SPOONER, Jordan (js4416)

Imperial College London

Department of Computing

Academic Year **2019-2020**

Page created Thu Feb 20 02:15:20 GMT 2020

499 fbelard 6 c4 js4416 v1 001190035913	Electronic submission **Mon - 17 Feb 2020 18:47:53** js4416

Exercise Information

Module: 499 Modal Logic for Strategic Reasoning in AI
Exercise: 6 (CW)
Title: Coursework2
FAO: Belardinelli, Francesco (fbelard)

Issued: Wed - 05 Feb 2020
Due: Wed - 19 Feb 2020
Assessment: Individual
Submission: Electronic

Student Declaration - Version 1

- I declare that this final submitted version is my unaided work.

Signed: (electronic signature) Date: 2020-02-17 18:32:23

For Markers only: (circle appropriate grade)

SPOONER, Jordan (js4416)	01201572	c4	2020-02-17 18:32:23	**A* A B C D E F**

Q1. (a) For a model M, path π and LTL formulas φ and ψ:

$(M,\pi) \models \varphi R \psi$ iff there is some $i \geq 0$ such that $(M, \pi[i..\infty]) \models \varphi$ and for all $0 \leq j \leq i$ we have $(M, \pi[j..\infty]) \models \psi$, or for all $k \geq 0$, $(M, \pi[k..\infty]) \models \psi$.

(b) $$\varphi R \psi \equiv \underbrace{(\neg(\top U \neg\psi))}_{\psi \text{ holds forever}} \lor \underbrace{(\psi U (\psi \land \varphi))}_{\psi \text{ until and including when } \varphi \text{ holds}}$$

or

(c) $(M,\pi) \models (\neg(\top U \neg\psi)) \lor (\psi U (\psi \land \varphi))$

(by defn 1.4) iff $(M,\pi) \models \neg(\top U \neg\psi)$ or $(M,\pi) \models \psi U (\psi \land \varphi)$

(by 1.4) iff $(M,\pi) \not\models \top U \neg\psi$ or $\underbrace{(M,\pi) \models \psi U (\psi \land \varphi)}_{*}$

(by 1.4) iff $\neg[(M, \pi[k..\infty]) \models \neg\psi$ for some $k \geq 0$, and $(M, \pi[l..\infty]) \models \top$ for all $0 \leq l < k]$, or $*$

(by 1.4) iff $\neg[(M, \pi[k..\infty]) \models \neg\psi$ for some $k \geq 0]$, or $*$

(by 1.4) iff $\neg[(M, \pi[k..\infty]) \not\models \psi$ for some $k \geq 0]$, or $*$

($\neg\exists\neg X \equiv \neg\neg\forall X \equiv \forall X$) iff $\underbrace{\text{for all } k \geq 0, (M, \pi) \models \psi}_{\dagger}$ or $*$

(by rewriting) iff $\oplus$ or $(M, \pi) \models \psi \, U \, (\psi \wedge \varphi)$

(1.4) iff $\oplus$ or $(M, \pi[i..\infty]) \models (\psi \wedge \varphi)$ for some $i \geqslant 0$

and $(M, \pi[j..\infty]) \models \psi$ for all $0 \leq j < i$

(1.4) iff $\oplus$ or $(M, \pi[i..\infty]) \models \psi$ and
$(M, \pi[i..\infty]) \models \varphi$ for some $i \geqslant 0$
and $(M, \pi[j..\infty]) \models \psi$ for all $0 \leq j < i$.

(rewrite) iff $(M, \pi[i..\infty]) \models \varphi$ for some $i \geqslant 0$ and
$(M, \pi[j..\infty]) \models \psi$ for all $0 \leq j \leq i$,
or for all $k \geqslant 0$ $(M, \pi[k..\infty]) \models \psi$.

which is the condition provided in part (a).

(d) $\bot R \psi \equiv (\neg(\top U \neg\psi)) \vee (\psi \, U \, (\psi \wedge \bot))$

(from parts (a) to (c)) $\equiv (\neg F \neg \psi) \vee (\psi \, U (\psi \wedge \bot))$

(by defn F) $\equiv G\psi \vee (\psi \, U (\psi \wedge \bot))$

(by defn G) $\equiv G\psi \vee (\underbrace{\psi \, U \bot})$

(since $X \wedge \top \equiv X$) $\equiv G\psi \vee \bot$ ← by defn 1.4, this would require $\lambda[i..\infty] \models \bot$ for some $i \geqslant 0$ to hold for some path λ. Clearly no such λ exists, so it is equivalent to $\bot$

(since $X \vee \bot \equiv X$) $\equiv G\psi$

Q2. (i) $(M, q) \models EF\phi$

iff $(M, q) \models E(\top U \phi)$ (given)

iff for some path λ starting from q, $(M, \lambda) \models \top U \phi$ (defn 1.7)

iff for some path λ starting from q, $(M, \lambda[j]) \models \phi$ for some $j \geq 0$ and $(M, \lambda[k]) \models \top$ for all $0 \leq k < j$ (defn 1.8)

iff for some path λ starting from q, for some $j \geq 0$ $(M, \lambda[j]) \models \phi$.

(since $(M, s) \models \top$) $\circledast$

(ii) $(M, q) \models AF\phi$

iff $(M, q) \models A(\top U \phi)$ (given)

iff for every path λ starting from q, $(M, \lambda) \models \top U \phi$ (1.7)

iff for every path λ starting from q, for some $j \geq 0$ $(M, \lambda[j]) \models \phi$.

(by the same reasoning as in $\circledast$)

(iii) $(M, q) \models EG\phi$

iff $(M, q) \models \neg AF \neg \phi$ (given)

iff $(M, q) \models \neg A(\top U \neg \phi)$ (given)

iff $(M, q) \not\models A(\top U \neg \phi)$ (defn 1.7)

iff it is not the case that for all paths λ starting from q, $(M, \lambda) \models \top U \neg \phi$ (1.7)

(†)

iff for some path λ starting from q, $(M, \lambda) \not\models \top U \neg \phi$ ($\neg \exists X \equiv \forall \neg X$)

iff for some path λ starting from q, it is not the case that $(M, \lambda[j]) \models \neg \phi$ for some $j \geq 0$ and $(M, \lambda[k]) \models \top$ for all $0 \leq k < j$. (1.8)

iff for some path λ starting from q, it is not the case that $(M, \lambda[j]) \models \neg \phi$ for some $j \geq 0$ ($(M, s) \models \top$)

iff for some path λ starting from q, for all $j \geq 0$ $(M, \lambda[j]) \not\models \neg \phi$ ($\neg \exists X \equiv \forall \neg X$)

iff for some path λ starting from q for all $j \geq 0$ $(M, \lambda[j]) \models \phi$ (1.7 and $\neg\neg X \equiv X$)

(iv) $(M, q) \models AG\phi$

iff $(M, q) \models \neg EF \neg \phi$ (given)

iff $(M, q) \models \neg E(\top U \neg \phi)$ (given)

iff $(M, q) \not\models E(\top U \neg \phi)$ (1.7)

iff there is some path λ starting from q s.t. $(M, \lambda) \models \top U \neg \phi$

iff for all paths λ starting from q, $(M, \lambda) \not\models \top U \neg \phi$ (1.7) (switch quantifiers)

iff for all paths λ starting from q, for all $j \geq 0$ $(M, \lambda[j]) \models \phi$. (by the same reasoning as in (†))

Q3. (a)

Consider an arbitrary CTL formula ϕ
According to the definition in lecture 5, ϕ can take seven forms:

$$\phi ::= a \mid \neg\phi_1 \mid \phi_1 \wedge \phi_2 \mid EX\phi_1 \mid AX\phi_1 \mid E(\phi_1 \cup \phi_2) \mid A(\phi_1 \cup \phi_2)$$

where ϕ_1 and ϕ_2 are CTL formulas.

We show by induction that ϕ is a formula CTL*.

Specifically, our inductive hypothesis is that any CTL formula is a CTL* (state) formula.

① <u>Base case</u>: ϕ is an atom a

By definition 1, an atom a is a (state) formula of CTL* as required.

② <u>Inductive case 1</u>: ϕ is $\neg\phi_1$

By our I.H., assume ϕ_1 is a (state) formula CTL*.

By defn 1 (line 1), $\neg\phi_1$ is therefore also a (state) formula in CTL*.

③ Inductive case 2 : Φ is $\phi_1 \wedge \phi_2$

By the IH, assume Φ_1 and ϕ_2 are (state) formulas

By defn 1, $\Phi_1 \wedge \Phi_2$ is then a (state) formula in CTL* as required.

④ Inductive case 3 : Φ is $EX\Phi_1$

By the IH, assume Φ_1 is a (state) formula of CTL*.

(*) Then by defn 1 (line 2) Φ_1 is a path formula also.
By defn 1 (line 2), $X\Phi_1$ is a path formula also.
By defn 1 (line 1), $EX\Phi_1$ is a (state) formula of CTL* as required.

⑤ Inductive case 4 : Φ is $AX\phi_1$

By the IH, assume Φ_1 is a (state) formula of CTL*.

By (*), $X\Phi_1$ is a path formula of CTL*

By defn 1 (line 1), $AX\Phi_1$ is a (state) formula of CTL*.

⑥ Inductive case 5 : ϕ is $E(\Phi_1 \cup \Phi_2)$

(+) By the IH, assume Φ_1 and Φ_2 are (state) formulas of CTL*
By defn 1, they are also path formulas of CTL*.
By defn 1, $\Phi_1 \cup \Phi_2$ is a path formula of CTL*.

By defn 1, $E(\Phi_1 \cup \Phi_2)$ is a (state) formula of CTL*.

(7) Inductive case 6: ϕ is $A(\phi_1 \cup \phi_2)$

By (+), we get $\phi_1 \cup \phi_2$ is a path formula of CTL*.

By defn 1, $A(\phi_1 \cup \phi_2)$ is a (state) formula of CTL* as required.

Hence every formula ϕ of CTL is a (state) formula in CTL*.

(b) Consider the CTL* formula

$$Ea$$

This is a formula of CTL*, derived as follows:

$$\phi \to E\psi \to E\phi \to Ea$$

It is not a formula of CTL since $E\psi$ can only be accepted if ψ is of the form $X\phi$ or $\phi \cup \phi$; and 'a' is clearly neither.

Hence there exists a formula of CTL* that does not belong to CTL.

Q4. We denote entailment in CTL as $\models$ and entailment in CTL* as $\models^*$.

We must show that every construction in CTL (i.e. those seen in the previous question) have the same semantics in CTL*.

Once again, we will show this by induction.

Let M be an arbitrary model, q be an arbitrary state, and ϕ be an arbitrary CTL* (state) formula which is also a (state) formula in CTL.

Our inductive hypothesis is that $(M, q) \models^* \phi$ iff $(M, q) \models \phi$.

We will use ϕ_i to denote CTL* (state) formulas which are also (state) formulas of CTL.

Inductive case 3 ϕ takes the form $EX\phi_1$

$(M, q) \models^* EX\phi_1$

$\circledast$

iff for some path λ starting from q, $(M, \lambda) \models^* X\phi_1$ (defn 2)

iff for some path λ starting from q, $(M, \lambda[1..\infty]) \models^* \phi_1$ (defn 2)

iff for some path λ starting from q, $(M, \lambda[1]) \models^* \phi_1$ (defn 2, since ϕ_1 is a state formula)

iff for some path λ starting from q, $(M, \lambda[1]) \models \phi_1$ (I.H., since $\lambda[1]$ is a state)

iff for some path λ starting from q, $(M, \lambda) \models X\phi_1$ (defn 1.8)

iff $(M, \lambda) \models EX\phi_1$ (defn 1.7)

as required.

Inductive case 4 ϕ takes the form $AX\phi_1$

$(M, q) \models^* AX\phi_1$

iff for all paths λ starting from q, $(M, \lambda) \models^* X\phi_1$ (defn 2)

iff for all paths λ starting from q, $(M, \lambda) \models X\phi_1$ (same as $\circledast$)

iff $(M, \lambda) \models AX\phi_1$ (defn 1.7)

as required.

Base case ϕ is an atom a

$$(M, q) \models^* a \text{ iff } q \in V(a) \quad \text{(defn 2)}$$
$$\text{iff } (M, q) \models a \quad \text{(defn 1.7)}$$

as required.

Inductive case 1 ϕ is of the form $\neg \phi_1$

$$(M, q) \models^* \neg \phi_1 \text{ iff } (M, q) \not\models^* \phi_1 \quad \text{(defn 2)}$$
$$\text{iff } (M, q) \not\models \phi_1 \quad \text{(by I.H.)}$$
$$\text{iff } (M, q) \models \neg \phi_1 \quad \text{(by defn 1.7)}$$

as required.

Inductive case 2 ϕ takes the form $\phi_1 \wedge \phi_2$

$$(M, q) \models^* \phi_1 \wedge \phi_2$$
$$\text{iff } (M, q) \models^* \phi_1 \text{ and } (M, q) \models^* \phi_2 \quad \text{(defn 2)}$$
$$\text{iff } (M, q) \models \phi_1 \text{ and } (M, q) \models \phi_2 \quad \text{(by I.H.)}$$
$$\text{iff } (M, q) \models \phi_1 \wedge \phi_2 \quad \text{(by defn 1.7)}$$

as required.

Inductive case 5 ϕ takes the form $E(\phi_1 \cup \phi_2)$

$(M, q) \models^* E(\phi_1 \cup \phi_2)$

iff for some path λ starting from q, (defn 2)
$(M, \lambda) \models^* \phi_1 \cup \phi_2$

iff for some path λ starting from q,
$(M, \lambda[i..\infty]) \models^* \phi_1$ for some $i \geq 0$, and
$(M, \lambda[j..\infty]) \models^* \phi_2$ for all $0 \leq j < i$. (defn 2)

$\oplus$ iff for some path λ starting from q
$(M, \lambda[i]) \models^* \phi_1$ for some $i \geq 0$ and
$(M, \lambda[j]) \models^* \phi_2$ for all $0 \leq j < i$ (defn 2, since ϕ_1 and ϕ_2 are state formulas)

iff for some path λ starting from q
$(M, \lambda[i]) \models \phi_1$ for some $i \geq 0$ and
$(M, \lambda[j]) \models \phi_2$ for all $0 \leq j < i$ (I. H., since $\lambda[i]$ and $\lambda[j]$ are states)

iff for some path λ starting from q
$(M, \lambda) \models \phi_1 \cup \phi_2$ (defn 1.8)

iff $(M, q) \models E(\phi_1 \cup \phi_2)$ as required. (defn 1.7)

Inductive case 6 ϕ takes the form $A(\phi_1 \cup \phi_2)$

$(M, q) \models^* A(\phi_1 \cup \phi_2)$

iff for all paths λ starting from q, (defn 2)
$(M, \lambda) \models^* \phi_1 \cup \phi_2$

iff for all paths λ starting from q,
$(M, \lambda) \models \phi_1 \cup \phi_2$ (same as $\oplus$)

iff $(M, q) \models A(\phi_1 \cup \phi_2)$ (defn 1.7)
as required.

Note that we have shown the I.H. holds over all formulas of CTL, as per the definition on slide 142. Also this is a strict subset of the formulas of CTL*, by the previous question. Hence the formulas of CTL* that are also formulas of CTL have the same semantics in both logics, as required.

Q5. (a)

Consider an arbitrary formula ϕ of CTL, an arbitrary model M and an arbitrary state s.

By question 3(a), $\phi' = \phi$ a formula of CTL*.
By question 4, $(M, s) \models \phi$ iff $(M, s) \models^* \phi'$, as required.
Hence CTL* is more expressive than CTL.

(b) Consider the formula of CTL*:

$$\phi = A(\top \, U \, (a \wedge Xa))$$

$$\equiv AF(a \wedge Xa) \quad \text{(by defn of F, slide 142)}$$

ϕ is equivalent to the LTL formula $F(a \wedge Xa)$ (slide 126)

ϕ is not expressible in CTL (slide 208)

Hence we have that CTL* is strictly more expressive than CTL, as required.

Q6. We do induction over defn 2.

Given models $M = (St, \rightarrow, V)$ and $M' = (St', \rightarrow', V')$, arbitrary states $t \in St$ and $t' \in St'$, arbitrary paths $\pi \in (St, \rightarrow)$ and $\pi' \in (St', \rightarrow')$, arbitrary state formula ϕ and arbitrary path formula ψ, such that $\underbrace{(M, t) \approx (M', t')}_{\oplus}$ and $\underbrace{(M, \pi) \approx (M', \pi')}_{\ddagger}$, our I.H. is that:

$(M, t) \models \phi$ iff $(M', t') \models \phi$

and $(M, \pi) \models \psi$ iff $(M', \pi') \models \psi$

<u>Base case</u> ϕ takes the form p.

$$
\begin{aligned}
(M, t) \models p \quad & \text{iff} \quad t \in V(p) && \text{(defn 2)} \\
& \text{iff} \quad t' \in V'(p) && \text{($\oplus$ and defn 3a)} \\
& \text{iff} \quad (M', t') \models p && \text{(defn 2)}
\end{aligned}
$$

as required

<u>Inductive case 1</u> ϕ takes the form $\neg \phi_1$

$$
\begin{aligned}
(M, t) \models \neg \phi_1 \quad & \text{iff} \quad (M, t) \not\models \phi_1 && \text{(defn 2)} \\
& \text{iff} \quad (M', t') \not\models \phi_1 && \text{(I.H.)} \\
& \text{iff} \quad (M', t') \models \neg \phi_1 && \text{(defn 2)}
\end{aligned}
$$

as required

Inductive case 2 ϕ takes the form $\phi_1 \wedge \phi_2$

$(M, t) \models \phi_1 \wedge \phi_2$

iff $(M, t) \models \phi_1$ and $(M, t) \models \phi_2$ (defn 2)

iff $(M', t') \models \phi_1$ and $(M', t') \models \phi_2$ (I.H.)

iff $(M', t') \models \phi_1 \wedge \phi_2$ (defn 2)

as required.

For the next inductive cases, we first prove the following:

Lemma 1 Given a path λ in M, starting at some state $\lambda[0]$, and a model M' such that $(M, \lambda[0]) \approx (M', \lambda'[0])$, (*) $(M, \lambda) \approx (M', \lambda')$, and vice versa.

Proof ① Forwards:

We must show that there exists a λ' s.t. $(M, \lambda[i]) \approx (M', \lambda'[i])$ for all $i \geq 0$.

We must show that there exists a bisimulation B between M and M' s.t. $B(\lambda[i], \lambda'[i])$ for all $i \geq 0$.

By (*), we know there exists a bisimulation B between M and M' s.t. $B(\lambda[0], \lambda'[0])$.

We now show by induction over $\mathbb{N}$ that $B(\lambda[i], \lambda'[i])$ for all $i \geq 0$.

Our I.H. is that $B(\lambda[k], \lambda'[k])$.

Base case $B(\lambda[0], \lambda'[0])$ holds as stated earlier.

Inductive case Assume $B(\lambda[k], \lambda'[k])$

we know that $\lambda[k] \in St$ and $\lambda[k] \to \lambda[k+1]$

By defn 3(b) (forth), there exists some state, call it $\lambda'[k+1] \in St'$ s.t. $B(\lambda[k], \lambda'[k+1])$, as required.

② Backwards: the proof is symmetrical, it relies on defn 3(c) (back) instead of 3(b) (forth).

Inductive case 3 ϕ takes the form $E\psi_1$

$(M, t) \models E\psi_1$

iff for some path λ starting from t,

$(M, \lambda) \models \psi_1$ (defn 2)

Now we note that $(M, t) \approx (M', t')$ by (†), and that $\lambda[0] = t$. If we have $\lambda'[0] = t'$, we can therefore apply Lemma 1 to get that $(M, \lambda) \approx (M', \lambda')$ and vice versa. Hence, this is equivalent to the condition that:

for some path λ' starting from t',

$(M', \lambda') \models \psi_1$ (by the I.H.)

iff $(M', \lambda') \models E\psi_1$ (defn 2)

as required

Inductive case 4 ϕ takes the form $A\psi_1$

$(M, t) \models A\psi_1$

iff for all paths λ starting from t,

$(M, \lambda) \models \psi_1$ (defn 2)

iff for all paths λ' starting from t'

$(M', \lambda') \models \psi_1$ (same reasoning as above)

iff $(M', t') \models A\psi_1$ (defn 2).

Inductive case 5 ψ takes the form ϕ_1

$(M, \pi) \models \phi_1$

iff $(M, \pi[0]) \models \phi_1$ (defn 2)

Note that $(M, \pi) \approx (M', \pi')$ by (‡). Hence by the definition of bisimilar paths, $(M, \pi[0]) \approx (M', \pi'[0])$, so this condition is equivalent to:

$(M', \pi'[0]) \models \phi_1$ (by the I.H.)

iff $(M', \pi') \models \phi_1$ (defn 2)

Inductive case 6 ψ takes the form $\neg\psi_1$

$(M, \pi) \models \neg\psi_1$

iff $(M, \pi) \not\models \psi_1$ (defn 2)

iff $(M', \pi') \not\models \psi_1$ (I.H.)

iff $(M', \pi') \models \neg\psi_1$ (defn 2)

as required.

Inductive case 7 ψ takes the form $\psi_1 \wedge \psi_2$

$$(M, \pi) \models \psi_1 \wedge \psi_2$$

iff $(M, \pi) \models \psi_1$ and $(M, \pi) \models \psi_2$ (defn 2)

iff $(M', \pi') \models \psi_1$ and $(M', \pi') \models \psi_2$ (I.H.)

iff $(M', \pi') \models \psi_1 \wedge \psi_2$ (defn 2)

as required.

Inductive case 8 ψ takes the form $X \psi_1$

$$(M, \pi) \models X \psi_1$$

iff $(M, \pi[1..\infty]) \models \psi_1$ (defn 2)

Note that $(M, \pi) \approx (M', \pi')$ by $\oplus$. Hence by the definition of a bisimilar path, $(M, \pi[i]) \approx (M', \pi'[i])$ for all $i \geq 0$. Again, by defn, we therefore have that $(M, \pi[1..\infty]) \approx (M', \pi'[1..\infty])$. Hence this condition is equivalent to

$(M', \pi'[1..\infty]) \models \psi_1$ (I.H.)

iff $(M', \pi') \models X \psi_1$ as required (defn 2)

Inductive case 9 ψ takes the form $\psi_1 \cup \psi_2$

$(M, \pi) \models \psi_1 \cup \psi_2$

iff $(M, \pi[i..\infty]) \models \psi_1$ for some $i \geq 0$

and $(M, \pi[j..\infty]) \models \psi_2$ for all $0 \leq j < i$

(defn 2)

By a similar argument as before

$(M, \pi[k..\infty]) \approx (M', \pi'[k..\infty])$ for any $k \geq 0$

Hence this is equivalent to the condition

$(M', \pi'[i..\infty]) \models \psi_1$ for some $i \geq 0$

and $(M', \pi'[j..\infty]) \models \psi_2$ for all $0 \leq j < i$

(I.H.)

iff $(M', \pi') \models \psi_1 \cup \psi_2$ (defn 2)

as required.

Hence we have shown that for all CTL* formulas,

if $(M, t) \approx (M', t')$:

$(M, t) \models \phi$ iff $(M', t') \models \phi$

That is, the truth of CTL* formulas is preserved by bisimulations.

Assume $t \in M$ and $t' \in M'$ are CTL-equivalent.

Q7. We must show that (M, t) and (M', t') are bisimilar.

We must show that there exists a bisimulation B from M to M' and that $B(t, t')$.

Specifically, we show that the CTL-equivalence relation (which we denote a $\leftrightsquigarrow$) is such a bisimulation.

① $\leftrightsquigarrow$ is a bisimulation from M to M'

Consider any states $u \in St$ and $u' \in St'$ such that $u \leftrightsquigarrow u'$. (*)

ⓐ For all atoms p, we must show $u \in V(p)$ iff $u' \in V'(p)$

$u \in V(p)$
iff $(M, u) \models p$ (defn 2)
iff $(M', u') \models p$ (*)
iff $u' \in V'(p)$ (defn 2) as required

ⓑ We must show that if $v \in St$ and $u \to v$, then there is $v' \in St'$ such that $u' \to' v'$ and $v \leftrightsquigarrow v'$.

- Assume there exists a $v \in St$ s.t. $u \to v$. (1)
- Assume for the purpose of contradiction that there is no $v' \in St'$ s.t. $u' \to' v'$ and $v \leftrightsquigarrow v'$. (2)
- Let $S' = \{w' \in St' \mid u' \to' w'\}$.
  Note that $(M, u) \models EXT$ (since $u \to v$ (1)) and $u \leftrightsquigarrow u'$ so $(M', u') \models EXT$. So S' is non-empty and finite.

- By (2), for every $w_i \in S'$ there exists a CTL formula ϕ_i such that $(M, v) \models \phi_i$ but $(M', w_i') \not\models \phi_i$.
- It follows that $(M, u) \models EX(\phi_1 \wedge \dots \wedge \phi_n)$ and that $(M', u') \not\models EX(\phi_1 \wedge \dots \wedge \phi_n)$.
- But that contradicts that $u \leftrightarrow u'$ (⊛).
- Hence the forth condition holds.

(c) The back condition can be shown similarly.

(2) We must show that $B(t, t')$

This holds directly from the definition of B and that t and t' are CTL-equivalent.

Q8. We show that CTL and CTL* have the same distinguishing power.

Forwards :
- Assume (M, t) and (M', t') satisfy the same formulas of CTL.
- That is, for any CTL formula ϕ, $(M, t) \models \phi$ iff $(M', t') \models \phi$.
- That is, $t \in M$ and $t' \in M'$ are CTL-equivalent.
- By Q7, $(M, t) \simeq (M', t')$.
- By Q6, truth of CTL* formulas is preserved by bisimulations. That is, $(M, t) \models^* \phi^*$ iff $(M', t') \models^* \phi^*$ for any CTL* formula ϕ^*.
- Hence (M, t) and (M', t') satisfy the same formulas of CTL*, as required.

Backwards :
- Assume (M, t) and (M', t') satisfy the same formulas of CTL*.
- That is, for any CTL* formula ϕ^* $(M, t) \models^* \phi^*$ iff $(M', t') \models^* \phi^*$.
- By Q5a, for every CTL formula ϕ there exists a CTL* formula ϕ' such that ϕ and ϕ' are equivalent : $(M, t) \models \phi$ iff $(M, t) \models^* \phi'$ and $(M', t') \models \phi$ iff $(M', t') \models^* \phi'$.
- By the previous two points, we conclude that $(M, t) \models \phi$ iff $(M', t') \models \phi$ for every CTL formula ϕ.
- That is (M, t) and (M', t') satisfy the same CTL formulas, as required.

Distinguishing power is determined by a logic's ability to discern between particular models, whilst expressiveness refers to the definability of certain properties by formulas of a logic.

Since CTL* is strictly more expressive than CTL, but has the same distinguishing power, this means that although CTL* can express properties which CTL cannot (such as AF $(a \land X a)$ - in all paths, there are two consecutive nodes satisfying a), it is always possible to write a CTL formula which is satisfied in the same finite models as those satisfied by a CTL* formula. Specifically, any non-bisimilar models (as per defn 3) can be distinguished, and so any set of non-bisimilar models could simply be distinguished by a disjunction of CTL formulas.