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CO499: Coursework 2

Temporal Logics

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1. a) Let M be a model and π a path.

We can define the operator R as:

$(M, \pi) \models \varphi R \psi$ iff either there is some $i \geq 0$ such that $(M, \pi[i \ldots \infty]) \models \varphi$ and for all $0 \leq j \leq i$ $(M, \pi[j \ldots \infty]) \models \psi$, or for all $k \geq 0$ $(M, \pi[k \ldots \infty]) \models \psi$.

b) We can formalize the meaning of R as:

$$\varphi R \psi = \psi U (\psi \wedge \varphi) \vee G \psi$$

But since we should only use X and U we transform the 2nd disjunct:

$$G \psi = \neg F \neg \psi = \neg(\text{true } U \neg \psi)$$

So finally we have: $\varphi R \psi = \psi U (\psi \wedge \varphi) \vee \neg(\text{true } U \neg \psi)$

c) $(M, \pi) \models \varphi R \psi$ iff $(M, \pi) \models \psi U (\psi \wedge \varphi) \vee \neg(\text{true } U \neg \psi)$ (by definition)

iff either $(M, \pi) \models \psi U (\psi \wedge \varphi)$ or $(M, \pi) \models \neg(\text{true } U \neg \psi)$ (by def of $\vee$)

iff there is some $i \geq 0$ such that $(M, \pi[i \ldots \infty]) \models \psi \wedge \varphi$ and for all $0 \leq j < i$ $(M, \pi[j \ldots \infty]) \models \psi$ (by def of U) or $(M, \pi) \not\models \text{true } U \neg \psi$ (by def of $\neg$)

iff either there is some $\underline{i \geq 0}$ such that $\underline{(M, \pi[i \ldots \infty]) \models \psi}$ and $(M, \pi[i \ldots \infty]) \models \varphi$ (by def of $\wedge$) and for all $\underline{0 \leq j < i \, (M, \pi[j \ldots \infty]) \models \psi}$

or it is not the case that there exists $k \geq 0$ such that ~~[illegible]~~ $(M, \pi[k \ldots \infty]) \models \neg \psi$, and for all $0 \leq l < k$, $(M, \pi[l \ldots \infty]) \models \text{true}$ (by def of U)

iff either there is some $i \geq 0$ such that $(M, \pi[i \ldots \infty]) \models \varphi$
and for all $0 \leq j \leq i$ $(M, \pi[j \ldots \infty]) \models \psi$ (combine underlined parts from previous iff)

or it is not the case that there exists $k \geq 0$ such that $(M, \pi[k \ldots \infty]) \not\models \psi$ (def of $\neg$)
(last condition 'and for all $0 \leq l < k$, $(M, \pi[l \ldots \infty]) \models true$ is valid so we can omit it)

iff either there is some $i \geq 0$ such that $(M, \pi[i \ldots \infty]) \models \varphi$
and for all $0 \leq j \leq i$ $(M, \pi[j \ldots \infty]) \models \psi$,
or for all $k \geq 0$ $(M, \pi[k \ldots \infty]) \models \psi$.

d) If we express $\perp R \psi$ in terms of the LTL formula we defined in (b) we get:

~~$\psi\, U\, false\, \wedge$~~

$$\psi\, U (\psi \wedge false) \vee \neg(true\, U \neg\psi) \quad (1)$$

If we ~~go~~ look at the ~~den~~ transformations in (c) we can observe that the first disjunct of (1) $(\psi U(\psi \wedge false))$ would correspond to:

there is some $i \geq 0$ such that $(M, \pi[i \ldots \infty]) \models false$
and for all $0 \leq j \leq i$ $(M, \pi[j \ldots \infty]) \models \psi$

Now clearly $(M, \pi[i \ldots \infty]) \models false$ would never be true for any path π and therefore that 1st disjunct will never be true either.
Now if we look again at (b) the second disjunct of (1) $(\neg true\, U \neg\neg\psi)$ would correspond to:

for all $k \geq 0$ $(M, \pi[k \ldots \infty]) \models \psi$ (2)

Since the 1st disjunct of LTL formula corresponding to $\bot R \psi$ can never be true that $\bot R \psi$ will be satisfied iff the second disjunct in the corresponding formula is true or in other words exactly when (2). What we have in (2), however, is exactly the semantics for $G\psi$. Therefore we can conclude that $\bot R \psi$ will be satisfied on a path iff $G\psi$ is satisfied on that path or in other words $G\psi$ can be expressed as ~~in terms of~~ $\bot R \psi$.

2.

$$EF\phi = E(\text{true}\, U\, \phi)$$
$$AF\phi = A(\text{true}\, U\, \phi)$$
$$EG\phi = \neg AF\neg\phi$$
$$AG\phi = \neg EF\neg\phi$$

$(M,q) \models EF\phi$ iff for some path λ from q, for some $j \geq 0$ $(M, \lambda[j]) \models \phi$

$\Rightarrow$ $(M,q) \models EF\phi$ iff $(M,q) \models E(\text{true}\, U\phi)$

iff for some path λ from q, $(M,\lambda) \models \text{true}\, U\phi$

iff for some path λ from q, $(M,\lambda[i]) \models \phi$ for some $i \geq 0$ and $\underline{(M,\lambda[j]) \models \text{true for all } 0 \leq j < i}$

iff for some path λ from q, $(M,\lambda[i]) \models \phi$ for some $i \geq 0$ (underlined part is always valid and we can omit it)

iff for some path λ from q, for some $j \geq 0$ $(M,\lambda[j]) \models \phi$ (can rename i to j as it is arbitrary)

$(M, q) \models AF\phi$ iff for every path λ from q, for some $j \geq 0$, $(M, \lambda[j]) \models \phi$

$(M, q) \models AF\phi$ iff $(M, q) \models A(\text{true}\ U \phi)$

iff for every path λ from q, $(M, \lambda) \models \text{true}\ U \phi$

iff for every path λ from q, $(M, \lambda[i]) \models \phi$ for some $i \geq 0$, and $\underline{(M, \lambda[j]) \models \text{true} \text{ for all } 0 \leq j < i}$

iff for every path λ from q, $(M, \lambda[i]) \models \phi$ for some $i \geq 0$ (we can omit underlined part as it is valid)

iff for every path λ from q, for some $j \geq 0$, $(M, \lambda[j]) \models \phi$ (just rename i to j as it is arbitrary)

$(M, q) \models EG\phi$ iff for some path λ from q, for all $j \geq 0$, $(M, \lambda[j]) \models \phi$

$(M, q) \models EG\phi$ iff $(M, q) \models \neg AF\neg\phi$

iff $(M, q) \not\models AF\neg\phi$

iff $(M, q) \not\models A(\text{true}\ U \neg\phi)$

iff it is not the case that, $(M, q) \models A(\text{true}\ U \neg\phi)$

iff it is not the case that, for every path λ from q, $(M, \lambda) \models \text{true}\ U \neg\phi$

iff it is not the case that, for every path λ from q, $(M, \lambda[i]) \models \neg\phi$ for some $i \geq 0$, and $\underline{(M, \lambda[j]) \models \text{true for all } 0 \leq j < i}$

iff it is not the case that, for every path λ from q, $(M, \lambda[i]) \models \neg\phi$ for some $i \geq 0$ (underlined part is valid and therefore can be removed)

iff it is not the case that, for every path λ from q, $(M, \lambda[i]) \not\models \phi$ for some $i \geq 0$

iff it is not the case that, for every path λ from q, it is not the case that $(M, \lambda[i]) \models \phi$ for some $i \geq 0$

iff for some path λ from q, $(M, \lambda[i]) \models \phi$ for ~~some~~ all $i \geq 0$ (transform the ~-underlined not, every into some and the ≈-underlined not, some into all)

iff for some path λ from q, for all $j \geq 0$, $(M, \lambda[j]) \models \phi$ (rename the arbitrary i to j)

$(M,q) \models AG\phi$ iff for every path λ from q, for all $j \geq 0$ $(M, \lambda[j]) \models \phi$

$(M,q) \models AG\phi$ iff $(M,q) \models \neg EF \neg\phi$

iff $(M,q) \not\models EF\neg\phi$

iff $(M,q) \not\models E(\text{true}\ U \neg\phi)$

iff it is not the case that, $(M,q) \models E(\text{true}\ U \neg\phi)$

iff it is not the case that, for some path λ from q, $(M,\lambda) \models \text{true}\ U \neg\phi$

iff it is not the case that, for some path λ from q, $(M, \lambda[i]) \models \neg\phi$ for some $i \geq 0$, and $\underline{(M, \lambda[j]) \models \text{true} \text{ for all } 0 \leq j < i}$

iff it is not the case that, for some path λ from q, $(M, \lambda[i]) \models \neg\phi$ for some $i \geq 0$

(we can omit the underlined part as it is always valid condition)

iff it is not the case that, for some path λ from q, $(M, \lambda[i]) \not\models \phi$ for some $i \geq 0$

iff it is not the case that, for some path λ from q, it is not the case that $(M, \lambda[i]) \models \phi$ for some $i \geq 0$

iff for every path λ from q, $(M, \lambda[i]) \models \phi$ for some $i \geq 0$ (transform the underlined not, some into every ~~not some, not into every~~ and the underlined not, some into all)

iff for ~~some~~ every path λ from q, for all $j \geq 0$, $(M, \lambda[j]) \models \phi$ (rename the arbitrary i to j)

3.

CTL* defined in Backus-Naur form:

(ϕ - state formula, ψ - path formula, p - atom)

$$\phi ::= p \mid \neg\phi \mid \phi \wedge \phi \mid E\psi \mid A\psi \quad (1)$$

$$\psi ::= \phi \mid \neg\psi \mid \psi \wedge \psi \mid X\psi \mid \psi U \psi \quad (2)$$

a) Let's look at how state and path formulas are constructed in CTL and whether corresponding equivalent construction methods exist in CTL*:

For CTL

- State formulas ϕ, ψ:
  - $a \in AP$ (atom) - covered by 1st expansion in (1)
  - $\neg\phi$ - covered by 2nd expansion in (1)
  - $\phi \wedge \psi$ - covered by 3rd expansion in (1) (the 2 ϕ's in the expansion can of course expand into different formulas)
  - $E\phi$ - covered by 4th expansion in (1)
  - $A\phi$ - covered by 5th expansion in (1)
- Path formulas ϕ:
  - $X\phi$ - covered by 4th and 1st expansion in (2) ($\psi \to X\psi \to X\phi$)
  - ~~$\phi U \psi$~~
  - $\phi U \psi$ - covered by 5th and 1st expansion in (2) ($\psi \to \psi U \psi \to \phi U \phi$ and these 2 ϕ's obviously further expand into different path formulas)

And in CTL just as in CTL* formulas are all and only the state formulas

We see that every rule for constructing a CTL formula also exists in the Backus-Naur form of CTL* and therefore every CTL formula is also a CTL* formula or in other words CTL is a syntactic fragment of CTL*.

(b) We can show that the formula ~~AXXp~~ AXXp is in CTL* but does not belong to CTL.
If utilize the Backus-Naur form of CTL* given above we can construct AXXp as follows:

$\Phi \to A\psi \to AX\psi \to AXX\psi \to AXX\Phi \to AXXp$ (p is an atom)

So AXXp is indeed in CTL*.

AXXp is not in CTL though; it is easy to observe that it has 2 consecutive temporal operators (2 Xs) and in CTL we need a quantifier between them. If we try to construct it in CTL we can go as far as

$\Phi \to A\phi \to AX\Phi$

but now the last Φ has to be expanded into either $A\phi$ or $E\phi$ which introduces the necessary quantifier between two consecutive temporal operators.

From (a) and (b) it follows that CTL is a strict fragment of CTL* i.e. all CTL formulas are CTL* formulas but there are CTL* formulas that are not in CTL.

4.

Definition 2 (Semantics of CTL*)

M - model   s - state π - path ϕ, ϕ' - state formulas

ψ, ψ' - path formulas

$(M, s) \models p$ iff $s \in V(p)$ (1)

$(M, s) \models \neg\phi$ iff $(M, s) \not\models \phi$ (2)

$(M, s) \models \phi \wedge \phi'$ iff $(M, s) \models \phi$ and $(M, s) \models \phi'$ (3)

$(M, s) \models E\psi$ iff for some path π starting from s, $(M, \pi) \models \psi$ (4)

$(M, s) \models A\psi$ iff for all paths π starting from s, $(M, \pi) \models \psi$ (5)

$(M, \pi) \models \phi$ iff $(M, \pi[0]) \models \phi$, where $\pi[0]$ is the initial state in path π (6)

$(M, \pi) \models \neg\psi$ iff $(M, \pi) \not\models \psi$ (7)

$(M, \pi) \models \psi \wedge \psi'$ iff $(M, \pi) \models \psi$ and $(M, \pi) \models \psi'$ (8)

$(M, \pi) \models X\psi$ iff $(M, \pi[1 \ldots \infty]) \models \psi$ (9)

$(M, \pi) \models \psi U \psi'$ iff $(M, \pi[i \ldots \infty]) \models \psi$ for some $i \geq 0$, and $(M, \pi[j \ldots \infty]) \models \psi$ for all $0 \leq j < i$ (10)

And from Def 1.7 and 1.8 in lecture 5:

M - model $M = \langle St, \rightarrow, V \rangle$

ϕ, Ψ - state formulas γ - path formula

$(M, q) \models a$ iff $q \in V(a)$ (11)

$(M, q) \models \neg\phi$ iff $(M, q) \not\models \phi$ (12)

$(M, q) \models \phi \wedge \Psi$ iff $(M, q) \models \phi$ and $(M, q) \models \Psi$ (13)

$(M, q) \models E\gamma$ iff for some path λ starting from q, $(M, \lambda) \models \gamma$ (14)

$(M, q) \models A\gamma$ iff for all paths λ starting from q, $(M, \lambda) \models \gamma$ (15)

$(M, \lambda) \models X\phi$ iff $(M, \lambda[1]) \models \phi$ (16)

$(M, \lambda) \models \phi U \psi$ iff $(M, \lambda[i]) \models \psi$ for some $i \geq 0$ and $(M, \lambda[j]) \models \phi$ for all $0 \leq j < i$ (17)

If we restrict Def. 2 to formulas in CTL only naturally we'll see some changes in the semantics in Def. 2.

First of all (6) is no longer needed as in CTL we only interpret state formulas on states.

The semantics in (1) through (5) remain the same and as we can see they match the ones from Lecture 5 in (11) to (15) respectively (of course we ignore the naming differences, i.e. using λ instead of π for denoting ~~a~~ path as those do not affect the meaning). (7) and (8) are not present in Def. 1.7 and 1.8 but ~~those~~ they are simple semantics that arise from the definitions of $\neg$ and $\wedge$.

(9) and (10) will have to change, however, as we restrict to CTL only. In ~~the~~ particular ψ would have to be replaced with ϕ as in CTL we only apply temporal operators (X and U) to state formulas in order to create path formulas. ~~Those path formulas are also~~

Additionally the "state-part" of a path formula (i.e. ϕ when we have $X\phi$) is only interpreted on a state so $\pi[1\ldots\infty]$ changes to just $\pi[1]$ and $\pi[i\ldots\infty]$ to $\pi[i]$ and so on. So the modified (9) will look like:

$$(M,\pi) \models X\phi \text{ iff } (M,\pi[1]) \models \phi \qquad (18)$$

and the modified (10):

$$(M,\pi) \models \phi \cup \phi' \text{ iff } (M,\pi[i]) \models \phi' \text{ for some } i \geq 0$$
$$\text{and } (M,\pi[j]) \models \phi \text{ for all } 0 \leq j < i \quad (19)$$

We can easily spot now that (18) and (19) look exactly like (16) and (17) respectively from Def. 1.8 (with just different variable names).

Now we have shown that after ~~to~~ restricting ~~to~~ Def. 2 to just CTL formulas (6) is not needed, (1)-(5) remain the same and match (11)-(15) from Def. 1.7 and (18) and (19) which are modified versions of (9) and (10) match (16) and (17) respectively from Def 1.8.

Therefore if we restrict the semantics of CTL^* (Def. 2) to only CTL formulas we obtain ~~exactly the~~ the same truth conditions as in Def. 1.7 and Def. 1.8 from Lecture 5.

5. a) As we saw in 4. the semantics of CTL* can be reduced to CTL and in general since CTL is a fragment of of CTL* any CTL formula Φ is also a formula in CTL* so

for every ϕ in ~~CTL*~~ CTL there is a Φ^* in CTL* such that

for model M and starting state s

$$(M, s) \models \phi \text{ iff } (M, s) \models \phi^*$$

with ϕ^* being ϕ itself.

b) CTL cannot express

" Everybody will always be safe, from some moment on ;

whereas CTL* can:

$$\bigwedge_{i \in People} AFG\, safe_i$$

6. Models $M = (St, \rightarrow, V)$ and $M' = (St', \rightarrow', V')$

$(M, t) \approx (M', t')$, Φ - state formula

$(M, t) \models \phi$ iff $(M', t') \models \phi$

$\Rightarrow$

Let's have $(M, t) \models \phi$. We can show that $(M', t') \models \phi$.

$(M, t) \approx (M', t') \Rightarrow B(t, t')$ for some bisimulation B

Let's look at what ϕ might be. If ϕ is simply an atom, say p, we have:

$(M, t) \models p \Rightarrow t \in V(p)$

but since $B(t, t')$ and $t \in V(p) \Rightarrow t' \in V'(p)$ (1)

~~[illegible]~~

From (1) $\Rightarrow (M', t') \models \phi$. This can be seen as a proof to a "base" case, i.e. when ϕ is just an atom.

If ϕ is instead a negation of some other formula $(\neg\phi')$ or perhaps a conjuction $(\phi' \wedge \phi'')$ we can reason accordingly in a recursive manner, i.e.:

$(M, t) \models \neg\phi' \Rightarrow (M, t) \not\models \phi'$ and prove that $(M', t') \not\models \phi'$.

and accordingly for conjunction.