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Academic Year 2019-2020
Page created Thu Feb 20 02:15:21 GMT 2020

CATE

499 fbelard 6
j4 elt16 v1

0011900036028

Electronic submission
Tue - 18 Feb 2020 17:58:03

elt16

Exercise Information

Module:	499 Modal Logic for Strategic Reasoning in AI	**Issued:**	Wed - 05 Feb 2020
Exercise:	6 (CW)	**Due:**	Wed - 19 Feb 2020
Title:	Coursework2	**Assessment:**	Individual
FAO:	Belardinelli, Francesco (fbelard)	**Submission:**	Electronic

Student Declaration - Version 1

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Signed: (electronic signature) Date: 2020-02-18 17:57:29

For Markers only: (circle appropriate grade)

TYE, Emma (elt16)	01201320	j4	2020-02-18 17:57:29	**A* A B C D E F**

Modal Logic - Coursework 2

Emma Tye

February 18, 2020

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(a)

$$\begin{aligned}\pi \models \varphi \,\mathrm{R}\, \psi \quad & \text{iff if } \exists i \geq 0 \text{ such that } \pi[i,\infty] \not\models \psi, \\ & \text{then } \exists j \geq 0 \text{ such that } \pi[j,\infty] \models \varphi \text{ and } \forall 0 \leq k \leq j \;\, \pi[k,\infty] \models \psi\end{aligned}$$

(b) $(\psi \,\mathrm{U}(\psi \wedge \varphi)) \vee G\psi$

(c) $\pi \models \varphi \,\mathrm{R}\, \psi \iff \pi \models (\psi \,\mathrm{U}(\psi \wedge \varphi)) \vee G\psi$

- ($\implies$): Let π be an arbitrary path. Assume $\pi \models \varphi \,\mathrm{R}\, \psi$.

  If $\exists i \geq 0$ such that $\pi[i,\infty] \not\models \psi$, then $\exists j \geq 0$ such that $\pi[j,\infty] \models \varphi$ and $\forall 0 \leq k \leq j \;\pi[k,\infty] \models \psi$.

  Assume $\exists i \geq 0$ such that $\pi[i,\infty] \not\models \psi$. Then $\exists j \geq 0$ such that $\pi[j,\infty] \models \varphi$. We also have that $\pi[j,\infty] \models \psi$, since $\pi[k,\infty] \models \psi$ for all $0 \leq k \leq j$. Hence, we have that $\pi[j,\infty] \models \psi \wedge \varphi$.

  But as $\pi[k,\infty] \models \psi$ for all $0 \leq k \leq j$, we certainly have that $\pi[k,\infty] \models \psi$ for all $0 \leq k < j$. So, by semantics of until U, we have that $\pi \models \psi \,\mathrm{U}(\psi \wedge \varphi)$. But by semantics of or, we have that $\pi \models (\psi \,\mathrm{U}(\psi \wedge \varphi)) \vee \mathrm{G}\,\psi$.

  Assume there doesn't exist an $i \geq 0$ such that $\pi[i,\infty] \not\models \psi$, i.e. $\pi[i,\infty] \models \neg\psi$. So by semantics of until U, $\pi \not\models \chi \,\mathrm{U}\, \neg\psi$ for any formula χ - therefore $\pi \not\models \top \,\mathrm{U}\, \neg\psi$. So $\pi \models \neg(\top \,\mathrm{U}\, \neg\psi)$ and hence $\pi \models \mathrm{G}\,\psi$. By semantics of or, we have that $\pi \models (\psi \,\mathrm{U}(\psi \wedge \varphi)) \vee \mathrm{G}\,\psi$.

- ($\impliedby$): Let π be an arbitrary path. Assume $\pi \models (\psi \,\mathrm{U}(\psi \wedge \varphi)) \vee G\psi$.

  Assume $\pi \models \psi \,\mathrm{U}(\psi \wedge \varphi)$. Then there exists an $i \geq 0$ such that $\pi[i,\infty] \models \psi \wedge \varphi$, and forall $0 \leq j < i$, we have $\pi[j,\infty] \models \psi$. Therefore, we have that $\pi[i,\infty] \models \psi$ and $\pi[i,\infty] \models \varphi$. So we must have that for all $0 \leq j \leq i$ $\pi[j,\infty] \models \psi$.

  So we can rename variables to give $\exists j \geq 0$ such that $\pi[j,\infty] \models \varphi$ and $\forall 0 \leq k \leq j$, $\pi[k,\infty] \models \psi$. If B is true, then $A \implies B$ is true no matter the truth of A, so we have that if $\exists i \geq 0$ such that $\pi[i,\infty] \not\models \psi$, then $\exists j \geq 0$ such that $\pi[j,\infty] \models \varphi$ and $\forall 0 \leq k \leq j$, $\pi[k,\infty] \models \psi$. Hence, $\pi \models \varphi \,\mathrm{R}\, \psi$.

  Assume $\pi \models \mathrm{G}\,\psi$. Then $\pi \models \neg(\top \,\mathrm{U}\, \neg\psi) \iff \pi \not\models \top \,\mathrm{U}\, \neg\psi$. So we do not have that there exists an $i \geq 0$ such that $\pi[i,\infty] \models \neg\psi$ and forall $0 \leq j < i$, $\pi[j,\infty] \models \top$. But $\lambda \models \top$ is always true for any path λ, so we must have that there is no $i \geq 0$ such that $\pi[i,\infty] \models \neg\psi \iff \pi[i,\infty] \not\models \psi$.

  If A is false, then $A \implies B$ is true no matter the truth of B, so we have that if $\exists i \geq 0$ such that $\pi[i,\infty] \not\models \psi$, then $\exists j \geq 0$ such that $\pi[j,\infty] \models \varphi$ and $\forall 0 \leq k \leq j$, $\pi[k,\infty] \models \psi$. Hence, $\pi \models \varphi \,\mathrm{R}\, \psi$.

(d) We have that $\bot \,\mathrm{R}\, \psi \equiv (\psi \,\mathrm{U}(\psi \wedge \bot)) \vee \mathrm{G}\,\psi$, from (c). Let π be an arbitrary path.

$$\begin{aligned}\pi \models (\psi \,\mathrm{U}(\psi \wedge \bot)) \vee \mathrm{G}\,\psi &\iff \pi \models \psi \,\mathrm{U}(\psi \wedge \bot) \text{ or } \pi \models \mathrm{G}\,\psi \\ \pi \models \psi \,\mathrm{U}(\psi \wedge \bot) &\iff \text{there exists } i \geq 0 \text{ such that } \pi[i,\infty] \models \psi \wedge \bot \text{ and} \\ & \qquad \text{forall } 0 \leq j < i,\ \pi[j,\infty] \models \psi \\ \pi[i,\infty] \models \psi \wedge \bot &\iff \pi[i,\infty] \models \psi \text{ and } \pi[i,\infty] \models \bot\end{aligned}$$

But $\lambda \models \bot$ is always false for any path λ, so $\pi[i,\infty] \not\models \psi \wedge \bot$ for any $i \geq 0$, hence $\pi \not\models \psi \,\mathrm{U}(\psi \wedge \bot)$.

So

$$\begin{aligned}
\pi \models \bot \,\mathrm{R}\, \psi &\iff \pi \models (\psi \,\mathrm{U}(\psi \wedge \bot)) \vee \mathrm{G}\, \psi \\
&\iff \text{false or } \pi \models \mathrm{G}\, \psi \\
&\iff \pi \models \mathrm{G}\, \psi
\end{aligned}$$

2

- $(M, q) \models \mathrm{E\,F}\,\Phi$ iff for some path λ from q, for some $j \geq 0$, $(M, \lambda[j]) \models \Phi$

$$\begin{aligned}
(M, q) \models \mathrm{E\,F}\,\Phi &\iff (M, q) \models \mathrm{E}(\top\,\mathrm{U}\,\Phi) \\
&\iff \text{for some path } \lambda \text{ from } q,\ (M, \lambda) \models \top\,\mathrm{U}\,\Phi \\
(M, \lambda) \models \top\,\mathrm{U}\,\Phi &\iff \text{for some } j \geq 0,\ (M, \lambda[j]) \models \Phi \text{ and for all } 0 \leq k < j,\ (M, \lambda[k]) \models \top
\end{aligned}$$

But $(M, p) \models \top$ is true for any state p, so we have

$$\begin{aligned}
(M, \lambda) \models \top\,\mathrm{U}\,\Phi &\iff \text{for some } j \geq 0,\ (M, \lambda[j]) \models \Phi \\
(M, q) \models \mathrm{E\,F}\,\Phi &\iff \text{for some path } \lambda \text{ from } q, \text{ for some } j \geq 0,\ (M, \lambda[j]) \models \Phi
\end{aligned}$$

- $(M, q) \models \mathrm{A\,F}\,\Phi$ iff for every path λ from q, for some $j \geq 0$, $(M, \lambda[j]) \models \Phi$

$$\begin{aligned}
(M, q) \models \mathrm{A\,F}\,\Phi &\iff (M, q) \models \mathrm{A}(\top\,\mathrm{U}\,\Phi) \\
&\iff \text{for all paths } \lambda \text{ from } q,\ (M, \lambda) \models \top\,\mathrm{U}\,\Phi \\
(M, \lambda) \models \top\,\mathrm{U}\,\Phi &\iff \text{for some } j \geq 0,\ (M, \lambda[j]) \models \Phi \text{ and for all } 0 \leq k < j,\ (M, \lambda[k]) \models \top
\end{aligned}$$

But $(M, p) \models \top$ is true for any state p, so we have

$$\begin{aligned}
(M, \lambda) \models \top\,\mathrm{U}\,\Phi &\iff \text{for some } j \geq 0,\ (M, \lambda[j]) \models \Phi \\
(M, q) \models \mathrm{A\,F}\,\Phi &\iff \text{for all paths } \lambda \text{ from } q, \text{ for some } j \geq 0,\ (M, \lambda[j]) \models \Phi
\end{aligned}$$

- $(M, q) \models \mathrm{E\,G}\,\Phi$ iff for some path λ from q, for all $j \geq 0$, $(M, \lambda[j]) \models \Phi$

$$\begin{aligned}
(M, q) \models \mathrm{E\,G}\,\Phi &\iff (M, q) \models \neg\,\mathrm{A}(\top\,\mathrm{U}\,\neg\Phi) \\
&\iff (M, q) \not\models \mathrm{A}(\top\,\mathrm{U}\,\neg\Phi) \\
&\iff \text{not for all paths } \lambda \text{ from } q,\ (M, \lambda) \models \top\,\mathrm{U}\,\neg\phi \\
&\iff \text{for some path } \lambda \text{ from } q,\ (M, \lambda) \not\models \top\,\mathrm{U}\,\neg\Phi \\
(M, \lambda) \models \top\,\mathrm{U}\,\neg\Phi &\iff \text{for some } j \geq 0,\ (M, \lambda[j]) \models \neg\Phi \text{ and for all } 0 \leq k < j,\ (M, \lambda[k]) \models \top
\end{aligned}$$

But $(M, p) \models \top$ is true for any state p, so we have

$$\begin{aligned}
(M, \lambda) \models \top\,\mathrm{U}\,\neg\Phi &\iff \text{for some } j \geq 0,\ (M, \lambda[j]) \models \neg\Phi \\
&\iff \text{for some } j \geq 0,\ (M, \lambda[j]) \not\models \Phi \\
(M, \lambda) \not\models \top\,\mathrm{U}\,\neg\Phi &\iff \text{not for some } j \geq 0,\ (M, \lambda[j]) \not\models \Phi \\
&\iff \text{for all } j \geq 0, \text{ not } (M, \lambda[j]) \not\models \Phi \\
&\iff \text{for all } j \geq 0,\ (M, \lambda[j]) \models \Phi
\end{aligned}$$

Hence

$$(M, q) \models \mathrm{E\,G}\,\Phi \iff \text{for some path } \lambda \text{ from } q, \text{ for all } j \geq 0,\ (M, \lambda[j]) \models \Phi$$

- $(M, q) \models \mathrm{A\,G}\,\Phi$ iff for all paths λ from q, for all $j \geq 0$, $(M, \lambda[j]) \models \Phi$

$$
\begin{aligned}
(M, q) \models \mathrm{A\,G}\,\Phi &\iff (M, q) \models \neg\,\mathrm{E}(\top\,\mathrm{U}\,\neg\Phi) \\
&\iff (M, q) \not\models \mathrm{E}(\top\,\mathrm{U}\,\neg\Phi) \\
&\iff \text{not for some path } \lambda \text{ from } q,\ (M, \lambda) \models \top\,\mathrm{U}\,\neg\Phi \\
&\iff \text{for all paths } \lambda \text{ from } q,\ (M, \lambda) \not\models \top\,\mathrm{U}\,\neg\Phi \\
(M, \lambda) \models \top\,\mathrm{U}\,\neg\Phi &\iff \text{for some } j \geq 0,\ (M, \lambda[j]) \models \neg\Phi \text{ and for all } 0 \leq k < j,\ (M, \lambda[k]) \models \top
\end{aligned}
$$

But $(M, p) \models \top$ is true for any state p, so we have

$$
\begin{aligned}
(M, \lambda) \models \top\,\mathrm{U}\,\neg\Phi &\iff \text{for some } j \geq 0,\ (M, \lambda[j]) \models \neg\Phi \\
&\iff \text{for some } j \geq 0,\ (M, \lambda[j]) \not\models \Phi \\
(M, \lambda) \not\models \top\,\mathrm{U}\,\neg\Phi &\iff \text{not for some } j \geq 0,\ (M, \lambda[j]) \not\models \Phi \\
&\iff \text{for all } j \geq 0, \text{ not } (M, \lambda[j]) \not\models \Phi \\
&\iff \text{for all } j \geq 0,\ (M, \lambda[j]) \models \Phi
\end{aligned}
$$

Hence

$$
(M, q) \models \mathrm{A\,G}\,\Phi \iff \text{for all paths } \lambda \text{ from } q, \text{ for all } j \geq 0,\ (M, \lambda[j]) \models \Phi
$$

3

(a) Take Φ a CTL formula. We will prove that Φ is a CTL* formula by induction on the structure of CTL formulae.

- Let $\Phi = p$, where $p \in AP$. Then Φ is a CTL* formula by definition.

- Let $\Phi = \neg\Psi$. Assume Ψ is a CTL* formula for the inductive hypothesis.

  Then $\neg\Psi$ is a CTL* formula by definition, and hence Φ is a CTL* formula.

- Let $\Phi = \Psi \wedge \Omega$. Assume Ψ and Ω are CTL* formulae for the inductive hypothesis.

  Then $\Psi \wedge \Omega$ is a CTL* formula by definition, and hence Φ is a CTL* formula.

- Let $\Phi = \mathrm{E}\,\mathrm{X}\,\Psi$. Assume Ψ is a CTL* formula for the inductive hypothesis.

  Since Ψ is a CTL* state formula, we have that Ψ is also a CTL* path formula, hence $\mathrm{X}\,\Psi$ is a CTL* path formula. Therefore, $\mathrm{E}\,\mathrm{X}\,\Psi$ is a CTL* state formula, so Φ is a CTL* formula.

- Let $\Phi = \mathrm{E}(\Psi\,\mathrm{U}\,\Omega)$. Assume Ψ and Ω are CTL* formulae for the inductive hypothesis.

  Since Ψ and Ω are CTL* state formulae, they are also CTL* path formulae. So $\Psi\,\mathrm{U}\,\Omega$ is a CTL* path formula, hence $\mathrm{E}(\Psi\,\mathrm{U}\,\Omega)$ is a CLT* state formula and so Φ is a CTL* state formula.

- Let $\Phi = \mathrm{A}\,\mathrm{X}\,\Psi$. Assume Ψ is a CTL* formula for the inductive hypothesis.

  Since Ψ is a CTL* state formula, we have that Ψ is also a CTL* path formula, hence $\mathrm{X}\,\Psi$ is a CTL* path formula. Therefore, $\mathrm{A}\,\mathrm{X}\,\Psi$ is a CTL* state formula, so Φ is a CTL* formula.

- Let $\Phi = \mathrm{A}(\Psi\,\mathrm{U}\,\Omega)$. Assume Ψ and Ω are CTL* formulae for the inductive hypothesis.

  Since Ψ and Ω are CTL* state formulae, they are also CTL* path formulae. So $\Psi\,\mathrm{U}\,\Omega$ is a CTL* path formula, hence $\mathrm{A}(\Psi\,\mathrm{U}\,\Omega)$ is a CLT* state formula and so Φ is a CTL* state formula.

(b) Let $\Phi = \mathrm{A}\,p$.

Φ is a CTL* formula: p is a CTL* state formula, hence it is also a CTL* path formula. Therefore, $\mathrm{A}\,p$ is a CTL* state formula by definition.

Φ is not a CTL formula: p is a CTL state formula, but it is not a CTL path formula - path formulas must be of the form $\mathrm{X}\,\Psi$ or $\Psi\,\mathrm{U}\,\Omega$. But by the definition of CTL, A can only prefix a path formula, hence $\mathrm{A}\,p$ is not a CTL formula.

4

Let M be a model and s a state in that model.

Take Φ a CTL formula. We will show that $(M,s) \models^{\mathrm{CTL}} \Phi \iff (M,s) \models^{\mathrm{CTL}^*} \Phi$, by induction over the structure of CTL formulae.

- Let $\Phi = p$, where $p \in AP$. Then

$$\begin{aligned}
(M,s) \models^{\mathrm{CTL}} \Phi &\iff s \in V(p) && \text{by def. of CTL semantics} \\
&\iff (M,s) \models^{\mathrm{CTL}^*} \Phi && \text{by def. of CTL* semantics}
\end{aligned}$$

- Let $\Phi = \neg\Psi$. For all states q in M, assume $(M,q) \models^{\mathrm{CTL}} \Psi \iff (M,q) \models^{\mathrm{CTL}^*} \Psi$ for the inductive hypothesis. Then

$$\begin{aligned}
(M,s) \models^{\mathrm{CTL}} \Phi &\iff (M,s) \not\models^{\mathrm{CTL}} \Psi && \\
&\iff (M,s) \not\models^{\mathrm{CTL}^*} \Psi && \text{inductive hypothesis} \\
&\iff (M,s) \models^{\mathrm{CTL}^*} \Phi && \text{by def. of CTL* semantics}
\end{aligned}$$

- Let $\Phi = \Psi \wedge \Omega$. For all states q in M, assume $(M,q) \models^{\mathrm{CTL}} \Psi \iff (M,q) \models^{\mathrm{CTL}^*} \Psi$ and $(M,q) \models^{\mathrm{CTL}} \Omega \iff (M,q) \models^{\mathrm{CTL}^*} \Omega$ for the inductive hypothesis. Then

$$\begin{aligned}
(M,s) \models^{\mathrm{CTL}} \Phi &\iff (M,s) \models^{\mathrm{CTL}} \Psi \text{ and } (M,s) \models^{\mathrm{CTL}} \Omega && \text{by def. of CTL semantics} \\
&\iff (M,s) \models^{\mathrm{CTL}^*} \Psi \text{ and } (M,s) \models^{\mathrm{CTL}^*} \Omega && \text{inductive hypothesis} \\
&\iff (M,s) \models^{\mathrm{CTL}^*} \Phi && \text{by def. of CTL* semantics}
\end{aligned}$$

- Let $\Phi = \mathrm{E}\,\mathrm{X}\,\Psi$. For all states q in M, assume $(M,q) \models^{\mathrm{CTL}} \Psi \iff (M,q) \models^{\mathrm{CTL}^*} \Psi$ for the inductive hypothesis. Then

$$\begin{aligned}
(M,s) \models^{\mathrm{CTL}} \Phi &\iff \text{for some path } \lambda \text{ starting from } s,\ (M,\lambda) \models^{\mathrm{CTL}} \mathrm{X}\,\Psi && \text{by def. of CTL semantics} \\
&\iff \text{for some path } \lambda \text{ starting from } s,\ (M,\lambda[1]) \models^{\mathrm{CTL}} \Psi && \text{by def. of CTL semantics} \\
&\iff \text{for some path } \lambda \text{ starting from } s,\ (M,\lambda[1]) \models^{\mathrm{CTL}^*} \Psi && \text{inductive hypothesis} \\
&\iff \text{for some path } \lambda \text{ starting from } s,\ (M,\lambda[1..\infty][0]) \models^{\mathrm{CTL}^*} \Psi && \text{re-arranging indexes} \\
&\iff \text{for some path } \lambda \text{ starting from } s,\ (M,\lambda[1..\infty]) \models^{\mathrm{CTL}^*} \Psi && \text{by def. of CTL* semantics} \\
&\iff \text{for some path } \lambda \text{ starting from } s,\ (M,\lambda) \models^{\mathrm{CTL}^*} \mathrm{X}\,\Psi && \text{by def. of CTL* semantics} \\
&\iff (M,s) \models^{\mathrm{CTL}^*} \Phi && \text{by def. of CTL* semantics}
\end{aligned}$$

- Let $\Phi = \mathrm{E}(\Psi\,\mathrm{U}\,\Omega)$. For all states q in M, assume $(M,q) \models^{\mathrm{CTL}} \Psi \iff (M,q) \models^{\mathrm{CTL}^*} \Psi$ and

$(M,q) \models^{\mathrm{CTL}} \Omega \iff (M,q) \models^{\mathrm{CTL}^*} \Omega$ for the inductive hypothesis. Then

$$
\begin{aligned}
(M,s) \models^{\mathrm{CTL}} \Phi &\iff \text{for some path } \lambda \text{ starting from } s,\ (M,\lambda) \models^{\mathrm{CTL}} \Psi \mathbin{\mathrm{U}} \Omega && \text{by def. of CTL semantics}\\
&\iff \text{for some path } \lambda \text{ starting from } s, \text{ for some } i \geq 0,\\
&\qquad (M,\lambda[i]) \models^{\mathrm{CTL}} \Omega \text{ and } (M,\lambda[j]) \models^{\mathrm{CTL}} \Psi \text{ for all } 0 \leq j < i && \text{by def. of CTL semantics}\\
&\iff \text{for some path } \lambda \text{ starting from } s, \text{ for some } i \geq 0,\\
&\qquad (M,\lambda[i]) \models^{\mathrm{CTL}^*} \Omega \text{ and } (M,\lambda[j]) \models^{\mathrm{CTL}^*} \Psi \text{ for all } 0 \leq j < i && \text{inductive hypothesis}\\
&\iff \text{for some path } \lambda \text{ starting from } s, \text{ for some } i \geq 0,\\
&\qquad (M,\lambda[i..\infty][0]) \models^{\mathrm{CTL}^*} \Omega \text{ and}\\
&\qquad (M,\lambda[j..\infty][0]) \models^{\mathrm{CTL}^*} \Psi \text{ for all } 0 \leq j < i && \text{re-arranging indexes}\\
&\iff \text{for some path } \lambda \text{ starting from } s, \text{ for some } i \geq 0,\\
&\qquad (M,\lambda[i..\infty]) \models^{\mathrm{CTL}^*} \Omega \text{ and}\\
&\qquad (M,\lambda[j..\infty]) \models^{\mathrm{CTL}^*} \Psi \text{ for all } 0 \leq j < i && \text{by def. of CTL* semantics}\\
&\iff \text{for some path } \lambda \text{ starting from } s,\ (M,\lambda) \models^{\mathrm{CTL}^*} \Psi \mathbin{\mathrm{U}} \Omega && \text{by def. of CTL* semantics}\\
&\iff (M,s) \models^{\mathrm{CTL}^*} \Phi
\end{aligned}
$$

- Let $\Phi = \mathrm{A}\,\mathrm{X}\,\Psi$. For all states q in M, assume $(M,q) \models^{\mathrm{CTL}} \Psi \iff (M,q) \models^{\mathrm{CTL}^*} \Psi$ for the inductive hypothesis. Then

$$
\begin{aligned}
(M,s) \models^{\mathrm{CTL}} \Phi &\iff \text{for all paths } \lambda \text{ starting from } s,\ (M,\lambda) \models^{\mathrm{CTL}} \mathrm{X}\,\Psi && \text{by def. of CTL semantics}\\
&\iff \text{for all paths } \lambda \text{ starting from } s,\ (M,\lambda[1]) \models^{\mathrm{CTL}} \Psi && \text{by def. of CTL semantics}\\
&\iff \text{for all paths } \lambda \text{ starting from } s,\ (M,\lambda[1]) \models^{\mathrm{CTL}^*} \Psi && \text{inductive hypothesis}\\
&\iff \text{for all paths } \lambda \text{ starting from } s,\ (M,\lambda[1..\infty][0]) \models^{\mathrm{CTL}^*} \Psi && \text{re-arranging indexes}\\
&\iff \text{for all paths } \lambda \text{ starting from } s,\ (M,\lambda[1..\infty]) \models^{\mathrm{CTL}^*} \Psi && \text{by def. of CTL* semantics}\\
&\iff \text{for all paths } \lambda \text{ starting from } s,\ (M,\lambda) \models^{\mathrm{CTL}^*} \mathrm{X}\,\Psi && \text{by def. of CTL* semantics}\\
&\iff (M,s) \models^{\mathrm{CTL}^*} \Phi && \text{by def. of CTL* semantics}
\end{aligned}
$$

- Let $\Phi = \mathrm{A}(\Psi \mathbin{\mathrm{U}} \Omega)$. For all states q in M, assume $(M,q) \models^{\mathrm{CTL}} \Psi \iff (M,q) \models^{\mathrm{CTL}^*} \Psi$ and $(M,q) \models^{\mathrm{CTL}} \Omega \iff (M,q) \models^{\mathrm{CTL}^*} \Omega$ for the inductive hypothesis. Then

$$
\begin{aligned}
(M,s) \models^{\mathrm{CTL}} \Phi &\iff \text{for all paths } \lambda \text{ starting from } s,\ (M,\lambda) \models^{\mathrm{CTL}} \Psi \mathbin{\mathrm{U}} \Omega && \text{by def. of CTL semantics}\\
&\iff \text{for all paths } \lambda \text{ starting from } s, \text{ for some } i \geq 0,\\
&\qquad (M,\lambda[i]) \models^{\mathrm{CTL}} \Omega \text{ and } (M,\lambda[j]) \models^{\mathrm{CTL}} \Psi \text{ for all } 0 \leq j < i && \text{by def. of CTL semantics}\\
&\iff \text{for all paths } \lambda \text{ starting from } s, \text{ for some } i \geq 0,\\
&\qquad (M,\lambda[i]) \models^{\mathrm{CTL}^*} \Omega \text{ and } (M,\lambda[j]) \models^{\mathrm{CTL}^*} \Psi \text{ for all } 0 \leq j < i && \text{inductive hypothesis}\\
&\iff \text{for all paths } \lambda \text{ starting from } s, \text{ for some } i \geq 0,\\
&\qquad (M,\lambda[i..\infty][0]) \models^{\mathrm{CTL}^*} \Omega \text{ and}\\
&\qquad (M,\lambda[j..\infty][0]) \models^{\mathrm{CTL}^*} \Psi \text{ for all } 0 \leq j < i && \text{re-arranging indexes}\\
&\iff \text{for all paths } \lambda \text{ starting from } s, \text{ for some } i \geq 0,\\
&\qquad (M,\lambda[i..\infty]) \models^{\mathrm{CTL}^*} \Omega \text{ and}\\
&\qquad (M,\lambda[j..\infty]) \models^{\mathrm{CTL}^*} \Psi \text{ for all } 0 \leq j < i && \text{by def. of CTL* semantics}\\
&\iff \text{for all paths } \lambda \text{ starting from } s,\ (M,\lambda) \models^{\mathrm{CTL}^*} \Psi \mathbin{\mathrm{U}} \Omega && \text{by def. of CTL* semantics}\\
&\iff (M,s) \models^{\mathrm{CTL}^*} \Phi
\end{aligned}
$$

5

(a) From question 3, any CTL formula is also a CTL* formula. From question 4, we have that $(M, s) \models^{\text{CTL}} \Phi \iff (M, s) \models^{\text{CTL}^*} \Phi$, so you can just take the same formula but in the CTL* context.

(b) Consider the CTL* formula $\Phi = \mathrm{A\,F\,G}\,p$, where F, G are the usual abbreviations. This is equivalent to the LTL formula $\mathrm{F\,G}\,p$ by Theorem 1.12 in Lecture 5. It is also easy to see they're equivalent: $M \models^{\text{CTL}^*} \mathrm{A\,F\,G}\,p$ iff for every initial state s_0 in M, for all paths λ starting from s_0, $(M, \lambda) \models^{\text{CTL}^*} \mathrm{F\,G}\,p$, and $M \models^{\text{LTL}} \mathrm{F\,G}\,p$ iff for every initial state s_0 in M, for all paths λ starting from s_0, $(M, \lambda) \models^{\text{LTL}} \mathrm{F\,G}\,p$, and path semantics are defined almost identically for CTL* and LTL.

But from Lecture 5, we saw that there is no equivalent CTL formula for this LTL formula. The equivalent formula would be of the form $\mathrm{A\,F\,A\,G}\,p$, but taking the model from slide 214 in Lecture 5 shows that $\mathrm{A\,F\,A\,G}\,p$ and $\mathrm{F\,G}\,p$ are not equivalent.

6

Lemma: Let N, N' be models. Let u, u' be states in those models, such that (N, u) and (N', u') are bisimilar. Then for any path π in N starting from u, there exists a bisimilar path π' in N' starting from u'.

Proof. Let π be an arbitrary path in N. Construct the path π' in N' by

1. $\pi'[0] = u'$

2. $\pi'[i+1] = s'$, where $B(\pi[i+1], s')$ and $\pi'[i] \to s'$ (choose random s' if there are multiple satisfying this)

This is a valid path and is bisimilar to π - since $B(\pi[0], \pi'[0])$, and by definition $B(\pi[j], \pi'[j])$ for all $0 < j$, it is sufficient to show that an s' satisfying the conditions shown always exists.

Take $0 \leq i$ arbitrary.

We have that $\pi[i] \to \pi[i+1]$ since π is a path. By the forth property of bisimulations, there must exist an s' such that $\pi'[i] \to s'$ and $B(\pi[i+1], s')$, as $(N, \pi[i]) \cong (N', \pi'[i])$.

$\square$

Let M, M' be models, t, t' states in those models (respectively) such that $(M, t) \cong (M, t')$. Let Φ be a CTL* state formula.

Assume for the inductive hypothesis that:

1. For any state s in M, for any state subformula of Φ, say Ψ, if $(M, s) \cong (M', s')$ for some $s' \in M'$, then $(M, s) \models^{\text{CTL}^*} \Psi \iff (M', s') \models^{\text{CTL}^*} \Psi$.

2. For any path π in M, for any path formula ϕ, if $(M, \pi) \cong (M', \pi')$ for some path π' in M', then $(M, \pi) \models^{\text{CTL}^*} \phi \iff (M', \pi') \models^{\text{CTL}^*} \phi$.

- Let $\Phi = p$. Then

$$\begin{aligned}
(M, t) \models^{\text{CTL}^*} \Phi &\iff t \in V(p) && \text{by def. of CTL* semantics} \\
&\iff t' \in V'(p) && \text{by def. of bisimulation} \\
&\iff (M', t') \models^{\text{CTL}^*} \Phi && \text{by def. of CTL* semantics}
\end{aligned}$$

- Let $\Phi = \neg\Psi$. Then

$$\begin{aligned}
(M, t) \models^{\text{CTL}^*} \Phi &\iff (M, t) \not\models^{\text{CTL}^*} \Psi && \text{by def. of CTL* semantics} \\
&\iff (M', t') \not\models^{\text{CTL}^*} \Psi && \text{inductive hypothesis 1} \\
&\iff (M', t') \models^{\text{CTL}^*} \Phi && \text{by def. of CTL* semantics}
\end{aligned}$$

- Let $\Phi = \Psi \wedge \Omega$. Then

$$
\begin{aligned}
(M,t) \models^{\mathrm{CTL}^*} \Phi &\iff (M,t) \models^{\mathrm{CTL}^*} \Psi \text{ and } (M,t) \models^{\mathrm{CTL}^*} \Omega && \text{by def. of CTL* semantics} \\
&\iff (M',t') \models^{\mathrm{CTL}^*} \Psi \text{ and } (M',t') \models^{\mathrm{CTL}^*} \Omega && \text{inductive hypothesis 1} \\
&\iff (M',t') \models^{\mathrm{CTL}^*} \Phi && \text{by def. of CTL* semantics}
\end{aligned}
$$

- Let $\Phi = \mathrm{E}\,\phi$. Then

$$
(M,t) \models^{\mathrm{CTL}^*} \Phi \iff \text{for some path } \lambda \text{ starting from } t,\ (M,\lambda) \models^{\mathrm{CTL}^*} \phi \quad \text{by def. of CTL* semantics}
$$

By the Lemma, letting $M = N$ and $M' = N'$, there is a path λ' in M' starting from t' that is bisimilar to λ.

$$
\begin{aligned}
(M,t) \models^{\mathrm{CTL}^*} \Phi &\iff \text{for some path } \lambda \text{ starting from } t,\ (M',\lambda') \models^{\mathrm{CTL}^*} \phi \\
&\qquad \text{where } \lambda \cong \lambda' && \text{inductive hypothesis 2} \\
&\implies \text{for some path } \lambda' \text{ starting from } t',\ (M',\lambda') \models^{\mathrm{CTL}^*} \phi && \text{Lemma} \\
&\iff (M',t') \models^{\mathrm{CTL}^*} \Phi && \text{by def. of CTL* semantics}
\end{aligned}
$$

So this proves the iff in one direction. But if we assume that $(M',t') \models^{\mathrm{CTL}^*} \Phi$, then we can use this exact same proof, but swapping round every instance of M, t, λ for M', t', λ' (i.e. taking $M' = N$, $M = N'$ in the Lemma), and hence get that

$$
\begin{aligned}
(M',t') \models^{\mathrm{CTL}^*} \Phi &\iff \text{for some path } \lambda' \text{ starting from } t',\ (M',\lambda') \models^{\mathrm{CTL}^*} \phi && \text{by def. of CTL* semantics} \\
&\iff \text{for some path } \lambda' \text{ starting from } t',\ (M,\lambda) \models^{\mathrm{CTL}^*} \phi \\
&\qquad \text{where } \lambda' \cong \lambda && \text{inductive hypothesis 2} \\
&\implies \text{for some path } \lambda \text{ starting from } t,\ (M,\lambda) \models^{\mathrm{CTL}^*} \phi && \text{Lemma} \\
&\iff (M,t) \models^{\mathrm{CTL}^*} \Phi && \text{by def. of CTL* semantics}
\end{aligned}
$$

- Let $\Phi = \mathrm{A}\,\phi$. Then

The Lemma tells us that for every path λ starting from t in M, there exists a bisimilar path λ' in M' starting from t'. So if we prove a property about every path λ' in M' starting from t', then since every λ in M is bisimilar to one of these λ', we can prove that property about every λ.

Hence, we can prove the iff in one direction:

$$
\begin{aligned}
(M',t') \models^{\mathrm{CTL}^*} \Phi &\iff \text{for all paths } \lambda' \text{ starting from } t',\ (M',\lambda') \models^{\mathrm{CTL}^*} \phi && \text{by def. of CTL* semantics} \\
&\iff \text{for all paths } \lambda' \text{ starting from } t',\ (M,\lambda) \models^{\mathrm{CTL}^*} \phi \\
&\qquad \text{where } \lambda \cong \lambda' && \text{inductive hypothesis 2} \\
&\implies \text{for all paths } \lambda \text{ starting from } t,\ (M,\lambda) \models^{\mathrm{CTL}^*} \phi && \text{Lemma} \\
&\iff (M,t) \models^{\mathrm{CTL}^*} \Phi && \text{by def. of CTL* semantics}
\end{aligned}
$$

But, again, we can just swap round the M, t, λ and M', t', λ', since the Lemma is a property about all models, and get the other direction for free:

$$
\begin{aligned}
(M,t) \models^{\mathrm{CTL}^*} \Phi &\iff \text{for all paths } \lambda \text{ starting from } t,\ (M,\lambda) \models^{\mathrm{CTL}^*} \phi && \text{by def. of CTL* semantics} \\
&\iff \text{for all paths } \lambda \text{ starting from } t,\ (M',\lambda') \models^{\mathrm{CTL}^*} \phi \\
&\qquad \text{where } \lambda \cong \lambda' && \text{inductive hypothesis 2} \\
&\implies \text{for all paths } \lambda' \text{ starting from } t',\ (M',\lambda') \models^{\mathrm{CTL}^*} \phi && \text{Lemma} \\
&\iff (M',t') \models^{\mathrm{CTL}^*} \Phi && \text{by def. of CTL* semantics}
\end{aligned}
$$

Let M, M' be models, λ and λ' paths in those models (respectively) such that $(M, \lambda) \cong (M', \lambda')$. Let ϕ be a CTL* path formula.

Assume for the inductive hypothesis that:

1. For any path π in M, for any state subformula of ϕ, say ψ, if $(M, \pi) \cong (M', \pi')$ for some π' a path in M', then $(M, \pi) \models^{\text{CTL}^*} \psi \iff (M', \pi') \models^{\text{CTL}^*} \psi$.

2. For any state formula Φ, for any state t in M, if $(M, t) \cong (M', t')$ for some state t' in M', then $(M, t) \models^{\text{CTL}^*} \Phi \iff (M', t') \models^{\text{CTL}^*} \Phi$.

- Let $\phi = \Phi$. Then

$$(M, \lambda) \models^{\text{CTL}^*} \phi \iff (M, \lambda[0]) \models^{\text{CTL}^*} \Phi \qquad \text{by def. of CTL* semantics}$$

  Since λ and λ' are bisimilar, we must have that $\lambda[0]$ and $\lambda'[0]$ are bisimilar by the definition of bisimilarity, so

$$\begin{aligned} (M, \lambda) \models^{\text{CTL}^*} \phi &\iff (M', \lambda'[0]) \models^{\text{CTL}^*} \Phi && \text{inductive hypothesis 2} \\ &\iff (M', \lambda') \models^{\text{CTL}^*} \phi && \text{by def. of CTL* semantics} \end{aligned}$$

- Let $\phi = \neg\psi$. Then

$$\begin{aligned} (M, \lambda) \models^{\text{CTL}^*} \phi &\iff (M, \lambda) \not\models^{\text{CTL}^*} \psi && \text{by def. of CTL* semantics} \\ &\iff (M', \lambda') \not\models^{\text{CTL}^*} \psi && \text{inductive hypothesis 1} \\ &\iff (M', \lambda') \models^{\text{CTL}^*} \phi && \text{by def. of CTL* semantics} \end{aligned}$$

- Let $\phi = \psi \wedge \omega$. Then

$$\begin{aligned} (M, \lambda) \models^{\text{CTL}^*} \phi &\iff (M, \lambda) \models^{\text{CTL}^*} \psi \text{ and } (M, \lambda) \models^{\text{CTL}^*} \omega && \text{by def. of CTL* semantics} \\ &\iff (M', \lambda') \models^{\text{CTL}^*} \psi \text{ and } (M', \lambda') \models^{\text{CTL}^*} \omega && \text{inductive hypothesis 1} \\ &\iff (M', \lambda') \models^{\text{CTL}^*} \phi && \text{by def. of CTL* semantics} \end{aligned}$$

- Let $\phi = \mathrm{X}\,\psi$. Then

$$(M, \lambda) \models^{\text{CTL}^*} \phi \iff (M, \lambda[1..\infty]) \models^{\text{CTL}^*} \psi \qquad \text{by def. of CTL* semantics}$$

  Since λ and λ' are bisimilar, $\lambda[1..\infty]$ and $\lambda'[1..\infty]$ must also be bisimilar - if they aren't, then there's an index $i \geq 1$ such that $(M, \lambda[i]) \ncong (M', \lambda'[i])$, hence λ and λ' wouldn't be bisimilar.

  So

$$\begin{aligned} (M, \lambda) \models^{\text{CTL}^*} \phi &\iff (M', \lambda'[1..\infty]) \models^{\text{CTL}^*} \psi && \text{inductive hypothesis 1} \\ &\iff (M', \lambda') \models^{\text{CTL}^*} \phi && \text{by def. of CTL* semantics} \end{aligned}$$

- Let $\phi = \psi\,\mathrm{U}\,\omega$. Then

$$\begin{aligned} (M, \lambda) \models^{\text{CTL}^*} \phi \iff & (M, \lambda[i..\infty]) \models^{\text{CTL}^*} \omega \text{ for some } i \geq 0, \\ & \text{and } (M, \lambda[j..\infty]) \models^{\text{CTL}^*} \psi \text{ for all } 0 \leq j < i \quad \text{by def. of CTL* semantics} \end{aligned}$$

By a similar argument as in the previous point, $(M, \lambda[k..\infty]) \cong (M', \lambda'[k..\infty])$ for any $0 \leq k$. So certainly $(M, \lambda[i..\infty]) \cong (M', \lambda'[i..\infty])$ for any $i \geq 0$, and $(M, \lambda[j..\infty]) \cong (M', \lambda'[j..\infty])$ for any $0 \leq j < i$.

Hence

$$\begin{aligned}
(M, \lambda) \models^{\mathrm{CTL}^*} \phi \iff & (M', \lambda'[i..\infty]) \models^{\mathrm{CTL}^*} \omega \text{ for some } i \geq 0, \\
& \text{and } (M', \lambda'[j..\infty]) \models^{\mathrm{CTL}^*} \psi \text{ for all } 0 \leq j < i && \text{inductive hypothesis 1} \\
\iff & (M', \lambda') \models^{\mathrm{CTL}^*} \psi \,\mathrm{U}\, \omega && \text{by def. of CTL* semantics}
\end{aligned}$$

7

We will prove that CTL-equivalence is a bisimulation.

Let M, M' be models and t, t' be states those models (respectively). Assume t, t' are CTL-equivalent.

(a) **Atoms are preserved**

Since t, t' are CTL-equivalent, $(M,t) \models^{\text{CTL}} p \iff (M',t') \models^{\text{CTL}} p$ (since p is a CTL formula), so this condition is trivially proved.

(b) **Forth**

Assume that $t \to u$, for a state u in M. Assume for a contradiction that there is no u' in M' such that $t' \to u'$ and u, u' are CTL-equivalent.

Take an atom p. Either $u \in V(p)$, or $u \notin V(p)$. In the first case, let $\Phi = p$, otherwise let $\Phi = \neg p$ - so $(M,u) \models^{\text{CTL}} \Phi$. Hence, $(M,t) \models^{\text{CTL}} \mathrm{E\,X}\,\Phi$.

Therefore we must have that $(M',t') \models^{\text{CTL}} \mathrm{E\,X}\,\Phi$. This implies that there is a path starting from t' (satisfying $\mathrm{X}\,\Phi$), hence there exists some u' such that $t' \to u'$.

Take the set $S' = \{u' \mid t' \to u'\}$. We have just shown that this set is non-empty. Since the states of M and M' are finite, and S' is a subset of the states of M', it must also be finite.

Since we assumed that no element of S' is CTL-equivalent with u, for every $u'_i \in S'$, there must be a formula Φ_i such that $(M,u) \models^{\text{CTL}} \Phi_i$ but $(M',u'_i) \not\models^{\text{CTL}} \Phi_i$.

So $(M,u) \models^{\text{CTL}} \Phi_1 \wedge ... \wedge \Phi_n$, but $(M',u'_i) \not\models^{\text{CTL}} \Phi_1 \wedge ... \wedge \Phi_n$ for any $u'_i \in S'$.

Hence $(M,t) \models^{\text{CTL}} \mathrm{E\,X}(\Phi_1 \wedge ... \wedge \Phi_n)$ but $(M',t') \not\models^{\text{CTL}} \mathrm{E\,X}(\Phi_1 \wedge ... \wedge \Phi_n)$, which is a contradiction.

(c) **Back**

Assume that $t' \to u'$, for a state u' in M'. Assume for a contradiction that there is no u in M such that $t \to u$ and u and u' are CTL equivalent.

Take an atom p. Either $u' \in V'(p)$, or $u' \notin V'(p)$. In the first case, let $\Phi = p$, otherwise let $\Phi = \neg p$ - so $(M',u') \models^{\text{CTL}} \Phi$. Hence, $(M',t') \models^{\text{CTL}} \mathrm{E\,X}\,\Phi$.

Therefore we must have that $(M,t) \models^{\text{CTL}} \mathrm{E\,X}\,\Phi$. This implies that there is a path starting from t (satisfying $\mathrm{X}\,\Phi$), hence there exists some u such that $t \to u$.

Let $S = \{u \mid t \to u\}$. We have just shown that this set is non-empty. Since the states of M and M' are finite, and S is a subset of the states of M, S is finite.

Since we assumed no element of S is CTL-equivalent with u', for every $u_i \in S$, there must be a formula Φ_i such that $(M',u') \models^{\text{CTL}} \Phi_i$ but $(M,u_i) \not\models^{\text{CTL}} \Phi_i$.

So $(M',u') \models^{\text{CTL}} \Phi_1 \wedge ... \wedge \Phi_n$, but $(M,u_i) \not\models^{\text{CTL}} \Phi_1 \wedge ... \wedge \Phi_n$ for any $u_i \in S$.

Hence $(M',t') \models^{\text{CTL}} \mathrm{E\,X}(\Phi_1 \wedge ... \wedge \Phi_n)$ but $(M',t') \not\models^{\text{CTL}} \mathrm{E\,X}(\Phi_1 \wedge ... \wedge \Phi_n)$, which is a contradiction

8

We will show that (M,t) and (M',t') are CTL-equivalent if and only if they are CTL* equivalent.

- ($\Longrightarrow$): Assume that (M,t) and (M,t') are CTL-equivalent.

  By question 7, (M,t) and (M,t') are bisimilar. But by question 6, CTL* formulae are preserved across bisimulations, so (M,t) and (M',t') are CTL* equivalent.

- ($\Longleftarrow$): Assume that (M,t) and (M,t') are CTL*-equivalent.

  By question 5, CTL* is more expressive than CTL, so if CTL* formulae are preserved then CTL formulae are preserved, hence (M,t) and (M',t') are CTL equivalent.

Although CTL* is strictly more expressive than CTL, their distinguishing power is the same. So any property that characterises a model can be written as a CTL formula.