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# Department of Computing

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## Exercise Information

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## Student Declaration - Version 1

- I declare that this final submitted version is my unaided work.

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**For Markers only:** (circle appropriate grade)

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| **VAN BAKEL, Isaac (iv915)** | 01058658 | j4 | 2020-02-19 17:55:37 | **A* A B C D E F** |

# Modal Logic CW 2

iv915

January 2020

1. (a) For a path $\pi$, then $\pi \models \phi \; R \; \psi$ iff either

   - $\pi[i..\infty] \models \phi$ for some $i \geq 0$ and $\pi[j..\infty] \models \psi$ for all $0 \leq j \leq i$, or 2
   - $\pi[i..\infty] \models \psi$ for all $i \geq 0$

   (b)

   $$(\psi \; U \; (\psi \wedge \phi)) \vee \neg(\top \; U \; \neg\psi)$$

   1

   Solution could have been simplified futher

   (c) $\pi \models (\psi \; U \; (\psi \wedge \phi)) \vee \neg(\top \; U \; \neg\psi)$ if and only if $\pi \models \psi \; U \; (\psi \wedge \phi)$ or $\pi \models \neg(\top \; U \; \neg\psi)$

   $\pi \models \psi \; U \; (\psi \wedge \phi)$, iff (from lecture) $\pi[i..\infty] \models \psi \wedge \phi$ for some $i \geq 0$, and $\pi[j..\infty] \models \psi$ for all $0 \leq j < i$. Then $\pi[i..\infty] \models \psi \wedge \phi$ iff $\pi[i..\infty] \models \psi$ and $\pi[i..\infty] \models \phi$. Clearly, this is true iff $\pi[i..\infty] \models \phi$, and $\pi[j..\infty] \models \psi$ for $0 \leq j \leq i$ - which is precisely the first condition.

   $\pi \models \neg(\top \; U \; \neg\psi)$ iff $\pi \not\models \top \; U \; \neg\psi$. $\pi \not\models \top \; U \; \neg\psi$ if, for all $i \geq 0$ such that $\pi[i..\infty] \models \neg\psi$, then there exists $0 \leq j < i$ such that $\pi[j..\infty] \not\models \top$. But by the semantics of $\top$, this cannot be the case - so in fact there exists no such $i \geq 0$ such that $\pi[i..\infty] \models \neg\psi$ - or equivalently $\pi[i..\infty] \not\models \psi$. So this is true iff $\forall i \geq 0$, then $\pi[i..\infty] \models \psi$ - which is precisely the second condition. 3

   Solution correct and very well explained

   (d) We already know that $G\psi \equiv \neg(\top \; U \; \neg\psi)$ (from lecture) - so it would suffice to show that $\psi \; U \; (\psi \wedge \bot) \equiv \bot$. Using $\psi \wedge \bot \equiv \bot$, this can be simplified to showing that $\psi \; U \; \bot \equiv \bot$.

   Let $\lambda$ be any path such that $\lambda \models \psi \; U \; \bot$. Then for some $i \geq 0$, $\lambda[i..\infty] \models \bot$ and for all $0 \leq j < i$, $\lambda[j..\infty] \models \psi$ - but this is a contradiction, so no such $\lambda$ exists. Therefore, the paths satisfying $\psi \; U \; \bot$ and $\bot$ in any model are the same - so $\psi \; U \; \bot \equiv \bot$. 2

   Solution is correct and explained but could have been presented with more clarity. Please clearly separate steps in solutions

2. We first show that $(M, \pi) \models (\top \; U \; \Phi)$ iff for some $j \geq 0$, $(M, \pi[j]) \models \Phi$.

   By definition of $U$, then $(M, \pi) \models (\top \; U \; \Phi)$ iff for some $j \geq 0$, $(M, \pi[j]) \models \Phi$ and $\forall 0 \leq i < j$, $(M, \pi[i]) \models \top$. But by the definition of $\top$, then this second condition always holds - so $(M,\pi) \models (\top\ U\ \Phi)$ iff for some $j \geq 0$, $(M,\pi[j]) \models \Phi$.

- $(M,q) \models EF\Phi$ iff for some path $\lambda$ from $q$, then $(M,\lambda) \models \top\ U\ \Phi$ - but this is iff for some $j \geq 0$, $M(,\lambda[j]) \models \Phi$. 2
  So $(M,q) \models EF\Phi$ iff for some path $\lambda$ from $q$, for some $j \geq 0$, then $(M,\lambda[j]) \models \Phi$.
- $(M,q) \models AF\Phi$ iff for every path $\lambda$ from $q$, then $(M,\lambda) \models \top\ U\ \Phi$ - but this is iff for some $j \geq 0$, $M(,\lambda[j]) \models \Phi$. 2
  So $(M,q) \models AF\Phi$ iff for every path $\lambda$ from $q$, for some $j \geq 0$, then $(M,\lambda[j]) \models \Phi$.
- $(M,q) \models EG\Phi$ iff $(M,q) \not\models AF\neg\Phi$ - so iff it is not true that for every path $\lambda$ from $q$, for some $j \geq 0$, $(M,\lambda[j]) \models \neg\Phi$.
  Then this is not true iff for some path $\lambda$ from $q$, every $j \geq 0$, $(M,\lambda[j]) \not\models \neg\Phi$ - so $(M,\lambda[j]) \models \Phi$. 2
  So $(M,q) \models EG\Phi$ iff for some path $\lambda$ from $q$, for every $j \geq 0$, then $(M,\lambda[j]) \models \Phi$.
- $(M,q) \models AG\Phi$ iff $(M,q) \not\models EF\neg\Phi$ - so iff it is not true that for some path $\lambda$ from $q$, for some $j \geq 0$, $(M,\lambda[j]) \models \neg\Phi$.
  Then this is not true iff for every path $\lambda$ from $q$, every $j \geq 0$, $(M,\lambda[j]) \not\models \neg\Phi$ - so $(M,\lambda[j]) \models \Phi$. 2
  So $(M,q) \models EG\Phi$ iff for every path $\lambda$ from $q$, for every $j \geq 0$, then $(M,\lambda[j]) \models \Phi$.

3. (a) We show this by structural induction on the sufficient connectives $\neg, \wedge, A, E, X, U$, over both path and state formulae.

   If $p \in \mathrm{AP}$ is a state formula of CTL, then by construction $p$ is a state formula of CTL*.

   Assume $\neg\Phi$ is a state formula of CTL. Then $\Phi$ is a state formula of CTL, so $\Phi$ is a state formula of CTL*. So by construction $\neg\Phi$ is a state formula of CTL*.

   Assume $\Phi \wedge \Psi$ is a state formula of CTL. Then $\Phi, \Psi$ are state formulae of CTL - so they are state formulae of CTL*. So by construction, $\Phi \wedge \Psi$ is a state formula of CTL*.

   Assume $E\phi$ is a state formula of CTL. Then $\phi$ is a path formula of CTL - so it is a path formula of CTL*. So by construction $E\psi$ is a path formula of CTL*.

   Assume $A\phi$ is a state formula of CTL. Then $\phi$ is a path formula of CTL - so it is a path formula of CTL*. So by construction $A\psi$ is a path formula of CTL*.

Assume $X\Phi$ is a path formula of CTL. Then $\Phi$ is a state formula of CTL, and so $\Phi$ is a state formula of CTL*. So by construction $X\Phi$ is a path formula of CTL*.

Assume $\Phi\ U\ \Psi$ is a path formula of CTL. Then $\Phi, \Psi$ are state formulae of CTL - so they are state formula of CTL*. So by construction $\Phi\ U\ \Psi$ is a path formula of CTL*.

So by induction, all state and path formulae of CTL are state and path formulae of CTL*, respectively.

Since the formulae of CTL are all and exactly the state formulae, and all state formulae of CTL are state formulae of CTL*, and the formulae of CTL* are all and exactly the state formulae - then CTL is a syntactic fragment of CTL*. 3

(b) $E((\neg Xp) \wedge q)$ - this does not belong to CTL since the contents of the $E$ is a path formula containing conjunction and propositional atoms, both not allowed in CTL. 2

4. Denote the semantic judgement of CTL as $\models_{\text{CTL}}$. We prove by induction that if $\Phi, \psi \in \text{CTL}$, then $(M,s) \models \Phi \iff (M,s) \models_{\text{CTL}} \Phi$, and $(M,\pi) \models \psi \iff (M,\pi) \models_{\text{CTL}} \psi$.

   For the state formula $p$, then the result is trivial, since $(M,s) \models p \iff s \in V(p) \iff (M,s) \models_{\text{CTL}} p$.

   For the state formula $\neg\Phi$, then $(M,s) \models \neg\Phi \iff (M,s) \not\models \Phi$ - and by the inductive hypothesis - $\iff (M,s) \not\models_{\text{CTL}} \Phi \iff (M,s) \models_{\text{CTL}} \neg\Phi$.

   For the state formula $\Phi \wedge \Phi'$, then $(M,s) \models \Phi \wedge \Phi'$ iff $(M,s) \models \Phi$ and $(M,s) \models \Phi'$. By the inductive hypothesis, this is iff $(M,s) \models_{\text{CTL}} \Phi$ and $(M,s) \models_{\text{CTL}} \Phi'$, so iff $(M,s) \models_{\text{CTL}} \Phi \wedge \Phi'$.

   For the state formula $E\psi$, then $(M,s) \models E\psi$ iff for some path $\pi$ starting from $s$, $(M,\pi) \models \psi$. By the inductive hypothesis, this is iff for some path $\pi$ starting from $s$, $(M,\pi) \models_{\text{CTL}} \psi$, and so iff $(M,s) \models_{\text{CTL}} E\psi$.

   For the state formula $A\psi$, then $(M,s) \models A\psi$ iff for every path $\pi$ starting from $s$, $(M,\pi) \models \psi$. By the inductive hypothesis, this is iff for every path $\pi$ starting from $s$, $(M,\pi) \models_{\text{CTL}} \psi$, and so iff $(M,s) \models_{\text{CTL}} A\psi$.

   For the path formula $X\psi$ then for $X\psi$ to be a formula of the CTL fragment, then $\psi = \Phi$. Then $(M,\pi) \models X\Phi$ iff $(M,\pi[1..]) \models \Phi$ - and by definition, this is iff $(M,\pi[1]) \models \Phi$. By the inductive hypothesis, then this is iff $(M,\pi[1]) \models_{\text{CTL}} \Phi$ - which is iff $(M,\pi) \models_{\text{CTL}} X\Phi$.

For the path formula $\psi\ U\ \psi'$, the for this to be a formula of the CTL fragment, then $\psi = \Phi, \psi' = \Phi'$. Then $(M, \pi) \models \Phi\ U\ \Phi'$ iff $(M, \pi[i..\infty]) \models \Phi'$ for some $i \geq 0$, and for all $0 \leq j < i$, then $(M, \pi[j..\infty]) \models \Phi$. By definition this is iff $(M, \pi[i]) \models \Phi'$ for some $i \geq 0$, and for all $0 \leq j < i$, then $(M, \pi[j]) \models \Phi$. By the inductive definition, then this is iff $(M, \pi[i]) \models_{\text{CTL}} \Phi'$ for some $i \geq 0$, and for all $0 \leq j < i$, then $(M, \pi[j]) \models_{\text{CTL}} \Phi$ - which is iff $(M, \pi) \models_{\text{CTL}} \Phi\ U\ \Phi'$. 5

5. (a) Take $\Phi' = \Phi$. By Question 4, the semantics of formulae of CTL* which lie inside the CTL fragment are the same as for the corresponding CTL formula. Therefore, 2

$$(M, s) \models \Phi \text{ (in CTL*)} \iff (M, s) \models \Phi \text{ (in CTL)}$$

   (b) We consider the LTL formula $F(p \wedge Xp)$, which from lectures is not expressible in CTL. 1

   No attempt to justify the example

   This formula is equivalent to the state formula $AF(p \wedge Xp)$ in CTL*.

6. We use the fact that if there exists a bisimulation between $M$ and $M'$, then for every path $\pi \in M$, there exists a bisimilar path $\pi' \in M'$ such that $\pi[0]$ is bisimilar to $\pi'[0]$.

   We can construct a finite prefix of such a path of length $n$ for any choice of $n$, simply taking $\pi'[0]$ to be any world bisimilar to $\pi[0]$ and repeatedly applying the forth condition.

   Using this fact, we can perform the proof by mutual induction.

   For the state formula $p$, as $(M, t)$ and $(M', t')$ are bisimilar, then

$$(M, t) \models p \iff t \in V(p) \iff t' \in V'(p) \iff (M', t') \models p$$

   For the state formula $\neg\Phi$, then $(M, t) \models \neg\Phi$ iff $(M, t) \not\models \Phi$. By the inductive hypothesis, this is iff $(M', t') \not\models \Phi$, so iff $(M', t') \models \neg\Phi$.

   For the state formula $\Phi \wedge \Phi'$, then $(M, t) \models \Phi \wedge \Phi'$ iff $(M, t) \models \Phi$ and $(M, t) \models \Phi'$. By the inductive hypothesis, this is iff $(M', t') \models \Phi$ and $(M', t') \models \Phi'$ - so iff $(M', t') \models \Phi \wedge \Phi'$.

   For the state formula $E\psi$, then $(M, t) \models E\psi$ only if for some $\pi$ starting from $t$, then $(M, \pi) \models \psi$. Then by the above result, there exists $\pi' \in M'$ starting from $t'$ such that $\pi$ and $\pi'$ are bisimilar - so by the inductive hypothesis $(M', \pi') \models \psi$ - and as $\pi'[0] = t'$, then $(M', t') \models E\psi$. Since this argument is symmetric, the result is an iff.

For the state formula $A\psi$, then $(M, t) \models A\psi$ only if for every $\pi$ starting from $t$, then $(M, \pi) \models \psi$. Let $\pi' \in M'$ be a path starting from $t'$- then by the above result there exists $\pi \in M$ starting from $t$ such that $\pi$ is bisimilar to $\pi'$. But as $\pi$ starts from $t$, then $(M, \pi) \models \psi$ - so by the inductive hypothesis, $(M', \pi') \models \psi$. Since $\pi'$ was arbitrary, then $(M', t') \models A\psi$. Since this argument is symmetric, the result is an iff.

For the path formula $\Phi$, then $(M, \pi) \models \Phi$ iff $(M, \pi[0]) \models \Phi$. As $\pi'$ is bisimilar to $\pi$, then $\pi'[0]$ is bisimilar to $\pi[0]$ - so by the inductive hypothesis, this is iff $(M', \pi'[0]) \models \Phi$, and so iff $(M', \pi') \models \Phi$.

For the path formulae $\neg\psi, \psi \wedge \psi'$, the proof is the same as for the state formulae version.

For the path formula $X\psi$, then $(M, \pi) \models X\psi$ iff $(M, \pi[1..\infty]) \models \psi$. As $\pi$ is bisimilar to $\pi'$, then $\pi[1..\infty]$ is bisimilar to $\pi'[1..\infty]$ - so by the inductive hypothesis, this is iff $(M', \pi'[1..\infty]) \models \psi$, and so iff $(M', \pi') \models \Phi$.

For the path formula $\psi \ U \ \psi'$, then $(M, \pi) \models \psi \ U \ \psi'$ only if for some $i \geq 0$, $(M, \pi[i..\infty]) \models \psi'$ and for all $0 \leq j < i$, $(M, \pi[j..\infty]) \models \psi$. As $\pi$ is bisimilar to $\pi'$, then all of these subpaths are bisimilar to their corresponding subpaths in $\pi'$ - so by the inductive hypothesis, this is only if $(M', \pi'[i..\infty]) \models \psi'$ and for all $0 \leq j < i$, $(M', \pi'[j..\infty]) \models \psi$. So $(M', \pi') \models \psi \ U \ \psi'$. Since this argument is symmetric, the result is an iff. 6

All works well justified and written very well. Well Done!

7. ( $\implies$ ) - as in the proof of Theorem 35, it suffices to show that $\leftrightsquigarrow$ is a bisimulation. Assume $u \in M, u' \in M'$ such that $u \leftrightsquigarrow u'$.

   Then

   $$u \in V(p) \iff (M, u) \models p \iff (M', u') \models p \iff u' \in V'(p)$$

   Assume $v \in M$ such that $u \to v$. Assume for a contradiction that there exists no $v' \in M'$ such that $u' \to v'$ and $v \leftrightsquigarrow v'$. By the assumption that the set of states of $M'$ is finite, then the set

   $$S = \{v' \in M \mid u' \to v'\}$$

   is finite. As we only consider infinite paths, it must also be nonempty. Therefore, for each $w \in S$, we can take $\Phi_w$ some state formula in CTL such that $(M, v) \models \Phi_w$ and $(M', w) \not\models \Phi_w$. Then the assertion $\bigwedge_{w \in S} \Phi_w$ has the property that $(M, v) \models \bigwedge_{w \in S} \Phi_w$.

   Considering some path starting at $u$ for which the next state is $v$, we then have that $(M, u) \models EX \bigwedge_{w \in S} \Phi_w$. As this is a formula of CTL, then by the assumption that $u \leftrightsquigarrow u'$, then $(M', u') \models EX \bigwedge_{w \in S} \Phi_w$ - so for some path $\pi$ starting from $u'$, then $(M', \pi[1]) \models \bigwedge_{w \in S} \Phi_w$. But then $\pi[1] \in S$, so in particular $(M', \pi[1]) \not\models \Phi_{\pi[1]}$, so $(M', \pi[1]) \not\models \bigwedge_{w \in S} \Phi_w$ - so we have a contradiction. So in fact there exists some $v' \in M'$ such that $u' \to v'$ and $v \leftrightsquigarrow v'$. So the forth condition is satisfied.

Since the above argument is symmetric, the back condition is also satisfied.

So $\leftrightsquigarrow$ is a bisimulation, and by assumption $t \leftrightsquigarrow t'$, so $(M, t)$ and $(M', t')$ are bisimilar.

$(\Longleftarrow)$ - follows from 4 and 6. If $(M, t)$ and $(M', t')$ are bisimilar, then they satisfy the same CTL* formulae - and so they satisfy the same CTL
formulae in the CTL* fragment, and so they satisfy the same CTL formu- 6
lae.

8. $(\Longrightarrow)$ - Assume $(M, t)$ and $(M', t')$ satisfy the same formulae of CTL. Then by 7, they are bisimilar. Then by 6, $(M, t) \models \Phi \iff (M', t') \models \Phi$, for $\Phi \in$ CTL* - so they satisfy the same formulae of CTL*.

   $(\Longleftarrow)$ - Assume $(M, t)$ and $(M', t')$ satisfy the same formulae of CTL*.

   Then $(M, t) \models_{\text{CTL}} \Phi$, for $\Phi \in$ CTL if and only $(M, t) \models \Phi$ as a formula in the syntactic fragment of CTL* - if and only if $(M', t') \models \Phi$ as a formula in the syntactic fragment, if and only if $(M', t') \models_{\text{CTL}} \Phi$.

   While more formulae could be expressible in CTL* than in CTL, those formulae are not more expressive in a way that can distinguish between
   bisimilar states - and can only distinguish between states which are not 5
   bisimilar, perhaps for the reason that they are already not bisimilar.

46 **Out of** **49**

| 1 | | | |
|---|---|---|---|
| a/**2** | b/**2** | c/**3** | d/**3** |
| | Solution correct and in fully simplified form | Solution correct and very well explained | Required condition shown but relation between the required conditions not made explicit |
| 2 | 1 | 3 | 2 |

| 2 | | | |
|---|---|---|---|
| a/**2** | b/**2** | c/**2** | d/**2** |
| 2 | 2 | 2 | 2 |

| 3 | |
|---|---|
| a/**3** | b/**2** |
| 3 | 2 |

| 4 |
|---|
| **/5** |
| 5 |

| 5 | |
|---|---|
| a/**2** | b/**2** |
| | No attempt to justify the example |
| 2 | 1 |

| 6 | 7 | 8 |
|---|---|---|
| **/6** | **/6** | **/5** |
| All works well justified and written very well. Well Done! | | |
| 6 | 6 | 5 |