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• I declare that this final submitted version is my unaided work.

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1. a. $\pi \models \varphi R \psi$ iff $\pi[i..\infty] \models \psi$ for all $i \geq 0$ or $\pi[i] \models \varphi$ and $\pi[j...\infty]$ for all $0 \leq j \leq i$ for some $i \geq 0$

b. $\varphi R \psi \equiv (G\psi) \vee (\psi \cup (\varphi \wedge \psi))$

c. $G\psi \vee (\psi \cup (\varphi \wedge \psi)) \overset{def\,1.4}{\Leftrightarrow} \pi[i...\infty] \models \psi$ for all $i \geq 0$

or $\pi[i...\infty] \models \varphi \wedge \psi$ for some $i \geq 0$

and $\pi[j...\infty] \models \psi$ for all $0 \leq j < i$

$\Leftrightarrow \pi[i...\infty] \models \psi$ for all $i \geq 0$

or $\pi[i...\infty] \models \varphi$ and $\pi[i...\infty] \models \psi$ for some $i \geq 0$

and $\pi[j...\infty] \models \psi$ for all $0 \leq j < i$

$\Leftrightarrow \pi[i...\infty] \models \psi$ for all $i \geq 0$ or for some $i \geq 0$

$\pi[i...\infty] \models \varphi$ and $\pi[j...\infty] \models \psi$ for all $0 \leq j \leq i$

So the truth conditions match.

d. $\bot R \psi \equiv (\psi \cup (\bot \wedge \psi)) \vee (G\psi) \equiv \underbrace{(\psi \cup \bot)}_{\equiv \bot} \vee (G\psi) \equiv G\psi$

(because $\bot \vee \phi \equiv \phi$)

(and because $\psi \cup \bot \equiv \bot$ since there is no i such that $\lambda[i...\infty]$ holds)

2. i. $(M,\xi) \vDash EF\phi \Leftrightarrow (M,\xi) \vDash E(\text{true} \cup \phi) \overset{\text{def}}{\Leftrightarrow}$ for some path λ from ξ $(M,\lambda) \vDash \text{true} \cup \phi \Leftrightarrow$ for some path λ starting from ξ, for some $j \geq 0$ $(M, \lambda[i]) \vDash \phi$ and $\underbrace{(M, \lambda[j']) \vDash \text{true for all } 0 \leq j' < j}_{\text{true}}$

$\Leftrightarrow$ for some path λ starting from ξ, for some $j \geq 0$ $(M, \lambda[j]) \vDash \phi$

ii. $(M,\xi) \vDash AF\phi \Leftrightarrow (M,\xi) \vDash A(\text{true} \cup \phi) \overset{\text{def}}{\Leftrightarrow}$ for every path λ from ξ, $(M,\lambda) \vDash \text{true} \cup \phi \Leftrightarrow$ for every path λ from ξ $(M, \lambda[i]) \vDash \phi$ for some $j \geq 0$ and $\underbrace{(M, \lambda[j']) \vDash \text{true fo}}_{\text{true}}$ all $0 \leq j' < j$

$\Leftrightarrow$ for every path λ from ξ $(M, \lambda[j]) \vDash \phi$ for some $j \geq$

iii. $(M,\xi) \vDash EG\phi \Leftrightarrow (M,\xi) \vDash \neg AF\neg\phi \Leftrightarrow (M,\xi) \nvDash AF\neg\phi$ $\overset{\text{def}}{\Leftrightarrow}$ it is not the case that for every path λ from ξ $(M,\lambda[j]) \vDash \neg$ for some $j \geq 0 \Leftrightarrow$ for some path λ from ξ $(M,\lambda[j]) \nvDash \neg\phi$ for all $j \geq 0 \Leftrightarrow$ for some path λ from ξ $(M,\lambda[j]) \vDash \phi$ for all $j \geq 0$

iv. $(M,\xi) \vDash AG\phi \Leftrightarrow (M,\xi) \vDash \neg EF\neg\phi \Leftrightarrow (M,\xi) \nvDash EF\neg\phi \overset{\text{def}}{\Leftrightarrow}$ it is not the case that for some path λ starting from ξ, for some $j \geq 0$ $(M, \lambda[j]) \vDash \neg\phi \Leftrightarrow$ for all paths λ starting from ξ for all $j \geq 0$ $(M, \lambda[j]) \nvDash \neg\phi \Leftrightarrow$ for all paths λ starting from ξ for all $j \geq 0$ $(M, \lambda[j]) \vDash \phi$

3.a. We have the following CTL syntax

$$\phi = a \mid \neg\phi \mid \phi\wedge\phi \mid EX\phi \mid AX\phi \mid E(\phi U \phi) \mid A(\phi U \phi)$$

To show that CTL is a syntactic fragment of CTL* we must show that every CLT formula is also a CLT* formul
We use structural induction on the state formulas.

If ϕ is a CTL* formula (inductive hypothesis) and a CTL formula then so are $\neg\phi$, $\phi\wedge\phi$ as these are exactly the same in the definitions.

If ϕ, ϕ' is a CTL formules and also a CTL* formulas (inductive hypothesis) then we consider:

1. $\phi_1 = EX\phi$ is a CTL* formula by the definition as it has the form $\underline{E\psi}$; $\psi = X\psi'$; $\psi' = \phi$.
2. $\phi_2 = AX\phi$ is a CTL* formula by the definition as it has the form $\underline{A\psi}$; $\psi = X\psi'$; $\psi' = \phi$.
3. $\phi_3 = E(\phi U \phi')$ is a CTL* formula by the definitio as it has the form $\underline{E\psi}$ where $\psi = \psi' U \psi''$, $\psi' = \phi$, $\psi'' = \phi$
4. $\phi_4 = A(\phi U \phi')$ is a CTL* formula by the definitio as it has the form $\underline{A\psi}$ where $\psi = \psi' U \psi''$, $\psi' = \phi$, $\psi'' = \phi'$.

b. $E(Xa \wedge Xb) \in CTL^*$ but $\notin CTL$ (a,b are atoms)

4. By restricting, we obtain:

State formulas:

$(M,s) \models p$ iff $s \in V(p)$ which is equivalent to 1 from Def. 1.7

$(M,s) \models \neg\phi$ iff $(M,s) \not\models \phi$ which is equivalent to 2 from Def. 1.7.

$(M,s) \models \phi \wedge \phi'$ iff $(M,s) \models \phi$ and $(M,s) \models \phi'$ which is equivalent to 3 from Def. 1.7.

$(M,s) \models E\psi \Leftrightarrow$ for some path π starting from s, $(M,\pi) \models \psi$ which is equivalent to 4 from Def 1.7

$(M,s) \models A\psi \Leftrightarrow$ for all paths π starting from s, $(M,\pi) \models \psi$ which is equivalent to 5 from Def. 1.7.

Path formulas:

$(M,\pi) \models X\psi \Leftrightarrow (M,\pi[1 \ldots \infty]) \models \psi$

we restrict ψ to ϕ to respect CTL syntax

$(M,\pi) \models X\phi \Leftrightarrow (M,\pi[1]) \models \phi$ which is equivalent to 1 from Def. 1.8.

$(M,\pi) \models \psi U \psi' \Leftrightarrow (M,\pi[i \ldots \infty]) \models \psi'$ for some $i \geq 0$ and $(M,\pi[j \ldots \infty]) \models \psi$ for all $0 \leq j < i$

we restrict ψ and ψ' to ϕ, ϕ' to respect CTL syntax

$(M,\pi) \models \phi U \phi' \Leftrightarrow (M,\pi[i]) \models \phi'$ for some $i \geq 0$ and $(M,\pi[j]) \models \phi$ for all $0 \leq j < i$ which is equivalent to 2 from Def 1.8.

5. a. By 3a we have shown that every formula in CTL is a formula in CTL*. By 4 we have shown that we obtain we obtain the same definitions as CTL by restricting the CTL* semantics accordingly. So every CTL formula is equivalent to that same formula in CTL*.

5.b. We consider the following LTL formula:

$$F(a \wedge Xa) \quad (1)$$

We know it is not expressible in CTL.
Consider the CTL* formula:

$$A(\text{true}\ U\ (a \wedge Xa)) \quad (2)$$

We prove that formula (1) and (2) are equivalent.

Formula (1)

$M \models F(a \wedge Xa) \overset{\text{Def 1.5}}{\Leftrightarrow} (M, s) \models F(a \wedge Xa)$ for every (initial) state s in M. $\overset{\text{Def 1.5}}{\Leftrightarrow} \lambda \models F(a \wedge Xa)$ for every path λ in M starting from every (initial) state in M $\Leftrightarrow$ for every path λ in M starting from every (initial) state in M for some $i \geq 0$ $\lambda[i \ldots \infty] \models a \wedge Xa \Leftrightarrow$ for every path λ in M starting from every (initial) state in M for some $i \geq 0$ $\lambda[i \ldots \infty] \models a \wedge \lambda[i \ldots \infty] \models Xa$ $\Leftrightarrow$ for every path λ in M starting from every (initial) state in M for some $i \geq 0$ $\lambda[i] \models a \wedge \lambda[i+1] \models a$.

Formula (2)

$M \models A(\text{true}\ U\ (a \wedge Xa)) \Leftrightarrow (M, s) \models A(\text{true}\ U(a \wedge Xa))$ for every s in M $\Leftrightarrow$ for all paths π starting from all states s in M we have $(M, \pi) \models \text{true}\ U\ (a \wedge Xa) \Leftrightarrow$ for all paths π from all states s in M we have $(M, \pi[i \ldots \infty]) \models (a \wedge Xa)$ for some $i \geq 0$ $\underbrace{\text{and } (M, \pi[j \ldots \infty]) \models \text{true for all } 0 \leq j < i}_{\text{trivial / redundant}}$ $\Leftrightarrow$ for all paths π starting from every state s in M for some $i \geq 0$ $(M, \pi[i \ldots \infty]) \models a$ and $(M, \pi[i \ldots \infty]) \models Xa \Leftrightarrow$ for all paths π starting from every state s in M for some $i \geq 0$ $(M, \pi[i]) \models a$ and $(M, \pi[i+1]) \models a$.

So the two formulas (1), (2) are equivalent, so (1) is expressible in CTL* but not in CTL.

6. $(M,t) \approx (M',t')$ and $(M,\pi) \approx (M',\pi')$

We use structural induction on the CTL* syntax.

(*) First we prove that if $(M,t) \approx (M',t')$ then for any path π starting from t there is a path π' starting from t' such that $(M,\pi) \approx (M',\pi')$. (Reverse proof is identical using (c

We use induction:

Base Case: $\pi[0] = t$ and $\pi'[0] = t'$ so $(M,\pi[0]) \approx (M',\pi'[0])$,

Inductive Hypothesis: $(M,\pi[i]) \approx (M',\pi'[i])$

We have $\pi[i+1] \in S_t$, $\pi'[i+1] \in S'_{t'}$ $\pi[i] \to \pi[i+1]$ and $\pi'[i] \to \pi'[i+1]$ and $(M,\pi[i+1]) \approx (M',\pi'[i+1])$ from (b) in Definition 3.

(**) Viceversa (Identical proof, using Def. 3. (c) instead of (b))

Now we start the structural induction.

Base Case (atoms)

$$(M,t) \models p \overset{\text{Def. 2.}}{\Leftrightarrow} t \in V(p) \overset{\text{Def 3.}}{\Leftrightarrow} t' \in V'(p) \overset{\text{Def 2}}{\Leftrightarrow} (M',t') \models p$$

Case $\neg\phi$ - Inductive Hypothesis: $(M,t) \models \phi$ iff $(M',t') \models \phi$

$$(M,t) \models \neg\phi \overset{\text{Def 2}}{\Leftrightarrow} (M,t) \not\models \phi \overset{\text{I.H.}}{\Leftrightarrow} (M',t') \not\models \phi \overset{\text{Def 2}}{\Leftrightarrow} (M',t') \models \neg\phi$$

Case $\phi \wedge \phi'$ - Inductive Hypothesis: $(M,t) \models \phi$ iff $(M',t') \models \phi$
$(M,t) \models \phi'$ iff $(M',t') \models \phi'$

$$(M,t) \models \phi \wedge \phi' \overset{\text{Def 2}}{\Leftrightarrow} (M,t) \models \phi \text{ and } (M,t) \models \phi' \overset{\text{I.H.}}{\Leftrightarrow} (M',t') \models \phi \text{ and } (M',t') \models \phi' \overset{\text{Def 2}}{\Leftrightarrow} (M',t') \models \phi \wedge \phi'$$

Case $E\psi$ - Inductive Hypothesis: $(M,\lambda) \models \psi$ iff $(M',\lambda') \models \psi$ for all λ, λ' such that $(M,\lambda) \approx (M',\lambda')$

$(M,t) \models E\psi \overset{\text{Def 2}}{\Leftrightarrow}$ for some path π starting from t $(M,\pi) \models \psi$
$\overset{\text{I.H.}}{\underset{(*)}{\Leftrightarrow}}$ for some path π' starting from t' (with $(M,\pi) \sim (M,\pi')$ taken from (*))
$(M,\pi') \models \psi \overset{\text{Def 2}}{\Leftrightarrow} (M,t') \models E\psi$

<u>Case $A\psi$</u> - Inductive Hypothesis: $(M,\pi) \models \psi$ iff $(M',\pi') \models \psi$
for all π, π' such that $(M,\pi) \approx (M',\pi')$

$(M,t) \models A\psi \overset{Def2}{\Longleftrightarrow}$ for all paths π starting from t $(M,\pi) \models \psi$ (1)

Consider an arbitrary path π' starting from t'.
Using (**) $\Rightarrow$ there is a path π starting from t such that $(M',\pi') \approx (M,\pi)$.
From (1) $\Rightarrow (M,\pi) \models \psi \underset{I.H.}{\Longleftrightarrow} (M',\pi') \models \psi$
Since π' was arbitrary we have that: for all paths π' starting in t' $(M',\pi') \models \psi \overset{Def2}{\Longleftrightarrow} (M',t') \models A\psi$

<u>Case $\neg\psi$</u> - Inductive Hypothesis: $(M,\pi) \models \psi$ iff $(M',\pi') \models \psi$
$(M,\pi) \models \neg\psi \overset{Def2}{\Longleftrightarrow} (M,\pi) \not\models \psi \overset{I.H.}{\Longleftrightarrow} (M',\pi') \not\models \psi \overset{Def2}{\Longleftrightarrow} (M',\pi') \models \neg\psi$

<u>Case $\psi \wedge \psi'$</u> - Inductive Hypothesis: $(M,\pi) \models \psi$ iff $(M',\pi') \models \psi$
$(M,\pi) \models \psi'$ iff $(M',\pi') \models \psi'$
$(M,\pi) \models \psi \wedge \psi' \overset{Def2}{\Longleftrightarrow} (M,\pi) \models \psi$ and $(M,\pi) \models \psi' \overset{IH}{\Longleftrightarrow} (M',\pi') \models \psi$ and $(M',\pi') \models \psi' \overset{Def2}{\Longleftrightarrow} (M',\pi') \models \psi \wedge \psi'$

<u>Case $X\psi$</u> - Inductive Hypothesis: $(M,\pi) \models \psi$ iff $(M',\pi') \models \psi$
$(M,\pi) \models X\psi \overset{Def2}{\Longleftrightarrow} (M,\pi[1...\infty]) \models \psi \underset{(***)}{\overset{IH}{\Longleftrightarrow}} (M',\pi'[1...\infty]) \models \psi \overset{Def2}{\Longleftrightarrow} (M',\pi') \models X\psi$

<u>Case $\psi U \psi'$</u> - Inductive Hypothesis: $(M,\pi) \models \psi$ iff $(M',\pi') \models \psi$
$(M,\pi) \models \psi'$ iff $(M',\pi') \models \psi'$

$(M,\pi) \models \psi U \psi' \overset{Def2}{\Longleftrightarrow} (M,\pi[i...\infty]) \models \psi'$ for some $i \geq 0$ and $(M,\pi[j...\infty]) \models \psi$ for all $0 \leq j < i \underset{(***)}{\overset{IH}{\Longleftrightarrow}} (M',\pi'[i...\infty]) \models \psi'$ for some $i \geq 0$ and $(M,\pi'[j...\infty]) \models \psi$ for all $0 \leq j < i \overset{Def2}{\Longleftrightarrow} (M',\pi') \models \psi U \psi'$

(***) $(M,\pi) \approx (M',\pi') \Longleftrightarrow (M,\pi[i...\infty]) \approx (M',\pi[i...\infty])$ for all $i \geq 0$

Proof: $(M,\pi) \approx (M',\pi') \overset{Def2}{\Longleftrightarrow}$ for every $j \geq 0$ $(M,\pi[j]) \approx (M',\pi'[j])$
Choose arbitrary $i \geq 0$. We have for every $j \geq i$
$(M,\pi[j]) \approx (M',\pi'[j]) \Rightarrow (M,\pi[i...\infty]) \approx (M',\pi'[i...\infty])$
Since i was arbitrary the result holds for all $i \geq 0$

6. Since we have proven that for arbitrary bisimilar (M, t) and (M', t') and (M, π) and (M', π') the same state and path formulas hold respectively, then we can conclude that the truth of CTL* formulas is preserved by bisimulations. This is because a bisimulation between M and M' applies to every state.

7. We prove that CTL equivalence is a bisimulation.

Condition (a) is trivial: equivalent states satisfy the same atoms.

Condition (b)

Choose $u \in S_t$ with $t \to u$.
Assume for a contradiction that:
for no $u' \in S_{t'}$ with $t' \to u'$
u is CTL equivalent with u' (*)

Let $S' = \{v' \in S_{t'} \mid t' \to v'\}$ (non-empty & finite by (***))
For every $v_i' \in S'$ there exists a formula ϕ_i such that $(M, u) \vDash \phi_i$ but $(M', v_i') \nvDash \phi_i$ (by assumption (*))
(**) $\Rightarrow$ So $(M, t) \vDash E(\phi_1 \wedge \ldots \wedge \phi_n)$ and $(M', t') \nvDash E(\phi_1 \wedge \ldots \wedge \phi_n)$
This contradicts the CTL-equivalence between t and t'.

(**) If $t \to u$ and for $k \le i \le n$ $(M, u) \vDash \phi_i$ then
$(M, u) \vDash (\phi_1 \wedge \ldots \wedge \phi_n) \overset{\text{Def 2}}{\Longleftrightarrow} (M, \pi) \vDash (\phi_1 \wedge \ldots \wedge \phi_n)$ for any path π starting in u. ($\pi[0] = u$)
Consider the path π' with $\pi'[0] = t$ and $\pi'[1 \ldots \infty] = \pi$ (some π)
Then for π' starting in t $(M, \pi') \vDash (\phi_1 \wedge \ldots \wedge \phi_n)$
$\overset{\text{Def 2}}{\Longleftrightarrow} (M, t) \vDash (\phi_1 \wedge \ldots \wedge \phi_n)$

(***) If t is CTL-equivalent with t' and we have $t \to u$ then $(M, t) \vDash E\top$ and then $(M', t') \vDash E\top$ so there must be a $u' \in S_{t'}$ with $t' \to u'$. So $S' = \{v \in S_{t'} \mid t' \to v'\}$ must be non-empty. Also $S' \subseteq S_{t'}$ so it must be finite as $S_{t'}$ is finite.

8. • We first prove that if (M,t) and (M',t') satisfy the same formulas in CTL* then they satisfy the same formulas in CTL.

Assume that there is some formula ϕ in CTL such that $(M,t) \models \phi$ but $(M',t') \not\models \phi$ (for contradiction)

By 5a we know that there is an equivalent formula ϕ' in CTL* so $(M,t) \models \phi'$ but $(M',t') \not\models \phi'$.

This contradicts (M,t) and (M',t') satisfying the same formulas.

So there is no formula in CTL such that $(M,t) \models \phi$ and $(M',t') \not\models \phi$. So (M,t) and (M',t') satisfy the same formulas in CTL as well.

• Next we prove that if (M,t) and (M',t') satisfy the same formulas in CTL they also satisfy the same formulas in CTL*.

By 7, if (M,t) and (M',t') satisfy the same formulas they are CTL-equivalent and therefore bisimilar.

From 6 we know that the truth of CTL* formulas is preserved by bisimulations. So any formula that holds in (M,t) holds in (M',t') and viceversa. So (M,t) and (M',t') satisfy the same formulas in CTL*.

Even though CTL* is strictly more expressive then CTL so ther are formulas in CTL* that cannot be expressed in CTL, the two logics partition the sets of states in the same way. That is, CTL equivalence is the same relation as CTL* equivalence despite the fact that CTL* equivalence entails aditional constraints.