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Academic Year **2019-2020**

Page created Thu Feb 20 02:15:22 GMT 2020

CATE

499 fbelard 6 t5 frw19 v1 0 0 1 1 9 0 0 3 6 1 6 7	Electronic submission **Wed - 19 Feb 2020 15:52:32** frw19

Exercise Information

Module:	499 Modal Logic for Strategic Reasoning in AI	**Issued:**	Wed - 05 Feb 2020
Exercise:	6 (CW)	**Due:**	Wed - 19 Feb 2020
Title:	Coursework2	**Assessment:**	Individual
FAO:	Belardinelli, Francesco (fbelard)	**Submission:**	Electronic

Student Declaration - Version 1

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Signed: (electronic signature) Date: 2020-02-19 15:51:52

For Markers only: (circle appropriate grade)

WARD, Francis (frw19)	01819196	t5	2020-02-19 15:51:52	**A*** **A** **B** **C** **D** **E** **F**

Coursework 2: Temporal Logics

Imperial College London

Department of Computing

Modal Logic for Strategic Reasoning

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Date: February 19, 2020

1

a) $\phi R\psi$

$$(M,\lambda) \models \phi R\psi \text{ iff } \lambda[i,...,\infty] \models \psi, \forall i \geq 0, \text{ or, } (\exists j \lambda[j,...,\infty] \models \phi \text{ and } \lambda[i,...,\infty] \models \psi \forall 0 \leq i \leq j) \quad (1)$$

b)

$$\phi R\psi = \neg(\neg\phi \cup \neg\psi) \quad (2)$$

c)

We wish to show that $(M,\lambda) \models \phi R\psi \iff (M,\lambda) \models \neg(\neg\phi \cup \neg\psi)$.
$(M,\lambda) \models \neg(\neg\phi \cup \neg\psi)$ iff $\neg i \geq 0(\lambda[i,...,\infty] \models \neg\psi \geq 0$ and $\lambda[j,...,\infty] \models \neg\phi \forall 0 \leq j < i)$
equivalently $\neg\exists i \geq 0($ such that $\lambda[i,...,\infty] \nvDash \psi$ and $\lambda[j,...\infty] \nvDash \phi \forall 0 \leq j < i)$
This is equivalent to
$\forall i \geq 0 \neg(\lambda[i,...,\infty] \nvDash \psi$ and $\lambda[j,...\infty] \nvDash \phi \forall 0 \leq j < i)$
by De Morgan's law we have
$\forall i \geq 0(\neg\lambda[i,...,\infty] \nvDash \psi$ or $\neg\lambda[j,...\infty] \nvDash \phi \forall 0 \leq j < i)$
i.e.
$\forall i \geq 0(\lambda[i,...,\infty] \models \psi$ or $\exists 0 \leq j < i \lambda[j,...\infty] \models \phi)$
which is equivalent to
$\forall i \geq 0 \lambda[i,...,\infty] \models \psi$ or $\exists 0 \leq j \leq i(\lambda[j,...\infty] \models \phi and \lambda[j,...,\infty] \models \psi)$ and this is our definition from a) as required.

d)

We have by definition $\lambda \models G\psi$ iff $\lambda[i,...,\infty] \models \psi \forall i \geq 0$ And $\lambda \models \bot R\psi \iff \forall i \geq 0 \lambda[i,...,\infty] \models \psi$ or $\exists 0 \leq j \leq i(\lambda[j,...\infty] \models \bot and \lambda[j,...,\infty] \models \psi)$
but clearly there is no such j that satisfies $\lambda[j,...\infty] \models \bot$. So we have: $\lambda \models \bot R\psi \iff \forall i \geq 0 \lambda[i,...,\infty] \models \psi$ as required.

2

i. $(M,q) \models EF\phi$ iff $\exists\lambda[q]$ (a path starting at q) such that
$(M,\lambda[q]) \models F\phi$ i.e. $\exists\lambda[q]$ such that $(M,\lambda[q]) \models (\top \cup \phi)$ Well, this is true iff $\exists\lambda[q] =: \lambda$ such that ,

$$(M,\lambda[i]) \models \phi \text{ for some } i \geq 0 \text{ and } (M,\lambda[j]) \models \top \forall 0 \leq j \leq i \quad (3)$$

Clearly this is equivalent to

$$\exists\lambda[q] \text{ such that } (M,\lambda[i]) \models \phi \text{ for some } i \geq 0 \quad (4)$$

as required.
ii.

$$(M,q) \models AF\phi \text{ iff } \forall\lambda[q],(M,\lambda) \models F\phi \tag{5}$$

Now just follow the same steps as in i. to show that the statement holds.
iii.
$EG\phi$ is equivalent to $\neg AF\neg\phi \equiv \neg A(\top \cup \neg\phi)$. Hence,

$$(M,q) \models EG\phi \tag{6}$$

iff

$$(M,q) \models \neg A(\top \cup \neg\phi) \tag{7}$$

which is true iff

$$\forall\lambda[q],(M,\lambda) \not\models (\top \cup \neg\phi) \tag{8}$$

i.e.

$$\exists\lambda[q],(M,\lambda) \models (\top \cup \neg\phi) \tag{9}$$

This holds iff $\exists\lambda$ starting at q such that

$$\exists i \geq 0, \text{ with } (M,\lambda[i]) \models \neg\phi \text{ and } (M,\lambda[j]) \models \top\forall 0 \leq j \leq i \tag{10}$$

Now, this is clearly equivalent to

$$\exists\lambda\forall i \geq 0,(M,\lambda[i]) \models \phi \tag{11}$$

iv.
$(M,q) \models AG\phi$ iff $(M,q) \models \neg EF\neg\phi$ By definition this holds iff, $\neg(\exists\lambda[q].(M,\lambda) \not\models F\phi)$ i.e. $\forall\lambda[q],(M,\lambda[q]) \models F\phi$ equivalently, $\forall\lambda[q],(M,\lambda) \models (\top \cup \phi)$. Now we can similarly follow the final two steps in iii. to show the result.

3

a)

Our CTL formulas are

$$\Phi ::= p|\neg\Phi|\Phi \wedge \Phi|EX\Phi|AX\Phi|E(\Phi \cup \Phi)|A(\Phi \cup \Phi) \tag{12}$$

We trivially have that $p|\neg\Phi|\Phi \wedge \Phi|$ are formulas of CTL* by definition 1.
Do we have $EX\Phi$? We can see that $X\Phi$ is a path formula of CTL* by definition 1 (as Φ is itself a path formula). Hence, $EX\Phi$ is a state formula of CTL* (i.e. a formula). Following the same reasoning we can show that $AX\Phi$, $E(\Phi \cup \Phi)$, and $A(\Phi \cup \Phi)$ are also all state formulas of CTL* (since $\Phi \cup \Phi$ is a path formula). Thus, every formula of CTL is also a formula of CTL*. □

b)

Consider, $AFGp \equiv A(\top \cup (\perp \cup p))$. This is indeed a formula of CTL* (take $\perp := p \wedge \neg p$ and $\top := \neg\perp$). By definition 1, $\perp$ and $\top$ are state formulas of CTL*, so they are also path formulas, so $\perp \cup p$ is a path formula, therefore $\top \cup (\perp \cup p)$ is a path formula. Finally $A(\top \cup (\perp \cup p))$ is a state formula of CTL*.
But, state formulas in CTL are not in general path formulas. In particular, $\perp \cup p$ is a path formula but not a state formula. So $\top \cup (\perp \cup p)$ is not a path formula of CTL and so $A(\top \cup (\perp \cup p))$ is not a (state) formula of CTL. □

4

To recover CTL from CTL* we restrict the quantifiers so that each temporal quantifier is preceded directly by a path quantifier. Equivalently we restrict the formulas of CTL* to CTL. Our formula of CTL are:

$$\Phi ::= p|\neg\Phi|\Phi \wedge \Phi|EX\Phi|AX\Phi|E(\Phi \cup \Phi)|A(\Phi \cup \Phi) \tag{13}$$

In particlar, compared to CTL* we are excluding the path formulas

$$\psi = \Phi|\neg\psi|\psi \wedge \psi \tag{14}$$

and keeping only

$$\psi = X\psi|\psi \cup \psi \tag{15}$$

Clearly, satisfaction on the state formulas is completely equivalent in CTL as in CTL* (the definitions are identical). So we need only show that satisfaction of $X\psi$ and $\psi \cup \psi'$ is preserved.
By definition 2

$$(M,\pi) \models X\psi \text{ iff } (M,\pi[1,...,\infty]) \models \psi \text{ iff } (M,\pi[1]) \models \psi \tag{16}$$

so we have recovered satisfaction of $X\psi$. Now consider

$$(M,\pi) \models \psi \cup \psi' \text{ iff } (M,\pi[i,...,\infty]) \models \psi' \text{ for some } i \geq 0 \text{ and } (M,\pi[j,...,\infty]) \models \psi \forall 0 \leq j \leq i \tag{17}$$

following definition 2 this is

$$(M,\pi) \models \psi \cup \psi' \text{ iff } (M,\pi[i]) \models \psi' \text{ for some } i \geq 0 \text{ and } (M,\pi[j]) \models \psi \forall 0 \leq j \leq i \tag{18}$$

and we have recovered satisfaction of until. □

5

a)

By question 3 CTL is a strict fragment of CTL* i.e. for every formula Φ of CTL, Φ is also a formula of CTL*. Furthermore, by 4, for the formulas in CTL, CTL and CTL* are semantically equivalent, i.e. they have the same truth conditions.So the Φ' we are looking for is just Φ.

b)

Take the example FGp from lecture 5. This is an LTL formula hence also a CTL* formula. But there is no equivalent CTL formula by the Clarke Draghicescu lemma and the example shown in the lecture slides.

6

Proceed by induction on the structure of Φ and ψ. Since (M,t) and (M',t') are bisimilar we have from definition 3 a) that $\forall p \in AP, t \in V(p)$ iff $t' \in V'(p)$. So we have that $(M,t) \models p$ iff $(M',t') \models p$, and hence trivially that $(M,t) \models \neg\Phi$ iff $(M',t') \models \neg\Phi$ and $(M,t) \models \Phi \wedge \Phi$ iff $(M',t') \models \Phi \wedge \Phi$.

To show that $(M,t) \models E\psi$ iff $(M',t') \models E\psi$ consider that $(M,t) \models E\psi$ iff $\exists\pi$ starting from t such that $(M,\pi) \models \psi$, well if there is such a π then, by the forth property of bisimulation we can find a corresponding bisimilar state in M' for each state in π such that the relations between states in the path are preserved, hence we can construct a π' from these states and this π' is bisimilar to π. Satisfaction is preserved between these paths since they are state-wise bisimilar and we have shown that bismulations between states preserve truth. We can similarly show that $\models A\psi$ iff $(M',t') \models A\psi$.

Now consider satisfaction on paths, $(M,\pi) \models \Phi$ iff $(M,\pi[0]) \models \Phi$ (from definition 2) and likewise for π' and $\pi'[0]$. But $\pi[0]$ and $\pi'[0]$ are bisimilar by definition 3 so we have $(M,\pi[0]) \models \Phi$ iff $(M',\pi'[0]) \models \Phi$ by above and hence $(M,\pi) \models \Phi$ iff $(M',\pi') \models \Phi$. Then we trivially have the equivalences for satisfaction of $\neg\psi$ and $\psi \wedge \psi'$.

Is satisfaction of $X\psi$ preserved by bisimulations? Well, since $\pi \approx \pi'$ we also have $\pi[1,...,\infty] \approx \pi'[1,...,\infty]$. $(M,\pi) \models X\psi$ iff $(M,\pi[1,...,\infty]) \models \psi$ iff $(M,\pi[1]) \models \psi$ by definition 2, and by above $(M,\pi[1]) \models \psi$ iff $(M',\pi'[1]) \models \psi$. So $(M,\pi) \models X\psi$ iff $(M',\pi') \models X\psi$.

Now consider the truth of $\psi \cup \psi'$. We can similarly show that this is preserved by bisimulations by reducing the definition to satisfaction on states which we have shown is preserved.

So the truth of CTL* formulas is preserved by bisimulations.

7

We wish to show that if $(M,t) \equiv (M',t')$ in CTL then they $(M,t) \approx (M',t')$. By definition of equivalence we have that for any formula Φ if $(M,t) \models \Phi$ then $(M',t') \models \Phi$, assume they are equivalent and we will show that the 3 properties of bisimulation from question 6 hold. a) is trivial as equivalent worlds satisfy the same atoms. To show b) assume that the forth condition does not hold, i.e. there is some $v \in M$ and $t \to v$ with no $v' \in M'$ such that $t' \to v'$ and $v \approx v'$.
Now let $S' = \{u' \in M' | t' \to u'\}$ this is nonempty as the relation $\to$ is serial and the sets of states in M and M' are finite by assumption.
Now, by our previous assumption $\forall u_i' \in S' \exists$ a formula ψ_i such that $(M,v) \models \psi_i$ but $(M',u_i') \nvDash \psi_i$ (as u_i not bisimular to v). But then
$(M,t) \models EX(\wedge_i \psi_i)$ but $(M',t') \nvDash EX(\wedge_i \psi_i)$ and we have derived a contradiction and the forth property must hold.
We can similaly prove the back property and hence that if $(M,t) \equiv (M',t')$ then they are bisimilar. □

8

What facts do we have? 5: CTL* ¿ CTL; 6: Truth of CTL* is preserved by bisimulations; 7: $(M,t) \equiv_{CTL} (M',t') \Rightarrow (M,t) \approx (M',t')$.
We have to show that $(M,t) \equiv_{CTL} (M',t')$ iff $(M,t) \equiv_{CTL*} (M',t')$
First $\Rightarrow$ direction: Assume (M,t) and (M',t') satisfy the same formulas in CTL, i.e. they are equivalent in CTL. Then by 7 they are bisimular. Now, since thye are bisimilar, by 6 we have that they are equivalent in CTL*.
Now, $\Leftarrow$ direction: Assume (M,t) and (M',t') satisfy the same formulas in CTL*, well by 5 CTL is a strict fragment of CTL* and we know that every CTL formula is also a formula of CTL* - so this direction is trivial. □
It is perhaps surprising thatthe satisfaction of formulas in CTL restricts which formulas can be satisfied in CTL*, even though CTL* some formulas cannot be expressed in CTL.