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CATE

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Exercise Information

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Student Declaration - Version 1

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CO499 CW2

~~1(a) (M, φ R ψ~~

1a) $(M, \pi) \models \varphi R \psi$ iff

$\nexists j$ st. $(M, \pi[j]) \models \varphi$ and $\forall\, 0 \leq i$ $(M, \pi[i]) \models \psi$ OR

$\exists j \geq 0$ st. $(M, \pi[j]) \models \varphi$ and $\forall\, 0 \leq i \leq j$ $(M, \pi[i]) \models \psi$

b) $[[(X\psi) U \varphi] \wedge \psi] \vee (\neg(\text{true}\ U \neg\psi))$

c) $(M, \pi) \models \neg(\text{true}\ U \neg\psi)$

iff

$\forall j \geq 0$

$(M, \pi[j]) \not\models \neg\psi$

iff

$\forall j \geq 0$

$(M, \pi[j]) \models \psi$ $(\square)$

$(M, \pi) \models [[(X\psi) U \varphi] \wedge \psi]$

iff

$\exists j \geq 0$ st.

$(M, \pi[j]) \models \varphi$ and $\forall\, 0 \leq i < j$ ~~$(M, \pi[i]) \models X\psi$ and~~

$(M, \pi[i]) \models X\psi$

and

$(M, \pi) \models \psi$

iff

$\exists j \geq 0$ st.

$(M, \pi[j]) \models \varphi$ and $\forall\, 0 \leq i \leq j$ $(M, \pi[i]) \models \psi$

(Δ)

Putting (Δ), $(\square)$ together we see that these are the same conditions as in (a)

d) ~~GA~~ $(M, \pi) \models G\varphi$ iff $\forall i \geq 0$ $(M, \pi[i]) \models \varphi$

$(M, \pi) \models \bot R \varphi$ iff $\forall i \geq 0$ $(M, \pi[i]) \models \varphi$

since $\nexists j \geq 0$ s.t. $(M, \pi[j]) \models \bot$

$\therefore G\varphi \equiv \bot R \varphi$

2a) $(M, q) \models EF\phi$ iff for some path λ from q

$(M, \lambda) \models true \, U \phi$

iff

$\exists j \geq 0$ st.

$(M, \lambda[j]) \models \phi$

$\therefore (M, q) \models EF\phi$ if for some path λ from q and some $j \geq 0$ $(M, \lambda[j]) \models \phi$

$(M, q) \models AF\phi$

iff

$\forall$ paths λ from ~~q~~ q we have

$(M, \lambda) \models true \, U \phi$

iff

for all paths λ from q, $\exists j \geq 0$ st.

$(M, \lambda[j]) \models \phi$

$(M, q) \models EG\phi$ iff

$(M, q) \models \neg AF \neg \phi$

iff

$\exists$ a path λ from q st.

$(M, \lambda) \models \neg F \neg \phi$

iff

$\nexists j \geq 0$ st. $(M, \lambda[j]) \models \neg \phi$

iff

$\forall j \geq 0$

$(M, \lambda[j]) \models \phi$

$\therefore (M, q) \models EG\phi$ iff for some path λ from q, for all $j \geq 0$

$(M, \lambda[j]) \models \phi$

$(M, q) \models AG\phi$ iff $(M, q) \models \neg EF \neg \phi$

iff

for all paths λ from q

we have

$(M, \lambda) \models \neg F \neg \phi$

iff

$\forall j \geq 0$ we have

$(M, \lambda[j]) \models \phi$

so $(M, q) \models AG\phi$ iff for all paths λ from q

for all $j \geq 0$ ~~$(M, q) \models$~~ $(M, \lambda[j]) \models \phi$

3a) If ϕ is a formula of CTL, then either

ϕ is an atom or:

ϕ is of the form $\neg\psi$ where ψ is a state formula

or ϕ is the conjunction of 2 state formulas.

or

ϕ is of the form $E\psi$ or $A\psi$ for some path formula ψ.

If ϕ is a path formula then ϕ is of the form

$X\psi$ or $\psi U \varphi$ for state formulas ψ, φ

So Any CTL formula fits into the BNF for CTL* so any ~~CTL*~~ CTL formula is a CTL* formula

b) $AXX\phi$ is a CTL* formula but not a CTL formula because in a CTL formula, every temporal operator must be immediately preceeded by exactly one path quantifier. In the formula $AXX\phi$ we have 2 consecutive X operators.

4.) The definition of satisfaction for state formulas is exactly the same in Def 2 and Def 1.7, 1.8.

For path formulas for CTL in Def 1.8, we have

$(M, \lambda) \models X\phi$ iff $(M, \lambda[1]) \models \phi$

where ϕ is a state fmla.

In definition 2, we have

$(M, \lambda) \models X\psi$ iff $(M, \lambda[1, \ldots \infty]) \models \psi$

where ψ is a path fmla

However by Part 6 of def 2

Satisfaction of state formulas on paths is defined in the semantics of CTL*

So by defn 2 $(M, \lambda) \models X\phi$ iff

$(M, \lambda[1, \ldots, \infty]) \models \phi$

iff

$(M, \lambda[1, \ldots, \infty][0]) \models \phi$

iff

$(M, \lambda[1]) \models \phi$ by part 6 (here ϕ is a state fmla)

So we recover the conditions of 1.8

Similary by definition 2 we have
$(M, \lambda) \models \phi \cup \psi$

iff

$(M, \lambda[i \ldots \infty]) \models \psi$ for some $i \geq 0$ and

$(M, \lambda[j, \ldots \infty]) \models \phi \ \forall \ 0 \leq j < i$ (where ϕ, ψ are state fmlas)
Like before using ~~def~~ satisfiability of state fmlas on paths
we get by part 6 of definition 2

$(M, \lambda[i \ldots \infty]) \models \psi \iff$
$(M, \lambda[i \ldots \infty][0]) \models \psi$

$\iff$

$(M, \lambda[i]) \models \psi$

and similarly for ϕ
so by definition 2 we have
$(M, \lambda) \models \phi \cup \psi \iff$

$\exists i$ st. $(M, \lambda[i]) \models \psi$ and $\forall \ 0 \leq j < i$
$(M, \lambda[j]) \models \phi$.

this is the ~~des~~ condition of 1.8

5a) We know that every formula in CTL is a formula in CTL^* by (3a). Moreover ~~the~~ the semantics of CTL^* reduce to the semantics of CTL when restricted to CTL formulas by (4)
So for a CTL formula ϕ, setting $\phi' = \phi$ we get the desired result.

b) Consider the CTL^* formula $AFGa$.

we know that for any state formula ϕ

$AF\phi \equiv \phi \vee AXAF\phi$

so $\phi \equiv AF\phi \wedge \neg AXAF\phi$

so if $AFGa$ can be written as a CTL formula φ

then $\exists$ a CTL fmla $\chi \equiv \varphi \wedge (\neg AX\varphi)$

where $\chi \equiv Ga$

so $\exists$ a CTL fmla χ st.

for (M, s_0)

$(M, s_0) \models Ga$ iff $(M, s_0) \models \chi$

but $(M, s_0) \models Ga$ iff $(M, s_0) \models AGa$

so $\chi \equiv AGa$

but we know from lectures that

$AFAGa \not\equiv AFGa$. Consider the model

we have $s_0 \models AFGa$ but $s_0 \not\models AFAGa$

since

$(s_0)^\omega \not\models FAGa$ because $s_0 \not\models AGa$

because $s_1 \not\models a$

so $AFGa$ cannot be written as a CTL fmla.

b) We prove by induction on connectives, quantifiers and operators
Let ϕ be a state fmla
Say ϕ is an ~~at~~ atom p

then
$(M,t) \models p$ iff $(M',t') \models p$ since
$t \in V(p)$ iff $t' \in V'(p)$ due to bisimilarity

If ϕ is of the form $\neg\psi$
Assume $(M,t) \models \psi$ iff $(M',t') \models \psi$
then $(M,t) \models \neg\psi$ iff
$(M,t) \not\models \psi$ iff $(M',t') \not\models \psi$ iff
$(M',t') \models \neg\psi$

If ϕ is of the form $\psi \wedge \chi$
Assume $(M,t) \models \psi$ iff $(M',t') \models \psi$ and likewise for χ.
$(M,t) \models \psi \wedge \chi$ iff $[(M,t) \models \psi$ and $(M,t) \models \chi]$ iff

$[(M',t') \models \psi$ and $(M',t') \models \chi]$ iff

$(M',t') \models \psi \wedge \chi$

Say
ϕ is of the form $E\psi$, where ψ is a path formula

$(M,t) \models E\psi \Rightarrow \exists$ a path π starting at t st.

$(M,\pi) \models \psi$

~~Rst~~ We have $R(t, \pi[1])$, then by the forth condition of a bisimulation, we know that
$\exists\ t'_1 \in M'$ st. $R'(t', t'_1)$ and $B(\pi[1], t'_1)$

Again $R(\pi[1], \pi[2])$ so by the forth condition $\exists\, t_2' \in M'$ st. $R'(t_1', t_2')$ and $B(\pi[2], t_2')$
we can continue this to obtain a path
$t', t_1', t_2' \ldots = \pi'$
~~st. π is b~~ π is bisimilar to π'
Similarly by using the back condition repeatedly for every path in M' ~~we~~ starting from t' we can obtain a bisimilar path in M, starting from t.

It remains to prove that the truth of path formulae is preserved on bisimilar paths, we do that first, before going back to state formulae.
We do this by induction on the number of connectives and temporal operators.

Say a path formula α is of the form $X\psi$
and we know that given bisimilar paths π, π' in M and M' resp.

$(M, \pi) \models \psi$ iff $(M', \pi') \models \psi$

~~Say~~

So
$(M, \pi) \models X\psi$ iff $(M, \pi[1..\infty]) \models \psi$

iff
$(M', \pi'[1 \ldots, \infty]) \models \psi$ (since $\pi[1 \ldots, \infty]$ is bisimilar to $\pi'[1 \ldots, \infty]$)

$\Longleftarrow$

$(M', \pi') \models X\psi$

Say $\cancel{\phi}^{\chi}$ is of the form $\phi \cup \psi$
Assume $(M, \pi) \models \phi$ iff $(M', \pi') \models \phi$ and likewise for ψ
$(M, \pi) \models \phi \cup \psi$

iff

$\exists j \geq 0$ st. $(M, \pi[j \ldots, \infty]) \models \psi$

and $\forall\ 0 \leq i < j$
$(M, \pi[i \ldots, \infty]) \models \phi$

iff

$(M', \pi'[j \ldots, \infty]) \models \psi$

and

$\forall\ 0 \leq i < j$

$(M', \pi'[i \ldots, \infty]) \models \phi$

iff

$(M', \pi') \models \phi \cup \psi$

The case for conjunction, negation of path fmlas is similar to state fmlas

So bisimilar paths preserve truth of path formulas.

Now considering state fmlas again,

Say we have $(M, t) \models E\psi$ and $B(t, t')$

$\Rightarrow \exists$ a path π starting at t st.

$(M, \pi) \models \psi$

we know $\exists$ a bisimilar path π' starting at t' in M'

so

$(M', \pi') \models \psi$

$\Rightarrow (M', t') \models E\psi$

and the converse direction follows similarly from the fact that every path in M' has a bisimilar path in M

If ϕ is of the form $A\psi$

then:

$\cancel{M, t \models A\psi}$ $(M, t) \models A\psi$

$\Leftrightarrow$ $\cancel{M \not\models E\neg\psi}$ $(M, t) \not\models E\neg\psi$

$\Leftrightarrow (M', t') \not\models E\neg\psi$

$\Leftrightarrow (M', t') \models A\psi$

So the truth of CTL formulas is preserved by bisimulations.

7.) ~~Say~~ $t \in M$, $t' \in M'$ are CTL equivalent.

then clearly for ~~all atoms p,~~ any arbitrary atom p

$(M,t) \models p$ iff $(M, t') \models p$

$\therefore t \in V(p)$ iff $t' \in V'(p)$

~~Say for a c~~

Define the relation $B \subseteq St \times St'$ by
~~B(a,a')~~ $B(a,a')$ iff a, a' are CTL equiv.
We show that B is a bisimulation and $B(t,t')$.
We have already verified condition 1.
For the forth condition
Say for a contradiction

$R(t,u)$ but $\nexists$ $u' \in M'$ st. $R(t',u')$ and $B(u,u')$

Let $S' = \{ w' \in M' \mid R(t', w') \}$
then for every ~~w~~ $w'_i \in S'$, $\exists$ a CTL fmla φ_i

st. $(M, u) \models \varphi_i$ but $(M', w'_i) \not\models \varphi_i$

$\Rightarrow (M,t) \models EX(\varphi_1 \wedge \cdots \wedge \varphi_n)$ but $(M',t') \not\models EX(\varphi_1 \wedge \cdots \varphi_n)$

This contradicts the fact that (t,t') are CTL equiv.

Similarly for the back condition, ~~say we~~ say for a condriction

$R'(t', v')$ and $B(t, t')$
but $\nexists v$ st. $R(t, v)$ and $B(v, v')$

Let $S' = \{w \in M \mid R(t, w)\}$
then $\forall w_i \in S'$ $\exists$ a CTL fmla φ_i st.

$(M', v') \models \varphi_i$ but $(M, w_i) \not\models \varphi_i$

$\therefore (M', t') \models EX(\varphi_1 \wedge \varphi_2 \ldots \wedge \varphi_n)$

but $(M, t) \not\models EX(\varphi_1 \wedge \varphi \ldots \wedge \varphi_n)$

this contradicts the fact that t, t' are CTL equiv.

Hence the back condition is also verified

so $(M, t), (M', t')$ are CTL equivalent

$\Rightarrow t, t'$ are bisimilar

8.) $(M, t), (M', t')$ are CTL equiv iff they are Bisimilar because bisimilarity preserves the truth of CTL* fmlas (Hence CTL fmlas) and CTL equiv $\Rightarrow$ bisimilarity by (7)

$(M, t), (M' t')$ are bisimilar $\Rightarrow (M, t), (M', t')$ are CTL* equiv
by (6)

$(M, t), (M', t')$ are CTL^* equiv
$\Rightarrow$ ~~&~~ they are CTL equiv
$\Rightarrow$ they are bisimilar by (7)

so.

$(M^*, t), (M', t')$ satisfy the same fmlas of CTL

iff

they are bisimilar

iff

$(M, t), (M', t')$ are CTL^* equiv

so we have CTL equivalence iff we have CTL^* equivalence.

This is ~~lik~~ akin to how a Polynomial over $\mathbb{R}$ Can be uniquely specified by ~~a~~ sequence of real numbers, but to write down its roots you need complex numbers which is a more expressive language.

So in the domain of Poly's over $\mathbb{R}$

$\mathbb{R}$ and $\mathbb{C}$ have the same distinguishing power even though ~~$\mathbb{R}$ is~~ $\mathbb{C}$ is more expressive than $\mathbb{R}$