^{[i]}_{t|t} \to 0$ as $t \to +\infty$.
From (6.3) and (6.30), it follows that

$$\mathbf{x}_{t+1} = (\mathbf{A} + \mathbf{B}\mathbf{K})\mathbf{x}_t + \mathbf{B}(\hat{\mathbf{u}}_{t|t} - \mathbf{K}\hat{\mathbf{x}}_{t|t}) + \mathbf{B}_{\mathbf{w}}\mathbf{w}_t$$

where $(\mathbf{A} + \mathbf{B}\mathbf{K})$ is stable in view of Assumption 2. Now define the stationary process Δ_t described by

$$\Delta_{t+1} = (\mathbf{A} + \mathbf{B}\mathbf{K})\Delta_t + \mathbf{B}_{\mathbf{w}}\mathbf{w}_t$$

with zero mean and variance $\mathbf{X}_{ss}$. Resorting to the separation principle, we can define $\tilde{\mathbf{x}}_t = \mathbf{x}_t - \Delta_t$, which evolves according to

$$\tilde{\mathbf{x}}_{t+1} = (\mathbf{A} + \mathbf{B}\mathbf{K})\tilde{\mathbf{x}}_t + \mathbf{B}(\hat{\mathbf{u}}_{t|t} - \mathbf{K}\hat{\mathbf{x}}_{t|t})$$

On the other hand, $\mathbf{B}(\hat{\mathbf{u}}_{t|t} - \mathbf{K}\hat{\mathbf{x}}_{t|t})$ is asymptotically vanishing, and therefore $\tilde{\mathbf{x}}_t \to 0$ as $t \to 0$. Therefore $\mathbb{E}\{\|\mathbf{x}_t\|^2\} = \mathbb{E}\{\|\tilde{\mathbf{x}}_t\|^2\} + \mathbb{E}\{\|\Delta_t\|^2\} \to \mathbb{E}\{\|\Delta_t\|^2\} = tr(\mathbf{X}_{ss})$.

Finally since, for all $t > 0$, $\hat{x}^{[i]}_{t|t} \in \hat{\mathbb{X}}^{[i]}_t$, $\bar{x}^{[i]}_t - \hat{x}^{[i]}_{t|t} \in \mathbb{Z}_i$ then $\bar{x}^{[i]}_t \in \bar{\mathbb{X}}^{[i]}_t$. Recall that sets $\hat{\mathbb{X}}^{[i]}_k$ and $\hat{\mathbb{U}}^{[i]}_k$ are computed according to (6.22), where $\bar{\mathbb{X}}^{[i]}_k$ and $\bar{\mathbb{U}}^{[i]}_k$ are defined in (6.15) and (6.16). In turn, they depend on the values of $\tilde{X}^{[i]}_k$ and $\tilde{U}^{[i]}_k = K_i\tilde{X}^{[i]}_k K_i^T$, respectively. From Lemma 5 and the Cantelli inequality, then (6.2a) is verified. Also, since $\hat{u}^{[i]}_{t|t} \in \hat{\mathbb{U}}^{[i]}_t$, then $\bar{u}^{[i]}_t \in \bar{\mathbb{U}}^{[i]}_t$. In turn, (6.2b) is also proved.

6.5 Simulation example

In this section we consider as an example the system described in [12] to show the effectiveness of the proposed distributed approach. In particular, the goal is to control the levels, h_1, h_2, h_3 and h_4, of the four tank system depicted in Figure.6.1 acting on the voltages of the two pumps, v_1 and v_2. The continuous-time dynamics of the system is given by

$$\begin{aligned}
\frac{dh_1}{dt} &= -\frac{a_1}{A_1}\sqrt{2gh_1} + \frac{a_4}{A_4}\sqrt{2gh_4} + \frac{\gamma_1 k_1}{A_1}v_1 \\
\frac{dh_2}{dt} &= -\frac{a_2}{A_2}\sqrt{2gh_2} + \frac{(1-\gamma_1)k_1}{A_2}v_1 \\
\frac{dh_3}{dt} &= -\frac{a_3}{A_3}\sqrt{2gh_3} + \frac{a_2}{A_2}\sqrt{2gh_2} + \frac{\gamma_2 k_2}{A_3}v_2 \\
\frac{dh_4}{dt} &= -\frac{a_4}{A_4}\sqrt{2gh_4} + \frac{(1-\gamma_2)k_2}{A_4}v_2
\end{aligned} \tag{6.33}$$

where A_i and a_i are the cross-section of Tank i and the cross section of the outlet hole of Tank i, respectively, and the parameters $\gamma_1 \in (0,1)$ and $\gamma_2 \in (0,1)$ represent the fraction (fixed) of water that flows inside the lower tanks. The coefficients k_1 and k_2 represent the conversion parameters from the voltage applied to the pump to the flux of water. The values are chosen as in [12]. The system is linearized around the equilibrium point $\bar{v}_1 = \bar{v}_2 = 3V$, $\bar{h}_1 = 12.263cm$, $\bar{h}_2 = 1.409cm$, $\bar{h}_3 = 12.783cm$ and $\bar{h}_4 = 1.634cm$ and then discretized using a sampling time of 1 s. The resulting dynamics is partitioned into two different subsystems collecting tanks 1-2 and tanks 3-4, respectively.

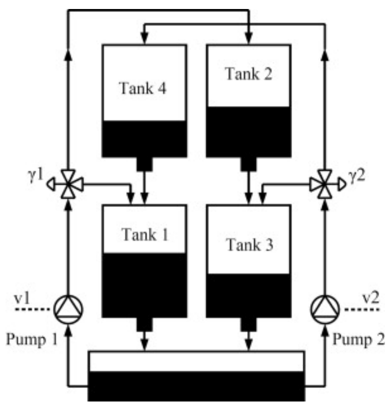


Figure 6.1: *Four tank system.*

With this choice, the system matrices that appear in (6.1) are

$$A_{11} = \begin{bmatrix} 0.98 & 0 \\ 0 & 0.97 \end{bmatrix},\ A_{12} = \begin{bmatrix} 0 & 0.04 \\ 0 & 0 \end{bmatrix},\ B_{u,1} = \begin{bmatrix} 0.08 \\ 0.03 \end{bmatrix},\ B_{w,1} = I_2$$
$$A_{21} = \begin{bmatrix} 0 & 0.03 \\ 0 & 0 \end{bmatrix},\ A_{22} = \begin{bmatrix} 0.99 & 0 \\ 0 & 0.96 \end{bmatrix},\ B_{u,2} = \begin{bmatrix} 0.06 \\ 0.05 \end{bmatrix},\ B_{w,2} = I_2$$

The disturbances are assumed to be two additive independent Gaussian noises with zero means and variances $W_1 = W_2 = 1$. The initial condition are chosen as $x_0^{[1]} = [0.274,\ 0.067]^T$ and $x_0^{[2]} = [0.203,\ 0.254]^T$. Moreover, the constraints on the inputs and states are defined by

$$\begin{aligned} x_{min}^{[1]} &= \left[-12.263, -1.409\right]^T,\ x_{max}^{[1]} = \left[40, 40\right]^T + x_{min}^{[1]} \\ x_{min}^{[2]} &= \left[-12.783, -1.634\right]^T,\ x_{max}^{[2]} = \left[40, 40\right]^T + x_{min}^{[2]} \\ u_{min}^{[1]} &= u_{min}^{[2]} = -3,\ u_{max}^{[1]} = u_{max}^{[2]} = 3 \end{aligned}$$

and, in line with the proposed approach, they are imposed as individual chance constraints following (6.2). The single violation probabilities are fixed to a common value for the state constraints, $p^x = 0.2$ and to $p^u = 0.1$ for the input. It is set $Q_1 = Q_2 = I_2$ amd $R_1 = R_2 = 0.1$. Also, matrices K_i and S_i are defined to satisfy Assumption2 and (6.26).

A set of 100 simulations is run, to show the effectiveness of the approach. Results are depicted in Figure 6.2 where it is possible to verify that the proposed technique correctly drives the state of the systems to a neighborhood of the origin.

6.6 Comments

In this chapter, a novel Stochastic Distributed Predictive Control technique has been presented for linear discrete-time systems with additive, possibly unbounded, disturbances and subject to individual chance-constraints on the state and the input. In particular, the p-SMPC approach described in Chapter 3 has been extended to the distributed framework and the guaranteed feasibility and convergence properties have been recovered by means of a proper choice of the initial conditions for the problem and its terminal constraints. An example showed the viability of the approach.
Despite the presence of stochastic constraints, the usage of robust set inclusion techniques renders the proposed algorithm quite conservative, and increases the design complexity. However, this Chapter has to be intended

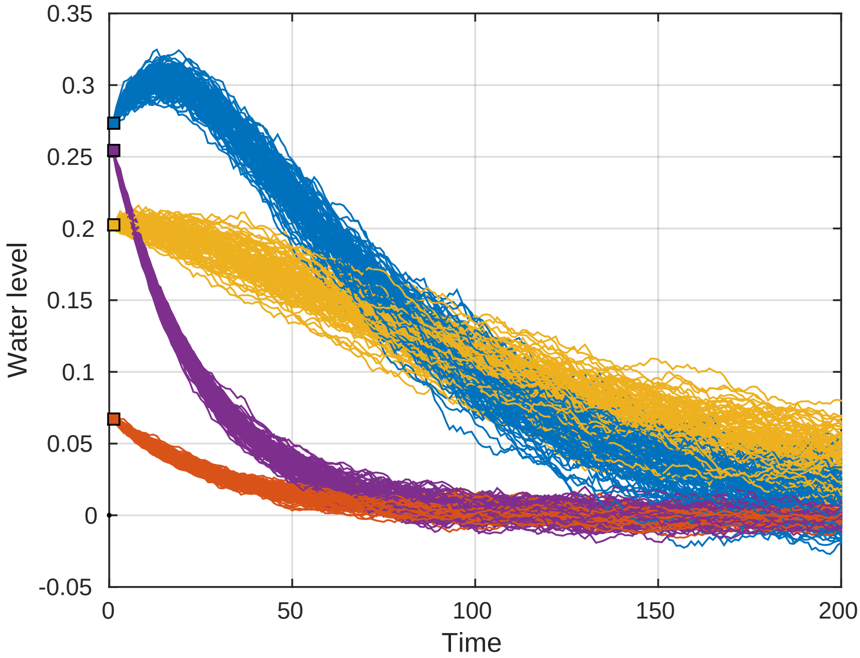


Figure 6.2: *Example of application of the distributed p-SMPC approach. The colored lines represent the trajectories of the system states (the levels) over time in* 100 *different simulations.*

as a first step in the direction of applying SMPC methods into distributed control problems, with the aim of coping with disturbances with possibily unbounded support and probabilistic constraints on the system variables.

CHAPTER 7

Distributed Predictive Control for tracking reference signals: a robust approach.

Many decentralized and distributed MPC algorithms have been recently developed, see for example the book [92] and the review papers [32, 142]. Most of these methods consider the so-called regulation problem: given a large-scale dynamical system made by a number of (interconnected or independent) subsystems, the problem is to asymptotically steer to zero the state of all the subsystems by coordinating the local control actions with a minimum amount of transmitted information. However, in this setting, the solution of the tracking problem with distributed MPC is much more difficult. In fact, the decentralization constraints do not allow to follow the standard approach, based on the reformulation of the tracking problem as a regulation one by computing, at any set-point change of the output, the corresponding state and control target values. For this reason, and to the best of our knowledge, only the cooperative distributed MPC algorithm described in [50] is nowadays available, while a different approach based on a distributed reference governor has been proposed in [149].

In this chapter, a new distributed MPC method for the solution of the tracking problem is proposed for systems made by the collection of M subsystems. The algorithm is developed according to the hierarchical structure depicted in Figure 7.1 and in particular

a) at the higher layer, given the required output reference signals $y^{[i]}_{set-point}$, $i = 1, ..., M$, feasible trajectories $\tilde{x}^{[i]}$, $\tilde{u}^{[i]}$, and $\tilde{y}^{[i]}$ of the local state, input and output variables are computed for each subsystem $\mathcal{S}_i$. Notably, these trajectories are computed according to the prescribed distributed information pattern, also adopted at the lower layer

b) at the lower layer, M regulators are designed with the distributed algorithm developed in [47] for the solution of the local regulation problems.

The overall algorithm guarantees that the controlled outputs reach the prescribed reference values whenever possible, or their nearest feasible value when feasibility problems arise due to the constraints. A preliminary version of this algorithm is presented in [43] where, however, a less general control problem is considered and more restrictive conditions are required. In particular, DPC assumes that the future state and control reference trajectories are transmitted by each subsystem to its neighbors, and the differences between these reference trajectories and the true ones are interpreted as disturbances to be rejected. As such, the robust MPC approach introduced in [102] is used for the development of the distributed control laws. In addition, the proposed algorithm also allows one to consider the presence of possible unfeasible reference signals, i.e. of set-points that cannot be reached due to state and/or control constraints. This problem is solved by resorting to the ideas described in [84] where, in case of unfeasible references, it is suggested to steer the output variables to the nearest (in some sense) value compatible with the process constraints.

Notes on the notation adopted in this chapter

In order to be consistent with the well-established literature on distributed predictive control, e.g. [88], the notation adopted in this chapter is slightly different from the one used in the rest of the Thesis. In the sequel a quick description is given, however, whenever the meaning is not clarified by the context, more details will be found locally.
A matrix is Schur if all its eigenvalues lie in the interior of the unit circle. The short-hand $\mathbf{v} = (v_1, \ldots, v_s)$ denotes a column vector with s (not necessarily scalar) components $v_1, \ldots, v_s$. The symbols $\oplus$ and $\ominus$ denote

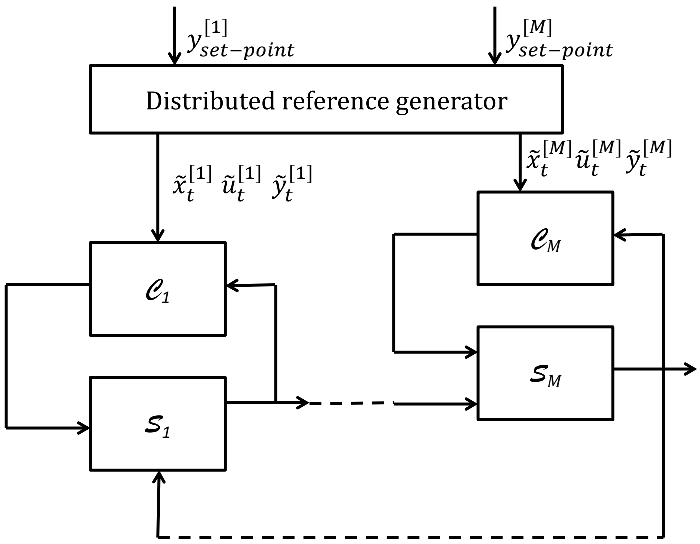


Figure 7.1: *Overall control architecture.*

the Minkowski sum and Pontryagin difference, respectively, [137], while $\bigoplus_{i=1}^{M} A_i = A_1 \oplus \cdots \oplus A_M$. A generic q-norm ball centered at the origin in the $\mathbb{R}^{dim}$ space is defined as follows $\mathcal{B}^{(dim)}_{q,\varepsilon}(0) := \{x \in \mathbb{R}^{dim} : \|x\|_q \leq \varepsilon\}$. For a discrete-time signal s_t and $a, b \in \mathbb{N}$, $a \leq b$, $(s_a, s_{a+1}, \ldots, s_b)$ is denoted with $s_{[a:b]}$.

7.1 Interacting subsystems

Consider M dynamically interacting subsystems which, according to the notation used in [88], are described by

$$x^{[i]}_{t+1} = A_{ii}\, x^{[i]}_t + B_{ii} u^{[i]}_t + E_i s^{[i]}_t \tag{7.1a}$$

$$y^{[i]}_t = C_i x^{[i]}_t \tag{7.1b}$$

$$z^{[i]}_t = C_{zi} x^{[i]}_t + D_{zi} u^{[i]}_t \tag{7.1c}$$

where $x^{[i]}_t \in \mathbb{R}^{n_i}$ and $u^{[i]}_t \in \mathbb{R}^{m_i}$ are the states and inputs, respectively, of the i-th subsystem, while $y^{[i]}_t \in \mathbb{R}^{p_i}$ is its output, with $p_i \leq m_i$. In line with the interaction-oriented models introduced in [88], the coupling input and output vectors $s^{[i]}_t$ and $z^{[i]}_t$, respectively, are defined to characterize the interconnections among the subsystems, i.e.,

$$s_t^{[i]} = \sum_{j=1}^{M} L_{ij} z_t^{[j]} \tag{7.2}$$

We say that subsystem j is a *dynamic neighbor* of subsystem i if and only if $L_{ij} \neq 0$, and we denote as $\mathcal{N}_i$ the set of dynamic neighbors of subsystem i (which excludes i).
The input and state variables are subject to the "local" constraints $u_t^{[i]} \in \mathbb{U}_i \subseteq \mathbb{R}^{m_i}$ and $x_t^{[i]} \in \mathbb{X}_i \subseteq \mathbb{R}^{n_i}$, respectively, where the sets $\mathbb{U}_i$ and $\mathbb{X}_i$ are convex. Furthermore, we allow for n_c linear constraints involving the output variables of more than one subsystem: the h-th constraint inequality is

$$\sum_{j=1}^{M} H_h^{[j]} y_t^{[j]} \leq l_h \tag{7.3}$$

where $H_h^{[j]} \in \mathbb{R}^{1\times p_j}$ are row vectors, which can be possibly equal to zero, for some values of j. Without loss of generality, the outputs involved in coupling constraints are accounted for as coupling outputs, i.e., for all $h = 1, \dots, n_c$ and for all $j = 1, \dots, M$, there exists a matrix $H_h^{[j]z}$ such that

$$H_h^{[j]} y_t^{[j]} = H_h^{[j]z} z_t^{[j]} \tag{7.4}$$

under (7.1b) and (7.1c).
We say that the h-th inequality is a constraint on subsystem i if $H_h^{[i]} \neq 0$, and we denote the set of constraints on subsystem i as $\mathcal{C}_i = \{h \in \{1, \dots, n_c\}$: the h-th inequality is a constraint on $i\}$, and with $n_c^{[i]}$ the number of elements of $\mathcal{C}_i$. Subsystem $j \neq i$ is a *constraint neighbor* of subsystem i if there exists $\bar{h} \in \mathcal{C}_i$ such that $H_{\bar{h}}^{[j]} \neq 0$. For all $h = 1, \dots, n_c$, we finally denote with $\mathcal{S}_h$ the set of subsystems for which the h-th inequality is a constraint, i.e., $\mathcal{S}_h = \{i : H_h^{[i]} \neq 0\}$ and with $n_h = |\mathcal{S}_h|$ the cardinality of $\mathcal{S}_h$.

Collecting the subsystems (7.1) for all $i = 1, \dots, M$, we obtain the collective dynamical model

$$\mathbf{x}_{t+1} = \mathbf{A}\,\mathbf{x}_t + \mathbf{B}\mathbf{u}_t \tag{7.5a}$$
$$\mathbf{y}_t = \mathbf{C}\,\mathbf{x}_t \tag{7.5b}$$

where $\mathbf{x}_t = (x_t^{[1]}, \dots, x_t^{[M]}) \in \mathbb{R}^n$, $n = \sum_{i=1}^M n_i$, $\mathbf{u}_t = (u_t^{[1]}, \dots, u_t^{[M]}) \in \mathbb{R}^m$, $m = \sum_{i=1}^M m_i$, and $\mathbf{y}_t = (y_t^{[1]}, \dots, y_t^{[M]}) \in \mathbb{R}^p$, $p = \sum_{i=1}^M p_i$, are the

collective state, input, and output vectors, respectively. The state transition matrices $A_{11} \in \mathbb{R}^{n_1 \times n_1}, \dots, A_{MM} \in \mathbb{R}^{n_M \times n_M}$ of the M subsystems are the diagonal blocks of $\mathbf{A}$, whereas the dynamic coupling terms between subsystems correspond to the non-diagonal blocks of $\mathbf{A}$, i.e., $A_{ij} = E_i L_{ij} C_{zj}$, with $j \neq i$. Correspondingly, B_{ii}, $i = 1, \dots, M$, are the diagonal blocks of $\mathbf{B}$, whereas the influence of the input of a subsystem upon the state of different subsystems is represented by the off-diagonal terms of $\mathbf{B}$, i.e., $B_{ij} = E_i L_{ij} D_{zj}$, with $j \neq i$. The collective output matrix is defined as $\mathbf{C}$ =diag$(C_{11}, \dots, C_{MM})$.
Concerning system (7.5a) and its partition, the following main assumption on decentralized stabilizability is introduced:

Assumption 6. *There exists a block-diagonal matrix* $\mathbf{K}$ *=diag*$(K_1, \dots, K_M)$*, with* $K_i \in R^{m_i \times n_i}$*,* $i = 1, \dots, M$ *such that: (i)* $\mathbf{F} = \mathbf{A} + \mathbf{BK}$ *is Schur, (ii)* $F_{ii} = (A_{ii} + B_{ii} K_i)$ *is Schur,* $i = 1, \dots, M$*.*

Remark 4. *The design of the stabilizing matrix* $\mathbf{K}$ *can be performed according to the procedure proposed in [47] or by resorting to an LMI formulation, see [11], based on well known results in decentralized control, see e.g. [153].*

The following standard assumption is made.

Assumption 7.

$$\textit{rank}\left(\begin{bmatrix} I_n - \mathbf{A} & -\mathbf{B} \\ \mathbf{C} & 0 \end{bmatrix}\right) = n + p$$

.

7.2 Control system architecture

The higher layer of the hierarchical scheme of Figure 7.1 is itself made by two sub-layers, as shown in Figure 7.2: a *reference output trajectory layer* computes in a distributed way the output reference trajectories $\tilde{y}^{[i]}$ given the "ideal" set-points $y^{[i]}_{set-point}$, while a *reference state and input trajectory layer* determines the corresponding state and control trajectories $\tilde{x}^{[i]}$ and $\tilde{u}^{[i]}$. At the lower layer of the structure of Figure 7.1, a distributed *robust MPC layer* is designed to drive the real state and input trajectories $x^{[i]}_t$ and $u^{[i]}_t$ of the subsystems as close as possible to $\tilde{x}^{[i]}_t$, $\tilde{u}^{[i]}_t$, while satisfying the constraints. Notably, at each level, information is required to be transmitted only among neighboring subsystems.

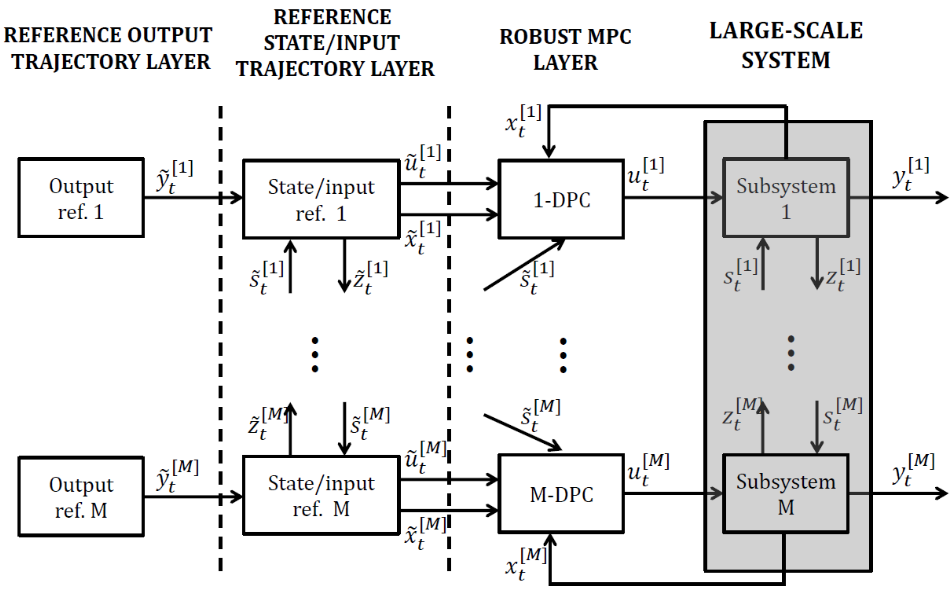


Figure 7.2: *Distributed architecture for tracking reference signals.*

The reference output trajectory layer

At any time t, the reference trajectories $\tilde{y}^{[i]}_{t+k}$, $k = 1, \ldots, N-1$, are regarded as an argument of an optimization problem itself (see the following Paraghaph IV.b) rather than a fixed parameter, similarly to the approach in [84]. However, in the considered distributed context, too rapid changes of the output reference trajectory of a given subsystem could greatly affect the performance and the behavior of the other subsystems. Therefore, in order to limit the rate of variation, it is required that, for all $i = 1, \ldots, M$, for all $t \geq 0$

$$\tilde{y}^{[i]}_{t+1} \in \tilde{y}^{[i]}_t \oplus \mathcal{B}^{(p_i)}_{q,\varepsilon}(0) \tag{7.6}$$

The reference output trajectory management layer is also committed to defining suitable update laws for $\tilde{y}^{[i]}_t$ in such a way that, for all time steps, (7.3) is verified for all $h = 1, \ldots, n_c$.

The reference state and input trajectory layer

Given, at any time step t, the future reference trajectories $\tilde{y}^{[i]}_k$, $k = t, \ldots, t+N-1$, in order to define the state and control reference trajectories $(\tilde{x}^{[i]}_t, \tilde{u}^{[i]}_t)$, consider the systems

$$\tilde{x}^{[i]}_{t+1} = A_{ii}\,\tilde{x}^{[i]}_t + B_{ii}\tilde{u}^{[i]}_t + E_i\tilde{s}^{[i]}_t \tag{7.7a}$$

$$\tilde{e}^{[i]}_{t+1} = \tilde{e}^{[i]}_t + \tilde{y}^{[i]}_{t+1} - C_i\tilde{x}^{[i]}_t \tag{7.7b}$$

where, similarly to (7.1c) and (7.2)

$$\tilde{z}^{[i]}_t = C_{zi}\tilde{x}^{[i]}_t + D_{zi}\tilde{u}^{[i]}_t \tag{7.7c}$$

$$\tilde{s}^{[i]}_t = \sum_{j\in\mathcal{N}_i} L_{ij}\tilde{z}^{[j]}_t \tag{7.7d}$$

Define $\chi^{[i]}_t = (\tilde{x}^{[i]}_t, \tilde{e}^{[i]}_t)$,

$$\mathcal{A}_{ij} = \begin{cases} \begin{bmatrix} A_{ii} & 0 \\ -C_i & I_{p_i} \end{bmatrix} & \text{if } j = i \\ \\ \begin{bmatrix} A_{ij} & 0 \\ 0 & 0 \end{bmatrix} & \text{if } j \neq i \end{cases}, \mathcal{B}_{ij} = \begin{bmatrix} B_{ij} \\ 0 \end{bmatrix}, \mathcal{G}_i = \begin{bmatrix} 0 \\ I_{p_i} \end{bmatrix} \tag{7.7e}$$

and consider the control law

$$\tilde{u}^{[i]}_t = \mathcal{K}_i\chi^{[i]}_t \tag{7.7f}$$

where $\mathcal{K}_i = \begin{bmatrix} K^x_i & K^e_i \end{bmatrix}$. Letting $\mathcal{F}_{ij} = \mathcal{A}_{ij} + \mathcal{B}_{ij}\mathcal{K}_j$, the dynamics of $\chi^{[i]}_t$ is therefore defined by

$$\chi^{[i]}_{t+1} = \mathcal{F}_{ii}\chi^{[i]}_t + \sum_{j\in\mathcal{N}_i} \mathcal{F}_{ij}\chi^{[j]}_t + \mathcal{G}_i\tilde{y}^{[i]}_{t+1} \tag{7.8}$$

The gain matrix $\mathcal{K}_i$ is to be determined as follows: denoting by $\mathcal{A}$ and $\mathcal{B}$ the matrices whose block elements are $\mathcal{A}_{ij}$ and $\mathcal{B}_{ij}$, respectively, and $\mathcal{K}$ =diag$(\mathcal{K}_1, \dots, \mathcal{K}_M)$, the following assumption must be fulfilled

Assumption 8. *The matrix $\mathcal{F} = \mathcal{A} + \mathcal{B}\mathcal{K}$ is Schur stable.* □

The synthesis of the $\mathcal{K}_i$'s can be performed provided that Assumption 7 is verified and according to the procedures proposed in Remark 4. Moreover, when $p_i > m_i$ for some i, the available degrees of freedom can be used to select $\mathcal{K}_i$'s fulfilling some additional optimization criteria.
Define, for all $i = 1, \dots, M$ and for all $t \geq 0$, $\chi^{[i]ss}_t = (x^{[i]ss}_t, e^{[i]ss}_t)$, as the steady-state condition for (7.8) corresponding to the reference outputs $\tilde{y}^{[i]}_t$ assumed constant, i.e., $\tilde{y}^{[i]}_{t+1} = \tilde{y}^{[i]}_t$, and satisfying for all $i = 1, \dots, M$

$$\chi^{[i]ss}_t = \mathcal{F}_{ii}\chi^{[i]ss}_t + \sum_{j\in\mathcal{N}_i} \mathcal{F}_{ij}\chi^{[j]ss}_t + \mathcal{G}_i\tilde{y}^{[i]}_t \tag{7.9}$$

In view of (7.7) and Assumption 8, $C_i x_t^{[i]ss} = \tilde{y}_{t+1}^{[i]}$ and $\mathcal{F}$ is Schur stable. Then a solution to the system (7.9) exists and is unique. Collectively define $\boldsymbol{\chi}_t^{ss} = (\chi_t^{[1]ss}, \dots, \chi_t^{[M]ss})$, $\boldsymbol{\chi}_t = (\chi_t^{[1]}, \dots, \chi_t^{[M]})$, and $\tilde{\mathbf{y}}_t = (\tilde{y}_t^{[1]}, \dots, \tilde{y}_t^{[M]})$. From (7.7e)-(7.9) we can collectively write

$$\boldsymbol{\chi}_{t+1}^{ss} - \boldsymbol{\chi}_t^{ss} = (I_{n+p} - \mathcal{F})^{-1}\mathcal{G}(\tilde{\mathbf{y}}_{t+1} - \tilde{\mathbf{y}}_t) \tag{7.10}$$

where $\mathcal{G}$ =diag$(\mathcal{G}_1, \dots, \mathcal{G}_M)$. Therefore

$$\boldsymbol{\chi}_{t+1} - \boldsymbol{\chi}_{t+1}^{ss} = \mathcal{F}(\boldsymbol{\chi}_t - \boldsymbol{\chi}_t^{ss}) - (I_{n+p} - \mathcal{F})^{-1}\mathcal{F}\mathcal{G}(\tilde{\mathbf{y}}_{t+1} - \tilde{\mathbf{y}}_t) \tag{7.11}$$

which can be rewritten as

$$\boldsymbol{\chi}_{t+1} - \boldsymbol{\chi}_{t+1}^{ss} = \mathcal{F}(\boldsymbol{\chi}_t - \boldsymbol{\chi}_t^{ss}) + \tilde{\mathbf{w}}_t \tag{7.12}$$

where $\tilde{\mathbf{w}}_t$ can be seen as a bounded disturbance. In fact, in view of (7.6), $\tilde{\mathbf{w}}_t \in \tilde{\mathbb{W}} = -(I_{n+p} - \mathcal{F})^{-1}\mathcal{F}\mathcal{G}\prod_{i=1}^{M} \mathcal{B}_{q,\varepsilon}^{(p_i)}(0)$. Under Assumption 8, for the system (7.12) there exists a possibly non-rectangular Robust Positive Invariant (RPI) set $\boldsymbol{\Delta}^{\chi}$ such that, if $\boldsymbol{\chi}_t - \boldsymbol{\chi}_t^{ss} \in \boldsymbol{\Delta}^{\chi}$, then it is guaranteed that $\boldsymbol{\chi}_{t+k} - \boldsymbol{\chi}_{t+k}^{ss} \in \boldsymbol{\Delta}^{\chi}$ for all $k \geq 0$. This, in turn, implies that the convex sets Δ_i^{χ} exist and can be defined in such a way that $\boldsymbol{\Delta}^{\chi} \subseteq \prod_{i=1}^{M} \Delta_i^{\chi}$, so that, for any initial condition $\boldsymbol{\chi}_0 - \boldsymbol{\chi}_0^{ss} \in \boldsymbol{\Delta}^{\chi}$, for all $t \geq 0$ it holds that

$$\chi_t^{[i]} - \chi_t^{[i]ss} \in \Delta_i^{\chi} \tag{7.13}$$

The robust distributed MPC layer

The DPC algorithm described in [47] is used to drive the real state and input trajectories $x_t^{[i]}$ and $u_t^{[i]}$ as close as possible to their references $\tilde{x}_t^{[i]}$, $\tilde{u}_t^{[i]}$. Specifically, by adding suitable constraints to the distributed MPC problem formulation, for each subsystem and for all $t \geq 0$ it is possible to guarantee that the actual coupling output trajectories lie in specified time-invariant neighborhoods of their reference trajectories. More formally, if $z_t^{[i]} \in \tilde{z}_t^{[i]} \oplus \mathcal{Z}_i$, where $\mathcal{Z}_i$ is compact, convex and $0 \in \mathcal{Z}_i$, in view of (7.7d) it is guaranteed that $s_t^{[i]} \in \tilde{s}_t^{[i]} \oplus \mathcal{S}_i$, where $\mathcal{S}_i = \bigoplus_{j \in \mathcal{N}_i} L_{ij}\mathcal{Z}_j$. In this way, (7.1a) can be written as

$$x_{t+1}^{[i]} = A_{ii}\, x_t^{[i]} + B_{ii}u_t^{[i]} + E_i\tilde{s}_t^{[i]} + E_i(s_t^{[i]} - \tilde{s}_t^{[i]}) \tag{7.14}$$

where $E_i(s_t^{[i]} - \tilde{s}_t^{[i]})$ can be seen as a bounded disturbance, while $E_i\tilde{s}_{t+k}^{[i]}$ can be interpreted as an input, known in advance over the prediction horizon $k = 0, \dots, N-1$.

For the statement of the individual MPC sub-problems, henceforth called i-DPC problems, define the i-th subsystem nominal model associated to equation (7.14)

$$\hat{x}_{t+1}^{[i]} = A_{ii}\,\hat{x}_t^{[i]} + B_{ii}\hat{u}_t^{[i]} + E_i\tilde{s}_t^{[i]} \tag{7.15}$$

and let

$$\hat{z}_t^{[i]} = C_{zi}\,\hat{x}_t^{[i]} + D_{zi}\hat{u}_t^{[i]} \tag{7.16}$$

The control law for the i-th subsystem (7.14), for all $t \geq 0$, is assumed to be given by

$$u_t^{[i]} = \hat{u}_t^{[i]} + K_i(x_t^{[i]} - \hat{x}_t^{[i]}) \tag{7.17}$$

where K_i satisfies Assumption 6. Letting $\varepsilon_t^{[i]} = x_t^{[i]} - \hat{x}_t^{[i]}$ from (7.14), (7.15) and (7.17) it follows that

$$\varepsilon_{t+1}^{[i]} \;=\; F_{ii}\varepsilon_t^{[i]} + w_t^{[i]} \tag{7.18}$$

where

$$w_t^{[i]} = E_i(s_t^{[i]} - \tilde{s}_t^{[i]}) \tag{7.19}$$

is a bounded disturbance since $s_t^{[i]} - \tilde{s}_t^{[i]} \in \mathcal{S}_i$. It follows that

$$w_t^{[i]} \in \mathbb{W}_i = E_i\mathcal{S}_i \tag{7.20}$$

Since $w_t^{[i]}$ is bounded and F_{ii} is Schur, there exists an RPI $\mathcal{E}_i$ for (7.18) such that, for all $\varepsilon_t^{[i]} \in \mathcal{E}_i$, then $\varepsilon_{t+1}^{[i]} \in \mathcal{E}_i$. Therefore at time $t+1$, in view of (7.1c) and (7.16), it holds that $z_{t+1}^{[i]} - \hat{z}_{t+1}^{[i]} = (C_{zi}+D_{zi}K_i)\varepsilon_{t+1}^{[i]} \in (C_{zi}+D_{zi}K_i)\mathcal{E}_i$. In order to guarantee that, at time $t+1$, $z_{t+1}^{[i]} - \tilde{z}_{t+1}^{[i]} \in \mathcal{Z}_i$ can be still verified by adding suitable constraints to the optimization problems, the following assumption must be fulfilled.

Assumption 9. *For all $i = 1, \dots, M$, there exists a positive scalar ρ_i such that*

$$(C_{zi} + D_{zi}K_i)\mathcal{E}_i \oplus \mathcal{B}_{q,\rho_i}(0) \subseteq \mathcal{Z}_i \tag{7.21}$$

If Assumption 9 is fulfilled define, for all $i = 1, \dots, M$, the convex neighborhood of the origin Δ_i^z satisfying

$$\Delta_i^z \subseteq \mathcal{Z}_i \ominus (C_{zi} + D_{zi}K_i)\mathcal{E}_i \tag{7.22}$$

and consider the constraint $\hat{z}^{[i]}_{t+1} - \tilde{z}^{[i]}_{t+1} \in \Delta^z_i$, in such a way that

$$z^{[i]}_{t+1}-\tilde{z}^{[i]}_{t+1} = z^{[i]}_{t+1}-\hat{z}^{[i]}_{t+1}+\hat{z}^{[i]}_{t+1}-\tilde{z}^{[i]}_{t+1} \in (C_{zi}+D_{zi}K_i)\mathcal{E}_i \oplus \Delta^z_i \subseteq \mathcal{Z}_i \quad (7.23)$$

as required at all time steps $t \geq 0$.

7.3 The distributed predictive control algorithm

The overall design problem is composed by a preliminary centralized off-line design and an on-line solution of the M i-DPC problems, as now detailed.

Off-line design

The off-line design consists of the following procedure:
1) compute the matrices $\mathbf{K}$ and $\mathcal{K}$ satisfying Assumptions 6 and 8 (see Remark 4);
2) define $\mathcal{B}^{(p_i)}_{q,\varepsilon}(0)$, compute $\mathbf{\Delta}^\chi$ (an RPI for (7.12)) and Δ^χ_i (for the computation of RPIs see [137]);
3) compute the RPI sets $\mathcal{E}_i$ for the subsystems (7.18) and the sets Δ^z_i satisfying (7.22) and (7.23);
4) compute $\hat{\mathbb{X}}_i \subseteq \mathbb{X}_i \ominus \mathcal{E}_i$, $\hat{\mathbb{U}}_i \subseteq \mathbb{U}_i \ominus K_i\mathcal{E}_i$, the positively invariant set Σ_i for the equation

$$\delta x^{[i]}_{t+1} = F_{ii}\delta x^{[i]}_t \quad (7.24)$$

such that

$$(C_{zi} + D_{zi}K_i)\Sigma_i \subseteq \Delta^z_i \quad (7.25a)$$

and the convex sets $\mathbb{Y}_i$ such that

$$\begin{aligned}\begin{bmatrix} I_{n_i} & 0 \\ K^x_i & K^e_i \end{bmatrix} \left(\Gamma_i(I_{n+p} - \mathcal{F})^{-1}\mathcal{G}\prod_{j=1}^M \mathbb{Y}_j \oplus \Delta^\chi_i\right) \oplus \\ \oplus \begin{bmatrix} I_{n_i} \\ K_i \end{bmatrix} \Sigma_i \subseteq \hat{\mathbb{X}}_i \times \hat{\mathbb{U}}_i\end{aligned} \quad (7.25b)$$

where Γ_i is the matrix, of suitable dimensions, that selects the subvector $\chi^{[i]}_t$ out of $\boldsymbol{\chi}_t$. Specifically, $\mathbb{Y}_i$ is the set associated to $\tilde{y}^{[i]}_t$ such that the corresponding steady-state state and input satisfy the control and state constraints defined by $\hat{\mathbb{X}}_i$ and $\hat{\mathbb{U}}_i$.
Concerning the set-theoretical conditions guaranteeing the design of the

control scheme, it is worth mentioning that, at the price of a more conservative scheme and slower settling times (e.g., small parameter ε), equations (7.25) can always be verified. On the other hand, it is not always possible to select sets $\mathcal{Z}_i$ such that (7.21) is verified; as investigated in [47] it consists in a network-wide small gain condition.

On-line design

The on-line design is based on the solution of the following distributed and independent optimization problems. It is worth remarking that the reference output layer and the MPC problem consist in two independent optimization problems. This enhances the reliability of the approach and reduces its computational load, at the price of limiting the rate of variation of the output reference to the value that guarantees constraint satisfaction in all possible conditions, contrarily to [84].
1) Computation of the reference outputs. The output reference trajectories $\tilde{y}^{[i]}_{t+N}$ are computed to minimize the distance from the ideal set-points $y^{[i]}_{set-point}$ and to fulfill the constraints, including the coupling ones (7.3). Concerning the latter, define

$$\tilde{l}_h = l_h - \sum_{j\in\mathcal{S}_h}\left\{\max_{\chi\in\Delta_j^{\chi}} H_h^{[j]}\begin{bmatrix} C_j & 0\end{bmatrix}\chi + \max_{z\in\mathcal{Z}_j} H_h^{[j]z} z\right\} \tag{7.26}$$

and, for all $t \geq 0$

$$k_{h,t+N-1} = \sum_{j\in\mathcal{S}_h} H_h^{[j]}\tilde{y}^{[j]}_{t+N-1} \tag{7.27}$$

Then, it is possible to show (see the Appendix) that

$$H_h^{[i]}\tilde{y}^{[i]}_{t+N} \leq \tilde{l}_h - \sum_{j\in\mathcal{S}_h\setminus\{i\}} H_h^{[j]}\tilde{y}^{[j]}_{t+N-1} - \frac{(n_h-1)}{n_h}(\tilde{l}_h - k_{h,t+N-1}) \tag{7.28}$$

guarantees that (7.3) holds and recursive feasibility. Therefore, (7.28) can be used in place of (7.3) in the following optimization problem associated with the reference output trajectory layer.

$$\min_{\bar{y}^{[i]}_{t+N}} \; V_i^y(\bar{y}^{[i]}_{t+N}, t) \tag{7.29}$$

subject to

$$\bar{y}^{[i]}_{t+N} - \tilde{y}^{[i]}_{t+N-1} \in \mathcal{B}^{(p_i)}_{q,\varepsilon}(0) \tag{7.30}$$

$$\bar{y}^{[i]}_{t+N} \in \mathbb{Y}_i \tag{7.31}$$

and (7.28), where

$$V^y_i(\bar{y}^{[i]}_{t+N}) = \gamma\|\bar{y}^{[i]}_{t+N} - \tilde{y}^{[i]}_{t+N-1}\|^2 + \|\bar{y}^{[i]}_{t+N} - y^{[i]}_{set-point}\|^2_{T_i}$$

The weight T_i must verify the inequality

$$T_i > \gamma I_{p_i} \tag{7.32}$$

while γ is an arbitrarily small positive constant.
2) Computation of the control variables. The i-DPC problem solved by the i-th robust MPC layer unit is defined as follows:

$$\min_{\hat{x}^{[i]}_t, \hat{u}^{[i]}_{[t:t+N-1]}} V^N_i(\hat{x}^{[i]}_t, \hat{u}^{[i]}_{[t:t+N-1]}) \tag{7.33}$$

where

$$V^N_i(\hat{x}^{[i]}_t, \hat{u}^{[i]}_{[t:t+N-1]}) = \sum_{k=t}^{t+N-1} \|\hat{x}^{[i]}_k - \tilde{x}^{[i]}_k\|^2_{Q_i} + \|\hat{u}^{[i]}_k - \tilde{u}^{[i]}_k\|^2_{R_i} + \|\hat{x}^{[i]}_{t+N} - \tilde{x}^{[i]}_{t+N}\|^2_{P_i} \tag{7.34}$$

subject to (7.15) and, for $k = t, \dots, t+N-1$,

$$x^{[i]}_t - \hat{x}^{[i]}_t \in \mathcal{E}_i \tag{7.35a}$$

$$\hat{z}^{[i]}_k - \tilde{z}^{[i]}_k \in \Delta^z_i \tag{7.35b}$$

$$\hat{x}^{[i]}_k \in \hat{\mathbb{X}}_i \tag{7.35c}$$

$$\hat{u}^{[i]}_k \in \hat{\mathbb{U}}_i \tag{7.35d}$$

and to the terminal constraint

$$\hat{x}^{[i]}_{t+N} - \tilde{x}^{[i]}_{t+N} \in \Sigma_i \tag{7.36}$$

The weights Q_i and R_i in (7.34) must be taken as positive definite matrices while, in order to prove the convergence properties of the proposed approach, select the matrices P_i as the solutions of the (fully independent) Lyapunov equations

$$F^T_{ii} P_i F_{ii} - P_i = -(Q_i + K^T_i R_i K_i) \tag{7.37}$$

At time t, the $(\hat{x}^{[i]}_{t|t}, \hat{u}^{[i]}_{[t:t+N-1]|t}, \bar{y}^{[i]}_{t+N|t})$ is the solution to the i-DPC problem and $\hat{u}^{[i]}_{t|t}$ is the input to the nominal system (7.15).

Remark 5. *Note that the problems* (7.29) *and* (7.33) *are independent of each other. In fact,* (7.29) *does not depend on* $\hat{x}^{[i]}_t$ *and* $\hat{u}^{[i]}_{[t:t+N-1]}$. *Moreover, both the cost function* V_i^N *and the constraints* (7.35) *are independent of* $\bar{y}^{[i]}_{t+N|t}$. □

According to (7.17), the input to the system (7.1a) is

$$u^{[i]}_t = \hat{u}^{[i]}_{t|t} + K_i(x^{[i]}_t - \hat{x}^{[i]}_{t|t}) \tag{7.38}$$

Moreover, set $\tilde{y}^{[i]}_{t+N} = \bar{y}^{[i]}_{t+N|t}$ and compute the references $\tilde{e}^{[i]}_{t+N}$ and $\tilde{x}^{[i]}_{t+N+1}$ with (7.7b) and (7.7a), respectively. Finally set $\tilde{u}^{[i]}_{t+N} = K^x_i \tilde{x}^{[i]}_{t+N} + K^e_i \tilde{e}^{[i]}_{t+N}$ from (7.7f).
Denoting by $\hat{x}^{[i]}_{k|t}$ the state trajectory of system (7.15) stemming from $\hat{x}^{[i]}_{t|t}$ and $\hat{u}^{[i]}_{[t:t+N-1]|t}$, at time t it is also possible to compute $\hat{x}^{[i]}_{t+N|t}$. The properties of the proposed distributed MPC algorithm for tracking can now be summarized in the following result.

Theorem 4. *Let Assumptions 6-9 be verified and the tuning parameters be selected as previously described. If at time* $t = 0$ *a feasible solution to the constrained problems* (7.29), (7.33) *exists and* $k_{h,N-1} \leq \tilde{l}_h$ *(see* (7.26), (7.27)*) for all* $h = 1, \dots, n_c$ *then, for all* $i = 1, \dots, M$

I) Feasible solutions to (7.29), (7.33) *exist for all* $t \geq 0$, *i.e., constraints* (7.28), (7.30), (7.31) *and* (7.35), (7.36), *respectively, are verified. Furthermore, the constraints* $(x^{[i]}_t, u^{[i]}_t) \in \mathbb{X}_i \times \mathbb{U}_i$ *and for all* $i = 1, \dots, M$, *and* (7.3) *for all* $h = 1, \dots, n_h$, *are fulfilled for all* $t \geq 0$.

II) If coupling constraints (7.3) *are absent, then the resulting MPC controller asymptotically steers the* i*-th system to the admissible set-point* $y^{[i]}_{feas.set-point}$, *where* $y^{[i]}_{feas.set-point}$ *is the solution to*

$$y^{[i]}_{feas.set-point} = \underset{y^{[i]} \in \mathbb{Y}_i}{\text{argmin}} \, \|y^{[i]} - y^{[i]}_{set-point}\|^2_{T_i} \tag{7.39}$$

When coupling static constraints are present, the convergence to the nearest feasible solution to the prescribed set-point may be prevented for some initial conditions. These situations are denoted *deadlock solutions* in [149]. Future work will be specifically devoted to this issue.

7.4 Proof of the main theorem

Proof of recursive feasibility of problem (7.28)-(7.31)

Assume that, at step t, a solution $\bar{y}^{[i]}_{t+N|t}$ to (7.29) exists for all $i = 1, \dots, M$ and that $k_{h,t+N-1} \leq \tilde{l}_h$ for all $h = 1 \dots, n_c$. First note that, since $k_{h,t+N-1} \leq \tilde{l}_h$, $\sum_{j\in\mathcal{S}_h} H^{[j]}_h \tilde{y}^{[j]}_{t+N-1} \leq \tilde{l}_h$. Recall that, in view of the relationship between the matrices $H^{[i]}_h$, $H^{[i]z}_h$, and C_i used in (7.23), for all $i = 1, \dots, M$

$$H^{[i]}_h y^{[i]}_{t+N-1} = H^{[i]z}_h z^{[i]}_{t+N-1} \in H^{[i]z}_h \tilde{z}^{[i]}_{t+N-1} \oplus H^{[i]z}_h \mathcal{Z}_i$$

Since, in view of (7.4), $H^{[i]z}_h \tilde{z}^{[i]}_{t+N-1} = H^{[i]}_h C_i \tilde{x}^{[i]}_{t+N-1}$, we obtain

$$H^{[i]}_h y^{[i]}_{t+N-1} \in H^{[i]}_h C_i \tilde{x}^{[i]}_{t+N-1} \oplus H^{[i]z}_h \mathcal{Z}_i$$

Recall that $C_i \tilde{x}^{[i]}_{t+N-1} = \begin{bmatrix} C_i & 0 \end{bmatrix} \chi^{[i]}_{t+N-1}$ and that, from (7.13), we have $\begin{bmatrix} C_i & 0 \end{bmatrix} \chi^{[i]}_{t+N-1} \in \begin{bmatrix} C_i & 0 \end{bmatrix} (\chi^{[i]ss}_{t+N-1} \oplus \Delta^{\chi}_i)$. Since $\begin{bmatrix} C_i & 0 \end{bmatrix} \chi^{[i]ss}_{t+N-1} = \tilde{y}^{[i]}_{t+N-1}$, it follows that

$$H^{[i]}_h y^{[i]}_{t+N-1} \quad \in \quad H^{[i]}_h \tilde{y}^{[i]}_{t+N-1} \oplus H^{[i]}_h \begin{bmatrix} C_i & 0 \end{bmatrix} \Delta^{\chi}_i \oplus H^{[i]z}_h \mathcal{Z}_i$$

From the definition of $\tilde{l}_h$, it is easy to see that the tightened constraint $k_{h,t+N-1} \leq \tilde{l}_h$ implies (7.3). Furthermore, the fulfillment of (7.28) at time t implies that

$$\begin{aligned} k_{h,t+N} = \sum_{i\in\mathcal{S}_h} H^{[i]}_h \tilde{y}^{[i]}_{t+N} &\leq -(n_h-1)k_{h,t+N-1} + n_h \tilde{l}_h \\ &- (n_h - 1)(\tilde{l}_h - k_{h,t+N-1}) \leq \tilde{l}_h \end{aligned} \tag{7.40}$$

which, in turn, implies that (7.3) will be verified also at time $t + N$. Finally we prove that a solution to (7.29) exists at step $t + 1$ for all $i = 1, \dots, M$. In fact, taking $\bar{y}^{[i]}_{t+N+1} = \bar{y}^{[i]}_{t+N|t} = \tilde{y}^{[i]}_{t+N}$ one has $\bar{y}^{[i]}_{t+N|t} - \tilde{y}^{[i]}_{t+N} = 0 \in \mathcal{B}^{(p_i)}_{q,\varepsilon(0)}$ and $\bar{y}^{[i]}_{t+N|t} \in \mathbb{Y}_i$, hence verifying (7.30) and (7.31), respectively. Furthermore

$$\begin{aligned} H^{[i]}_h \tilde{y}^{[i]}_{t+N+1} &= H^{[i]}_h \tilde{y}^{[i]}_{t+N} \\ &\leq \tilde{l}_h - \textstyle\sum_{j\in\mathcal{S}_h\setminus\{i\}} H^{[j]}_h \tilde{y}^{[j]}_{t+N} - \frac{(n_h-1)}{n_h}(\tilde{l}_h - k_{h,t+N}) \end{aligned}$$

In fact $k_{h,t+N} \leq \tilde{l}_h - \frac{(n_h-1)}{n_h}(\tilde{l}_h - k_{h,t+N})$ in view of the fact that $k_{h,t+N} \leq \tilde{l}_h$, as it is proved in (7.40).

Proof of convergence for the reference management layer

In absence of coupling constraints (7.3), since at time $t+1$, $\bar{y}^{[i]}_{t+N+1} = \tilde{y}^{[i]}_{t+N}$ is a feasible solution, in view of the optimality of the solution $\bar{y}^{[i]}_{t+N+1|t+1}$

$$V_i^y(\bar{y}^{[i]}_{t+N+1|t+1}, t+1) \leq V_i^y(\bar{y}^{[i]}_{t+N|t}, t+1) \leq \|\bar{y}^{[i]}_{t+N|t} - y^{[i]}_{set-point}\|^2_{T_i} \tag{7.41}$$

In view of the fact that $\bar{y}^{[i]}_{t+N+1|t+1} = \tilde{y}^{[i]}_{t+N+1}$ for all t, we can write the term $V_i^y(\bar{y}^{[i]}_{t+N+1|t+1}, t+1) = \gamma\|\tilde{y}^{[i]}_{t+N+1} - \tilde{y}^{[i]}_{t+N}\|^2 + \|\tilde{y}^{[i]}_{t+N+1} - y^{[i]}_{set-point}\|^2_{T_i}$, and rewrite (7.41) as $\|\tilde{y}^{[i]}_{t+N+1} - y^{[i]}_{set-point}\|^2_{T_i} \leq \|\tilde{y}^{[i]}_{t+N} - y^{[i]}_{set-point}\|^2_{T_i} - \gamma\|\tilde{y}^{[i]}_{t+N+1} - \tilde{y}^{[i]}_{t+N}\|^2$. From this we infer that, as $t \to \infty$, $\tilde{y}^{[i]}_{t+N+1} - \tilde{y}^{[i]}_{t+N} \to 0$ and

$$\|\tilde{y}^{[i]}_{t+N} - y^{[i]}_{set-point}\|^2_{T_i} \to \text{const} \tag{7.42}$$

Assume, by contradiction, that $\|\tilde{y}^{[i]}_{t+N} - y^{[i]}_{set-point}\|^2_{T_i} \to \bar{c}_i$, with $\bar{c}_i > c_i^o$, where

$$c_i^o = \|y^{[i]}_{feas.set-point} - y^{[i]}_{set-point}\|^2_{T_i} \tag{7.43}$$

Note that this implies that $\tilde{y}^{[i]}_{t+N} \neq y^{[i]}_{feas.set-point}$.
Assume that, given $\bar{t}$, for all $t \geq \bar{t}$ the optimal solution to (7.29) is $\bar{y}^{[i]}_{t+N|t} = \bar{y}^{[i]}$ where $\|\bar{y}^{[i]} - y^{[i]}_{set-point}\|^2_{T_i} = \bar{c}_i$. It results that $V_i^y(\bar{y}^{[i]}_{t+N|t}, t) = \bar{c}_i$. On the other hand, an alternative solution is given by $\bar{\bar{y}}^{[i]}_{t+N}$, where $\bar{\bar{y}}^{[i]}_{t+N} = \lambda_i \bar{y}^{[i]} + (1-\lambda_i) y^{[i]}_{feas.set-point}$, with $\lambda_i \in [0, 1)$. This solution is feasible provided that (I) $\bar{\bar{y}}^{[i]}_{t+N} - \bar{y}^{[i]} \in \mathcal{B}^{p_i}_{q,\varepsilon}(0)$ which can be verified if $(1-\lambda_i)$ is sufficiently small, (II) $\bar{\bar{y}}^{[i]}_{t+N} \in \mathbb{Y}_i$ which is also satisfied if $(1-\lambda_i)$ is sufficiently small (since $\mathbb{Y}_i$ is convex and $\bar{y}^{[i]} \neq y^{[i]}_{feas.set-point}$).
According to this alternative solution

$$V_i^y(\bar{\bar{y}}^{[i]}_{t+N}, t) = \gamma\|\bar{\bar{y}}^{[i]}_{t+N} - \bar{y}^{[i]}\|^2 + \|\bar{\bar{y}}^{[i]}_{t+N} - y^{[i]}_{set-point}\|^2_{T_i}$$

Now if (7.32) is verified, then $V_i^y(\bar{\bar{y}}^{[i]}_{t+N}, t) < V_i^y(\bar{y}^{[i]}, t)$. This contradicts the assumption that $\|\tilde{y}^{[i]}_{t+N} - y^{[i]}_{set-point}\|^2_{T_i} \to \bar{c}_i$, with $\bar{c}_i > c_i^o$. Therefore, the only asymptotic solution compatible with (7.29), is that corresponding with $\tilde{y}^{[i]}_{t+N} = y^{[i]}_{feas.set-point}$.

It is now proved that $\tilde{y}_t^{[i]} \to y_{feas.set-point}^{[i]}$ for $t \to \infty$. In view of Assumption 7, this implies that $\mathcal{C}_i \chi_t^{[i]} = C_i \tilde{x}_t^{[i]} \to y_{feas.set-point}^{[i]}$ for all $i = 1, \dots, M$.

Proof of recursive feasibility of the i-DPC problem

Assume that, at step t, a solution to (7.33) exists for all $i = 1, \dots, M$, i.e., $(\hat{x}_{t|t}^{[i]}, \hat{u}_{[t:t+N-1]|t}^{[i]})$. Next we prove that, at step $t+1$, a solution to (7.33) exists for all $i = 1, \dots, M$. To do so, we prove that the tuple $(\hat{x}_{t+1|t}^{[i]}, \hat{u}_{[t+1:t+N]|t}^{[i]})$ satisfies the constraints (7.15), (7.35a)-(7.36) and is therefore a feasible (possibly suboptimal) solution to (7.33). Here $\hat{u}_{[t+1:t+N]|t}^{[i]}$ is obtained with

$$\hat{u}_{t+N|t}^{[i]} = \tilde{u}_{t+N}^{[i]} + K_i(\hat{x}_{t+N|t}^{[i]} - \tilde{x}_{t+N}^{[i]}) \tag{7.44}$$

First, note that, in view of the robust positive invariance of sets $\mathcal{E}_i$ with respect to equation (7.18), $i = 1, \dots, M$, $x_{t+1}^{[i]} - \hat{x}_{t+1|t}^{[i]} \in \mathcal{E}_i$, and therefore (7.35a) is verified. Furthermore, in view of the feasibility of (7.35b)-(7.35d) at step t, it follows that (7.35b)-(7.35d) are satisfied at step $t+1$ for $k = t+1, \dots, t+N-1$ and, from (7.36) and (7.25a),

$$\begin{aligned} \hat{z}_{t+N|t}^{[i]} - \tilde{z}_{t+N}^{[i]} &= (C_{zi} + D_{zi}K_i)(\hat{x}_{t+N|t}^{[i]} - \tilde{x}_{t+N}^{[i]}) \\ &\in (C_{zi} + D_{zi}K_i)\Sigma_i \subseteq \Delta_i^z \end{aligned}$$

Hence constraint (7.35b) is verified for $k = t+N$. Furthermore

$$\begin{bmatrix} \hat{x}_{t+N|t}^{[i]} \\ \hat{u}_{t+N|t}^{[i]} \end{bmatrix} \in \begin{bmatrix} \tilde{x}_{t+N}^{[i]} \\ \tilde{u}_{t+N}^{[i]} \end{bmatrix} \oplus \begin{bmatrix} I_{n_i} \\ K_i \end{bmatrix} \Sigma_i$$

where, from (7.13)

$$\begin{bmatrix} \tilde{x}_{t+N}^{[i]} \\ \tilde{u}_{t+N}^{[i]} \end{bmatrix} \in \begin{bmatrix} I_{n_i} & 0 \\ K_i^x & K_i^e \end{bmatrix} \left(\chi_{t+N}^{[i]ss} \oplus \Delta_i^\chi \right) \tag{7.45}$$

In turn, in view of (7.9) and similarly to (7.10),

$$\chi_{t+N}^{[i]ss} \in \Gamma_i (I_{n+p} - \mathcal{F})^{-1} \mathcal{G} \prod_{i=1}^{M} \mathbb{Y}_j \tag{7.46}$$

This eventually implies that, in view of (7.25b)

$$\begin{bmatrix}\hat{x}^{[i]}_{t+N|t}\\ \hat{u}^{[i]}_{t+N|t}\end{bmatrix} \in \begin{bmatrix} I_{n_i} & 0\\ K^x_i & K^e_i\end{bmatrix}\left(\Gamma_i(I_{n+p}-\mathcal{F})^{-1}\mathcal{G}\prod_{i=1}^M \mathbb{Y}_j \oplus \Delta^\chi_i\right)\oplus \\ \oplus \begin{bmatrix} I_{n_i}\\ K_i\end{bmatrix}\Sigma_i \subseteq \hat{\mathbb{X}}_i \times \hat{\mathbb{U}}_i$$

which verifies constraints (7.35c) and (7.35d) for $k = t + N$. Note that, in view of (7.7a), (7.15), and (7.44)

$$\hat{x}^{[i]}_{t+N+1|t} - \tilde{x}^{[i]}_{t+N+1} = F_{ii}(\hat{x}^{[i]}_{t+N|t} - \tilde{x}^{[i]}_{t+N}) \tag{7.47}$$

and therefore $\hat{x}^{[i]}_{t+N+1|t} - \tilde{x}^{[i]}_{t+N+1} \in \Sigma_i$ in view of the definition of Σ_i as a positively invariant set for (7.24), hence verifying (7.36). Therefore also the constraint (7.36) is verified at step $t + 1$.

Proof of convergence for the robust MPC layer

At time t the pair $(\hat{x}^{[i]}_{t|t}, \hat{u}^{[i]}_{[t:t+N-1]|t})$ is a solution to (7.33), leading to the optimal cost $V_i^{*N}(t)$. Since $(\hat{x}^{[i]}_{t+1|t}, \hat{u}^{[i]}_{[t+1:t+N]|t})$ is a feasible solution to (7.33) at time $t+1$, by optimality $V_i^{*N}(t+1) \leq V_i^N(\hat{x}^{[i]}_{t+1|t}, \hat{u}^{[i]}_{[t+1:t+N]|t})$ and, by applying standard arguments in MPC [138] $V_i^{*N}(t+1) \leq V_i^{*N}(t) - (\|\hat{x}^{[i]}_{t|t} - \tilde{x}^{[i]}_t\|^2_{Q_i} + \|\hat{u}^{[i]}_{t|t} - \tilde{u}^{[i]}_t\|^2_{R_i})$, so that, for all $i = 1, \ldots, M$, $\hat{x}^{[i]}_{t|t} \to \tilde{x}^{[i]}_t$ and $\hat{u}^{[i]}_{t|t} \to \tilde{u}^{[i]}_t$ as $t \to \infty$. Now, consider the model (7.5a) and, collectively, the model (7.7a) and equation (7.17). We have that

$$\begin{array}{rcl} \mathbf{x}_{t+1} & = & \mathbf{A}\,\mathbf{x}_t + \mathbf{B}\left(\hat{\mathbf{u}}_{t|t} + \mathbf{K}(\mathbf{x}_t - \hat{\mathbf{x}}_{t|t})\right)\\ \tilde{\mathbf{x}}_{t+1} & = & \mathbf{A}\,\tilde{\mathbf{x}}_t + \mathbf{B}\tilde{\mathbf{u}}_t + \mathbf{B}\mathbf{K}(\tilde{\mathbf{x}}_t - \tilde{\mathbf{x}}_t)\end{array} \tag{7.48}$$

for all $t \geq 0$. Denote $\Delta\mathbf{x}_t = \mathbf{x}_t - \tilde{\mathbf{x}}_t$, $\Delta\hat{\mathbf{x}}_t = \hat{\mathbf{x}}_{t|t} - \tilde{\mathbf{x}}_t$ and $\Delta\hat{\mathbf{u}}_t = \hat{\mathbf{u}}_{t|t} - \tilde{\mathbf{u}}_t$. From (7.48), $\Delta\mathbf{x}_{t+1} = \mathbf{F}\Delta\mathbf{x}_t + \mathbf{B}\left(\Delta\hat{\mathbf{u}}_t - \mathbf{K}\Delta\hat{\mathbf{x}}_t\right)$. Since $\mathbf{B}\left(\Delta\hat{\mathbf{u}}_t - \mathbf{K}\Delta\hat{\mathbf{x}}_t\right) \to 0$ as $t \to \infty$, in view of Assumption 6 it holds that $\Delta\mathbf{x}_t \to 0$ as $t \to \infty$, which implies that asymptotically $C_i x^{[i]}_t \to C_i\tilde{x}^{[i]}_t$.

7.5 Control of unicycle robots

In this section the proposed algorithm is applied to the problem of positioning a number of mobile robots in specified positions, while guaranteeing collision avoidance. In principle, two main limitations have been

detected for the application of the discussed control technique to the considered problem. First, as it is widely known, the model of unicycle robots is nonlinear and nonholonomic constraints affect its dynamics, so that it is not possible to control these vehicles using linear models obtained through linearization. However, by resorting to a feedback linearization procedure, previously discussed in [121], a linear model of the robots will be used to describe the system's dynamics.
Furthermore, collision avoidance constraints (on the surface) should be in principle non-convex and described using nonlinear inequalities. This would prevent the application of our algorithm, since the allowed coupling constraints correspond to linear inequalities of the form (7.3). To circumvent this problem, suitable linear constraints are defined to replace non-convex ones. Experimental results will be shown to witness the viability of our approach.

7.5.1 The model of unicycle robots

The continuous-time dynamics of a single robot is described by a modified version of the first-order kinematic model [121]

$$\dot{x} = v\cos\phi \tag{7.49a}$$
$$\dot{y} = v\sin\phi \tag{7.49b}$$
$$\dot{\phi} = \omega \tag{7.49c}$$
$$\dot{v} = a \tag{7.49d}$$

where the pair (x, y) represent the cartesian position of the robot on its working area, ϕ denotes its orientation angle, and v is its linear velocity. The linear acceleration a and the angular velocity ω are considered as control inputs to be selected through the proposed algorithm.

By resorting to a feedback linearization procedure (see [121]) a linear model of the robots can be derived to describe the system's dynamics. Namely, define $\eta_1 = x$, $\eta_2 = \dot{x}$, $\eta_3 = y$, $\eta_4 = \dot{y}$, and the dynamics resulting from (7.49) is

$$\dot{\eta}_1 = \eta_2 \tag{7.50a}$$
$$\dot{\eta}_2 = a\cos\phi - v\omega\sin\phi \tag{7.50b}$$
$$\dot{\eta}_3 = \eta_4 \tag{7.50c}$$
$$\dot{\eta}_4 = a\sin\phi + v\omega\cos\phi \tag{7.50d}$$

Now define two new "fictitious" input variables $a_x = a\cos\phi - v\omega\sin\phi$ and $a_y = a\sin\phi + v\omega\cos\phi$ that represent the acceleration along the two cartesian axes. In this way, from (7.50) the model (7.49) is transformed in a set of two decoupled double integrators with inputs a_x and a_y.
To recover the real inputs (ω, a) from (a_x, a_y) it is sufficient to compute

$$\begin{bmatrix}\omega\\ a\end{bmatrix} = \frac{1}{v}\begin{bmatrix}\sin\phi & -\cos\phi\\ v\cos\phi & -v\sin\phi\end{bmatrix}\begin{bmatrix}a_x\\ a_y\end{bmatrix} \tag{7.51}$$

Note that, for obtaining (7.51), it is assumed that $v \neq 0$. This singularity point must be accounted for when designing control laws on the equivalent linear model [121].

The linear model from (7.50) is then discretized with sampling time τ, obtaining the following matrices for the dynamics

$$A_{ii} = A = \begin{bmatrix}1 & \tau & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & \tau\\ 0 & 0 & 0 & 1\end{bmatrix}, \quad B_{ii} = B = \begin{bmatrix}\frac{\tau^2}{2} & 0\\ \tau & 0\\ 0 & \frac{\tau^2}{2}\\ 0 & \tau\end{bmatrix} \tag{7.52}$$

The output variables are the cartesian coordinates, x and y, i.e., η_1 and η_3 in (7.50) and thus

$$C_i = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}$$

Note that this case study is characterized by (i) no dynamically coupling terms, i.e., $E_i = 0$ and $L_{ij} = 0$ for all $i, j = 1, \ldots, M$; (ii) static coupling constraints on the position variables guaranteeing collision avoidance. Therefore, set $C_{zi} = C_i$ and $D_{zi} = 0$.

7.5.2 The experimental setup

The experimental set-up consists of three e-puck mobile robots [106]. To simplify the application of the algorithm, the control law is designed on a portable computer communicating with the e-puck robots through wireless connection. The measurement system consists of a camera, installed on the top of the 130×80 cm^2 working area. Position and orientation of each robot are detected using two colored circles, placed on the top of each agent, see Figure 7.3. The linear velocity is reconstructed from measured data using

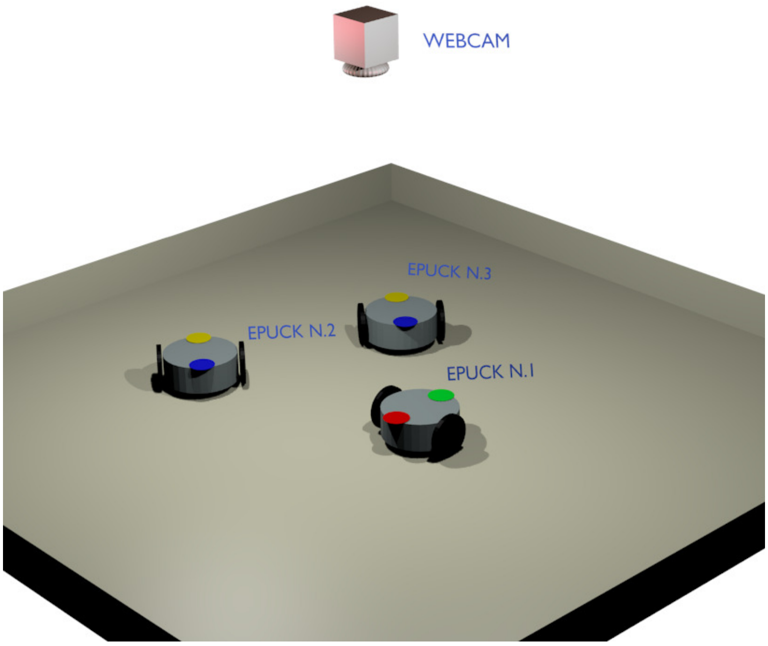


Figure 7.3: *Sketch of the experimental set-up.*

a filtering procedure.

Note that the fact that no coupling terms are present (i.e., $E_i = 0$ and $\mathcal{F}_{ij} = 0$ for all i, j) greatly simplifies the design phase. More specifically, with reference to the off-line design steps outlined in Section 7.3

1) Assumption 6 and Assumption 8 correspond to solve centralized small-scale (e.g., eigenvalue assignment) problems.

2) Similarly to the previous step, Δ_i^{χ} can be computed as the RPI set for

$$\chi_{t+1}^{[i]} - \chi_{t+1}^{[i]ss} = \mathcal{F}_{ii}(\chi_t^{[i]} - \chi_t^{[i]ss}) + \tilde{w}_t^{[i]}$$

with $\tilde{w}_t^{[i]} = -(I_6 - \mathcal{F}_i)^{-1}\mathcal{F}_i\mathcal{G}_i\mathcal{B}_{q,\varepsilon}^{(2)}$ and $\varepsilon = 5$ cm.

3) Since $E_i = 0$, $w_t^{[i]} = 0$ in (7.18), $\mathcal{E}_i$ is an arbitrarily small positively invariant set, and Δ_i^z can be chosen arbitrarily.

4) Since, in (7.25b), we have that

$$\Gamma_i(I_{n+p} - \mathcal{F})^{-1}\mathcal{G}\prod_{i=1}^{M}\mathbb{Y}_j = (I_6 - \mathcal{F}_i)^{-1}\mathcal{G}_i\mathbb{Y}_i$$

this step can be verified in a decentralized fashion.

In this experiment, only the collision avoidance with respect to other agents is considered. However, note that the fixed obstacle case, generically denoted with the term obstacle avoidance, can be easily derived with similar considerations. To handle collisions in the problem each agent is considered as a circular object characterized by the time-varying position of its center and a fixed radius R_i. In general, collision avoidance constraints are non-convex and described using nonlinear inequalities and thus particular attention is needed to handle them inside the proposed framework.
Assume for example a 2D circular obstacle centered in $\bar{z}$ with radius $\bar{d}$, the non collision requirements corresponds to the satisfaction of a minimum distance constraint, and thus the feasibility region $\mathcal{O}^C$ is

$$\mathcal{O}^C = \{z \mid \|z - \bar{z}\|_2 > \bar{d}\} \tag{7.53}$$

that is clearly non-convex. To circumvent this problem, suitable linear constraints are defined to replace non-convex ones. In particular, each of them is obtained by tracing a line stemming from the center of the i-th robot and tangent to a circumference whose center corresponds to the position of the neighboring robot j, and whose radius is $\bar{d} = R_i + R_j$. An example of the approximation is shown in Figure 7.4. Between the two computed constraints ($a_1^T z \leq b_1$ and $a_2^T z \leq b_2$ in this simple example) only one must be selected and in particular we choose the one that allows the agent to stay closer to its final goal. Of course this procedure introduces some conservatism, since the new feasibility region is much smaller than the original one, however it allows to efficiently solve the optimization problem. A more efficient solution, inspired from the work in [2], will be presented in the next chapter for the stochastic framework.

For all $i = 1, 2, 3$, in the cost functions V_i^y and V_i^N we set $\gamma = 1$, $T_i = 4I_2$ and $Q_i = I_4$, $R_i = 0.01I_2$, respectively.
In the reported real experiment the three robots are initially placed (at time $t = 1$) at positions $(28, 52)$, $(39, 16)$, and $(90, 39)$ - all coordinates are in cm. Figure 7.5 shows the evolution of their motion in reaching the goal positions - i.e., $(86, 13)$, $(77, 55)$, and $(20, 39)$ - at time $t = 45$ s while fulfilling collision avoidance constraints.

7.6 Control of dynamically coupled subsystems

Consider now the four-tank system (see Figure 7.6) described in [71] and used as a benchmark to test several control algorithms, both centralized and

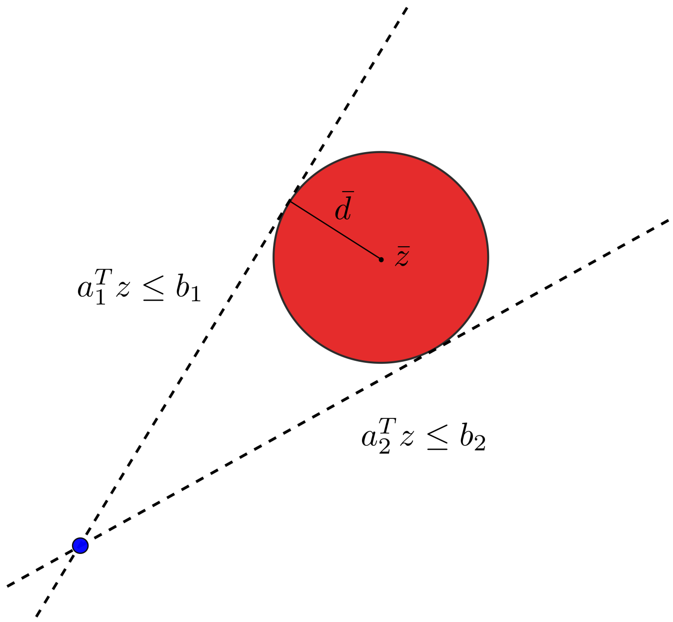


Figure 7.4: *Linear approximation of the avoidance constraint. The obstacle (red ball) is approximated choosing one of the two tangent lines stemming from the current position of the i-th agent (blue point).*

distributed, e.g., in [1,35,51,103]. The goal is to control the water levels h_1 and h_3 of tanks 1 and 3 using the pump command voltages v_1 and v_2. The variables $x^{[i]}$, $i = 1, ..., 4$ and $u^{[j]}$, $j = 1, 2$ are the variations of the levels h_i and of the voltages v_j with respect to the corresponding nominal working points. The obtained linearized and discretized system with sampling time $\tau = 0.5$ s has the form (7.5) with $n = 4$, $p = m = 2$ where

$$\mathbf{A} = \begin{bmatrix} 0.9921 & 0 & 0 & 0.0206 \\ 0 & 0.9835 & 0 & 0 \\ 0 & 0.0165 & 0.9945 & 0 \\ 0 & 0 & 0 & 0.9793 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 0.0417 & 2.47 \cdot 10^{-4} \\ 0.0156 & 0 \\ 1.30 \cdot 10^{-4} & 0.0311 \\ 0 & 0.0235 \end{bmatrix}, \ \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The system is decomposed into two subsystems: the first one is composed by Tank 1 and Tank 2, and the second one by Tank 3 and Tank 4. We set $x^{[1]} = (x_1, x_2)$, $u^{[1]} = u_1$, $y^{[1]} = y_1$, $x^{[2]} = (x_3, x_4)$, $u^{[2]} = u_2$, and $y^{[2]} =$

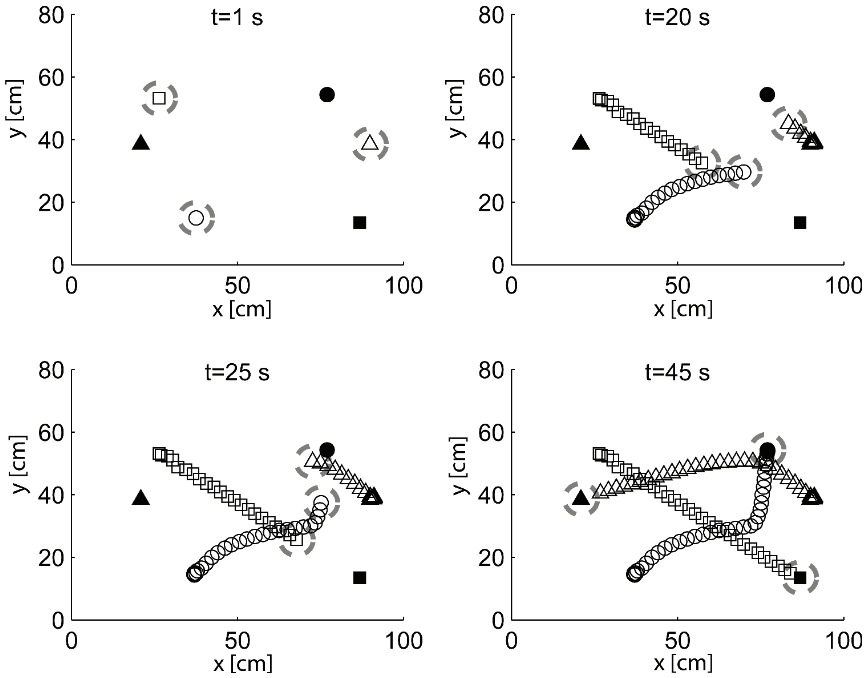


Figure 7.5: *Plots of the robot trajectories. Robot 1: □; robot 2: ○; robot 3: △. Symbols with white surface denote the position of the robots, while symbols with black surface denote the goal positions. Large circles with grey dashed line denote the area occupied by the robots.*

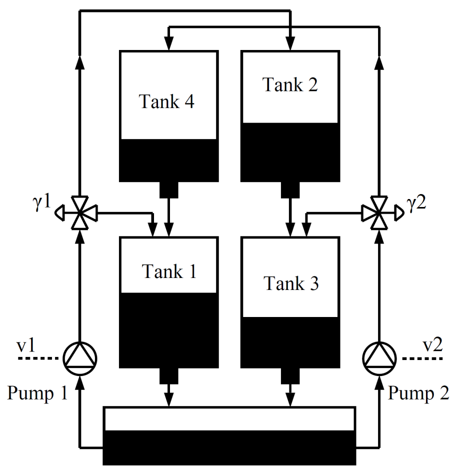


Figure 7.6: *Schematic representation of the four-tanks system*

y_2. The constraints on the system's variables are $x^{[1]}_{min} = \begin{bmatrix}-12.4 & -1.4\end{bmatrix}^T$, $x^{[1]}_{max} = \begin{bmatrix}27.6 & 38.6\end{bmatrix}^T$, $x^{[2]}_{min} = \begin{bmatrix}-12.7 & -1.8\end{bmatrix}^T$, $x^{[2]}_{max} = \begin{bmatrix}27.3 & 38.2\end{bmatrix}^T$, $u^{[1]}_{min} = -3$, $u^{[1]}_{max} = 3$, $u^{[2]}_{min} = -3$, $u^{[2]}_{max} = 3$.
The matrices K_i and $\mathcal{K}_i$ fulfilling Assumption 6 and Assumption 7 have been computed as described in Remark 4.
The weights used are $Q_1 = Q_2 = I_2$, $R_1 = R_2 = 1$, $T_1 = T_2 = 1$, $\gamma = 10^{-6}$, $N = 3$. In the simulations, the reference trajectories $y^{[i]}_{set-point}$, $i = 1, 2$ are piece-wise constant, see Figure 7.7. The results achieved are depicted in Figure 7.7. Notably, the set-point $y^{[1]}_{set-point} = 2.5$ results infeasible to our algorithm, and hence the system output $y^{[1]}_t$ converges to the nearest feasible value.

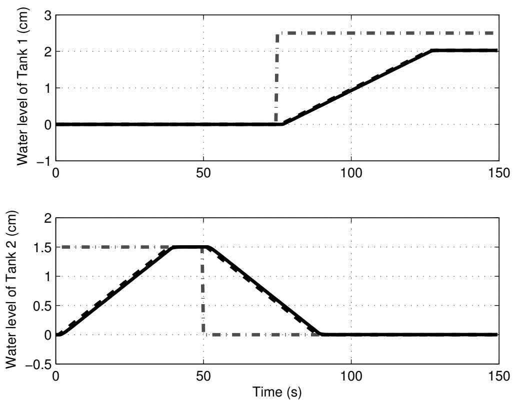


Figure 7.7: *Trajectories of the output variables $y^{[1]}$ (above) and $y^{[2]}$ (below) (black solid lines) and reference outputs $\tilde{y}^{[1]}$ (above) and $\tilde{y}^{[2]}$ (below) (black dashed lines). Grey dash-dotted lines: desired set-points $y^{[1,2]}_{set-point}$.*

7.7 Comments

In this chapter a novel distributed scheme for tracking reference signals has been proposed. As in all existing noncooperative schemes, a degree of conservativity is brought about when addressing couplings among subsystems and for obtaining simple problems at the reference generator level. On the other hand, the main advantages are: scalability of the online implementation, limited transmission and computational load (also in view of the fact that the reference generator layer is independent from the robust MPC layer, and hence computations can be performed in a parallelized fashion), and simplicity of implementation. The algorithm has also proven to be very flexible, i.e., in the unicycle robot application, where it has already been successfully used for the solution of leader-following and/or formation control problems. In the next chapter an extension of the proposed algorithm for tracking reference signals under probabilistic constraints is considered for the simpler case of dynamically decoupled systems and with the aim to use it in the unicycle coordination problem described here.

CHAPTER 8

Stochastic Distributed Predictive Control for tracking of independent systems with coupling constraints

In the previous chapter we presented a distributed predictive control approach for tracking reference signals that is based on the use of robust/worst-case techniques to handle couplings and uncertainties and satisfy local and coupling deterministic constraints. With the aim of obtaining similar results, we consider in this chapter a distributed predictive control approach for tracking reference signals in a stochastic framework, i.e. in presence of probabilistic constraints, as discussed in Chapter 2. In particular, inspired by the work in [127] and some results presented in Chapter 6, an algorithm is presented for the case of dynamically independent systems with coupling chance constraints and then reformulated specifically in the case of the unicycle coordination problem introduced in Section 7.5. In order to show the viability of the approach in a real test case, some results from [105] are presented and discussed.

8.1 Problem statement

Similarly to the previous chapters, consider a set of M systems, denoted by $\mathcal{S}_i$, $i = 1, \dots, M$, each of which is described by the independent dynamics

$$x_{t+1}^{[i]} = A_{ii}x_t^{[i]} + B_{ii}u_t^{[i]} + B_{w,i}w_t^{[i]} \tag{8.1}$$

where, for the i-th subsystem, $x_t^{[i]} \in \mathbb{R}^{n_i}$ is the state, $u_t^{[i]} \in \mathbb{R}^{m_i}$ is the input and $w_t^{[i]} \in \mathbb{R}^{p_i}$ is a zero-mean white noise with variance W_i and possibly unbounded support. In addition, we assume that the uncertainties $w_t^{[i]}$ and $w_t^{[j]}$ are uncorrelated for all $j \neq i$. In order to derive a tracking controller, it is useful to define the output of subsystem $\mathcal{S}_i$ as

$$z_t^{[i]} = C_i x_t^{[i]} \tag{8.2}$$

where $z_t^{[i]} \in \mathbb{R}^{m_i}$ for all $i = 1, \dots, M$. In line with the probabilistic MPC algorithm presented in Chapter 3 we assume perfect state information and the stabilizability of the pairs (A_{ii}, B_{ii}) and $(A_{ii}, \tilde{B}_{w,i})$ with $\tilde{B}_{w,i}W_i\tilde{B}_{w,i}^T = \tilde{B}_{w,i}$. Note also that, due to the assumption of independent dynamics, the decentralized stabilizability required in the general distributed framework, amounts here to the stabilizability of the single subsystems. Finally, as in Assumption 7 of the previous chapter, we define the matrix

$$S_i = \begin{bmatrix} I - A_{ii} & -B_{ii} \\ C_i & 0 \end{bmatrix}$$

and we assume that $\text{rank}(S_i) = n_i + m_i$ for all $i = 1, \dots, M$.

The purpose is to develop a stochastic distributed control scheme guaranteeing that both local and collective constraints are verified while achieving tracking properties. However, being in a probabilistic framework, some remarks are in order

- Concerning the tracking property, if $\tilde{z}_G^{[i]}$ is the desired goal for the output variable $z_t^{[i]}$, we aim to guarantee that $\mathbb{E}[z_t^{[i]}] \to \tilde{z}_G^{[i]}$ as $t \to \infty$.
- As for the constraints, we must allow for a probability of constraint violation. Indeed, based on the definitions given in Chapter 2 and similarly to the solution adopted in Chapter 3, for each subsystem $\mathcal{S}_i$ the local constraints are defined as a set of individual chance constraints

$$\mathcal{P}\left\{b_{r,i}^T x_t^{[i]} \leq x_{r,i}^{max}\right\} \geq 1 - p_{r,i}^x,\ \forall r = 1, \dots, n_{r,i} \tag{8.3a}$$

$$\mathcal{P}\left\{c_{s,i}^T u_t^{[i]} \leq u_{s,i}^{max}\right\} \geq 1 - p_{s,i}^u,\ \forall s = 1, \dots, n_{s,i} \tag{8.3b}$$

In addition, we define a set of collective probabilistic constraints that involve the states of possibly all the subsystems at the same time, i.e.,

$$\mathcal{P}\left\{\sum_{i=1}^{M} d_{l,i} x_t^{[i]} \leq 1\right\} \geq 1 - p_l^c,\ \forall l = 1, \ldots, n_c \tag{8.3c}$$

In line with the notation adopted in the previous chapters, in (8.3) the terms $b_{r,i}$, $c_{s,i}$ and $d_{l,i}$ are constant vectors, while the maximal allowed probabilities of constraint violation $p_{r,i}^x$, $p_{s,i}^u$ and p_l^c are design parameters. Concerning (8.3a), the sets

$$\mathbb{X}_i = \{x \mid b_{r,i}^T x_t^{[i]} \leq x_{r,i}^{max},\ \forall r = 1, \ldots, n_{r,i}\},\ \forall\, i = 1, \ldots, M$$

and

$$\mathbb{U}_i = \{u \mid c_{s,i}^T u_t^{[i]} \leq u_{s,i}^{max},\ \forall s = 1, \ldots, n_{s,i}\},\ \forall\, i = 1, \ldots, M$$

are assumed to be bounded and containing the origin in their interior. Finally, based on the structure in (8.3c) and using the terminology introduced in Chapter 7, we say that systems $\mathcal{S}_i$ and $\mathcal{S}_j$ are *constraint neighbors* if there exists $l \in [1, \ldots, n_c]$ such that both $d_{l,i} \neq 0$ and $d_{l,j} \neq 0$. The set of neighbors of subsystem $\mathcal{S}_i$ is denoted again by $\mathcal{N}_i$.

8.1.1 Local controller structure

According to the p-SMPC approach presented in Chapter 3, the local control law for the i-th subsystem is composed by an open-loop and a feedback term and takes the form

$$u_t^{[i]} = \bar{u}_t^{[i]} + K_t^{[i]}(x_t^{[i]} - \bar{x}_t^{[i]}) \tag{8.4}$$

where $\bar{u}_t^{[i]}$ and $\bar{x}_t^{[i]}$ are the input and the state, respectively, of the nominal system $\bar{\mathcal{S}}_i$, defined as

$$\bar{x}_{t+1}^{[i]} = A_{ii}\bar{x}_t^{[i]} + B_{ii}\bar{u}_t^{[i]} \tag{8.5a}$$

$$\bar{z}_t^{[i]} = C_i \bar{x}_t^{[i]} \tag{8.5b}$$

and $\bar{z}_t^{[i]}$ represents the nominal output. Note that, also in this case, by properly initializing the nominal system and given that the uncertainty in (8.1) has zero mean, it is possible to prove that $\bar{x}_t^{[i]}$ and $\bar{u}_t^{[i]}$ represents the expected values of the state and the input, respectively. In (8.4) both the gain $K_t^{[i]}$ and the open-loop term $\bar{u}_t^{[i]}$ need to be computed as the solutions of a

suitable local MPC optimization problem.
Contrarily to the case discussed in Chapter 6, thanks to the assumption of independent dynamics, the expected values of the states of each subsystem are now accessible in a decentralized fashion and, thus, they can be directly used in our setup.

8.1.2 Reformulation of the probabilistic constraints

Besides minimizing local cost functions and satisfying local constraints (8.3a) and (8.3b) we require that collective goals are attained, i.e., that the coupling constraints (8.3c) are verified at all time instants. As in the previous chapter, in order to ensure the required global properties, we assume that information are transmitted between agents at each time instant, and in particular that subsystem $\mathcal{S}_i$ is able to communicate with its neighbors $\mathcal{S}_j$, $j \in \mathcal{N}_i$. More details on the type and the amount of information to be transmitted will be given later. As in the rest of the Thesis, the following notation will be adopted to denote the optimal predictions along the horizon. Given a generic variable z and a generic matrix Z , at any time step t we will denote by $z_{t+k|t}$ and $Z_{t+k|t}$, $k \geq 0$, the specific values of z_{t+k} and Z_{t+k} computed based on the knowledge (e.g. measurements) available at time t.

According to the discussion in Chapter 3, it is possible to consider $x_t^{[i]}$ (respectively, $u_t^{[i]}$) as a stochastic variable with mean $\bar{x}_t^{[i]}$ (respectively, $\bar{u}_t^{[i]}$) and variance $X_t^{[i]}$ (respectively, $U_t^{[i]}$) and use the second order description of the system variables to enforce probabilistic constraints. In particular, constraints (8.3) are verified if the following inequalities can be guaranteed for all $t \geq 0$

$$b_{r,i}^T \bar{x}_t^{[i]} \leq x_{r,i}^{max} - \sqrt{b_{r,i}^T X_t^{[i]} b_{r,i}}\, f(p_i^x) \qquad \forall r = 1, \ldots, n_{r_i} \tag{8.6a}$$

$$c_{s,i}^T \bar{u}_t^{[i]} \leq u_{s,i}^{max} - \sqrt{c_{s,i}^T U_t^{[i]} c_{s,i}}\, f(p_i^u) \qquad \forall s = 1, \ldots, n_{s_i} \tag{8.6b}$$

$$\sum_{i=1}^{\nu} d_{z,i}^T \bar{x}_t^{[i]} \leq 1 - \sqrt{\sum_{i=1}^{\nu} d_{z,i}^T X_t^{[i]} d_{z,i}}\, f(p^c) \qquad \forall l = 1, \ldots, n_c \tag{8.6c}$$

As described in Section 2.2.3 the function $f(p)$ depends on the particular distribution of the uncertainty and is defined as $f(p) = \sqrt{(1-p)/p}$ if the distribution is not known or $f(p) = \mathcal{N}^{-1}(1-p)$ if the uncertainty is Gaussian (and thus, due to linearity, also the state and input variables are

Gaussian).
Note that the two local constraints, (8.6a) and (8.6b), are derived exactly as described in Chapter 3, while the collective constraint (8.6c) is obtained in view of the uncorrelation of the noises and the fact that the subsystems are independent from each other by assumption.

8.1.3 Application of local and coupling constraints

As in the p-SMPC approach, local constraints (8.6a) and (8.6b) can be directly enforced in the MPC optimization problem by each local controller. On the other hand, constraint (8.6c) should be solved in a centralized way due to the fact that it includes, at the same time, the decision variables of more than one subsystem. An alternative is to handle (8.6c) in a distributed fashion, as proposed for example in [127]. Adopting an approach similar to the one described in Chapter 7, this is achieved by means of two elements. Firstly, at time t, the couplings are handled by using the optimal state trajectories (in terms of expected value and variance in this case) computed at the previous time instant, $t-1$, and received by the neighboring agents. Secondly, it is ensured that these trajectories at time t maintain similar properties with respect to the ones that have been previously transmitted by means of additional constraints to be applied both on the expected value and the variance of the state. Such additional constraints on the predictions, that need to be verified for all $k = 0, \ldots, N-1$, are obtained by considering the optimal distance from a certain constraint at the previous time instant, splitting it between the agents involved in the constraint and, finally, allowing each of them to deviate from their optimal predicted trajectory of at most the resulting value. In particular, and for all $l = 1, \ldots, n_c$ and $i = 1, \ldots, M$

$$d_{l,i}^T(\bar{x}_{t+k}^{[i]} - \bar{x}_{t+k|t-1}^{[i]}) \leq \frac{1}{\nu_l}\delta_l(t+k|t-1) \tag{8.7a}$$

$$X_{t+k}^{[i]} \preceq X_{t+k|t-1}^{[i]} \tag{8.7b}$$

where ν_l is the number of subsystems such that $d_{l,i} \neq 0$ (i.e., the number of subsystems constrained by the l-th coupling constraint). Consistently with the proposed probabilistic approach, for constraints (8.6c), the value $\delta_l(t+k|t-1)$ is computed as follows

$$\delta_l(t+k|t-1) = 1 - \sum_{i=1}^{M} d_{l,i}^T \bar{x}_{t+k|t-1}^{[i]} - \sqrt{\sum_{i=1}^{M} d_{l,i}^T X_{t+k|t-1}^{[i]} d_{l,i}^T}\; f(p_l^c) \tag{8.8}$$

and represents the distance, with respect to the optimal trajectories obtained at time $t-1$ by all the agents from the violation of the l-th constraint at time $t+k$. At this point some comments are in order. In particular, note that the values $\delta_l(t+k|t-1)$, $\forall k = 0, \dots, N-1$ and $l = 1, \dots, n_c$ are completely determined at time t by the optimal expected value and variance trajectories of the neighboring agents computed at time $t-1$. These information are thus the ones we need to transmit between neighboring agents in order to to solve the coupling in a distributed way. Of course, similarly to the approach discussed in Chapter 7 this choice introduces a source of conservatism but allows to completely decouple the collective constraints and to parallelize the solution of the problem.

8.2 SDPC for tracking: formulation and properties

As discussed in Chapter 7, in order to guarantee the tracking property, at each time step we need to add to the problem an extra degree of freedom, consisting of the virtual output, $\tilde{z}_t^{[i]}$, that is actually tracked by the local (probabilistic) MPC controller while moving towards its final goal $z_G^{[i]}$. In line with the multi-layer strategy, starting from the output reference, that is kept constant along the prediction horizon, we need to define the constant reference trajectories for the state and the input to be tracked by the state and the input of the nominal model, $\bar{x}_{t+k}^{[i]}$ and $\bar{u}_{t+k}^{[i]}$, respectively, for all $k = 0, \dots, N$. In Chapter 7 this is done by the so-called reference state/input trajectory layer through the definition of a new system that acts as an observer fed by the desired output reference to track. However, in the case considered here, we aim to simplify the formulation and thus we resort to the following equivalence

$$\begin{bmatrix} \tilde{x}_t^{[i]} \\ \tilde{u}_t^{[i]} \end{bmatrix} = S_i^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tilde{z}_t^{[i]} = M_i \tilde{z}_t^{[i]} \tag{8.9}$$

where the matrix S_i is always full-rank by assumption.

Consider now the problem of generating the virtual output reference signal, $\tilde{z}_t^{[i]}$, and the problem of tracking the generated state and input references $\tilde{x}_t^{[i]}$ and $\tilde{u}_t^{[i]}$. Differently from the previous chapter, in which the control scheme is composed by several levels, the goal here is to solve the reference generation and reference tracking problems together in one single optimization. Consistently with the stochastic framework, introduced in the previous section, the local MPC controller we want to design to perform this task is similar to the p-SMPC described in this Thesis. In particular, the main ingredients are described below.

8.2.1 Cost function

As for the local cost function, we consider here, similarly to Chapter 3, the following expression for all $i = 1, \ldots, M$, i.e.,

$$\begin{aligned} J(t)^{[i]} &= \mathbb{E}[\sum_{k=t}^{t+N-1} \|x_k^{[i]} - \tilde{x}_t^{[i]}\|_{Q_i}^2 + \|u_k^{[i]} - \tilde{u}_t^{[i]}\|_{R_i}^2 + \|x_{t+N}^{[i]} - \tilde{x}_t^{[i]}\|_{P_i}^2] \\ &\quad + \|\tilde{z}_t^{[i]} - \tilde{z}_G^{[i]}\|_{T_i}^2 \\ &= J_m^{[i]}(\bar{x}_t^{[i]}, \bar{u}_{t \ldots t+N-1}^{[i]}, \tilde{z}_t^{[i]}) + J_v^{[i]}(X_t^{[i]}, K_{t \ldots t+N-1}^{[i]}) \end{aligned} \tag{8.10}$$

where the two components that account respectively for the expected value and the variance of the system variables are defined as

$$\begin{aligned} J_m^{[i]} = & \sum_{k=t}^{t+N-1} \|\bar{x}_k^{[i]} - \tilde{x}_t^{[i]}\|_{Q_i}^2 + \|\bar{u}_k^{[i]} - \tilde{u}_t^{[i]}\|_{R_i}^2 + \|\bar{x}_{t+N}^{[i]} - \tilde{x}_t^{[i]}\|_{P_i}^2 \\ & + \|\tilde{z}_t^{[i]} - \tilde{z}_G^{[i]}\|_{T_i}^2 \end{aligned} \tag{8.11a}$$

$$J_v^{[i]} = \sum_{k=t}^{t+N-1} \mathrm{tr}\{(Q_i + K_k^{[i]T} R_i K_k^{[i]}) X_k^{[i]}\} + \mathrm{tr}\{P_i X_{t+N}^{[i]}\} \tag{8.11b}$$

The main difference with respect to the approach proposed in Chapter 3 is that $J_m^{[i]}$ is now formulated for tracking and that there is an extra term that accounts for the evolution of the virtual reference to the final goal $\tilde{z}_G^{[i]}$. Consistently with the p-SMPC case, for all $i = 1, \ldots, M$ we compute the terminal weight P_i as

$$(A_{ii} + B_{ii}\bar{K}_i)^T P_i (A_{ii} + B_{ii}\bar{K}_i) - P_i = -Q_i - \bar{K}_i^T R_i \bar{K}_i \tag{8.12}$$

where the constant gain $\bar{K}_i$ can be any stabilizing gain for the i-th nominal model, $\bar{\mathcal{S}}_i$ in (8.5a).

8.2.2 Terminal constraints

As in standard MPC schemes, terminal constraints are enforced to guarantee recursive feasibility properties. Accordingly to the discussion in Chapter 3 and 5, a simple way to handle terminal constraints in the probabilistic framework is to involve the expected value and the variance of the state variables. Similarly to equation (3.8), the covariance matrix of the state of the i-th subsystem, $X_t^{[i]}$, under the control law (8.4) evolves as

$$X_{t+1}^{[i]} = (A_{ii} + B_{ii}K_t^{[i]})X_t^{[i]}(A_{ii} + B_{ii}K_t^{[i]})^T + B_{w,i}W_iB_{w,i}^T \tag{8.13}$$

Based on equation (8.13) and similarly to the result in(3.23), we define, for each $i = 1, \dots, M$ the steady state covariance matrix $\bar{X}_i$ as the solution of the Lyapunov equation

$$(A_{ii} + B_{ii}\bar{K}_i)^T\bar{X}_i(A_{ii} + B_{ii}\bar{K}_i) - \bar{X}_i = -B_{w,i}\bar{W}_iB_{w,i}^T \tag{8.14}$$

where the covariance of the disturbance is chosen as $\bar{W}_i \succeq W_i$. Similarly to equation (3.25), using the steady state covariance $\bar{X}_i$, we define for each subsystem the tightened convex set of local constraints $\bar{\mathbb{X}}_i$ as

$$\bar{\mathbb{X}}_i = \left\{ \bar{x}^{[i]} \mid b_{r,i}^T\bar{x}^{[i]} \leq x_{r,i}^{max} - \sqrt{b_{r,i}^T\bar{X}_ib_{r,i}}\, f(p_i^x), \forall r = 1, \dots, n_{r_i} \right\} \tag{8.15}$$

Moreover, the steady state covariance matrix of the control input is obtained as $\bar{U}_i = \bar{K}_i\bar{X}_i\bar{K}_i^T$ and, based on this, we define the local probabilistic set $\bar{\mathbb{U}}_i$ as follows

$$\bar{\mathbb{U}}_i = \left\{ \bar{u}^{[i]} \mid c_{s,i}^T\bar{u}^{[i]} \leq u_{s,i}^{max} - \sqrt{c_{s,i}^T\bar{U}_ic_{s,i}}\, f(p_i^u), \forall s = 1, \dots, n_{s_i} \right\} \tag{8.16}$$

However, with respect to the definition of the terminal constraints in Chapter 3, the sets (8.15) and (8.16) are not sufficient, in the tracking framework, to compute a terminal set that guarantees the desired properties and further considerations are required.

In particular, we adopt here a zero terminal constraint strategy for the choice of the expected value of the state, i.e., we require that

$$x_{t+N}^{[i]} = \begin{bmatrix} I & 0 \end{bmatrix} M_i\tilde{z}_t^{[i]} \tag{8.17}$$

Furthermore, with this choice, the term $\tilde{z}_t^{[i]}$ must be selected in order to satisfy the local tightened constraints in (8.15) and in (8.16) and thus we require that

$$M_i\tilde{z}_t^{[i]} \in \lambda\left(\bar{\mathbb{X}}_i \times \bar{\mathbb{U}}_i\right) \tag{8.18}$$

where $\lambda \in (0, 1)$. Finally, $\tilde{z}_t^{[i]}$, $\forall i = 1, \dots, M$ must verify also the collective constraints in (8.6c) and, to this end, we resort again to a distributed solution. In particular, we assume that, at time t, each agent $\mathcal{S}_i$, $i = 1, \dots, M$

can use the optimal solutions obtained at time $t-1$ by its neighbors, namely $\tilde{z}^{[j]}_{t-1|t-1}$, $j \in \mathcal{N}_i$, to compute

$$\delta^{[z]}_l(t-1) = 1 - \sum_{i=1}^{M} d^T_{z,i} \begin{bmatrix} I & 0 \end{bmatrix} M_i \tilde{z}^{[i]}_{t-1|t-1} - \sqrt{\sum_{i=1}^{M} d^T_{z,i} \bar{X}_i d_{z,i}}\, f(p^c_l) 1 \tag{8.19}$$

and then to define the terminal constraint

$$d^T_{i,z} \begin{bmatrix} I & 0 \end{bmatrix} M_i (\tilde{z}^{[i]}_t - \tilde{z}^{[i]}_{t-1|t-1}) \leq \frac{1}{\nu_l} \delta^{[z]}_l(t-1) \tag{8.20}$$

Concerning the variance, similarly to Chapter 3 we require for all $i = 1, \ldots, M$ that the following inequality holds

$$X^{[i]}_{t+N} \preceq \bar{X}^{[i]}_i \tag{8.21}$$

8.2.3 Choice of the initial conditions

In order to apply a state-feedback p-SMPC like controller, as described in Chapter 3, we need to consider the initial conditions for the expected value and the variance of the state, namely $\bar{x}^{[i]}_{t|t}$ and $X^{[i]}_{t|t}$, as extra degrees of freedom of the optimization problem and select them according to the following strategy

Strategy 1 - Reset of the initial state: in order to use all the information available at time t from measures, set $\bar{x}^{[i]}_{t|t} = x^{[i]}_t$, $X^{[i]}_{t|t} = 0$.
Strategy 2 - Prediction: in order to maintain feasibility of the problem when the first strategy is not applicable (due to the possibly unbounded disturbance that can drive the state outside the feasibility region), set $\bar{x}^{[i]}_{t|t} = \bar{x}^{[i]}_{t|t-1}$, $X^{[i]}_{t|t} = X^{[i]}_{t|t-1}$, i.e., use the past optimal predictions.

For further details on the rationale behind this choice and on the implementation issues please see Chapter 3.

8.2.4 The i-th MPC controller

Based on the ingredients discussed so far, the local deterministic optimization problem for the i-th subsystem $\mathcal{S}_i$ (denoted by i-pSMPC) can now be stated as follows

i**-pSMPC problem:** at any time instant t solve the minimization problem

$$\min_{\substack{\bar{u}^{[i]}_{t\ldots t+N-1},K^{[i]}_{t\ldots t+N-1}\\ \tilde{z}^{[i]}_t,(\bar{x}^{[i]}_t,X^{[i]}_t)}} J^{[i]}_m(\bar{x}^{[i]}_t,\bar{u}^{[i]}_{t\ldots t+N-1},\tilde{z}^{[i]}_t)+J^{[i]}_v(X^{[i]}_t,K^{[i]}_{t\ldots t+N-1}) \tag{8.22}$$

subject to the dynamics (8.5a) and (8.13), to the local constraints (8.6a), (8.6b) and the collective constraints (8.7) for all $k=0,\ldots,N-1$, to the terminal constraints (8.17), (8.18), (8.20) and (8.21) and to the initialization strategy

$$(\bar{x}^{[i]}_{t|t},X^{[i]}_{t|t})\in\{(x^{[i]}_t,0),(\bar{x}^{[i]}_{t|t-1},X^{[i]}_{t|t-1})\}$$

As a solution to the i-pSMPC problem we obtain the optimal sequences of open-loop terms and gains, respectively $\bar{u}^{[i]}_{t\ldots t+N-1}$ and $K^{[i]}_{t\ldots t+N-1}$, and the optimal values $\bar{x}^{[i]}_{t|t}$,$X^{[i]}_{t|t}$ and $\tilde{z}^{[i]}_{t|t}$. Thus, consistently with (8.4), at time step t we apply the following input

$$u^{[i]}_t=\bar{u}^{[i]}_{t|t}+K^{[i]}_{t|t}(x^{[i]}_t-\bar{x}^{[i]}_{t|t}) \tag{8.23}$$

Moreover, in view of (8.5a), (8.13) and (8.5b), we can compute the optimal sequences at time t, i.e., $\bar{x}^{[i]}_{t\ldots t+N-1|t}$, $X^{[i]}_{t\ldots t+N-1|t}$ and $\bar{z}^{[i]}_{t\ldots t+N-1|t}$, to be transmitted to the other agents accordingly with the neighboring scheme.

Note that, in line with the discussion in Chapter 3, the i-pSMPC optimization problem can be simplified by choosing offline a fixed gain, i.e., setting $K^{[i]}_t=\bar{K}_i,\ \forall t\geq 0,\ i=1,\ldots,M$. In view of this, the computational load associated to each i-pSMPC problem would be the same of a small-scale nominal MPC problem and, moreover, the nonlinearity in the constraints does not carry about implementation problems (for further details on the implementation in the complete case, see Chapter 3). However, even in this case the term J_v in (8.10) cannot be discarded in the optimization because, although it does not depend anymore on the control gain sequence $K^{[i]}_{t\ldots t+N-1}$, it still depends upon the initial condition $X^{[i]}_t$.

8.3 Stochastic Distributed Control of a fleet of unicycle robots

In this section we discuss the application of the algorithm described in this chapter to the problem of controlling the fleet of mobile robots presented in Section 7.5. With respect to the robust setup, the worst-case considerations are relaxed into the probabilistic framework by allowing a certain probability of violating the obstacle/collision avoidance constraints. Besides the

the apparent naivety of the application, the setup allows to show, through a simple distributed control problem (due to the independent dynamics), the effectiveness of such techniques and once again, stress the fact that probabilistic approaches represent the only way we have to handle possibly unbounded disturbances.

In view of the special structure of the mobile robots coordination problem, where the inter-robot collision avoidance represents the only coupling term, several details must be considered and the optimization problem needs to be re-defined.

8.3.1 Model of unicycle robots

The model of the i-th subsystem is the unicycle model described in Section 7.5, i.e.,

$$\dot{x} = v\cos\phi \tag{8.24a}$$
$$\dot{y} = v\sin\phi \tag{8.24b}$$
$$\dot{\phi} = \omega \tag{8.24c}$$
$$\dot{v} = a \tag{8.24d}$$

The system dynamics are linearized through a feedback-linearization loop (see e.g. [121]) and discretized with sample time τ so that the model of the subsystem $\mathcal{S}_i$ is the one in (8.1) with

$$A_{ii} = \begin{bmatrix} 1 & \tau & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \tau \\ 0 & 0 & 0 & 1 \end{bmatrix},\ B_{ii} = \begin{bmatrix} \frac{\tau^2}{2} & 0 \\ \tau & 0 \\ 0 & \frac{\tau^2}{2} \\ 0 & \tau \end{bmatrix},\ C_i = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},\ B_{w,i} = I_4$$

where the uncertainty is used to consider unmodeled dynamics, actuation disturbances, etc. Further details on the identification of the uncertainty distributions will be given later in the example section.

8.3.2 Obstacle and collision avoidance

In Section 7.5 the collision (and obstacle) avoidance property has been quickly discussed and a solution to replace nonconvex distance constraints with linear, although conservative, constraints has been proposed. However, in order to introduce the concept of avoidance in the probabilistic framework, here a different setup, taken from [2], is presented. More specifically, our goal is to define an algorithm such that, for any fixed obstacle h with known position or any other moving agent $j = 1, \ldots, M,\ j \neq i$, the collision with the i-th robot is avoided with a certain probability. To this end, and in order to simplify the setup, each obstacle (robot) is assumed to be circular, centered in z_h^o ($z_t^{[i]}$) and with radius R_h^o (R_i). Moreover, for each robot i we define the set of n_i^o proximal obstacles as $\mathcal{O}_i \subseteq \{1, \ldots, n_i^o\}$, i.e., the set of obstacles that are to be accounted for by robot i and the set of proximal neighbors as $\mathcal{C}_i \subseteq \mathcal{N} \backslash \{i\}$, i.e., the set of other agents with which collision avoidance must be enforced. For consistency we assume that $j \in \mathcal{C}_i$ if and only if $i \in \mathcal{C}_j$.

The obstacle and collision avoidance properties will be satisfied by simply incorporating suitable constraints into the local optimization during the computation of the reference trajectories to be pursued by the robots. Note that, in view of the non-convexity of both types of constraints, they cannot be simply expressed in the forms (8.3c), and thus further discussions are needed.

In order to obtain linear bounds that approximate the obstacle and collision avoidance constraints, firstly consider the generic minimum distance constraint

$$\|z - \bar{z}\|_2 \geq \bar{d} \tag{8.25}$$

where $\bar{z}$ and $\bar{d}$ are given. As proposed in [2], define as an outer approximation of the circle a symmetric polytope $\mathbb{P}$, that is defined using r linear inequalities, i.e.,

$$\mathbb{P} = \{z \mid h_k^T(z - \bar{z}) \leq \bar{d}\} \tag{8.26}$$

The idea is now to note that the violation of any of the inequalities $h_k^T(z - \bar{z}) \leq \bar{d}$, i.e., the fact that there exists an index $\bar{k} \in \{1, \ldots, r\}$ such that $h_{\bar{k}}^T(z - \bar{z}) > \bar{d}$, implies that (8.25) is verified and thus that we are in a collision situation. With this in mind two strategies for handling collisions with fixed obstacles and other moving agents, respectively, are presented. Both of them are inspired from the results in [2] and extended to the adopted probabilistic framework using the approach in [127].

8.3.3 Fixed obstacle avoidance

Consider now the problem of avoiding the collisions with the h-th fixed obstacle, centered in z_h^o and with radius R_h^o. Denoting with R_i the radius of the i-th unicycle robot positioned at time t in $z_t^{[i]}$, to prevent collision with at most probability p^o_{hi}, we ideally require that

$$\mathcal{P}\left\{\|z^{[i]}_t - z^o_h\|_2 \geq d^o_{hi}\right\} \tag{8.27}$$

where the overall radius is defined as $d^o_{hi} = R_i + R^o_h$. However, due to the lack of convexity of (8.27), we need to use the strategy described above to derive a linear approximation. In particular, define the outer approximating polytope $\mathbb{P}^o_{hi}$, described by a set of r^o_{hi} linear inequalities

$$\mathbb{P}^o_{hi} = \{(h^{[o,hi]}_k)^T(z^{[i]}_t - z^o_h) \leq d^o_{hi}, \quad \forall\, k = 1,\ldots,r^o_{hi}\} \tag{8.28}$$

Now, similarly to (8.3a), the probabilistic obstacle avoidance constraint in (8.27) is reformulated as the existence of at least a value $\bar{l} \in [1,\ldots,r^o_{hi}]$ such that $\mathcal{P}\left\{-(h^{[0,hi]}_{\bar{l}})^T(z^{[i]}_t - z^o_h) \geq -d^o_{hi}\right\}$. This, similarly to (8.6a) and based on the results discussed in Chapter 3, is verified if

$$(h^{[o,hi]}_l)^T(\bar{z}^{[i]}_t - z^o_h) \geq d^o_{hi} + \sqrt{(h^{[o,hi]}_l)^T Z^{[i]}_t (h^{[o,hi]}_l)}\; f(p^o_{hi}) \tag{8.29}$$

where the expected value of the position of the robot is $\bar{z}^{[i]}_t = C\bar{x}^{[i]}_t$ and its variance is defined as $Z^{[i]}_t = CX^{[i]}_t C^T$.
Inspired by the approach in [2] we need to define along the prediction horizon, i.e., for all $k = t,\ldots,t+N-1$, the function

$$\begin{aligned}\delta^o_{hi,l}(k|t-1) = (h^{[o,hi]}_l)^T(\bar{z}^{[i]}_{k|t-1} - z^o_h) - d^o_{hi}\\ - \sqrt{(h^{[o,hi]}_l)^T Z^{[i]}_{k|t-1} h^{[o,hi]}_l}\; f(p^o_{hi})\end{aligned} \tag{8.30}$$

and the index

$$\bar{l}^o_{hi}(k|t-1) = \underset{l\in[1,\ldots,r^o_{hi}]}{\operatorname{argmax}} \quad \delta^o_{hi,l}(k|t-1) \tag{8.31}$$

Note that both (8.30) and (8.31) are completely defined by the knowledge of the optimal trajectory of robot i computed at the previous time instant $t-1$. Based on the computed values, at time step t, the following linear constraint, to be guaranteed by robot i, may be used to replace (8.27)

$$(h^{[o,hi]}_{\bar{l}^o_{hi}(k|t-1)})^T(\bar{z}^{[i]}_k - \bar{z}^{[i]}_{k|t-1}) \geq -\delta^o_{hi,\bar{l}^o_{hi}(k|t-1)}(k|t-1) \tag{8.32}$$

8.3.4 Inter-robot collision avoidance

The inter-robot collision avoidance issue is more challenging than the plain obstacle avoidance one, since pairs of moving objects must coordinate their actions in a distributed computation scheme. In other words, while the obstacle avoidance requirement generates only local constraints to be applied on the i-th agent, the inter-robot collision avoidance requires the use of collective constraints. However, similar considerations can be used to handle the non convex distance constraints and the strategy described in [127] applied to handle collective constraints.

To prevent collision between robots i and j, centered respectively in $z_t^{[i]}$ and $z_t^{[j]}$, with at most probability p_{ij}, at time t, we ideally require that

$$\mathcal{P}\left\{\|z_t^{[i]} - z_t^{[j]}\|_2 \geq d_{ij}\right\} \leq p_{ij} \tag{8.33}$$

where $d_{ij} = R_i + R_j$. Again, we start defining the outer approximating polytope, $\mathbb{P}_{ij}$, circumscribing the circle centered at $z_t^{[j]}$ with radius d_{ij}, that is represented by a set of r_{ij} linear inequalities, i.e.,

$$\mathbb{P}_{ij} = \{(h_k^{[ij]})^T(z_t^{[i]} - z_t^{[j]}) \leq d_{ij} \quad \forall\, k = 1, \ldots, r_{ij}\} \tag{8.34}$$

In this way, as in the fixed obstacle case, inter-robot collision avoidance can be stated as the existence of at least a value of the index $\bar{l} \in [1, \ldots, r_{ij}]$ such that $\mathcal{P}\left\{-(h_{\bar{l}}^{[ij]})^T(z_t^{[i]} - z_t^{[j]}) \geq -d_{ij}\right\} \leq p_{ij}$. This is verified if

$$(h_l^{[hi]})^T(\bar{z}_t^{[i]} - \bar{z}_t^{[j]}) \geq d_{ij} + \sqrt{(h_l^{[ij]})^T(Z_{k|t-1}^{[i]} + Z_{k|t-1}^{[j]})h_l^{[ij]}}\, f(p_{ij}) \tag{8.35}$$

In order to apply the constraint to the problem we need to define along the horizon, i.e., for all $k = t, \ldots, t+N-1$ the following function

$$\begin{aligned}\delta_{ij,l}(k|t-1) = (h_l^{[ij]})^T(\bar{z}_{k|t-1}^{[i]} - \bar{z}_{k|t-1}^{[j]}) - d_{ij} \\ - \sqrt{(h_l^{[ij]})^T(Z_{k|t-1}^{[i]} + Z_{k|t-1}^{[j]})h_l^{[ij]}}\, f(p_{ij})\end{aligned} \tag{8.36}$$

and in turn we need to compute the index corresponding to the maximum value of $\delta_{ij,l}(k|t-1)$ as

$$\bar{l}_{ij}(k|t-1) = \underset{l \in [1,\ldots,r_{ij}]}{\operatorname{argmax}} \quad \delta_{ij,l}(k|t-1) \tag{8.37}$$

Note that the values (8.36) and (8.37) depend from the optimal trajectories of both the agents i and j but also that these values are computed at time $t-1$ and thus are completely known at time t under the assumption that the neighboring agents are able to communicate their future steps. According to the procedure described in Section 8.1, in order to apply the avoidance constraint (8.33) at time step t, the following linear constraint, to be guaranteed by robot i and j, may be used along the prediction horizon, i.e. for all $k = 0, \dots, N-1$

$$(h^{[ij]}_{\bar{l}_{ij}(k|t-1)})^T(\bar{z}^{[i]}_k - \bar{z}^{[i]}_{k|t-1}) \geq -\frac{1}{2}\delta_{ij,\bar{l}_{ij}(k|t-1)}(k|t-1) \tag{8.38a}$$

$$(h^{[ij]}_{\bar{l}_{ij}(k|t-1)})^T(\bar{z}^{[j]}_{k|t-1} - \bar{z}^{[j]}_k) \geq -\frac{1}{2}\delta_{ij,\bar{l}_{ij}(k|t-1)}(k|t-1) \tag{8.38b}$$

The main difference with (8.32) is that in the inter-robot collision avoidance case, the single avoidance constraint computed for robot i affects also the set of constraints to be applied to the robot j. However, applying the proposed strategy, the resulting optimization problem for agent i is completely independent from the degrees of freedom of the neighboring agents, thus allowing for a distributed implementation at the price of a slight conservatism due to the limitation imposed to the evolution of the output variable between two consecutive steps.

8.3.5 Terminal Constraints

As discussed in Section 8.2.2, also in the framework of distributed control for mobile robots it is useful to adopt a zero terminal constraint-type strategy. In particular, for the state at the end of the horizon, $\bar{x}^{[i]}_{t+N}$, i.e., we require that

$$\bar{x}^{[i]}_{t+N} = \begin{bmatrix} I & 0 \end{bmatrix} M_i \tilde{z}^{[i]}_t \tag{8.39}$$

Concerning $\tilde{z}^{[i]}_t$ it must, naturally, first be selected in such a way that

$$\begin{bmatrix} I & 0 \end{bmatrix} M_i \tilde{z}^{[i]}_t \in \lambda \bar{\mathbb{X}}_i \tag{8.40}$$

where the set $\tilde{\mathbb{X}}_i$ represents the local constraints to be satisfied and is defined as

$$\bar{\mathbb{X}}_i = \left\{ \bar{x}^{[i]} \mid b^T_{r,i}\bar{x}^{[i]} \leq 1 - \sqrt{b^T_{r,i}\bar{X}^{[i]}b_{r,i}}\, f(p^x_{r,i}),\ r = 1, \dots, n_r \right\} \tag{8.41}$$

Secondly, it must be selected in order to prevent collisions. This is done by considering the properties of the solution to the problem obtained at the previous time step, $t-1$, i.e., $\tilde{z}^{[i]}_{t-1|t-1}$. Concerning obstacle avoidance between the i-th robot and the h-th obstacle, this requires the definition of

$$\begin{aligned} \tilde{\delta}^o_{hi,l}(t-1) = (h^{[o,hi]}_l)^T(\tilde{z}^{[i]}_{t-1|t-1} - z^o_h) - d^o_{hi} \\ - \sqrt{(h^{[o,hi]}_l)^T \bar{Z}^{[i]} h^{[o,hi]}_l}\, f(p^o_{hi}) \end{aligned} \tag{8.42}$$

where $\bar{Z}^{[i]}$ is the steady state value of the covariance matrix of the output and, in turn, is computed starting from the steady state covariance matrix of the state from (8.14) as $\bar{Z}^{[i]} = C\bar{X}^{[i]}C^T$. In addition, the index corresponding to the maximum value of $\tilde{\delta}^o_{hi,l}(t-1)$ is computed as

$$\tilde{l}^o_{hi}(t-1) = \underset{l\in[1,\ldots,r^o_{hi}]}{\text{argmax}} \quad \tilde{\delta}^o_{hi,(t-1)}(t-1) \tag{8.43}$$

and at time step t, the following linear constraint, to be guaranteed by robot i, must be fulfilled

$$(h^{[o,hi]}_{\tilde{l}^o_{hi}(t-1)})^T(\tilde{z}^{[i]}_t - \tilde{z}^{[i]}_{t-1|t-1}) \geq -\tilde{\delta}^o_{hi,\tilde{l}^o_{hi}(t-1)}(t-1) \tag{8.44}$$

in order to safely choose $\tilde{z}^{[i]}_t$ at the end of the horizon.
Similarly, concerning inter-robot collision avoidance, consider the steady state variances of robot i and j, namely $\bar{Z}^{[i]}$ and $\bar{Z}^{[j]}$, and define the function

$$\begin{aligned}\tilde{\delta}_{ij,l}(t-1) = (h^{[ij]}_l)^T(\tilde{z}^{[i]}_{t-1|t-1} - \tilde{z}^{[j]}_{t-1|t-1}) - d_{ij} \\ - \sqrt{(h^{[ij]}_l)^T(\bar{Z}^{[i]} + \bar{Z}^{[j]})h^{[ij]}_l}\, f(p_{ij})\end{aligned} \tag{8.45}$$

and the corresponding index

$$\tilde{l}_{ij}(t-1) = \underset{l\in[1,\ldots,r_{ij}]}{\text{argmax}} \quad \tilde{\delta}_{ij,l}(t-1) \tag{8.46}$$

At time step t, the following linear constraint, to be guaranteed by both the robot i and j, must be fulfilled

$$(h^{[ij]}_{\tilde{l}_{ij}(t-1)})^T(\tilde{z}^{[i]}_t - \tilde{z}^{[i]}_{t-1|t-1}) \geq -\frac{1}{2}\tilde{\delta}_{ij,\tilde{l}_{ij}(t-1)}(t-1) \tag{8.47a}$$

$$(h^{[ij]}_{\tilde{l}_{ij}(t-1)})^T(\tilde{z}^{[j]}_{t-1|t-1} - \tilde{z}^{[j]}_t) \geq -\frac{1}{2}\tilde{\delta}_{ij,\tilde{l}_{ij}(t-1)}(t-1) \tag{8.47b}$$

8.3.6 Statement of the problem and sketch of the algorithm

We are now in the position to derive the main optimization problem, solved by the local controller embedded in the i-th robot. With respect to the algorithm presented in the previous section and with the aim of reducing the computational load and simplifying the application we fix the value of the controller gain, i.e. we set $K^{[i]}_t = \bar{K}_i$, where $\bar{K}_i$ is a generic stabilizing gain for the nominal system.

i-pSMPC problem: at any time instant t solve

$$\min_{\bar{u}^{[i]}_{t\dots t+N-1}, \bar{x}^{[i]}_t, X^{[i]}_t, \tilde{z}^{[i]}_t} \quad J^{[i]}_m(\bar{x}^{[i]}_t, \bar{u}^{[i]}_{t\dots t+N-1}, \tilde{z}^{[i]}_t) + J^{[i]}_v(X^{[i]}_t) \tag{8.48}$$

subject to the dynamics (8.5a) and (8.13), constraints (8.6a), obstacle avoidance constraints (8.32) for all $h \in \mathcal{O}_i$, inter-robot collision avoidance constraints (8.38) for all $j \in \mathcal{C}_j$ and for all $k = 0, \dots, N-1$ the initial constraint

$$(\bar{x}^{[i]}_{t|t}, X^{[i]}_{t|t}) \in \{(x^{[i]}_t, 0), (\bar{x}^{[i]}_{t|t-1}, X^{[i]}_{t|t-1})\},$$

the terminal constraints (8.39), (8.40), (8.44) and (8.47).
Similarly to the general case presented in the previous section, as a solution to the i-pSMPC problem we obtain the optimal sequences for the open-loop term, $\bar{u}^{[i]}_{t\dots t+N-1}$, and the optimal values $\bar{x}^{[i]}_{t|t}$,$X^{[i]}_{t|t}$ and $\tilde{z}^{[i]}_{t|t}$. Thus, consistently with (8.4), at time step t we apply the following input

$$u^{[i]}_t = \bar{u}^{[i]}_{t|t} + \bar{K}_i(x^{[i]}_t - \bar{x}^{[i]}_{t|t}) \tag{8.49}$$

Moreover, the optimal sequences, in view of (8.5a), (8.13) and (8.5b), allow to define the optimal predictions sequences at time t to be transmitted to the other agents accordingly with the neighboring scheme, i.e., $\bar{x}^{[i]}_{t\dots t+N-1|t}$, $X^{[i]}_{t\dots t+N-1|t}$ and $\tilde{z}^{[i]}_{t\dots t+N-1|t}$, respectively.

8.4 Simulation results

In this section two simulation examples are presented to show the efficacy of the adopted approach. In particular, the first example shows the behavior of a single robot while avoiding a fixed obstacle to reach a desired goal. In the second example two different agents are considered and the results of the inter-robot collision avoidance strategy are presented and discussed.
With respect to the algorithm described in the previous section, a summary of the implementation steps is given below for the general multi-agent case.

Coordination algorithm

Initialization
At time $t = 0$ and for all $i = 1, \dots, M$

I) Set $\tilde{z}^{[i]}_0 = z^{[i]}_0$, where $z^{[i]}_0$ is the initial position of robot i

II) Set $\bar{z}^{[i]}_{k|0} = z^{[i]}_0, \forall k = 1, \dots, N$

III) Set $X^{[i]}_{0|0} = 0$ and compute $X^{[i]}_{k|0}$, $\forall k = 1, \dots, N$ using (8.13), with a fixed gain $K^{[i]}_k = \bar{K}_i$. Concerning this, check that the tuning knobs N and $\bar{W}_i$ are compatible with the fulfillment of the terminal constraint (8.21), i.e., $X^{[i]}_{N|0} \preceq \bar{X}_i$

IV) Compute $Z^{[i]}_{k|0} = CX^{[i]}_{k|0}C^T, \forall k = 1, \dots, N$

Online implementation
At each time step $t > 0$

Ia) Update the set of proximal obstacles $\mathcal{O}_i$

Ib) Update the set of proximal neighbors $\mathcal{C}_i$

Ic) Receive, for all $j \in \mathcal{C}_i$, the quantities $\tilde{z}^{[j]}_{t-1|t-1}$ and R_j, and the sequences $\bar{z}^{[j]}_{t\dots t+N-1|t-1}$,$Z^{[j]}_{t\dots t+N-1|t-1}$

II) Solve the i-rSMPC problem and compute the sequences $\bar{z}^{[i]}_{t\dots t+N-1|t}$, $Z^{[i]}_{t\dots t+N-1|t}$

III) Apply input $u^{[i]}_t$ computed as in (8.4)

8.4.1 Single robot probabilistic obstacle avoidance

The first simulation is run considering a single unicycle robot and a single fixed obstacle with the aim to analyze the effect of the probabilistic avoidance constraints in terms of number of collisions detected on a bunch of different simulations (note that in this case the i-pSMPC controller is reduced to a standard p-SMPC controller for tracking). The obstacle is centered in $z^o_1 = (5, 5)$ with radius $R^o_1 = 2$, the goal is in $z^G_1 = (9, 9)$ and, without loss of generality, the robot is supposed to be a point with initial conditions $x^{[1]}_t = (0, 0, 0, 0)$ and $X^{[1]}_t = 0$. The disturbance acting on the robot is assumed to be a zero mean Gaussian white noise with variance $W_1 = 0.01\, I_4$. The parameters of the i-pSMPC controller are chosen as $Q = I_4$, $R = I_2$, $T = 100I_2$, the gain $\bar{K}_1$ is fixed and computed with LQ,

together with the terminal weight P_1. The avoidance constraint is allowed to be violated up to $p_1^o = 0.2$, the state is left unconstrained and the input constraints are selected to ensure that $\|u^{[1]}\|_\infty \leq 1$ is violated with probability at most $p_1^u = 0.1$.
In Figure.8.1 the computed trajectories corresponding to 100 different simulations are shown. The measured frequency of collision is 0.17, close to the desired value p_1^o.

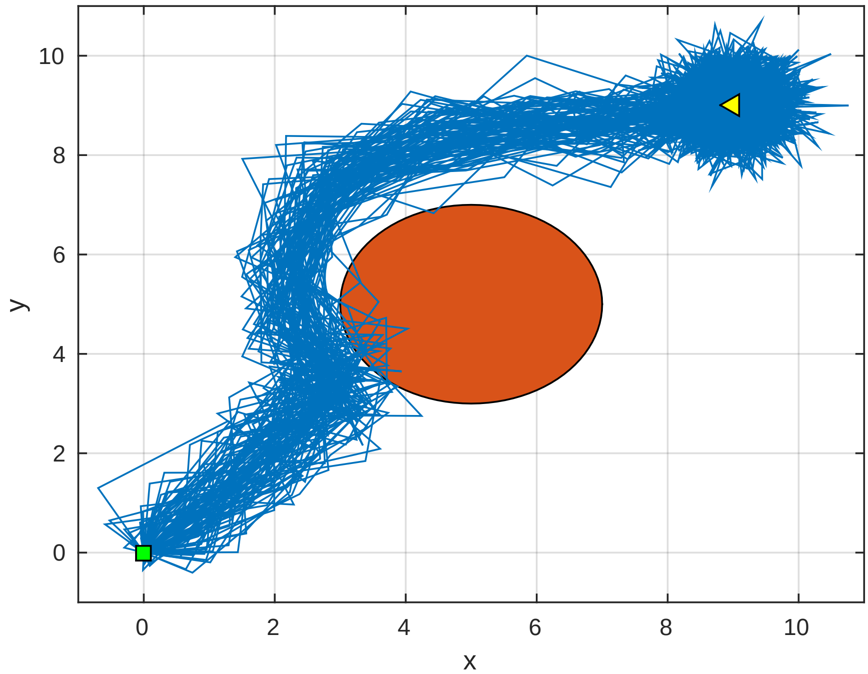


Figure 8.1: *Example of probabilistic obstacle avoidance. Trajectories of the robot over* 100 *different simulations starting from the green square and pointing to the yellow triangle. The measured collision frequency is* 0.17.

8.4.2 Multi-robot collision avoidance

The second simulation is a collision avoidance test between two different agents each running the i-pSMPC algorithm. Also in this case, the aim is to show the effectiveness of the approach and to discuss the probabilistic properties of the solution in terms of detected collisions over different simulations. To this end, the setup is constructed so that the two robots meet in the center of the working area and thus need to start an avoidance maneuver as shown in Figure 8.2. The red dashed lines denote the optimal trajectories predicted by the controller.

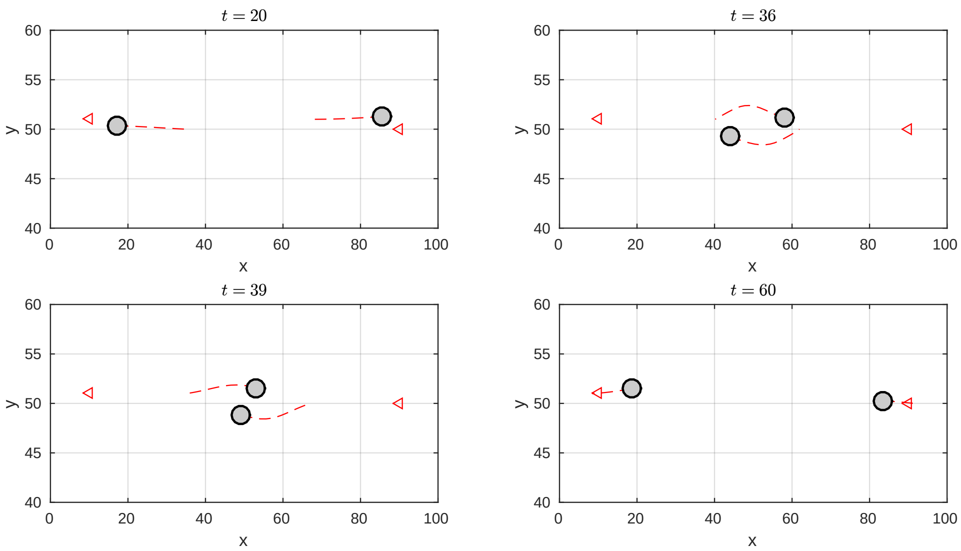


Figure 8.2: *Example of collision avoidance between two unicycles. The grey circles represent the two robots, the red triangles are the two goals, respectively* $z_G^{[1]} = (90, 50)$ *and* $z_G^{[2]} = (10, 51)$ *and the red dashed lines are the optimal predicted trajectories at the current instant.*

The unicycles are assumed to be circular with radius $R_1 = R_2 = 1.5$. The two goals are selected as $z_G^{[1]} = (90, 50)$ and $z_G^{[2]} = (10, 51)$ and the initial conditions for the two robots set to $x_t^{[1]} = (0, 0, 50, 0)$, $x_t^{[2]} = (100, 0, 50, 0)$ and $X_t^{[1]} = X_t^{[2]} = 0$. The disturbances acting on the two agents are independent zero-mean Gaussian white noises with variances $W_1 = W_2 = 0.1 I_4$. The parameters of the two i-pSMPC controllers are $Q = I_4$, $R = I_2$, $T = 100\, I_2$. As in the previous example, the control gains for both the agents, $\bar{K}_1$ and $\bar{K}_2$ are fixed and computed with LQ, together with the terminal weights P_1 and P_2. The collision is allowed with a probability up to $p_{12} = 0.3$, and as in the previous example the state is left unconstrained while the input constraints are selected to ensure that $\|u^{[1]}\|_\infty \leq 1$ is violated with probability at most $p_1^u = 0.1$.
In Figure 8.3 the evolution over time of the relative distance between the two robots is shown for 250 different simulations. The measured frequency of detected collisions (that happen around the time instant $t = 40$, as apparent from Figure 8.2) is 0.29 , thus confirming the efficacy of the approach.

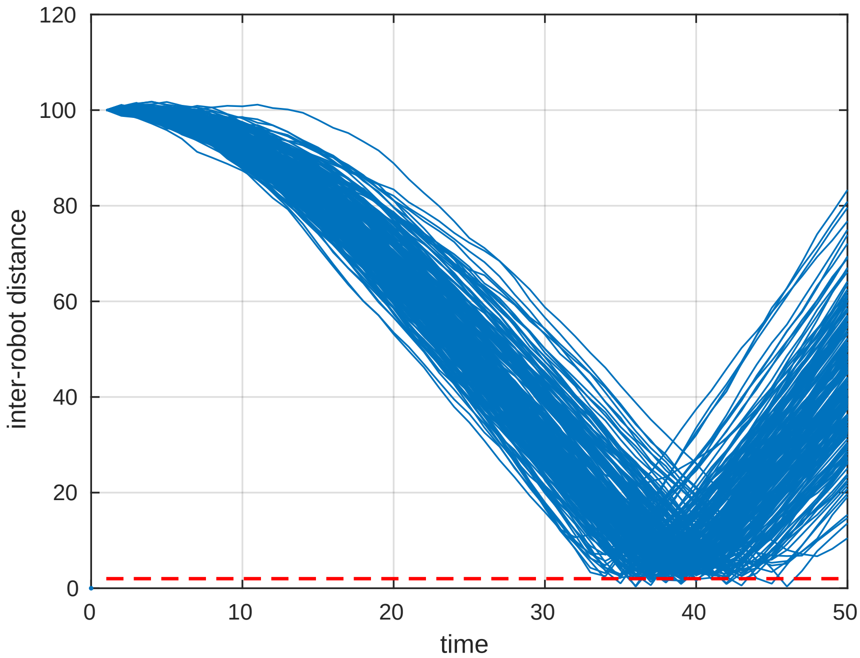


Figure 8.3: *Example of probabilistic collision avoidance between two unicycles. Blue lines represent the relative distance over time for* 250 *different extractions of the uncertainty. The red dotted line is the minimum distance to avoid collisions set to* 3 *cm The measured collision frequency is* 0.29.

8.5 Application to a real coordination problem

In this section the approach discussed in this chapter is applied to a real coordination problem run on the experimental setup described in Section 7.5. Firstly, in order to identify the model disturbance acting on the i-th robot, several open-loop tests are made obtaining the measurements in Figure 8.4. In particular, their mean results to be zero and the distribution nearly Gaussian as depicted in Figure 8.5, thus motivating the choice made in the previous section. The estimated covariance matrix is block diagonal

$$W \simeq \begin{bmatrix} 0.011 & -0.0017 & 0 & 0 \\ -0.0017 & 0.033 & 0 & 0 \\ 0 & 0 & 0.011 & -0.0017 \\ 0 & 0 & -0.0017 & 0.033 \end{bmatrix}$$

Using the identified disturbance model, two different tests are made. Obviously, in both the cases, the goal is not to check the probabilistic behavior

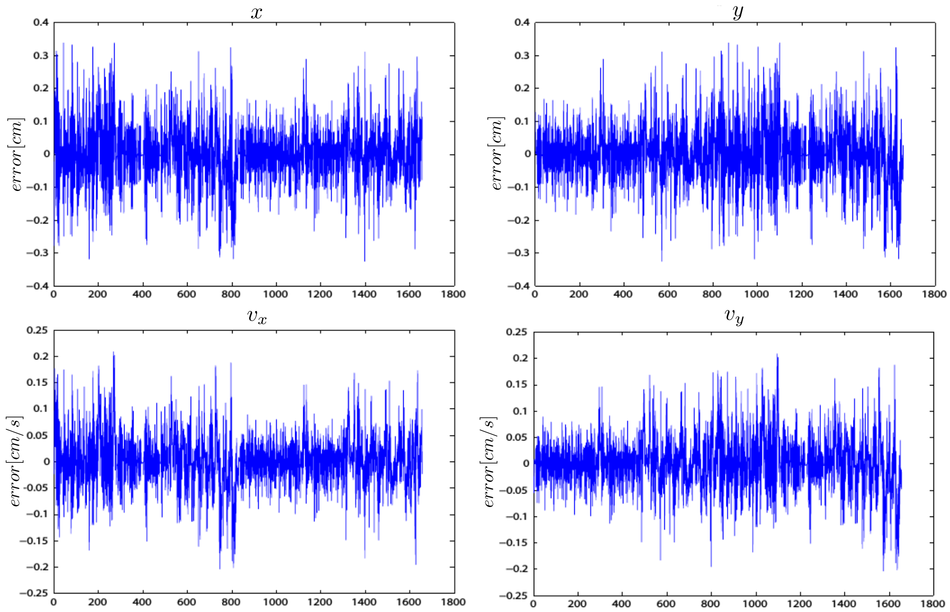


Figure 8.4: *Measured disturbance samples in one of the tests. The two figures on the top represent the disturbance on the position while the bottom figures represent the disturbance on the speed.*

of the system but just to show the efficacy of the approach that allows to solve the problem even in the case of unbounded disturbances for which a worst-case technique like the one used in Chapter 7 is not applicable. In practice, the collision probability will be set to a sufficiently small value.

The first test is similar to the second simulation example in which two unicycles are required to switch their position thus generating a possible collision situation close to the center of the working area. The result is shown in Figure 8.6 by means of a series of pictures of the working plane taken at different time instants.

The second example is a complete coordination task in which three agents need to reach their own parking spots while avoiding collisions between each other and with fixed obstacles. The resulting behavior is described in Figure 8.7 where the red circles represent virtual obstacles along the path.

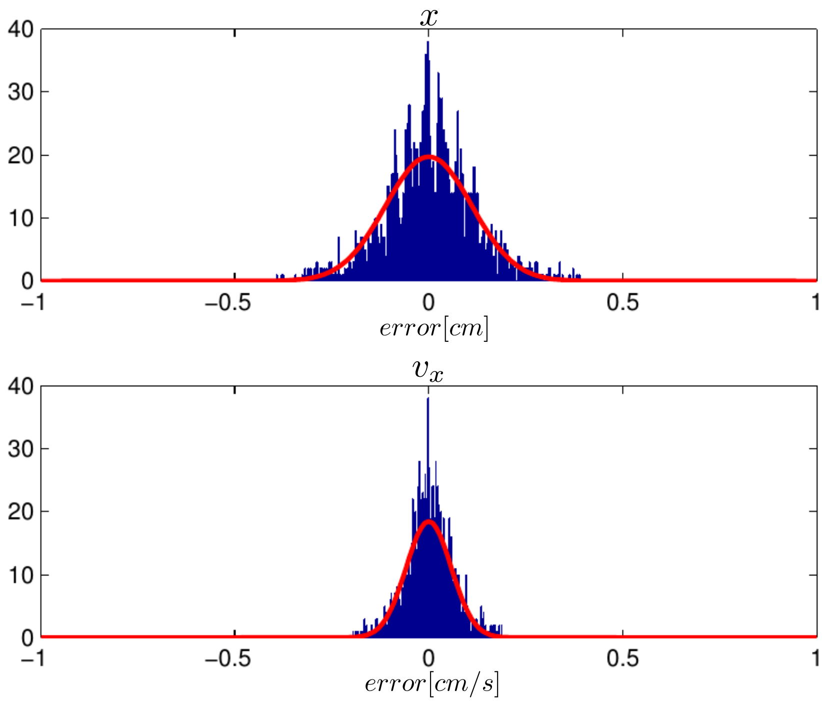


Figure 8.5: *Example of distribution of the measured disturbance samples. The first figure shows the disturbance on the position along the x axis and the second figure the disturbance on the velocity along the x axis.*

8.6 Comments

In this chapter we described a distributed stochastic MPC algorithm for tracking reference signals in the case of dynamically decoupled subsystems and probabilistic local and coupling constraints, based on the preliminary work in [127]. The algorithm is then specialized to the mobile robots coordination problem where the previous results are used, together with the procedure suggested in [2], for avoiding collisions with fixed obstacle and within the agents. Two simulation experiments has shown the efficacy of the proposed approach in terms of probabilistic characterization of the controlled system trajectories, i.e. the number of detected collisions over multiple simulations, both in the case of fixed obstacles avoidance and in the case of an inter-robot coordination task. A third example originally presented in [105] is reported in order to show the effectiveness of the proposed approach in a real unicycles coordination problem.

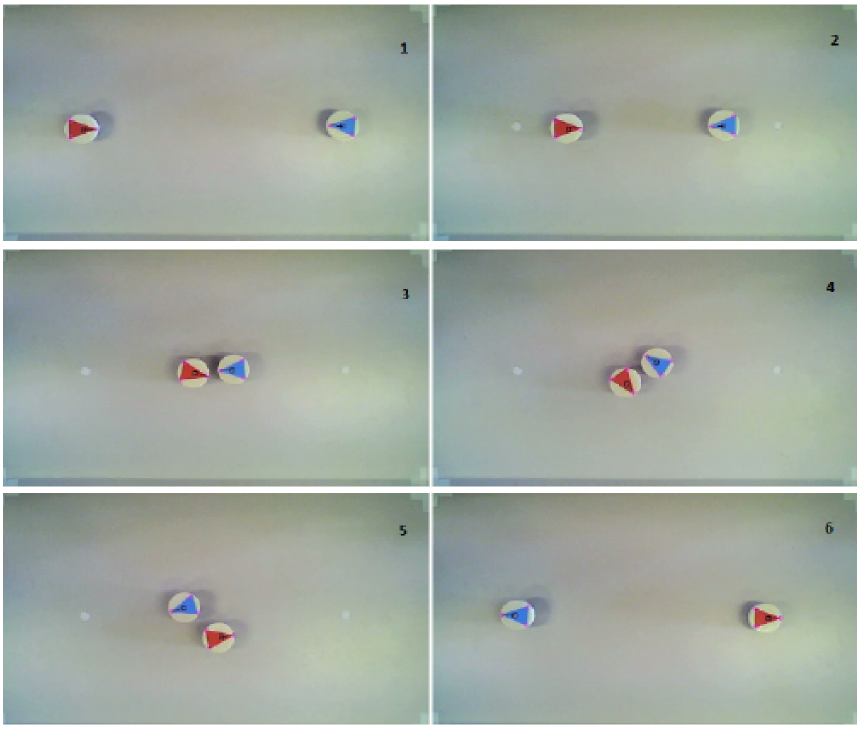


Figure 8.6: *Coordination test with two unicycles (red and blue). For both of them the aim is to reach their own local goals while avoiding collision in the central zone with high probability.*

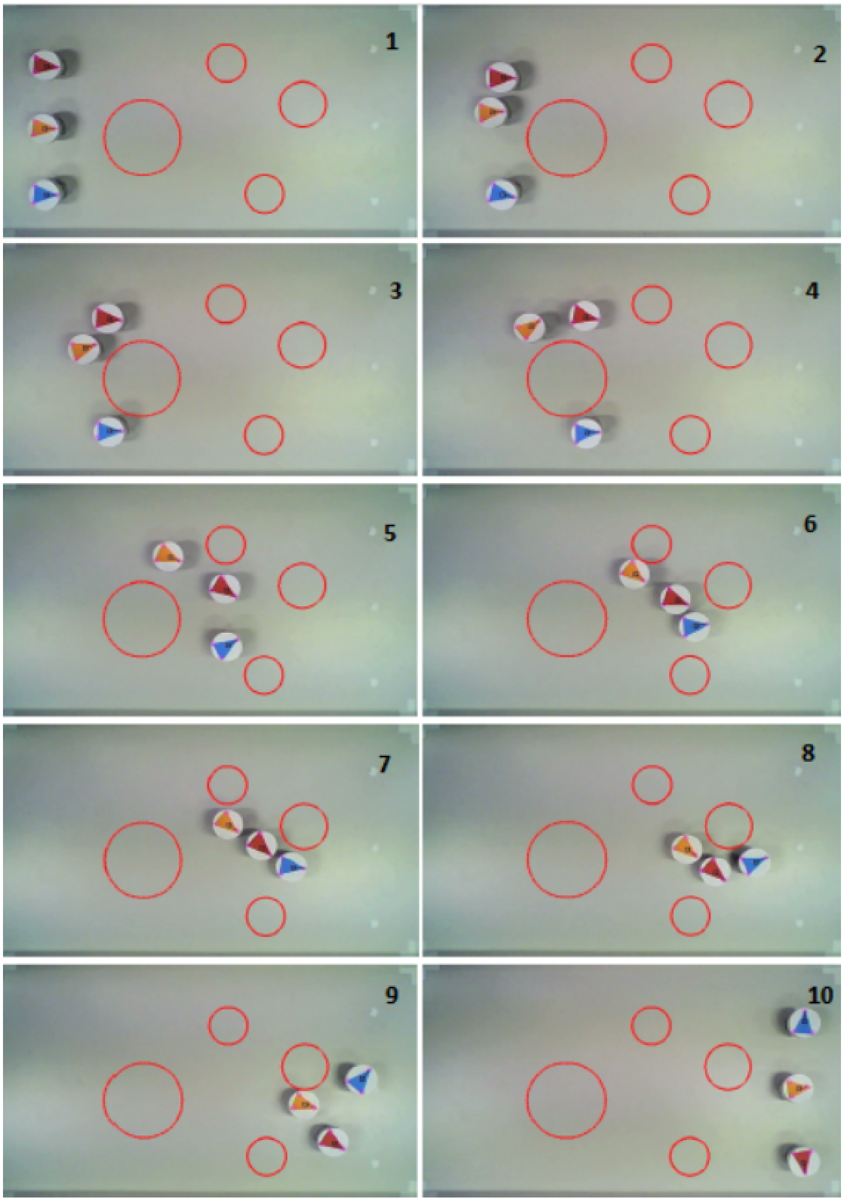


Figure 8.7: *Example of collision free navigation in presence of several agents (blue, yellow and red) and fixed (virtual) obstacles. The robots, equipped with the i-pSMPC controllers, are able to reach their goal while avoiding collisions and resolving conflicts.*

CHAPTER 9

Conclusions

In this Thesis, Stochastic Model Predictive Control approaches for linear discrete-time systems affected by additive, possibly unbounded, stochastic disturbances and subject to probabilistic constraints have been discussed. After a detailed analysis of the problem ingredients, a description of the main challenges, due to the stochastic setup, and a general discussion on the many algorithms available in the literature, are reported. Then, a novel technique, named p-SMPC, has been presented for systems subject to a set of independent chance constraints both on the state and the input.

Initially, the algorithm has been developed for systems with measurable states. An analytic technique for handling probabilistic constraints has been described in the case of disturbances with Gaussian distribution and disturbances with unknown distribution, but known second order moments. With respect to other algorithms available in the literature, that assume bounded uncertainties, a proper choice of the terminal constraints, together with a switching strategy for the selection of the initial conditions of the MPC optimization problem, guarantee the recursive feasibility and the convergence of the algorithm even in the case of unbounded disturbances. This result, together with the reduced online computational load, comparable to the one of standard nominal MPC techniques, represent the main advantages of the proposed approach. On the other hand, the main drawback relies on the fundamental assumption that constraints on the input are probabilistic. This in principle, does not allow to consider saturations on the control variables, however, since the level of constraint violation is a design parameter, one can overcome the problem by choosing this parameter as sufficiently low. The application of the proposed algorithm to several simulation examples has been discussed and comparisons with other interesting algorithms in the literature have been made to show the viability of the approach. Later, the p-SMPC technique has been extended to the case of systems with not measurable states and an output-feedback stochastic algorithm has been presented, that exhibits the same feasibility and convergence properties of the state-feedback one. A detailed discussion on the main implementation issues has been added to give a clear overview of the pros and cons of the algorithm. Eventually a simulation example has shown the efficacy of the approach.

In the second part of this Thesis, the focus has been moved to distributed control problems with the aim of extending the novel p-SMPC technique into a distributed framework in order to allow considering local and collective probabilistic constraints in presence of possibly unbounded uncertainties on each subsystem. Firstly, the case of dynamically coupled subsystems subject to local probabilistic constraints has been considered and a distributed predictive controller for regulation has been developed. The main advantage of the proposed approach relies on its capability to extend the feasibility and convergence properties of the centralized case also to the distributed setup. The main drawback is related to the conservative nature of the approach, due to the many approximation introduced in the design phase. Far from being perfect, the proposed approach has to be interpreted as a first step in the direction of developing efficient Distributed SMPC techniques. Secondly, the problem of designing a distributed predictive controller for tracking reference signals has been addressed. This has been initially solved in a deterministic setup, relying on a multi-level scheme based on well-known robust MPC algorithms to handle the uncertainties. Finally, a simplified version of this setup has been extended to the stochastic case and, in particular, a distributed algorithm has been presented to solve the tracking problem of a system composed by several dynamically decoupled agents subject to local and collective probabilistic constraints. The algorithm, inspired by some results already available in the literature, has been quickly discussed and then specialized for solving a mobile robot coordination problem. Concerning the coordination algorithm, simulations and real experiments have been added to show the viability and the flexibility of the approach.

Several extensions to the work presented in this Thesis are possible. Concerning the p-SMPC algorithm, an approach for enforcing hard constraints on the input, while preserving the nice feasibility and convergence properties, should be studied and the effect of different control law parametrizations needs to be investigated. Moreover, the proposed approach can be extended to encompass different uncertainty models, for example the case of systems affected by multiplicative disturbances, or different type of stochastic constraints, for example the so-called joint chance-constraints. Concerning the topic of Distributed Model Predictive Control, while for the robust deterministic case results are well-established, in the stochastic setup a lot of work has still to be done. The stochastic distributed control algorithm for regulation should be revised, less conservative solutions need to be defined to handle the couplings and stronger theoretical results should to be derived. As for the Stochastic Distributed Control algorithm for tracking, the extension to the case of dynamically coupled systems could be the next step.

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