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Distributed algorithm for ε-generalized Nash equilibria with uncertain coupled constraints☆

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ABSTRACT

In this paper, we design a distributed algorithm to seek generalized Nash equilibria with uncertain coupled constraints. It is hard to find the exact equilibria directly, because the parameters in the coupled constraint come from general convex sets, which may not have analytic expressions. To solve the problem, we first approximate general convex sets by inscribed polyhedrons and transform the approximate problem into a variational inequality by robust optimization. Then, with help of convex set geometry and metric spaces, we prove that the solution to the variational inequality induces an ε-generalized Nash equilibrium of the original game in the worst case. Furthermore, we propose a distributed algorithm to seek an ε-generalized Nash equilibrium, and show the convergence analysis with Lyapunov functions and variational inequalities. Finally, we illustrate the effectiveness of the distributed algorithm by a numerical example.

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1. Introduction

Multi-agent systems have attracted a large amount of attention for the last decade, due to its broad applications in social science and engineering systems. More recently, distributed Nash equilibrium (NE) seeking in games, where players make decisions by computing with local data and communicating through networks, has become an emerging research topic (Gadjov & Pavel, 2018; Grammatico, Parise, Colombino, & Lygeros, 2015; Koshal, Nedić, & Shanbhag, 2016; Ye & Hu, 2017; Zeng, Chen, Liang, & Hong, 2019). Many distributed algorithms have been proposed for NE seeking, such as asynchronous gossip-based distributed algorithms (Salehisadaghiani & Pavel, 2016), distributed algorithms with gradient estimation and search for time-varying NE seeking (Ye & Hu, 2015), passivity-based distributed algorithm design (Gadjov & Pavel, 2018), and strategic decision-making distributed algorithms for energy resources (Gharesifard, Başar, & Domínguez-García, 2015).

In practice, coupled constraints frequently occur because the involved players may have a shared resource such as communication bandwidth and network energy. Generalized Nash equilibrium (GNE) or the variational GNE as its refinement (Kulkarni & Shanbhag, 2012) was adopted as proper solutions for games with coupled constraints (see Facchinei & Kanzow, 2010 for a historical review and Fischer, Herrich, & Schönefeld, 2014 for computational methods). A GNE is regarded as an acceptable solution since no one is capable to decrease the cost unilaterally among the feasible sets, in which each player's decision variables are dependent on others' actions due to the coupled constraints. Hereupon, various distributed algorithms were proposed and improved for GNE seeking. Specifically, GNE seeking in a class of generalized convex games was considered with network delay in Zhu and Frazzoli (2016), while a distributed GNE via operator splitting methods with guaranteeing convergence was derived in Yi and Pavel (2017) under fixed step-sizes. As for aggregative games, a variational GNE was computed in a distributed way for quadratic aggregative games with affine coupling constraints in Paccagnan, Gentile, Parise, Kamgarpour, and Lygeros (2016), while aggregative games with coupled constraints were investigated based on non-smooth tracking dynamics in Liang, Yi, and Hong (2017). Moreover, asynchronous distributed algorithms

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for seeking GNE were proposed in Yi and Pavel (2019) under full or partial-decision information, while another asynchronous forward-backward distributed algorithm for GNE seeking was discussed in Cenedese, Belgioioso, Grammatico, and Cao (2019).

Nevertheless, there inevitably exist uncertainties in practical games like electric vehicle charging (Yang, Xie, & Vasilakos, 2015) and security resource allocation (Nikoofal & Zhuang, 2012). One way to handle uncertainties in games is to utilize robust optimization (see more details in Bertsimas, Brown, & Caramanis, 2011). The concept of robust games was first proposed in Aghassi and Bertsimas (2006), where players used a robust optimization approach to contend with uncertainties in players' payoffs. Afterwards, researchers turned their attention to robust games and showed interest in various games with uncertainties. For instance, Hu and Fukushima (2013) discussed the existence, uniqueness and computation of robust NE in a multi-leader-follower game with uncertainties in objective functions, while (Miao, Pajic, & Pappas, 2013) employed a stochastic game model to minimize the worst-case detection cost in a finite horizon. Not restricted to theoretical study, the framework of robust games has been applied to deal with practical engineering scenarios. A robust Stackelberg game model was used to obtain noncooperative and cooperative optimization in Yang et al. (2015) under demand uncertainty of electric vehicle charging; a robust game model was employed in security resource allocation in Nikoofal and Zhuang (2012), where uncertainty is defined in bounded distribution-free intervals; a robust game approach was taken in Pita, John, Maheswaran, Tambe, Yang, and Kraus (2012) to incorporate better models of human decision-making. However, most previous works concentrated on the uncertainties in payoff functions or strategy variables and did not study the uncertainties in the parameters of the accompanied constraints. In addition, distributed algorithms for robust games deserve further investigation, considering the status that distributed NE seeking thrives a lot.

The above motivates this paper to study distributed seeking GNE of a game with linear coupled constraints, where constraint parameters come from general uncertain convex sets. Challenges come from the general uncertain convex sets, since they may not be equipped with exact analytic expressions due to the complexity of uncertainty modeling. Thereby, we use inscribed polyhedrons to approximate the general uncertain convex sets, and employ robust optimization to transform the original problem in the worst case to a variational inequality (VI) problem. Then we design a distributed algorithm for seeking an ε-GNE, followed by the convergence analysis and illustration.

The main contributions of this paper are listed in the following.

- We consider robust games with coupled constraints where constraint parameters are in general uncertain convex sets. We propose to seek ε-GNE by approximating the parameter uncertainty sets. The uncertainty in game models with coupled constraints has not been considered in previous works (Liang, Wang, & Yin, 2020; Liang et al., 2017; Paccagnan et al., 2016). Moreover, the parameter uncertainty sets are quite general but not restricted to special structures (Wang, Peng, Jin, & Zhao, 2014; Zeng, Yi, & Hong, 2018).
- After a proper approximation of the parameter uncertainty sets by inscribed polyhedrons, we transform the original game in the worst case into a variational inequality problem with the help of robust optimization. By virtue of variational inequalities, convex set geometry, and metric spaces, we show that a solution to the variational inequality contains an ε-GNE of the original game in the worst case.
- Based on the transformed variational inequality, we propose a distributed continuous-time algorithm for seeking an ε-GNE, with projection maps and gradient descent methods. Then we give its convergence analysis by Lyapunov stability theory.

The rest of this paper is organized as follows: Section 2 provides notations and preliminary knowledge, while Section 3 formulates a distributed game problem with coupled constraints, where parameters of constraints are in general uncertain convex sets. Then Section 4 transforms the original problem in the worst case into a variational inequality problem after a proper approximation, and shows that solutions to the variational inequality induce ε-GNE of the original game. Section 5 proposes a distributed continuous-time algorithm to solve for the ε-GNE and gives the convergence analysis followed by a numerical example. Finally, Section 6 concludes the paper.

2. Preliminaries

In this section, we list some notations needed in this paper and introduce related preliminary knowledge about graphs, convex sets, variational inequalities, and the Hausdorff metric.

Denote $\mathbb{R}^n$ (or $\mathbb{R}^{m\times n}$) as the set of n-dimension (or m-by-n) real column vectors (or real matrices), and I_n as the $n\times n$ identity matrix. Let $\mathbf{1}_n$ (or $\mathbf{0}_n$) be the n-dimensional column vector with all elements of 1 (or 0). Denote $A\otimes B$ as the Kronecker product of matrices A and B. Take $col(x_1,\dots,x_n)=(x_1^{\mathrm{T}},\dots,x_n^{\mathrm{T}})^{\mathrm{T}}$ and $\|\cdot\|$ as the Euclidean norm. Denote $\ker(A)$ as the kernel of the matrix A, $\mathrm{Im}(A)$ as the image space of the matrix A and $\mathrm{span}(v)$ as the spanning subspace by vector v. Denote $\mathbf{B}_\rho(x)\subseteq\mathbb{R}^n$ as a ball with the center at point x and the radius ρ.

An undirected graph $\mathcal{G}$ is defined by $\mathcal{G}(\mathcal{V},\mathcal{E})$, where $\mathcal{V}=\{1,\dots,n\}$ is the set of nodes and $\mathcal{E}\subset\mathcal{V}\times\mathcal{V}$ is the set of edges. $\mathcal{A}=[a_{i,j}]\in\mathbb{R}^{n\times n}$ is the *adjacency matrix* such that $a_{i,j}=a_{j,i}>0$ if $\{j,i\}\in\mathcal{E}$ and $a_{i,j}=0$ otherwise. The *Laplacian matrix* is $L=\mathcal{D}-\mathcal{A}$, where $\mathcal{D}\in\mathbb{R}^{n\times n}$ is diagonal with $\mathcal{D}_{i,i}=\sum_{j=1}^n a_{i,j}$, $i\in\{1,\dots,n\}$. Specifically, if the graph $\mathcal{G}$ is connected, the *Laplacian matrix* L satisfies: $L=L^{\mathrm{T}}\geq 0$, $\mathrm{rank}\,L=n-1$, and $\ker(L)=\{k1_n:k\in\mathbb{R}\}$.

For a convex set $\Omega\subseteq\mathbb{R}^n$, a projection map $P_\Omega:\mathbb{R}^n\to\Omega$ is defined as

$$P_\Omega(x)=\mathrm{argmin}_{y\in\Omega}\|x-y\|.$$

The following two basic properties hold:

$$(x-P_\Omega(x))^{\mathrm{T}}(P_\Omega(x)-y)\geq 0,\quad \forall y\in\Omega,$$
$$\|P_\Omega(x)-P_\Omega(y)\|\leq\|x-y\|,\quad \forall x,y\in\mathbb{R}^n.$$

For $x\in\Omega$, denote the normal cone to Ω at x by

$$\mathcal{N}_\Omega(x)=\left\{v\in\mathbb{R}^n: v^{\mathrm{T}}(y-x)\leq 0,\quad \forall y\in\Omega\right\},$$

and the tangent cone to Ω at x by

$$\mathcal{T}_\Omega(x)=\{\lim_{k\to\infty}\frac{x_k-x}{t_k}:x_k\in\Omega,t_k>0,x_k\to x,t_k\to 0\}.$$

A function $f:\mathbb{R}^n\to\mathbb{R}$ is said to be convex if

$$f(\lambda x_1+(1-\lambda)x_2)\leq\lambda f(x_1)+(1-\lambda)f(x_2),$$

for any $x_1,x_2\in\mathbb{R}^n$ and $\lambda\in(0,1)$. Given $\omega>0$, a vector-valued function F is ω-strongly monotone on C if

$$(x-y)^{\mathrm{T}}(f(x)-f(y))\geq\omega\|x-y\|^2,\ \forall x,y\in C.$$

Given a set $C\subseteq\mathbb{R}^n$ and a map $F:C\to\mathbb{R}^n$, the variational inequality $\mathrm{VI}(C,F)$ is to find a vector $x\in C$ such that

$$(y-x)^{\mathrm{T}}F(x)\geq 0,\ \forall y\in C,$$

with its solution denoted by SOL(C, F). When C is closed and convex, according to Facchinei and Pang (2007),

$$x \in \mathrm{SOL}(C, F) \Leftrightarrow 0_n \in F(x) + \mathcal{N}_C(x).$$

Let $X, Y \subseteq \mathbb{R}^n$ be two non-empty sets. For $z \in \mathbb{R}^n$ and $X \subseteq \mathbb{R}^n$, denote $dist(z, X)$ as the distance between z and X, *i.e.*,

$$dist(z, X) = \inf_{x \in X} \|z - x\|.$$

Define the Hausdorff metric of $X, Y \subseteq \mathbb{R}^n$ by

$$\delta_H(X, Y) = max\{\sup_{x \in X} dist(x, Y), \sup_{y \in Y} dist(y, X)\}.$$

The Hausdorff metric makes the integration of all compact sets of $\mathbb{R}^n$ into a metric space.

3. Problem formulation

Consider N players indexed by $\mathcal{P} = \{1, \ldots, N\}$. For each $i \in \mathcal{P}$, the ith player has an action variable x_i in an action set $\Omega_i \subseteq \mathbb{R}^n$. Denote $\boldsymbol{\Omega} = \prod_{i=1}^N \Omega_i$, $\boldsymbol{x} = col\{x_1, \ldots, x_N\} \in \mathbb{R}^{nN}$ as the profile of all the actions, and $\boldsymbol{x}_{-i}$ as the vector formed by all the action variables except the ith player's.

The ith player has a payoff function $J_i(x_i, \boldsymbol{x}_{-i}) : \mathbb{R}^{nN} \to \mathbb{R}$ depending on both x_i and $\boldsymbol{x}_{-i}$. All players' action variables need to satisfy *a coupled inequality constraint*, where parameters in constraints are given in general uncertain convex sets. Given $\boldsymbol{x}_{-i}$, the ith player intends to solve

$$\begin{aligned} &\min_{x_i \in \Omega_i} J_i(x_i, \boldsymbol{x}_{-i}) \\ &\text{s.t.} \quad \sum_{j=1}^N a_j^{\mathrm{T}} x_j \le b_0, \quad a_j \in \mathcal{U}_j \subseteq \mathbb{R}^n, \quad \forall j \in \mathcal{P}, \end{aligned} \tag{1}$$

where $\mathcal{U}_i$ is convex and compact. Note that the inequality constraint must be satisfied for any $a_j \in \mathcal{U}_j$.

Definition 1. For a given $\varepsilon \ge 0$, a profile $\boldsymbol{x}^*$ is said to be an ε-generalized Nash equilibrium (ε-GNE) of game (1) if for every $i \in \mathcal{P}$, $x_i^* \in \Omega_i$, $(x_i^*, \boldsymbol{x}_{-i}^*)$ satisfying the constraint in (1) and

$$J_i(x_i^*, \boldsymbol{x}_{-i}^*) \le J_i(x_i, \boldsymbol{x}_{-i}^*) + \varepsilon,$$

for any $x_i \in \Omega_i$ and $(x_i, \boldsymbol{x}_{-i}^*)$ satisfying the constraint in (1). Particularly, $\boldsymbol{x}^*$ is said to be a generalized Nash equilibrium (GNE) when $\varepsilon = 0$.

Some basic assumptions, widely used in Paccagnan et al. (2016), Yi and Pavel (2017) and Zhu and Frazzoli (2016), are listed for the wellposedness of (1).

Assumption 1.

- For $i \in \mathcal{P}$, $J_i(\cdot)$ is Lipschitz continuous w.r.t. $\boldsymbol{x}$, while $J_i(\cdot, \boldsymbol{x}_{-i})$ is differentiable w.r.t. x_i. Moreover, the pseudo-gradient
$$F(\boldsymbol{x}) \triangleq col\{\nabla_{x_1} J_1(\cdot, \boldsymbol{x}_{-1}), \ldots, \nabla_{x_N} J_N(\cdot, \boldsymbol{x}_{-N})\}$$
is ω-strongly monotone on set $\boldsymbol{\Omega}$.
- Set $\boldsymbol{\Omega}$ is closed and convex. In addition, there exists $\boldsymbol{x} \in$ rint$(\boldsymbol{\Omega})$ such that
$$\sum_{j=1}^N a_j^{\mathrm{T}} x_j < b_0, \quad a_j \in \mathcal{U}_j \subseteq \mathbb{R}^n, \quad \forall j \in \mathcal{P}.$$

It is the main task of this paper to find a GNE of (1) in the worst case. However, it is very challenging to do so directly because the constraint parameter a_i is arbitrarily chosen in a general convex set $\mathcal{U}_i$, which may not have an analytic expression. Therefore, we have to consider how to seek an ε-GNE of (1) in the worst case by a proper approximation.

In our distributed problem, each player only has access to its own local payoff function and decision set. The coupled constraint is not fully available to all players, e.g., the ith player can only access $a_i^{\mathrm{T}} x_i$ and the parameter uncertainty set $\mathcal{U}_i$, rather than $\sum_{j=1}^N a_j^{\mathrm{T}} x_j$. Moreover, each player can observe the action variables that have direct impact on his payoff function (moreover, the gradient). To fulfill the task, players can exchange local information through a communication network $\mathcal{G}$ with the following assumptions.

Assumption 2.

- The undirected graph $\mathcal{G}$ is connected.
- The decision variable x_j is observable by the ith player, if $J_i(x_i, \boldsymbol{x}_{-i})$ depends explicitly on x_j, for any $j \in \mathcal{P}$.

4. Problem approximation

In this section, we provide a scheme to approximate the original problem (1) in the worst case with the following four steps, that is,

(i) Employ inscribed polyhedrons to approximate parameter uncertainty convex sets $\mathcal{U}_i$ for $i \in \mathcal{P}$;
(ii) Handle the coupled constraint with parameter uncertainties in the worst case;
(iii) With the approximation by inscribed polyhedrons, derive a variational inequality from the original problem in the worst case;
(iv) Show that a solution to the variational inequality induces an ε-GNE of the original game (1).

An inscribed polyhedron of a convex set is defined that all its vertices are on the boundary of the convex set. Denote $\boldsymbol{\mathcal{U}} = \prod_{i=1}^N \mathcal{U}_i$ and $\boldsymbol{R}_k = \prod_{i=1}^N R_{k_i}^i$, where $R_{k_i}^i$ is an inscribed polyhedron of $\mathcal{U}_i$ with k_i vertices, expressed as

$$R_{k_i}^i = \{a_i \in \mathbb{R}^n : D^i a_i \le d^i\}.$$

Here $D^i \in \mathbb{R}^{m_i \times n}$ with normalized rows, $d^i \in \mathbb{R}^{m_i}$, and m_i actually refers to the number of hyperplanes enclosing the inscribed polyhedron for $i \in \mathcal{P}$. With the help of approximation by inscribed polyhedrons, we can investigate the worst-case solution based on robust optimization (Bertsimas et al., 2011) and robust games (Aghassi & Bertsimas, 2006).

Remark 1. In fact, there are various ways to approximate a convex set, such as polyhedrons, ellipsoids and Cardinalities. Here we choose polyhedrons because they can be expressed explicitly by linear inequalities, which provide simple mathematical derivation and make the distributed algorithms concise. Furthermore, although the analytic expressions of the convex sets may not be obtained directly and clearly, in some situations one can sample exactly a few points on the boundary of the convex set. These points form an inscribed polyhedron naturally and this is another important reason for choosing inscribed polyhedrons.

With help of the approximation by inscribed polyhedrons, we consider the following game in the worst case, and study the relation between the following game and the original game (1).

$$\begin{aligned} &\min_{x_i \in \Omega_i} J_i(x_i, \boldsymbol{x}_{-i}) \\ &\text{s.t.} \quad \sum_{j=1}^N a_j^{\mathrm{T}} x_j \le b_0, \quad a_j \in R_{k_j}^j, \quad \forall j \in \mathcal{P}, \end{aligned} \tag{2}$$

We employ ideas in robust optimization to handle game (2) in the worst case, and show the process of transformation in the following theorem.

Theorem 1. *Under Assumption 1, a vector $\boldsymbol{x}^*$ is a GNE of the game (2) in the worst case if and only if there exists a vector $\boldsymbol{\lambda} = col\{\lambda_1, \dots, \lambda_N\}$ with $\lambda_i \in \mathbb{R}_+^{m_i}$ such that $\{\boldsymbol{x}^*, \boldsymbol{\lambda}^*\}$ is a GNE to the following game.*

$$\min_{x_i \in \Omega_i, \lambda_i \in \mathbb{R}_+^{m_i}} J_i(x_i, \boldsymbol{x}_{-i})$$

$$s.t. \sum_{j=1}^{N} \lambda_j^{\mathrm{T}} d^j \le b_0, \quad D^{j\mathrm{T}}\lambda_j - x_j = \mathbf{0}_n, \quad \forall j \in \mathcal{P}. \tag{3}$$

Proof. With $a_i \in R_{k_i}^i$, the coupled constraint in (2) in the worst case is

$$\sum_{j=1}^{N} \max_{a_j \in R_{k_j}^j} a_j^{\mathrm{T}} x_j \le b_0. \tag{4}$$

Separate the subproblem in (4) as an independent optimization problem:

$$\max_{a_j} a_j^{\mathrm{T}} x_j \quad \text{s.t.} \quad D^j a_j \le d^j. \tag{5}$$

The dual problem of (5) (here a_j is the variable of optimization and x_j is a coefficient vector) is

$$\min_{\lambda_j} \lambda_j^{\mathrm{T}} d^j$$

$$\text{s.t.} \quad D^{j\mathrm{T}}\lambda_j - x_j = \mathbf{0}_n, \ \lambda_j \ge \mathbf{0}_{m_j}, \tag{6}$$

where $\lambda_j \in \mathbb{R}^{m_j}$ is the dual variable. Since $R_{k_j}^j$ is nonempty, the duality gap vanishes and the maximum of $a_j^T x_j$ in (5) is equal to the minimum of $\lambda_j^T d^j$ in (6). Hence, the coupled constraint (4) becomes

$$\sum_{j=1}^{N} \min_{\lambda_j \in \mathbb{R}_+^{m_j}} \lambda_j^{\mathrm{T}} d^j \le b_0,$$

$$\text{s.t.} \quad D^{j\mathrm{T}}\lambda_j - x_j = \mathbf{0}_n, \quad \forall j \in \mathcal{P}. \tag{7}$$

Referring to section 2.2 in Bertsimas et al. (2011), we can remove "min" on the left-hand side of (7) because if there exists at last one $\boldsymbol{\lambda}$ that satisfies (7), the minimum does as well, which yields (3).

On the other hand, if there exist $\boldsymbol{\lambda}$ and $\boldsymbol{x}$ that satisfy (3), then (7) holds naturally. Due to the equivalent transformation between (4) and (7), solutions to problem (3) contain equilibria of game (2) in the worst case. ■

Then we discuss the equilibria to game (3), in order to obtain the equilibria to game (2). Since both $\boldsymbol{x}$ and $\boldsymbol{\lambda}$ are variables in the transformed game (3), take variable $z_i = col\{x_i, \lambda_i\} \in \mathbb{R}^{n+m_i}$, $\boldsymbol{z}_{-i}$ as all the vectors except z_i and $\boldsymbol{z} = col\{z_1, \dots, z_N\} \in \mathbb{R}^{nN+M}$, where $M = \sum_{j=1}^{N} m_j$. For the ith player, let

$$\Xi_i(\boldsymbol{z}_{-i}) \triangleq \{z_i \in \Theta_i | \sum_{j=1}^{N} A_j z_j \le b_0, \ C_i z_i = \mathbf{0}_n\},$$

where $\Theta_i = \Omega_i \times \mathbb{R}_+^{m_i}$, and

$$A_i = [\mathbf{0}_n^{\mathrm{T}}, d^{i\mathrm{T}}] \in \mathbb{R}^{1\times(n+m_i)},$$

$$C_i = [-I_n, D^{i\mathrm{T}}] \in \mathbb{R}^{n\times(n+m_i)}.$$

Moreover, denote $\boldsymbol{\Theta} = \prod_{i=1}^{N} \Theta_i$ and

$$\boldsymbol{\Xi} = \{\boldsymbol{z} \in \boldsymbol{\Theta} | \sum_{j=1}^{N} A_j z_j \le b_0, \ C_i z_i = \mathbf{0}_n, \ \forall i \in \mathcal{P}\}.$$

For the notations about the pseudo-gradient, let

$$\boldsymbol{G}(\boldsymbol{z}) \triangleq col\{G_1(z_1, \boldsymbol{z}_{-1}), \dots, G_N(z_N, \boldsymbol{z}_{-N})\} \in \mathbb{R}^{nN+M},$$

where $G_i(z_i, \boldsymbol{z}_{-i}) \triangleq col\{\nabla_{x_i} J_i(\cdot, \boldsymbol{x}_{-i}), \mathbf{0}_{m_i}\} \in \mathbb{R}^{n+m_i}$, for $i \in \mathcal{P}$.

Define a variational inequality problem

$$\mathrm{VI}(\boldsymbol{\Xi}, \boldsymbol{G}(\boldsymbol{z})).$$

It follows from Facchinei and Pang (2007) that if $\boldsymbol{z} \in \mathrm{SOL}(\boldsymbol{\Xi}, \boldsymbol{G}(\boldsymbol{z}))$, then $\boldsymbol{z}$ is a GNE of the transformed problem (3). Furthermore, for the distributed design, we investigate a variational GNE with the same multiplier of the coupled linear inequality constraint, which is widely discussed in Liang et al. (2017), Paccagnan et al. (2016) and Kulkarni and Shanbhag (2012). Under Assumptions 1 and 2, we give the first order conditions (referring to Facchinei & Kanzow, 2010) for a variational GNE of (3), which also serves as a solution $\boldsymbol{z} \in \mathrm{SOL}(\boldsymbol{\Xi}, \boldsymbol{G}(\boldsymbol{z}))$.

$$0 \in \boldsymbol{G}(\boldsymbol{z}) + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{\gamma} + \boldsymbol{C}^{\mathrm{T}}\boldsymbol{\zeta} + \mathcal{N}_{\boldsymbol{\Theta}}(\boldsymbol{z}), \tag{8a}$$

$$0 \le -(\boldsymbol{A}\boldsymbol{z} - \boldsymbol{b})^{\mathrm{T}} \cdot \mathbf{1}_N, \quad 0 = (\boldsymbol{A}\boldsymbol{z} - \boldsymbol{b})^{\mathrm{T}}\boldsymbol{\gamma}, \tag{8b}$$

$$0 = \boldsymbol{C}\boldsymbol{z}, \quad 0 = L\boldsymbol{\gamma}, \tag{8c}$$

where

$$\boldsymbol{A} = Diag(A_1, \dots, A_N) \in \mathbb{R}^{N\times(nN+M)},$$

$$\boldsymbol{b} = col\{b_1, \dots, b_N\} \in \mathbb{R}^N, \quad \text{with} \sum_{i=1}^{N} b_i = b_0,$$

$$\boldsymbol{C} = Diag(C_1 \dots, C_N) \in \mathbb{R}^{nN\times(nN+M)},$$

and multipliers $\boldsymbol{\gamma} = col\{\gamma_1, \dots, \gamma_N\} \in \mathbb{R}_+^N$, $\boldsymbol{\zeta} = col\{\zeta_1, \dots, \zeta_N\} \in \mathbb{R}^{nN}$. Here L is the Laplacian matrix of the network $\mathcal{G}$. Since L is symmetric and positive semidefinite with $\ker(L) = \{k1_n : k \in \mathbb{R}\}$, the last equation $0 = L\boldsymbol{\gamma}$ is equivalent to

$$\gamma_1 = \gamma_2 = \dots = \gamma_N,$$

which conforms with the definition of variational GNE.

By solving the first order conditions (8) of the variational inequality $\mathrm{VI}(\boldsymbol{\Xi}, \boldsymbol{G}(\boldsymbol{z}))$, we derive a variational GNE of game (3). Furthermore, this solution contains a GNE of game (2) in the worst case according to Theorem 1. Then, it is time to reveal the relationship between the variational GNE $\boldsymbol{z} \in \mathrm{SOL}(\boldsymbol{\Xi}, \boldsymbol{G}(\boldsymbol{z}))$ satisfying (8) and the original game (1) in the worst case.

Theorem 2. *Under Assumptions 1 and 2, the variational GNE of game (1) in the worst case exists and is unique. Moreover, a solution to the variational inequality $VI(\boldsymbol{\Xi}, \boldsymbol{G}(\boldsymbol{z}))$ satisfying (8) induces an ε-GNE of game (1) in the worst case.*

Before proving Theorem 2, we need more discussion about the inscribed polyhedrons and the solutions to the variational inequalities. Under Assumption 1, the pseudo-gradient of the original payoff functions is ω-strongly monotone w.r.t. $\boldsymbol{x}$, which implies that $\boldsymbol{z}^* \in \mathrm{SOL}(\boldsymbol{\Xi}, \boldsymbol{G}(\cdot))$ contains a unique $\boldsymbol{x}^*$, but the optimal $\boldsymbol{\lambda}^*$ may not be unique. Moreover, if we fix the form of payoff functions J_i for $i \in \mathcal{P}$, then with a different polyhedron approximation, the resulting variational inequality solution is different. Thus, we can write $\boldsymbol{x}^* = \boldsymbol{x}^*(\boldsymbol{R}_k)$ since $\boldsymbol{R}_k$ determines $\boldsymbol{\Xi}$.

Suppose that

$$\boldsymbol{R}_{k_1} = \prod_{i=1}^{N} R_{k_{1,i}}^i, \quad \boldsymbol{R}_{k_2} = \prod_{i=1}^{N} R_{k_{2,i}}^i \tag{9}$$

are two inscribed polyhedrons of $\boldsymbol{\mathcal{U}}$. For $i \in \mathcal{P}$, denote V_1^i and $V_2^i = V_1^i \cup \{v_0^i\}$ as sets of vertices in $R^i_{k_{1,i}}$ and $R^i_{k_{2,i}}$, respectively, where v_0^i denotes an additional vertex. The following lemma shows how two different inscribed polyhedrons influence the solution to VI($\boldsymbol{\Xi}, \boldsymbol{G}(\boldsymbol{z})$), whose proof can be found in Appendix A.

Lemma 1. *Under Assumption 1, for any $\varepsilon > 0$, there exists $\eta > 0$ such that if $\delta_H(R^i_{k_{1,i}}, R^i_{k_{2,i}}) \le \eta$ for $i \in \mathcal{P}$, then the two variational inequality solutions satisfy*

$$\|\boldsymbol{x}^*(\boldsymbol{R}_{k_1}) - \boldsymbol{x}^*(\boldsymbol{R}_{k_2})\| \le \varepsilon.$$

The following lemma reveals the Hausdorff metric between a convex set and its inscribed polyhedrons.

Lemma 2 (Bronstein, 2008)**.** *Given an arbitrary inscribed polyhedron R_k of the convex set U, the upper bound of the Hausdorff metric between U and R_k satisfies*

$$\delta_H(R_k, U) \le \frac{c(U)}{k^{2/(n-1)}}, \tag{10}$$

where $c(U)$ is constant related with the curvature of U, n serves as the dimension of the Euclidean space, and k refers to the number of vertices in R_k.

Now it is time for the proof of Theorem 2.

Proof. We introduce $\boldsymbol{R}_{k_1}$ and $\boldsymbol{R}_{k_2}$ defined in (9) as two profiles of inscribed polyhedrons of $\boldsymbol{\mathcal{U}}$, and suppose that $\delta_H(R^i_{k_{1,i}}, R^i_{k_{2,i}}) \le \eta$ for $i \in \mathcal{P}$. Denote $\boldsymbol{R}_{k_1+N} = \prod_{i=1}^N R^i_{k_{1,i}+1}$, where vertices in $R^i_{k_{1,i}+1}$ consist of all vertices in $R^i_{k_{1,i}}$ and one different vertex in $R^i_{k_{2,i}}$. Similarly, denote $\boldsymbol{R}_{k_2+N}$, $\boldsymbol{R}_{k_1+2N}$, $\boldsymbol{R}_{k_2+2N}$ and so on and $\boldsymbol{R}_{k_1+k_2}$ as the profile of polyhedrons whose vertices consist of all the vertices in both $\boldsymbol{R}_{k_1}$ and $\boldsymbol{R}_{k_2}$. Due to the Hausdorff metric on convex and compact sets,

$$\delta_H(R^i_{k_{1,i}}, R^i_{k_{1,i}+1}) \le \delta_H(R^i_{k_{1,i}}, R^i_{k_{2,i}}) \le \eta.$$

Also,

$$\begin{aligned}
&\|\boldsymbol{x}^*(\boldsymbol{R}_{k_1}) - \boldsymbol{x}^*(\boldsymbol{R}_{k_2})\| \\
\le&\|\boldsymbol{x}^*(\boldsymbol{R}_{k_1}) - \boldsymbol{x}^*(\boldsymbol{R}_{k_1+k_2})\| + \|\boldsymbol{x}^*(\boldsymbol{R}_{k_2}) - \boldsymbol{x}^*(\boldsymbol{R}_{k_1+k_2})\| \\
\le&\|\boldsymbol{x}^*(\boldsymbol{R}_{k_1}) - \boldsymbol{x}^*(\boldsymbol{R}_{k_1+N})\| \\
&+ \|\boldsymbol{x}^*(\boldsymbol{R}_{k_1+N}) - \boldsymbol{x}^*(\boldsymbol{R}_{k_1+2N})\| \\
&+ \cdots + \|\boldsymbol{x}^*(\boldsymbol{R}_{k_1+k_2-N}) - \boldsymbol{x}^*(\boldsymbol{R}_{k_1+k_2})\| \\
&+ \|\boldsymbol{x}^*(\boldsymbol{R}_{k_2}) - \boldsymbol{x}^*(\boldsymbol{R}_{k_2+N})\| \\
&+ \|\boldsymbol{x}^*(\boldsymbol{R}_{k_2+N}) - \boldsymbol{x}^*(\boldsymbol{R}_{k_2+2N})\| \\
&+ \cdots + \|\boldsymbol{x}^*(\boldsymbol{R}_{k_1+k_2-N}) - \boldsymbol{x}^*(\boldsymbol{R}_{k_1+k_2})\|.
\end{aligned}$$

We learn from the above inequalities that the difference between the two variational inequality solutions $\|\boldsymbol{x}^*(\boldsymbol{R}_{k_1}) - \boldsymbol{x}^*(\boldsymbol{R}_{k_2})\|$ has been decomposed. We only need to investigate $\|\boldsymbol{x}^*(\boldsymbol{R}_{k_1}) - \boldsymbol{x}^*(\boldsymbol{R}_{k_1+N})\|$ since all terms above share a similar structure. For any $\varepsilon > 0$, it follows from Lemma 1 that there exists η such that if $\delta_H(R^i_{k_{1,i}}, R^i_{k_{1,i}+1}) \le \eta$, then $\|\boldsymbol{x}^*(\boldsymbol{R}_{k_1}) - \boldsymbol{x}^*(\boldsymbol{R}_{k_1+N})\| \le \varepsilon$, which implies

$$\|\boldsymbol{x}^*(\boldsymbol{R}_{k_1}) - \boldsymbol{x}^*(\boldsymbol{R}_{k_2})\| \le k\varepsilon.$$

That is to say, $\boldsymbol{x}^*(\boldsymbol{R}_k)$ is continuous with respect to $\boldsymbol{R}_k$ in the Hausdorff metric. In addition, it follows from Lemma 2 that $\lim_{k\to\infty} \delta_H(\boldsymbol{R}_{k_1}, \boldsymbol{\mathcal{U}}) = 0$. Therefore, there exist a unique $\boldsymbol{x}^*(\boldsymbol{\mathcal{U}})$ such that

$$\lim_{k\to\infty} \boldsymbol{x}^*(\boldsymbol{R}_k) = \boldsymbol{x}^*(\boldsymbol{\mathcal{U}}),$$

which conforms that the variational GNE of game (1) exists and is unique.

Based on the definition of ε-GNE, we need to investigate the difference between $J_i(\boldsymbol{x}^*(\boldsymbol{R}_k))$ and $J_i(x_i', \boldsymbol{x}^*_{-i}(\boldsymbol{R}_k))$, where the ith player's strategy is $x_i^*(\boldsymbol{R}_k)$ (based on the equilibrium with polyhedron $\boldsymbol{R}_k$) and x_i' (arbitrarily chosen from Ω_i), respectively. Meanwhile, other players' strategies remain the same $\boldsymbol{x}^*_{-i}(\boldsymbol{R}_k)$. When $\delta_H(R^i_{k_i}, \mathcal{U}_i) \le \eta$ for $i \in \mathcal{P}$, it follows form Lemma 1 that

$$\begin{aligned}
&J_i(\boldsymbol{x}^*(\boldsymbol{R}_k)) - J_i(x_i', \boldsymbol{x}^*_{-i}(\boldsymbol{R}_k)) \\
=&J_i(\boldsymbol{x}^*(\boldsymbol{R}_k)) - J_i(\boldsymbol{x}^*(\boldsymbol{\mathcal{U}})) \\
&+ J_i(\boldsymbol{x}^*(\boldsymbol{\mathcal{U}})) - J_i(x_i', \boldsymbol{x}^*_{-i}(\boldsymbol{\mathcal{U}})) \\
&+ J_i(x_i', \boldsymbol{x}^*_{-i}(\boldsymbol{\mathcal{U}})) - J_i(x_i', \boldsymbol{x}^*_{-i}(\boldsymbol{R}_k)) \\
\le&\|J_i(x_i', \boldsymbol{x}^*_{-i}(\boldsymbol{\mathcal{U}})) - J_i(x_i', \boldsymbol{x}^*_{-i}(\boldsymbol{R}_k))\| \\
&+ \|J_i(\boldsymbol{x}^*(\boldsymbol{R}_k)) - J_i(\boldsymbol{x}^*(\boldsymbol{\mathcal{U}}))\| \\
&+ J_i(\boldsymbol{x}^*(\boldsymbol{\mathcal{U}})) - J_i(x_i', \boldsymbol{x}^*_{-i}(\boldsymbol{\mathcal{U}})) \\
\le&c\|\boldsymbol{x}^*(\boldsymbol{R}_k) - \boldsymbol{x}^*(\boldsymbol{\mathcal{U}})\| + c\|\boldsymbol{x}^*_{-i}(\boldsymbol{\mathcal{U}}) - \boldsymbol{x}^*(\boldsymbol{R}_k)\| + 0 \\
\le&2c\varepsilon,
\end{aligned}$$

where c is the Lipschitz constant of J_i. This completes the proof. ■

It is not hard to find in the proofs that the accuracy of the ε-GNE is influenced by the Lipschitz constants of payoff functions $J_i(\boldsymbol{x})$, geometric structures of convex sets $\mathcal{U}_i$ (referring to the constant $c(U)$ with $U = \mathcal{U}_i$ in Lemma 2), and number of vertices of the approximate inscribed polyhedrons. However, when we pick polyhedrons with more vertices for higher accuracy, we derive more corresponding hyperplanes, which lead to more rows of matrices D^i, vectors d^i, and λ_i.

5. Distributed algorithm and analysis

In this section, based on the analysis about the approximation, we propose a distributed continuous-time algorithm for solutions to (8) to derive ε-GNE of the original game (1), and show the convergence analysis followed by a numerical example.

In the multi-agent framework, since each player has the own choice for approximation, inscribed polyhedron $R^i_{k_i}$ is the private information of the ith player. Thus, according to the expression of $R^i_{k_i}$, A_i, b_i and C_i are all regarded as private knowledge. Under this condition, we propose a distributed algorithm for seeking solutions $\boldsymbol{z} \in \mathrm{SOL}(\boldsymbol{\Xi}, \boldsymbol{G}(\boldsymbol{z}))$ to (8) in Algorithm 1.

Algorithm 1 (for each $i \in \mathcal{P}$)

Initialization:

$$z_i(0) \in \Theta_i,\ \nu_i(0) \in \mathbb{R},\ \gamma_i(0) \in \mathbb{R}_+,\ \zeta_i(0) \in \mathbb{R}^n.$$

Dynamics renewal:

$$\begin{aligned}
\dot{z}_i =&P_{\mathcal{T}_{\Theta_i}(z_i)}(-G_i(z_i, \boldsymbol{z}_{-i}) - A_i^{\mathrm{T}}\gamma_i - C_i^{\mathrm{T}}\zeta_i), \\
\dot{\nu}_i =&\sum_{j=1}^N a_{ij}(\gamma_i - \gamma_j), \\
\dot{\gamma}_i =&P_{\mathcal{T}_{\mathbb{R}_+}(\gamma_i)}\Big[A_i z_i - b_i - \sum_{j=1}^N a_{ij}(\gamma_i - \gamma_j) - \sum_{j=1}^N a_{ij}(\nu_i - \nu_j)\Big], \\
\dot{\zeta}_i =&C_i z_i.
\end{aligned}$$

where $P_{\mathcal{T}_{\theta_i}(z_i)}(\cdot)$ and $P_{\mathcal{T}_{\mathbb{R}_+}(\gamma_i)}(\cdot)$ are projection maps on the tangent cone about θ_i at z_i and on the tangent cone about $\mathbb{R}_+$ at γ_i, respectively, and $a_{i,j}$ is the (i, j)th element of the adjacency matrix.

In the distributed continuous-time Algorithm 1, the ith player calculates local decision variable $z_i \in \theta_i$, local multiplier $\zeta_i \in \mathbb{R}^n$

for equality constraints, local multiplier $\gamma_i \in \mathbb{R}_+$ for the estimation of the same shadow price γ^*, and local auxiliary variable $\nu_i \in \mathbb{R}$ to reach consensus of the local multipliers γ_i.

Equivalently, Algorithm 1 can be expressed in a compact form as follows, with the defined compact notations in (8).

$$\begin{cases} \dot{\boldsymbol{z}} = P_{\mathcal{T}_{\boldsymbol{\Theta}}(\boldsymbol{z})}(-\boldsymbol{G}(\boldsymbol{z}) - \boldsymbol{A}^{\mathrm{T}}\boldsymbol{\gamma} - \boldsymbol{C}^{\mathrm{T}}\boldsymbol{\zeta}), \\ \dot{\boldsymbol{\nu}} = L\boldsymbol{\gamma}, \\ \dot{\boldsymbol{\gamma}} = P_{\mathcal{T}_{\mathbb{R}^N_+}(\boldsymbol{\gamma})}(\boldsymbol{A}\boldsymbol{z} - \boldsymbol{b} - L\boldsymbol{\gamma} - L\boldsymbol{\nu}), \\ \dot{\boldsymbol{\zeta}} = \boldsymbol{C}\boldsymbol{z}. \end{cases} \tag{11}$$

Remark 2. Note that (11) is a non-smooth ordinary differential equation due to the projected operator on tangent cones. By Aubin and Cellina (1984), the right-side of (11) is locally bounded and upper semi-continuous with values in a nonempty, convex set. It follows from Brogliato and Daniilidis (2006) that the solution to algorithm (11) has been guaranteed and the solution satisfies $\boldsymbol{z}(t) \in \boldsymbol{\Theta}$ for all $t \geq 0$ if $\boldsymbol{z}(0) \in \boldsymbol{\Theta}$.

Lemma 3. *Under Assumptions 1 and 2, $\boldsymbol{z}^*$ is a solution to VI($\boldsymbol{\Xi}$, $\boldsymbol{G}(\boldsymbol{z})$) satisfying (8) if and only if there exist $\boldsymbol{\nu}^* \in \mathbb{R}^N$, $\boldsymbol{\gamma}^* \in \mathbb{R}^N_+$ and $\boldsymbol{\zeta}^* \in \mathbb{R}^{nN}$ satisfying*

$$0 = P_{\mathcal{T}_{\boldsymbol{\Theta}}(\boldsymbol{z}^*)}(-\boldsymbol{G}(\boldsymbol{z}^*) - \boldsymbol{A}^{\mathrm{T}}\boldsymbol{\gamma}^* - \boldsymbol{C}^{\mathrm{T}}\boldsymbol{\zeta}^*), \tag{12a}$$

$$0 = P_{\mathcal{T}_{\mathbb{R}^N_+}(\boldsymbol{\gamma}^*)}(\boldsymbol{A}\boldsymbol{z}^* - \boldsymbol{b} - L\boldsymbol{\gamma}^* - L\boldsymbol{\nu}^*), \tag{12b}$$

$$0 = L\boldsymbol{\gamma}^*, \quad 0 = \boldsymbol{C}\boldsymbol{z}^*. \tag{12c}$$

Proof. By properties of tangent cones and normal cones to a nonempty closed convex set (see in Brogliato & Daniilidis, 2006), for any two vectors $u, v \in \mathbb{R}^n$, $v = P_{\mathcal{T}_\Omega(x)}(u)$ is equivalent to

$$v \in \mathcal{T}_\Omega(x),\ u - v \in \mathcal{N}_\Omega(x),\ (u - v)^{\mathrm{T}} v = 0,$$

which implies that (12a) and (8a) are equivalent. In addition, since (8c) and (12c) are equivalent naturally, we only need to discuss (8b) and (12b).

First, (12b) induces

$$(\boldsymbol{A}\boldsymbol{z}^* - \boldsymbol{b} - L\boldsymbol{\gamma}^* - L\boldsymbol{\nu}^*)^{\mathrm{T}}(\boldsymbol{\gamma} - \boldsymbol{\gamma}^*) \leq 0 \quad \forall \boldsymbol{\gamma} \in \mathbb{R}^N_+.$$

Together with $L\boldsymbol{\gamma}^* = 0$, we have

$$-(\boldsymbol{A}\boldsymbol{z}^* - \boldsymbol{b})^{\mathrm{T}}\boldsymbol{\gamma}^* + (\boldsymbol{A}\boldsymbol{z}^* - \boldsymbol{b} - L\boldsymbol{\nu}^*)^{\mathrm{T}}\boldsymbol{\gamma} \leq 0.$$

On the one hand, let $\boldsymbol{\gamma}$ big enough and it leads to $\boldsymbol{A}\boldsymbol{z}^* - \boldsymbol{b} - L\boldsymbol{\nu}^* \leq 0$, otherwise it leads to a contradiction. Because $\boldsymbol{1}_N$ is the zero eigenvector of L, we have $(\boldsymbol{A}\boldsymbol{z}^* - \boldsymbol{b})^{\mathrm{T}} \cdot \boldsymbol{1}_N \leq 0$. Moreover, it follows $\boldsymbol{\gamma}^* \geq 0$ that

$$(\boldsymbol{A}\boldsymbol{z}^* - \boldsymbol{b} - L\boldsymbol{\nu}^*)^{\mathrm{T}}\boldsymbol{\gamma}^* \leq 0,$$

which implies $(\boldsymbol{A}\boldsymbol{z}^* - \boldsymbol{b})^{\mathrm{T}}\boldsymbol{\gamma}^* \leq 0$.

On the other hand, let $\boldsymbol{\gamma} = 0$, which yields $(\boldsymbol{A}\boldsymbol{z}^* - \boldsymbol{b})^{\mathrm{T}}\boldsymbol{\gamma}^* \geq 0$. Hence, $(\boldsymbol{A}\boldsymbol{z}^* - \boldsymbol{b})^{\mathrm{T}}\boldsymbol{\gamma}^* = 0$, which implies (8b). Next, suppose that (8) holds. Since $0 \geq (\boldsymbol{A}\boldsymbol{z}^* - \boldsymbol{b})^{\mathrm{T}} \cdot \boldsymbol{1}_N$, there exists an $\boldsymbol{w} \in \mathbb{R}^N_+$ such that

$$0 = (\boldsymbol{A}\boldsymbol{z}^* - \boldsymbol{b} + \boldsymbol{w})^{\mathrm{T}} \cdot \boldsymbol{1}_N.$$

Because $\ker(L) = \mathrm{span}\{\boldsymbol{1}_N\}$ and $\mathbb{R}^N = \ker(L) \oplus \mathrm{Im}(L)$, there exists $\boldsymbol{\nu}^* \in \mathrm{Im}(L)$ such that

$$L\boldsymbol{\nu}^* = \boldsymbol{A}\boldsymbol{z}^* - \boldsymbol{b} + \boldsymbol{w},$$

which implies $L\boldsymbol{\nu}^* \geq \boldsymbol{A}\boldsymbol{z}^* - \boldsymbol{b}$. Together with (8b),

$$-(\boldsymbol{A}\boldsymbol{z}^* - \boldsymbol{b})^{\mathrm{T}}\boldsymbol{\gamma}^* + (\boldsymbol{A}\boldsymbol{z}^* - \boldsymbol{b} - L\boldsymbol{\nu}^*)^{\mathrm{T}}\boldsymbol{\gamma} \leq 0,\ \forall \boldsymbol{\gamma} \in \mathbb{R}^N_+,$$

and (12b) holds. This completes the proof. ■

Lemma 3 shows the equivalence between equilibria of algorithm (11) and solutions to VI($\boldsymbol{\Xi}$, $\boldsymbol{G}(\boldsymbol{z})$) satisfying (8). Now we discuss the convergence of algorithm (11).

Theorem 3. *Under Assumptions 1 and 2, for any given initial condition $(\boldsymbol{z}(0), \boldsymbol{\nu}(0), \boldsymbol{\gamma}(0), \boldsymbol{\zeta}(0)) \in \boldsymbol{\Theta} \times \mathbb{R}^N \times \mathbb{R}^N_+ \times \mathbb{R}^{nN}$, the trajectory $(\boldsymbol{z}(t), \boldsymbol{\nu}(t), \boldsymbol{\gamma}(t), \boldsymbol{\zeta}(t))$ of (11) is bounded and $\boldsymbol{z}(t)$ converges to a solution to VI($\boldsymbol{\Xi}$, $\boldsymbol{G}(\boldsymbol{z})$) satisfying (8).*

Proof. Set a Lyapunov candidate function:

$$V(\boldsymbol{z}, \boldsymbol{\nu}, \boldsymbol{\gamma}, \boldsymbol{\zeta}) = \frac{1}{2}\|\boldsymbol{z} - \boldsymbol{z}^*\|^2 + \frac{1}{2}\|\boldsymbol{\nu} - \boldsymbol{\nu}^*\|^2 + \frac{1}{2}\|\boldsymbol{\gamma} - \boldsymbol{\gamma}^*\|^2 + \frac{1}{2}\|\boldsymbol{\zeta} - \boldsymbol{\zeta}^*\|^2,$$

where $(\boldsymbol{z}^*, \boldsymbol{\nu}^*, \boldsymbol{\gamma}^*, \boldsymbol{\zeta}^*)$ is an equilibrium point satisfying (12). It is clear that $V \geq 0$ and $V = 0$ if and only if $(\boldsymbol{z}, \boldsymbol{\nu}, \boldsymbol{\gamma}, \boldsymbol{\zeta}) = (\boldsymbol{z}^*, \boldsymbol{\nu}^*, \boldsymbol{\gamma}^*, \boldsymbol{\zeta}^*)$. According to projections on tangent cones,

$$(-\boldsymbol{G}(\boldsymbol{z}) - \boldsymbol{A}^{\mathrm{T}}\boldsymbol{\gamma} - \boldsymbol{C}^{\mathrm{T}}\boldsymbol{\zeta} - \dot{\boldsymbol{z}})^{\mathrm{T}}(\boldsymbol{z}' - \boldsymbol{z}) \leq 0, \quad \forall \boldsymbol{z}' \in \boldsymbol{\Theta},$$
$$(\boldsymbol{A}\boldsymbol{z} - \boldsymbol{b} - L\boldsymbol{\gamma} - L\boldsymbol{\nu} - \dot{\boldsymbol{\gamma}})^{\mathrm{T}}(\boldsymbol{\gamma}' - \boldsymbol{\gamma}) \leq 0, \quad \forall \boldsymbol{\gamma}' \in \mathbb{R}^N_+.$$

Hence, along the dynamics (11),

$$\begin{aligned} \frac{dV}{dt} &= (\boldsymbol{z} - \boldsymbol{z}^*)^{\mathrm{T}}\dot{\boldsymbol{z}} + (\boldsymbol{\nu} - \boldsymbol{\nu}^*)^{\mathrm{T}}\dot{\boldsymbol{\nu}} + (\boldsymbol{\gamma} - \boldsymbol{\gamma}^*)^{\mathrm{T}}\dot{\boldsymbol{\gamma}} + (\boldsymbol{\zeta} - \boldsymbol{\zeta}^*)^{\mathrm{T}}\dot{\boldsymbol{\zeta}} \\ &\leq (\boldsymbol{z} - \boldsymbol{z}^*)^{\mathrm{T}}(-\boldsymbol{G}(\boldsymbol{z}) - \boldsymbol{A}^{\mathrm{T}}\boldsymbol{\gamma} - \boldsymbol{C}^{\mathrm{T}}\boldsymbol{\zeta}) \\ &\quad + (\boldsymbol{\gamma} - \boldsymbol{\gamma}^*)^{\mathrm{T}}(\boldsymbol{A}\boldsymbol{z} - \boldsymbol{b} - L\boldsymbol{\gamma} - L\boldsymbol{\nu}) \\ &\quad + (\boldsymbol{\nu} - \boldsymbol{\nu}^*)^{\mathrm{T}}L\boldsymbol{\gamma} + (\boldsymbol{\zeta} - \boldsymbol{\zeta}^*)^{\mathrm{T}}\boldsymbol{C}\boldsymbol{z}. \end{aligned} \tag{13}$$

In addition, it follows from (12a) and (12b) that

$$0 \leq (-\boldsymbol{G}(\boldsymbol{z}^*) - \boldsymbol{A}^{\mathrm{T}}\boldsymbol{\gamma}^* - \boldsymbol{C}^{\mathrm{T}}\boldsymbol{\zeta}^*)^{\mathrm{T}}(\boldsymbol{z}^* - \boldsymbol{z}), \tag{14a}$$

$$0 \leq (\boldsymbol{A}\boldsymbol{z}^* - \boldsymbol{b} - L\boldsymbol{\gamma}^* - L\boldsymbol{\nu}^*)^{\mathrm{T}}(\boldsymbol{\gamma}^* - \boldsymbol{\gamma}). \tag{14b}$$

Substitute (14) into (13), and moreover, since $L\boldsymbol{\gamma}^* = 0$ and $\boldsymbol{C}\boldsymbol{z}^* = 0$,

$$\frac{dV}{dt} \leq -(\boldsymbol{z} - \boldsymbol{z}^*)^{\mathrm{T}}(\boldsymbol{G}(\boldsymbol{z}) - \boldsymbol{G}(\boldsymbol{z}^*)) - \boldsymbol{\gamma}^{\mathrm{T}}L\boldsymbol{\gamma}. \tag{15}$$

Then we learn from (15) that $\frac{dV}{dt} \leq 0$ because $\mathcal{J}_i(\cdot, \boldsymbol{z}_{-i})$ is ω-strongly convex and L is positive semidefinite. Hence, the trajectory of algorithm (11) is bounded and any finite equilibrium satisfying (12) is Lyapunov stable.

Let us investigate the set

$$I_v \triangleq \left\{(\boldsymbol{z}, \boldsymbol{\nu}, \boldsymbol{\gamma}, \boldsymbol{\zeta}) : \frac{d}{dt}V = 0\right\},$$

with Q as its largest invariant subset. It follows from the invariance principle (Theorem 2.41 of Haddad & Chellaboina, 2011) that $(\boldsymbol{z}(t), \boldsymbol{\nu}(t), \boldsymbol{\gamma}(t), \boldsymbol{\zeta}(t)) \to Q$ as $t \to \infty$, and Q is a positive invariant set. Consider a trajectory $(\bar{\boldsymbol{z}}, \bar{\boldsymbol{\nu}}, \bar{\boldsymbol{\gamma}}, \bar{\boldsymbol{\zeta}})$ in Q. From (15), we have

$$I_v \subseteq \left\{(\boldsymbol{z}, \boldsymbol{\nu}, \boldsymbol{\gamma}, \boldsymbol{\zeta}) : \boldsymbol{z} = \boldsymbol{z}^*, L\boldsymbol{\gamma} = 0\right\},$$

which implies $\dot{\bar{\boldsymbol{z}}} = 0$, $\dot{\bar{\boldsymbol{\nu}}} = 0$, $\dot{\bar{\boldsymbol{\gamma}}} = const$ and $\dot{\bar{\boldsymbol{\zeta}}} = const$. Because the trajectory is bounded, it leads to a contradiction if $\dot{\bar{\boldsymbol{\gamma}}} \neq 0$ or $\dot{\bar{\boldsymbol{\zeta}}} \neq 0$. Hence, any point in Q is an equilibrium point of algorithm (11), which satisfies (12). By the LaSalle invariance principle and Lyapunov stability of the equilibrium point, system (11) converges to an equilibrium point. According to Lemma 3, $\boldsymbol{z}(t)$ converges to a solution to VI($\boldsymbol{\Xi}$, $\boldsymbol{G}(\boldsymbol{z})$) satisfying (8). ■

Since the payoff function $J_i(\cdot, \boldsymbol{x}_{-i})$ has strongly monotone pseudo-gradient for $i \in \mathcal{P}$, game (1) has a unique optimal solution $\boldsymbol{x}^*$ in the worst case. However, as the expansion of both $\boldsymbol{x}^*$ and

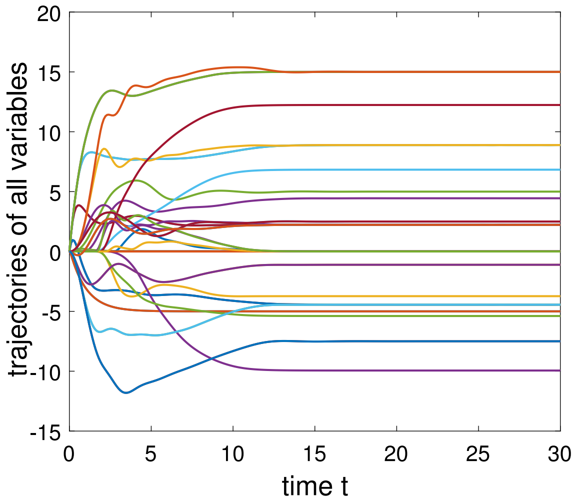

Fig. 1. Trajectories of all variables.

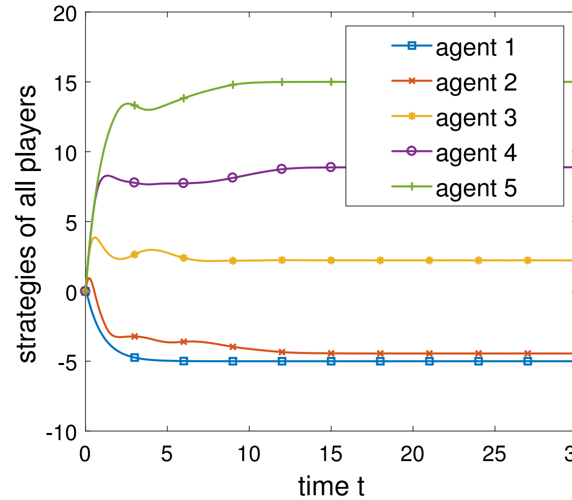

Fig. 2. Trajectories of all players' strategies.

Table 1
Numerical values of ε from different approximations.

Polyhedrons	Triangle	Rectangle	Hexagon	Octagon
Values of ε	31.76	17.45	2.35	0.34

$\boldsymbol{\lambda}^*$, $\boldsymbol{z}^*$ is non-unique and relies on the initial conditions since $\boldsymbol{\lambda}^*$ is non-unique. Moreover, as discussed in the end of Section 4, choosing polyhedrons with more edges directly results in higher-dimension dynamics, which brings more communication burden. Hence, we need to consider the trade-off between accuracy and data size.

Before the end of this section, we provide an illustrative example.

Example 1. Consider a game with $N = 5$ players. For $i \in \mathcal{P} = \{1, \ldots, 5\}$, strategy $x_i \in \mathbb{R}^2$ of each player satisfies $x_{ik} \in [-5, 15]$, $k = \{1, 2\}$. The payoff function of the ith player is

$$J_i(x_i, \boldsymbol{x}_{-i}) = \frac{1}{2}x_i^{\mathrm{T}}x_i + \sum_{j=1}^{N} x_i^{\mathrm{T}}x_j - p_i^{\mathrm{T}}x_i,$$

where $p_i = 10(i-1)\mathbf{1}_2 \in \mathbb{R}^2$. Players need to meet a coupled inequality constraint $\sum_{j=1}^{N} \alpha^{\mathrm{T}}x_j \leq b_0$, where $\alpha \in \mathbf{B}_1(1, 1)$ and $b_0 = 75$. We adopt a ring graph as the communication network $\mathcal{G}$:

$$1 \rightleftarrows 2 \rightleftarrows 3 \rightleftarrows 4 \rightleftarrows 5 \rightleftarrows 1.$$

We present trajectories by using inscribed rectangles to approximate $\mathbf{B}_1(1, 1)$. The trajectories of all variables in algorithm (11) are shown in Fig. 1. The trajectories of one dimension of each strategy x_i are shown in Fig. 2.

Fig. 3 shows different strategy trajectories of one fixed player with inscribed triangles, rectangles, hexagons and octagons to approximate $\mathbf{B}_1(1, 1)$, respectively. The vertical axis represents the value of the convergent ε-GNE. As can be seen from Fig. 3, equilibria with different polyhedrons get closer when we choose more accurate approximation. Moreover, according to the definition of ε-GNE, the numerical values of ε from different types of approximation are listed in Table 1. Clearly, the value of ε decreases when we increase the edges of polyhedrons, which conforms with Theorem 2.

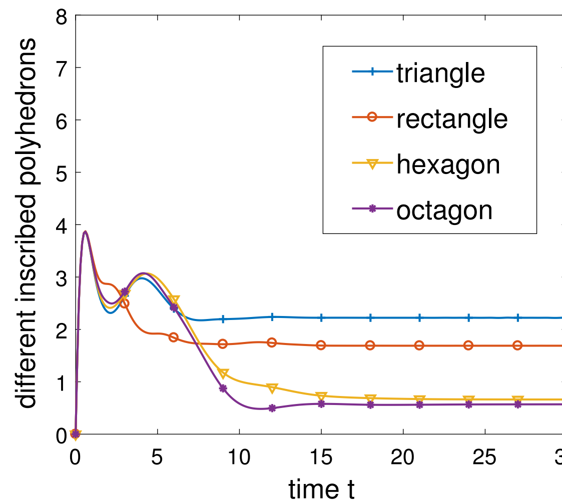

Fig. 3. Trajectories of approximation by different inscribed polyhedrons.

6. Conclusion

A distributed game with coupled inequality constraints has been investigated in this paper, where parameters in constraints have general convex uncertainties. A distributed algorithm in the worst case has been proposed by utilizing projection operators and variational inequalities. The equilibria of the algorithm have been rigorously proved to be ε-GNE of the original problem with general uncertainties and the convergence of the algorithm has been shown by employing variational inequalities and Lyapunov functions.

In the future, there are many possible extensions of the current research. First, we plan to apply this technique to complicated or nonlinear coupled constraints, not limited to linear ones. Also, we will explore powerful tools to help us quantitatively analyze the effectiveness of the approximation. Additionally, we will study the ε-Nash equilibria for other games with uncertainties.

Acknowledgments

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Appendix A. Proof of Lemma 1

We will prove the conclusion of Lemma 1 in three steps.

Step 1: Take an inscribed polyhedron of the convex and compact set $U \subseteq \mathbb{R}^n$ as

$$R^1_{k_1} = \{a \in \mathbb{R}^n : D^1 a \le d^1\},$$

with V_1 as the set of vertices on the boundary of U. Similarly, take another inscribed polyhedron as

$$R^2_{k_2} = \{a \in \mathbb{R}^n : D^2 a \le d^2\},$$

whose vertices consist of $V_2 = V_1 \cup \{v_0\}$, with v_0 as an additional vertex on the boundary of U. We first prove that for any $\varepsilon > 0$ and D^2_i as any row of matrix D^2, there exist $\eta > 0$ and a corresponding row $D^1_{j(i)}$ of matrix D^1 such that if $\delta_H(R^1_{k_1}, R^2_{k_2}) \le \eta$, then $\|D^2_i - D^1_{j(i)}\| \le \varepsilon$.

Clearly, $\delta_H(R^1_{k_1}, R^2_{k_2}) = dist(R^1_{k_1}, v_0)$ since $R^2_{k_2}$ is convex and v_0 belongs to the boundary of the convex set U. Suppose that there are m_1 rows of D^1 and d^1, m_2 rows of D^2 and d^2, and V_0 consists of the vertices constructing the hyperplanes together with v_0 in $R^2_{k_2}$. Without loss of generality, to the boundary of $R^1_{k_1}$, vertices in V_0 construct one hyperplane noted as $\mathcal{H}_0$. Hence,

$$dist(R^1_{k_1}, v_0) = dist(v_0, \mathcal{H}_0).$$

Since normalized vectors D^1_i (or D^2_i) represent normal vectors of hyperplanes enclosing $R^1_{k_1}$ (or $R^2_{k_2}$), we assume that the first m_1-1 rows of D^1 are same to the first $m_1 - 1$ rows of D^2. Thus, we only need to investigate the difference between $D^1_{m_1}$ and the last $(m_2 - m_1 + 1)$ rows of D^2. Specifically, the two matrices can be written by row as

$$D^1 = \begin{bmatrix} D^0_1 \\ \vdots \\ D^0_{m_1-1} \\ D^1_{m_1} \end{bmatrix}, \quad D^2 = \begin{bmatrix} D^0_1 \\ \vdots \\ D^0_{m_1-1} \\ D^2_{m_1} \\ \vdots \\ D^2_{m_2} \end{bmatrix}.$$

Note that the angle between two hyperplanes is equal to the angle between the two corresponding normal vectors. For $m_1 \le i \le m_2$, denote θ_i as the angle formed by D^2_i and $D^1_{m_1}$. With $h_0 = dist(v_0, rbd(\mathcal{H}_0))$, we have

$$\sin\theta_i \le \frac{dist(v_0, \mathcal{H}_0)}{dist(v_0, rbd(\mathcal{H}_0))} = \frac{\delta_H(R^1_{k_1}, R^2_{k_2})}{h_0}.$$

Meanwhile, for $m_1 \le i \le m_2$, there exist $M_i \in SO(n)$ such that $D^2_i = D^1_{m_1} M_i$ and

$$\|D^2_i - D^1_{m_1}\| = \|(M_i - I)D^1_{m_1}\|.$$

When the rotation angle is small enough, $\sin\theta_i \sim \theta_i$ and we use the expansion at M_i, referring to Hall (2003, Theorem 2.21), to derive

$$M_i \sim I + \theta_i W_i, \quad W_i \in \mathfrak{g}(SO(n)),$$

where W_i is constant and $\mathfrak{g}(\cdot)$ represents its Lie algebra. Let $\eta = \varepsilon h_0/\|W_i\|$ and this yields the conclusion.

Step 2: Consider VI($\varXi$, $\boldsymbol{G}(\boldsymbol{z}) + \beta\boldsymbol{z}$) with $\beta > 0$. Since $G(\boldsymbol{z}) + \beta\boldsymbol{z}$ is strongly monotone, it follows from Facchinei and Pang (2007, Theorem 2.3.3) that SOL($\varXi$, $\boldsymbol{G}(\boldsymbol{z})+\beta\boldsymbol{z}$) exists uniquely. For $\beta_1, \beta_2 > 0$, $\|\beta_1\boldsymbol{z} - \beta_2\boldsymbol{z}\| \le c\|\beta_1 - \beta_2\|$ in a neighbor of $\boldsymbol{z}$. Thus, it follows from Facchinei and Pang (2007, Corollary 5.1.5) that

$$\|\mathrm{SOL}(\varXi, \boldsymbol{G}(\boldsymbol{z}) + \beta_1\boldsymbol{z}) - \mathrm{SOL}(\varXi, \boldsymbol{G}(\boldsymbol{z}) + \beta_2\boldsymbol{z})\| \to 0,$$

as $\beta_1 \to \beta_2$. Denote $\varXi_1 = \varXi(\boldsymbol{R}_{k_1})$ and $\varXi_2 = \varXi(\boldsymbol{R}_{k_2})$. Define $\boldsymbol{z}^{1*} = \lim_{\beta_n \to 0} \mathrm{SOL}(\varXi_1, \boldsymbol{G}(\boldsymbol{z}) + \beta_n\boldsymbol{z})$ for any sequence $\{\beta_n\} \to 0$. Then $\boldsymbol{z}^{1*} \in \mathrm{SOL}(\varXi_1, \boldsymbol{G}(\boldsymbol{z}))$. The discussion of $\boldsymbol{z}^{2*}$ is similar.

Step 3: If $\delta_H(R^i_{k_{1,i}}, R^i_{k_{1,i}}) \le \eta$, from Step 1 that there exist constants h_0 and $\|W_\tau\|$ such that the τth row of $D^{1,i}$ and $D^{2,i}$ satisfies

$$\|D^{1,i}_\tau - D^{2,i}_{j(\tau)}\| \le \eta\|W_\tau\|/h_0 = O(\eta).$$

Correspondingly, $\|d^{1,i} - d^{2,i}\| \le O(\eta)$. Furthermore, since $\delta_H(R^i_{k_{1,i}}, R^i_{k_{2,i}}) \le \eta$, $\forall i \in \mathcal{P}$, and $\boldsymbol{R}_k$ determines $\varXi$, we get $\varXi_2 \to \varXi_1$ as $\eta \to 0$. Notice that SOL($\varXi$, $\boldsymbol{G}(\boldsymbol{z}) + \beta\boldsymbol{z}$) exists as an isolated solution, and $\boldsymbol{G}(\boldsymbol{z})+\beta\boldsymbol{z}$ is locally Lipschitz continuous with respect to $\boldsymbol{z}$ but constant with respect to $\boldsymbol{R}_k$. Following from Facchinei and Pang (2007, *Proposition 5.4.1*),

$$\mathrm{SOL}(\varXi_1, \boldsymbol{G}(\boldsymbol{z}) + \beta\boldsymbol{z}) \to \mathrm{SOL}(\varXi_2, \boldsymbol{G}(\boldsymbol{z}) + \beta\boldsymbol{z}),$$

as $\eta \to 0$. Therefore, for any $\varepsilon > 0$, there exist $\beta > 0$ and $\eta > 0$ such that

$$\begin{aligned}
&\|\boldsymbol{x}^*(\boldsymbol{R}^1_{k_1}) - \boldsymbol{x}^*(\boldsymbol{R}^2_{k_2})\| \\
\le&\|\boldsymbol{z}^{1*} - \boldsymbol{z}^{2*}\| \\
\le&\|\boldsymbol{z}^{1*} - \mathrm{SOL}(\varXi_1, \boldsymbol{G}(\boldsymbol{z}) + \beta\boldsymbol{z})\| \\
&+ \|\mathrm{SOL}(\varXi_1, \boldsymbol{G}(\boldsymbol{z}) + \beta\boldsymbol{z}) - \mathrm{SOL}(\varXi_2, \boldsymbol{G}(\boldsymbol{z}) + \beta\boldsymbol{z})\| \\
&+ \|\mathrm{SOL}(\varXi_2, \boldsymbol{G}(\boldsymbol{z}) + \beta\boldsymbol{z}) - \boldsymbol{z}^{2*}\| \\
=&\|\mathrm{SOL}(\varXi_1, \boldsymbol{G}(\boldsymbol{z})) - \mathrm{SOL}(\varXi_1, \boldsymbol{G}(\boldsymbol{z}) + \beta\boldsymbol{z})\| \\
&+ \|\mathrm{SOL}(\varXi_1, \boldsymbol{G}(\boldsymbol{z}) + \beta\boldsymbol{z}) - \mathrm{SOL}(\varXi_2, \boldsymbol{G}(\boldsymbol{z}) + \beta\boldsymbol{z})\| \\
&+ \|\mathrm{SOL}(\varXi_2, \boldsymbol{G}(\boldsymbol{z}) + \beta\boldsymbol{z}) - \mathrm{SOL}(\varXi_2, \boldsymbol{G}(\boldsymbol{z}))\| \\
\le&\frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon = \varepsilon,
\end{aligned}$$

which completes the proof. ■

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