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A generalization of linear positive systems with applications to nonlinear systems: Invariant sets and the Poincaré–Bendixson property[☆]

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ABSTRACT

The dynamics of linear positive systems maps the positive orthant to itself. In other words, it maps a set of vectors with zero sign variations to itself. What linear systems map the set of vectors with k sign variations to itself? We address this question using tools from the theory of cooperative dynamical systems and the theory of totally positive matrices. This yields a generalization of positive linear systems called k-positive linear systems, that reduces to positive systems for $k = 1$. We describe applications of this new type of systems to the analysis of nonlinear dynamical systems. In particular, we show that such systems admit certain explicit invariant sets, and for the case $k = 2$ establish the Poincaré–Bendixson property for any bounded trajectory.

1. Introduction

Positive dynamical systems appear in many fields of science where the state-variables represent quantities that can only take nonnegative values (Farina & Rinaldi, 2000). In compartmental systems (Sandberg, 1978) every state-variable represents the density of "particles" in a compartment, and this cannot be negative. In chemical reaction networks the state-variables represent reactant concentrations. Another important example are models describing the evolution of probabilities (e.g. Markov chains) (Haag, 2017).

The dynamics of such systems map the nonnegative orthant $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x_i \geq 0 \text{ for all } i\}$ to itself (and also $\mathbb{R}^n_- := -\mathbb{R}^n_+$ to itself). Intuitively speaking, the dynamics map vectors with zero sign variations to vectors with zero sign variations.

Here, we suggest a generalization called a *k-positive linear system*. Such a system maps the set of vectors with at most $k-1$ sign variations to itself. For the case $k = 1$ this reduces to a positive linear system. But for $k \geq 2$ the system may be k-positive even if it is not a positive system in the usual sense.

Positive linear systems are important in their own right, and are an active area of research (see, e.g. the recent tutorial by Rantzer & Valcher, 2018), but also play an important role in the context of nonlinear systems. Indeed, if the variational system associated with the nonlinear system (see the exact definition below) is a positive linear time-varying (LTV) system then the nonlinear system is *cooperative* and this has far reaching consequences (Smith, 1995). We generalize this by defining *k-cooperative systems* as systems with a variational system that is a k-positive LTV. We describe the implications of this on the asymptotic behavior of the nonlinear system. In particular, we strengthen a seminal result of Sanchez (2009) to prove the Poincaré-Bendixson property for any trajectory of a 2-cooperative system that remains in a compact set. We use the nested structure of the invariant sets of a 2-cooperative system to prove a result that is considerably stronger than the one in Sanchez (2009). We believe that these results provide new tools for analyzing the asymptotic behavior of nonlinear dynamical systems. For a recent application to an important closed-loop system from systems biology, see Margaliot and Sontag (2019a).

We begin with motivating the general ideas in a slightly simplified setting. More general and rigorous statements are given in the next sections. For $B \in \mathbb{R}^{n\times m}$ we write $B \geq 0$ $[B \gg 0]$ if every entry of B is nonnegative [positive]. Recall that $P \in \mathbb{R}^{n\times n}$ is called *Metzler* if every off-diagonal entry of P is nonnegative.

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Consider the LTV system

$$\dot{x}(\tau) = A(\tau)x(\tau), \; x(t_0) = x_0, \tag{1}$$

with $A : (a, b) \to \mathbb{R}^{n\times n}$ a continuous matrix function. The associated LTV matrix differential system is:

$$\dot{\Phi}(\tau) = A(\tau)\Phi(\tau), \; \Phi(t_0) = I. \tag{2}$$

For any pair (t_0, t) the solution $x(t)$ of (1) at time t is given by $x(t) = \Phi(t, t_0)x(t_0)$, where $\Phi(t, t_0)$ is the solution of (2) at time t. We refer to $\Phi(t, t_0)$ as the *transition matrix* from time t_0 to time t of (1).

The system (1) is said to be positive on the time interval (a, b) if for any pair (t_0, t) with $a < t_0 < t < b$ and any $x(t_0) \in \mathbb{R}^n_+$ we have $x(t) \in \mathbb{R}^n_+$. Equivalently, $\Phi(t, t_0) \geq 0$ for all $a < t_0 < t < b$. It is well-known that this holds if and only if (iff) $A(\tau)$ is Metzler for all $a < \tau < b$. Thus, we have the following set of equivalent conditions:

- The LTV (1) is positive on the time interval (a, b);
- All the minors of order one of $\Phi(t, t_0)$ are nonnegative for all $a < t_0 < t < b$;
- $A(\tau)$ is Metzler for all $a < \tau < b$.

Our goal here is to introduce a generalization called a *k-positive system*. This is an LTV that maps the set of vectors with at most $k-1$ sign variations to itself. In particular, the standard positive system is a 1-positive system. We show that the following is a set of equivalent conditions:

- The LTV (1) is k-positive on the time interval (a, b);
- All the minors of order k of the transition matrix $\Phi(t, t_0)$ are nonnegative for all $a < t_0 < t < b$;
- $A^{[k]}(\tau)$ is Metzler for all $a < \tau < b$.

Here $A^{[j]}(\tau)$ denotes the j'th additive compound of $A(\tau)$ (see e.g., Muldowney, 1990). In particular $A^{[1]} = A$, so for $k = 1$ we obtain the set of conditions described above for a positive LTV. We provide for every k a simple condition on the structure of $A(t)$ guaranteeing that $A^{[k]}(t)$ is Metzler. Thus, our results do not require computing the transition matrix. Specifically, we show that an LTV is $(n-1)$-positive iff it is a competitive system (up to a coordinate transformation). For $1 < k < n-1$ it is k-positive with k even iff it is 2-positive, and if it is k-positive with k odd then it is 1-positive.

Positive LTVs play an important role in the analysis of time-varying *nonlinear* dynamical systems. To explain this, consider the time-varying nonlinear system:

$$\dot{x}(t) = f(t, x(t)), \tag{3}$$

whose trajectories evolve on a convex state-space $\Omega \subseteq \mathbb{R}^n$. Assume that f is C^1 with respect to x, and denote its Jacobian by $J(t, x) := \frac{\partial}{\partial x} f(t, x)$. For $p \in \Omega$, let $x(t, p)$ denote the solution of (3) at time t with $x(0) = p$. For $p, q \in \Omega$, let $z(t) := x(t, p) - x(t, q)$, that is, the difference at time t between the solutions emanating at time zero from p and from q. Then

$$\dot{z}(t) = A^{pq}(t)z(t), \tag{4}$$

where $A^{pq}(t) := \int_0^1 J(t, rx(t, p) + (1-r)x(t, q))\mathrm{d}r$. Eq. (4) is called the *variational system*, as it describes how a variation in the initial condition evolves with time.

If $A^{pq}(t)$ is Metzler for all $t \geq 0$ and all $p, q \in \Omega$ then (4) is a positive LTV. Then we conclude that

$$p \leq q \implies x(t, p) \leq x(t, q) \text{ for all } t \geq 0, \tag{5}$$

i.e., (3) is a cooperative dynamical system. Note that if $0 \in \Omega$ and 0 is an equilibrium point of (3) then (5) implies that $\mathbb{R}^n_+$ is an invariant set of (3). Cooperative systems have a well-ordered behavior. For example, in the time-invariant case and when the state-space Ω is compact almost every trajectory converges to an equilibrium point (Smith, 1995).

Intuitively speaking, (5) can be stated as follows: if $p-q$ has zero sign variations then $x(t, p) - x(t, q)$ has zero sign variations for all $t \geq 0$. We call (3) a *k-cooperative* system if the associated variational system is k-positive. This means that if $p - q$ has no more than $k-1$ sign variations then so does $x(t, p) - x(t, q)$ for all $t \geq 0$. We then describe the implications of this to the solutions of (3). In particular, we show that such systems admit special invariant sets, and that 2-cooperative systems satisfy a Poincaré-Bendixson property.

The next section reviews definitions and tools from the theory of totally positive (TP) matrices that are needed later on. These include the rigorous definitions of the number of sign variations in a vector, the variation diminishing properties of sign-regular matrices, and compound matrices. The next four sections describe our main results. Section 3 defines the new notions of a k-positive and a strongly k-positive LTV as systems that leave certain sets invariant. Section 4 provides explicit conditions for a system to be k-positive. Section 5 analyzes the geometrical structure of the invariant sets of k-positive systems, and shows that they are solid cones that include a linear subspace of dimension k, but no linear subspace of a higher dimension. However, these cones are not necessarily convex. Applications to nonlinear systems are given in Section 6. We show that if the variational system associated with the nonlinear system is k-positive then the nonlinear system admits certain invariant sets that can be described *explicitly*. Invariant sets play a significant role in many control-theoretic and engineering applications (see e.g., the survey by Blanchini, 1999 and the more recent PhD thesis by Song, 2015), yet analytic verification that a set is invariant is a non-trivial problem (Horváth et al., 2016). We also show that 2-cooperative systems satisfy a Poincaré-Bendixson property: a nonempty compact omega limit set which does not contain an equilibrium is a closed orbit. The final section concludes and describes topics for further research.

We use small [capital] letters to denote column vectors [matrices]. For $A \in \mathbb{R}^{n\times m}$, A' denotes the transpose of A. For a vector $y \in \mathbb{R}^n$, y_i is the i'th entry of y. For two integers $i \leq j$ we use the notation $[i, j]$ for the set $\{i, i+1, \ldots, j\}$. For a set S, $\text{int}(S)$ is the interior of S, and $\text{clos}(S)$ denotes its closure. For a square matrix A, $\text{tr}(A)$ is the trace of A. For $v_1, \ldots, v_n \in \mathbb{R}$, we use $\text{diag}(v_1, \ldots, v_n)$ to denote the diagonal matrix with diagonal entries $v_1, \ldots, v_n$.

2. Preliminaries

We begin by reviewing linear mappings that do not increase the number of sign variations in a vector.

2.1. Number of sign variations in a vector

For $y \in \mathbb{R}^n$ with no zero entries the number of sign variations in y is $\sigma(y) := |\{i \in [1, n-1] : y_i y_{i+1} < 0\}|$. For example, $\sigma(\begin{bmatrix} -4.2 & 3 & -0.5 \end{bmatrix}') = 2$.

In the more general case where the vector may include zero entries, we recall two definitions for the number of sign variations from the theory of TP matrices. For $y \in \mathbb{R}^n$, $s^-(y) = 0$ if $y = 0$, and otherwise $s^-(y) := \sigma(\bar{y})$, where $\bar{y}$ is the vector obtained from y by deleting all its zero entries. Let $s^+(y) := \max_{z\in S(y)} \sigma(z)$, where $S(y)$ includes all the vectors obtained by replacing every zero entry in y by either $+1$ or -1. For example, for

$$y = \begin{bmatrix} -1 & 1 & 0 & 0 & -3.5 \end{bmatrix}', \tag{6}$$

$$s^-(y) = \sigma(\begin{bmatrix} -1 & 1 & -3.5 \end{bmatrix}') = 2,$$
$$s^+(y) = \sigma(\begin{bmatrix} -1 & 1 & -1 & 1 & -3.5 \end{bmatrix}') = 4.$$

It follows from these definitions that $0 \le s^-(y) \le s^+(y) \le n-1$ for all $y \in \mathbb{R}^n$.

Let $\mathcal{V} := \{x \in \mathbb{R}^n : s^-(x) = s^+(x)\}$. It is not difficult to show that

$$\mathcal{V} = \{x \in \mathbb{R}^n : x_1 \neq 0, x_n \neq 0, \\ \text{if } x_i = 0 \text{ for some } i \in [2, n-1] \text{ then } x_{i-1}x_{i+1} < 0\}. \tag{7}$$

For example, for $n = 3$ the vector $x := \begin{bmatrix} 1 & \varepsilon & -1 \end{bmatrix}'$ satisfies $s^-(x) = s^+(x)$ for all $\varepsilon \in \mathbb{R}$, and x satisfies the condition in (7) for all $\varepsilon \in \mathbb{R}$.

There is a useful duality relation between s^- and s^+. Let $D := \operatorname{diag}(1, -1, \dots, (-1)^{n-1})$. Then

$$s^-(x) + s^+(Dx) = n-1 \text{ for all } x \in \mathbb{R}^n, \tag{8}$$

see e.g. Pinkus (2010, Ch. 3). For example, for $n = 5$ and the vector y in (6), we have $s^-(y) = 2$, $s^+(Dy) = s^+([-1\ -1\ 0\ 0\ -3.5]') = 2$, so $s^-(y) + s^+(Dy) = 4$.

Next we review the variation diminishing property (VDP) of certain matrices.

2.2. Sign regularity and the VDP

Consider $A \in \mathbb{R}^{n\times m}$, and pick $k \in [1, \min(n,m)]$. The matrix A is said to be *sign-regular of order k* (denoted SR_k) if all its minors of order k are nonnegative or all are nonpositive. It is called *strictly sign-regular of order k* (denoted SSR_k) if it is sign-regular of order k, and all the minors of order k are non-zero. In other words, all minors of order k are non-zero and have the same sign. For example, if all the entries of A are nonnegative [positive] then it is SR_1 [SSR_1]. The matrix is called *sign-regular* (SR) if it is SR_k for all k, and *strictly sign-regular* (SSR) if it is SSR_k for all k. For example, $\begin{bmatrix} 1 & 1/4 \\ 40 & 2 \end{bmatrix}$ is SSR_1 because all its 1×1 minors are positive, SSR_2 because its single 2×2 minor is negative, and thus it is SSR.

SR and SSR matrices are important in various fields. The most prominent examples are totally nonnegative (TN) [totally positive (TP)] matrices, that is, matrices with all minors nonnegative [positive]. Such matrices have beautiful properties and have found applications in statistics, computer graphics, approximation theory, and more (Fallat & Johnson, 2011; Fallat et al., 2017; Gantmacher & Krein, 2002; Pinkus, 2010).

An important property of TN and TP matrices is that multiplying a vector by such a matrix can only decrease the number of sign variations (see, e.g., Fallat & Johnson, 2011, Chapter 1), referred to as the VDP, if $A \in \mathbb{R}^{n\times m}$ is TN then $s^-(Ax) \le s^-(x)$ for all $x \in \mathbb{R}^m$, and if A is TP then $s^+(Ax) \le s^-(x)$ for all $x \in \mathbb{R}^m \setminus \{0\}$.

There is a renewed interest in such VDPs in the context of dynamical systems. Margaliot and Sontag (2019b) showed that strong results on the asymptotic behavior of nonlinear time-varying tridiagonal cooperative dynamical systems derived by Smillie (1984) and Smith (1991) follow from the fact that the transition matrix $\Phi(t, t_0)$ corresponding to their variational system is TP for all $t > t_0$ (see also Weiss & Margaliot, 2018). In other words, the variational system is a *totally positive differential system* (TPDS) (Schwarz, 1970). These transition matrices are real, square, and non-singular. Another recent paper showed that the transition matrix satisfies a VDP with respect to the *cyclic* number of sign variations iff it is SSR_k for all *odd* k (Ben-Avraham et al., 2019). Alseidi et al. (2019) studied the spectral properties of matrices that are SSR_k for some order k and introduced the notion of a totally positive discrete-time system. Katz et al. (2020) recently generalized this to the notion of an oscillatory discrete-time system.

The next result describes the equivalence between SSR_k and a special kind of VDP.

Theorem 1 (Ben-Avraham et al., 2019). *Let $A \in \mathbb{R}^{n\times n}$ be a nonsingular matrix. Pick $k \in [1, n]$. Then the following two conditions are equivalent:*

(a) If $x \in \mathbb{R}^n \setminus \{0\}$ with $s^-(x) \le k-1$ then $s^+(Ax) \le k-1$.
(b) A is SSR_k.

Example 1. For the particular case $k = 1$ Theorem 1 implies that for a nonsingular matrix $A \in \mathbb{R}^{n\times n}$ the following properties are equivalent:

(a) For any $x \in \mathbb{R}^n \setminus \{0\}$ with $s^-(x) = 0$ the entries of Ax are either all positive or all negative;
(b) The entries of A are either all positive or all negative.

Theorem 1 does not imply in general that $s^+(Ax) \le s^-(x)$. However if A is square and TP (and thus nonsingular) then Condition (b) holds for any k and this implies the following. Pick $x \in \mathbb{R}^n \setminus \{0\}$, and let k be such that $s^-(x) = k-1$. Then $s^+(Ax) \le k-1$, i.e., $s^+(Ax) \le s^-(x)$ and this recovers the VDP of (square) TP matrices.

For our purposes, we also need the next result that states an analogue of Theorem 1 for SR_k matrices.

Theorem 2. *Let $A \in \mathbb{R}^{n\times n}$ be a nonsingular matrix. Pick $k \in [1, n]$. Then the following two conditions are equivalent:*

(a) For any vector $x \in \mathbb{R}^n$ with $s^-(x) \le k-1$, we have

$$s^-(Ax) \le k-1. \tag{9}$$

(b) A is SR_k.

The proof follows from a standard continuity argument and is given, for the sake of completeness, in the Appendix.

For example, for the particular case $k = 1$ this implies that for a nonsingular matrix $A \in \mathbb{R}^{n\times n}$ the following properties are equivalent:

(a) For any $x \in \mathbb{R}^n$ with $s^-(x) = 0$ the entries of Ax are either all nonpositive or all nonnegative;
(b) The entries of A are either all nonpositive or all nonnegative.

Remark 1. Recall that a vector $x \in \mathbb{R}^n$ is called *totally nonzero* if $x_i \neq 0$ for all $i \in [1, n]$. Let TNV_k denote the set of all totally nonzero vectors $x \in \mathbb{R}^n$ with $\sigma(x) = k$ (and then of course $s^-(x) = s^+(x) = k$ as well). Johnson and Pena (2007) studied the set of nonsingular matrices that map TNV_k to itself. However, these matrices are quite different from the ones studied in this paper, due to the requirement that every entry of Ax must be nonzero.

Another important property of TN matrices, that will be used below to analyze the geometry of the invariant sets of k-positive systems, is their spectral structure. All the eigenvalues of a TN matrix are real and nonnegative, and the corresponding eigenvectors have special sign patterns. A matrix $A \in \mathbb{R}^{n\times n}$ is called *oscillatory* if it is TN and there exists an integer $k \ge 1$ such that A^k is TP (Gantmacher & Krein, 2002). The special spectral structure is particularly evident in the case of oscillatory matrices.

Theorem 3 (Gantmacher & Krein, 2002; Pinkus, 1996). *If $A \in \mathbb{R}^{n\times n}$ is an oscillatory matrix then its eigenvalues are all real, positive, and distinct. Order the eigenvalues as $\lambda_1 > \lambda_2 > \dots > \lambda_n > 0$, and let $u^k \in \mathbb{R}^n$ denote the eigenvector corresponding to λ_k. Then for any $1 \le i \le j \le n$ and any real scalars $c_i, \dots, c_j$, that are not all zero,*

$$i-1 \le s^-(\sum_{k=i}^{j} c_k u^k) \le s^+(\sum_{k=i}^{j} c_k u^k) \le j-1. \tag{10}$$

This implies in particular that $s^-(u^i) = s^+(u^i) = i-1$ for all $i \in [1, n]$.

Example 2. Consider the oscillatory matrix $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}$. Its eigenvalues are $\lambda_1 = 4$, $\lambda_2 = 2$, $\lambda_3 = 1$, with corresponding eigenvectors $u^1 = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}'$, $u^2 = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}'$, and $u^3 = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}'$. Note that $s^-(u^k) = s^+(u^k) = k-1$ for all $k \in [1, 3]$.

In the context of dynamical systems, the question is not when does a static mapping satisfy a VDP, but rather when does the transition matrix of the system satisfies a VDP for all time. As shown by Schwarz (1970), this can be analyzed using the dynamics of *compound matrices* (Muldowney, 1990).

2.3. Compound matrices

Given $A \in \mathbb{R}^{n\times n}$ and $k \in [1, n]$, consider the $\binom{n}{k}^2$ minors of order k of A. Each minor is defined by a set of row indices $1 \le i_1 < i_2 < \dots < i_k \le n$ and column indices $1 \le j_1 < j_2 < \dots < j_k \le n$. This minor is denoted by $A(\alpha|\beta)$, where $\alpha := \{i_1, \dots, i_k\}$ and $\beta := \{j_1, \dots, j_k\}$. With a slight abuse of notation we will sometimes treat such ordered sequences as sets. For example, for $A = \begin{bmatrix} 4 & 5 & 6 \\ -1 & 4 & -2 \\ 0 & 3 & -3 \end{bmatrix}$, $\alpha = \{1, 3\}$, and $\beta = \{2, 3\}$, we have $A(\alpha|\beta) = \det \begin{bmatrix} 5 & 6 \\ 3 & -3 \end{bmatrix} = -33$.

For $A \in \mathbb{R}^{n\times n}$ and $k \in [1, n]$ the k'th *multiplicative compound matrix* $A^{(k)}$ of A is the $\binom{n}{k} \times \binom{n}{k}$ matrix that includes all these minors ordered lexicographically. For example, for $n = 3$ and $k = 2$, $A^{(2)}$ is the 3×3 matrix

$$\begin{bmatrix} A(\{1,2\}|\{1,2\}) & A(\{1,2\}|\{1,3\}) & A(\{1,2\}|\{2,3\}) \\ A(\{1,3\}|\{1,2\}) & A(\{1,3\}|\{1,3\}) & A(\{1,3\}|\{2,3\}) \\ A(\{2,3\}|\{1,2\}) & A(\{2,3\}|\{1,3\}) & A(\{2,3\}|\{2,3\}) \end{bmatrix}.$$

Note that $A^{(1)} = A$ and $A^{(n)} = \det(A)$.

Remark 2. A matrix A is SR_k iff all the entries of $A^{(k)}$ are either all nonnegative or all nonpositive. In the first case $A^{(k)}$ maps the cone $\mathbb{R}_+^{\binom{n}{k}}$ to itself. Kushel (2012) studied matrices A such that for any k the matrix $A^{(k)}$ preserves a proper cone.

The Cauchy–Binet formula (see, e.g., Fallat & Johnson, 2011, Ch. 1) asserts that $(AB)^{(k)} = A^{(k)}B^{(k)}$. This justifies the term multiplicative compound.

The k'th *additive compound matrix* of A is defined by $A^{[k]} := \frac{d}{d\varepsilon}(I + \varepsilon A)^{(k)}|_{\varepsilon=0}$. This implies that

$$(I + \varepsilon A)^{(k)} = I + \varepsilon A^{[k]} + o(\varepsilon). \tag{11}$$

Example 3. Consider the case $n = 3$ and $k = 2$. Then

$$(I + \varepsilon A)^{(2)} = \begin{bmatrix} 1+\varepsilon a_{11} & \varepsilon a_{12} & \varepsilon a_{13} \\ \varepsilon a_{21} & 1+\varepsilon a_{22} & \varepsilon a_{23} \\ \varepsilon a_{31} & \varepsilon a_{32} & 1+\varepsilon a_{33} \end{bmatrix}^{(2)}$$
$$= \begin{bmatrix} 1+\varepsilon(a_{11}+a_{22}) & \varepsilon a_{23} & -\varepsilon a_{13} \\ \varepsilon a_{32} & 1+\varepsilon(a_{11}+a_{33}) & \varepsilon a_{12} \\ -\varepsilon a_{31} & \varepsilon a_{21} & 1+\varepsilon(a_{22}+a_{33}) \end{bmatrix}$$
$$+\, o(\varepsilon),$$

so (11) gives

$$A^{[2]} = \begin{bmatrix} a_{11}+a_{22} & a_{23} & -a_{13} \\ a_{32} & a_{11}+a_{33} & a_{12} \\ -a_{31} & a_{21} & a_{22}+a_{33} \end{bmatrix}. \tag{12}$$

The Cauchy–Binet formula can be used to prove that $(A+B)^{[k]} = A^{[k]} + B^{[k]}$, thus justifying the term additive compound.

The additive compound arises naturally when studying the dynamics of the multiplicative compound. For a time-varying matrix $Y(t)$ let $Y^{(k)}(t) := (Y(t))^{(k)}$. Suppose that $Y(t)$ evolves according to $\frac{d}{dt}Y(t) = A(t)Y(t)$. Then a Taylor approximation yields

$$\begin{aligned} Y^{(k)}(t+\varepsilon) &= (Y(t) + \varepsilon A(t)Y(t))^{(k)} + o(\varepsilon) \\ &= (I + \varepsilon A(t))^{(k)} Y^{(k)}(t) + o(\varepsilon), \end{aligned}$$

and combining this with (11) gives

$$\frac{d}{dt} Y^{(k)}(t) = A^{[k]}(t) Y^{(k)}(t), \tag{13}$$

where $A^{[k]}(t) := (A(t))^{[k]}$. Thus, the dynamics of all the minors of order k of $Y(t)$, stacked in the matrix $Y^{(k)}(t)$, is also described by a linear dynamical system, with the matrix $A^{[k]}(t)$.

For any $k \in [1, n]$, the matrix $A^{[k]}$ can be given explicitly in terms of the entries a_{ij} of A.

Lemma 1. *The entry of $A^{[k]}$ corresponding to $(\alpha|\beta) = (i_1, \dots, i_k | j_1, \dots, j_k)$ is:*

- $\sum_{\ell=1}^{k} a_{i_\ell i_\ell}$ *if $i_\ell = j_\ell$ for all $\ell \in [1, k]$;*
- $(-1)^{\ell+m} a_{i_\ell j_m}$ *if all the indices in α and β coincide except for a single index $i_\ell \ne j_m$; and*
- 0, *otherwise.*

For a proof of this result, see e.g., Schwarz (1970) or Fiedler (2008).

The first case in Lemma 1 corresponds to diagonal entries of $A^{[k]}$. All the other entries of $A^{[k]}$ are either zero or an entry of A multiplied by either plus or minus one.

Example 4. Consider the case $n = 4$, i.e., $A = \{a_{ij}\}_{i,j=1}^4$. Then Lemma 1 yields

$$A^{[2]} = \begin{bmatrix} a_{11}+a_{22} & a_{23} & a_{24} & -a_{13} & -a_{14} & 0 \\ a_{32} & a_{11}+a_{33} & a_{34} & a_{12} & 0 & -a_{14} \\ a_{42} & a_{43} & a_{11}+a_{44} & 0 & a_{12} & a_{13} \\ -a_{31} & a_{21} & 0 & a_{22}+a_{33} & a_{34} & -a_{24} \\ -a_{41} & 0 & a_{21} & a_{43} & a_{22}+a_{44} & a_{23} \\ 0 & -a_{41} & a_{31} & -a_{42} & a_{32} & a_{33}+a_{44} \end{bmatrix}, \tag{14}$$

and

$$A^{[3]} = \begin{bmatrix} a_{11}+a_{22}+a_{33} & a_{34} & -a_{24} & a_{14} \\ a_{43} & a_{11}+a_{22}+a_{44} & a_{23} & -a_{13} \\ -a_{42} & a_{32} & a_{11}+a_{33}+a_{44} & a_{12} \\ a_{41} & -a_{31} & a_{21} & a_{22}+a_{33}+a_{44} \end{bmatrix}.$$

The entry in the first row and third column of $A^{[3]}$ corresponds to $(\alpha|\beta) = (\{1,2,3\}|\{1,3,4\})$, and since α and β coincide except for the entry $\alpha_{i_2} = 2$ and $\beta_{j_3} = 4$, this entry is $(-1)^{2+3} a_{i_2 j_3} = -a_{24}$. It is useful to index compound matrices using α, β. For example, we write $A^{[3]}(\{1,2,3\}|\{1,3,4\}) = -a_{24}$.

We note two special cases of (13). For $k = 1$, $Y^{(1)}$ is the matrix that contains the first-order minors of Y, that is, $Y^{(1)} = Y$, and Lemma 1 gives $A^{[1]} = A$, so (13) becomes $\dot{Y} = AY$. For $k = n$, $Y^{(n)}$ is the matrix that contains all the $n \times n$ minors of Y, that is, $\det Y$, and using Lemma 1 yields $\frac{d}{dt}(\det Y(t)) = \operatorname{tr}(A(t)) \det Y(t)$, which is the Abel–Jacobi–Liouville identity (see, e.g. Byrnes, 1999).

For our purposes, it is important to determine whether for a given $A \in \mathbb{R}^{n\times n}$ the matrix $A^{[k]}$ is Metzler or not. This can be done using Lemma 1. The next result demonstrates this. We require the following definition.

Definition 1. Let M_2^n denote the set of matrices $A \in \mathbb{R}^{n\times n}$ satisfying:

(a) $a_{1n}, a_{n1} \le 0$;
(b) $a_{ij} \ge 0$ for all i, j with $|i-j| = 1$;
(c) $a_{ij} = 0$ for all i, j with $1 < |i-j| < n-1$.

For example, for $n = 5$ the matrices in M_2^5 are those with the sign pattern $\begin{bmatrix} * & \geq 0 & 0 & 0 & \leq 0 \\ \geq 0 & * & \geq 0 & 0 & 0 \\ 0 & \geq 0 & * & \geq 0 & 0 \\ 0 & 0 & \geq 0 & * & \geq 0 \\ \leq 0 & 0 & 0 & \geq 0 & * \end{bmatrix}$, where $*$ denotes "do not care".

Lemma 2. *Let $A \in \mathbb{R}^{n\times n}$ with $n > 2$. Then $A^{[2]}$ is Metzler iff $A \in M_2^n$.*

Example 5. Consider the case $n = 4$. In this case $A^{[2]}$ is given in (14) and it is straightforward to verify that $A^{[2]}$ is Metzler iff $a_{12}, a_{23}, a_{34}, a_{21}, a_{32}, a_{43} \geq 0$, $a_{13} = a_{24} = a_{31} = a_{42} = 0$, and $a_{14}, a_{41} \leq 0$, that is, iff $A \in M_2^4$.

Proof of Lemma 2. It follows from Lemma 1 that for any $i \neq j$ the entry a_{ij} or $(-a_{ij})$ appears as an offdiagonal entry of $A^{[2]}$ iff one of the following cases holds for some $p \in [1, n]$:

(1) if $i < p < j$ then $A^{[2]}(\{i, p\}|\{p, j\}) = -a_{ij}$;
(2) if $p < i$ and $p < j$ then $A^{[2]}(\{p, i\}|\{p, j\}) = a_{ij}$;
(3) if $p > i$ and $p > j$ then $A^{[2]}(\{i, p\}|\{j, p\}) = a_{ij}$;
(4) if $j < p < i$ then $A^{[2]}(\{p, i\}|\{j, p\}) = -a_{ij}$.

Consider the case $i = 1$ and $j = n$. Then only case (1) applies and we conclude that $-a_{1n}$ (but not a_{1n}) appears in $A^{[2]}$, so if $a_{1n} > 0$ then $A^{[2]}$ is not Metzler. A similar argument using case (4) shows that $-a_{n1}$ appears in $A^{[2]}$, so if $a_{n1} > 0$ then $A^{[2]}$ is not Metzler.

Pick $i, j \in [1, n]$ with $|i - j| = 1$. Then cases (1) and (4) do not apply, whereas cases (2) and (3) imply that a_{ij} appears in $A^{[2]}$. This entry must be nonnegative, or else $A^{[2]}$ is not Metzler.

Pick $i, j \in [1, n]$ with $1 < |i - j| < n - 1$. Then it can be shown using cases (1)-(4) that both a_{ij} and $-a_{ij}$ appear in $A^{[2]}$ and thus if $a_{ij} \neq 0$ then $A^{[2]}$ is not Metzler. We conclude that if $A \notin M_2^n$ then $A^{[2]}$ is not Metzler. But the arguments above also show that if $A \in M_2^n$ then $A^{[2]}$ is Metzler. This completes the proof of Lemma 2. $\square$

Let $\mathbb{M} \subset \mathbb{R}^{n\times n}$ [$\mathbb{M}^+ \subset \mathbb{R}^{n\times n}$] denote the set of matrices that are tridiagonal, and with nonnegative [positive] entries on the super- and sub-diagonals. One implication of Lemma 2 is that $A^{[1]} = A$ and $A^{[2]}$ are both Metzler iff $A \in \mathbb{M}$. If, in addition, we require A to be irreducible then this holds iff $A \in \mathbb{M}^+$ (Margaliot & Sontag, 2019b). Schwarz (1970) showed that the transition matrix $\exp(At)$ is TP for all $t > 0$ iff $A \in \mathbb{M}^+$.

We are now ready to define a generalization of a positive LTV system.

3. *k*-positive linear systems

For any $k \in [1, n]$, define the sets $P_-^k := \{z \in \mathbb{R}^n : s^-(z) \leq k - 1\}$, and $P_+^k := \{z \in \mathbb{R}^n : s^+(z) \leq k - 1\}$. It is not difficult to show that P_-^k is closed, P_+^k is open. Note that

$$P_-^1 = \mathbb{R}_+^n \cup \mathbb{R}_-^n, \quad P_+^1 = \operatorname{int}\mathbb{R}_+^n \cup \operatorname{int}\mathbb{R}_-^n, \tag{15}$$

and that

$$\begin{aligned} &P_+^k = \operatorname{int}(P_-^k) \text{ for all } k \in [1, n-1], \\ &P_-^1 \subset P_-^2 \subset \ldots \subset P_-^n = \mathbb{R}^n, \\ &P_+^1 \subset P_+^2 \subset \ldots \subset P_+^n = \mathbb{R}^n. \end{aligned} \tag{16}$$

Remark 3. Oliva et al. (1993) studied diffeomorphisms $f : \mathbb{R}^n \to \mathbb{R}^n$ whose Jacobian $J(x)$ is an oscillatory matrix for all $x \in \mathbb{R}^n$, and defined sets that are closely related to P_-^k and P_+^k. In Section 5 below we analyze the geometrical structure of P_-^k, and in particular show that they are cones of rank k (see also Krasnoselskii et al., 1989, Ch. 1).

Fix a time interval $-\infty \leq a < b \leq \infty$. Consider the time-varying linear system:

$$\dot{x}(t) = A(t)x(t), \quad x(t_0) = x_0, \tag{17}$$

where $A(\cdot) : (a, b) \to \mathbb{R}^{n\times n}$ is a locally (essentially) bounded measurable matrix function and $t_0 \in (a, b)$. It is well-known that this implies that (17) admits a unique absolutely-continuous solution (Sontag, 1998). This solution satisfies $x(t) = \Phi(t, t_0)x(t_0)$, where $\Phi(t, t_0)$ (sometimes written $\Phi(t)$ for brevity) is the solution at time t of the matrix differential equation:

$$\dot{\Phi}(s) = A(s)\Phi(s), \quad \Phi(t_0) = I, \tag{18}$$

We are now ready to define the main notion studied in this paper.

Definition 2. Fix $k \in [1, n]$. We say that (17) is *k-positive* on the time interval (a, b) if P_-^k is an invariant set of the dynamics, that is, for any pair $a < t_0 < t < b$ and any $x(t_0) \in P_-^k$ we have $x(t) \in P_-^k$.

Eq. (15) implies that a 1-positive system is a positive system.

The next result provides a necessary and sufficient condition for (17) to be k-positive in terms of the k'th additive compound $A^{[k]}(t)$.

Theorem 4. *The system (17) is k-positive on (a, b) iff $A^{[k]}(s)$ is Metzler for almost all $s \in (a, b)$.*

Proof of Theorem 4. Theorem 2 implies that k-positivity is equivalent to $\Phi(t, t_0)$ being SR_k for all $a < t_0 < t < b$, that is, either $\Phi^{(k)}(t, t_0) \geq 0$ or $\Phi^{(k)}(t, t_0) \leq 0$ for all $a < t_0 < t < b$. By (13) and (18),

$$\frac{d}{ds}\Phi^{(k)}(s) = A^{[k]}(s)\Phi^{(k)}(s), \quad \Phi^{(k)}(t_0) = I. \tag{19}$$

By continuity, this implies that $\Phi(t, t_0)$ is SR_k for all $a < t_0 < t < b$ iff

$$\Phi^{(k)}(t, t_0) \geq 0 \text{ for all } a < t_0 < t < b. \tag{20}$$

It is well-known (see e.g., Margaliot & Sontag, 2019b, Lemma 2) that the solution of (19) satisfies (20) iff $A^{[k]}(s)$ is Metzler for almost all $s \in (a, b)$. $\square$

Example 6. Consider (17) with the constant matrix $A = \begin{bmatrix} -1 & 2 & -2 & 1 \\ 3 & 0 & 1 & -1 \\ -4 & 1.5 & 2 & 4 \\ 1 & -1 & 2 & 5 \end{bmatrix}$. Lemma 1 yields $A^{[3]} = \begin{bmatrix} 1 & 4 & 1 & 1 \\ 2 & 4 & 1 & 2 \\ 1 & 1.5 & 6 & 2 \\ 1 & 4 & 3 & 7 \end{bmatrix}$, so $A^{[3]}$ is Metzler. Hence, the system is 3-positive, and the set $P_-^3 = \{x \in \mathbb{R}^4 : s^-(x) \leq 2\}$ is an invariant set of the dynamics. Fig. 1 depicts $s^-(x(t))$, with $x(0) = \begin{bmatrix} 0.34 & -0.54 & -1.06 & 0.49 \end{bmatrix}'$, for $t \in [0, 2.5]$. Note that $s^-(x(0)) = 2$. It may be seen that $s^-(x(t))$ both decreases and increases yet, as expected, $s^-(x(t)) \leq 2$ for all $t \geq 0$.

For a given A, the additive compounds $A^{[1]}, \ldots, A^{[n]}$ are related. In particular, Schwarz (1970) showed that if $A^{[1]}$ and $A^{[2]}$ are Metzler then $A^{[k]}$ is Metzler for every $k \in [1, n]$. Combining this with Definition 2 and Theorem 4 yields the following result.

Corollary 1. *If the system (17) is 1-positive and 2-positive then it is k-positive for all $k \in [1, n]$.*

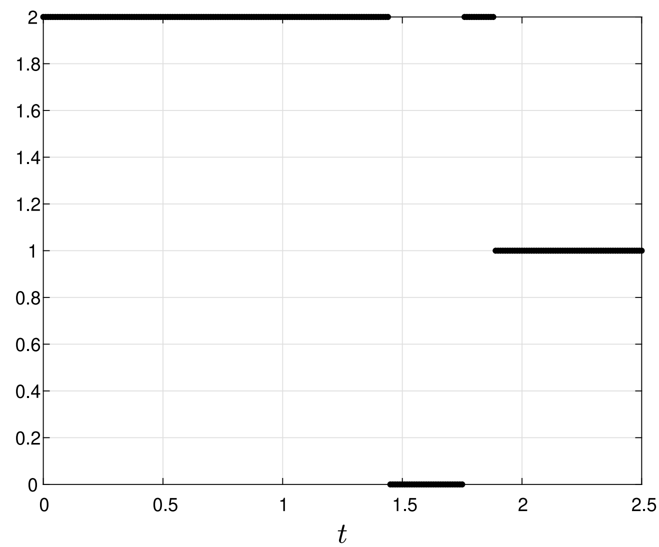

Fig. 1. $s^-(x(t))$ as a function of t for $x(t)$ in Example 6.

We now turn to define a stronger notion of k-positivity.

Definition 3. Fix $k \in [1, n]$. We say that (17) is *strongly k-positive* on (a, b) if for any $a < t_0 < t < b$ we have $x(t_0) \in P_-^k \setminus \{0\} \implies x(t) \in P_+^k$.

In other words, the dynamics maps $P_-^k \setminus \{0\}$ to P_+^k.

To provide a sufficient condition for strongly k-positivity, we recall one possible definition for irreducibility of a measurable matrix function (Walter, 1997). Let $J := (a, b)$. A measurable set $M \subset J$ is said to be *dense at a* if the set $M \cap [a, a+\varepsilon]$ has positive measure for every $\varepsilon > 0$. For measurable functions $f, g : J \to \mathbb{R}$ and $a \in J$, we write $f > g$ at a^+ if the set $\{t \in J | f(t) > g(t)\}$ is dense at a. A measurable matrix function $C : J \to \mathbb{R}^{n\times n}$ is said to be *irreducible at a^+* if for every two nonempty index sets $\alpha, \beta \subset \{1, \dots, n\}$, with $\alpha \cup \beta = \{1, \dots, n\}$, and $\alpha \cap \beta = \emptyset$, there exist indices $k \in \alpha, j \in \beta$ such that $c_{jk} > 0$ at a^+.

The next result provides a sufficient condition for strong k-positivity.

Theorem 5. *Suppose that $A^{[k]}(s)$ is Metzler for almost all $s \in (a, b)$, and that for any $a < t_0 < t < b$ there exists $t_0 \le \tau < t$ such that $A^{[k]}(s)$ is irreducible at τ^+. Then (17) is strongly k-positive on (a, b).*

Proof of Theorem 5. It is well-known (Walter, 1997) that the assumptions in the statement of the theorem imply that for any $a < t_0 < t < b$ the solution of (19) satisfies $\Phi^{(k)}(t, t_0) \gg 0$. In particular, $\Phi(t, t_0)$ is SSR_k. Pick $x(t_0) \in P_-^k \setminus \{0\}$. Theorem 1 implies that $s^+(x(t)) = s^+(\Phi(t, t_0)x(t_0)) \le k - 1$, so $x(t) \in P_+^k$. $\square$

For the case where $A(t)$ is continuous in t it is possible to give a necessary and sufficient condition for strongly k-positivity.

Theorem 6. *Let $A(\cdot) : (a, b) \to \mathbb{R}^{n\times n}$ be a continuous matrix function. The system (17) is strongly k-positive on (a, b) iff the following two conditions hold: $A^{[k]}(\tau)$ is Metzler for all $\tau \in (a, b)$, and for any interval $[p, q]$, with $a < p < q < b$, there exists $t^* \in [p, q]$ such that $A^{[k]}(t^*)$ is irreducible.*

Proof. Ben-Avraham et al. (2019, Lemma 2) shows that the conditions above are equivalent to the condition $\Phi^{(k)}(t, t_0) \gg 0$ for all $a < t_0 < t < b$. Combining this with Theorem 1 completes the proof. $\square$

Our next goal is to study systems that are strongly k-positive for several values of k. Since we are interested in asymptotic properties, we assume from here on that the time interval is $(a, b) = (a, \infty)$.

Proposition 7. *Assume that there exists $k \in [1, n-1]$ such that (17) is strongly i-positive for all $i \le k$. Then for any $x(t_0) \in P_-^k \setminus \{0\}$ and any set of times $t_0 < t_1 < t_2 < \cdots$ we have*

$$\begin{aligned} s^-(x(t_0)) &\ge s^+(x(t_1)) \ge s^-(x(t_1)) \ge s^+(x(t_2)) \\ &\ge s^-(x(t_2)) \ge s^+(x(t_3)) \ge \dots, \end{aligned} \tag{21}$$

and no more than $k - 1$ inequalities here are strict. Furthermore, there exists a time $\tau \ge t_0$ such that

$$x(t) \in \mathcal{V} \text{ for all } t \ge \tau. \tag{22}$$

Note that (21) implies that both $s^-(x(t))$ and $s^+(x(t))$ are integer-valued Lyapunov functions as they are bounded below (by zero) and non-increasing along any trajectory $x(t)$ emanating from $x(t_0) \in P_-^k \setminus \{0\}$.

Proof of Proposition 7. Pick $x(t_0) \in P_-^k \setminus \{0\}$. Let $v := s^-(x(t_0))$. Then $v \le k - 1$ and $x(t_0) \in P_-^{v+1} \setminus \{0\}$. Since the system is strongly $(v+1)$-positive, $x(t_1) \in P_+^{v+1}$, that is, $s^+(x(t_1)) \le v = s^-(x(t_0))$. In particular, $w := s^-(x(t_1)) \le s^+(x(t_1)) \le v$. Since the system is strongly $(w+1)$-positive, $x(t_2) \in P_+^{w+1}$, that is, $s^+(x(t_2)) \le w = s^-(x(t_1))$. Continuing in this manner yields (21).

Since s^-, s^+ take values in $[0, k-1]$, no more than $k - 1$ inequalities in (21) can be strict. Let τ_i denote the (up to $k - 1$) time points where $s^+(x(\tau_{\ell+1})) < s^-(x(\tau_\ell))$. Then (22) holds for $\tau := \max_i \tau_i$. $\square$.

Proposition 7 implies in particular that if the system is strongly i-positive for all $i \in [1, n-1]$ then (22) holds for any $x(t_0) \neq 0$. This recovers an important result in Schwarz (1970), which states that for a TPDS Eq. (22) holds for any $x(t_0) \neq 0$.

Theorem 4 provides a condition on $A^{[k]}$ ensuring that the linear system (17) is k-positive. We now turn to express this condition in terms of A.

4. Explicit algebraic conditions for k-positivity

We begin by considering the case $k = n - 1$.

4.1. $(n-1)$-positive systems

Given $A \in \mathbb{R}^{n\times n}$, when is $A^{[n-1]}$ Metzler? To address this question we require the following definition.

Definition 4. Let M_{n-1}^n denote the set of matrices $A \in \mathbb{R}^{n\times n}$ satisfying $a_{ij} \ge 0$ for all i, j such that $i - j$ is odd, and $a_{ij} \le 0$ for all $i \neq j$ such that $i - j$ is even.

For example, for $n = 4$ the matrices in M_3^4 are those with the sign pattern: $\begin{bmatrix} * & \ge 0 & \le 0 & \ge 0 \\ \ge 0 & * & \ge 0 & \le 0 \\ \le 0 & \ge 0 & * & \ge 0 \\ \ge 0 & \le 0 & \ge 0 & * \end{bmatrix}$, where $*$ denotes "do not care". In particular, the matrix A in Example 6 satisfies $A \in M_3^4$.

Lemma 3. *Let $A \in \mathbb{R}^{n\times n}$ with $n > 2$. Then $A^{[n-1]}$ is Metzler iff $A \in M_{n-1}^n$.*

Proof of Lemma 3. It follows from Lemma 1 that an off-diagonal entry of $A^{[k]}$ corresponding to $(\alpha|\beta) = (i_1, \dots, i_k | j_1, \dots, j_k)$ can be nonzero only if all the indices in α and β coincide, except for a single index $i_\ell \neq j_m$, and then $A^{[k]}(\alpha, \beta) = (-1)^{\ell+m} a_{i_\ell j_m}$. We use this to determine when a_{pq} (or $-a_{pq}$) appears on an off-diagonal entry of $A^{[n-1]}$. We consider only pairs (p, q) with $q \geq p$, as the case $p \geq q$ follows by symmetry. It is clear that if $p = q$ then $a_{pq} = a_{pp}$ does not appear as an off-diagonal entry of $A^{[n-1]}$. Pick p, q with $1 \leq p < q \leq n$. Suppose that a_{pq} or $-a_{pq}$ appears as an off-diagonal entry of $A^{[n-1]}$ corresponding to $(\alpha|\beta)$. This implies that $\alpha = \{i_1, i_2, \dots, i_{n-1}\}$ and $\beta = \{j_1, j_2, \dots, j_{n-1}\}$ coincide except for a single index $i_\ell \neq j_m$, with $i_\ell = p$ and $j_m = q$. Thus, $\alpha = \{1, 2, \dots, n\} \setminus \{q\}$ and $\beta = \{1, 2, \dots, n\} \setminus \{p\}$. Since $p < q$, this gives $i_\ell = i_p$ and $j_m = j_{q-1}$, so $A^{[n-1]}(\alpha|\beta) = (-1)^{p+q-1} a_{pq}$. By symmetry, we conclude that for any $p \neq q$ we have that $(-1)^{p+q-1} a_{pq}$ is an off-diagonal entry of $A^{[n-1]}$. Hence, $A^{[n-1]}$ is Metzler iff $(-1)^{p+q-1} a_{pq} \geq 0$ for all $p \neq q$. $\square$

Remark 4. For $F \in \mathbb{R}^{n\times n}$, let $\tilde{F}$ denote the matrix with entries $\tilde{f}_{ij} := (-1)^{i+j} f_{n+1-i,n+1-j}$ for all $i, j \in [1, n]$. Schwarz (1970) proved that if $A \in \mathbb{R}^{n\times n}$ then $A^{[n-1]} = \tilde{B}$, where $B := \operatorname{tr}(A)I - A'$. This implies that $A^{[n-1]}$ is Metzler iff $(-1)^{i+j+1} a_{n+1-j,n+1-i} \geq 0$ for all $i \neq j$. This provides an alternative proof of Lemma 3.

Example 7. Consider the case $n = 3$ and $k = n - 1 = 2$. Then

$$B := \operatorname{tr}(A)I - A' = \begin{bmatrix} a_{22}+a_{33} & -a_{21} & -a_{31} \\ -a_{12} & a_{11}+a_{33} & -a_{32} \\ -a_{13} & -a_{23} & a_{11}+a_{22} \end{bmatrix},$$

so

$$A^{[2]} = \tilde{B} = \begin{bmatrix} a_{11}+a_{22} & a_{23} & -a_{13} \\ a_{32} & a_{11}+a_{33} & a_{12} \\ -a_{31} & a_{21} & a_{22}+a_{33} \end{bmatrix},$$

and this agrees with (12).

Recall that the system $\dot{x} = Ax$ is called a *competitive system* if $(-A)$ is Metzler (Smith, 1995). The next result shows that $(n-1)$-positive systems are just competitive systems in disguise.

Lemma 4. *Let $D := \operatorname{diag}(1, -1, 1, \dots, (-1)^{n-1})$, and let $P \in \mathbb{R}^{n\times n}$ denote the permutation matrix* $P := \begin{bmatrix} 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & \dots & 0 & 1 & 0 \\ & & \vdots & & & \\ 1 & 0 & \dots & 0 & 0 & 0 \end{bmatrix}$. *Note that $D^{-1} = D$ and $P^{-1} = P$. Consider the system $\dot{x}(t) = Ax(t)$, and let $y(t) := -DPx(t)$, so that $\dot{y}(t) = By(t)$, with $B := DPAPD$. The following two conditions are equivalent.*

(1) $A \in M^n_{n-1}$, i.e. $\dot{x} = Ax$ is $(n-1)$-positive;
(2) the matrix $(-B)$ is Metzler, i.e. $\dot{y} = By$ is competitive.

Proof. Let $C := PAP$. Then $c_{ij} = a_{n+1-i,n+1-j}$. Since $-B = -DCD$,

$$\begin{aligned} -b_{ij} &= (-1)^{i+j+1} c_{ij} \\ &= (-1)^{i+j+1} a_{n+1-i,n+1-j} \\ &= (-1)^{n+1+i-(n+1-j)+1} a_{n+1-i,n+1-j}, \end{aligned}$$

and the definition of M^n_{n-1} implies that $-b_{ij} \geq 0$ for all $i \neq j$ iff $A \in M^n_{n-1}$. $\square$

Remark 5. Thus, 1-positive systems are cooperative systems, and $(n-1)$-positive systems are competitive systems, so the notion of a k-positive system provides a generalization of both cooperative and competitive systems.

We now turn to consider $A^{[k]}$ with $k \neq n-1$. The case $k = n$ is trivial as $A^{[n]}$ is a scalar, so the associated linear dynamical system is always cooperative. The case $k = 1$ is also clear as $A^{[1]} = A$. Thus, we only need to consider the case $k \in [2, n-2]$.

4.2. *k*-positive systems for some $k \in [2, n-2]$

We begin by defining a special set of periodic Jacobi matrices.

Definition 5. For any $k \in [2, n-2]$ let M^n_k denote the set of matrices $A \in \mathbb{R}^{n\times n}$ satisfying:

(a) $(-1)^{k-1} a_{1n}, (-1)^{k-1} a_{n1} \geq 0$;
(b) $a_{ij} \geq 0$ for all i, j with $|i - j| = 1$;
(c) $a_{ij} = 0$ for all i, j with $1 < |i - j| < n - 1$.

For example, the matrices in M^5_3 are those with the sign pattern: $\begin{bmatrix} * & \geq 0 & 0 & 0 & \geq 0 \\ \geq 0 & * & \geq 0 & 0 & 0 \\ 0 & \geq 0 & * & \geq 0 & 0 \\ 0 & 0 & \geq 0 & * & \geq 0 \\ \geq 0 & 0 & 0 & \geq 0 & * \end{bmatrix}$, where $*$ denotes "do not care". Note that the definition of M^n_k implies that $M^n_i = M^n_j$ for any $i, j \in [2, n-2]$ that have the same parity.

The next result generalizes Lemma 2.

Theorem 8. *Let $A \in \mathbb{R}^{n\times n}$ with $n > 2$. Then for any $k \in [2, n-2]$ the matrix $A^{[k]}$ is Metzler iff $A \in M^n_k$.*

Proof. We already proved this result for $k = 2$. Fix $k \in [3, n-2]$. It follows from Lemma 1 that an off-diagonal entry of $A^{[k]}$ corresponding to $(\alpha|\beta) = (i_1, \dots, i_k | j_1, \dots, j_k)$ can be nonzero only if all the indices in α and β coincide, except for a single index $i_\ell \neq j_m$, and then this entry is $A^{[k]}(\alpha, \beta) = (-1)^{\ell+m} a_{i_\ell j_m}$. We use this to determine when a_{ij} (or $-a_{ij}$) appears on an off-diagonal entry of $A^{[k]}$. We consider only pairs (i, j) with $j \geq i$, as the case $i \geq j$ follows by symmetry.

Case 1. If $j = i$ then $a_{ij} = a_{ii}$ does not appear in any off-diagonal entry of $A^{[k]}$. This explains the "do not care"s in the definition of M^n_k.

Case 2. If $j = i + 1$ then $a_{ij} = a_{i,i+1}$ will appear in an off-diagonal entry $(\alpha|\beta)$ of $A^{[k]}$ if all the entries of α and β coincide except that i appears in α but not in β, and $i + 1$ appears in β but not in α. But this implies that i and $i + 1$ appear in the same entry of α and β, that is, $\ell = m$ and the off-diagonal entry of $A^{[k]}$ is $(-1)^{2\ell} a_{i,i+1} = a_{i,i+1}$. Hence, $A^{[k]}$ is not Metzler if $a_{i,i+1} < 0$.

Case 3. Suppose that $1 < j - i < n - 1$ and $j = i+2$ (so $i+2 \leq n$). We now show that both $a_{i,i+2}$ and $-a_{i,i+2}$ appear on off-diagonal entries of $A^{[k]}$. It is not difficult to show that since $k + 2 \leq n$ and $i + 2 \leq n$, there exists an integer x such that

$$1 \leq x \leq i \text{ and } i - k + 1 \leq x \leq n - k - 1. \tag{23}$$

Then for

$$\alpha := \{x, \dots, i-1, i, \widehat{i+1}, \widehat{i+2}, i+3, \dots, x+k+1\},$$
$$\beta := \{x, \dots, i-1, \hat{i}, \widehat{i+1}, i+2, i+3, \dots, x+k+1\},$$

where $\hat{j}$ means that j is *not* included in the set, we have $A^{[k]}(\alpha|\beta) = a_{i,i+2}$, so $a_{i,i+2}$ appears on an off-diagonal entry of $A^{[k]}$. Note that (23) guarantees that α $[\beta]$ includes i $[i+2]$.

Similarly, it is not difficult to show that since $2 \leq k \leq n-1$ and $i + 2 \leq n$, there exists an integer x such that

$$1 \leq x \leq i \text{ and } i - k + 2 \leq x \leq n - k. \tag{24}$$

Then for

$$\alpha := \{x, \dots, i-1, i, i+1, \widehat{i+2}, i+3, i+4, \dots, x+k\},$$
$$\beta := \{x, \dots, i-1, \hat{i}, i+1, i+2, i+3, \dots, x+k\},$$

we have $A^{[k]}(\alpha|\beta) = -a_{i,i+2}$, so $-a_{i,i+2}$ also appears on an off-diagonal entry of $A^{[k]}$. Hence, $A^{[k]}$ is not Metzler if $a_{i,i+2} \neq 0$. Note that (24) guarantees that α [β] includes i [$i+2$].

Case 4. Suppose that $1 < j-i < n-1$ and $j > i+2$. Then it can be shown as in Case 3 that both $a_{i,j}$ and $-a_{i,j}$ appear on off-diagonal entries of $A^{[k]}$. Hence, $A^{[k]}$ is not Metzler if $a_{ij} \neq 0$.

Case 5. Suppose that $j-i = n-1$, that is, $i = 1$ and $j = n$. Then $a_{ij} = a_{1n}$ appears in an entry $(\alpha|\beta)$ of $A^{[k]}$ only when $\alpha = \{1, i_2, \dots, i_k\}$ and $\beta = \{j_1, \dots, j_{k-1}, n\}$, with $i_{p+1} = j_p$ for all $p \in [1, k-1]$, and then $A^{[k]}(\alpha|\beta) = (-1)^{1+k}a_{1n}$. Hence, $A^{[k]}$ is not Metzler if $(-1)^{k-1}a_{1n} < 0$.

Summarizing the cases above, we conclude that if $A \notin M_k^n$ then $A^{[k]}$ is not Metzler. But the analysis above actually covers all the cases where an entry a_{ij} appears as an off-diagonal entry of $A^{[k]}$, and this completes the proof of Theorem 8. □

Combining Theorems 4 and 8 yields the following result.

Corollary 2. *For any $k \in [2, n-1]$ the LTV (17) is k-positive on (a, b) iff $A(s) \in M_k^n$ for almost all $s \in (a, b)$.*

Using the explicit structure of a k-positive system yields a generalization of Corollary 1.

Corollary 3. *If there exist $i, j \in [1, n-2]$, with i even and j odd such that (17) is i-positive and j-positive then (17) is k-positive for all $k \in [1, n]$.*

Proof. Since the system is i-positive with i even, Definition 5 implies that: $a_{1n}, a_{n1} \leq 0$, the super- and sub-diagonals of A include non-negative entries, and all other off-diagonal entries are zero. The system is also j-positive with j odd. If $j = 1$ then A is Metzler, so we conclude that $a_{1n} = a_{n1} = 0$. If $j > 1$ then Definition 5 implies that $a_{1n}, a_{n1} \geq 0$, so again $a_{1n} = a_{n1} = 0$. We conclude that A is tridiagonal and Metzler, so (17) is k-positive for all $k \in [1, n]$. □

5. Geometrical structure of the invariant sets

A natural question is what is the structure of the invariant sets P_-^k and P_+^k? It is clear that these sets are cones, as $s^-(x) = s^-(\alpha x)$ and $s^+(x) = s^+(\alpha x)$ for all $\alpha \in \mathbb{R} \setminus \{0\}$. However, these sets are *not* convex cones. For example, for $n = 2$ the vectors $x = \begin{bmatrix} 1 & 1 \end{bmatrix}'$, $y = \begin{bmatrix} -1 & -1 \end{bmatrix}'$ satisfy $x, y \in P_+^1$, yet $\frac{1}{2}(x+y) = \begin{bmatrix} 0 & 0 \end{bmatrix}' \notin P_+^1$. Similarly, for $n = 3$ the vectors $x = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}'$, $y = \begin{bmatrix} 0 & -1 & 1 \end{bmatrix}'$ satisfy $x, y \in P_-^2$, yet $\frac{1}{2}(x+y) = \begin{bmatrix} 1/2 & -1 & 1/2 \end{bmatrix}' \notin P_-^2$.

Recall that a dynamical system is called *monotone* if its flow is order-preserving with respect to the (partial) order $\leq$ induced by a closed, convex and pointed cone K, that is, $x \leq y$ iff $y - x \in K$. The convexity of K implies that $x \leq y,\ y \leq z \implies x \leq z$, and the fact that K is pointed yields $x \leq y,\ y \leq x \implies x = y$. Since P_-^k, P_+^k are not convex, this suggests that k-positive systems are not monotone. Fortunately, these sets, although not convex, do possess a useful structure.

5.1. P_-^k is a cone of rank k

A set $C \subseteq \mathbb{R}^n$ is called a *cone of rank k* (see e.g. Krasnoselskii et al., 1989; Sanchez, 2009) if: C is closed, $x \in C$ implies that $\alpha x \in C$ for all $\alpha \in \mathbb{R}$, and C contains a linear subspace of dimension k and no linear subspace of higher dimension. For example, it is straightforward to see that $\mathbb{R}_+^2 \cup (-\mathbb{R}_+^2)$ (and, more generally, $\mathbb{R}_+^n \cup (-\mathbb{R}_+^n)$) is a cone of rank 1.

A cone C of rank k is called *solid* if its interior is nonempty, and *k-solid* if there is a linear subspace W of dimension k such that $W \setminus \{0\} \subseteq \text{int}(C)$. In the context of dynamical systems, such cones are important because trajectories of dynamical systems that are confined to C can be projected to the linear subspace W (Sanchez, 2009). Roughly speaking, if this projection is one-to-one then the trajectories must satisfy the same properties as trajectories in a k-dimensional space. Krasnoselskii et al. (1989, Ch. 1) showed that the set P_-^k is a k-solid cone. The next result slightly strengthens this. Also, the proof, unlike that in Krasnoselskii et al. (1989), uses the elegant spectral properties of oscillatory matrices.

Lemma 5. *For any $k \in [1, n-1]$ the set P_-^k is a k-solid cone, and its complement*

$$(P_-^k)^c := \text{clos}(\mathbb{R}^n \setminus P_-^k) \tag{25}$$

is an $(n-k)$-solid cone.

Proof. Pick $k \in [1, n-1]$. It follows from the definition of s^- that P_-^k is closed. If $x \in P_-^k$, that is, $s^-(x) \leq k-1$ then clearly $\alpha x \in P_-^k$ for all $\alpha \in \mathbb{R}$. The set P_-^k cannot contain a linear subspace of dimension $k+1$, as using a linear combination of $k+1$ independent vectors in $\mathbb{R}^n$ one can generate a vector y such that $s^-(y) \geq k$. Let $A \in \mathbb{R}^{n\times n}$ be an oscillatory matrix, and denote its eigenvalues and eigenvectors as in Theorem 3. Then (10) implies that for any $c_1, \dots c_k \in \mathbb{R}$, that are not all zero,

$$s^-(\sum_{p=1}^{k} c_p u^p) \leq s^+(\sum_{p=1}^{k} c_p u^p) \leq k-1. \tag{26}$$

We conclude that $W := \text{span}\{u^1, \dots, u^k\} \subseteq P_-^k$, and that $W \setminus \{0\} \subseteq P_+^k$.

Now pick $x \in W$. Suppose that $x \in \partial P_-^k$. Then by the definition of s^-, x includes a zero entry, say, x_i and there exists $\varepsilon \in \mathbb{R} \setminus \{0\}$, with $|\varepsilon|$ arbitrarily small, such that the vector $\tilde{x}$ obtained from x by setting x_i to ε satisfies $s^-(\tilde{x}) > k-1$. Thus, $s^+(x) > k-1$. But now (26) gives $x = 0$. We conclude that $W \cap \partial P_-^k = \{0\}$. This shows that $W \setminus \{0\} \subseteq \text{int}(P_-^k)$, so P_-^k is a k-solid cone.

We now turn to prove the assertion for $(P_-^k)^c$. By definition, this set is closed, and $x \in (P_-^k)^c$ implies that $\alpha x \in (P_-^k)^c$ for all $\alpha \in \mathbb{R}$. Eq. (10) implies that for any $c_{k+1}, \dots c_n \in \mathbb{R}$, that are not all zero,

$$k \leq s^-(\sum_{p=k+1}^{n} c_p u^p). \tag{27}$$

In other words, for $W^c := \text{span}\{u^{k+1}, \dots, u^n\}$ we have $W^c \setminus \{0\} \subseteq \mathbb{R}^n \setminus P_-^k$. Combining this with (25) implies that $W^c \subseteq (P_-^k)^c$.

Pick $x \in W^c$, that is, $x = \sum_{p=k+1}^n d_p u^p$, for some $d_{k+1}, \dots, d_n \in \mathbb{R}$. Suppose that $x \in \partial((P_-^k)^c)$. Since P_-^k is closed, we conclude that $x \in \partial P_-^k$. Thus, $x \in \{0\} \cup \{x \in \mathbb{R}^n : s^-(x) = k-1\}$. If $x \neq 0$ then at least one of the d_is is not zero, so (27) yields $k \leq s^-(x) = k-1$. We conclude that $x = 0$, so $W^c \setminus \{0\} \in \text{int}((P_-^k)^c)$. Thus, $(P_-^k)^c$ is an $(n-k)$-solid cone. □

Our next goal is to derive an explicit decomposition for the sets P_-^k, P_+^k.

5.2. P_-^k is the union of convex sets

For any $k \in [1, n]$, let $Q_-^k := \{z \in \mathbb{R}^n : s^-(z) = k-1\}$. For example $Q_-^1 = \mathbb{R}_+^n \cup \mathbb{R}_-^n$, and $Q_-^2 = F \cup (-F)$, where F is the set of all vectors with the sign pattern: $\geq 0, \dots, \geq 0, \leq 0, \dots, \leq 0$, with at least one entry positive and one entry negative. Note that $x \in Q_-^k$ implies that $\alpha x \in Q_-^k$ for all $\alpha \in \mathbb{R} \setminus \{0\}$.

Any $y \in Q_-^k$ can be decomposed into k disjoint and consecutive sets of entries, where each set is composed of entries that are all nonnegative [nonpositive] and at least one entry is positive [negative]. For example, the vector $y = \begin{bmatrix} 0 & 1 & 2 & 0 & -2 & 0 & 1 & 2 \end{bmatrix}'$ satisfies $y \in Q_-^3$ and can be decomposed into three sets: the first is 0, 1, 2, 0, the second is -2, 0, and the third is 1,2. We use this idea to derive a decomposition of Q_-^k. We require the following definition.

Definition 6. For a vector $v = \begin{bmatrix} v_1 & \dots & v_k \end{bmatrix}'$ with integer entries such that

$$1 \le v_1 < v_2 < \cdots < v_k = n, \tag{28}$$

let $C_-^k(v) \subseteq \mathbb{R}^n$ denote the set of vectors $y \in \mathbb{R}^n$ satisfying:

- $y_1, \dots, y_{v_1} \ge 0$, with at least one of these entries positive;
- $y_{v_1+1} < 0$, and $y_{v_1+2}, \dots, y_{v_2} \le 0$;
- $y_{v_2+1} > 0$, and $y_{v_2+2}, \dots, y_{v_3} \ge 0$; and so on until
- $(-1)^{k-1} y_{v_{k-1}+1} > 0$, and $(-1)^{k-1} y_{v_{k-1}+1}, \dots, (-1)^{k-1} y_{v_k} \ge 0$ (recall that $v_k = n$).

For example, for $n = 4$, $k = 3$, and $v = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix}'$,

$$C_-^3(v) = \{y \in \mathbb{R}^4 : y_1 \ge 0, y_2 \ge 0, y_1 y_2 \ne 0, y_3 < 0, y_4 > 0\}.$$

Note that $C_-^k(v)$ is a convex cone. In fact, $C_-^k(v)$ is an orthant in $\mathbb{R}^n$, and if $i \ne j$ then $C_-^i(v)$ and $C_-^j(v)$ are different orthants.

It is clear that $y \in Q_-^k$ iff $y \in C_-^k(v) \cup (-C_-^k(v))$ for some $v \in \mathbb{R}^k$ satisfying (28). The number of different vectors v that satisfy (28) is $\binom{n-1}{k-1}$, as we fix $v_k = n$. Combining this with the definitions of P_-^k and Q_-^k yields the following characterization of P_-^k as the union of convex cones.

Proposition 9. *For any $s \in [1, n]$ we have $P_-^s = \bigcup_{k=1}^s Q_-^k$, where*

$$Q_-^k = \bigcup_{i=1}^{\binom{n-1}{k-1}} C_-^k(v^i) \cup (-C_-^k(v^i)),$$

and v^i, $i \in [1, \binom{n-1}{k-1}]$, are all the different vectors that satisfy (28).

Example 8. Consider again the trajectory $x(t)$ of the system in Example 6 with $x(0) = \begin{bmatrix} 0.34 & -0.54 & -1.06 & 0.49 \end{bmatrix}'$. Recall that here $s^-(x(t)) \le 2$ for all $t \ge 0$. Note that $x(0) \in C_-^3(\begin{bmatrix} 1 & 3 & 4 \end{bmatrix}')$. An analysis of this trajectory shows that it crosses through the following cones: $C_-^3(\begin{bmatrix} 1 & 3 & 4 \end{bmatrix}') \to C_-^3(\begin{bmatrix} 2 & 3 & 4 \end{bmatrix}') \to C_-^1(\begin{bmatrix} 4 \end{bmatrix}) \to C_-^3(\begin{bmatrix} 1 & 2 & 4 \end{bmatrix}') \to C_-^2(\begin{bmatrix} 2 & 4 \end{bmatrix}')$. Note that all these cones belong to P_-^3.

Remark 6. The duality relation (8) and the fact that $D^{-1} = D$ implies that

$$\begin{aligned} DP_-^k &:= \{Dx : x \in \mathbb{R}^n,\ s^-(x) \le k-1\} \\ &= \{x \in \mathbb{R}^n : s^-(Dx) \le k-1\} \\ &= \{x \in \mathbb{R}^n : s^+(x) \ge n-k\} \\ &= \mathbb{R}^n \setminus \{x \in \mathbb{R}^n : s^+(x) < n-k\} \\ &= \mathbb{R}^n \setminus \{x \in \mathbb{R}^n : s^+(x) \le n-k-1\} \\ &= \mathbb{R}^n \setminus P_+^{n-k}. \end{aligned} \tag{29}$$

Thus, the results above on the structure of P_-^k, $k \in [1, n-1]$ can be transformed to characterizations of P_+^j, $j \in \{n-1, n-2, \dots, 1\}$, using (29). For example, since $P_-^1 = \mathbb{R}_+^n \cup \mathbb{R}_-^n$, (29) implies that $P_+^{n-1} = \mathbb{R}^n \setminus ((D\mathbb{R}_+^n) \cup (D\mathbb{R}_-^n))$. In other words, P_+^{n-1} is the set of all vectors except for those with either the sign pattern $\begin{bmatrix} \ge 0 & \le 0 & \ge 0 & \dots \end{bmatrix}'$ or the sign pattern $\begin{bmatrix} \le 0 & \ge 0 & \le 0 & \dots \end{bmatrix}'$.

Note that (29) implies that in general the sets P_-^k and P_+^j have a different structure. For example, P_-^k is closed for every k so (29) implies that P_+^j is open for every j. Also, $0 \in P_-^k$ for all $k \in [1, n]$, so $0 \notin P_+^k$ for all $k \in [1, n-1]$.

The next section describes several applications of the notion of k-positive linear systems to the asymptotic analysis of nonlinear dynamical systems.

6. Applications to nonlinear dynamical systems

We begin by considering time-varying nonlinear systems, and then results for the time-invariant case follow as a special case.

Consider the time-varying nonlinear dynamical system:

$$\dot{x}(t) = f(t, x(t)), \tag{30}$$

whose trajectories evolve on a convex invariant set $\Omega \subseteq \mathbb{R}^n$.

We assume throughout that f is C^1 with respect to its second variable x, and that for all $z \in \Omega$ the map $t \to f(t, z)$ is measurable and essentially bounded. Denote the Jacobian of f with respect to its second variable by $J(t, x) := \frac{\partial}{\partial x} f(t, x)$.

For any initial condition $x_0 \in \Omega$ and any initial time $t_0 \in (a, b)$ we assume throughout that (30) admits a unique solution for all $t \ge t_0$ and denote this solution by $x(t, t_0, x_0)$. In what follows we take $t_0 = 0$ and write $x(t, x_0)$ for $x(t, 0, x_0)$.

The application of k-positive linear systems to (30) is based on the variational system associated with (30). To define this, fix $p, q \in \Omega$. Let $z(t) := x(t, p) - x(t, q)$, and for $r \in [0, 1]$, let $\gamma(r) := r x(t, p) + (1 - r) x(t, q)$. Then

$$\begin{aligned} \dot{z}(t) &= f(t, x(t, p)) - f(t, x(t, q)) \\ &= \int_0^1 \frac{\partial}{\partial r} f(t, \gamma(r)) \mathrm{d}r, \end{aligned}$$

and this gives the LTV:

$$\dot{z}(t) = A^{pq}(t) z(t), \tag{31}$$

with

$$A^{pq}(t) := \int_0^1 J(t, \gamma(r)) \mathrm{d}r. \tag{32}$$

This LTV is the *variational system* associated with (30).

Definition 7. We say that the nonlinear system (30) is *[strongly] k-cooperative* if the LTV (31) is [strongly] k-positive for all $p, q \in \Omega$.

The results above can be used to provide simple to verify sufficient conditions for [strongly] k-cooperativity of (30). The next two results demonstrate this.

Corollary 4. *Suppose that there exists $k \in [1, n-1]$ such that $J(t, z) \in M_k^n$ for almost all $t \in (a, b)$ and all $z \in \Omega$. Then (30) is k-cooperative on (a, b). If, furthermore, for any $z \in \Omega$ and any $a < t_0 < t < b$ there exists $\tau \in [t_0, t)$ such that $J(t, z)$ is irreducible at τ^+ then (30) is strongly k-cooperative on (a, b).*

The proof follows from the fact that, by the definition of M_k^n, if $F, G \in M_k^n$ then $F + G \in M_k^n$, and this is carried over to the integration in (32). Also, addition of two matrices in M_k^n cannot change a nonzero entry to a zero entry, so irreducibility is also carried over to the integral.

The next two examples describe specific examples of nonlinear systems that are k-cooperative for some k.

Example 9. Elkhader (1992) studied the nonlinear system

$$\dot{x}_1 = f_1(x_1, x_n),$$

$$\dot{x}_i = f_i(x_{i-1}, x_i, x_{i+1}), \quad i = 2, \dots, n-1,$$
$$\dot{x}_n = f_n(x_{n-1}, x_n). \tag{33}$$

It is assumed that the state-space $\Omega \subseteq \mathbb{R}^n$ is convex, that $f_i \in C^{n-1}$, $i = 1, \dots, n$, and that there exist $\delta_i \in \{-1, 1\}$, $i = 1, \dots, n$, such that

$$\delta_1 \frac{\partial}{\partial x_n} f_1(x) > 0,$$
$$\delta_2 \frac{\partial}{\partial x_1} f_2(x),\ \delta_3 \frac{\partial}{\partial x_3} f_2(x) > 0,$$
$$\vdots$$
$$\delta_{n-1} \frac{\partial}{\partial x_{n-2}} f_{n-1}(x),\ \delta_n \frac{\partial}{\partial x_n} f_{n-1}(x) > 0,$$
$$\delta_n \frac{\partial}{\partial x_{n-1}} f_n(x) > 0,$$

for all $x \in \Omega$. This is a generalization of the monotone cyclic feedback system analyzed in the seminal work of Mallet-Paret and Smith (1990). As noted by Elkhader (1992), we may assume without loss of generality that $\delta_2 = \dots = \delta_n = 1$ and $\delta_1 \in \{-1, 1\}$. Then the Jacobian of (33) has the form

$$J(x) = \begin{bmatrix} * & 0 & 0 & 0 & \dots & 0 & 0 & \operatorname{sgn}(\delta_1) \\ >0 & * & >0 & 0 & \dots & 0 & 0 & 0 \\ 0 & >0 & * & >0 & \dots & 0 & 0 & 0 \\ & & & \vdots & & & & \\ 0 & 0 & 0 & 0 & \dots & 0 & >0 & * \end{bmatrix},$$

for all $x \in \Omega$. Note that $J(x)$ is irreducible for all $x \in \Omega$. If $\delta_1 = 1$ then $J(x)$ is Metzler, so the system is strongly 1-cooperative. Consider the case $\delta_1 = -1$. Then $J(x) \in M_2^n$, so the system is strongly 2-cooperative. (If n is odd then $J(x) \in M_{n-1}^n$, so the system is also strongly competitive.) The main result in Elkhader (1992) is that when $\delta_1 = -1$ the omega-limit set of any bounded solution of (33) includes at least one equilibrium or a periodic orbit. Our main result in this section generalizes this in several ways: first, we allow f_1 [f_n] to depend also on x_2 [x_{n-2}]. Second, we require $f_i \in C^1$ for all i rather than $f_i \in C^{n-1}$ for all i, and third we require $J(x)$ to be irreducible, but not necessarily of the form assumed by Elkhader (1992).

Example 10. Our second example is a system with scalar nonlinearities:

$$\dot{x}(t) = C(t) \begin{bmatrix} f_1(x_1(t)) \\ f_2(x_2(t)) \\ \vdots \\ f_n(x_n(t)) \end{bmatrix}, \tag{34}$$

where $f_i : \mathbb{R} \to \mathbb{R}$, $i \in [1, n]$, are C^1 functions, and $C : (a, b) \to \mathbb{R}^{n \times n}$. Suppose that its trajectories evolve on a compact and convex state-space Ω. The Jacobian of (34) is

$$J(t, x) = C(t) \operatorname{diag}(f_1'(x_1), \dots, f_n'(x_n)), \tag{35}$$

where $f_i'(z) := \frac{d}{dz} f_i(z)$. Pick $p, q \in \Omega$ and consider the line $\gamma(r) := rp + (1-r)q$, $r \in [0, 1]$. Substituting (35) in (32) yields

$$A^{pq}(t) = C(t) \operatorname{diag}(g_1(p_1, q_1), \dots, g_n(p_n, q_n)), \tag{36}$$

where

$$g_i(p_i, q_i) := \begin{cases} \frac{f_i(p_i) - f_i(q_i)}{p_i - q_i}, & \text{if } p_i \neq q_i, \\ f_i'(q_i), & \text{if } p_i = q_i. \end{cases}$$

This implies that for any $k \in [1, n-1]$ it is straightforward to provide sufficient conditions guaranteeing that $A^{[k]}(t)$ is Metzler. To demonstrate this, assume for simplicity that $f_i'(z) > 0$ for all $z \in \mathbb{R}$, $i \in [1, n]$. Then the compactness of Ω implies that there exists $\delta > 0$ such that $g_i(p_i, q_i) \geq \delta$ for all $p, q \in \Omega$, $i \in [1, n]$. Now (36) implies that every entry of $A^{pq}(t)$ satisfies $a_{ij}(t) = c_{ij}(t) m(t)$ with $m(t) \geq \delta$ for all t. Thus, if $C(t) \in M_k^n$ for almost all t then so does $A^{pq}(t)$, and (34) is k-cooperative.

We now describe several applications of k-cooperativity of (30). The first is the existence of certain *explicit* invariant sets. The second application is less immediate and concerns the Poincaré-Bendixson property in strongly 2-cooperative systems.

6.1. Invariant sets

Proposition 10. *Suppose that (30) is k-cooperative. Then for any $p, q \in \Omega$ we have*

$$p - q \in P_-^k \implies x(t, p) - x(t, q) \in P_-^k \text{ for all } t \geq 0. \tag{37}$$

If furthermore $0 \in \Omega$ and 0 is an equilibrium point of (31), i.e. $f(t, 0) = 0$ for all t then

$$p \in P_-^k \implies x(t, p) \in P_-^k \text{ for all } t \geq 0. \tag{38}$$

Proof. Eq. (37) follows from the fact that k-positivity of (31) implies that for any $z(0) \in P_-^k$ we have $z(t) \in P_-^k$ for all $t \geq 0$. Taking $q = 0$ in (37) yields (38). □

If we strengthen the requirement to *strongly* k-cooperativity then we can strengthen (37) to

$$p - q \in P_-^k \setminus \{0\} \implies x(t, p) - x(t, q) \in P_+^k \text{ for all } t > 0,$$

and (38) to $p \in P_-^k \setminus \{0\} \implies x(t, p) \in P_+^k$ for all $t > 0$.

Note that Proposition 9 provides an *explicit* characterization of the invariant sets here as the union of convex sets.

Our next goal is to combine the results in Feng et al. (2017) and Sanchez (2009) with the facts that P_-^2 is 2-solid and its complement $(P_-^2)^c$ is $(n-2)$-solid to establish the Poincaré-Bendixson property for systems that are strongly 2-cooperative. The next remark states a key point that allows us to prove a result that is considerably stronger than that in Sanchez (2009).

Remark 7. Suppose that the nonlinear system $\dot{x} = f(x)$ is 2-cooperative, i.e. $J(x) \in M_2^n$ for all $x \in \Omega$. It follows from the definition of the sets M_k^n that $J(x) \in M_i^n$ for $i = 2, 4, 6, \dots$ and all $x \in \Omega$. Thus, the system is in fact $(2i)$-cooperative for all $i \geq 1$. Similarly, strongly 2-cooperativity implies strongly $(2i)$-cooperativity for all $i \geq 1$.

The framework of k-cooperative systems cannot be used to analyze stability nor boundness. Indeed, consider the LTI $\dot{x} = Ax$. The conditions for k-positivity do not depend on the diagonal entries of A, so the system is k-positive iff $\dot{x} = (cI + A)x$ is k-positive for any $c \in \mathbb{R}$. However, we will see below that 2-cooperativity has important implications on the possible asymptotic behavior of any *bounded* solution. For simplicity, we consider systems whose trajectories evolve on a compact set, so that every solution is bounded. Alternatively, the results hold for *any bounded* trajectory.

6.2. Poincaré-Bendixson property

We begin by recalling some definitions and results by Sanchez (2009). Let $C \subseteq \mathbb{R}^n$ be a k-solid cone. A set $S \subset \mathbb{R}^n$ is called *strongly ordered* if any $v, w \in S$, with $v \neq w$, satisfy $w - v \in \operatorname{int}(C)$. A map $M : \mathbb{R}^n \to \mathbb{R}^n$ is called *positive* if $MC \subseteq C$, and *strongly positive* if $M(C \setminus \{0\}) \subseteq \operatorname{int}(C)$. Consider the time-invariant dynamical system $\dot{x} = f(x)$ and the associated variational equation $\dot{z}(t)$

$= A^{pq}(t)z(t)$, with $A^{pq}(t) := \int_0^1 J(rx(t,p) + (1-r)x(t,q))dr$. The nonlinear system is said to be *C-cooperative* if $A^{pq}(t)$ is strongly positive for all p, q in the state-space and all $t > 0$. A solution $x(t, x_0)$ is called *pseudo-ordered* if there exists a time $\tau \geq 0$ such that $\dot{x}(\tau, x_0) \in \text{int}(C)$. Note that since $z(t) := \dot{x}(t)$ satisfies the variational equation, this implies that $\dot{x}(t, x_0) \in \text{int}(C)$ for all $t \geq \tau$.

The main result in Sanchez (2009) establishes a strong Poincaré-Bendixson property for pseudo-ordered solutions of a C-cooperative system.

Theorem 11 (*Sanchez, 2009*)**.** *Suppose that the dynamical system $\dot{x} = f(x)$ is C-cooperative with respect to a 2-solid cone $C \subseteq \mathbb{R}^n$ whose complement $\text{clos}(\mathbb{R}^n \setminus C)$ is $(n-2)$-solid. Let $x(t, x_0)$ be a solution with a compact omega-limit set $\omega(x_0)$ and suppose that $\dot{x}(\tau, x_0) \in C$ for some $\tau \geq 0$. If $\omega(x_0)$ does not include an equilibrium then it is a closed orbit.*

An important tool in the proof of this result is $\mathcal{P} : \mathbb{R}^n \to W$ the linear projection onto W, parallel to the complement W^c, where W is a 2-dimensional subspace contained in C. Sanchez (2009) proved that if the pseudo-ordered solution is a closed orbit γ then γ is strongly ordered, and deduces that the projection $\mathcal{P}$ of γ is one-to one. He then uses the closing lemma (Arnaud, 1998) to extend the results to pseudo-ordered solutions that are not necessarily closed orbits.

We can now state the main result in this section.

Theorem 12. *Suppose that $\dot{x} = f(x)$ is strongly 2-cooperative. Let $x(t, x_0)$ be a solution with a compact omega-limit set $\omega(x_0)$. If $\omega(x_0)$ does not include an equilibrium then it is a closed orbit.*

This result is considerably stronger than Theorem 11, as it applies to *any* solution with a compact omega-limit set and not only to pseudo-ordered solutions. Note also that the explicit analysis of the set M_2^n can be immediately used to provide a simple condition for strong 2-cooperativity in terms of the sign pattern of the Jacobian $J(x) := \frac{\partial}{\partial x} f(x)$. Note also that we have an explicit expression for a set of vectors that span a 2-dimensional subspace in P_-^2 (in terms of eigenvectors of an oscillatory matrix) and thus an explicit expression for the linear projection $\mathcal{P}$.

The proof of Theorem 12 requires several auxiliary results. The next two results analyze solutions that are closed orbits.

Lemma 6. *Suppose that $\dot{x} = f(x)$ is strongly 2-cooperative. Let γ be a closed orbit corresponding to a periodic solution i.e. $x(t+T, x_0) = x(t, x_0)$ for all $t \geq 0$, where $T > 0$ is the minimal period. Fix an even integer $k \geq 2$. If $\dot{x}(\tau, x_0) \in P_-^k$ for some $\tau \geq 0$ then*

$$x(t_2, x_0) - x(t_1, x_0) \in P_+^k \text{ for all } 0 < t_2 - t_1 < T. \tag{39}$$

Conversely, if $\dot{x}(\tau, x_0) \notin P_-^k$ for all $\tau \geq 0$ then

$$x(t_2, x_0) - x(t_1, x_0) \notin P_-^k \text{ for all } 0 < t_2 - t_1 < T. \tag{40}$$

Proof. Since the system is strongly 2-cooperative, it is in fact strongly $(2i)$-cooperative for all $i \geq 1$. Fix an even integer $k \geq 2$. Suppose that there exists $\tau \geq 0$ such that $\dot{x}(\tau, x_0) \in P_-^k$. Pick $t > \tau$. Since the system is strongly k-cooperative, $\dot{x}(t, x_0) \in P_+^k$, so

$$x(t+\varepsilon, x_0) - x(t, x_0) \in P_+^k \tag{41}$$

for all $\varepsilon > 0$ sufficiently small. Seeking a contradiction, assume that there exist two distinct points $p, q \in \gamma$ such that $p - q \notin P_+^k$. Let τ_1, τ_2 be such that $0 < \tau_2 - \tau_1 < T$, $x(\tau_1, x_0) = q$ and $x(\tau_2, x_0) = p$. Note that by adding a multiple of T to τ_1, τ_2 we may assume that $\tau_1, \tau_2 > \tau$. Combining this with (41) implies that we may actually assume that

$$p - q \in \partial P_+^k \subset P_-^k \tag{42}$$

and since P_+^k is an open set, $p - q \notin P_+^k$. Let $z(t) := x(t,p) - x(t,q)$. Then $\dot{z}(t) = M(t)z(t)$, with $M(t) := \int_0^1 J(rx(t,p) + (1-r)x(t,q))dr$. Note that $M(t)$ satisfies the same sign pattern as J does. Thus, if $z(\tau) \in P_-^k$ for some $\tau \geq 0$ then $z(t) \in P_+^k$ for all $t > \tau$. Eq. (42) implies that $z(0) \in P_-^k$, so $z(t) \in P_+^k$ for all $t > 0$ and in particular $z(T) \in P_+^2$. Thus, $p - q \in P_+^k$. This contradiction implies that for any $p, q \in \gamma$ with $p \neq q$ we have $p - q \in P_+^k$, and this proves (39).

To prove (40), assume that $\dot{x}(t, x_0) \notin P_-^k$ for all t, i.e. $s^-(\dot{x}(t, x_0)) > k - 1$ for all t. Fix $t \geq 0$. Then

$$s^-(x(t+\varepsilon, x_0) - x(t, x_0)) > k - 1 \tag{43}$$

for all $\varepsilon > 0$ sufficiently small. Thus, $x(t+\varepsilon, x_0) - x(t, x_0) \notin P_-^k$ for all $\varepsilon > 0$ sufficiently small. Seeking a contradiction, assume that there exist two distinct points $p, q \in \gamma$ such that $p - q \in P_-^k$. Let τ_1, τ_2 be such that $0 < \tau_2 - \tau_1 < T$, $x(\tau_1, x_0) = q$ and $x(\tau_2, x_0) = p$. Combining this with (43) implies that we may actually assume that $p - q \in \partial P_-^k$, so $p - q \notin P_+^k$, and arguing as above yields a contradiction that proves (40). □

Lemma 7. *Suppose that the system $\dot{x} = f(x)$ is strongly 2-cooperative. Let γ be a closed orbit corresponding to a periodic solution $x(t+T, x_0) = x(t, x_0)$ for all $t \geq 0$, where $T > 0$ is the minimal period. Then there exists an odd integer $\ell \geq 1$ such that*

$$\begin{aligned} \ell - 1 &\leq s^-(x(t_2, x_0) - x(t_1, x_0)) \\ &\leq s^+(x(t_2, x_0) - x(t_1, x_0)) \leq \ell \end{aligned} \tag{44}$$

for all $0 < t_2 - t_1 < T$.

Proof. We consider several cases.

Case 1. Suppose that there exist $\tau \geq 0$ and $k \in \{1, 2\}$ such that $\dot{x}(\tau, x_0) \in P_-^k$. Then (16) implies that $\dot{x}(\tau, x_0) \in P_-^2$ (i.e. $x(t, x_0)$ is pseudo-ordered). Lemma 6 implies that any two distinct points $p, q \in \gamma$ satisfy $p - q \in P_+^2$, so (44) holds with $\ell = 1$.

Case 2. Suppose that Case 1 does not hold, and that there exist $\tau \geq 0$ and $k \in \{3, 4\}$ such that $\dot{x}(\tau, x_0) \in P_-^k$. Then Lemma 6 implies that for any $p, q \in \gamma$ with $p \neq q$ we have

$$p - q \in P_+^4. \tag{45}$$

Since we assume that Case 1 does not hold, $s^-(\dot{x}(t, x_0)) > 1$ for all t, so Lemma 6 implies that $p - q \notin P_-^2$. Combining this with (45), we conclude that $2 \leq s^-(p-q) \leq s^+(p-q) \leq 3$, so (44) holds with $\ell = 3$.

The next case is when Cases 1 and 2 do not hold, and there exist $\tau \geq 0$ and $k \in \{5, 6\}$ such that $\dot{x}(\tau, x_0) \in P_-^k$. A similar argument in this case (and all other cases) completes the proof. □

The next result describes an important application of Lemma 7. Let $e^i \in \mathbb{R}^n$ denote the ith canonical vector in $\mathbb{R}^n$. Let $W^{1n} := \text{span}\{e^1, e^n\}$. This is a two-dimensional subspace that is contained in P_-^2.

Lemma 8. *Suppose that the conditions in Lemma 7 hold. Then the orthogonal projection of γ to W^{1n} is one-to-one.*

Proof. Seeking a contradiction, assume that there exist $p, q \in \gamma$, with $p \neq q$, such that $p_1 - q_1 = p_n - q_n = 0$. It is easy to see that this implies that $s^+(p-q) \geq 2 + s^-(p-q)$. However, this contradicts (44). □

Proof of Theorem 12. Using the fact that strong 2-cooperativity implies strong $2i$-cooperativity for every i, we showed that any periodic solution (and not only pseudo-ordered periodic solutions) can be projected to a two-dimensional subspace in a one-to-one way. Now the remainder of the proof of Theorem 12 follows from the proof of Theorem 11, which appears in Sanchez (2009) as Thm. 1. □

7. Conclusion

Positive dynamical systems are typically defined as systems whose flow maps $\mathbb{R}^n_+$ to $\mathbb{R}^n_+$. In fact, the flow maps the 1-solid cone $P^1_- = \mathbb{R}^n_+ \cup \mathbb{R}^n_-$ to itself. The important asymptotic properties of positive systems follow from the fact that they admit an invariant 1-solid cone. Roughly speaking, this implies that a trajectory can be projected to a one-dimensional subspace and that this projection is generically one-to-one. Hence almost every trajectory that remains in a compact set converges to an equilibrium. The reason that $\mathbb{R}^n_+$ (and $\mathbb{R}^n_-$) are also invariant sets of positive systems is only because the only way to cross from $\mathbb{R}^n_+$ to $\mathbb{R}^n_-$ (or vice versa) is through the origin.

Using tools from the theory of TP matrices and TPDSs, we introduced a generalization called a k-positive LTV. This is a system in the form $\dot{x}(t) = A(t)x(t)$ whose dynamics maps the k-solid cone P^k_- to itself. We showed how this property can be analyzed using the minors of order k of the transition matrix of the LTV. In the case where the matrix in the LTV is a continuous function of time we derived a necessary and sufficient condition for k-positivity in terms of the k'th additive compound of the matrix $A(t)$. This condition is straightforward to verify and, in particular, does not require to calculate the corresponding transition matrix. We also provided an explicit description of every set P^k_- as the union of certain convex cones.

The results for LTVs were applied to define and analyze k-cooperative nonlinear time-varying dynamical systems, that is, systems with a k-positive variational system. Our results provide new tools for the analysis of nonlinear dynamical systems.

We believe that out results can be extended in several directions. First, the theory of positive and cooperative systems has been applied to many types of dynamical systems including those described by ODEs, PDEs, systems with time-delay, difference equations, and more. A promising direction for further research is to extend the notion and applications of k-positivity and k-cooperativity to additional types of dynamical systems, such as those mentioned above, and to dynamical systems that evolve on manifolds (Mostajeran & Sepulchre, 2017). Another possible research direction is the extension of k-positivity to *control systems*.

We analyzed k-positivity with respect to the set P^k_-. Obviously, it is possible that $\dot{x} = Ax$ is not k-positive yet there exists an invertible matrix T such that the dynamical system for $y(t) := Tx(t)$ is k-positive. A systematic analysis of when this is possible can greatly extend the applications of the theory.

Finally, Example 8 illustrates that although P^k_- is a union of the convex sets $C^k_-(v^i)$ and $-C^k_-(v^i)$, we do not know how the solution actually evolves from one convex set to another. A deeper understanding of the sign changes that can take place along the solution may yield stronger analysis results.

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Appendix

Proof of Theorem 2. Suppose that A is nonsingular and SR_k. For $y \in \mathbb{R}$, let $F(y)$ denote the $n \times n$ matrix whose i,j entry is $\exp(-(i-j)^2 y)$. For example, for $n = 3$, $F(y) = \begin{bmatrix} 1 & \exp(-y) & \exp(-4y) \\ \exp(-y) & 1 & \exp(-y) \\ \exp(-4y) & \exp(-y) & 1 \end{bmatrix}$. It is well-known that $F(y)$ is TP for all $y > 0$ (Gantmacher & Krein, 2002, Ch. II), and clearly $\lim_{y\to\infty} F(y) = I$. Fix $y > 0$ and let $F := F(y)$, and $B := FA$. Let α, β denote two sets of k integers $1 \le i_1 < \cdots < i_k \le n$ and $1 \le j_1 < \cdots < j_k \le n$, respectively. The Cauchy–Binet formula yields $B(\alpha|\beta) = \sum_\gamma F(\alpha|\gamma)A(\gamma|\beta)$, where the sum is over all $\gamma = \{p_1, \ldots, p_k\}$, with $1 \le p_1 < \cdots < p_k \le n$. Since F is TP and the minors of order k of A are either all nonnegative or all nonpositive and they are not all zero (as A is nonsingular), we conclude that B is SSR_k. Pick $x \in \mathbb{R}^n$ such that $s^-(x) \le k-1$. If $x = 0$ then clearly $s^-(Bx) \le k-1$. If $x \ne 0$ then Theorem 1 implies that $s^+(Bx) \le k-1$. We conclude that $s^-(Bx) \le k-1$. Taking $y \to \infty$ and using the fact that P^k_- is closed yields (9).

To prove the converse implication, suppose that condition (a) holds, that is, for any $x \in \mathbb{R}^n$ with $s^-(x) \le k-1$, we have $s^-(Ax) \le k-1$. Pick $x \in \mathbb{R}^n \setminus \{0\}$ with $s^-(x) \le k-1$. Since A is nonsingular, $Ax \ne 0$. For any $y > 0$ the matrix $F(y)$ is TP, so $s^+(F(y)Ax) \le s^-(Ax)$, and applying condition (a) yields $s^+(F(y)Ax) \le k-1$. Theorem 1 implies that $F(y)A$ is SSR_k. Taking $y \to \infty$ and using continuity of the determinant, we conclude that A is SR_k. This completes the proof of Theorem 2. □

References

Alseidi, R., Margaliot, M., & Garloff, J. (2019). On the spectral properties of nonsingular matrices that are strictly sign-regular for some order with applications to totally positive discrete-time systems. *Journal of Mathematical Analysis and Applications*, *474*, 524–543.

Arnaud, M.-C. (1998). Le closing lemma en topologie C^1. *Memoires de la Societe Mathematique de France*, *74*, 1–120, URL http://eudml.org/doc/94925.

Ben-Avraham, T., Sharon, G., Zarai, Y., & Margaliot, M. (2019). Dynamical systems with a cyclic sign variation diminishing property. *Institute of Electrical and Electronics Engineers. Transactions on Automatic Control*, *65*(3), 941–954.

Blanchini, F. (1999). Set invariance in control. *Automatica*, *35*(11), 1747–1767.

Byrnes, C. I. (1999). On the global analysis of linear systems. In J. Baillieul, & J. C. Willems (Eds.), *Mathematical control theory* (pp. 99–139). New-York: Springer-Verlag.

Elkhader, A. S. (1992). A result on a feedback system of ordinary differential equations. *Journal of Dynamics and Differential Equations*, *4*(3), 399–418.

Fallat, S. M., & Johnson, C. R. (2011). *Totally nonnegative matrices*. Princeton, NJ: Princeton University Press.

Fallat, S., Johnson, C. R., & Sokal, A. D. (2017). Total positivity of sums, hadamard products and hadamard powers: Results and counterexamples. *Linear Algebra and its Applications*, *520*, 242–259.

Farina, L., & Rinaldi, S. (2000). *Positive linear systems: Theory and applications*. John Wiley.

Feng, L., Wang, Y., & Wu, J. (2017). Semiflows "monotone with respect to high-rank cones" on a Banach space. *SIAM Journal on Mathematical Analysis*, *49*(1), 142–161.

Fiedler, M. (2008). *Special matrices and their applications in numerical mathematics* (2nd ed.). Mineola, NY: Dover Publications.

Gantmacher, F. R., & Krein, M. G. (2002). *Oscillation matrices and kernels and small vibrations of mechanical systems*. Providence, RI: American Mathematical Society, Translation based on the 1941 Russian original.

Haag, G. (2017). *Modelling with the master equation*. Cham, Switzerland: Springer.

Horváth, Z., Song, Y., & Terlaky, T. (2016). Invariance conditions for nonlinear dynamical systems. In B. Goldengorin (Ed.), *Springer Optimization and Its Applications*: *vol. 115*, *Optimization and Its Applications in Control and Data Sciences* (pp. 265–280). Springer, chapter 8.

Johnson, C. R., & Pena, J. M. (2007). Matrices that preserve vectors of fixed sign variation. *Linear and Multilinear Algebra*, *55*(6), 521–533.

Katz, R., Margaliot, M., & Fridman, E. (2020). Entrainment to subharmonic trajectories in oscillatory discrete-time systems. *Automatica*, *116*(108919).

Krasnoselskii, M. A., Lifshitz, E. A., & Sobolev, A. V. (1989). *Positive linear systems: The method of positive operators*. Berlin: Heldermann Verlag.

Kushel, O. Y. (2012). Cone-theoretic generalization of total positivity. *Linear Algebra and its Applications*, *436*(3), 537–560.

Mallet-Paret, J., & Smith, H. L. (1990). The Poincaré-Bendixson theorem for monotone cyclic feedback systems. *Journal of Dynamics and Differential Equations*, *2*(4), 367–421.

Margaliot, M., & Sontag, E. D. (2019a). Compact attractors of an antithetic integral feedback system have a simple structure. http://dx.doi.org/10.1101/868000, bioRxiv, URL https://www.biorxiv.org/content/early/2019/12/08/868000.

Margaliot, M., & Sontag, E. D. (2019b). Revisiting totally positive differential systems: A tutorial and new results. *Automatica*, *101*, 1–14.
Mostajeran, C., & Sepulchre, R. (2017). Differential positivity with respect to cones of rank $k \geq 2$. In *Proc. 20th IFAC World Congress, Vol. 50* Toulouse, France, (pp. 7439–7444).
Muldowney, J. S. (1990). Compound matrices and ordinary differential equations. *The Rocky Mountain Journal of Mathematics*, *20*(4), 857–872.
Oliva, W. M., Kuhl, N. M., & Magalhães, L. T. (1993). Diffeomorphisms of $\mathbb{R}^n$ with oscillatory Jacobians. *Publicacions Matemàtiques*, 255–269.
Pinkus, A. (1996). Spectral properties of totally positive kernels and matrices. In M. Gasca, & C. A. Micchelli (Eds.), *Total positivity and its applications* (pp. 477–511). Dordrecht: Springer Netherlands.
Pinkus, A. (2010). *Totally positive matrices*. Cambridge, UK: Cambridge University Press.
Rantzer, A., & Valcher, M. E. (2018). A tutorial on positive systems and large scale control. In *Proc. 57th IEEE Conf. on Decision and Control* Miami Beach, FL, USA, (pp. 3686–3697).
Sanchez, L. A. (2009). Cones of rank 2 and the Poincaré-Bendixson property for a new class of monotone systems. *Journal of Differential Equations*, *246*(5), 1978–1990.
Sandberg, I. W. (1978). On the mathematical foundations of compartmental analysis in biology, medicine, and ecology. *IEEE Transactions on Circuits and Systems*, *25*(5), 273–279.
Schwarz, B. (1970). Totally positive differential systems. *Pacific Journal of Mathematics*, *32*(1), 203–229.
Smillie, J. (1984). Competitive and cooperative tridiagonal systems of differential equations. *SIAM Journal on Mathematical Analysis*, *15*, 530–534.
Smith, H. L. (1991). Periodic tridiagonal competitive and cooperative systems of differential equations. *SIAM Journal on Mathematical Analysis*, *22*(4), 1102–1109.
Smith, H. L. (1995). Monotone dynamical systems: An introduction to the theory of competitive and cooperative systems. In *Mathematical Surveys and Monographs*: *vol. 41*, Providence, RI: Amer. Math. Soc..
Song, Y. (2015). *Optimization theory and dynamical systems: Invariant sets and invariance preserving discretization methods* (PhD thesis), Lehigh University.
Sontag, E. D. (1998). *Mathematical control theory: Deterministic finite dimensional systems* (2nd ed.). New York: Springer.
Walter, W. (1997). On strongly monotone flows. *Annales Polonici Mathematici*, *66*(1), 269–274.
Weiss, E., & Margaliot, M. (2018). A generalization of smillie's theorem on strongly cooperative tridiagonal systems. In *Proc. 57th IEEE Conf. on Decision and Control* Miami, FL, (pp. 3080–3085).

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