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Heterophilious Dynamics Enhances Consensus*

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Abstract. We review a general class of models for self-organized dynamics based on alignment. The dynamics of such systems is governed solely by interactions among individuals or "agents," with the tendency to adjust to their "environmental averages." This, in turn, leads to the formation of clusters, e.g., colonies of ants, flocks of birds, parties of people, rendezvous in mobile networks, etc. A natural question which arises in this context is to ask when and how clusters emerge through the self-alignment of agents, and what types of "rules of engagement" influence the formation of such clusters. Of particular interest to us are cases in which the self-organized behavior tends to concentrate into *one* cluster, reflecting a *consensus* of opinions, *flocking* of birds, fish, or cells, *rendezvous* of mobile agents, and, in general, concentration of other traits intrinsic to the dynamics.

Many standard models for self-organized dynamics in social, biological, and physical sciences assume that the intensity of alignment increases as agents get closer, reflecting a common tendency to align with those who think or act alike. Moreover, "similarity breeds connection" reflects our intuition that increasing the intensity of alignment as the difference of positions decreases is more likely to lead to a consensus. We argue here that the converse is true: when the dynamics is driven by local interactions, it is more likely to approach a consensus when the interactions among agents *increase* as a function of their difference in position. *Heterophily*, the tendency to bond more with those who are different rather than with those who are similar, plays a decisive role in the process of clustering. We point out that the number of clusters in heterophilious dynamics *decreases* as the heterophily dependence among agents increases. In particular, sufficiently strong heterophilious interactions enhance consensus.

Key words. agent-based models, self-alignment, heterophilious dynamics, clusters, consensus, flocking, active sets, connectivity of graphs, mean-field limits, kinetic equations, hydrodynamics

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Contents

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1. Introduction.

Nature and human societies offer many examples of self-organized behavior. Ants form colonies to coordinate the construction of a new nest, birds form flocks which fly in the same direction, mobile networks are sought to form a coordinated rendezvous, and human crowds form parties to reach a consensus when choosing a leader. The self-organized aspect of such systems is their dynamics, governed solely by interactions among its individuals or "agents," which tend to cluster into colonies, flocks, parties, etc. A natural question which arises in this context is to ask when and how clusters emerge through the self-interactions of agents, and what types of "rules of engagement" influence the formation of such clusters. Of particular interest to us are cases in which the self-organized behavior tends to concentrate into *one* cluster, reflecting a *consensus* of opinions, *flocking* of birds, fish, or cells, *rendezvous* of mobile networks, and, in general, *concentration* around other positions intrinsic to the self-organized dynamics. Generically, we will refer to this process as concentration around an emerging consensus.

Many models have been introduced to appraise the emergence of consensus. Representative examples can be found in [12, 34, 36, 49, 63, 88, 107, 112], and we refer the reader to a more comprehensive list of references surveyed in section 9. The starting point for our discussion is a general framework which embeds several types of models describing self-organized dynamics. We consider the evolution of N agents, each of which is identified by its "position" $\mathbf{p}_i(t) \in \mathbb{R}^d$. The position $\mathbf{p}_i(t)$ may account for opinion, velocity, or other attributes of agent "i" at time t. Each agent adjusts its position according to the position of its neighbors:

$$\frac{d}{dt}\mathbf{p}_i = \alpha \sum_{j \neq i} a_{ij}(\mathbf{p}_j - \mathbf{p}_i), \qquad a_{ij} \geq 0. \tag{1.1}$$

This provides a rather general description for processes of *alignment*. Here, $\alpha > 0$ is a scaling parameter and the coefficients a_{ij} quantify the strength of influence between agents i and j: the larger a_{ij} is, the more weight is given to agent j to align itself with agent i, based on the difference of their positions $\mathbf{p}_i - \mathbf{p}_j$. The underlying fundamental assumption here is that agents react not to the *positions* of others, but to their differences relative to other agents. In particular, the a_{ij}'s themselves are allowed to depend on the relative differences $\mathbf{p}_i - \mathbf{p}_j$. Indeed, we consider nonlinear models (1.1) where

$$a_{ij} = a_{ij}(\mathcal{P}(t)), \qquad \mathcal{P}(t) := \{\mathbf{p}_k(t)\}_k.$$

We emphasize the nonlinear aspect of the alignment models (1.1): the intricate aspect of such models is the nonlinear dependence of the influence matrix on the dynamics, $a_{ij} = a_{ij}(\mathcal{P}(t))$. We ignore two other important processes involved in self-organized dynamics as advocated in the pioneering work of Reynolds [93], namely, the short-range repulsion (or avoidance) and the long-range cohesion (or attraction), and we refer to recent works driven by the balance of these two processes such as [8, 47, 50, 79, 84, 104]. Our purpose here is to shed light on the role of mid-range alignment, which covers the important zone "trapped" between the short-range attraction and long-range repulsion.

We distinguish between two main classes of self-alignment models. In the global case, the rules of engagement are such that every agent is influenced by every other agent, $a_{ij} > \eta > 0$. The dynamics in this case is driven by *global* interactions. We have a fairly good understanding of the large time dynamics of such models; an incomplete list of recent works in this direction includes [11, 17, 36, 44, 60, 61, 63, 69, 75, 88] and the references therein. Global interactions which are sufficiently strong lead to *unconditional consensus* in the sense that *all* initial configurations of agents concentrate around an emerging limit state, the "consensus" $\mathbf{p}^\infty$,

$$\mathbf{p}_i(t) \stackrel{t\to\infty}{\longrightarrow} \mathbf{p}^\infty.$$

Section 2 contains an overview of the concentration dynamics in such global models from the perspective of the general framework of (1.1).

In more realistic models, however, interactions between agents are limited to their local neighbors [1, 4, 35, 71, 93]. The behavior of *local* models, where some of the a_{ij} may vanish, requires a more intricate analysis. In the general scenario for such local models, discussed in section 3, agents tend to concentrate into one or more separate *clusters*. The particular case in which agents concentrate into one cluster, that is, the emergence of a consensus or a flock, depends on the propagation of *uniform connectivity* of the underlying (weighted) graph associated with the adjacency matrix, $\{a_{ij}\}$. This issue is explored in section 4, where we show that connectivity implies consensus. Thus, the question of consensus for local models is turned into the question of persistence of connectivity over time. Note that even if the initial configuration is assumed connected, then there is still a possibility of losing connectivity as the a_{ij}'s may vary in time together with the positions $\mathcal{P}(t)$. The open question of tracing the propagation of connectivity in time for the *general* class of local models (1.1) plays an important role in many applications, beyond the implication of emerging consensus. As an example we mention engineering applications to sensor-based networks, from automatic traffic control and wireless communication to production systems and mobile robot networks, as seen, e.g., in [64, 71, 89, 91, 94, 113, 112] and the references therein.

Many standard models for self-organized dynamics in social, biological, and physical sciences assume that the dependence of a_{ij} decreases as a function of $|\mathbf{p}_i - \mathbf{p}_j|$, where $|\cdot|$ is a problem-dependent proper metric to measure a difference of positions, opinions, etc. The statement that "birds of a feather flock together" reflects a common tendency to align with those who think or act alike [69, 77, 83]. Moreover, "similarity breeds connection" reflects the intuitive scenarios in which the influence coefficients a_{ij} increase as the difference of positions $|\mathbf{p}_i - \mathbf{p}_j|$ decreases: the more the a_{ij}'s increase, the more likely this will lead to a consensus. However, we argue here that the converse is true: for a self-organized dynamics driven by local interactions, it is more likely to approach a consensus when the interaction among agents *increases* as a function of their difference $|\mathbf{p}_i - \mathbf{p}_j|$. *Heterophily*, the tendency to bond, more with the different rather than with those who are similar, plays a decisive role in the clustering of (1.1). The consensus in heterophilious dynamics is explored in the second part of the paper in terms of local interactions of the form $a_{ij} = \phi(|\mathbf{p}_i - \mathbf{p}_j|)$, where $\phi(\cdot)$ is a compactly supported influence function which *is increasing* over its support. In section 5 we report on our extensive numerical simulations which confirm the counterintuitive phenomenon in which the number of clusters *decreases* as the heterophilious dependence increases; in particular, if ϕ is increasing fast enough, then the corresponding dynamics concentrate into one cluster, that is, heterophilious dynamics enhances consensus. We mention in passing the scenario of "extreme heterophily" advocated in [71, 113, 112], where distributed coordination is governed by a local influence function $\phi(\cdot)$ which grows to infinity as it approaches the right edge of its support, in order to create an energy barrier which enforces connectivity and hence consensus. We are not unaware that this phenomenon of enhanced consensus in the presence of heterophilious interactions may have intriguing consequences in areas other than social networks, e.g., global bonding in atomic scales, avoiding materials' fractures in mesoscopic scales, or "cloud" formations in macroscopic scales.

In the rest of the paper, we address a few important extensions of the self-alignment models outlined above. These extensions are still a work in progress and we by no means try to be comprehensive. In section 6 we turn our attention to nearest neighbor dynamics. Careful 3D observations made by the StarFlag project [26, 25, 24] showed that interactions of birds are driven by *topological* neighborhoods, involving a fixed number of nearby birds, instead of geometric neighborhoods involving a fixed radius of interaction. Here we prove that in the simplest case of two nearest neighbor dynamics, connectivity propagates in time and consensus follows for influence functions which are nondecreasing on their compact support. In section 7 we turn our attention to *fully discrete* models for self-alignment. The large time evolution in discrete time-steps, e.g., the opinion of dynamics in [11, 12, 75], may depend on the time-step Δt. Here, we show that the semidiscrete framework for global and local self-alignment outlined in sections 2–5 can be extended, mutatis mutandis, to the fully discrete case. In particular, we recover a decreasing Lyapunov functional, a fully discrete analogue of the semidiscrete clustering analysis in section 4.2. Finally, in section 8 we discuss the passage from the agent-based description to mean-field limits as the number of agents, or "particles," tends to be large enough. There is a growing literature on kinetic descriptions of such models; see [20, 21, 22, 49, 51, 61] and the references therein. Here we focus our attention on the hydrodynamic descriptions of self-organized opinion dynamics and flocking. The closing section 9 is devoted to a more detailed discussion on the broader subject of self-organized dynamics. Since a comprehensive review of this multidisciplinary subject is beyond the scope of this paper, in particular, we include a selection of references, classified into several complementary categories of different disciplines, models, scales, approaches, and patterns.

1.1. Examples of Opinion Dynamics and Flocking. Models for self-organized dynamics (1.1) have appeared in a large variety of different contexts, including load balancing in computer networks, evolution of languages, gossiping, algorithms for sensor networks, emergence of flocks, herds, schools, and other biological "clustering," pedestrian dynamics, ecological models, peridynamic elasticity, multiagent robots, models for opinion dynamics, economic networks, and more; a detailed list of references is surveyed in section 9.

To demonstrate the general framework for self-alignment dynamics (1.1), we shall work with two main concrete examples. The first models *opinions dynamics*. In these models, N agents, each with a vector of opinions quantified by $\mathbf{p}_i \rightsquigarrow \mathbf{x}_i \in \mathbb{R}^d$, interact with each other according to the first-order system

$$\frac{d}{dt}\mathbf{x}_i = \alpha \sum_{j\neq i} a_{ij}(\mathbf{x}_j - \mathbf{x}_i), \quad a_{ij} = \frac{\phi(|\mathbf{x}_j - \mathbf{x}_i|)}{N}. \tag{1.2a}$$

Here, $0 < \phi < 1$ is the scaled *influence function* which acts on the "difference of opinions" $|\mathbf{x}_i - \mathbf{x}_j|$. The metric $|\cdot|$ needs to be properly interpreted, adapted to the specific context of the problem at hand. Another model for interaction of opinions is

$$\frac{d}{dt}\mathbf{x}_i = \alpha \sum_{j\neq i} a_{ij}(\mathbf{x}_j - \mathbf{x}_i), \quad a_{ij} = \frac{\phi_{ij}}{\sum_k \phi_{ik}}, \quad \phi_{ij} := \phi(|\mathbf{x}_j - \mathbf{x}_i|). \tag{1.2b}$$

The classical Krause model for opinion dynamics [75, 12] is a time-discretization of (1.2b), which will be discussed in section 7. Observe that the adjacency matrix $\{a_{ij}\}$ in the first model (1.2a) is symmetric, while in the second model (1.2b) it is not.

Another branch of models has been proposed to describe *flocking*. These are second-order models where the observed property is the velocity of birds, $\mathbf{p}_i \mapsto \mathbf{v}_i \in \mathbb{R}^d$, which are coupled to their location $\mathbf{x}_i \in \mathbb{R}^d$. The flocking model of Cucker and Smale (C-S) has received considerable attention in recent years [36, 37, 61, 17, 60],

$$\frac{d}{dt}\mathbf{v}_i = \alpha \sum_{j\neq i} a_{ij}(\mathbf{v}_j - \mathbf{v}_i), \quad a_{ij} = \frac{\phi(|\mathbf{x}_j - \mathbf{x}_i|)}{N}, \quad \text{where} \quad \frac{d}{dt}\mathbf{x}_i = \mathbf{v}_i. \tag{1.3a}$$

In the C-S model, alignment is carried out by isotropic averaging. In [88] we advocated a more realistic alignment-based model for flocking, where alignment is based on the relative influence, similar to (1.2b),

$$\frac{d}{dt}\mathbf{v}_i = \alpha \sum_{j\neq i} a_{ij}(\mathbf{v}_j - \mathbf{v}_i), \quad a_{ij} = \frac{\phi_{ij}}{\sum_k \phi_{ik}}, \quad \text{with} \quad \phi_{ij} := \phi(|\mathbf{x}_j - \mathbf{x}_i|). \tag{1.3b}$$

Again, the C-S model is based on a symmetric adjacency matrix, $\{a_{ij}\}$, while symmetry is lost in (1.3b), i.e., $a_{ij} \neq a_{ji}$.

The models for opinion and flocking dynamics (1.2) and, respectively, (1.3) can be written in the unified form

$$\frac{d}{dt}\mathbf{p}_i = \alpha \sum_{j=1}^{N} a_{ij}(\mathbf{p}_j - \mathbf{p}_i), \qquad a_{ij} = \frac{1}{\sigma_i}\phi(|\mathbf{x}_i - \mathbf{x}_j|). \tag{1.4}$$

In the opinion dynamics, $\mathbf{p} \mapsto \mathbf{x}$; in the flocking dynamics, $\mathbf{p} \mapsto \dot{\mathbf{x}}$. The degree is $\sigma_i = N$ in the symmetric models, or $\sigma_i = \sum_{j\neq i} \phi(|\mathbf{x}_i - \mathbf{x}_j|)$ in the nonsymmetric models. The local versus global behavior of these models hinges on the behavior of the influence function, ϕ. If the support of ϕ is large enough to cover the convex hull of $\mathcal{P}(0) = \{\mathbf{p}_k(0)\}_k$, then global interactions will yield unconditional consensus or flocking. On the other hand, if ϕ is locally supported, then the group dynamics in (1.4) depends on the connectivity of the underlying graph, $\{a_{ij}\}$. In particular, if the overall connectivity is lost over time, then each connected component may lead to a separate cluster. Heterophilious self-organized dynamics is characterized by a locally supported influence function, ϕ, which is *increasing* as a function of the mutual differences, $\phi_{ij} = \phi(|\mathbf{x}_i - \mathbf{x}_j|)$. The more heterophilious the dynamics is, in the sense that its influence function has a steeper increase over its compact support, the more it tends to concentrate in the sense of approaching a smaller number of clusters. In particular, heterophilious dynamics is more likely to lead to a consensus as demonstrated, for example, in Figure 1 (and further documented in Figures 8 and 13). Observe that the *only* difference between the two models depicted in Figure 1 is that the influence in the immediate neighborhood (of radius $r \leq 1/\sqrt{2}$) was decreased, from $\phi = 1\chi_{[0,1]}$ (at the top) into $\phi = 0.1\chi_{[0,1/\sqrt{2}]} + \chi_{[1/\sqrt{2},1]}$ (at the bottom): this was sufficient to enhance the four-party clustering on the top to become a consensus shown on the bottom.

2. Global Interactions and Unconditional Emergence of Consensus. In this section we derive explicit conditions for global self-organized dynamics (1.1) to concentrate around an emerging consensus. Our starting point is a *convexity argument* which is valid for any adjacency matrix $A = \{a_{ij}\}$, whether symmetric or not. We begin by noting that, without loss of generality, A may be assumed to be row-stochastic,

$$\sum_j a_{ij} = 1, \qquad i = 1, \ldots, N. \tag{2.1}$$

Indeed, by rescaling α if necessary we have $\sum_{j\neq i} a_{ij} \leq 1$, and (2.1) holds when we set $a_{ii} := 1 - \sum_{j\neq i} a_{ij} \geq 0$. We rewrite (1.1) in the form

$$\frac{d}{dt}\mathbf{p}_i = \alpha\left(\overline{\mathbf{p}}_i - \mathbf{p}_i\right), \qquad \overline{\mathbf{p}}_i := \sum_{j=1}^{N} a_{ij}\mathbf{p}_j. \tag{2.2}$$

Thus, if we let $\Omega(t)$ denote the convex hull of the properties $\{\mathbf{p}_k\}_k$, then, according to (2.2), $\mathbf{p}_i$ is relaxing to the average value $\overline{\mathbf{p}}_i \in \Omega(t)$, while the boundary of Ω is a *barrier* for the dynamics, as shown in Figure 2. It follows that the positions in the general self-organized model (1.1) remain bounded.

PROPOSITION 2.1. *The convex hull of* $\mathbf{p}(t)$ *is decreasing in time in the sense that the convex hull,* $\Omega(t) := Conv\left(\{\mathbf{p}_i(t)\}_{i\in[1,N]}\right)$*, satisfies*

$$\Omega(t_2) \subset \Omega(t_1), \quad t_2 > t_1 \geq 0. \tag{2.3}$$

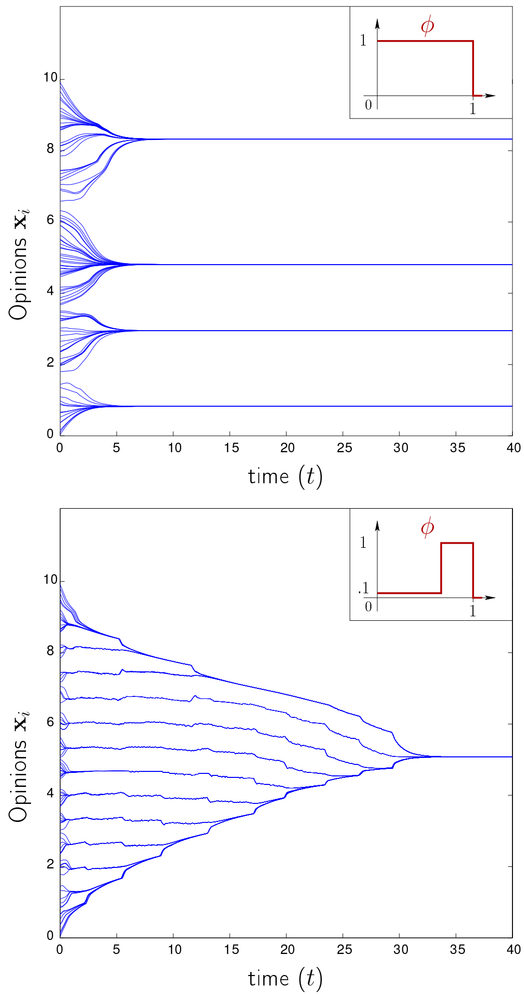

Fig. 1 *Evolution in time of the consensus model for two different influence functions* ϕ *(top:* $\phi(r) = \chi_{[0,1]}$*; bottom:* $\phi(r) = .1\chi_{[0,1/\sqrt{2}]} + \chi_{[1/\sqrt{2},1]}$*). By diminishing the influence of close neighbors (bottom), we enhance the emergence of a consensus. Simulations are started with the same initial condition (*100 *agents uniformly distributed on* $[0, 10]$*).*

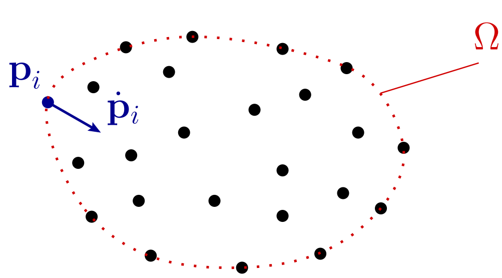

Fig. 2 *The convex hull* Ω *of the positions* $\mathbf{p}_i$*.*

Moreover, we have

$$\max_i |\mathbf{p}_i(t)| \leq \max_i |\mathbf{p}_i(0)|. \tag{2.4}$$

Proof. We verify (2.4) for a general vector norm $|\cdot|$ which we characterize in terms of its dual $|\mathbf{w}|_* = \sup_{\mathbf{w}\neq 0}\langle \mathbf{w}, \mathbf{z}\rangle/|\mathbf{z}|$, so that $|\mathbf{p}| = \sup\langle \mathbf{p}, \mathbf{w}\rangle/|\mathbf{w}|_*$. Let $\mathbf{w} = \mathbf{w}(t)$ denote the maximal dual vector of $\mathbf{p}_i(t)$, so that $\langle \mathbf{p}_i, \mathbf{w}\rangle = |\mathbf{p}_i|$; then

$$\langle \dot{\mathbf{p}}_i, \mathbf{w}\rangle = \alpha\left(\langle \overline{\mathbf{p}}_i, \mathbf{w}\rangle - \langle \mathbf{p}_i, \mathbf{w}\rangle\right) \leq \alpha(|\overline{\mathbf{p}}_i| - |\mathbf{p}_i|).$$

Since $\langle \mathbf{p}_i, \dot{\mathbf{w}}\rangle \leq 0$, we have

$$\frac{d}{dt}|\mathbf{p}_i(t)| = \langle \dot{\mathbf{p}}_i, \mathbf{w}\rangle + \langle \mathbf{p}_i, \dot{\mathbf{w}}\rangle \leq \alpha(|\overline{\mathbf{p}}_i(t)| - |\mathbf{p}_i(t)|),$$

and, finally, $|\overline{\mathbf{p}}_i(t)| \leq \max_i |\mathbf{p}_i(t)|$ yields (2.4). ◻

REMARK 2.2. *Since the models of opinion dynamics and flocking dynamics* (1.4) *are translation invariant in the sense of admitting the family of solutions* $\{\mathbf{p}_i - \mathbf{c}\}$, *then, for any fixed state* $\mathbf{c}$, *Proposition* 2.1 *implies*

$$\max_i |\mathbf{p}_i(t) - \mathbf{c}| \leq \max_i |\mathbf{p}_i(0) - \mathbf{c}|.$$

Consensus and flocking are achieved when the decreasing $\Omega(t)$ shrinks to a limit point $\Omega(t) \overset{t\to\infty}{\longrightarrow} \{\mathbf{p}^\infty\}$,

$$\max_i |\mathbf{p}_i(t) - \mathbf{p}^\infty| \overset{t\to\infty}{\longrightarrow} 0.$$

There are various approaches, not unrelated, to deriving conditions which ensure unconditional consensus or flocking. We shall mention two: an L^∞ contraction argument and an L^2 energy method based on spectral analysis.

2.1. An L^∞ Approach: Contraction of Diameters. Proposition 2.1 tells us that $\{\mathbf{p}_i(t)\}_i$ remain uniformly bounded and the diameter, $\max_{ij}|\mathbf{p}_i(t) - \mathbf{p}_j(t)|$, is nonincreasing in time. In order to have concentration, however, we need to verify that the diameter of $\mathbf{p}(t)$ decays to zero. The next proposition quantifies this decay rate.

THEOREM 2.3. *Consider the self-organized model* (1.1) *with a row-stochastic adjacency matrix A* (2.1). *Let*

$$[\mathbf{p}] := \max_{ij} |\mathbf{p}_i - \mathbf{p}_j|$$

denote the diameter of the position vector $\mathbf{p}$. *Then the diameter satisfies the concentration estimate*

$$\frac{d}{dt}[\mathbf{p}(t)] \leq -\alpha\eta_{A(\mathcal{P}(t))}[\mathbf{p}(t)], \qquad \eta_A := \min_{ij}\sum_k \min\{a_{ik}, a_{jk}\}. \tag{2.5}$$

In particular, if there is a slow decay of the concentration factor so that $\int^\infty \eta_{A(\mathcal{P}(s))}\,ds = \infty$, *then the agents concentrate in the sense that*

$$\Theta(t) := \int^t \eta_{A(\mathcal{P}(s))}\,ds \overset{t\to\infty}{\longrightarrow} \infty \qquad \rightsquigarrow \qquad \lim_{t\to\infty}\max_{i,j}|\mathbf{p}_i(t) - \mathbf{p}_j(t)| = 0. \tag{2.6a}$$

Moreover, if the decay of the concentration factor is slow enough in the sense that

$\int^{\infty} \exp(-\alpha\Theta(s))ds < \infty$, *then there is an emerging consensus* $\mathbf{p}^{\infty} \in \Omega(0)$,
(2.6b)
$$\int^{\infty} e^{-\alpha\Theta(t)}dt < \infty \qquad \rightsquigarrow \qquad |\mathbf{p}_i(t) - \mathbf{p}^{\infty}| \lesssim e^{-\alpha\Theta(t)}[\mathbf{p}(0)] \qquad \textit{for all} \ \ i = 1, \ldots, N.$$

REMARK 2.4. *We note that Theorem* 2.3 *applies to any vector norm* $|\cdot|$.

Proof. We begin with the following estimate, which quantifies the contractivity of the row-stochastic A in the induced vector *semi*norm $[\,\cdot\,]$ (since this bound is solely due to the convexity of the row-stochastic A, we suppress the time-dependence of $\mathbf{p}$ and $\overline{\mathbf{p}} = A\mathbf{p}$),

$\boxed{\min_{ij}}$

$$(2.7) \qquad [A\mathbf{p}] \leq (1 - \eta_{_A})[\mathbf{p}], \qquad [\mathbf{p}] = \max_{ij} |\mathbf{p}_i - \mathbf{p}_j|, \quad 1 - \eta_{_A} = \frac{1}{2}\sum_k |a_{ik} - a_{jk}|.$$

The estimate (2.7) in its ℓ^1-dual form for column-stochastic matrices goes back to Dobrushin [46], and his so-called coefficient of ergodicity, $\eta_{_A}$, was later used to quantify the relative entropy in discrete Markov processes [29, 30] and the contractivity in models of opinion dynamics [75]. For completeness, we proceed with the proof for general vector norms $|\cdot|$. Fix any i and j, which are to be chosen later, and set $\eta_k := \min\{a_{ik}, a_{jk}\}$ so that $a_{ik} - \eta_k$ and $a_{jk} - \eta_k$ are nonnegative. Then, for arbitrary $\mathbf{w} \in \mathbb{R}^d$, we have

$$\begin{aligned}
\langle \overline{\mathbf{p}}_i - \overline{\mathbf{p}}_j, \mathbf{w}\rangle &= \sum_k a_{ik}\langle \mathbf{p}_k, \mathbf{w}\rangle - \sum_k a_{jk}\langle \mathbf{p}_k, \mathbf{w}\rangle \\
&= \sum_k (a_{ik} - \eta_k)\langle \mathbf{p}_k, \mathbf{w}\rangle - \sum_k (a_{jk} - \eta_k)\langle \mathbf{p}_k, \mathbf{w}\rangle \\
&\leq \sum_k (a_{ik} - \eta_k)\max_k \langle \mathbf{p}_k, \mathbf{w}\rangle - \sum_k (a_{jk} - \eta_k)\min_k \langle \mathbf{p}_k, \mathbf{w}\rangle \\
&= (1 - \eta_{_A})\left(\max_k \langle \mathbf{p}_k, \mathbf{w}\rangle - \min_k \langle \mathbf{p}_k, \mathbf{w}\rangle\right) \\
&\leq (1 - \eta_{_A})\max_{k\ell} \langle \mathbf{p}_k - \mathbf{p}_\ell, \mathbf{w}\rangle \leq \Big(1 - \eta_{_A}\Big)\max_{k,\ell} |\mathbf{p}_k - \mathbf{p}_\ell||\mathbf{w}|_*.
\end{aligned}$$

In the last step, we characterize the norm $|\cdot|$ by its dual $|\mathbf{w}|_* = \sup_{\mathbf{w}\neq 0}\langle \mathbf{w}, \mathbf{z}\rangle/|\mathbf{z}|$ so that $\langle \mathbf{z}, \mathbf{w}\rangle \leq |\mathbf{z}||\mathbf{w}|_*$. Now, choose i and j as a maximal pair such that $[\overline{\mathbf{p}}] = |\overline{\mathbf{p}}_i - \overline{\mathbf{p}}_j|$; we then have

$$[A\mathbf{p}] \equiv [\overline{\mathbf{p}}] = |\overline{\mathbf{p}}_i - \overline{\mathbf{p}}_j| = \sup_{\mathbf{w}\neq 0} \frac{\langle \overline{\mathbf{p}}_i - \overline{\mathbf{p}}_j, \mathbf{w}\rangle}{|\mathbf{w}|_*} \leq (1 - \eta_{_A})\max_{k,\ell}|\mathbf{p}_k - \mathbf{p}_\ell|,$$

and (2.7) now follows.

Next, we consider the discrete time-marching system associated with (1.1),

$$\frac{\mathbf{p}(t + \Delta t) - \mathbf{p}(t)}{\Delta t} = \alpha\left(A\mathbf{p}(t) - \mathbf{p}(t)\right).$$

Using (2.7) we obtain

$$[\mathbf{p}(t + \Delta t)] = [(1 - \alpha\Delta t)\mathbf{p}(t) + \alpha\Delta t\, A\mathbf{p}(t)] \leq (1 - \alpha\Delta t)[\mathbf{p}(t)] + \alpha\Delta t(1 - \eta_{_A})[\mathbf{p}(t)],$$

or, after rearrangement,

$$\frac{[\mathbf{p}(t + \Delta t)] - [\mathbf{p}(t)]}{\Delta t} \leq -\alpha\eta_{_A}[\mathbf{p}(t)],$$

and the desired bound (2.5) follows by letting $\Delta t \to 0$. In particular, we have

$$\max_{ij} |\mathbf{p}_i(t) - \mathbf{p}_j(t)| \leq \exp\left(-\alpha \int_0^t \eta_{A(\mathcal{P}(s))} ds\right) [\mathbf{p}(0)] \stackrel{t\to\infty}{\longrightarrow} 0, \tag{2.8}$$

which proves (2.6a). Moreover,

$$\begin{aligned} |\mathbf{p}_i(t_2) - \mathbf{p}_i(t_1)| &= \left| \int_{\tau=t_1}^{t_2} \dot{\mathbf{p}}_i(\tau)\, d\tau \right| \leq \alpha \max_{ij} \int_{\tau=t_1}^{t_2} |\mathbf{p}_i(\tau) - \mathbf{p}_j(\tau)|\, ds \\ &\leq \alpha \int_{\tau=t_1}^{t_2} \exp\left(-\alpha\Theta(\tau)\right) d\tau\, [\mathbf{p}(0)], \qquad \Theta(\tau) = \int_0^\tau \eta_{A(\mathcal{P}(s)}\, ds, \end{aligned}$$

which tends to zero, $|\mathbf{p}_i(t_2) - \mathbf{p}_i(t_1)| \to 0$ for $t_2 > t_1 \gg 1$, thanks to our assumption (2.6b). It follows that the limit $\mathbf{p}_i(t) \stackrel{t\to\infty}{\longrightarrow} \mathbf{p}_i^\infty$ exists, and hence all agents concentrate around the same limit position, an emerging consensus $\mathbf{p}^\infty \in \Omega(0)$. The concentration rate estimate (2.6b) follows from (2.8). ◻

Theorem 2.3 relates the emergence of consensus or flocking of $\dot{\mathbf{p}} = A\mathbf{p} - \mathbf{p}$ to the behavior of $\int^t \eta_{A(\mathcal{P}(s))} ds \uparrow \infty$, and to this end we seek lower bounds on the "concentration factor" η_A, which are easily checkable in terms of the entries of A. This brings us to the following definition.

Definition 2.5 (active sets [88]). *Fix $\theta > 0$. The active set, $\Lambda(\theta)$, is the set of agents which influence every other agent "more" than θ,*

$$\Lambda(\theta) := \{ j \mid a_{ij} \geq \theta \ for\ any \ \ i\}. \tag{2.9}$$

Observe that since a_{ij} changes in time, $a_{ij} = a_{ij}(\mathcal{P}(t))$, the number of agents in the active set $\Lambda(\theta)$ is a time-dependent quantity, denoted $\lambda(\theta) = \lambda(\theta, t) := |\Lambda(\theta, t)|$.

The straightforward lower bound $\eta_A \geq \max_\theta \theta \cdot \lambda(\theta)$ yields the following corollary.

Corollary 2.6. *The diameter of the self-organized model* (1.1) *with a stochastic adjacency matrix A, as in* (2.1)*, satisfies the concentration estimate*

$$\frac{d}{dt}[\mathbf{p}(t)] \leq -\alpha(\max_\theta \theta \cdot \lambda(\theta, t))\, [\mathbf{p}(t)]. \tag{2.10}$$

In particular, the lower bound $\eta_A \geq N \min_{ij} a_{ij}$, corresponding to $\theta = \min_{ij} a_{ij}$ with $\lambda(\theta, t) = N$, yields [61]

$$|\mathbf{p}(t) - \mathbf{p}^\infty| \lesssim \exp\left(-\alpha N \int_0^t m(s) ds\right) [\mathbf{p}(0)], \qquad m(s) := \min_{ij} a_{ij}(s). \tag{2.11}$$

Remark 2.7. *The bound* (2.10) *is an improvement of the "flocking" estimate* [88, Lemma 3.1]

$$\frac{d}{dt}[\mathbf{p}(t)] \leq -\alpha(\max_\theta \theta \cdot \lambda(\theta, t))^2\, [\mathbf{p}(t)].$$

Corollary 2.6 is a useful tool to verify consensus and flocking behavior for general adjacency matrices $A = \{a_{ij}\}$, whether symmetric or not. We demonstrate its application with the following sufficient condition for the emergence of a consensus in the opinion models (1.2). In either the symmetric or the nonsymmetric case,

$$a_{ij} = \left\{ \begin{array}{c} \dfrac{\phi_{ij}}{N} \\ \\ \dfrac{\phi_{ij}}{\sigma_i} \end{array} \right\} \geq \frac{\phi([\mathbf{x}(t)])}{N}, \qquad \sigma_i = \sum_k \phi_{ik} \leq N.$$

By Proposition 2.1, the diameter $[\mathbf{x}(t)]$ is nonincreasing, yielding the lower bound

$$Na_{ij}(\mathcal{P}(t)) = \frac{N}{\sigma_i}\phi(|\mathbf{x}_i(t) - \mathbf{x}_j(t)|) \geq \min_{r\leq[\mathbf{x}(t)]}\phi(r) \geq \min_{r\leq[\mathbf{x}(0)]}\phi(r),$$

which in turns implies the following exponentially fast convergence toward a consensus $\mathbf{x}^\infty$.

Proposition 2.8 (unconditional consensus). *Consider the models for opinion dynamics* (1.2) *with an influence function* $\phi(r) \leq 1$, *and assume that*

$$m := \min_{r\leq[\mathbf{x}(0)]}\phi(r) > 0. \tag{2.12}$$

Then there is an exponentially fast convergence toward an emerging consensus $\mathbf{x}^\infty$,

$$|\mathbf{x}_i(t) - \mathbf{x}^\infty| \lesssim e^{-\alpha m t}[\mathbf{x}(0)]. \tag{2.13}$$

Similar arguments apply for the flocking models (1.3): since $[\mathbf{v}(t)]$ is nonincreasing, then $[\mathbf{x}(t)] \leq [\mathbf{x}(0)] + t[\mathbf{v}(0)]$ and hence

$$Na_{ij}(\mathcal{P}(t)) = \frac{N}{\sigma_i}\phi(|\mathbf{x}_i(t) - \mathbf{x}_j(t)|) \geq \min_{r\leq[\mathbf{x}(t)]}\phi(r) \geq \min_{r\leq[\mathbf{x}(0)]+t[\mathbf{v}(0)]}\phi(r);$$

if $\phi(\cdot)$ is decreasing, then we can set $m(t) = \phi([\mathbf{x}(0)] + t[\mathbf{v}(0)])$ and unconditional flocking follows from for Corollary 2.6 for sufficiently strong interaction such that $\int^\infty \phi(s)ds = \infty$. In fact, a more precise statement of flocking is summarized in the following proposition.

Proposition 2.9 (unconditional flocking). *Consider the flocking dynamics* (1.3) *with a decreasing influence function* $\phi(r) \leq \phi(0) \leq 1$, *and assume that*

$$\int^\infty \phi(s)ds = \infty. \tag{2.14}$$

Then the diameter of positions remains uniformly bounded, $[\mathbf{x}(t)] \leq D_\infty < \infty$, *and there is an exponentially fast concentration of velocities around a flocking state* $\mathbf{v}^\infty$,

$$|\mathbf{v}_i(t) - \mathbf{v}^\infty| \lesssim e^{-\alpha m t}[\mathbf{v}(0)], \qquad m = \phi(D_\infty). \tag{2.15}$$

Proof. Unlike the first-order models for consensus, the diameter in second-order flocking models, $[\mathbf{x}(t)]$, may increase over time. The bound D_∞ stated in (2.15) places a uniform bound on the maximal active diameter. To derive such a bound, observe that in the second-order flocking models, the evolution of the diameter of velocities satisfies

$$\frac{d}{dt}[\mathbf{v}(t)] \leq -\alpha\phi([\mathbf{x}(t)])[\mathbf{v}(t)],$$

and is coupled with the evolution of positions $[\mathbf{x}(t)]$: since $\dot{\mathbf{x}} = \mathbf{v}$, we have

$$\frac{d}{dt}[\mathbf{x}(t)] \leq [\mathbf{v}(t)].$$

The last two inequalities imply that the energy functional introduced by Ha and Liu [60],

$$\mathcal{E}(t) := [\mathbf{v}(t)] + \alpha\int_0^{[\mathbf{x}(t)]}\phi(s)ds,$$

is decreasing in time,

$$\alpha \int_{[\mathbf{x}(0)]}^{[\mathbf{x}(t)]} \phi(s)\,ds \le [\mathbf{v}(0)] - [\mathbf{v}(t)] \le [\mathbf{v}(0)]. \tag{2.16}$$

This, together with our assumption (2.14), yields the existence of a finite $D_\infty > [\mathbf{x}(0)]$ such that

$$\alpha \int_{[\mathbf{x}(0)]}^{[\mathbf{x}(t)]} \phi(s)\,ds \le [\mathbf{v}(0)] \le \alpha \int_{[\mathbf{x}(0)]}^{D_\infty} \phi(s)\,ds. \tag{2.17}$$

Thus, the active diameter of positions does not exceed $[\mathbf{x}(t)] \le D_\infty$, and since ϕ is assumed decreasing, the minimal interaction is $Na_{ij} \ge \phi([\mathbf{x}(t)]) \ge \phi(D_\infty)$, which yields

$$\frac{d}{dt}[\mathbf{v}(t)] \le -\alpha\phi(D_\infty)[\mathbf{v}(t)].$$

This concludes the proof of (2.15). □

REMARK 2.10 (global interactions). *Proposition* 2.8 *derives an unconditional consensus under the assumption of* global interaction*, namely, according to* (2.12) *every agent interacts with every other agent as*

$$a_{ij} \ge \frac{1}{N}\phi(|\mathbf{x}_i - \mathbf{x}_j|) \ge \frac{m}{N} > 0.$$

Similarly, the unconditional flocking stated in Proposition 2.9 *requires global interactions, in the sense of having an influence function* (2.14) *which is supported over the entire flock. Indeed, if the influence function ϕ is compactly supported,* $\mathrm{supp}\{\phi\} = [0, R]$*, then assumption* (2.14) *tells us that*

$$[\mathbf{v}(0)] \le \alpha \int_{[\mathbf{x}(0)]}^{R} \phi(s)\,ds;$$

however, according to (2.16), $\alpha \int_{[\mathbf{x}(0)]}^{[\mathbf{x}(t)]} \phi(s)\,ds \le [\mathbf{v}(0)]$ *and hence the support of ϕ remains larger than the diameter of positions, $R \ge [\mathbf{x}(t)]$.*

Proposition 2.9 recovers the unconditional flocking results for the C-S model, $\phi(r) \propto (1+r)^{-2\beta}, \beta > 1/2$, obtained elsewhere in using spectral analysis, ℓ_1-, ℓ_2-, and ℓ_∞-based estimates [36, 61, 17, 60, 21]. The derivations are different, yet they all require the symmetry of the C-S influence matrix, $a_{ij} = \phi_{ij}/N$. Here, we unify and generalize the results, covering both the symmetric and nonsymmetric scenarios. In particular, we improve here the unconditional flocking result in the nonsymmetric model obtained in [88, Theorem 4.1]. Although the tools are different, notably, lack of conservation of momentum $\frac{1}{N}\sum_i \mathbf{v}_i(t)$ in the nonsymmetric case—we nevertheless end up with the same condition (2.14) for unconditional flocking.

2.2. Spectral Analysis of Symmetric Models. A more precise description of the concentration phenomenon is available for models governed by *symmetric* influence matrices, $a_{ij} = a_{ji}$, such as (1.2a) and (1.3a). Set $\mathbf{q}_i = \mathbf{p}_i - \langle \mathbf{p} \rangle$, where $\langle \mathbf{p} \rangle := 1/N \sum_i \mathbf{p}_i$ is the average (total momentum), which thanks to symmetry is conserved in time, $\dot{\langle \mathbf{p} \rangle}(t) \propto \sum_{ij} a_{ij}(\mathbf{p}_i - \mathbf{p}_j) = 0$, and hence the symmetric system (1.1) reads

$$\frac{d}{dt}\mathbf{q}_i(t) = \alpha \sum_{j=1}^{N} a_{ij}(\mathbf{q}_j - \mathbf{q}_i), \qquad \mathbf{q}_i := \mathbf{p}_i - \langle \mathbf{p} \rangle.$$

Let $L_A := I - A$ denote the *Laplacian matrix* associated with A, with ordered eigenvalues $0 = \lambda_1(L_A) \le \lambda_2(L_A) \le \cdots \lambda_N(L_A)$. The following estimate is at the heart of the matter (here, $|\cdot|$ denotes the usual Euclidean norm on $\mathbb{R}^d$):

(2.18)
$$\frac{1}{2}\frac{d}{dt}\sum_i |\mathbf{q}_i(t)|^2 = \alpha\sum_{i,j} a_{ij}\langle \mathbf{q}_j - \mathbf{q}_i, \mathbf{q}_i\rangle = -\frac{\alpha}{2}\sum_{ij} a_{ij}|\mathbf{q}_j - \mathbf{q}_i|^2 \le -\alpha\lambda_2(L_A)\sum_i |\mathbf{q}_i(t)|^2.$$

The second equality is a straightforward consequence of A being symmetric; the following inequality follows from the Courant–Fischer characterization of the second eigenvalue of L_A in terms of vectors $\mathbf{q}$ orthogonal to the first eigenvector $\mathbf{1} = (1, 1, \ldots, 1)^\top$,

$$\lambda_2(L_A) = \min_{\sum \mathbf{q}_k = 0} \frac{\langle L_A\mathbf{q}, \mathbf{q}\rangle}{\langle \mathbf{q}, \mathbf{q}\rangle} \le \frac{(1/2)\sum_{ij} a_{ij}|\mathbf{q}_i - \mathbf{q}_j|^2}{\sum_i |\mathbf{q}_i|^2}. \tag{2.19}$$

We end up with the following sufficient condition for the emergence of unconditional concentration.

THEOREM 2.11 (unconditional concentration in the symmetric case). *Consider the self-organized model* (1.1), (2.1) *with a symmetric adjacency matrix A. Then the following concentration estimate holds:*

(2.20)
$$\vee_{\mathbf{p}(t)} \le \exp\left(-\alpha\int^t \lambda_2(L_{A(\mathcal{P}(s))})ds\right)\vee_{\mathbf{p}(0)}, \qquad \vee^2_{\mathbf{p}(t)} := \frac{1}{N}\sum_i |\mathbf{p}_i(t) - \langle\mathbf{p}\rangle(0)|^2.$$

In particular, if the interactions remain "sufficiently strong" so that $\int^\infty \lambda_2(L_{A(\mathcal{P}(s))})ds = \infty$, then there is convergence toward consensus $\mathbf{p}_i(t) \to \mathbf{p}^\infty = \langle\mathbf{p}\rangle(0)$.

To apply Theorem 2.11, we need to trace effective lower bounds on $\lambda_2(L_A)$; there follow two examples which recover our previous results in section 2.1.

Example 1 (revisiting Theorem 2.3). If $\mathbf{r}$ is the Fiedler eigenvector associated with $\lambda_{N-1}(A)$ with $\mathbf{r} \perp \mathbf{1}$, then (2.7) implies

$$\lambda_{N-1}(A) = \frac{[A\mathbf{r}]}{[\mathbf{r}]} \le \sup_{\mathbf{p}\perp\mathbf{1}} \frac{[A\mathbf{p}]}{[\mathbf{p}]} \le 1 - \eta_{_A}.$$

We end up with the following lower bound for the Fiedler number:

$$\lambda_2(L_A) = 1 - \lambda_{N-1}(A) \ge 1 - (1 - \eta_{_A}) \ge \eta_{_A}.$$

Thus, Theorem 2.3 is recovered here as a special case of the sharp bound (2.20) in Theorem 2.11. The former has the advantage that it applies to nonsymmetric models, but, as remarked earlier, it is limited to models with global interactions; the latter can address the consensus of local, connected models; see section 6.

We remark in passing that while Theorem 2.3 employs the ℓ^∞-based diameter, $[\mathbf{p}] = [\mathbf{p}]_\infty = \max_{ij}|\mathbf{p}_i - \mathbf{p}_j|$, Theorem 2.11 is in fact the corresponding ℓ^2-based diameter, $[\mathbf{p}]_2^2 := \sum_{ij}|\mathbf{p}_i - \mathbf{p}_j|^2/(2N) = \vee_{\mathbf{p}}$.

Example 2 (revisiting Propositions 2.8 and 2.9). *A straightforward lower bound $\lambda_2(L_A) \ge N\min a_{ij}$ recovers Corollary* 2.6*:*

$$\vee_{\mathbf{p}(t)} \le \exp\left(-\alpha\int^t m(s)ds\right)\vee_{\mathbf{p}(0)}, \qquad m(t) := \min_{ij}\phi(|\mathbf{x}_i(t) - \mathbf{x}_j(t)|). \tag{2.21}$$

The characterization of concentration in Theorem 2.11 is sharp in the sense that the estimate (2.18) is sharp. Indeed, it is well known that positivity of the Fiedler number, $\lambda_2(L_A) > 0$, characterizes the *algebraic connectivity* of the graph associated with the adjacency matrix A [53, 87, 28]. Theorem 2.11 places a minimal requirement on the amount of *connectivity as a necessary condition* for consensus.[1] There are many characterizations for the algebraic connectivity of *static* graphs [28, 45, 53, 54, 56, 86, 87, 95]. In the present context of self-organized dynamics (1.1), however, the dynamics of $\dot{\mathbf{p}} = \alpha(A\mathbf{p} - \mathbf{p})$ dictates the connectivity of $A = A(\mathcal{P}(t))$, which in turn determines the clustering behavior of the dynamics due to the nonlinear dependence, $A = A(\mathcal{P}(t))$. Thus, the intricate aspect of the self-organized dynamics (1.1) lies in tracing its algebraic connectivity over time through the self-propelled mechanism in which the nonlinear dynamics and algebraic connectivity are tied together. This issue will be explored in the following sections dealing with clustering driven by *local* interactions.

3. Local Interactions and Clustering.

In this section we consider the self-organized dynamics (1.1) of a "crowd" of N agents, $\mathcal{P} = \{\mathbf{p}_i\}_{i=1}^N$, which do not interact globally: entries in their adjacency matrix may vanish, $a_{ij} \geq 0$. The dynamics is dictated by local interactions and its large time behavior leads to the formation of one or more *clusters*.

3.1. The Formation of Clusters.

A cluster $\mathcal{C}$ is a connected subset of agents, $\{\mathbf{p}_i\}_{i\in\mathcal{C}}$, which is separated from all other agents outside $\mathcal{C}$, namely,

$$\#1.\quad a_{ij} \neq 0 \quad \text{for all} \quad i,j \in \mathcal{C}; \quad \text{and} \quad \#2.\quad a_{ij} = 0 \quad \text{whenever} \quad i \in \mathcal{C} \text{ and } j \notin \mathcal{C}.$$

The important feature of such clusters is their self-contained dynamics, in the sense that

$$\frac{d}{dt}\mathbf{p}_i = \alpha \sum_{j\in\mathcal{C}} a_{ij}(\mathbf{p}_j - \mathbf{p}_i), \quad \sum_{j\in\mathcal{C}} a_{ij} = 1, \qquad i \in \mathcal{C}.$$

The dynamics of such self-contained clusters is covered by the concentration statements of global dynamics in section 2. In particular, if cluster $\mathcal{C}(t)$ remains connected and isolated for a sufficiently long time, then its agents will tend to concentrate around a local consensus,

$$\mathbf{p}_i(t) \stackrel{t\to\infty}{\longrightarrow} \mathbf{p}_{\mathcal{C}}^\infty \quad \text{for all} \quad i \in \mathcal{C}.$$

The intricate aspect, however, is the last *if* statement: the evolution of agents in a cluster $\mathcal{C}$ may become influenced by non-$\mathcal{C}$ agents, and, in particular, different clusters may merge over time.

In the following, we fix our attention on the particular models for opinion and flocking dynamics expressed in the unified framework (1.4):

$$\text{(3.1a)} \qquad \frac{d}{dt}\mathbf{p}_i = \alpha \sum_{j=1}^N a_{ij}(\mathbf{p}_j - \mathbf{p}_i), \qquad a_{ij} = a_{ij}(\mathbf{x}) = \frac{1}{\sigma_i}\phi(|\mathbf{x}_i - \mathbf{x}_j|).$$

Recall that $\mathbf{p} \mapsto \mathbf{x}$ in opinion dynamics, $\mathbf{p} \mapsto \dot{\mathbf{x}}$ in flocking dynamics, and σ_i is the degree,

$$\text{(3.1b)} \qquad \begin{cases} \sigma_i = N, & \text{symmetric model,} \\ \sigma_i = \sum_{j\neq i} \phi(|\mathbf{x}_i - \mathbf{x}_j|), & \text{nonsymmetric model.} \end{cases}$$

[1] We ignore possible cases in which the self-organized dynamics may regain connectivity under "cluster dynamics," namely, agents separated into disconnected clusters and merging into each other at a later stage.

We assume that the influence function ϕ is compactly supported,

$$\text{Supp}\{\phi(\cdot)\} = [0, R]. \tag{3.2}$$

A cluster $\mathcal{C} = \mathcal{C}(t) \subset \{1, 2, \ldots, N\}$ is specified by the finite diameter of the influence function ϕ such that the following two properties hold:

$$\#1. \quad \max_{i,j \in \mathcal{C}(t)} |\mathbf{x}_i(t) - \mathbf{x}_j(t)| \le R; \quad \text{and} \quad \#2. \quad \min_{i \in \mathcal{C}(t), j \notin \mathcal{C}(t)} |\mathbf{x}_i(t) - \mathbf{x}_j(t)| > R.$$

When the dynamics is global, $R \gg [\mathbf{x}(0)]$, then the whole crowd of agents can be considered as one connected cluster. Here we consider the *local* dynamics when R is small enough relative to the active diameter of the global dynamics: $R < [\mathbf{x}(0)]$ in the opinion dynamics (1.2), or $R < D_\infty$ in the flocking dynamics (1.3). The statements of global concentration toward a consensus state asserted in Propositions 2.8 and 2.9 do not apply. Instead, the local dynamics of agents leads them to concentrate in one or *several* clusters—see, for example, Figures 1, 5, and 8. Our primary interest is in the large time behavior of such clusters. The generic scenario is a crowd of agents which is partitioned into a collection of clusters, $\mathcal{C}_k, \ k = 1, \ldots, K$, such that

$$\begin{cases} \text{either} \quad |\mathbf{p}_i(t) - \mathbf{p}_j(t)| \overset{t\to\infty}{\longrightarrow} 0 & \text{if} \quad i, j \in \mathcal{C}_k \leftrightarrow |\mathbf{x}_i(t) - \mathbf{x}_j(t)| \le R, \\ \text{or} \qquad |\mathbf{x}_i(t) - \mathbf{x}_j(t)| > R & \text{if} \quad i \in \mathcal{C}_k, j \in \mathcal{C}_\ell, \quad k \neq \ell. \end{cases}$$

In this context, we raise the following two fundamental questions.

Question 1. Identify the class of initial configurations, $\mathcal{P}(0)$, which evolve into finitely many clusters, $\mathcal{C}_k, \ k = 1, \ldots, K$. In particular, characterize the number of such clusters K for $t \gg 1$.

Question 2. Assume that the initial configuration $\mathcal{P}(0))$ is connected. Characterize the initial configuration $\mathcal{P}(0)$ which evolves into one cluster, $K(t) = 1$ for $t \gg 1$, namely, the question of the emergence of consensus in the local dynamics.

A complete answer to each of these questions should provide an extremely interesting insight into local processes of self-organized dynamics, with many applications. In the next two sections we provide partial answers to these questions. We begin with the first result, which shows that if the solution of (3.1) has bounded time-variation, then it must be partitioned into a collection of clusters.

PROPOSITION 3.1 (formation of clusters). *Let $\mathcal{P}(t) = \{\mathbf{p}_k(t)\}_k$ be the solution of the opinion or flocking models* (3.1) *with compactly supported influence function* $\text{Supp}\{\phi(\cdot)\} = [0, R)$*, and assume it has a bounded time-variation*

$$\int^{\infty} |\dot{\mathbf{p}}_i(s)| ds < \infty. \tag{3.3}$$

Then $\mathcal{P}(t)$ approaches a stationary state, $\mathbf{p}^\infty$, which is partitioned into K clusters, $\{\mathcal{C}_k\}_{k=1}^K$, such that $\{1, 2, \ldots, N\} = \cup_{k=1}^K \mathcal{C}_k$ and

$$\begin{cases} \textit{either} \quad \mathbf{p}_i(t) \longrightarrow \mathbf{p}^\infty_{\mathcal{C}_k} \quad \textit{as } t \to \infty & \textit{for all} \ \ i \in \mathcal{C}_k, \\ \textit{or} \qquad |\mathbf{x}_i(t) - \mathbf{x}_j(t)| > R \quad \textit{for } t \gg 1 & \textit{if} \ \ i \in \mathcal{C}_k, j \in \mathcal{C}_\ell, \quad k \neq \ell. \end{cases} \tag{3.4}$$

REMARK 3.2. *Observe that if the solution decays fast enough—in particular, if $\mathbf{p}(t)$ decays exponentially fast, $|\mathbf{p}_i(t) - \mathbf{p}_i^\infty| \lesssim e^{-C(t-t_0)}$, $t \ge t_0 > 0$ (as in the unconditional consensus and flocking of global interactions discussed in section* 2)*, then it has a bounded time-variation.*

Proof. Assumption (3.3) implies

$$|\mathbf{p}_i(t_2) - \mathbf{p}_i(t_1)| \leq \int_{t_1}^{t_2} |\dot{\mathbf{p}}_i(s)| ds \ll 1 \quad \text{for} \quad t_2 > t_1 \gg 1,$$

hence each agent approaches its own stationary state, $\mathbf{p}_i(t) \stackrel{t\to\infty}{\longrightarrow} \mathbf{p}_i^\infty$. We claim that $\dot{\mathbf{p}}_i(t) \stackrel{t\to\infty}{\longrightarrow} 0$. To this end, we distinguish between the two cases of first-order opinion dynamics and second-order flocking dynamics. In opinion dynamics, $\mathbf{p} \mapsto \mathbf{x}$: since the expression on the right of (3.1a),

$$\dot{\mathbf{p}}_i(t) = \frac{\alpha}{\sigma_i(t)} \sum_j \phi(|\mathbf{x}_i(t) - \mathbf{x}_j(t)|)(\mathbf{p}_i(t) - \mathbf{p}_j(t)), \quad \sigma_i(t) = \sum_j \phi(|\mathbf{x}_i(t) - \mathbf{x}_j(t)|), \tag{3.5}$$

has a limit (involving $\mathbf{p}_i^\infty = \mathbf{x}_i^\infty$), it follows that $\lim_{t\to\infty} \dot{\mathbf{p}}_i(t)$ exists and by (3.3) it must be zero, $\dot{\mathbf{p}}_i(t) \to 0$. In the case of flocking dynamics, $\mathbf{p} \mapsto \dot{\mathbf{x}}$, and there are two types of pairs of agents (i, j): either they have the same limiting "velocity," $\mathbf{p}_i^\infty - \mathbf{p}_j^\infty = 0$, and, since ϕ is bounded,

$$\phi(|\mathbf{x}_i(t) - \mathbf{x}_j(t)|)(\mathbf{p}_i(t) - \mathbf{p}_j(t)) \stackrel{t\to\infty}{\longrightarrow} 0;$$

or, if $\mathbf{p}_i^\infty - \mathbf{p}_j^\infty \neq 0$, then

$$|\mathbf{x}_i^\infty - \mathbf{x}_j^\infty| \gtrsim |\mathbf{p}_i^\infty - \mathbf{p}_j^\infty| t > R, \qquad t \gg 1, \tag{3.6}$$

and hence

$$\phi(|\mathbf{x}_i(t) - \mathbf{x}_j(t)|)(\mathbf{p}_i(t) - \mathbf{p}_j(t)) = 0, \qquad t \gg 1.$$

In either case, the expression on the right of (3.5) vanishes as $t \to \infty$.

Now, take the scalar product of (3.5) against $\mathbf{p}_i$ and sum:

$$\sum_i \sigma_i \langle \dot{\mathbf{p}}_i, \mathbf{p}_i \rangle = \alpha \sum_{ij} \phi_{ij} \langle \mathbf{p}_j - \mathbf{p}_i, \mathbf{p}_i \rangle \equiv -\frac{\alpha}{2} \sum_{ij} \phi_{ij} |\mathbf{p}_j - \mathbf{p}_i|^2. \tag{3.7}$$

Since $\mathbf{p}_i \in \Omega(0)$, $\sigma_i \leq N$ are uniformly bounded and $\dot{\mathbf{p}}_i(t) \to 0$ on the left, it follows that the expression on the right tends to zero. In opinion dynamics ($\mathbf{p} \mapsto \mathbf{x}$), we can pass to the limit in the expression on the right, which yields

$$\phi(|\mathbf{x}_i^\infty - \mathbf{x}_j^\infty|)|\mathbf{p}_i^\infty - \mathbf{p}_j^\infty|^2 = 0 \quad \text{for all} \;\; i, j \leq N. \tag{3.8}$$

Thus, if $|\mathbf{x}_i^\infty - \mathbf{x}_j^\infty| > R$, then agents i and j are in separate clusters. Otherwise, when they are in the same cluster, say, $i, j \in \mathcal{C}_k$ so that $|\mathbf{x}_i^\infty - \mathbf{x}_j^\infty| < R$, then $\phi(|\mathbf{x}_i^\infty - \mathbf{x}_j^\infty|) > 0$, and by (3.8) they must share the same stationary state, $\mathbf{p}_i^\infty = \mathbf{p}_j^\infty =: \mathbf{p}_{\mathcal{C}_k}^\infty$, that is, (3.4) holds. In the case of flocking dynamics, $\mathbf{p} \mapsto \dot{\mathbf{x}}$, we either have one type of pairs, $|\mathbf{p}_i(t) - \mathbf{p}_j(t)| \stackrel{t\to\infty}{\longrightarrow} 0$, or a second type of pairs, (3.6), namely, (3.4) holds. ∎

We now turn our attention to the number of clusters, K.

3.2. How Many Clusters? Note that if $\mathbf{p}^\infty = (\mathbf{p}_1^\infty, \dots, \mathbf{p}_N^\infty)^\top$ is a stationary state of (3.1), then $\mathbf{p}^\infty$ is an eigenvector associated with the nonlinear eigenvalue problem,

$$A(\mathbf{x}^\infty)\mathbf{p}^\infty = \mathbf{p}^\infty,$$

corresponding to the eigenvalue $\lambda_N(A(\mathbf{x}^\infty)) = 1$. In fact, the number of stationary clusters can be directly computed from the multiplicity of leading spectral eigenvalues of $\lambda_N(A(\mathbf{x}^\infty))$.

PROPOSITION 3.3. *Assume that the crowd of N agents $\{\mathbf{p}_i(t)\}_{i=1}^N$ is partitioned into K clusters, $\{1,2,\ldots,N\} = \cup_{k=1}^{K(t)}\mathcal{C}_k$. Then the number of clusters, $K = K(t)$, equals the geometric multiplicity of $\lambda_N(A(\mathbf{x}(t)) = 1$,*

$$K(t) = \{\#\lambda_N(A(\mathbf{x}(t)) \mid \lambda_N(A(\mathbf{x}(t)) = 1\}. \tag{3.9}$$

Proof. We include the rather standard argument for completeness. Suppose that the dynamics of (3.1) at time t consists of $K = K(t)$ clusters, $\cup_{k=1}^{K(t)}\mathcal{C}_k$. Define the vector $\mathbf{r}^k = (r_1^k,\ldots,r_N^k)^\top$ such that

$$r_j^k = \begin{cases} 1 & \text{if } j \in \mathcal{C}_k, \\ 0 & \text{otherwise.} \end{cases}$$

We obtain

$$\left(A\mathbf{r}^k\right)_i = \sum_j a_{ij} r_j^k = \sum_{j\in\mathcal{C}_k} a_{ij}.$$

Using the facts that A is a stochastic matrix and that $a_{ij} = 0$ if $\mathbf{x}_i$ and $\mathbf{x}_j$ are not in the same cluster, we deduce

$$\sum_{j\in\mathcal{C}_k} a_{ij} = \left\{ \begin{array}{ll} 1 & \text{if } i \in \mathcal{C}_k \\ 0 & \text{otherwise} \end{array} \right\} = \mathbf{r}_i^k,$$

and therefore $A\mathbf{r}^k = \mathbf{r}^k$. Thus, associated with each cluster $\mathcal{C}_k$ there is an eigenvector $\mathbf{r}^k$ corresponding to $\lambda_N(A) = 1$. To conclude the proof, we have to show that there are no other vectors $\mathbf{r}$ satisfying $A\mathbf{r} = \mathbf{r}$. Indeed, assume that $A\mathbf{r} = \mathbf{r}$,

$$\sum_j a_{ij} r_j = r_i \qquad \text{for any } i.$$

Fix a cluster $\mathcal{C}_k$. Then for any $p \in \mathcal{C}_k$ we have

$$\sum_{j\in\mathcal{C}_k} a_{pj} r_j = r_p \qquad \text{for any } p \in \mathcal{C}_k.$$

Denote by r_q the maximal entry of r_j's on the left, corresponding to some $q \in \mathcal{C}_k$: since $\sum_{j\in\mathcal{C}_k} a_{pj} = 1$ with $a_{pj} > 0$, we deduce that for any $p \in \mathcal{C}_k$ we have $r_p = \sum a_{pj} r_j \leq \sum a_{pj} r_q = r_q$. Thus, the entries of $\mathbf{r}$ are constant on the cluster $\mathcal{C}_k$, so that $\mathbf{r} \propto \mathbf{r}^k$. ∎

3.3. Numerical Simulations with Local Dynamics. We illustrate the emergence of clusters with 1D and 2D simulations of the opinion dynamics model (1.2b),

$$\frac{d}{dt}\mathbf{x}_i = \sum_j \frac{\phi_{ij}}{\sum_k \phi_{ik}}(\mathbf{x}_j - \mathbf{x}_i), \qquad \mathbf{x}_i(t) \in \mathbb{R}^d. \tag{3.10}$$

The influence function, ϕ, is taken as the characteristic function of the interval $[0,1]$: $\phi(r) = \chi_{[0,1]}$, and we use the Runge–Kutta method of order 4 with a time-step of $\Delta t = .05$ for the time-discretization of the system of ODEs (3.10).

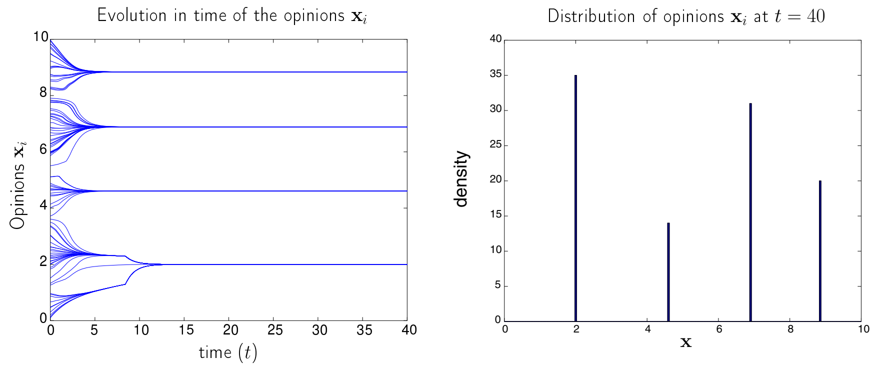

Fig. 3 *The opinion model* (1.2b) *with* $M = 100$ *agents and* $\phi = \chi_{[0,1]}$ *(left) and the histogram of the distribution of* $\mathbf{x}_i$ *at* $t = 40$ *unit time (right). We observe the formation of* 4 *clusters separated by a distance greater than* 1.

As a first example, we run a simulation of the 1D opinion model, $d = 1$, subject to an initial configuration of $N = 100$ agents uniformly distributed on the interval $[0, 10]$. In Figure 3 (left) we plot the evolution of the opinions $\mathbf{x}_i(t)$ in time. We observe the formation of four clusters after 15 unit time. The histogram of the distribution of agents at the final time $t = 40$ (see Figure 3 (right)) shows that the distance between the clusters is greater than 1, as predicted by Proposition 3.1. We also observe that the number of opinions contained in each cluster differs (respectively, 35, 14, 31, and 20 agents). Indeed, the larger cluster at $x \approx 2$ with 35 opinions is a merge between three *branches* (see Figure 3) with one branch in the middle connecting the two external branches. When the two external branches finally connect at $t \approx 8.5$ (their distance is less than 1), we observe an abrupt change in the dynamics followed by a merge of the three branches into a single cluster.

To analyze the cluster formation, we also look at the evolution of the eigenvalues of the matrix of interaction $A(\mathbf{x}(t))$ in (3.10), $a_{ij} = \phi_{ij}/\sum_k \phi_{ik}$. In Figure 4, we represent the evolution of the first eight eigenvalues of the matrix A. From $t = 0$ to $t \approx 3.5$, we observe that the first four first eigenvalues converge to 1, which accounts for the fact that only four clusters remain at this time. Then the matrix $A(\mathbf{x}(t))$ remains constant in time from $t \approx 3.5$ to $t \approx 8.5$. At $t \approx 8.5$, two *branches* (see Figure 3) reconnect, and the two eigenvalues λ_5 and λ_6 equal zero. This confirms Proposition 3.3 where the additional multiplicity of the spectral eigenvalue $\lambda_N(A(\mathbf{x}(t)) = 1$ indicates the formation of a new cluster.

Next we illustrate the dynamics of the 2D, $d = 2$, opinion model (1.2b). With this aim, we run the model starting with an initial condition of $N = 1000$ agents distributed uniformly on the square $[0, 10] \times [0, 10]$. We present, in Figure 5, several snapshots of the simulations at different times ($t = 0$, 2, 4, 6, 12, and 30 unit time). As in the 1D case, we first observe a fast transition to a cluster formation (from $t = 0$ to $t = 6$). However, at time $t = 12$, the dynamics has not yet converged to a stationary state, and we observe on the upper left that three branches are at a distance less than 1 from each other. This scenario is similar to the one observed in Figure 3 with the apparition of three *branches*. At $t = 30$, the three clusters on the upper left have finally merged and the system has reached a stationary state: each cluster is at a distance greater than 1 from all others.

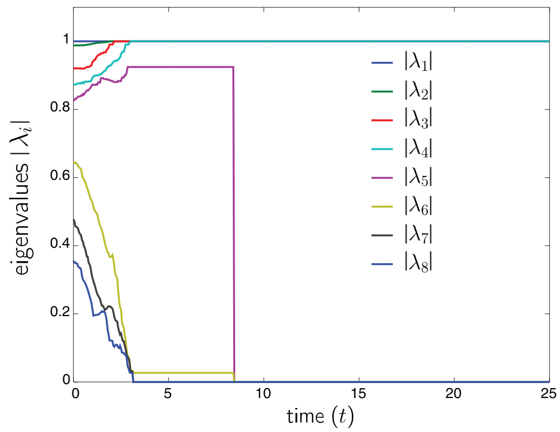

Fig. 4 *Absolute values of the eigenvalues of the matrix A (3.10) during the simulation given in Figure 3. The number of eigenvalues equal to 1 corresponds to the number of clusters.*

4. *K* = 1: Uniform Connectivity Implies Consensus. The emergence of a consensus in the opinion or flocking models (1.4) implies that the underlying graph associated with the dynamics must remain connected, namely, $|\mathbf{x}_i(t) - \mathbf{x}_i(t)| \ll R$ at least for $t \gg 1$. In this section we discuss the converse statement, namely, that uniform connectivity implies consensus. The implication of consensus in the symmetric case is based on a straightforward application of *algebraic connectivity* and is outlined in section 4.1. The corresponding question of consensus in nonsymmetric connected models is carried out in section 4.2 using an *energy method*. We emphasize that consensus in both cases depends on the time-dependent behavior of intensity of connectivity, beyond the mere graph connectivity. Recall that the graph associated with (1.1), $\mathcal{G}_A := (\mathcal{P}, A(\mathcal{P}))$, is connected if every two agents $\mathbf{p}_i(t)$ and $\mathbf{p}_j(t)$ are connected through a path $\Gamma_{ij} := \{k_1 = i < k_2 < \cdots < k_r = j\}$ of length $r_{ij} \leq N$. We measure the *uniform connectivity* by its "weakest link."

Definition 4.1 (uniform connectivity). *The self-organized dynamics* (1.1) *is connected if there exists $\mu(t) > 0$ such that, for all paths Γ_{ij},*

$$\min_{k_\ell \in \Gamma_{ij}} a_{k_\ell, k_{\ell+1}}(\mathcal{P}(t)) \geq \mu(t) > 0 \qquad \textit{for all } i, j. \tag{4.1}$$

In particular, if $\mu(t) \geq \mu > 0$, then we say that $\mathcal{P}(t)$ is uniformly connected.

Alternatively, uniform connectivity of (1.1) requires the existence of $\mu = \mu_A > 0$ independent of time, such that

$$\left(A^N(\mathcal{P}(t))\right)_{ij} \geq \mu^N > 0.$$

4.1. Consensus in Local Dynamics: Symmetric Models. We consider the *symmetric* dynamics (1.1) with associated graph $\mathcal{G}_A := (\mathcal{P}, A(\mathcal{P}))$. Fix the positions of any two agents $\mathbf{p}_i(t)$ and $\mathbf{p}_j(t)$ and their (shortest) connecting path Γ_{ij} of length r_{ij}. Thus, r_{ij} measures the degree of separation between agents (i, j), and if we let the maximal degree of separation denote the diameter of the graph, $diam(\mathcal{G}_A) := \max_{ij} r_{ij}$, then

$$|\mathbf{p}_i - \mathbf{p}_j|^2 \leq diam(\mathcal{G}_A) \sum_{k_\ell \in \Gamma_{ij}} |\mathbf{p}_{k_{\ell+1}} - \mathbf{p}_{k_\ell}|^2, \qquad diam(\mathcal{G}_A) \leq N.$$

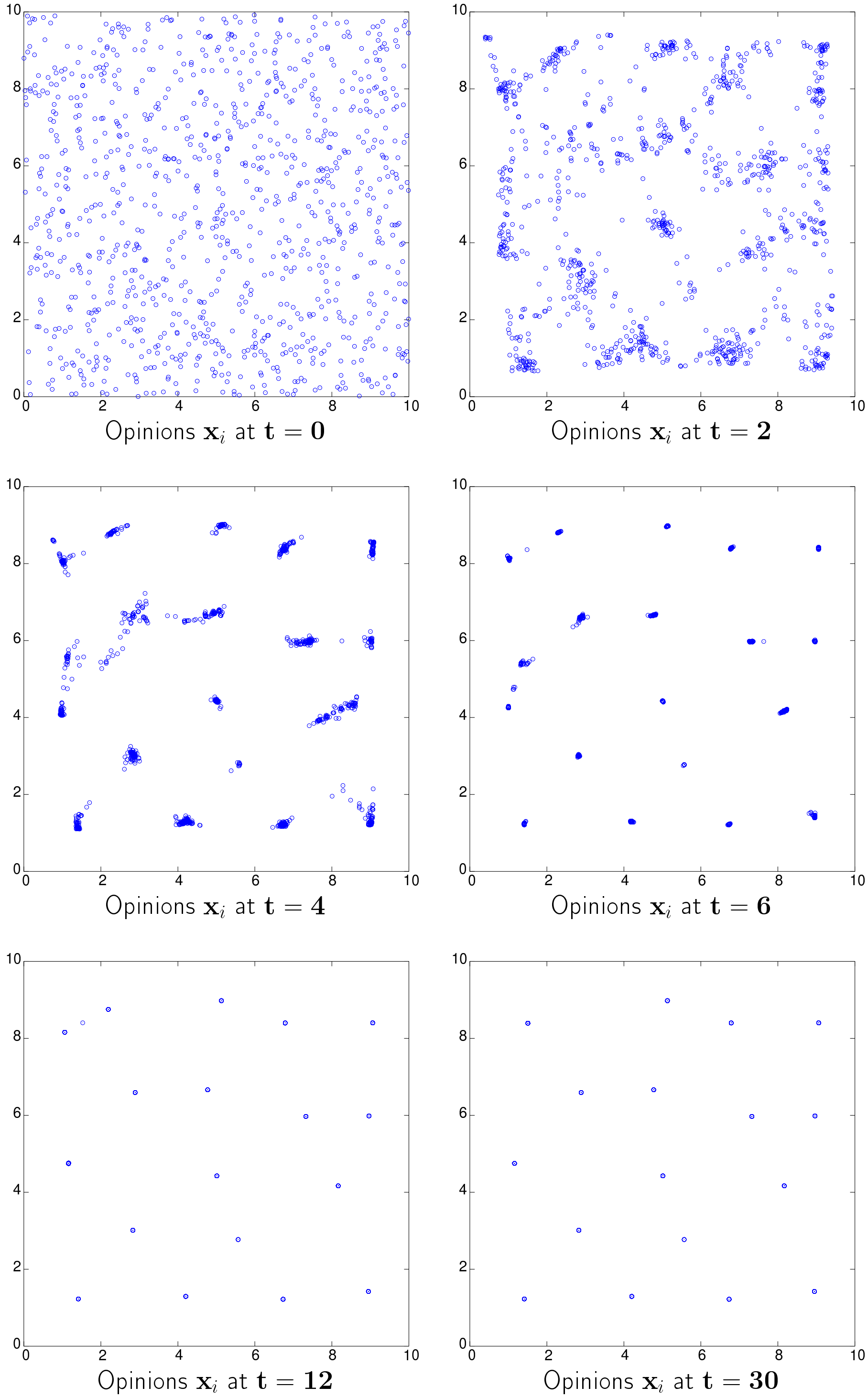

Fig. 5 *Simulation of the opinion model* (1.2b) *in* 2*D with* $M = 1000$ *agents and* $\phi = \chi_{[0,1]}$. *The dynamics converges to a cluster formation (17 clusters), with each cluster separated by a distance greater than* 1.

By uniform connectivity, $\mu \le a_{k_{\ell+1},k_\ell}$ along each path and hence

$$\frac{\mu}{diam(\mathcal{G}_A)}|\mathbf{p}_i - \mathbf{p}_j|^2 \le \sum_{k_\ell \in \Gamma_{ij}} a_{k_{\ell+1},k_\ell}|\mathbf{p}_{k_{\ell+1}} - \mathbf{p}_{k_\ell}|^2 \le \sum_{ij} a_{ij}|\mathbf{p}_i - \mathbf{p}_j|^2, \tag{4.2}$$

and summation over all pairs yields

$$\frac{\mu}{diam(\mathcal{G}_A)} \sum_{ij} |\mathbf{p}_i - \mathbf{p}_j|^2 \le N^2 \sum_{ij} a_{ij}|\mathbf{p}_i - \mathbf{p}_j|^2.$$

Now we recall our notation $\mathbf{q}_i := \mathbf{p}_i - \langle \mathbf{p} \rangle$: invoking (2.19) we find

$$\lambda_2(L_A) = \min_{\sum \mathbf{q}_k = 0} \frac{\langle L_A \mathbf{q}, \mathbf{q}\rangle}{\langle \mathbf{q}, \mathbf{q}\rangle} = \min_{\mathbf{p}} \frac{(1/2)\sum_{ij} a_{ij}|\mathbf{p}_i - \mathbf{p}_j|^2}{(1/2N)\sum_{ij} |\mathbf{p}_i - \mathbf{p}_j|^2} \ge \frac{\mu}{N diam(\mathcal{G}_A)}. \tag{4.3}$$

Thus, the scaled connectivity factor $\mu/(N diam(\mathcal{G}_A)) \ge \mu/N^2$ serves as a lower bound for the Fiedler number associated with the symmetric dynamics of (1.1) (counting the number of "maximal" edges, it yields the slightly sharper lower bound $\lambda_2 \ge 4\mu/N^2$ [87]).

Using Theorem 2.11 we conclude the following result.

THEOREM 4.2 (connectivity implies consensus: the symmetric case). *Let $\mathcal{P}(t) = \{\mathbf{p}_k(t)\}_k$ be the solution of a symmetric self-organized dynamics*

$$\frac{d}{dt}\mathbf{p}_i(t) = \alpha \sum_{j \ne i} a_{ij}(\mathcal{P}(t))(\mathbf{p}_j(t) - \mathbf{p}_i(t)), \qquad a_{ij} = a_{ji}.$$

If $\mathcal{P}(t)$ remains connected in time with "sufficiently strong" connectivity $\mu_{A(\mathcal{P}(s))} > 0$, then it approaches the consensus $\langle \mathbf{p}\rangle(0)$, namely,

$$\vee_{\mathbf{p}(t)} \lesssim \exp\left(-\frac{\alpha}{N^2}\int_0^t \mu_{A(\mathcal{P}(s))} ds\right) \vee_{\mathbf{p}(0)}, \qquad \vee^2_{\mathbf{p}(t)} := \frac{1}{N}\sum |\mathbf{p}_i(t) - \langle \mathbf{p}\rangle(0)|^2.$$

In particular, if $\mathcal{P}(t)$ remains uniformly connected in time (cf. (4.1)*), then it approaches an emerging consensus, $\mathbf{p}_i(t) \to \mathbf{p}^\infty = \langle \mathbf{p}\rangle(0)$, with a convergence rate*

$$\vee_{\mathbf{p}(t)} \lesssim e^{-\alpha \frac{\mu}{N^2} t} \vee_{\mathbf{p}(0)}\ . \tag{4.4}$$

It is important to notice that Theorem 4.2 requires the intensity of connectivity to be sufficiently strong: connectivity alone, with a rapidly decaying $\mu(t)$, is not sufficient for consensus as illustrated by the following counterexample.

Counterexample. Consider the symmetric dynamics (1.2a) with five agents, $x_1, \ldots, x_5$, subject to initial configuration

$$\mathbf{x}_1(0) = -\mathbf{x}_5(0), \quad \mathbf{x}_2(0) = -\mathbf{x}_4(0), \quad \mathbf{x}_3(0) = 0, \tag{4.5}$$

with $(\mathbf{x}_4(0), \mathbf{x}_5(0))$ to be specified below inside the box $\mathcal{D} := \{\frac{1}{2} < \mathbf{x}_4 < 1 < \mathbf{x}_5 < \frac{3}{2}\}$. We fix the influence function $\phi(r) = (1+r)^2(1-r)^2\chi_{[0,1]}$, compactly supported on $[0,1]$; note that $\phi'(0) = \phi'(1) = 0$. By symmetry, the initial ordering in (4.5) is preserved in time. In particular, $\mathbf{x}_3(t) \equiv 0$, and $(\mathbf{x}_4(t), \mathbf{x}_5(t)) \mapsto (x(t), y(t))$ preserve the original ordering, $\frac{1}{2} < x(t) < 1 < y(t)$; the symmetric opinion dynamics (1.2a) (with $\alpha = 5$ for simplicity) is reduced to

$$\begin{aligned} \dot{x} &= -\phi(|x|)x + \phi(|y-x|)(y-x), \\ \dot{y} &= \phi(|x-y|)(x-y). \end{aligned} \tag{4.6}$$

An equilibrium for the system is given by $x = y = 1$. The eigenvalues of the linearized system at $(1,1)$ are $\lambda_1 = 0$ and $\lambda_2 = -2$; therefore, the equilibrium is unstable. We would like to prove that there exists an initial condition $(x(0), y(0))$ close to $(1,1)$ which converges toward this unstable equilibrium. We use for this purpose a variant of the antifunnel theorem [68].

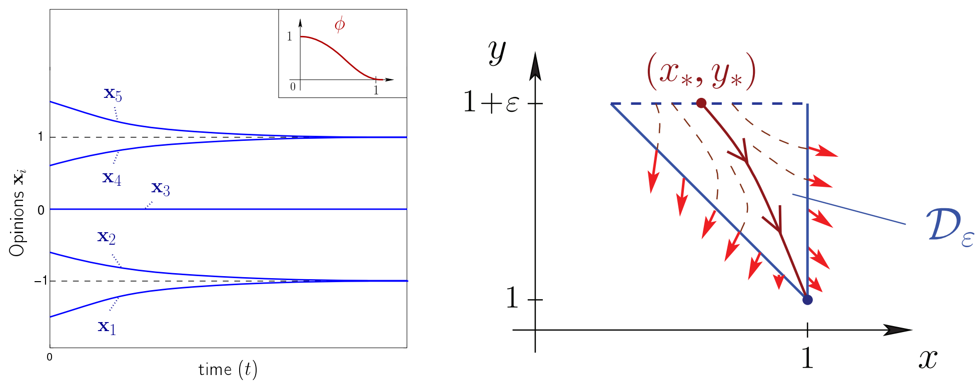

Fig. 6 Left: *A solution of the symmetric model that stays connected but does not converge to a consensus.* Right: *In phase space, the counterexample is a solution that stays in the antifunnel $\mathcal{D} - \varepsilon$ enclosed by the curves α on its left, β on its right, and γ on top.*

We study the phase portrait of the dynamical system (4.6) close to the unstable equilibrium $(1,1)$. Take ε such that $0 < \varepsilon < \frac{1}{2}$ and consider the three curves (see Figure 6)

$$\begin{aligned} \alpha(s) = (2-s, s), \quad \beta(s) = (1, s) &\text{ for } s \in (1, 1+\varepsilon], \\ \gamma(s) = (2-s, 1+\varepsilon) \qquad\qquad &\text{ for } s \in [1, 1+\varepsilon]. \end{aligned}$$

We denote by $\mathcal{D}_\varepsilon$ the domain enclosed by the three curves:

$$\mathcal{D}_\varepsilon = \{2 - y \le x \le 1\ ,\ 1 < y \le 1 + \varepsilon\}.$$

Notice that on the domain $\mathcal{D}_\varepsilon$, we have $\dot{y} < 0$. Thus, given a solution of (4.6) starting on γ, there are three possibilities: the solution exits the domain passing through the curves α, or it exits passing through β, or it converges to the equilibrium $(1,1)$.

To prove the existence of solutions in the third category, we notice that the curves α and β form an *antifunnel* for the dynamical system. Starting on the curve β, since $\dot{x} > 0$, the solution exits the domain $\mathcal{D}_\varepsilon$ (see Figure 6). Similarly, on the curve α, since $\dot{x} < \dot{y}$, the solution exits the domain $\mathcal{D}_\varepsilon$ as well.

We denote by γ_α the set of initial conditions contained in γ such that the solution exits through α. The set γ_a is nonempty since $(1-\varepsilon, 1+\varepsilon) \in \gamma_a$. Moreover, using the same arguments as in [68], we find that γ_α is open. Similarly, we denote by $\gamma_\beta \subset \gamma$ the set of initial conditions such that the solution exits through β and we deduce that

γ_β is open and nonempty. Since $\gamma_\alpha \cap \gamma_\beta = \emptyset$, by connectivity of the set γ, there exists (x_*, y_*) which does not belong to $\gamma_\alpha \cup \gamma_\beta$. Thus, the solution $(x(t), y(t))$ starting from (x_*, y_*) stays in between α and β:

$$2 - y(t) \leq x(t) \leq 1, \quad 1 \leq y(t) \leq 1 + \varepsilon \qquad \text{for all } t \geq 0.$$

Since $y(t)$ is decreasing and lower bounded, $y(t)$ converges: $y(t) \stackrel{t\to\infty}{\longrightarrow} y_\infty$. Moreover, the solution $(x(t), y(t))$ is globally Lipschitz, thus $\dot{y}$ converges to zero. Thus, combining (4.6) with $\phi(|y(t) - x(t)|) \geq m > 0$, we deduce that $x(t) \stackrel{t\to\infty}{\longrightarrow} y_\infty$. Since there is only one equilibrium in the domain $\overline{\mathcal{D}}_\varepsilon$, we necessarily have $y_\infty = 1$, and therefore $(x(t), y(t)) \stackrel{t\to\infty}{\longrightarrow} (1, 1)$.

4.2. Consensus in Local Nonsymmetric Opinion Dynamics. Next we consider the question of consensus for the nonsymmetric opinion model (1.2b).

THEOREM 4.3 (connectivity implies consensus: nonsymmetric opinion dynamics). *Let $\mathcal{P}(t) = \{\mathbf{p}_k(t)\}_k$ be the solution of the nonsymmetric opinion dynamics* (1.2b) *with compactly supported influence function,* $\text{Supp}\{\phi(\cdot)\} = [0, R)$*,*

$$\sigma_i \frac{d}{dt}\mathbf{x}_i(t) = \alpha \sum_j \phi_{ij}(\mathbf{x}_i(t) - \mathbf{x}_j(t)), \qquad \sigma_i = \sum_k \phi_{ik}.$$

If $\mathcal{P}(t)$ remains uniformly connected in time in the sense that each pair of agents (i, j) is connected through a path Γ_{ij} such that[2]

$$\min_{k_\ell \in \Gamma_{ij}} \phi(|\mathbf{x}_{k_\ell} - \mathbf{x}_{k_{\ell+1}}|) \geq \mu > 0 \qquad \textit{for all } i, j,$$

then it has bounded time-variation and, consequently, $\mathcal{P}(t)$ approaches an emerging consensus $\mathbf{x}_i(t) \to \mathbf{x}^\infty$ with a convergence rate

$$|\mathbf{x}_i(t) - \mathbf{x}^\infty| \lesssim e^{-\alpha m(t-t_0)}[\mathbf{x}(0)], \qquad m = \min_{r \leq R/2} \phi(r) > 0. \tag{4.7}$$

Proof. We introduce the energy functional

$$\mathcal{E}(t) := \alpha \sum_{i,j} \Phi(|\mathbf{x}_j(t) - \mathbf{x}_i(t)|), \qquad \Phi(r) := \int_{s=0}^r s\phi(s)ds, \tag{4.8a}$$

which is decreasing in time,

$$\begin{aligned} \frac{d}{dt}\mathcal{E}(t) &= \alpha \sum_{i,j} \phi_{ij} \langle \dot{\mathbf{x}}_j - \dot{\mathbf{x}}_i \,,\, \mathbf{x}_j - \mathbf{x}_i \rangle = -2\alpha \sum_{i,j} \phi_{ij} \langle \dot{\mathbf{x}}_i \,,\, \mathbf{x}_j - \mathbf{x}_i \rangle \\ &= -2 \sum_i \langle \dot{\mathbf{x}}_i \,, \alpha \sum_{j \neq i} \phi_{ij}\, (\mathbf{x}_j - \mathbf{x}_i) \rangle = -2 \sum_i \sigma_i |\dot{\mathbf{x}}_i|^2 \leq 0. \end{aligned} \tag{4.8b}$$

To upper-bound the expression on the right of (4.8a), sum (1.2b) against $\mathbf{x}_i$ to find

$$\begin{aligned} \frac{\alpha}{2} \sum_{i,j} \phi_{ij} |\mathbf{x}_i - \mathbf{x}_j|^2 &= -\alpha \sum_{i,j} \phi_{ij} \langle \mathbf{x}_i - \mathbf{x}_j, \mathbf{x}_i \rangle = \sum \sigma_i \langle \mathbf{x}_i, \dot{\mathbf{x}}_i \rangle \\ &\leq \sqrt{\sum_i \sigma_i |\mathbf{x}_i|^2} \sqrt{\sum_i \sigma_i |\dot{\mathbf{x}}_i|^2} \leq N \max_i |\mathbf{x}_i(0)| \sqrt{\sum_i \sigma_i |\dot{\mathbf{x}}_i|^2}. \end{aligned} \tag{4.9}$$

[2]Observe that here we measure connectivity in terms of the influence function ϕ_{ij} rather than the adjacency matrix a_{ij} as in (4.1); the two are equivalent up to an obvious scaling of the degree σ_i.

We end up with the energy decay

$$\frac{d}{dt}\mathcal{E}(t) \leq -\frac{1}{2}\alpha^2 C_0^2 \left(\sum_{i,j} \phi_{ij}|\mathbf{x}_i - \mathbf{x}_j|^2\right)^2, \qquad C_0 = \frac{1}{N \max_i |\mathbf{x}_i(0)|}. \tag{4.10}$$

Hence, since

$$\int^\infty \left(\sum_{i,j} \phi_{ij}(t)|\mathbf{x}_i(t) - \mathbf{x}_j(t)|^2\right)^2 dt < \frac{2}{\alpha^2 C_0^2}\mathcal{E}(0) < \infty,$$

the sum $\sum_{i,j} \phi_{ij}(t)|\mathbf{x}_i(t) - \mathbf{x}_j(t)|^2$ must become arbitrarily small at some point in time, namely, there exists $t_0 > 0$ such that

$$\sum_{i,j} \phi_{ij}(t_0)|\mathbf{x}_i(t_0) - \mathbf{x}_j(t_0)|^2 \leq \frac{\mu}{4N}R^2, \tag{4.11}$$

and, by uniform connectivity (cf. (4.2)),

$$\frac{\mu}{N}|\mathbf{x}_i(t_0) - \mathbf{x}_j(t_0)|^2 \leq \sum_{k_\ell \in \Gamma_{ij}} \phi_{k_\ell, k_{\ell+1}}(t_0)|\mathbf{x}_{k_\ell}(t_0) - \mathbf{x}_{k_{\ell+1}}(t)|^2 \leq \frac{\mu}{4N}R^2. \tag{4.12}$$

Thus, the dynamics at time t_0 concentrate so that its diameter, $[\mathbf{x}(t_0)] = \max_{i,j} |\mathbf{x}_i(t_0) - \mathbf{x}_j(t_0)| \leq R/2$, and, since $[\mathbf{x}(\cdot)]$ is nonincreasing in time, $[\mathbf{x}(t)] \leq R/2$ thereafter. Arguing along the lines of Proposition 2.8, we conclude that there is an exponential time decay,

$$N a_{ij} \geq \phi(|\mathbf{x}_i(t) - \mathbf{x}_j(t)|) \geq \min_{r \leq [\mathbf{x}(t)]} \phi(r) \geq \min_{r \leq R/2} \phi(r) = m, \qquad t > t_0,$$

and consensus follows from Corollary 2.6. ◻

The decreasing energy functional $\mathcal{E}(t)$ can be used to estimate the first "arrival" time of concentration t_0. To this end, observe that

$$\Phi(|\mathbf{x}_j - \mathbf{x}_i|) = \int_{s=0}^{|\mathbf{x}_j - \mathbf{x}_i|} s\phi(s)ds \leq M \int_{s=0}^{|\mathbf{x}_j - \mathbf{x}_i|} sds = M\frac{|\mathbf{x}_j - \mathbf{x}_i|^2}{2}, \quad M := \max_r \phi(r).$$

Using the assumption of uniform connectivity, there exists $\mu > 0$ and a path Γ_{ij} such that

$$|\mathbf{x}_j - \mathbf{x}_i|^2 \leq \frac{N}{\mu} \sum_{k_\ell \in \Gamma_{ij}} \phi_{k_\ell, k_{\ell+1}}|\mathbf{x}_{k_{\ell+1}} - \mathbf{x}_{k_\ell}|^2 \leq \frac{N}{\mu}\sum_{ij} \phi_{i,j}|\mathbf{x}_j - \mathbf{x}_i|^2.$$

Combining the last two inequalities, we can upper-bound the energy $\mathcal{E}$:

$$\mathcal{E} = \sum_{ij} \Phi_{ij} \leq \frac{MN^3}{4\mu}\sum_{ij} \phi_{i,j}|\mathbf{x}_j - \mathbf{x}_i|^2.$$

Hence, (4.10) implies the Riccati equation

$$\frac{d}{dt}\mathcal{E}(t) \leq -\frac{1}{2}\alpha^2 C_0^2 \left(\frac{4\mu}{MN^3}\mathcal{E}\right)^2 = -\frac{C\mu^2}{N^6}\mathcal{E}^2,$$

which shows the energy decay

$$\mathcal{E}(t) \lesssim \frac{1}{1 + \dfrac{C\mu^2 t}{N^6}}.$$

Thus, the arrival time of concentration t_0 (4.11) is of the order of at most $\mathcal{O}(N^7/\mu^3)$. This bound on the first arrival time can be improved.[3]

We close this section by noting that the lack of a consensus proof for our non-symmetric model of flocking dynamics (1.3b) is due to the lack of a proper decreasing energy functional.

5. Heterophilious Dynamics Enhances Consensus: Simulations. As we noted earlier, the large time behavior of local models for self-organized dynamics depends on the details of the interactions, $\{a_{ij}\}$, and, in the particular case of local models (3.1), on the profile of the compactly supported influence function ϕ. Here we explore how the profile of ϕ dictates cluster formation in the opinion dynamics model (1.2b). The numerical simulations presented in this section lead to the main conclusion that an *increasing* profile of ϕ reduces the number of clusters $\{\mathcal{C}_k\}_{k=1}^K$. In particular, if the profile of ϕ is increasing fast enough, then $K = 1$; thus, heterophilious dynamics enhances the emergence of consensus.

In what follows, we employ a compactly supported influence function ϕ which is a simple step function,

$$\phi(r) = \begin{cases} a & \text{for } r \leq \frac{1}{\sqrt{2}}, \\ b & \text{for } \frac{1}{\sqrt{2}} < r \leq 1, \\ 0 & \text{for } r > 1. \end{cases} \tag{5.1}$$

The essential quantity here is the ratio b/a, which measures the balance between the influence of "far" and "close" neighbors (see Figure 7). We initiate the opinion dynamics (1.2b) with random initial configuration $\{\mathbf{x}_i(0)\}_i$.

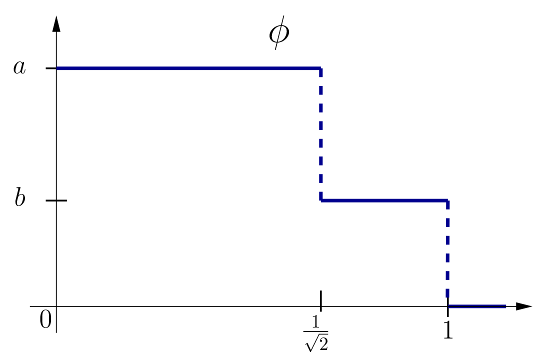

Fig. 7 *Influence functions ϕ used in the simulations. The larger b/a is, the more heterophilious is the dynamics.*

[3]In fact, the energy $\mathcal{E}(t)$ decays exponentially in time.

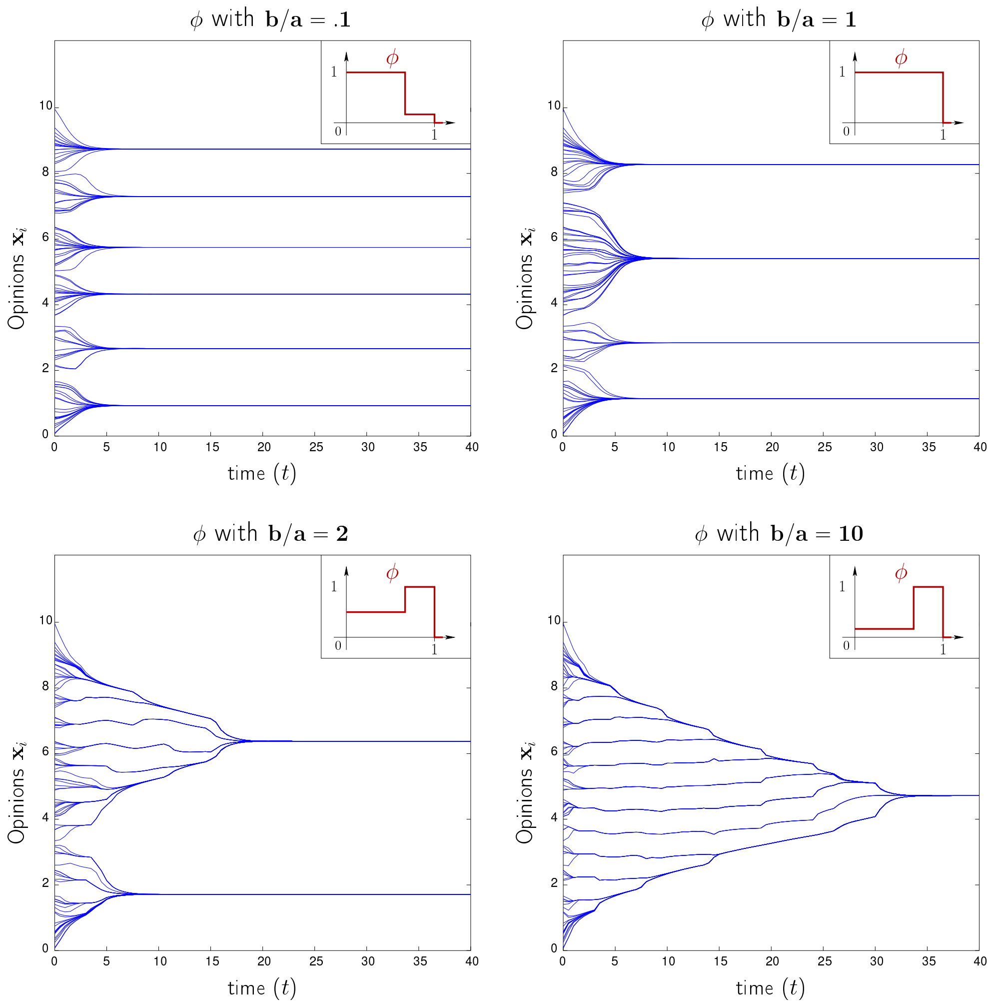

Fig. 8 *Simulation of the opinion dynamics model with different interacting functions ϕ. When the influence of close neighbors is reduced (i.e., b/a is large), the number of cluster decreases. For $b/a = 10$, the dynamics converges to a consensus.*

5.1. 1D Simulations. We begin with four simulations of the 1D opinion dynamics (1.2b) subject to 100 opinions distributed uniformly on $[0, 10]$, the same initial configuration as in Figure 3. To explore the impact of the influence step function (5.1) on the dynamics, we used four different ratios, $b/a = .1$, 1, 2, and 10. As b/a increases, we reduce the influence of the closer neighbors and increase the influence of neighbors further away; thus, increasing b/a reflects the tendency to "bond with the other." As observed in Figure 8, the increase in the ratios $b/a = .1$, 1, 2, and 10 reduces the corresponding number of limit clusters to $K = 6,\ 4,\ 2$, and, for $b/a = 10$, the dynamics converges to a consensus, $K = 1$. The simulations of Figure 8 indicate that reducing the influence of closer neighbors, and hence increasing the weight of the influence of neighbors further away, will favor increased *connectivity* and the emergence of consensus.

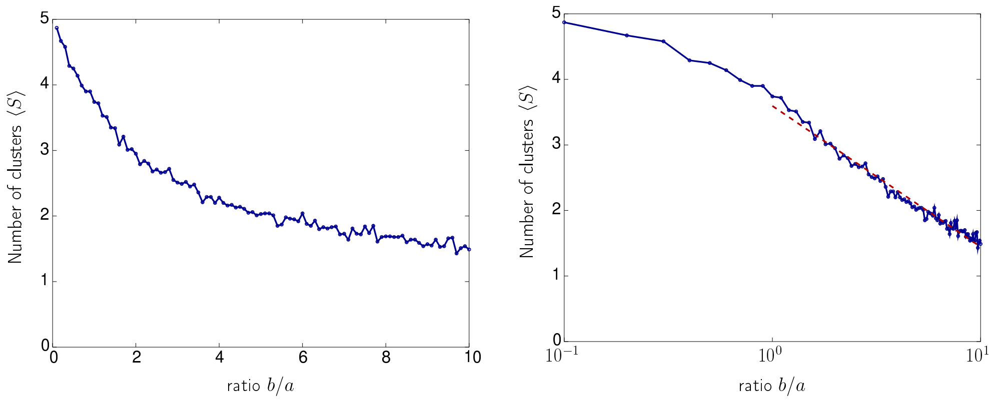

Fig. 9 *Average number of clusters $\langle S \rangle$ depending on the ratio b/a.* Left: *The larger b/a is, the fewer the number of clusters. The decay is logarithmic on* $[1, 10]$. Right: *For each value of b/a, we run 100 simulations to estimate the mean number of clusters $\langle S \rangle$. Simulations are run with $\Delta t = .05$ and a final time equal to $t = 100$ unit time.*

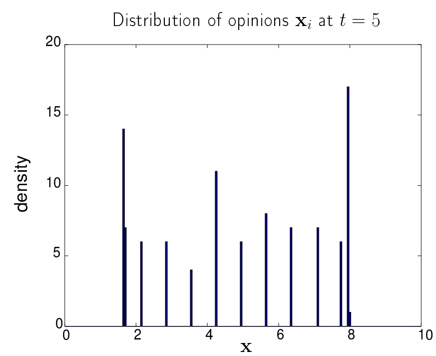

Fig. 10 *The distribution of $\{\mathbf{x}_i\}_i$ in the simulation of Figure 8 with $b/a = 10$ at time $t = 5$. The distance between two picks of density is around $1/\sqrt{2} \approx .7$ space units. This distance corresponds to the discontinuity of the function $\phi(r)$.*

For a systematic analysis of the cluster formation dependence on the ratio b/a, we made several simulations with random initial conditions for a given ratio b/a. Then, we take an average of the number of clusters, denoted by $\langle S \rangle$, at the end of each simulation ($t = 100$). To compute the number of clusters, we estimate the number of connected components of the matrix A (3.10) using a depth-first search algorithm. As observed in Figure 9, the number of clusters $\langle S \rangle$ decreases as b/a increases. Moreover, $\langle S \rangle$ approaches 1 when b/a approaches 10, implying that a consensus is likely to occur when b/a is large enough.

5.2. Clusters and Branches. We revisit the opinion model (1.2b) with an influence step function (5.1). As noted above, increasing b/a increases the probability of reaching a consensus. The simulations in Figure 8 with $b/a = 2$ and with $b/a = 10$ show the apparition of *branches*, where subgroups of agents have converged to the same opinion; however in contrast to clustering, these branches of opinions are still interacting with outsiders, which are at a distance strictly less than $R = 1$. In particular, when $b/a = 10$, the distribution of opinions $\{\mathbf{x}_i(t)\}_i$ aggregates to form the distinct branches seen in Figure 8: at $t \sim 5$, one can identify in Figure 10 the forma-

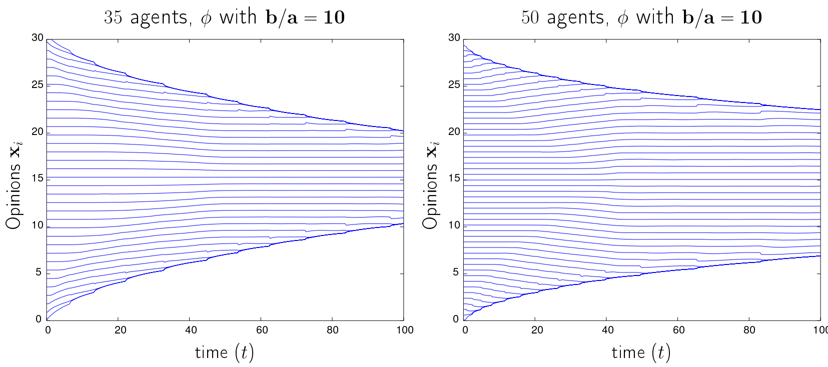

Fig. 11 *Initial condition with an equirepartition of* $\{\mathbf{x}_i\}_i$*:* $|\mathbf{x}_{i+1} - \mathbf{x}_i| = .9$ *(left) and* $|\mathbf{x}_{i+1} - \mathbf{x}_i| = .6$ *(right). Nearest neighbors readjust their distance to* $1/\sqrt{2} \approx .7$ *unit space; we observe a concentration of the trajectories in the left graph and a spread of the trajectories in the right graph.*

tion of ten branches separated by a distance of approximately .7 spatial units. Since the distance between two such branches is always less than the diameter $R = 1$ of ϕ, these branches do not qualify as isolated clusters as they continue to be influenced by "outsiders" from the nearby branches. Over time, these branches merge into each other before they emerge as one final cluster, the consensus, at $t \sim 33$. Thus, the decisive factor in the consensus dynamics is not the number of branches, but their large time connected components. Indeed, Figure 8 with $b/a = 10$ shows that the agents in the different branches remain in the same connected component at distance $\sim .7$, corresponding to the discontinuity of $\phi(\cdot)$, which experiences a jump from .1 to 1 at $1/\sqrt{2} \approx .7$.

To illustrate the apparition of the distance $1/\sqrt{2}$ between two nearest branches, we repeated the simulations, this time with special initial configurations where all the opinions are uniformly spaced with $|\mathbf{x}_{i+1} - \mathbf{x}_i| = d_*$ for $0 < d_* < 1$. As we observe in Figure 11, the agents $\{\mathbf{x}_i\}_i$ readjust their "opinion" such that the distance between nearest neighbors $|\mathbf{x}_{i+1} - \mathbf{x}_i|$ approaches $1/\sqrt{2}$ as $t \gg 1$.

5.3. 2D Simulations. We made several 2D simulations with different influence functions ϕ. As a first illustration, we made a 2D simulation with the same initial configuration used in Figure 5, but this time we used the influence step function ϕ in (5.1) with $b/a = 10$. In Figure 12, one can observe a concentration phenomenon (from $t = 0$ to $t = 2.5$)—the opinions aggregate into five final clusters, compared with the 17 clusters observed in Figure 5 with the influence function $\phi = \chi_{[0,1]}$. Thus, as in the 1D case, a more heterophilious influence function increases the clustering effect.

We also estimate the average number of clusters $\langle S \rangle$ depending on the ratio b/a. As observed in Figure 13, $\langle S \rangle$ is a decreasing function of b/a and once again the decay of $\langle S \rangle$ as a function of $b/a \in [0, 10]$ is *logarithmic*.

6. Heterophilious Dynamics with a Fixed Number of Neighbors. Careful observations of starling flocks led the Rome group [26, 25, 24] to the fundamental conclusion that their dynamics is driven by local interaction with a *fixed* number of nearest neighbors. This motivates our study of nearest neighbor models for opinion dynamics which take the form

$$\frac{d}{dt}\mathbf{x}_i = \alpha \sum_{\{j:\,|j-i|\le q\}} \frac{\phi_{ij}}{\sigma_i}(\mathbf{x}_j - \mathbf{x}_i), \qquad \mathbf{x}_i \in \mathbb{R}^d, \tag{6.1a}$$

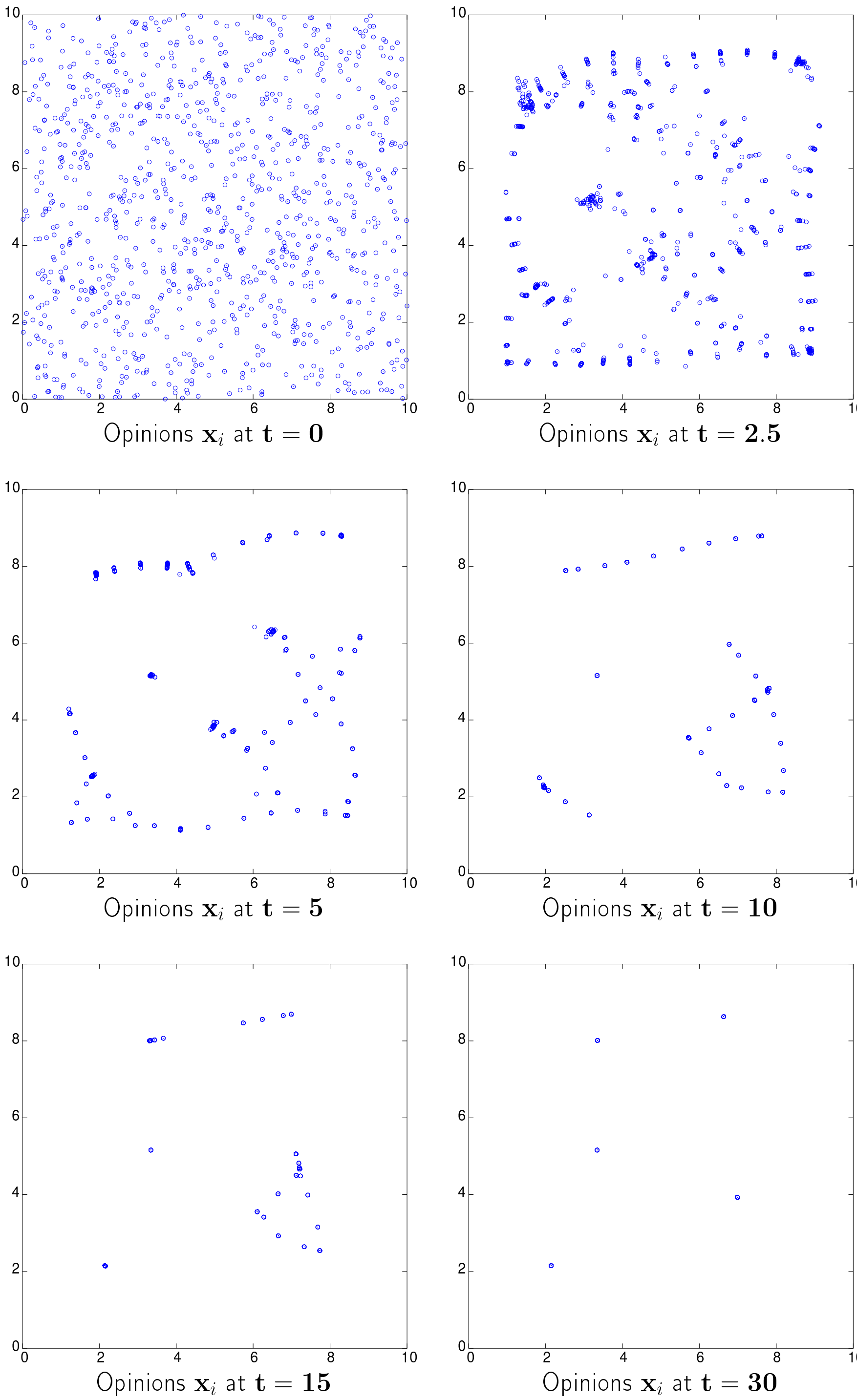

Fig. 12 *The heterophilious effect: Diminishing the influence of close neighbors relative to those further away increases the clustering effect. 2D simulation of the opinion model* (1.2b) *with $M = 1000$ agents using a step influence function $\phi = .1\,\chi_{[0,1/\sqrt{2}]} + \chi_{[1/\sqrt{2},1]}$ leads to five clusters which remain at the end of the simulation. This should be compared with* 17 *clusters with $\phi = \chi_{[0,1]}$ (see Figure* 5*).*

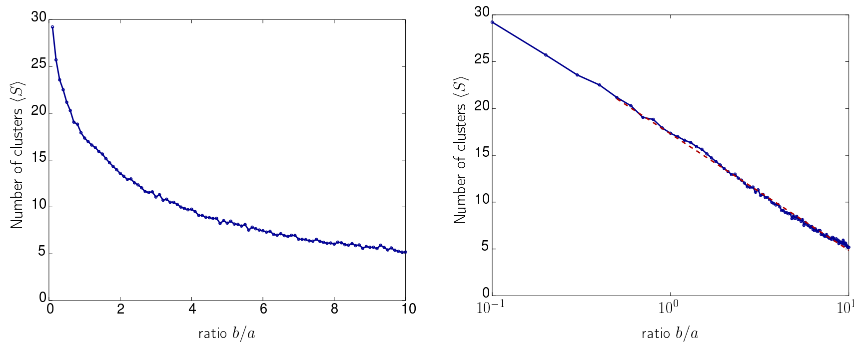

Fig. 13 *Average number of clusters $\langle S\rangle$ depending on the ratio b/a in $2D$ (left). As in the $1D$ case, the larger b/a is, the fewer the number of clusters, and the decay is logarithmic on* $[1,10]$ *(right). For each value of b/a, we made* 100 *simulations to estimate the mean number of clusters $\langle S\rangle$. Simulations were made with $\Delta t = .05$ and were recorded at the final time* $t = 100$.

where the degree σ_i is given by one of two forms, depending on the symmetric and nonsymmetric versions of the opinion dynamics in (1.2):

$$\begin{cases} \text{symmetric case :} & \sigma_i = \dfrac{1}{2q}; \\ \text{nonsymmetric case :} & \sigma_i = \displaystyle\sum_{\{j:|j-i|\le q\}} \phi_{ij}. \end{cases} \tag{6.1b}$$

Thus, each agent i is assumed to interact only with its $2q$ agents $i-q,\dots,i+q$. Typically, q is small (the observations in [26, 25, 24] report on six to seven active nearest neighbors). We analyze the connectivity of the particular case of *two nearest neighbors*, $q = 1$. Here we prove that such local models preserve connectivity and hence converge to a consensus, provided the influence function ϕ is increasing. This result supports our findings in section 5 that heterophilious dynamics is an efficient strategy to reach a consensus.

6.1. A Fixed Number of Neighbors with Global Influence Function. We begin by noting that the different approaches for consensus of global models apply in the present framework of local nearest neighbor models (6.1). For example, consider the nonsymmetric nearest neighbor model

$$\sigma_i \frac{d}{dt}\mathbf{x}_i = \alpha \sum_{\{j:|j-i|\le q\}} \phi_{ij}(\mathbf{x}_j - \mathbf{x}_i), \qquad \sigma_i = \sum_{\{j:|j-i|\le q\}} \phi_{ij}. \tag{6.2}$$

This admits an energy functional,

$$\mathcal{E}(t) := \alpha \sum_{\{i,j:|i-j|\le q\}} \Phi(|\mathbf{x}_j(t) - \mathbf{x}_i(t)|), \qquad \Phi(r) := \int_{s=0}^{r} s\phi(s)ds,$$

which is decreasing in time, $\mathcal{E}(t) \le \mathcal{E}(0)$, and we conclude the following result.

THEOREM 6.1 (global connectivity). *Consider the nearest neighbor model* (6.2) *with an influence function ϕ,* $\text{Supp}\{\phi(\cdot)\} = [0,R)$*, and assume $\alpha\Phi(R) > \mathcal{E}(0)$. Then*

$\min_{|i-j|\le q}\phi_{ij}(t) > m_\infty$, *where* $m_\infty := \min_{r<R}\phi(r)$. *Hence, the nearest neighbor dynamics* (6.2) *remains connected and consensus follows.*

Proof. Since $\mathcal{E}$ is decreasing in time,

$$\alpha\Phi(|\mathbf{x}_i(t)-\mathbf{x}_j(t)|) < \mathcal{E}(0) \le \alpha\Phi(R) \qquad \text{for any } |i-j|\le q,$$

and since $\Phi(r) = \int^r s\phi(s)ds$ is an increasing function, $|\mathbf{x}_i(t)-\mathbf{x}_j(t)| < R$, hence $\phi_{ij} > m_\infty$ and consensus follows. ∎

We note, however, that since $m_\infty \le \phi \le 1$, then $\Phi(r)$ has a quadratic bounds, $m_\infty r^2 \le 2\Phi(r) \le r^2$, and hence the assumption made in Theorem 6.1 implies

$$\alpha\Phi(R) > \mathcal{E}(0) \ \rightsquigarrow \ R^2 > m_\infty \sum_{|i-j|\le q} |\mathbf{x}_i - \mathbf{x}_j|^2.$$

In other words, the support of ϕ should be sufficiently large to cover a globally connected path in phase space.

6.2. Two-Neighbor Dynamics. In this section we prove uniform connectivity and hence convergence to a consensus of a symmetric *two* nearest neighbor model, as in (6.1):

(6.3)
$$\frac{d}{dt}\mathbf{x}_i = \frac{\alpha}{2}\Big(\kappa_{i+\frac12}(\mathbf{x}_{i+1}-\mathbf{x}_i)+\kappa_{i-\frac12}(\mathbf{x}_{i-1}-\mathbf{x}_i)\Big), \quad \kappa_{i+\frac12} := \begin{cases}0, & i=0,N,\\ \phi(|\mathbf{x}_{i+1}-\mathbf{x}_i)|), & 1\le i\le N.\end{cases}$$

We assume that the initial configuration of agents can be enumerated such that $\{\mathbf{x}_i(0)\}_i$ is connected:

$$\max_i |\mathbf{x}_{i+1}(0)-\mathbf{x}_i(0)| < R, \qquad \mathrm{Supp}\{\phi(\cdot)\} = [0,R). \tag{6.4}$$

The configuration of such "purely" local interactions applies to the 1D setup where each agent is initially connected to its left and right neighbors; we emphasize that these configurations are not necessarily restricted to the 1D setup.

Forward differencing of (6.3) implies that $\Delta_{i+\frac12} := \Delta_{i+\frac12}(t) := \mathbf{x}_{i+1}(t)-\mathbf{x}_i(t)$ satisfy

$$\begin{aligned}\frac{d}{dt}\Delta_{i+\frac12} &= \frac{\alpha}{2}\Big(\kappa_{i+\frac32}(\mathbf{x}_{i+2}-\mathbf{x}_{i+1})+\kappa_{i+\frac12}(\mathbf{x}_i-\mathbf{x}_{i+1})-\kappa_{i+\frac12}(\mathbf{x}_{i+1}-\mathbf{x}_i)-\kappa_{i-\frac12}(\mathbf{x}_{i-1}-\mathbf{x}_i)\Big)\\ &= \frac{\alpha}{2}\Big(\kappa_{i+\frac32}\Delta_{i+\frac32} \ - \ 2\kappa_{i+\frac12}\Delta_{i+\frac12} \ + \ \kappa_{i-\frac12}\Delta_{i-\frac12}\Big), \qquad i=1,2,\ldots,N-1.\end{aligned}$$

The missing Δ's for $i=\frac12$ and $i=N+\frac12$ are defined as $\Delta_{\frac12}=\Delta_{N+\frac12}=0$. Let $\Delta_{p+\frac12}$ denote the maximal difference $|\Delta_{p+\frac12}| = \max_i |\Delta_{i+\frac12}|$ measured in the ℓ_2-norm. Then

$$\begin{aligned}\frac12\frac{d}{dt}|\Delta_{p+\frac12}|^2 &= \frac{\alpha}{2}\Big(\kappa_{p+\frac32}\langle\Delta_{p+\frac32},\Delta_{p+\frac12}\rangle \ - \ 2\kappa_{p+\frac12}|\Delta_{p+\frac12}|^2 \ + \ \kappa_{p-\frac12}\langle\Delta_{p-\frac12},\Delta_{p+\frac12}\rangle\Big)\\ &\le \frac{\alpha}{2}\Big(\kappa_{p+\frac32} - 2\kappa_{p+\frac12} + \kappa_{p-\frac12}\Big)|\Delta_{p+\frac12}|^2.\end{aligned}$$

Now, if ϕ is a *nondecreasing* influence function, then

$$|\Delta_{p+\frac12}| \ge |\Delta_{i+\frac12}| \quad \rightsquigarrow \quad 2\kappa_{p+\frac12} = 2\phi(|\Delta_{p+\frac12}|) \ge \phi(|\Delta_{p-\frac12}|)+\phi(|\Delta_{p+\frac32}|),$$

and hence $|\Delta_{p+\frac12}(t)| = \max_i \phi(|\mathbf{x}_{i+1}(t)-\mathbf{x}_i(t)|) \le \max_i \phi(|\mathbf{x}_{i+1}(0)-\mathbf{x}_i(0)|)$. We deduce the following theorem.

THEOREM 6.2. *Consider the nearest neighbor dynamics* (6.3) *subject to initial configuration,* $\mathbf{x}(0)$*, which is connected (cf.* (6.4)*),*

$$\max_i |\mathbf{x}_{i+1}(0) - \mathbf{x}_i(0)| < R, \quad \mathrm{Supp}\{\phi(\cdot)\} = [0, R).$$

Assume that the influence function, ϕ*, is nondecreasing. Then the dynamics* (6.3) *remains connected and converges to a consensus,* $\mathbf{x}^\infty = \langle \mathbf{x} \rangle(0)$*,*

$$\sum_i |\mathbf{x}_i(t) - \langle \mathbf{x} \rangle(0)|^2 \lesssim \exp\left(-\frac{2\phi(0)t}{N}\right) \sum_i |\mathbf{x}_i(0) - \langle \mathbf{x} \rangle(0)|^2, \qquad \langle \mathbf{x} \rangle := \frac{1}{N} \sum_i \mathbf{x}_i. \tag{6.5}$$

It is important to notice that Theorem 6.2 requires a nondecreasing influence function. Indeed, the steeper the increase of ϕ is, the better the connectivity is. This is a concrete ramification of our main statement that heterophilious dynamics enhances consensus. Note that a two nearest neighbor dynamics driven by a decreasing ϕ will not guarantee consensus, as illustrated by the following counterexample.

Counterexample. We revisit the counterexample in section 4.1 of five agents symmetrically distributed around $\mathbf{x}_3(t) \equiv 0$ with $\frac{1}{2} < \mathbf{x}_4(t) < 1 < \mathbf{x}_5(t) < \frac{3}{2}$, governed by

$$\begin{aligned} \dot{\mathbf{x}}_4 &= -\phi(|\mathbf{x}_4|)\mathbf{x}_4 + \phi(|\mathbf{x}_5 - \mathbf{x}_4|)(\mathbf{x}_5 - \mathbf{x}_4), \\ \dot{\mathbf{x}}_5 &= \phi(|\mathbf{x}_5 - \mathbf{x}_5|)(\mathbf{x}_4 - \mathbf{x}_5), \end{aligned}$$

with a compactly supported influence function $\phi(r) = (1-r)^2(1+r)^2\chi_{[0,1]}$. Observe that this configuration amounts to a two nearest neighbor dynamics. Its concentration into three separate clusters $\{-1, 0, 1\}$ shown in Figure 6 requires a rapidly decreasing influence function (to be precise, $\phi(r)r \downarrow$ for $r \sim 1$), which is not covered by the two nearest neighbor heterophilious dynamics sought in Theorem 6.2.

Proof. The adjacency matrix associated with (6.3), $\dot{\mathbf{x}} = \alpha(A\mathbf{x} - \mathbf{x})$, is given by the tridiagonal matrix $A = \{a_{ij}\}$ given by

$$a_{ij} = \begin{cases} \dfrac{1}{2}\kappa_{\frac{i+j}{2}}, \quad \kappa_{\frac{i+j}{2}} = \phi(|\mathbf{x}_i - \mathbf{x}_j|), \qquad |i-j| = 1, \\[2ex] 1 - \dfrac{1}{2}\kappa_i, \quad \kappa_i := \phi_{i,i+1} + \phi_{i,i-1}, \quad i = j. \end{cases}$$

The corresponding Laplacian associated with A is given by

$$L_A = \frac{1}{2} \begin{bmatrix} -\kappa_1 & \kappa_{\frac{3}{2}} & & & & \\ \kappa_{\frac{3}{2}} & -\kappa_2 & \kappa_{\frac{5}{2}} & & & \\ & \kappa_{\frac{5}{2}} & -\kappa_3 & \kappa_{\frac{7}{2}} & & \\ & & \ddots & \ddots & \ddots & \\ & & & \kappa_{N-\frac{3}{2}} & -\kappa_{N-1} & \kappa_{N-\frac{1}{2}} \\ & & & & \kappa_{N-\frac{1}{2}} & -\kappa_N \end{bmatrix}. \tag{6.6}$$

Since $|\mathbf{x}_{i+1}(t) - \mathbf{x}_i(t)| < R$, the off-diagonal entries $\kappa_{i+\frac{1}{2}} = \phi(|\mathbf{x}_{i+1} - \mathbf{x}_i|) > 0$ and, hence, the graph $\mathcal{G}_A = (\{\mathbf{x}(t)\}, A(\mathbf{x}(t)))$ remains connected with $\mu = \min_i \kappa_{i+\frac{1}{2}}$ and $diam(\mathcal{G}_A) = N$. By (4.3) we find

$$\lambda_2(L_A) \geq \frac{\min_i \kappa_{i+\frac{1}{2}}}{N^2} \geq \frac{\phi(0)}{N^2}.$$

Using Theorem 2.11 (see (2.21)), we end up with

$$\sum_i |\mathbf{x}_i(t) - \langle \mathbf{x}\rangle(0)|^2 \lesssim \mathrm{e}^{-\frac{\alpha\phi(0)t}{N^2}} \sum_i |\mathbf{x}_i(0) - \langle \mathbf{x}\rangle(0)|^2,$$

which concludes the proof. ∎

REMARK 6.3. *The worst-case scenario for the decaying of* $|\mathbf{x}_i(t) - \langle \mathbf{x}\rangle|$ *is to have many opinions* $\mathbf{x}_i$ *concentrate at two extreme values, with just one path of opinion connecting the two extremes (see Figure* 14*).*

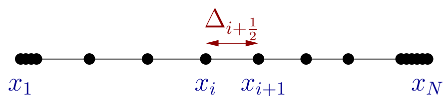

Fig. 14 *The worst-case scenario for the decaying of the norm of the vector* Δ*: the formation is connected, but there are two large groups with extreme values.*

7. Self-Alignment Dynamics with Discrete Time-Steps.

Models for opinion dynamics were originally introduced as discrete algorithms. In this section we therefore extend our results on the semidiscrete continuous opinion dynamics (1.2b) to the fully discrete case,

$$\frac{\mathbf{x}_i(t+\Delta t) - \mathbf{x}_i(t)}{\Delta t} = \alpha \frac{\sum_j \phi_{ij}(\mathbf{x}_j(t) - \mathbf{x}_i(t))}{\sum_j \phi_{ij}}. \tag{7.1}$$

In particular, for $\alpha = 1/\Delta t$ we find that $\mathbf{x}_i^n = \mathbf{x}_i(n\Delta t)$ satisfies the Krause model [11, 12, 75]

$$\mathbf{x}_i^{n+1} = \frac{\sum_j \phi_{ij}\mathbf{x}_j^n}{\sum_j \phi_{ij}}, \qquad \phi_{ij} = \phi(|\mathbf{x}_j^n - \mathbf{x}_i^n|). \tag{7.2}$$

In what follows, we study the properties of the discrete dynamics (7.2).

7.1. Consensus with Global Interactions.

Many results of the continuous dynamics (1.2b) can be translated to the discrete dynamics (7.2). For example, the convex hull of the opinions Ω (2.3) is still decreasing over time:

$$\Omega(n+1) \subset \Omega(n).$$

The discrete dynamics (7.2) will also converge to a consensus if initially all agents interact with each other. More precisely, arguing along the lines of Proposition 2.8 gives the following result.

THEOREM 7.1. *Assume that* $m = \min_{r\in[0,[\mathbf{x}(0)]]} \phi(r) > 0$. *Then the diameter of the discrete dynamics* (7.2) *satisfies*

$$[\mathbf{x}^n] \,\leq\, (1-m)^n [\mathbf{x}^0] \,\stackrel{n\to\infty}{\longrightarrow}\, 0, \tag{7.3}$$

and convergence to a consensus $\mathbf{x}_i^n \stackrel{n\to\infty}{\longrightarrow} \mathbf{x}^\infty \in \Omega(0)$ *follows.*

Proof. Using the contraction estimate (2.7) followed by the bound $\eta_{_A} \geq \max_\theta \theta \cdot \lambda(\theta)$ yields

$$[\mathbf{x}^{n+1}] \leq (1-\eta_{_A})[\mathbf{x}^n] \leq (1-\theta\cdot\lambda(\theta,t^n))[\mathbf{x}^n], \qquad a_{ij} = \frac{\phi_{ij}}{\sum_\ell \phi_{\ell j}}.$$

Fix $\theta = m/N$; then $\Lambda(\theta)$ includes all agents, $\lambda(\theta, t^n) = N$, and we conclude

$$[\mathbf{x}^{n+1}] \leq \big(1 - m\big)\, [\mathbf{x}^n],$$

which proves (7.3). ∎

7.2. Clustering with Local Interactions. As in the continuous dynamics, we would like to investigate the behavior of the discrete dynamics (7.2) with *local* interactions; in particular, we are interested in the formation of clusters. Our aim is to reproduce the discrete analogue of Proposition 3.1.

PROPOSITION 7.2. *Let $\mathcal{P}^n = \{\mathbf{x}^n_k\}_k$ be the solution of the discrete opinion dynamics* (7.2) *with compactly supported influence function* $\mathrm{Supp}\{\phi(\cdot)\} = [0, R)$. *Assume that it approaches a steady state fast enough so that*

$$\sum_{n=m}^{\infty} \sum_i |\mathbf{x}^{n+1}_i - \mathbf{x}^n_i| \stackrel{m\to\infty}{\longrightarrow} 0. \tag{7.4}$$

Then $\{\mathbf{x}^n\}$ approaches a stationary state, $\mathbf{x}^\infty$, which is partitioned into clusters, $\{\mathcal{C}_k\}_k$, such that $\{1, 2, \ldots, N\} = \cup_{k=1}^K \mathcal{C}_k$ and

$$\mathbf{x}^n_i \longrightarrow \mathbf{x}^\infty_{\mathcal{C}_k} \qquad \textit{for all} \ \ i \in \mathcal{C}_k. \tag{7.5}$$

Proof. By assumption (7.4),

$$|\mathbf{x}^{n_2}_i - \mathbf{x}^{n_1}_i| \leq \sum_{n=n_1}^{n_2 - 1} |\mathbf{x}^{n+1}_i - \mathbf{x}^n_i| \ll 1 \ \ \text{for} \ \ n_2 > n_1 \gg 1,$$

and hence $\mathbf{x}^n$ approaches a limit, $\mathbf{x}^n_i \stackrel{n\to\infty}{\longrightarrow} \mathbf{x}^\infty_i$. The discrete dynamics (7.2) can be written in the following form:

$$\sum_j \phi_{ij} \left(\mathbf{x}^{n+1}_i - \mathbf{x}^n_i\right) = \sum_j \phi_{ij} (\mathbf{x}^n_j - \mathbf{x}^n_i).$$

Taking the scalar product against $\mathbf{x}^n_i$, summing in i, and using the symmetry of ϕ_{ij} yields

$$\sum_{ij} \phi_{ij} \left\langle \left(\mathbf{x}^{n+1}_i - \mathbf{x}^n_i\right), \mathbf{x}^n_i \right\rangle = \sum_{ij} \phi_{ij} \langle \mathbf{x}^n_j - \mathbf{x}^n_i \,,\, \mathbf{x}^n_i \rangle = -\frac{1}{2} \sum_{ij} \phi_{ij} |\mathbf{x}^n_j - \mathbf{x}^n_i|^2.$$

Since ϕ_{ij}, $\mathbf{x}^n_i$, and, by assumption, the tail $\sum_m^\infty |\mathbf{x}^{n+1}_i - \mathbf{x}^n_i|$ are bounded, we conclude that the sum on the right converges to zero,

$$\phi_{ij} |\mathbf{x}^n_j - \mathbf{x}^n_i|^2 \stackrel{n\to\infty}{\longrightarrow} \phi(|\mathbf{x}^\infty_j - \mathbf{x}^\infty_i|) |\mathbf{x}^\infty_j - \mathbf{x}^\infty_i|^2 = 0.$$

Hence, either $\mathbf{x}^\infty_j$ and $\mathbf{x}^\infty_i$ are in separate clusters, $|\mathbf{x}^\infty_j - \mathbf{x}^\infty_i| > R$, or they are in the limiting point of the same cluster, say, $i, j \in C_\ell$, so that $\mathbf{x}^\infty_j = \mathbf{x}^\infty_i$. ∎

We now turn our attention to the convergence toward consensus for the discrete dynamics (7.2). As for the continuous dynamics (1.2), there exists a Lyapunov functional energy for the dynamics under the additional assumption that the influence function ϕ is nonincreasing. Consequently, we deduce the analogue of Theorem 4.3 for the discrete dynamics.

THEOREM 7.3. *Let $\mathcal{P}^n = \{\mathbf{x}_k^n\}_k$ be the solution of the discrete opinion dynamics* (7.2) *with nonincreasing, compactly supported influence function* $\mathrm{Supp}\{\phi(\cdot)\} = [0, R)$. *If $\mathcal{P}^n$ remains uniformly connected for any n, then $\mathcal{P}^n$ converges to a consensus.*

Proof. First, we prove that the energy functional $\mathcal{E}^n$ is also a Lyapunov function for the discrete dynamics:

$$\mathcal{E}^n := \sum_{ij} \Phi(|\mathbf{x}_j^n - \mathbf{x}_i^n|), \quad \Phi(r) = \int_0^r s\phi(s)ds. \tag{7.6}$$

Introducing $\varphi(r^2) = \Phi(r)$, we have $\varphi(r) = \int_0^{\sqrt{r}} s\phi(s)ds = \frac{1}{2}\int_0^r \phi(\sqrt{y})dy$. By assumption, ϕ is nonincreasing, thus φ is concave-down. Therefore,

$$\begin{aligned}\mathcal{E}^{n+1} - \mathcal{E}^n &= \sum_{ij} \varphi(|\mathbf{x}_j^{n+1} - \mathbf{x}_i^{n+1}|^2) - \varphi(|\mathbf{x}_j^n - \mathbf{x}_i^n|^2) \\ &\leq \frac{1}{2}\sum_{ij} \phi(|\mathbf{x}_j^n - \mathbf{x}_i^n|)\big(|\mathbf{x}_j^{n+1} - \mathbf{x}_i^{n+1}|^2 - |\mathbf{x}_j^n - \mathbf{x}_i^n|^2\big).\end{aligned}$$

Using $|\mathbf{a}|^2 - |\mathbf{b}|^2 = \langle \mathbf{a} - \mathbf{b}\,,\, \mathbf{a} + \mathbf{b}\rangle$, we deduce

$$\begin{aligned}\mathcal{E}^{n+1} - \mathcal{E}^n &\leq \frac{1}{2}\sum_{ij} \phi_{ij}\langle \Delta_t\mathbf{x}_j^n - \Delta_t\mathbf{x}_i^n\,,\, \mathbf{x}_j^{n+1} - \mathbf{x}_i^{n+1} + \mathbf{x}_j^n - \mathbf{x}_i^n\rangle \\ &= \sum_{ij} \phi_{ij}\langle \Delta_t\mathbf{x}_j^n\,,\, \mathbf{x}_j^{n+1} - \mathbf{x}_i^{n+1} + \mathbf{x}_j^n - \mathbf{x}_i^n\rangle,\end{aligned}$$

since $\phi_{ij} = \phi_{ji}$. Writing $\mathbf{x}_j^{n+1} = \mathbf{x}_j^n + \Delta_t\mathbf{x}_j^n$, we obtain

$$\mathcal{E}^{n+1} - \mathcal{E}^n \leq \sum_{ij} \phi_{ij}\langle \Delta_t\mathbf{x}_j^n\,,\, 2(\mathbf{x}_j^n - \mathbf{x}_i^n) + \Delta_t\mathbf{x}_i^n - \Delta_t\mathbf{x}_j^n\rangle.$$

Combining with the equality

$$\sum_j \phi_{ij}\Delta_t\mathbf{x}_i^n = \sum_j \phi_{ij}(\mathbf{x}_j^n - \mathbf{x}_i^n), \tag{7.7}$$

we conclude

$$\mathcal{E}^{n+1} - \mathcal{E}^n \leq \sum_{ij} \phi_{ij}\langle \Delta_t\mathbf{x}_i^n\,,\, -\Delta_t\mathbf{x}_i^n - \Delta_t\mathbf{x}_j^n\rangle = -\sum_{ij} \phi_{ij}|\Delta_t\mathbf{x}_i^n|^2,$$

where we use once again the symmetry of the coefficients ϕ_{ij}. Thus, $\mathcal{E}^n$ is decaying.

Now, we would like to combine the decay of $\mathcal{E}^n$ and the strong connectivity of $\mathcal{P}^n$. Noting $\sigma_i = \sum_j \phi_{ij}$, the equality (7.7) yields

$$\frac{1}{2}\sum_{i,j} \phi_{ij}|\mathbf{x}_j^n - \mathbf{x}_i^n|^2 = \sum_i \sigma_i\langle \mathbf{x}_i^n, \Delta_t\mathbf{x}_i^n\rangle \leq N\max_i |\mathbf{x}_i^0|\sqrt{\sum_i \sigma_i|\Delta_t\mathbf{x}_i^n|^2}.$$

Thus,

$$\mathcal{E}^{n+1} - \mathcal{E}^n \leq -C_0^2\left(\sum_{i,j} \phi_{ij}|\mathbf{x}_j^n - \mathbf{x}_i^n|^2\right)^2, \qquad C_0 = \frac{1}{2N\max_i|\mathbf{x}_i(0)|}.$$

Summing in n, we deduce that the sum $\sum_{i,j}\phi_{ij}|\mathbf{x}_j^n - \mathbf{x}_i^n|^2$ becomes arbitrarily small. To conclude, we proceed as in the proof of Theorem 4.3. $\square$

7.3. Numerical Simulations of Discrete Dynamics. In this section we illustrate the difference between the continuous opinion models (1.2b) and its discrete version (7.2). To this end, we run in parallel numerical simulations of the discrete and continuous models subject to the same initial conditions.

First, we run a simulation with an influence function $\phi = \chi_{[0,1]}$ (see Figure 15). Discrete and continuous dynamics are very similar, except that there are three *branches* in the continuous dynamics which are not present in the discrete dynamics. For this reason, at the end of the simulation, we count four clusters in the discrete dynamics and only three in the continuous version.

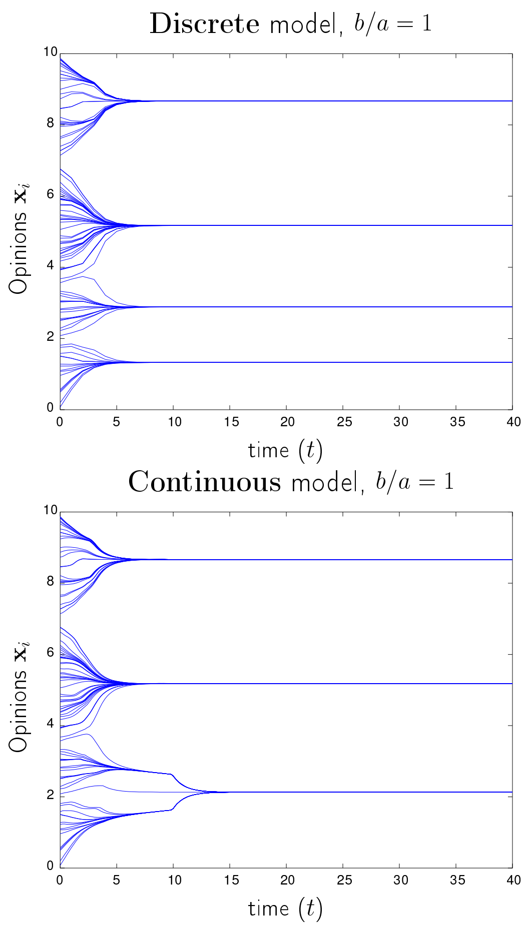

Fig. 15 *Simulations of the discrete (top) and continuous (bottom) dynamics with $\phi = \chi_{[0,1]}$, starting with the same initial condition. Although the two simulations are very similar, the discrete dynamics yields four clusters, whereas the continuous dynamics gives three. We use a time-discretization of $\Delta t = .05$ to simulate the continuous dynamics.*

Next we use the influence function (5.1) $\phi = a\chi_{[0,1/\sqrt{2}]} + b\chi_{[1/\sqrt{2},1]}$ with $b/a = 10$. Here, the discrete and continuous dynamics give very different results shown in Figure 16. As we have seen previously, the continuous dynamics converges to a distribution with uniformly spaced clusters and then reaches a consensus. In contrast, the discrete dynamics does not stabilize. Order between the opinions $\{\mathbf{x}_i\}_i$ is no longer

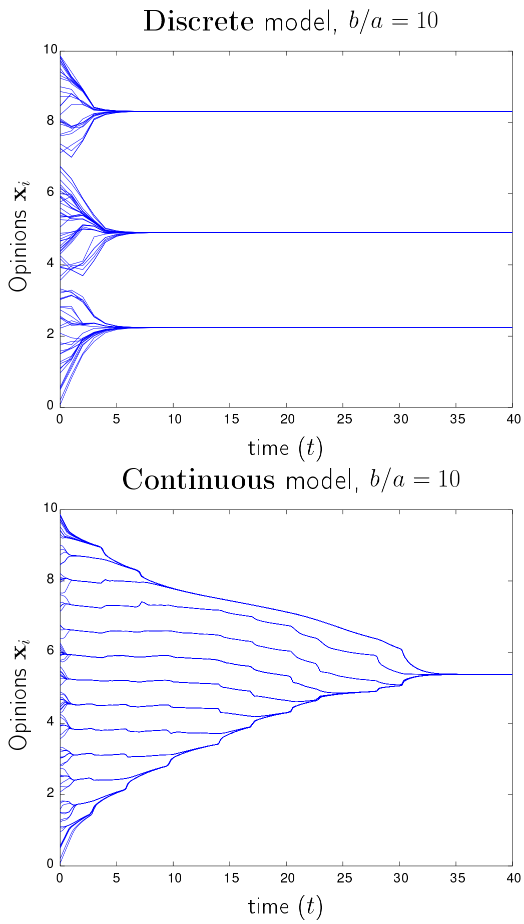

Fig. 16 *Simulations of the discrete (top) and continuous (bottom) dynamics with $\phi = .1\chi_{[0,1/\sqrt{2}]} + \chi_{[1/\sqrt{2},1]}$, starting with the same initial condition. In contrast with Figure 15, the two models produce very different output. There is no uniformly spaced formation in the discrete model: we only observe cluster formation.*

preserved, and trajectories do cross. Even though the total number of clusters has been diminished with $b/a = 10$ (from four to three clusters), the effect of the ratio b/a on the clustering formation is less pronounced in the discrete dynamics.

8. Mean-Field Limits: Self-Organized Hydrodynamics. When the number of agents N is large, it is convenient to describe the evolution of the resulting large dynamical systems as a mean-field equation. We limit ourselves to a few classic general references on this topic [27, 57, 101] and a few recent references in the contexts of opinion hydrodynamics [18, 105] and flocking hydrodynamics [21, 22, 40, 61, 73, 84, 88].

8.1. Opinion Hydrodynamics. To derive the mean-field limit of the opinion dynamics model (1.2b), we introduce the so-called empirical distribution $\rho(t, \mathbf{x})$,

$$\rho(t, \mathbf{x}) := \frac{1}{N} \sum_{j=1}^{N} \delta_{\mathbf{x}_j(t)}(\mathbf{x}),$$

where δ is a Dirac mass and $\{\mathbf{x}_j(t)\}_j$ is the solution of the consensus model (1.2b). Expressed in terms of this empirical distribution, the nonsymmetric model (1.2b) (with $\alpha = 1$) reads

$$\dot{\mathbf{x}}_i = \frac{\int_{\mathbf{y}} \phi(|\mathbf{y}-\mathbf{x}_i|)(\mathbf{y}-\mathbf{x}_i)\rho(t,\mathbf{y})\,d\mathbf{y}}{\int_{\mathbf{y}} \phi(|\mathbf{y}-\mathbf{x}_i|)\rho(t,\mathbf{y})\,d\mathbf{y}} = \frac{(\phi(|\mathbf{y}|)\mathbf{y} * \rho)(\mathbf{x}_i)}{(\phi(|\mathbf{y}|) * \rho)(\mathbf{x}_i)}. \tag{8.1}$$

This equation describes the characteristics of the density ρ. Indeed, integrating ρ against a test function φ yields[4]

$$\frac{d}{dt}(\rho,\varphi) = \frac{d}{dt}\left(\frac{1}{N}\sum_{j=1}^{N}\varphi(\mathbf{x}_j(t))\right) = \frac{1}{N}\sum_{j}^{N}\langle \dot{\mathbf{x}}_j(t), \nabla_{\mathbf{x}}\varphi(\mathbf{x}_j(t))\rangle.$$

Using the expression (8.1), we deduce

$$\begin{aligned}\frac{d}{dt}(\rho,\varphi) &= \frac{1}{N}\sum_{j=1}^{N}\left\langle \frac{\phi(|\mathbf{y}|)\mathbf{y} * \rho(\mathbf{x}_j)}{\phi(|\mathbf{y}|) * \rho(\mathbf{x}_j)}, \nabla_{\mathbf{x}}\varphi(\mathbf{x}_j(t))\right\rangle = \left(\rho, \left\langle \frac{\phi(|\mathbf{y}|)\mathbf{y} * \rho}{\phi(|\mathbf{y}|) * \rho}, \nabla_{\mathbf{x}}\varphi\right\rangle\right)\\ &= \left(-\nabla_{\mathbf{x}} \cdot \left(\frac{\phi(|\mathbf{y}|)\mathbf{y} * \rho}{\phi(|\mathbf{y}|) * \rho}\,\rho\right), \varphi\right).\end{aligned}$$

Thus, $\rho = \rho(t,\mathbf{x})$ satisfies a continuum transport equation,

$$\partial_t\rho + \nabla_{\mathbf{x}} \cdot \big(\rho\mathbf{u}\big) = 0 \quad \text{with} \quad \mathbf{u}(\mathbf{x}) = \frac{\int_{\mathbf{y}} \phi(|\mathbf{y}-\mathbf{x}|)(\mathbf{y}-\mathbf{x})\,\rho(\mathbf{y})\,d\mathbf{y}}{\int_{\mathbf{y}} \phi(|\mathbf{y}-\mathbf{x}|)\,\rho(\mathbf{y})\,d\mathbf{y}}. \tag{8.2a}$$

This is the hydrodynamic description of the agent-based opinion model (1.2b). Similarly, the opinion hydrodynamics of the corresponding *symmetric* model (1.2a) (with $\alpha = 1$) amounts to the aggregation model [8, 18]

$$\partial_t\rho + \nabla_{\mathbf{x}} \cdot \big(\rho\mathbf{u}\big) = 0 \quad \text{with} \quad \mathbf{u}(\mathbf{x}) = \nabla\Phi * \rho. \tag{8.2b}$$

We note that the transport equations (8.2) are nonlinear due to the dependence of the velocity field $\mathbf{u} = \mathbf{u}(\rho)$. The main features of the particle description for opinion dynamics (1.2) carry over to the hydrodynamic model (8.2). Thus, for example, the symmetric model (8.2b) preserves the center of mass, $\frac{d}{dt}\left(\int_{\mathbf{x}} \mathbf{x}\rho(t,\mathbf{x})\,d\mathbf{x}\right) = 0$, whereas the nonsymmetric model (8.2a) does not. We distinguish between the two cases of global and local interactions.

The existence of regular solutions of the *symmetric* aggregation model (8.2b) for bounded decreasing ϕ such that $|\phi'(r)r| \lesssim \phi(r)$ was proved in [8]. This holds independently, whether or not ϕ is global. Moreover, if the kernel ϕ is globally supported, then one can argue along the lines of the underlying agent-based model (8.1) to prove convergence of the hydrodynamics toward a consensus, that is, $\rho(t,\mathbf{x})$ converges to a single point asymptotically in time. If ϕ is compactly supported, however, then the velocity field $\mathbf{u}$ need not be continuous with respect to ρ due to the singularity when $\int_{\mathbf{y}} \phi(|\mathbf{y}-\mathbf{x}|)\,\rho(\mathbf{y})\,d\mathbf{y} = 0$. Then existence and uniqueness of solutions of the nonsymmetric model (8.2a) cannot be obtained through a standard Picard's iteration argument. The large time behavior of the dynamics in this local setup is completely

[4] $(\cdot,\cdot)$ denotes the duality bracket between distributions and test functions.

open. As in the agent-based dynamics, the generic solution $\rho(t,\mathbf{x})$ is expected to concentrate in finitely many clusters, or "islands"; in particular, under appropriate assumptions on the persistence of connectivity among these islands, one may expect a consensus. Preliminary simulations show that cluster formation tends to persist for the hydrodynamic model, but analytical justification remains open.

8.2. Flocking Hydrodynamics. We study the second-order flocking models (1.3) in terms of the empirical distribution $f^N(t,\mathbf{x},\mathbf{v}) := \frac{1}{N}\sum_{j=1}^N \delta_{\mathbf{x}_j(t)}(\mathbf{x}) \otimes \delta_{\mathbf{v}_j(t)}(\mathbf{v})$, where $\delta_{\mathbf{x}} \otimes \delta_{\mathbf{v}}$ is the usual Dirac mass on the phase space $\mathbb{R}^d \times \mathbb{R}^d$. Consider the nonsymmetric particle model system for flocking (1.3b): expressed in terms of f^N, it reads

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{v}_i,\ \frac{d\mathbf{v}_i}{dt} = \alpha F[f^N](\mathbf{x}_i,\mathbf{v}_i), \quad F[f](\mathbf{x},\mathbf{v}) := \alpha\frac{\int_{\mathbf{y},\mathbf{w}} \phi(|\mathbf{y}-\mathbf{x}|)\,(\mathbf{w}-\mathbf{v}_i)\, f(\mathbf{y},\mathbf{w})\, d\mathbf{y}d\mathbf{w}}{\int_{\mathbf{y}} \phi(|\mathbf{y}-\mathbf{x}|)\, f(\mathbf{y},\mathbf{w})\, d\mathbf{y}d\mathbf{w}},$$

which leads to Liouville's equation,

$$\partial_t f + \mathbf{v}\cdot\nabla_{\mathbf{x}} f + \nabla_{\mathbf{v}}\cdot(F[f]\,f) = 0. \tag{8.3}$$

Integrating the empirical distribution f^N in the velocity variable $\mathbf{v}$ yields the hydrodynamic description of flocking, expressed in terms of the density and momentum distributions of particles:

$$\rho(t,\mathbf{x}) = \int_{\mathbf{v}} f(t,\mathbf{x},\mathbf{v})\, d\mathbf{v} \qquad \left(\text{corresponding to } \frac{1}{N}\sum_{j=1}^N \delta_{\mathbf{x}_j(t)}(\mathbf{x})\right),$$

$$\rho(t,\mathbf{x})\mathbf{u}(t,\mathbf{x}) = \int_{\mathbf{v}} \mathbf{v} f(t,\mathbf{x},\mathbf{v})\, d\mathbf{v} \qquad \left(\text{corresponding to } \frac{1}{N}\sum_{j=1}^N \mathbf{v}_j(t)\delta_{\mathbf{x}_j(t)}(\mathbf{x})\right).$$

Integrating the kinetic equation (8.3) against the first moments $(1,\mathbf{v})$ yields the system (cf. [61, 22, 88])

$$\partial_t\rho + \nabla_{\mathbf{x}}\cdot(\rho\mathbf{u}) = 0, \tag{8.4a}$$

$$\partial_t(\rho\mathbf{u}) + \nabla_{\mathbf{x}}\cdot(\rho\mathbf{u}\otimes\mathbf{u} + \mathbf{P}) = \alpha\rho(\overline{\mathbf{u}} - \mathbf{u}). \tag{8.4b}$$

The expression on the right of (8.4b) reflects alignment: the tendency of agents with velocity $\mathbf{u}$ to relax toward the local average velocity, $\overline{\mathbf{u}}(\mathbf{x})$, dictated by the normalized influence function $a(\mathbf{x},\mathbf{y})$,

$$\overline{\mathbf{u}}(\mathbf{x}) := \int_{\mathbf{y}} a(\mathbf{x},\mathbf{y})\rho(\mathbf{y})\mathbf{u}(\mathbf{y})\, d\mathbf{y}, \quad \int_{\mathbf{y}} a(\mathbf{x},\mathbf{y})\rho(\mathbf{y})\, d\mathbf{y} = 1. \tag{8.4c}$$

This includes, in particular, the hydrodynamic description of the symmetric and nonsymmetric flocking models, given, respectively, by

$$a(\mathbf{x},\mathbf{y}) = \begin{cases} \phi(|\mathbf{y}-\mathbf{x}|), & \text{C-S model (1.3a)}, \\[2ex] \dfrac{\phi(|\mathbf{y}-\mathbf{x}|)}{\int_{\mathbf{y}}\phi(|\mathbf{y}-\mathbf{x}|)\rho(\mathbf{y})\, d\mathbf{y}}, & \text{nonsymmetric model (1.3b)}. \end{cases}$$

The system (8.4) is not closed since the equation for $\rho\mathbf{u}$ (8.4b) depends on the third moment of f, which is encoded in the pressure term $\mathbf{P} := \int_{\mathbf{v}}(\mathbf{v}-\mathbf{u})\otimes(\mathbf{v}-$

$\mathbf{u})f(t,\mathbf{x},\mathbf{v})\,d\mathbf{v}$. If we neglect the pressure (in other words, assume a monophase distribution, $f(t,\mathbf{x},\mathbf{v}) = \rho(t,\mathbf{x})\,\delta_{\mathbf{u}(t,\mathbf{x})}(\mathbf{v})$ so that $\mathbf{P} \equiv 0$), then the flocking hydrodynamics (8.4) is reduced to the closed system

$$\left\{\begin{aligned} &\partial_t\rho + \nabla_{\mathbf{x}}\cdot(\rho\mathbf{u}) = 0, \\ &\partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla_{\mathbf{x}})\mathbf{u} = \alpha(\overline{\mathbf{u}} - \mathbf{u}). \end{aligned}\right. \tag{8.5}$$

The question of an emerging flock in (8.5) follows along the lines of our discussion on the underlying agent-based models (1.3). The case of a *global* influence function is rather well understood: in particular, regularity of the 1D "incompressible" case, $\rho \equiv 1$, depends on an initial critical threshold [81, 96]. Flocking hydrodynamics governed by a *locally* supported influence function requires a more intricate analysis, due to the realistic presence of a vacuum [102]. The hydrodynamic description of self-organized dynamics gives rise to systems like (8.5) which involve nonlocal means. Questions of regularity and quantitative behavior of such systems provide a rich source for future studies.

9. Further Reading on Self-Organized Dynamics. In this paper we discussed fundamental aspects which arise in the context of flocking and opinion dynamics, as prototype models for self-organized dynamics. Specifically, we focused on the emerging large time behavior of self-alignment and we highlighted a few open questions aiming to attract further mathematical studies in this direction. The much broader subject of self-organized dynamics lies at the crossroads of several fields. A comprehensive review of the subject is beyond the scope of this paper, in particular, as it continues to attract an increasing amount of attention reported in a rapidly growing literature. Instead, we refer the interested reader to a selection of references outlined below. As with all multidisciplinary fields, the work on self-organized dynamics can be classified into several different categories. We shall mention five of them.

Different Disciplines. A natural classification is offered by the underlying topic. Many models of self-organized dynamics are driven by examples from biology: these include aggregation of bacteria and amoeba [6, 51, 59, 74, 98], dynamics of insects [14, 33], schools of fish [1, 67, 111], flocking of birds [4, 26, 25, 24, 36, 37, 61, 93, 103, 107], and related models in ecology [58]. Self-organized dynamics is found in many other areas, from pedestrian and traffic dynamics [64, 92], social networks and economics [48, 66, 69, 77, 83], complex networks [5, 44, 91], and opinion dynamics [7, 23, 42, 49, 50, 63, 75, 105, 110, 109], all the way to applications in marketing [2, 3], production networks [94], robotics [32, 71, 112], and materials [99, 85], and in somewhat more esoteric examples such as gossiping [13], collective motion at heavy metal concerts [100], and self-organized phases in the Tour de France [106].

Different Models. Together with the different contexts come different models of self-organized dynamics. We mention a few of the more notable ones: the Krause model for opinion dynamics [75] and the follow-up works in [12, 18, 63, 76, 82]; the Axelrod models for marketing [2] and the influential models for "flocking" (at various "levels") of Aoki, Reynolds, and Couzin [1, 33, 35, 80, 93, 111], Vicsek et al. [107], and the follow-up works in [40, 41, 70]; the C-S model [36, 37] and related works in [10, 21, 60, 61, 62, 73, 88, 97]; and the StarFlag project [4, 26, 25, 24].

Different Scales. Different models of self-organized dynamics are realized at different scales. As examples of agent-based models (also known as individual-based models (IBMs)), we mention [7, 33, 58, 75, 79, 90, 93]. Their mean-field limit leads to a kinetic description [20, 21, 49, 61, 105], and macroscopic averaging then leads to a hydrodynamic-scale description as in [15, 22, 40, 41, 51, 73, 74, 78, 84, 85, 102].

Different Approaches. In this paper, we have focused our attention on mathematical aspects which explain the large time behavior of self-alignment models. The study of general models for self-organized dynamics includes several different approaches. Classified by the tools of the trade, we mention statistical mechanics [10, 23, 101], clustering and spectral theory of graphs [15, 31, 32, 70, 90], optimization and control [39, 43, 44, 71, 72, 91, 112], game theory [5, 65], jump processes, *nonlinear* Markov chains, and stochastic analysis [15, 52, 66, 107].

Different Patterns. One of the most intriguing features of self-organized dynamics is the formation of different patterns. In this paper, we have limited ourselves to the simple pattern of "consensus" (or a "flock"), but the class of possible patterns is much richer. We mention the examples of swarming and mill-like vortices [17, 20, 22, 47, 50, 78, 79, 80, 98, 104], phase transition [55, 107], aggregation [15], biotic colonies [6, 74], lattices [89], leaders [34, 97], shocks [9, 102], and related issues which arise in the context of control and stability [11, 47, 72, 79].

Finally, we recommend several reviews on self-organization [16, 23, 51, 66, 109] and, in particular, the most recent comprehensive review of Vicsek and Zefeiris [108].

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