Fluid Dynamics

Third year Physics core course (9 lectures)

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Contents

1 Introduction **5**
1.1 What is Fluid Dynamics? . . . 5
1.2 Fluids, Liquids and Gases . . . 5

2 Kinematics **6**
2.1 Flow visualisation . . . 6
2.2 Lagrangian and Eulerian descriptions of motion . . . 8
2.2.1 Compressible and incompressible fluids . . . 8
2.2.2 Lagrangian description . . . 8
2.2.3 Eulerian description . . . 8
2.3 The material derivative . . . 9

3 Conservation Laws **10**
3.1 Navier-Stokes equation . . . 10
3.2 Drag and ram pressure . . . 14
3.3 Viscosity . . . 16
3.4 Reynolds number . . . 18
3.5 Bernoulli's principle . . . 18
3.5.1 Derivation of Bernoulli's principle . . . 19
3.6 Applications of Bernoulli's principle . . . 21
3.6.1 Venturis . . . 21
3.6.2 The Magnus effect . . . 22
3.6.3 Cavitation . . . 24

4 Types of flows **25**
4.1 Ideal flows . . . 25
4.2 Continuity Equation . . . 25
4.3 Irrotational Flows . . . 26
4.3.1 Incompressible flow . . . 26
4.3.2 Incompressible flow . . . 26
4.3.3 Laminar and turbulent flows . . . 26
4.4 Rotational flows . . . 27
4.4.1 Vorticity propagation . . . 27
4.4.2 Vorticity equation . . . 29
4.4.3 Types of vortex . . . 31
4.4.4 Flow past a cylinder . . . 33
4.4.5 Wingtip vortices . . . 34
4.4.6 Cyclones . . . 35

5 Geophysical Fluid Dynamics **39**
5.1 Introduction . . . 39

5.2 Equations of motion 40
5.2.1 Rotating frame 41
5.2.2 Equation of motion in a rotating frame 44
5.2.3 The centrifugal acceleration term 46
5.2.4 Rossby number 47
5.3 Geostrophic flow 48
5.4 Thermal wind 49

Recommended Literature:

No single textbook will be ideal to cover all the material. This lecture course has benefited from many sources, and below I list merely those sources which I found particularly useful.

1. G. K. Batchelor: **An introduction to fluid dynamics**, Cambridge University Press, 1973/2000: This is a classic textbook, more wide reaching than necessary for the present course, but excellent for looking up individual topics and studying them in more depth.

2. P. K. Kundu: **Fluid Mechanics**, Academic Press, 2002 (2nd Ed.). This is an excellent text book which much of the material presented in this course is derived from.

3. M. C. Potter and D. C. Wiggert: **Mechanics of fluids**, Brooks/Cole, 2002 (3rd Ed.). A text book with many good illustrations, more hands-on than the above two books.

4. G. K. Vallis: **Atmospheric and Oceanic Fluid Dynamics: Fundamentals and Large-scale Circulation**, Cambridge University Press, 2006: This book is specifically about the Earth's atmosphere and ocean, so dealing mostly with Geophysical Fluid Dynamics.

Recommended Web Links:

Fluid dynamics is often best illustrated through animations or films. Some important web resources with such material are listed below, many more exist.

1. **http://web.mit.edu/hml/ncfmf.html** : Films produced by the National Committee for Fluid Mechanics Films (NCFMF). They "look old" but are amongst the best illustrations available of the key concepts of Fluid dynamics.

A brief foreword

The following is the lecture script for the *Fluid dynamics* core year 3 Physics course of the Department of Physics at Imperial College London. The course was introduced in 2014 but not all material is new, I am indebted to Prof. Jeremy Chittenden who provided material from his original *Hydrodynamics and Shocks* course, some of which is used here.

Fluid Dynamics – while an applied discipline – is fundamental to so many areas of physics and other areas, from the study of small-scale liquid flow in organisms to water flow through pipes, the study of aerodynamics of cars and planes to understanding the circulation of oceans and atmospheres on our planet as well as other bodies in the solar system. The basic principles of Fluid Dynamics equally apply to Magnetohydrodynamics and allow us to understand large structures such as the Sun and solar wind. So, understanding the behaviour of mediums which behave like fluids is a truly universal discipline!

Please email me if anything doesn't appear to make sense, while I have made every effort to eliminate mistakes some may still be hidden in this document. Any changes that I make to this script will be uploaded to Blackboard and announced in the lectures.

The course – due to its brevity – can only skim over the main topics of Fluid Dynamics. Still, it will hopefully be a useful introduction to this fascinating (and useful!) subject and trigger the motivation for further study!

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Accompanying this script are *Lecture Slides* and 3 *Problem Sheet*. Problem Sheet questions relevant to any part of the script are indicated in grey boxes.

Indicators are added on the right margin which show what material is covered in what lecture. See those as a rough guidance only, things may shift slightly in practice.

Finally, you will notice that this script explains some aspects in more depth (and with more equations) than can reasonably be covered during the lectures. At the same time, the format of lectures allows me to explain some aspects more vividly than is possible in a script. Hence, I recommend that you consult both the script and the lectures (live and/or on Panopto) for an optimum learning outcome. Problem sheet are essential for exploring the theoretical concepts explored here with more concrete calculations.

1 Introduction

Lecture 1

1.1 What is Fluid Dynamics?

Fluid dynamics numerically treats the flow of material (fluids). The main purpose in this is to understand behaviour of fluids on a range of scales, from small (flow of blood in organisms, swimming of fish, the flight of birds, flow around cars and aeroplanes) to large (motion of atmospheric gases or ocean water). Fluid Dynamics is of interest both to fundamental scientists as well as to engineers, with the goal of understanding how nature works and of designing machines that most efficiently achieve any desired practical purpose.

1.2 Fluids, Liquids and Gases

Most substances exist in three possible states, solid, liquid and gas. Fluid dynamics is concerned with the state and behaviour of liquids and gases. Together, we often refer to these as fluids. There is no fundamental difference in the physics describing liquids and gases except for the issue of compressibility/incompressibility (see below).

Fluid dynamics limits itself to the study of liquids and gases that move under the action of shear stress. Substances that do not move under shear stress are subject of *rheology.*

Gases tend to be hydrostatically *compressible* whereas liquids tend to be hydrostatically *incompressible.* We shall see later (Section 4.3.1) that the behaviour of gases can to a reasonable approximation be treated as incompressible. *This lecture course shall therefore focus on the treatment of incompressible fluids.*

Fluid dynamics does not study individual particles – this would for most situations be impossible in practice – but the formalism rather looks at their *collective* behaviour. This assumes particles interacting with one another, most commonly via collisions, but in other theories such as Magneto-Hydrodynamics (MHD) this interaction can instead occur via electro-magnetic interaction between the (plasma-) particles. Assuming particle interaction by collisions or other forces, the Navier-Stokes equations are approximations of the Boltzmann equation, assuming a Maxwellian energy distribution of the particles. This course will not be treating the Boltzmann equation.

2 Kinematics

2.1 Flow visualisation

To begin with, we need to introduce a number of key concepts frequently encountered in Fluid Dynamics. Given that we are not looking at individual particles we need to define what structures of a fluid can be used for its description and visualisation.

Consider the situation in Figure 2.1. At any moment in time a point in a fluid element has a velocity vector with specific direction.

- *Pathlines* are the trajectories (paths) of fluid particles over a period of time. They are sometimes also called *flowlines*. If the flow is steady (not changing in time) all particles emanating from a point follow the same pathlines. You need to follow particles over time to determine their pathlines (as opposed to what is the case with streaklines, see below).
- *Streamlines* are curves formed by tangents in the direction of velocity flow vectors at any one time. This is illustrated in Figure 2.1. In other words, a streamline is a curve which is tangent everywhere to the local, instantaneous velocity field.
- *Streaklines* show the line formed by particles that are continuously introduced at a certain point, as illustrated in Figure 2.2. An example is dye continually released from the end of a fine tube, or smoke from a chimney. Streaklines thus represent a snapshot at one moment in time of the position of all particles that originated from the same point. If you take a photograph of a smoking chimney, the pattern of smoke that you see are streaklines.

For *steady* flow, the streamlines will always look the same, at every instant in time. In this case, the pathlines match the streamlines.

If the flow is *unsteady*, the streamlines will change from one instant in time to the next, so streamlines and flowlines do not match. This is illustrated in Figure 2.3 which shows a body moving from right to left in a fluid which at infinite distance from the object is stationary. The flow pattern of the fluid seen by an observer who is stationary with respect to the undisturbed fluid changes with time, so it appears as an unsteady flow to this observer. In front of the body and behind it, the streamlines (solid) are directed towards the left, the direction of motion of the body. On the sides (above and below the body) they are essentially lateral. If we look at the motion of an individual particle / fluid element (labelled with "P" in Figure 2.3) then its pathline is as shown in the figure, looping outward and forward again as the body passes by the fluid element.

Note that a flow which is steady in one reference frame isn't necessarily so in another.

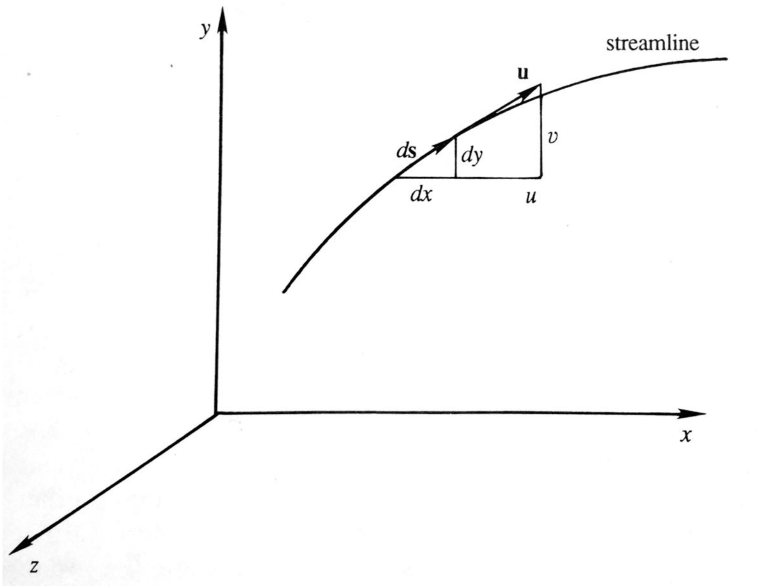

Figure 2.1: Streamline (Credit: Kundu, 2002)

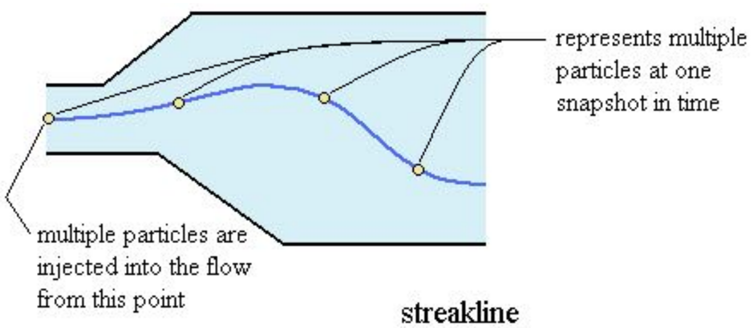

Figure 2.2: Streaklines (Credit: Engineering Archives)

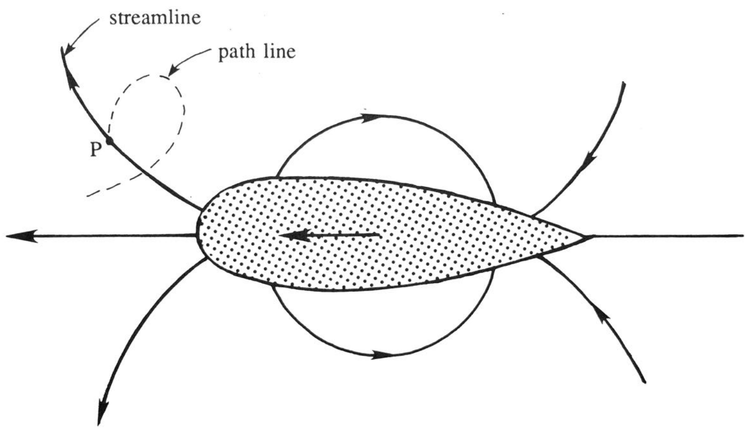

Figure 2.3: Several streamlines (solid) and a path line (dashed) due to a body moving in a fluid. (Credit: Kundu, 2002)

2.2 Lagrangian and Eulerian descriptions of motion

2.2.1 Compressible and incompressible fluids

When numerically describing a flow field, it is convenient to consider small parcels of fluid containing many molecules. If the fluid is *incompressible* the volume does not change in magnitude but it may deform. If the fluid is *compressible* the volume deforms and changes in magnitude. For subsonic flow compressible fluids can often be treated mathematically in the same way as incompressible fluids, as discussed in Section 4.3.1.

2.2.2 Lagrangian description

When studying particle motion we may define their position, velocity and acceleration at point (x_0, y_0, z_0, t) as $\underline{\mathbf{r}}(x_0, y_0, z_0, t)$, $\underline{\mathbf{u}}(x_0, y_0, z_0, t)$ and $\underline{\mathbf{a}}(x_0, y_0, z_0, t)$. Here, (x_0, y_0, z_0, t) represents the starting point of each particle. This is the *Lagrangian* description. While useful for small numbers of particles it becomes cumbersome for very large numbers of particles.

2.2.3 Eulerian description

As an alternative to following each particle individually, we may define specific points in space and observe the properties (velocities, accelerations) of particles passing these specific points. This is the *Eulerian* description of motion and flow properties. In Cartesian coordinates the velocity is expressed as $\underline{\mathbf{u}} = \underline{\mathbf{u}}(x, y, z, t)$ where x, y, z are the coordinates of the fixed points in space. At each point we can also define the change of velocity at each point with space, $\partial\underline{\mathbf{u}}/\partial x$, $\partial\underline{\mathbf{u}}/\partial y$, $\partial\underline{\mathbf{u}}/\partial z$ and with time, $\partial\underline{\mathbf{u}}/\partial t$. The region of flow investigated in this way is often referred to as the *flow field.*

Examples:

As an example to illustrate the different approaches, imagine you want to study traffic flow in London. You can either hire people to drive through town in cars and record the appropriate parameters (Lagrangian approach) or you can you can hire people to stand at certain intersections throughout town and record the required information (Eulerian approach). Both approaches should lead to the same result, each situation will have one approach be more suitable than the other. In fluid dynamics the Eulerian approach is used most commonly but in some instances (e.g., the study of drifting buoys to study ocean currents) the Lagrangian approach is used.

In section 2.1 we referred to *steady* flow. This is flow where the quantities of interest, e.g. the velocity field $\underline{\mathbf{u}}$, are independent of time at any fixed point, $\underline{\mathbf{u}} = \underline{\mathbf{u}}(x, y, z)$. Thereby, in a steady flow we have $\partial\underline{\mathbf{u}}/\partial t = 0$. This applies to the Eulerian description. Whereas in a steady flow the properties of fluid particles at any given point in space are constant with time, the properties of individual particles do in general vary with time, so in a Lagrangian description their properties could still vary with time (especially if the flowlines are curved). This illustrates for the case of steady flows that a Eulerian formulation may yield simpler expressions (since the time variation falls out).

2.3 The material derivative

In describing the change of any field variable C (e.g., temperature, velocity, stress) we seek to calculate the change of C at each Lagrangian point (x, y, z, t) by following a particle of fixed identity. This is in essence a Lagrangian concept which we try to express in Eulerian language.

Let the position vector be $\underline{\mathbf{r}} = (x, y, z)\ \underline{\hat{\mathbf{r}}}$. For arbitrary and independent changes, $d\underline{\mathbf{r}}$ and dt, the change in C becomes:

$$dC \;=\; \frac{\partial C}{\partial t}dt \;+\; \frac{\partial C}{\partial x}dx + \frac{\partial C}{\partial y}dy + \frac{\partial C}{\partial z}dz \qquad (2.1)$$

What if the increments are not independent and arbitrary but associated with a particle of fixed identity? Then $\underline{\mathbf{r}}$ and dt are no longer independent but related by

$$\begin{aligned} dx &= u_x\, dt \\ dy &= u_y\, dt \\ dz &= u_z\, dt \end{aligned} \qquad (2.2)$$

Substituting (2.2) into (2.1) gives:

$$\frac{dC}{dt} \;=\; \frac{\partial C}{\partial t} \;+\; u_x\frac{\partial C}{\partial x} + u_y\frac{\partial C}{\partial y} + u_z\frac{\partial C}{\partial z} \qquad (2.3)$$

It is important to distinguish dC/dt from $\partial C/\partial t$. This difference is often emphasised by writing it as DC/Dt instead of dC/dt, but both represent the same. The total rate of change d/dt (or D/Dt) is called the *material derivative* and consists of two parts, the *local* rate of change $\partial/\partial t$ and the *advective* change $u_i\,\partial/\partial r_i$ (where $u_i = u_x$ or u_y or u_z and $r_i = x$ or y or z). In other words, the rate of change in a moving frame equals the advective term + the rate of change at a fixed point. Sometimes the advective change is also referred to as the *convective* change.

We may write (2.3) in vector notation:

$$\frac{dC}{dt} \;=\; \frac{\partial C}{\partial t} \;+\; (\underline{\mathbf{u}} \cdot \nabla)C \tag{2.4}$$

The scalar product $(\underline{\mathbf{u}}\cdot\nabla)C$ is the magnitude of the velocity vector $\underline{\mathbf{u}}$ times the component of ∇C in the direction of $\underline{\mathbf{u}}$.

3 Conservation Laws

Lecture 2

3.1 Navier-Stokes equation

The Navier-Stokes momentum equation describes the motion of fluid under the influence of forces and acceleration. When speaking of "the" Navier-Stokes equation, the momentum equation is normally meant. However, there are several Navier-Stokes equations which all form a coupled set of equations, the momentum equation (which we discuss here) and the energy and continuity equations. We will briefly discuss the continuity equation (Section 4.2) but within this short course cannot discuss the Navier-Stokes energy equation (the Bernoulli equation, Section 3.5, is an energy equation but for highly idealised conditions only).

Consider pressure related forces on a fluid element as illustrated in Figure 3.2. We assume pressure to change with x only, so if the pressure acting at x differs from that at $x + \Delta x$ we obtain a net pressure – and thereby force – acting on the fluid element. Pressure is force per unit area, so $F = PA$ and in the x direction we have $A = \Delta y \Delta z$. The net force acting upon the fluid element in the x direction due to changing pressure in the fluid is:

$$F_x \;=\; m\,a_x \;=\; \rho\,\Delta x\,\Delta y\,\Delta z\,\frac{du_x}{dt} \;=\; P(x)\,\Delta y\,\Delta z \;-\; P(x+\Delta x)\,\Delta y\,\Delta z \tag{3.1}$$

where we also used that the mass density ρ is $\rho = m/V = m/(\Delta x \Delta y \Delta z)$ and the acceleration $a_x = du_x/dt$. Rearranging (3.1) gives:

$$\rho\,\frac{du_x}{dt} \;=\; \frac{P(x) \;-\; P(x+\Delta x)}{\Delta x} \tag{3.2}$$

Letting $\Delta x \to 0$ gives:

$$\rho\,\frac{du_x}{dt} \;=\; -\frac{\partial P}{\partial x} \tag{3.3}$$

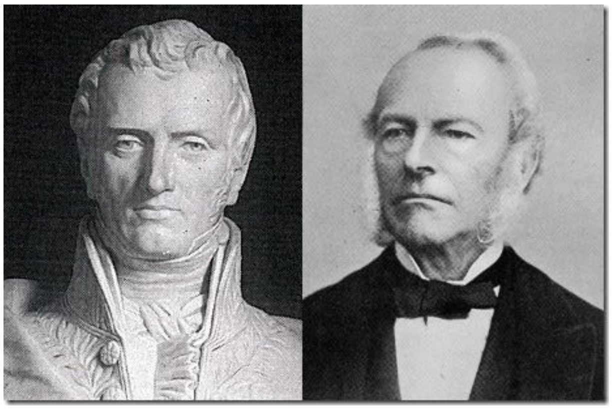

Figure 3.1: Claude-Louis Navier (1785–1836, left) and George Gabriel Stokes (1819–1903, right), the fathers of the Navier-Stokes equations

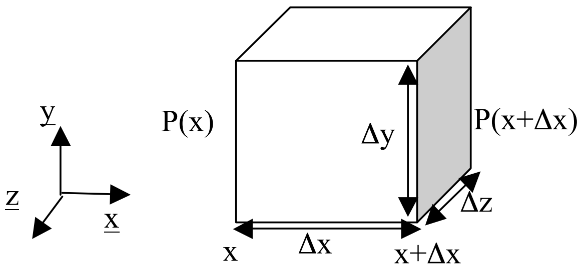

Figure 3.2: A fluid element under the influence of a pressure change in the x direction.

Written in 3D this gives:

$$\rho \frac{d\underline{\mathbf{u}}}{dt} = -\nabla P \tag{3.4}$$

This can be regarded as a momentum equation describing the change of velocity of a fluid as a result of internal changes of pressure. The right side of (3.4) is the acceleration (to be exact, the force per unit volume) which causes the velocity change (left side).

In order to apply (3.4) to the atmosphere or ocean we must include more acceleration terms on the right side, *gravity* and *viscosity*.

The contribution due to gravity is given by $-\rho\, \underline{\mathbf{g}}$ where $\underline{\mathbf{g}}$ is the gravity acceleration (on Earth the average magnitude of gravity acceleration is $g = 9.81$ m/s^2).

The contribution due to viscous forces is expressed as $\eta\ \nabla^2 \underline{\mathbf{u}}$ where η is the *dynamic viscosity* coefficient (unit: kg m^{-1}s^{-1}), a quantity dependent on the material and on temperature. (see Section 3.3).

Adding these terms yields the Navier-Stokes momentum equation:

$$\rho \frac{d\underline{\mathbf{u}}}{dt} = -\nabla P + \rho\, \underline{\mathbf{g}} + \eta\, \nabla^2 \underline{\mathbf{u}} \tag{3.5}$$

Finally, recall from Section 2.3 that the left side of (3.5) is the material derivative of velocity as previously expressed in equation (2.4). We may thus rewrite (3.5) as:

$$\boxed{\rho \frac{\partial \underline{\mathbf{u}}}{\partial t} + \rho\, (\underline{\mathbf{u}} \cdot \nabla)\underline{\mathbf{u}} = -\nabla P + \rho\, \underline{\mathbf{g}} + \eta\, \nabla^2 \underline{\mathbf{u}}} \tag{3.6}$$

which is the Navier-Stokes momentum equation in an inertial reference frame. The physical units on both sides (in S.I.) are (N/m^3), i.e., force per unit volume. The equation can also be divided by ρ on both sides to yield a balance of accelerations (m/s^2) instead.

Section 5.2.2 will introduce the Navier-Stokes equation for a non-inertial frame of reference, specifically a rotating frame of reference (such as Earth), adding two further acceleration terms on the right side, the centrifugal and Coriolis accelerations (Eq. 5.13).

The above form of the momentum equation can be rewritten in a slightly different way without changing its content. We may rewrite the advection term $(\underline{\mathbf{u}} \cdot \nabla)\underline{\mathbf{u}}$ using a standard vector identity

$$\nabla(\underline{\mathbf{A}} \cdot \underline{\mathbf{B}}) = \underline{\mathbf{A}} \times (\nabla \times \underline{\mathbf{B}}) + \underline{\mathbf{B}} \times (\nabla \times \underline{\mathbf{A}}) + (\underline{\mathbf{A}} \cdot \nabla)\underline{\mathbf{B}} + (\underline{\mathbf{B}} \cdot \nabla)\underline{\mathbf{A}} \tag{3.7}$$

So:

$$\nabla(\underline{\mathbf{u}} \cdot \underline{\mathbf{u}}) = \mathbf{2}\, \underline{\mathbf{u}} \times (\nabla \times \underline{\mathbf{u}}) + \mathbf{2}\, (\underline{\mathbf{u}} \cdot \nabla)\underline{\mathbf{u}} \tag{3.8}$$

Giving:

$$\begin{aligned}(\underline{\mathbf{u}}\cdot\nabla)\underline{\mathbf{u}} &= -\underline{\mathbf{u}}\times(\nabla\times\underline{\mathbf{u}}) + \frac{\mathbf{1}}{\mathbf{2}}\nabla(\underline{\mathbf{u}}\cdot\underline{\mathbf{u}}) \\ &= (\nabla\times\underline{\mathbf{u}})\times\underline{\mathbf{u}} + \frac{\mathbf{1}}{\mathbf{2}}\nabla(\underline{\mathbf{u}}\cdot\underline{\mathbf{u}})\end{aligned} \tag{3.9}$$

The term $\nabla\times\underline{\mathbf{u}}$ (the curl of the velocity vector) is known as the *vorticity* and describes how much spin the fluid has (see also Section 4.4) and is a microscopic measure of rotation at a given point in the fluid.

The Navier-Stokes momentum equation in an inertial reference frame may therefore alternatively be written in the form

$$\boxed{\rho\,\frac{\partial\underline{\mathbf{u}}}{\partial t} = -\nabla P + \rho\,\underline{\mathbf{g}} + \eta\,\nabla^2\underline{\mathbf{u}} - \frac{1}{2}\rho\,\nabla(u^2) - \rho\,(\nabla\times\underline{\mathbf{u}})\times\underline{\mathbf{u}}} \tag{3.10}$$

So, total force (per volume) = pressure gradient + gravity + viscosity + inertia + vorticity.

So, what happened between (3.6) and (3.10)? We simply split up the total advection term of (3.6) into two components, linear convection ("inertia") and rotational motion ("vorticity") in (3.10). This decomposition into the linear and rotational components or momentum transport is useful in some situations. In terms of core physics contents, though, both (3.6) and (3.10) are identical.

See also: Problem Sheet 2, Question 3

A word about the solution of Navier-Stokes equation:
For simplified forms (often steady state and linearised) of the Navier-Stokes momentum equation, analytical solutions exist. In more complex applications (the majority of cases), the equation is however solved *numerically* (i.e., computationally): the change ($\partial\underline{\mathbf{u}}/\partial\mathbf{t}$) is calculated, added to a background value via time integration, then for the next time step ($\partial\underline{\mathbf{u}}/\partial\mathbf{t}$) is recalculated, and so on. Various numerical schemes are applied, depending on the problem at hand. This is a separate topic of investigation beyond the scope of this course.

Little curiosity:
No one has yet proven that in three dimensions a solution of the Navier-Stokes equation always exist ("existence"), or if it does exist, that it does not contain any singularity (that it is "smooth"). These two are referred to as the Navier-Stokes existence and smoothness problems. In fact, the Clay Mathematics Institute has called this one of the seven most important open problems in mathematics and has offered a US$1,000,000 prize for a solution or a counter-example. Click here if you are curious:
http://www.claymath.org/millennium-problems/navier-stokes-equation

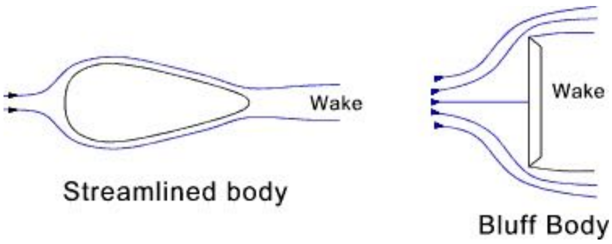

Figure 3.3: Illustration of a body experiencing low drag (left side) and large drag (right).

3.2 Drag and ram pressure

Consider the horizontal movement of a fluid along the x direction, ignoring gravity and viscosity. The momentum balance according to Equation 3.6 is then given by

$$\rho \frac{\partial u_x}{\partial t} + \rho u_x \frac{\partial u_x}{\partial x} = -\frac{\partial P}{\partial x} \tag{3.11}$$

If we assume the fluid to be incompressible (constant ρ) then we may rewrite the equation as

$$\frac{\partial}{\partial t}(\rho u_x) + \frac{\partial}{\partial x}\left(\frac{1}{2}\rho u_x^2\right) = -\frac{\partial P}{\partial x} \tag{3.12}$$

or

$$\frac{\partial}{\partial t}(\rho u_x) = -\frac{\partial P}{\partial x} - \frac{\partial}{\partial x}\left(\frac{1}{2}\rho u_x^2\right) \tag{3.13}$$

The first term on the right side is the gradient of thermal pressure P along x. The second term is commonly referred to as the *ram pressure* or *dynamic pressure* of the fluid. This is associated with the bulk movement of the liquid in the x direction and simply results from momentum conservation, or inertia, of the fluid in the x direction. Therefore, the ram pressure acts only in the direction of bulk movement of a fluid. In contrast, thermal pressure results from random motion of particles inside a fluid and acts isotropically.

The deceleration (drag) of an object moving through a fluid depends on how much the streamlines must deviate in flowing around object, as illustrated in Figure 3.3. Lecture 3

To calculate the drag on any object placed into a flow, we define a drag coefficient C_D such that

$$F_{drag} = -\frac{1}{2}\rho u^2 A C_D \tag{3.14}$$

Where A is the surface area perpendicular to the flow and F_{drag} is the drag force.

See also: Problem Sheet 1, Question 1

Thus, the expression for drag on an object (3.14) is very similar to the expression of ram pressure, but multiplied by the drag coefficient which depends on the geometry of the

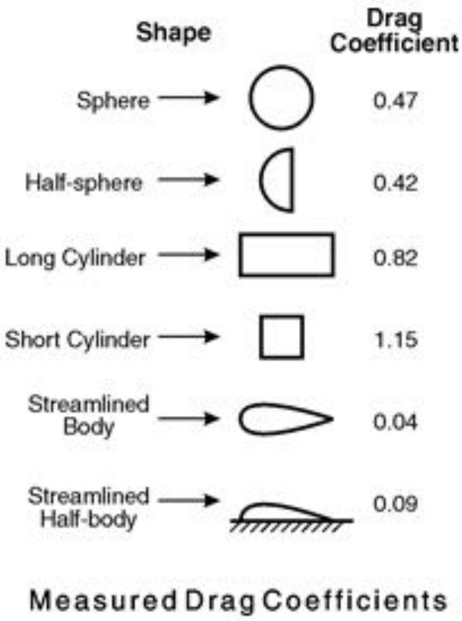

Figure 3.4: Measured drag coefficients C_D for various shapes in a fluid with Reynolds number $Re \approx 10^4$ (see Section 3.4 for the definition of Reynolds number).

object. Like ram pressure, the drag force increases with u^2 and thus strongly depends on the flow velocity, u. To understand this relationship, we may more accurately calculate drag by considering each colliding air molecule imparting a momentum $m\,u$ to an object. The number of particles per second is $\propto u$, so the momentum exchange per unit volume is $\propto \rho u^2$ but this assumes all momentum being lost. In fact, the fraction of momentum lost depends on the shape of the streamlines, expressed via C_D.

C_D (unitless) depends on the shape of the flow around the object and can be measured (e.g. in a wind tunnel) or calculated by computer modelling of airflow using the Navier-Stokes equation. In practice, C_D also needs to include other effects such as skin friction and viscosity. Examples of C_D values are shown in Figure 3.4.

We may write energy = force · distance and power = force · velocity = force · distance in 1 sec. Therefore, the power required to maintain constant velocity is

$$\text{Power} \;=\; \frac{1}{2}\,\rho\, u^3 A\, C_D \tag{3.15}$$

Examples:

A car cruising at 50 mph (22 m/s) may require only 10 horsepower (7.35 kW) depending on AC_D, but at 100 mph it requires 80 hp (59 kW).

Say, $AC_D = 1$ m^2, engine power ≈600 kW (805 horsepower!), air density = 1.2 kg/m^3. Since $u^3 = 2\ P/\rho AC_D$ we obtain $u = (1.2{\cdot}10^6/1.2/1)^{1/3} = 100$ m/s (224 mph).

Sky diver: $\frac{1}{2}\rho u^2 AC_D = mg$, m=100 kg, $AC_D = 0.5$ m^2, so:
$u = \sqrt{1000 \cdot 2/1.2/.5} = 58$ m/s = 130 mph.

3.3 Viscosity

Viscosity can be regarded as the internal "stickiness" of a fluid. It accounts for the energy losses associated with the transport of fluids in ducts, channels and pipes. Viscosity also plays a major role in the generation of turbulence.

Viscosity in *gases* arises principally from the molecular diffusion that transports momentum between layers of flow, i.e. high velocity molecules diffuse transverse to motion, transferring momentum to adjacent layers.

In *liquids*, the forces between molecules in layers of different velocity introduce shear stress.

We will infer a law to express the effect of flow speed gradients du_x/dy on a fluid, using the illustration of Figure 3.5 (for simplicity, we look at the x-component of the flow speed only). The speed distribution will tend with time towards the dashed line. Momentum in the x direction is transported downward (from larger y to smaller y). The fluid above AB will push forward (towards larger x) the liquid below AB. This *momentum flux* is equivalent to the existence of a shear stress in the liquid.

Stress is a measure of internal forces in a body between its constituent particles. Experimentally, it is found that the shear stress τ along AB is to a good approximation related to the flow speed gradient through

$$\tau \;=\; \eta\,\frac{du_x}{dy} \tag{3.16}$$

The physical unit of stress (see Eq. 3.16) is $[\tau]$ = kg m^{-1}s^{-2}. The unit of force (which we ultimately want) is however $[F]$ = kg m/s^2. This illustrates that the stress τ is not the same as the force (though they are related of course).

In fact, stress is like a tensor form of pressure – i.e., the force is the gradient of the stress. Stress is defined as force per unit area. The gradient of stress in the y direction is:

$$\frac{\partial\tau}{\partial y} \;=\; \eta\,\frac{\partial^2 u_x}{\partial y^2} \tag{3.17}$$

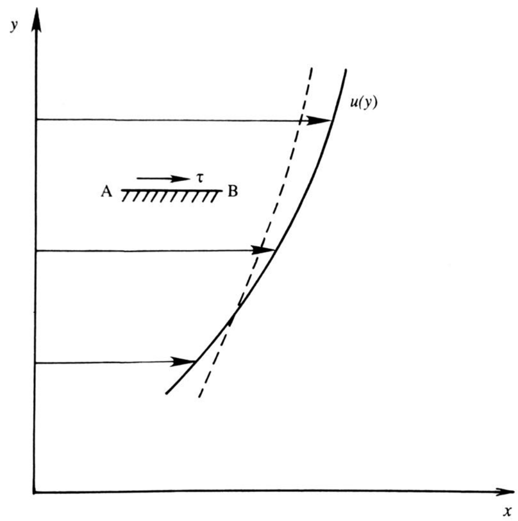

Figure 3.5: Shear stress τ on a surface AB. The 3 arrows represent velocity vectors in the x direction. Their differences in length are represented by the solid line linking the arrow heads. The dashed line represents x-speeds some time later. (Credit: Kundu, 2002)

The above is a force per unit volume. A layer of thickness Δy experiences a force per unit area of

$$\Delta y \frac{\partial \tau}{\partial y} = \eta \frac{\partial^2 u_x}{\partial y^2} \Delta y \tag{3.18}$$

This accelerates a mass per unit area of $\rho \Delta y$, so from $F = ma$ we have a deceleration of

$$\frac{\partial u_x}{\partial t} = \frac{\eta}{\rho} \frac{\partial^2 u_x}{\partial y^2} \tag{3.19}$$

Compare this to the diffusion equation (*Fick's Law*) for a gas of concentration n_i:

$$\frac{\partial n_i}{\partial t} = D \frac{\partial^2 n_i}{\partial x^2} \tag{3.20}$$

where D is the diffusion coefficient.

This illustrates that (3.19) is a one-dimensional diffusion equation, i.e. flow speed diffuses from the high speed region into the slow speed region, or momentum diffuses to reach a final state where momentum in a medium is uniformly distributed.

The diffusion of momentum inside a fluid is determined by the quantity

$$\mu \equiv \frac{\eta}{\rho} \tag{3.21}$$

where μ is the *kinematic viscosity* (unit: m^2/s) and ρ the mass density (unit: $\mathrm{kg/m}^3$). For water at 20°C, $\mu \approx 1.0 \cdot 10^{-6}$ m^2/s, for olive oil at 50°C, $\mu \approx 25.0 \cdot 10^{-6}$ m^2/s, for honey at 40°C, $\mu \approx 70.0 \cdot 10^{-6}$ m^2/s.

We can regard μ as a "diffusion coefficient" for momentum. Momentum diffusion inside a fluid is a fundamental consequence of individual particles interacting with one another and exchanging momentum.

See also: Problem Sheet 1, Question 2

3.4 Reynolds number

The dimensionless parameter Reynolds number (Re) compares the magnitude of the convective (inertia) term $\frac{1}{2}\rho\nabla u^2$ to the dissipative viscosity term $\eta\ \nabla^2\underline{\mathbf{u}}$:

$$Re \ = \ \frac{|\frac{1}{2}\rho\nabla u^2|}{|\eta\ \nabla^2\underline{\mathbf{u}}|} \tag{3.22}$$

We may simplify this by writing $\nabla \approx 1/L$ with L being a typical scale length of the system looked at:

$$\boxed{Re \ = \ \rho u\frac{L}{\eta} \ = \ u\frac{L}{\mu}} \tag{3.23}$$

where we also used equation (3.21) to replace η with μ. Note that the Reynolds number is used for scale analysis only and hence the factor (1/2) which appears in Eq. 3.22 can be ignored in Eq. 3.23.

See also: Problem Sheet 1, Questions 3 & 4

If Re is large ($Re \gg 1$) it implies that viscosity is unimportant, the flow is *laminar* or, if unstable, becomes *turbulent* (for $Re \leq 10^4 - 10^6$). Laminar and turbulent flows are discussed in Section 4.3.3.

If Re is small ($Re \ll 1$), viscosity damps motion. Note that $Re \propto L$ implies that often large scales are laminar, smaller scales are damped.

3.5 Bernoulli's principle

Lecture 4

The Bernoulli equation (first derived 1738 by the Swiss mathematician Daniel Bernoulli) describes conservation of energy appropriate for flowing fluids, that the total mechanical

Figure 3.6: Daniel Bernoulli (1700-1782) who derived the Bernoulli equation

energy of the flowing fluid or gas (energy associated with pressure, gravitational potential energy and kinetic energy of fluid motion) remains constant:

Energy per unit volume before = Energy per unit volume after

Bernoulli's equation is probably the most used equation in Fluid Dynamics. It is often also mis-used due to lack of understanding of its limitations, in particular the assumption that viscous effects are ignored.

There are several ways of deriving Bernoulli's principle, of which one is shown in the following.

3.5.1 Derivation of Bernoulli's principle

We can derive Bernoulli's equation from the Navier-Stokes equation which was given in (3.10) as

$$\rho\,\frac{\partial \underline{\mathbf{u}}}{\partial t} \;=\; -\nabla P - \rho\,\nabla\Phi + \eta\,\nabla^2\underline{\mathbf{u}} \;-\; \frac{1}{2}\,\rho\,\nabla(u^2) \;-\; \rho\,(\nabla\times\underline{\mathbf{u}})\times\underline{\mathbf{u}} \tag{3.24}$$

where we expressed gravitational acceleration $\underline{\mathbf{g}}$ in terms of a gradient of gravitational potential Φ, $\underline{\mathbf{g}} = -\nabla\Phi$.

We now make the following assumptions:

- Steady state: $\partial\underline{\mathbf{u}}/\partial\mathbf{t} = \mathbf{0}$
- Inviscid: $\eta\ \nabla^2\underline{\mathbf{u}} = 0$
- Irrotational: $\rho\,(\nabla \times \underline{\mathbf{u}}) \times \underline{\mathbf{u}} = \mathbf{0}$
- Incompressible: ρ constant

Thereby:

$$0 \;=\; -\nabla P - \rho\nabla\Phi \;-\; \frac{1}{2}\,\rho\nabla(u^2) \tag{3.25}$$

Giving:

$$\boxed{\nabla\left(P + \rho\Phi \;+\; \frac{1}{2}\,\rho u^2\right) \;=\; 0} \tag{3.26}$$

This is equivalent to stating that $(P + \rho\Phi + \frac{1}{2}\,\rho u^2)$ is constant. Relative to the surface, we may rewrite the gravity term in terms of gravitational acceleration g (used as a scalar now) and height h: $\Phi = gh$. The Bernoulli principle is then equivalent to stating that $(P + \rho g h + \frac{1}{2}\,\rho u^2)$ is constant. These terms all have the physical unit of energy per volume, illustrating that (3.26) in effect expresses energy conservation.

See also: Problem Sheet 2, Question 3 & Problem Sheet 3, Questions 2, 4&5.

For $u = 0$ this represents Pascal's principle[1], $\Delta P = \rho g(\Delta h)$ (ΔP being a change wrt background P).

If the flow is horizontal (ie. Φ is constant):

$$\boxed{\nabla\left(P + \frac{1}{2}\rho u^2\right) \;=\; 0} \tag{3.27}$$

Equations 3.26 and 3.27 are known as the Bernoulli equation (one with gravitational changes and the other without).

Bernoulli's principle is the most fundamental principle of hydrodynamics, it explains, lift, spin, waves, instability, etc..

While we assumed the flow to be irrotational, this is not strictly necessary. Even with rotation included $\underline{\mathbf{u}} \cdot ((\nabla \times \underline{\mathbf{u}}) \times \underline{\mathbf{u}}) = \mathbf{0}$, so $\underline{\mathbf{u}} \cdot \nabla(P + \rho\ \Phi + \frac{1}{2}\rho u^2) = 0$, so $P + \rho\ \Phi + \frac{1}{2}\rho u^2$ is constant along a stream line.

[1] Pascal's Principle: A change in pressure at any point in an enclosed fluid at rest is transmitted undiminished to all points in the fluid.

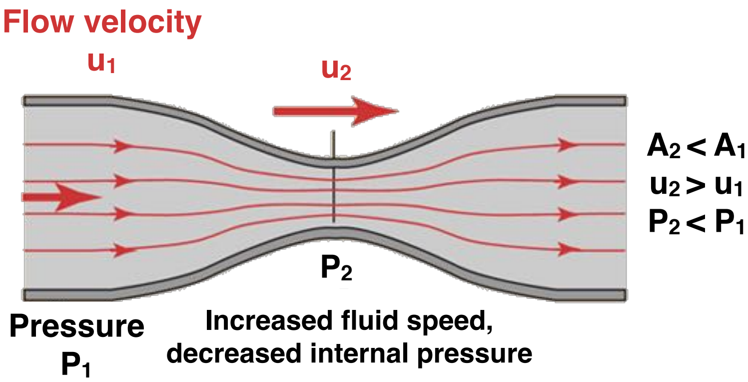

Figure 3.7: The change of fluid velocity in a tube resulting from changing cross section.

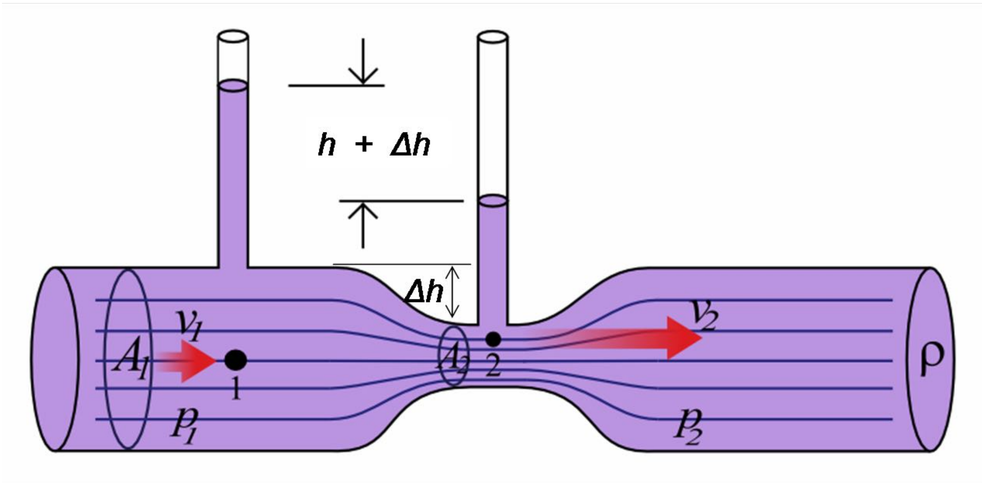

Figure 3.8: Flow of a fluid through a Venturi tube.

3.6 Applications of Bernoulli's principle

3.6.1 Venturis

Figure 3.8 shows flow of a fluid through a Venturi tube. Increased flow velocity through constriction gives rise to change in pressure through Bernoulli's principle. Flow velocity can be measured by measuring pressure e.g. by measuring height of water in a vertical tube attached (Pitot tube used to measure air speed on aircraft).

See also: Problem Sheet 2, Question 1

Alternatively, reduced pressure at a constriction can be used to suck another fluid (e.g. petrol) in order to mix it with the flowing fluid (e.g. air) as in a carburettor.

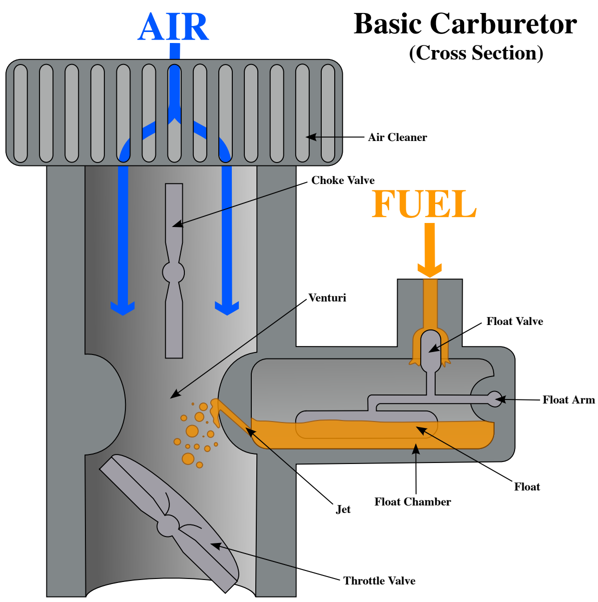

Figure 3.9: Carburettor

The carburettor (Figure 3.9) makes use of Bernoulli's principle. The throttle (accelerator) controls the flow of air being pulled into the engine (rather than the amount of fuel). The speed of air flow (its pressure) regulates the amount of fuel drawn into the airstream and engine.

See also: Problem Sheet 2, Question 2

3.6.2 The Magnus effect

Spinning objects generate rotation in fluids around them. Difference side-to-side in relative velocity between object surface and unperturbed stream velocity. The finite viscous drag allows angular momentum to be transferred from the object to the air. Thereby, the air near the surface of the ball moves with the velocity of rotation. This adds to the velocity of flow on one side and decreases flow velocity on other side. This is illustrated in Figure 3.10.

Due to Bernoulli's principle, $P + \frac{1}{2}\rho u^2$ is constant and there is an effective pressure difference side-to-side. If u is the average speed of the ball and Δu is the speed due to rotation (spin) of the ball then the pressure difference is

$$\frac{1}{2}\rho\,(u+\Delta u)^2 - \frac{1}{2}\rho\,(u-\Delta u)^2 = 2\rho\,u\Delta u \tag{3.28}$$

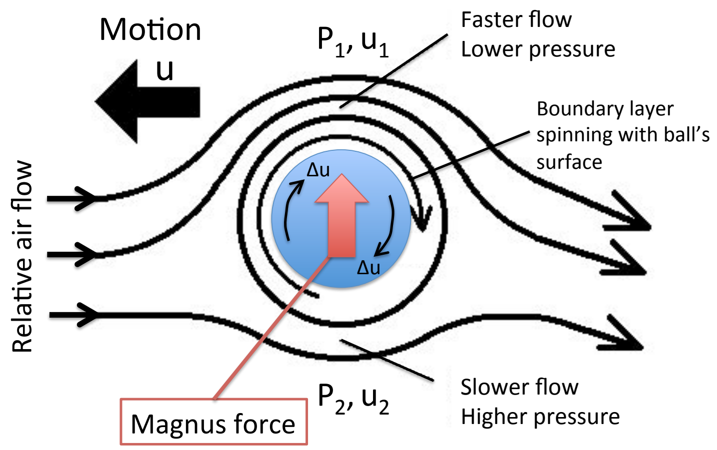

Figure 3.10: The Magnus effect on a ball spinning clockwise which moves to the left. The resulting Magnus force lifts the ball up.

Note that ρ is the density of the medium in which the ball moves (e.g., air) and not of the ball itself.

A little note on interpreting the Magnus effect:
You may justifiably ask yourself now how one side of the spinning object could possibly "communicate" with the other side of the spinning object, as is implied by Equation 3.28. The two sides don't directly communicate any local change in energy to one another, so the energy gained by one side isn't lost immediately by the other. However, they are both embedded in a fluid which moves at an average speed and has average properties, the fluid surrounding the regions of one or the other side of the object. The Magnus effect should be seen on the local scale of how the flow near the surface of the spinning object differs from the average flow slightly further away. The two sides of the object locally experience forces which then add up to a total force expressed via the pressure difference of Eq. 3.28.

Examples:

In soccer, a player may hit the ball and give it a spin. Thereby, the ball will be deflected from a straight line, confusing the goalkeeper.

See also: Problem Sheet 2, Question 5

In Tennis, backspin produces lift, whereas top spin caused ball to dive, allowing the player to hit the ball at a higher speed while still allowing it to hit the ground within the boundaries of the field.

In Golf: Backspin on golf balls produces lift, giving appearance of ball taking off from tee.

In reality, the boundary layer plays a role. Drag is actually lower for turbulent flow than for flow with smaller Reynolds number Re (cf Eq. 3.23) (because vortices detach later and are smaller). Fast freekicks don't decelerate as much as slow ones. As the ball decelerates, Re falls and flow becomes less turbulent – increasing the drag and changing the swerve.

For a golf ball, dimples are added to make the boundary layer more turbulent and reduce drag. (Dimples mean turbulent for $Re > 2 \cdot 10^4$, c.f. smooth football requires $Re > 4 \cdot 10^5$, see Section 4.1).

3.6.3 Cavitation

Cavitation describes the formation of vapour cavities (bubbles, voids) in a liquid where the pressure of the liquid falls below its vapour pressure. When the surrounding pressure is higher, the voids implode and may generate a shockwave. The collapsing voids which implode near a metal surface may generate cyclic stress through repeated implosion, causing surface fatigue of the metal.

Consider Bernoulli's principle for a sudden large change in flow speed e.g. forcing water through a constriction.

Change in pressure is $\frac{1}{2}\rho\,(u_1^2 - u_2^2)$. For $u_1 \ll u_2$ a change in pressure of 1 atmosphere (1 at = 0.98 bar; 1 bar = 10^5 Pa) requires $\frac{1}{2}\rho\, u_2^2 \approx 1 \cdot 10^5$, i.e. $u_2 \approx 14$ m/s ($\approx$30 mph).

For abrupt speed changes more than this the pressure in water becomes negative. In principle, water can retain a negative pressure (c.f tensile strength) but in practice impurities or surfaces introduce nucleation sites so that once pressure drops below vapour pressure, bubbles of water vapour start to form, (c.f. boiling). If these bubbles are generated in going through a constriction between two regions of atmospheric pressure, then once released the high external pressure causes the bubbles to implode. For example, in a barely closed tap (where the water is forced through a narrow constriction causing cavitation) collapse of the bubbles causes the hissing sound. The same effect can occur when water is forced to undergo rapid deviation due to impacting a solid surface, e.g. a bullet fired into water, or a rotating propeller blade.

For powerful swimming animals like dolphins and tuna, cavitation may be detrimental, because it limits their maximum swimming speed. Even if they have the power to swim faster, dolphins may have to restrict their speed because collapsing cavitation bubbles on their tail are too painful. Cavitation also slows tuna, but for a different reason. Unlike dolphins, these fish do not feel the painful bubbles, because they have bony fins without nerve endings. Nevertheless, they cannot swim faster because the cavitation bubbles create an air film around their fins that limits their speed. Lesions have been found on tuna that are consistent with cavitation damage.

4 Types of flows

Lecture 5

4.1 Ideal flows

Simplified solutions arise when terms in Navier-Stokes can be neglected. One of these is to assume *ideal flow*. It is incompressible, inviscid, irrotational, steady and has no thermal conduction.

- *Incompressible*: Density does not change, usually the case for liquids, but also for subsonic flow in gases normally change in density is small compared to change in speed (see 4.3.1).
- *Inviscid*: Negligible viscosity.
- *Irrotational*: Negligible vorticity (see Section 4.4). If a flow is initially uniform and it doesnt encounter any obstacle or force which is asymmetric, then it won't pick up any spin and its vorticity will remain zero.
- *Steady*: Velocity at each point in the flow is constant in time.
- *No thermal conduction*: Any pressure changes are primarily due to changes in velocity rather than heat flow.

Many of the principles (e.g. Bernoulli) may still be used for non-ideal flow but require some modifications.

4.2 Continuity Equation

We will briefly introduce the continuity equation as it plays an important role when discussing incompressible flow. The continuity equation expresses the conservation of mass:

$$\frac{\partial \rho}{\partial t} + \nabla \left(\rho \, \underline{\mathbf{u}} \right) \ = \ \mathbf{0} \tag{4.1}$$

The right term may be expanded to give:

$$\frac{\partial \rho}{\partial t} + \nabla \rho \cdot \underline{\mathbf{u}} + \rho \nabla \underline{\mathbf{u}} \ = \ \mathbf{0} \tag{4.2}$$

Following equation 2.4, this becomes:

$$\frac{d\rho}{dt} \ = \ -\rho \nabla \underline{\mathbf{u}} \tag{4.3}$$

4.3 Irrotational Flows

4.3.1 Incompressible flow

A flow is considered incompressible if the density ρ is constant, i.e.,

$$\frac{d\rho}{dt} = 0 \tag{4.4}$$

Using the continuity equation (4.3), this implies

$$\frac{d\rho}{dt} = 0 \qquad \Leftrightarrow \qquad \nabla\underline{\mathbf{u}} = \mathbf{0} \tag{4.5}$$

Thereby, incompressible flow has zero divergence.

See also: Problem Sheet 3, Question 1

4.3.2 Incompressible flow

The Navier-Stokes equation introduced so far (3.10) assumes density to be constant, i.e., the fluid to be incompressible. This approximation holds even for a compressible fluid (e.g., a gas) if we look at at sub-sonic phenomena, as will be argued in the following.

From the Bernoulli principle (see Section 3.5), $P + 1/2\rho u^2$ is constant. Consider a pressure perturbation caused by a speed pulse of $u = 50$ m/s (180 km/h). According to Bernoulli's principle this leads to a pressure change of $1/2\rho u^2 \approx 1.5 \cdot 10^3$ Pa (assuming a sea level air density of $\rho_{\text{air}} \approx 1.2$ kg/m^3). This is considerably smaller than the background surface pressure of 10^5 Pa. Hence, we may often deal with situations where changes in P are small compared with P_0. The relative simplicity of the incompressible Navier-Stokes equation (compared to the compressible version, not discussed here) raises the question of whether we cannot assume incompressibility in many situations.

Generally, a flow can still be considered incompressible if changes in P (change in ρ) are small compared to P$_0$ (ρ_0). Following Bernoulli's principle, it follows that for $P \ll P_0$ we also have $1/2\rho_0 u^2 \ll P_0$, so $u \ll \sqrt{P_0/\rho_0}$ and $u \ll c_s$ (recall: $c_s = \sqrt{2kT/m} = \sqrt{2P_0/\rho_0}$). Hence, the Mach number of the speed perturbation is $\ll 1$. Thus, low Mach-number situations are synonymous with low perturbations in P (ρ) relative to the background. As a result, incompressibility is a good approximation for sub-sonic phenomena even in a medium such as air (which we otherwise regard as being compressible).

4.3.3 Laminar and turbulent flows

Viscous flows can be divided into two categories, namely *laminar flow* and *turbulent flow*. Their basic difference can be seen from Osborne Reynolds' experiment (1883) which consisted in injecting a thin stream of dye into the flow of water through a tube,

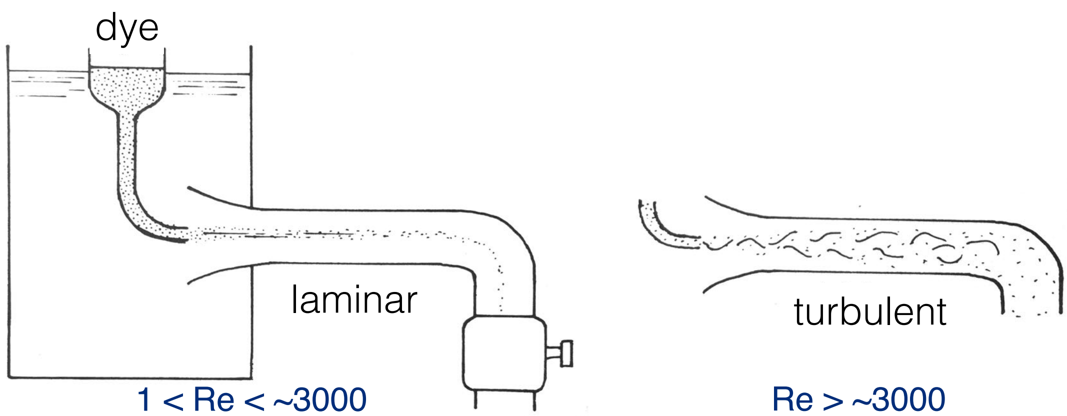

Figure 4.1: Reynolds' experiment to distinguish between laminar and turbulent flows (Adapted from: Kundu, 2002).

as illustrated in Figure 4.1.

At low rates of flow (left image), the dye stream follows a well-defined straight path, showing that the liquid moved in parallel layers ("laminas") with no macroscopic mixing across the layers. This is called *laminar flow.* In laminar flow there is no disruption between layers, no lateral mixing. Laminar flow within a pipe is parallel to the pipe walls.

As the flow rate was increased beyond a critical value (right image), the dye streak broke up into irregular motion and spread throughout the cross section of the tube, indicating the presence of macroscopic mixing motions perpendicular to the direction of the bulk water flow. This chaotic fluid motion is characteristic of *turbulent flow.*

Reynolds showed that the transition between laminar and turbulent flow occurred at a fixed value of $Re = u\ (d/\mu) \approx 3000$ (u being the average water speed, d the tube diameter and μ the kinematic viscosity).

4.4 Rotational flows

4.4.1 Vorticity propagation

In Section 3.1 we introduced the term $\underline{\omega} = \nabla \times \underline{\mathbf{u}}$ (the curl of the velocity vector) as the *vorticity* and argued that it describes how much spin the fluid has. We will in the following examine in more detail, how the vorticity introduced in one region of a flow can propagate to other regions.

In the case of a flow moving over a flat surface in the $+x$ direction (See Figure 4.2),

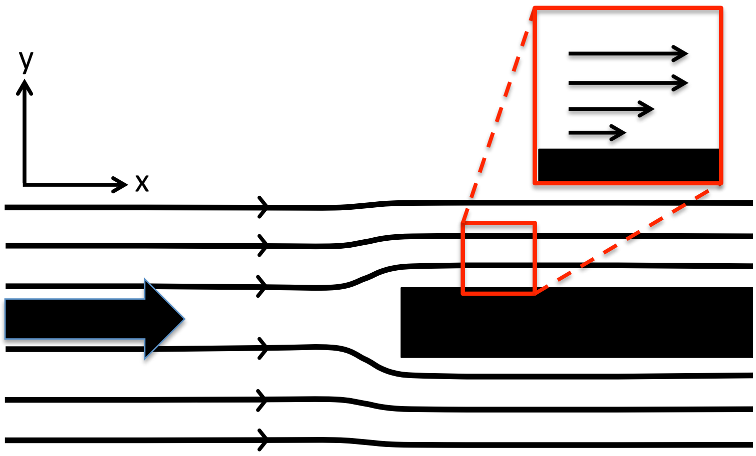

Figure 4.2: Flow of liquid around a flat surface. At the boundary layer, viscous drag adds vorticity to the flow, illustrated in the red box.

viscosity within the boundary layer causes drag between layers with different velocity. A fluid element now has different velocity above (see red inset in Figure 4.2 which zooms in on the boundary layer) and below which causes the fluid element to rotate. In other words the curl of the velocity vector (its vorticity) is finite:

$$\nabla \times \underline{\mathbf{u}} = \left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right)\underline{\hat{\mathbf{x}}} + \left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right)\underline{\hat{\mathbf{y}}} + \left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right)\underline{\hat{\mathbf{z}}} \tag{4.6}$$

Here, $\nabla \times \underline{\mathbf{u}}$ has a finite $(\partial u_x/\partial y)$ in the $\underline{\hat{\mathbf{z}}}$ direction representing rotation about a vector into the paper. In practice there is also a little $(\partial u_y/\partial x)$ in the $+\underline{\hat{\mathbf{z}}}$ direction.

Thereby, for the specific situation here: $\nabla \times \underline{\mathbf{u}} \approx (\partial u_y/\partial x - \partial u_x/\partial y)\underline{\hat{\mathbf{z}}} \approx -(\partial u_x/\partial y)\underline{\hat{\mathbf{z}}}$.

Relate this to the viscous deceleration (see Section 3.3, Eq. (3.19)):

$$\frac{\partial u_x}{\partial t} = \frac{\eta}{\rho}\frac{\partial^2 u_x}{\partial y^2} \tag{4.7}$$

Take $\partial/\partial y$:

$$\frac{\partial}{\partial y}\frac{\partial u_x}{\partial t} = \frac{\partial}{\partial y}\left(\frac{\eta}{\rho}\frac{\partial u_x^2}{\partial y^2}\right) \tag{4.8}$$

Assuming steady flow and swapping the derivatives gives:

$$\frac{\partial}{\partial t}\frac{\partial u_x}{\partial y} = \frac{\eta}{\rho}\frac{\partial^2}{\partial y^2}\frac{\partial u_x}{\partial y} \tag{4.9}$$

So,

$$\frac{\partial}{\partial t}\,(vorticity) \;=\; \frac{\eta}{\rho}\,\frac{\partial^2}{\partial y^2}\,(vorticity) \;=\; \mu\,\frac{\partial^2}{\partial y^2}\,(vorticity) \tag{4.10}$$

which is the same form of 1-D diffusion equation we discussed in the context of viscosity (cf. Section 3.3, Eq. (3.19)).

Earlier, we stated that if vorticity were initially zero then it would remain zero, and the above has not changed this. However, in the situation investigated here we are forcing a boundary condition where $u_x = 0$ near the surface layer. This introduces a spatial gradient into the flow velocity, and thereby vorticity, which then diffuses into the flow. So, the message of Eq. (4.10) is that vorticity, once introduced, will not be restricted to the location in the fluid where it is introduced but will diffuse throughout the rest of the fluid and be present away from the region where it originally occurred. Away from the boundary layer the vorticity does however decrease and the potential flow solution at sufficient distance again becomes a reasonable assumption.

As previously for the case of momentum diffusion, the rate or extent to which vorticity diffuses into the flow is set by the viscosity or the Reynolds number of the flow.

See also: Problem Sheet 3, Question 6

4.4.2 Vorticity equation

Lecture 6

Vorticity describes microscopic rotation at a given point in a fluid. The vorticity equation describes the change of vorticity $\underline{\omega} = \nabla \times \underline{\mathbf{u}}$ with time and is given by

$$\boxed{\frac{\partial \underline{\omega}}{\partial t} \,+\, (\underline{\mathbf{u}}\cdot\nabla)\,\underline{\omega} \;=\; (\underline{\omega}\cdot\nabla)\,\underline{\mathbf{u}} \,+\, \mu\,\nabla^2\,\underline{\omega}} \tag{4.11}$$

where μ is the kinematic viscosity coefficient. The equation describes how the vorticity of a fluid element evolves with time. The first term on the right side represents the stretching and tilting of vortex tubes due to velocity gradients while the second term describes the diffusion of vorticity due to viscosity. Vortex tubes can be regarded as bundles of vortex lines, as illustrated in Figure 4.3

We may derive the vorticity equation from the Navier-Stokes equation (3.10), which is

$$\rho\,\frac{\partial \underline{\mathbf{u}}}{\partial t} \;=\; -\nabla P \,+\, \rho\,\underline{\mathbf{g}} \,+\, \eta\nabla^2\underline{\mathbf{u}} \,-\, \frac{1}{2}\,\rho\,\nabla(u^2) \,-\, \rho\,(\nabla\times\underline{\mathbf{u}})\times\underline{\mathbf{u}} \tag{4.12}$$

or, after dividing both sides by ρ and replacing η with $\mu = \eta/\rho$:

$$\frac{\partial \underline{\mathbf{u}}}{\partial t} \;=\; -\frac{1}{\rho}\,\nabla P \,+\, \underline{\mathbf{g}} \,+\, \mu\nabla^2\underline{\mathbf{u}} \,-\, \frac{1}{2}\,\nabla(u^2) \,-\, (\nabla\times\underline{\mathbf{u}})\times\underline{\mathbf{u}} \tag{4.13}$$

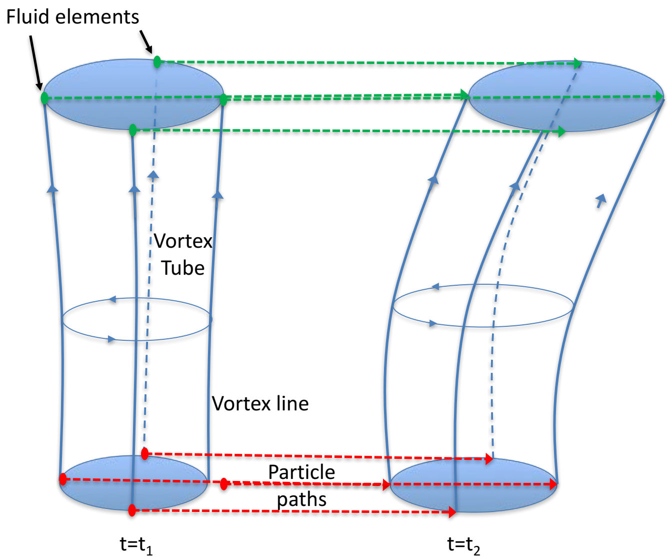

Figure 4.3: Evolution of a vortex tube. Solid dots correspond to fluid elements. Solid lines are vortex lines which point in the direction of the vortex vectors. Due to the shear in the velocity field, the vortex tube is stretched and tilted with time.

Now take the curl on both sides:

$$\nabla\times\frac{\partial\underline{\mathbf{u}}}{\partial t} = -\frac{1}{\rho}\nabla\times\nabla P + \nabla\times\underline{\mathbf{g}} + \mu\nabla\times(\nabla^2\underline{\mathbf{u}}) - \frac{1}{2}\nabla\times\big(\nabla(u^2)\big) - \nabla\times\big((\nabla\times\underline{\mathbf{u}})\times\underline{\mathbf{u}}\big) \tag{4.14}$$

Evaluate each term individually (from left to right):

- $\nabla\times\partial\underline{\mathbf{u}}/\partial t = \partial/\partial t(\nabla\times\underline{\mathbf{u}}) = \partial\underline{\omega}/\partial t$
- $\nabla\times\nabla P = 0$ since $\nabla\times\nabla a = 0$ for any scalar a
- $\nabla\times\underline{\mathbf{g}} = -\nabla\times\nabla\Phi = 0$ (where Φ is the gravitational potential), for the same reason as above with the pressure gradient term.
- $\mu\nabla\times(\nabla^2\underline{\mathbf{u}}) = \mu\ \nabla^2(\nabla\times\underline{\mathbf{u}}) = \mu\nabla^2\underline{\omega}$
- $\nabla\times\big(\nabla(u^2)\big) = 0$, for the same reason as above.

- $\nabla \times (\,(\nabla \times \underline{\mathbf{u}}) \times \underline{\mathbf{u}}) \;=\; \nabla \times (\underline{\omega} \times \underline{\mathbf{u}})$. To evaluate this term, we make use of the vector identity

  $$\nabla \times (\underline{\mathbf{A}} \times \underline{\mathbf{B}}) = \underline{\mathbf{A}}(\nabla \cdot \underline{\mathbf{B}}) \;-\; \underline{\mathbf{B}}(\nabla \cdot \underline{\mathbf{A}}) \;+\; (\underline{\mathbf{B}} \cdot \nabla)\underline{\mathbf{A}} \;-\; (\underline{\mathbf{A}} \cdot \nabla)\underline{\mathbf{B}} \tag{4.15}$$

  Using this given vector identity the term becomes:
  $\nabla \times (\underline{\omega} \times \underline{\mathbf{u}}) = \underline{\omega}(\nabla \cdot \underline{\mathbf{u}}) - \underline{\mathbf{u}}(\nabla \cdot \underline{\omega}) + (\underline{\mathbf{u}} \cdot \nabla)\underline{\omega} - (\underline{\omega} \cdot \nabla)\underline{\mathbf{u}}$
  For an incompressible fluid, the continuity equation implies that the flow is divergence free, $\nabla \cdot \underline{\mathbf{u}} = 0$ (see equation 4.5), so the first term is zero.
  $\nabla \cdot \underline{\omega} = \nabla \cdot (\nabla \times \underline{\mathbf{u}}) = 0$ always (divergence of curl of any vector field is always zero), so the second term is also zero.
  Therefore:
  $\nabla \times (\underline{\omega} \times \underline{\mathbf{u}}) = (\underline{\mathbf{u}} \cdot \nabla)\underline{\omega} - (\underline{\omega} \cdot \nabla)\underline{\mathbf{u}}$

So, the full equation becomes:

$$\frac{\partial \underline{\omega}}{\partial t} \;=\; \mu(\nabla^2 \underline{\omega}) \;-\; (\underline{\mathbf{u}} \cdot \nabla)\underline{\omega} + (\underline{\omega} \cdot \nabla)\underline{\mathbf{u}} \tag{4.16}$$

Rearranging gives the vorticity equation (4.11).

Note that the vorticity equation contains no gravity and no pressure terms. Mathematically, this is plausible (see above). The physical reason for this is that these forces act through the centre of mass of an element and thus generate no torque.

4.4.3 Types of vortex

A vortex is a circulating fluid with closed streamlines.

Consider vorticity in rotating flows. In cylindrical co-ordinates (r, z, θ) with flow in the θ direction we may write the vorticity as

$$\nabla \times \underline{\mathbf{u}} \;=\; \frac{1}{r}\,\frac{\partial (r u_\theta)}{\partial r}\,\hat{\underline{\mathbf{z}}} \tag{4.17}$$

We may describe 3 different types of vortices:

- *Forced (or rotational) vortex*: If angular frequency is constant (like a rotating plate or CD) then $u_\theta \propto r$ and there is large (constant) vorticity – this is a forced vortex. The vorticity $\nabla \times \underline{\mathbf{u}}$ is non-zero in a forced vortex. A forced vortex must have some outer boundary otherwise $u_\theta \to \infty$ as $r \to \infty$.

- *Free (or irrotational) vortex*: If angular momentum of a fluid element is conserved then $u_\theta \propto 1/r$ and therefore

  $$\nabla \times \underline{\mathbf{u}} \;=\; \frac{1}{r}\,\frac{\partial (r u_\theta)}{\partial r}\,\hat{\underline{\mathbf{z}}} \;=\; 0 \tag{4.18}$$

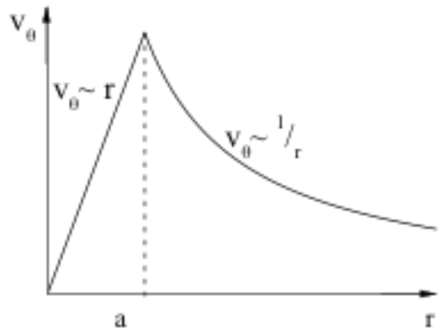

Figure 4.4: Rankine vortex model.

The circulation around a closed contour $\underline{\mathbf{S}}$ ($\int \underline{\mathbf{u}} \cdot d\underline{\mathbf{S}} = 2\pi r u_\theta(r)$) is finite and constant (this follows from Stoke's theorem). Here, the vorticity is zero, so it is an irrotational flow. A free vortex occurs in the absence of external forcing.

One way to visualise this is to imagine the fluid to be subdivided into small square "packets". In a free vortex, these packets move around the vortex axis but they don't rotate in themselves (due to the specific $1/r$ dependence of the velocity), and hence the vorticity is zero. If they moved around the vortex axis and also rotated in themselves then vorticity would be non-zero, as is the case for a forced vortex.

Conservation of angular momentum means that free vortices occur naturally in convergent rotating flows. With minimal dissipation, the structure is long lived, e.g. whirlpool in bath, wingtip vortices, hurricane, tornadoes.

One problem with the free vortex is that velocity $u_\theta \to \infty$ as $r \to 0$. This means that there is either a singularity (e.g. the hole in the middle of the bath whirlpool) or there is some central region or core where $u_\theta \to 0$ (u_θ cannot be finite at $r = 0$ as acceleration would be infinite), i.e. there is a forced vortex within the free vortex.

- The *Rankine vortex* is a vortex in a viscous fluid. A swirling flow in a viscous fluid can be characterised by a forced vortex in its central core, surrounded by a free vortex. The Rankine vortex assumes an inner circular region about the origin which is in solid rotation, while the outer region is free of vorticity, the speed being inversely proportional to the distance from the origin. This flow model has occasionally been used for the wind distribution in a hurricane and in a tornado. The change of u_θ with distance from the core is illustrated in Figure 4.4. This model in itself is problematic since the rate of change in $\partial/\partial r(u_\theta)$ is infinite at $r = a$. With finite viscosity the discontinuity at $r = a$ is smoothed.

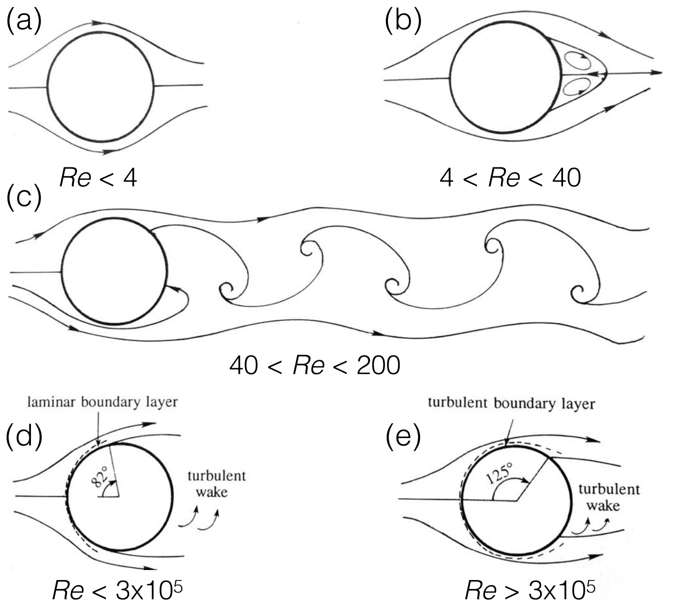

Figure 4.5: Regimes of flow over a circular cylinder for different Reynolds numbers Re (Adapted from Kundu, 2002).

4.4.4 Flow past a cylinder

Viscous flows can be characterised analytically for the two extreme cases of $Re \ll 1$ and $Re \gg 1$ but for the range in-between, experimental or numerical solutions are necessary. In the following, the flow past a circular cylinder is characterised over the entire possible range of Re.

Consider a uniform flow meeting a cylindrical obstacle, as shown in Figure 4.5. We know from (3.23) that the Reynolds number can be written as

$$Re = u\frac{L}{\mu} = u_\infty \frac{d}{\mu} \tag{4.19}$$

where u_∞ represents the undisturbed flow velocity of the liquid at large distance from the cylinder and d is the diameter of the cylinder. From this relationship we see that changing Re can be obtained for constant d and μ by changing the flow velocity u_∞.

At low velocity (low Re) (case (a) in Figure 4.5) viscosity dominates, a broad boundary layer with weak shear means the flow pattern is similar to the potential flow solution. For $Re \approx 20$ (case (b)) the flow lines close off and isolated vortices become visible behind the cylinder. Increasing Re further, the vortices become more elongated.

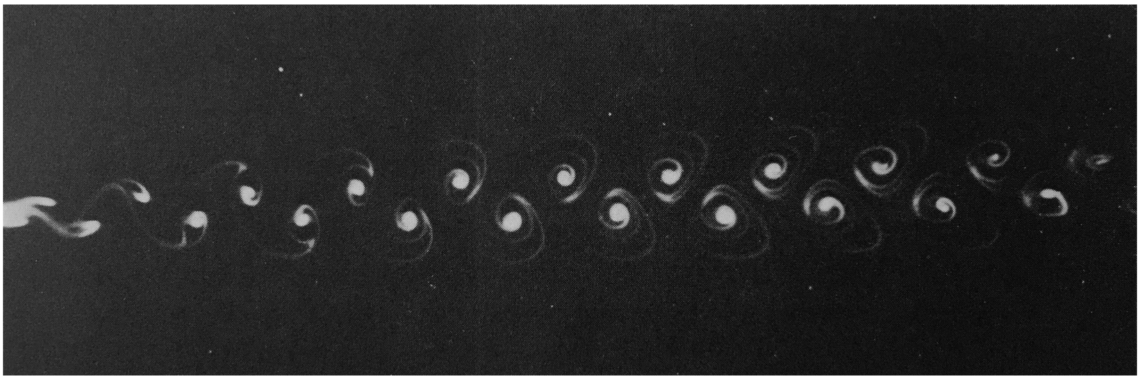

Figure 4.6: Karman vortex street downstream of a circular cylinder for $Re = 55$. (Credit: Taneda, 1965).

When $Re > 40$ (case (c)) the circulation in the vortex closes the loop away from the cylinder, the vortex detaches from the cylinder. Minute differences mean that this detachment occurs on one side first, the flow is then asymmetric, triggering vortex detachment on the opposite side and so on. The resulting structure is called a *Karman vortex street.* An example experimental image of a Karman vortex street is shown in Figure 4.6. The flow is no longer steady and the alternating pressure increase on the object can cause a destructive resonance (e.g. Tacoma narrows bridge).

Increasing Re still further, vortices break up into smaller structures. The frictional effects upstream of separation are confined to near the surface of the cylinder. For $Re < 3 \cdot 10^5$ (case (d)) the boundary layer remains laminar while the wake remains turbulent. The laminar boundary layer separates at around $82°$ from the ram point.

For $Re > 3 \cdot 10^5$ (case (e)) the laminar boundary layer becomes unstable and transitions to turbulence. For a circular cylinder the turbulent boundary layer separates at $125°$ from the ram point, generating a thinner wake and a pressure distribution more similar to that of a potential flow. So, we still have a boundary layer, but dominated by turbulence rather than viscosity.

4.4.5 Wingtip vortices

A free vortex can be formed at the wing tips of airplanes, where the flow from the high pressure region below wings to lower pressure region above wing generates rotation at the wingtips, as shown in Figure 4.7.

The long-lived nature of these free vortices can be a cause of problems in practice. The stability of vortex means this part of wake takes a long time to decay and limits aeroplane take off frequency at airports (2-3 mins depending on size).

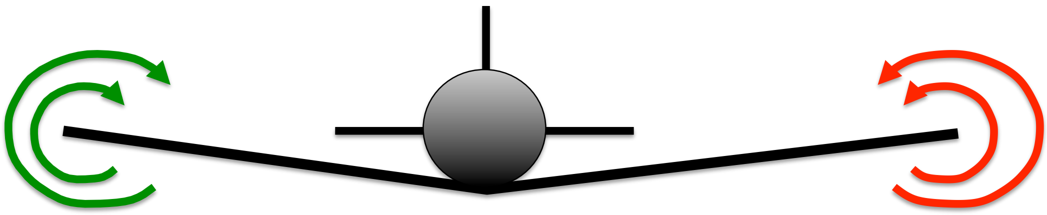

Figure 4.7: Vortices formed over the wing tips of an airplane.

The same problem applies to helicopter noise from rotor blades cutting through the vortex left by the previous blade.

4.4.6 Cyclones

Lecture 7

Cyclones are giant storms generated over warm oceans and are characterised by a sharp pressure drop in the centre (the "eye") which drives circular motion there (a forced vortex of ∼50 km diametre), surrounded by a vast structure moving in circular motion around the eye. The surrounding structure can be ∼500 km across and is a free vortex.

Cyclones form when warm, moist air over the ocean rises up from near the surface. As this moist air moves up, it generates an area of lower air pressure below near the surface. Near the surface, therefore, air from surrounding areas with higher air pressure pushes in to the low pressure area. As this new cool air warms up and becomes moist, it rises, too, so a cycle has been initiated near the surface. Further up, the warm, moist air rises and cools the water vapour, forming clouds. The entire cloud system spins and grows, fed by the oceans heat and water evaporating from the ocean surface. As the storm system rotates faster and faster, an "eye" forms in the centre, a low pressure region which represents the "engine" of the cyclone.

The Cyclone structure (Figure 4.8) mostly resembles that of a Rankine vortex, i.e. a forced vortex at the centre, the eye, and a free vortex outside. It is in practice complicated by the vertical dimension – some regions of warm damp air rising from ocean or land (energy source) & some regions of falling air with precipitation (energy dissipation), plus a central column of descending dry air (the eye). The Coriolis force also introduces finite vorticity. However, we may ignore these substructures to a first approximation and distinguish between two regions, the eye and everything else surrounding the eye.

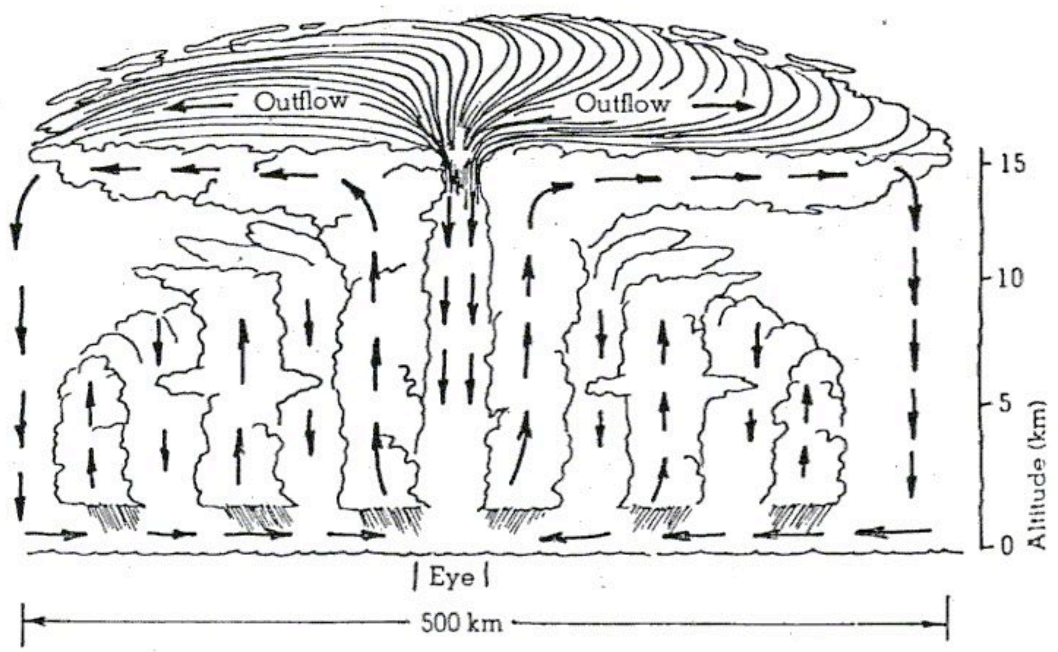

Figure 4.8: Structure of a cyclone.

The motion of a free vortex in reality is slowed down with time due to viscous drag. In a free vortex, therefore, some force is required to keep the fluid moving in a circle (and this doesn't make it a forced vortex, this forcing may affect only the boundary region close to the centre of the vortex while the rest is free). In a bath plug whirlpool this comes from gravity. In free vortices (e.g. wing tip vortices, cf. Section 4.4.5) this comes from pressure gradients which generate initial circular motion (at the wing tips).

Let's work out where the Bernoulli equation holds in a cyclone. First, let's look at the free vortex outside the eye. From Section 4.4.3 we know that a free vortex by definition has zero vorticity. From the Navier-Stokes equation (3.10) (neglecting gravity and viscosity):

$$\rho\frac{\partial \underline{\mathbf{u}}}{\partial t} = -\nabla P - \frac{1}{2}\,\rho\nabla u^2 - \rho(\nabla\times\underline{\mathbf{u}})\times\underline{\mathbf{u}} \tag{4.20}$$

In the frame moving with a fluid element the direction of the velocity $\underline{\mathbf{u}}$ is changing but at a fixed point $\underline{\mathbf{u}}$ is constant: $\partial\underline{\mathbf{u}}/\partial t = 0$ and $(\nabla\times\underline{\mathbf{u}}) = \mathbf{0}$ (free vortex).
Thus, $0 = -\nabla P - \nabla\frac{1}{2}\,\rho u^2$, i.e., Bernoulli's Principle ($\nabla(P+\frac{1}{2}\rho u^2) = 0$) applies in a free vortex, and thereby in the cyclone outside the eye.

Let's now look at the eye of a cyclone. There, the low pressure at the centre, generating a radial pressure gradient, balances centrifugal forces on rotating fluid elements:

$$\frac{dp}{dr} = \rho \frac{u^2}{r} \tag{4.21}$$

where p is the pressure and u represents the azimuthal wind speed in a cylindrical coordinate system, or "rotation speed" of fluid elements. Equation 4.21 allows us to calculate u if we know the radial pressure gradient across the eye of a cyclone. The equation is not relevant for the rest of the cyclone (outside of the eye) since that is a free vortex and has a negligible radial pressure gradient. The balance of Equ.4.21 is referred to as *cyclostrophic balance.*

Back to the question of whether the Bernoulli principle holds. We have seen that it holds outside the eye, but what about inside the eye? For Bernoulli to hold we need $p + 1/2\,\rho u^2$ to be invariant with distance r from the centre:

$$0 = \frac{d}{dr}\left(p + \frac{1}{2}\rho u^2\right) = \frac{dp}{dr} + \frac{1}{2}\rho \frac{d}{dr}\left(u^2\right) = \frac{dp}{dr} + \rho\, u \frac{du}{dr} \tag{4.22}$$

where we assumed ρ to be constant with r. We may insert Equation (4.21) into (4.22) to obtain

$$0 = \rho \frac{u^2}{r} + \rho\, u \frac{du}{dr} \tag{4.23}$$

which gives a relationship for the radial velocity gradient:

$$\frac{du}{dr} = -\frac{u}{r} \tag{4.24}$$

So, in order for the Bernoulli principle to hold in the eye of the cyclone we would need the radial velocity gradient to satisfy Equation (4.24). This implies a (-1/r) decrease of the velocity from the centre (where it is zero), which is opposite to what is observed for cyclones (an increase of velocity within the eye away from the centre, shown in Figure 4.9). So, the Bernoulli principle cannot hold in the eye of a cyclone. This is not an unexpected result since forced vortices have non-zero vorticity (see Section 4.4.3) and our original derivation of Bernoulli's principle (Section 3.5.1) assumed an irrotational flow, i.e., one with zero vorticity. Still, it's important to check these things explicitly.

The winds observed on the ground during the passage of a cyclone are shown in Figure 4.9. Note the decrease of wind speed during the passage of the eye of the cyclone (between 07:00 and 09:30 local time) as the observer fixed to the ground enters the eye of the cyclone during its passage. Moving away from the centre, the speed picks up again, as expected in a forced vortex. The whole shape of the curve in Figure 4.9 very much resembles that of a Rankine vortex (Figure 4.4) centred around 8-9 h time.

In weather cyclones the Coriolis force due to rotating frame of reference of the Earth's surface (see Section 5.2.2) also plays a role but the dominant balance is still between

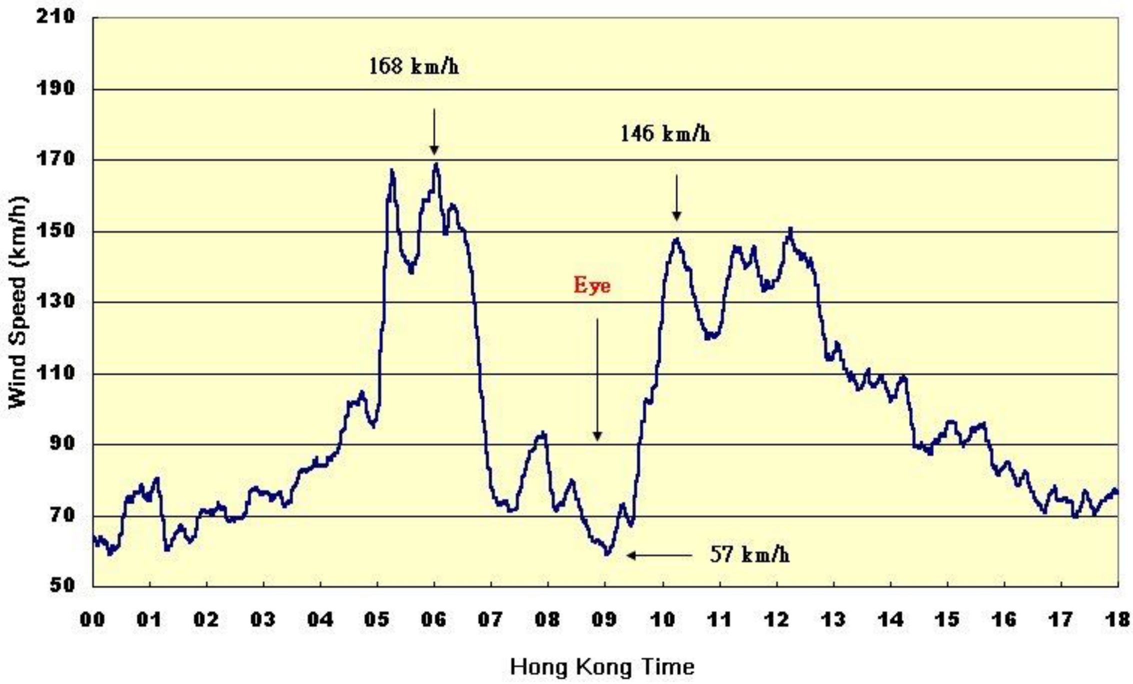

Figure 4.9: Wind speed record of Waglan Island during the direct hit of Typhoon York over Hong Kong on 16 September 1999. Notice the dramatic fall and rise in wind strength during the passage of the eye of the cyclone (Source: Hong Kong Observatory).

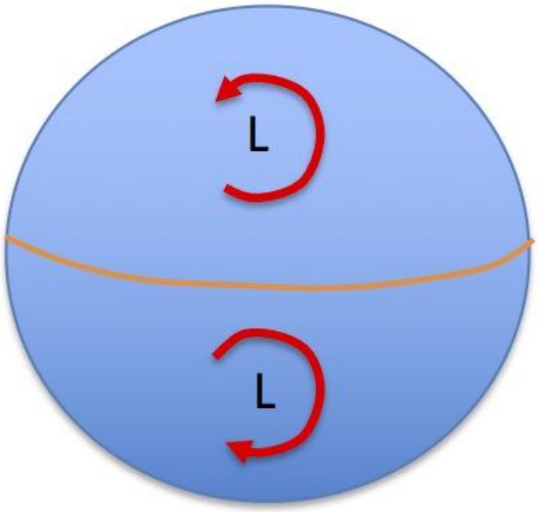

Figure 4.10: The orientation of cyclones on Earth. As a result of Coriolis forces acting during the formation process of a cyclone, winds blow anticlockwise in the northern hemisphere and clockwise in the southern hemisphere. Once a cyclone is established, Coriolis forces are less important than the radial pressure gradient which balances centrifugal forces.

pressure and centrifugal forces (Equation 4.21). Coriolis forces cause the initial circular motion of the wind system (they give it the "initial circular kick") and is responsible for the hemispheric asymmetries, with cyclones rotating anti-clockwise in the northern hemisphere and clockwise in the southern hemisphere, as illustrated in Figure 4.10.

The energy of a cyclone can be estimated fairly easily by calculating its kinetic energy. The result is in the order of 10^{17} J, corresponding to the energy produced in 1 day by all the Earth's power stations!

5 Geophysical Fluid Dynamics

Lecture 8

5.1 Introduction

Up until now, this course has discussed the behaviour of fluids on small scales, from small tubes to cyclones. These are small compared to the scale of our planet. Once we look at fluids on a planetary scale (ocean flow, atmosphere flow) we need to consider additional aspects for the mere fact that the Earth (or any other planet) rotates. Geophysical Fluid Dynamics (or GFD) treats dynamics of stratified and turbulent fluid on (big!) rotating spheres. Stratified fluids are those that contain variations of density in the radial (or vertical) direction. In oceans and atmospheres, fluids with lower densities

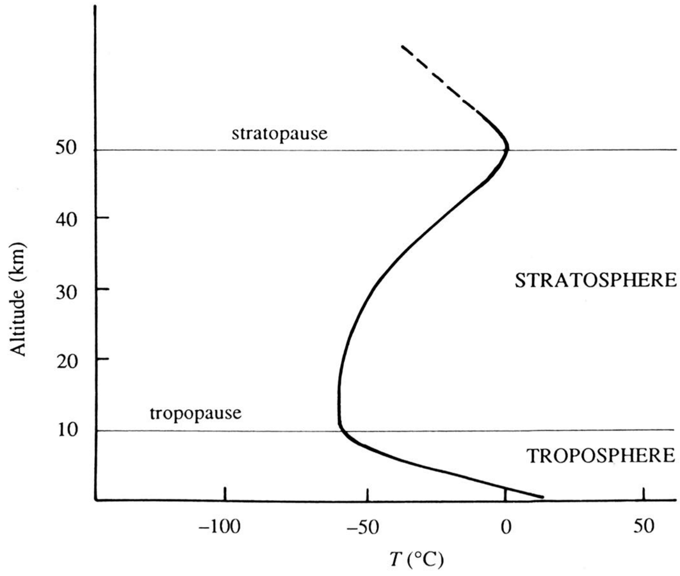

Figure 5.1: Vertical variation of temperatures in the Earth's atmosphere. (Credit: Kundu, 2002)

are always located above those with higher density. We will therefore begin by examining the vertical structure of the atmosphere and ocean.

As shown in Figure 5.1, the Earth's atmosphere can be vertically subdivided into different distinct regions based on the vertical temperature gradient. In the lower atmosphere (*troposphere*), the vertical temperature gradient is around −6.5 K/km, close to the *dry adiabatic lapse rate*. As a result, the troposphere is convectively stable. The temperature minimum at the top of the troposphere near around 12 km is a transition region called the *tropopause*, above which lies the *stratosphere* where temperature increases with height as a result of UV absorption by atmospheric ozone. The top of the stratosphere (near 50 km) is the *stratopause*, above which lie the *mesosphere* (50–80 km) and *thermosphere* (80–400 km) (not shown in Figure 5.1).

Figure 5.2 shows the depth structure of temperatures and densities in the ocean. The temperature increase with height as a result of solar heating near the surface. The vertical density change is mainly caused by temperature (to second order by salinity of the ocean).

5.2 Equations of motion

The quantitative treatment of GFD is primarily computational even relative to other branches of fluid dynamics since laboratory experiments can – due to their small spatial scale – properly address only a very small fraction of interesting questions (e.g.,

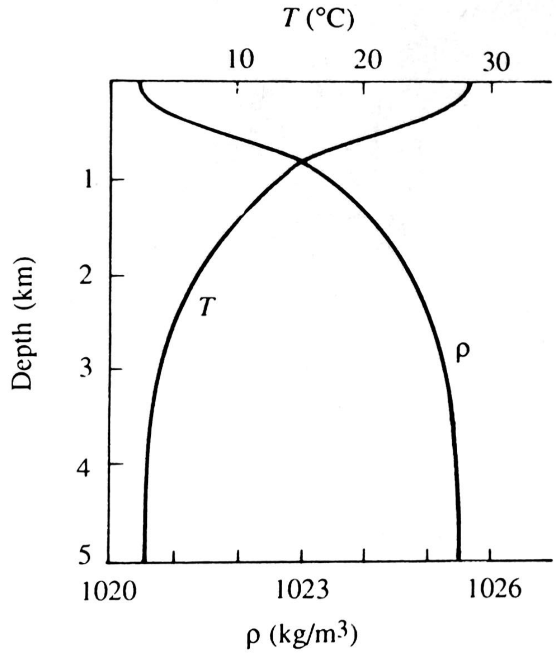

Figure 5.2: Vertical variation of temperature and density in the ocean. (Credit: Kundu, 2002)

small-scale waves, convection, microphysics). Numerical modelling of the ocean and atmosphere circulation and of atmospheric clouds are the largest computational problems in the history of science!

5.2.1 Rotating frame

Note: The following is a description of the mathematical treatment of rotating frames. It is presented here to complement understanding of the rotational terms in the Navier-Stokes momentum equation. The derivations in this section (5.2.1) are however not relevant for examination purposes.

The Navier-Stokes equation of motion discussed so far (Equation 3.10) were given for an inertial frame of reference. The Earth, a rotating planet, is not an inertial frame of reference. Hence, the momentum equation needs to be adjusted to account for the additional accelerations associated with a rotating frame of reference. When looking at small-scale problems (smaller than a typical planetary scale, say $L < 10$ km) then this is not necessary and to a good approximation we can use the inertial frame version of the equation (Eq. 3.10).

Consider the coordinate system depicted in Figure 5.3. There, the system (x_1, x_2, x_3) rotates at an angular velocity $\underline{\mathbf{\Omega}}$ with respect to a fixed frame (X_1, X_2, X_3). Any vector

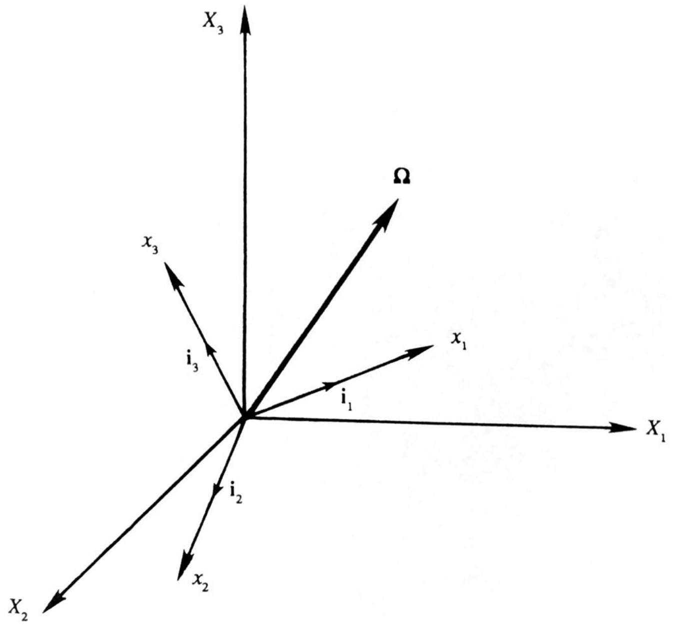

Figure 5.3: Coordinate frame (x_1, x_2, x_3) rotating at an angular velocity $\underline{\mathbf{\Omega}}$ with respect to a fixed frame (X_1, X_2, X_3). (Credit: Kundu, 2002).

$\underline{\mathbf{r}}$ can be written in the rotating frame as

$$\underline{\mathbf{r}} = r_1\,\underline{\mathbf{i_1}} + r_2\,\underline{\mathbf{i_2}} + r_3\,\underline{\mathbf{i_3}} \tag{5.1}$$

For an observer located in an inertial frame of reference, the rotating unit vectors $\underline{\mathbf{i}}_1$, $\underline{\mathbf{i}}_2$ and $\underline{\mathbf{i}}_3$ change with time, so the time derivative of $\underline{\mathbf{r}}$ is:

$$\begin{aligned}\left(\frac{d\,\underline{\mathbf{r}}}{dt}\right)_I &= \frac{d}{dt}\left(r_1\,\underline{\mathbf{i_1}} + r_2\,\underline{\mathbf{i_2}} + r_3\,\underline{\mathbf{i_3}}\right) \\ &= \underline{\mathbf{i_1}}\frac{dr_1}{dt} + \underline{\mathbf{i_2}}\frac{dr_2}{dt} + \underline{\mathbf{i_3}}\frac{dr_3}{dt} + r_1\frac{d\underline{\mathbf{i_1}}}{dt} + r_2\frac{d\underline{\mathbf{i_2}}}{dt} + r_3\frac{d\underline{\mathbf{i_3}}}{dt}\end{aligned} \tag{5.2}$$

where the index I denotes the inertial reference frame. The first three terms in Eq. (5.2) are the rate of change of $\underline{\mathbf{r}}$ to the observer in the rotating frame (denoted by index R):

$$\left(\frac{d\,\underline{\mathbf{r}}}{dt}\right)_I = \left(\frac{d\,\underline{\mathbf{r}}}{dt}\right)_R + r_1\frac{d\underline{\mathbf{i_1}}}{dt} + r_2\frac{d\underline{\mathbf{i_2}}}{dt} + r_3\frac{d\underline{\mathbf{i_3}}}{dt} \tag{5.3}$$

Next, we consider the changing unit vectors of the rotating system, shown in Figure 5.4. Each unit vector $\underline{\mathbf{i}}$ traces a cone with a radius of $sin\,\alpha$ (α being a constant angle, e.g. a latitude angle).

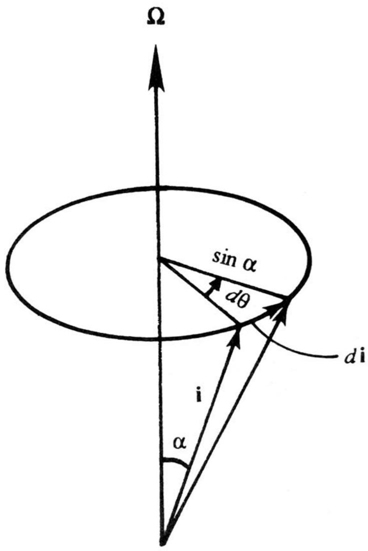

Figure 5.4: Rotation of a unit vector. (Credit: Kundu, 2002).

The scalar magnitude of the change of $\underline{\mathbf{i}}$ in a time interval dt is given by

$$|d\underline{\mathbf{i}}| \;=\; sin\,\alpha\; d\theta \qquad \Rightarrow \qquad \left|\frac{d\underline{\mathbf{i}}}{dt}\right| \;=\; sin\,\alpha \left(\frac{d\theta}{dt}\right) \;=\; \Omega\, sin\,\alpha \tag{5.4}$$

where $\Omega = |\underline{\boldsymbol{\Omega}}|$ is the magnitude of the rotation vector $\underline{\boldsymbol{\Omega}}$. The change of $\underline{\mathbf{i}}$ occurs in the plane perpendicular to the $(\underline{\boldsymbol{\Omega}},\underline{\mathbf{i}})$ plane. Thus, we have

$$\frac{d\,\underline{\mathbf{i}}}{dt} \;=\; \underline{\boldsymbol{\Omega}} \times \underline{\mathbf{i}} \tag{5.5}$$

for every rotating unit vector $\underline{\mathbf{i}}$. As a result, Eq. (5.3) becomes

$$\left(\frac{d\,\underline{\mathbf{r}}}{dt}\right)_I \;=\; \left(\frac{d\,\underline{\mathbf{r}}}{dt}\right)_R \;+\; \underline{\boldsymbol{\Omega}} \times \underline{\mathbf{r}} \tag{5.6}$$

Thereby, the velocities transform between the rotating and inertial coordinate systems as

$$\underline{\mathbf{u}}_I \;=\; \underline{\mathbf{u}}_R \;+\; \underline{\boldsymbol{\Omega}} \times \underline{\mathbf{r}} \tag{5.7}$$

In fact, we may regard Eq. (5.6) as the general transformation equation between the rotating and inertial coordinate systems and apply it to the velocities as well:

$$\left(\frac{d\underline{\mathbf{u}}_I}{dt}\right)_I \;=\; \left(\frac{d\underline{\mathbf{u}}_I}{dt}\right)_R \;+\; \underline{\boldsymbol{\Omega}} \times \underline{\mathbf{u}}_I \tag{5.8}$$

Inserting (5.7) on the right side of Eq. (5.8) gives for the velocity transformation

$$\begin{aligned}\left(\frac{d\underline{\mathbf{u}}_I}{dt}\right)_I &= \frac{d}{dt}\left(\underline{\mathbf{u}}_R + \underline{\mathbf{\Omega}} \times \underline{\mathbf{r}}\right)_R + \underline{\mathbf{\Omega}} \times \left(\underline{\mathbf{u}}_R + \underline{\mathbf{\Omega}} \times \underline{\mathbf{r}}\right) \\ &= \left(\frac{d\underline{\mathbf{u}}_R}{dt}\right)_R + \underline{\mathbf{\Omega}} \times \left(\frac{d\underline{\mathbf{r}}}{dt}\right)_R + \underline{\mathbf{\Omega}} \times \underline{\mathbf{u}}_R + \underline{\mathbf{\Omega}} \times (\underline{\mathbf{\Omega}} \times \underline{\mathbf{r}}) \\ &= \left(\frac{d\underline{\mathbf{u}}_R}{dt}\right)_R + 2\,\underline{\mathbf{\Omega}} \times \underline{\mathbf{u}}_R + \underline{\mathbf{\Omega}} \times (\underline{\mathbf{\Omega}} \times \underline{\mathbf{r}})\end{aligned} \tag{5.9}$$

Here, $2\,\underline{\mathbf{\Omega}} \times \underline{\mathbf{u}}_R$ denotes the Coriolis term and $\underline{\mathbf{\Omega}} \times (\underline{\mathbf{\Omega}} \times \underline{\mathbf{r}})$ is the Centrifugal term.

We will in a final step further simplify the Centrifugal term. Let $\underline{\mathbf{r}}_\perp$ be the perpendicular distance vector of the location to the axis of rotation (essentially the vector pointing along the line underneath "$sin\,\alpha$" in Fig. 5.4). Since $\underline{\mathbf{\Omega}}$ is perpendicular to $\underline{\mathbf{r}}_\perp$ we may write $(\underline{\mathbf{\Omega}} \times \underline{\mathbf{r}}) = (\underline{\mathbf{\Omega}} \times \underline{\mathbf{r}}_\perp)$. Note that this is not an approximation, it is only a different way of writing the same term, using $\underline{\mathbf{r}}_\perp$ instead of $\underline{\mathbf{r}}$

We can now apply the vector identity

$$\underline{\mathbf{A}} \times (\underline{\mathbf{B}} \times \underline{\mathbf{C}}) = \underline{\mathbf{B}}\,(\underline{\mathbf{A}} \cdot \underline{\mathbf{C}}) - \underline{\mathbf{C}}\,(\underline{\mathbf{A}} \cdot \underline{\mathbf{B}}) \tag{5.10}$$

to the last term of Eq. (5.9) to obtain

$$\begin{aligned}\underline{\mathbf{\Omega}} \times (\underline{\mathbf{\Omega}} \times \underline{\mathbf{r}}) &= \underline{\mathbf{\Omega}} \times (\underline{\mathbf{\Omega}} \times \underline{\mathbf{r}}_\perp) \\ &= (\underline{\mathbf{\Omega}} \cdot \underline{\mathbf{r}}_\perp)\,\underline{\mathbf{\Omega}} - (\underline{\mathbf{\Omega}} \cdot \underline{\mathbf{\Omega}})\,\underline{\mathbf{r}}_\perp \\ &= -\,(\underline{\mathbf{\Omega}} \cdot \underline{\mathbf{\Omega}})\,\underline{\mathbf{r}}_\perp \\ &= -\Omega^2\,\underline{\mathbf{r}}_\perp\end{aligned} \tag{5.11}$$

Thereby, Eq. (5.9) becomes

$$\left(\frac{d\underline{\mathbf{u}}_I}{dt}\right)_I = \left(\frac{d\underline{\mathbf{u}}_R}{dt}\right)_R + 2\,\underline{\mathbf{\Omega}} \times \underline{\mathbf{u}}_R - \Omega^2\,\underline{\mathbf{r}}_\perp \tag{5.12}$$

5.2.2 Equation of motion in a rotating frame

With the transformations between an inertial and a rotating frame of reference discussed in Section 5.2.1 and expressed in Eq. (5.9) we obtain for the Navier-Stokes equation (in its form given by Eq. (3.6)) in a rotating frame of reference such as planet Earth

$$\boxed{\frac{d\underline{\mathbf{u}}}{dt} = \frac{\partial \underline{\mathbf{u}}}{\partial t} + (\underline{\mathbf{u}} \cdot \nabla)\underline{\mathbf{u}} = -\frac{1}{\rho}\nabla P + \underline{\mathbf{g}} + \mu\nabla^2\underline{\mathbf{u}} + \Omega^2\underline{\mathbf{r}}_\perp - 2\,\underline{\mathbf{\Omega}} \times \underline{\mathbf{u}}} \tag{5.13}$$

where the dynamic viscosity η used in Eq. (3.6) was replaced with the kinematic viscosity μ using Eq. (3.21).

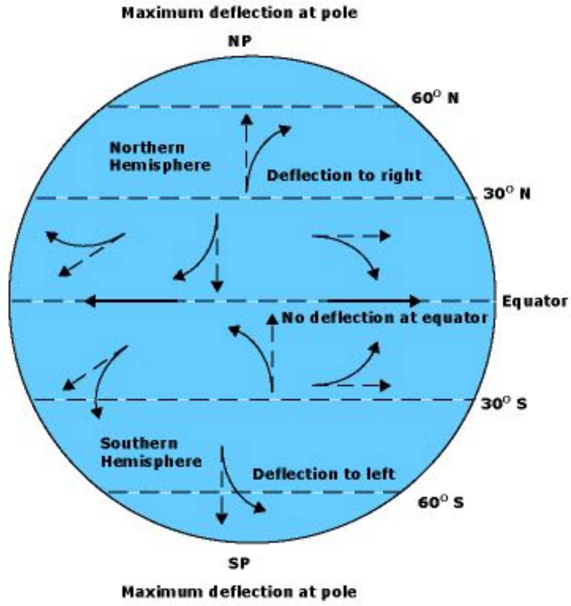

Figure 5.5: Schematic of the effects of Coriolis acceleration on atmospheric winds. The dashed arrows indicate the flow direction in a non-rotating system.

The term $(-2\,\underline{\mathbf{\Omega}} \times \underline{\mathbf{u}})$ in Eq. (5.13) represents the *Coriolis acceleration.* As shown in Figure 5.5, it deflects a particle traveling in the northern hemisphere to the right of its direction of travel (i.e., it diverts it in clockwise direction in the northern hemisphere, anti-clockwise in the southern hemisphere). Note that the Coriolis force is a pseudo-force that appears in a rotating frame of reference. Since it always acts at a right angle to a particle's velocity vector the Coriolis force does no work on the particle.

As previously when we went from equation (3.6) to (3.10), we may rewrite the Navier-Stokes momentum equation in a rotating frame of reference as

$$\boxed{\frac{\partial \underline{\mathbf{u}}}{\partial t} \;=\; -\frac{1}{\rho}\nabla P \;+\; \underline{\mathbf{g}} \;+\; \mu\nabla^2\underline{\mathbf{u}} \;-\; \frac{1}{2}\nabla(u^2) \;-\; (\nabla\times\underline{\mathbf{u}})\times\underline{\mathbf{u}} \;+\; \mathbf{\Omega^2}\underline{\mathbf{r}}_\perp \;-\; \mathbf{2}\;\underline{\mathbf{\Omega}}\times\underline{\mathbf{u}}} \tag{5.14}$$

If we apply the *thin sheet approximation* whereby vertical winds (u_z) are considered to be much smaller than horizontal winds, the Coriolis term can be simplified may its local Cartesian components (x: north-south, y: west-east, z: vertical) be written as

$$\begin{aligned}
(2\,\underline{\mathbf{\Omega}}\times\underline{\mathbf{u}})_x &= -(2\,\Omega\, sin\theta)\, u_y \;=\; -f\, u_y \\
(2\,\underline{\mathbf{\Omega}}\times\underline{\mathbf{u}})_y &= (2\,\Omega\, sin\theta)\, u_x \;=\; f\, u_x \\
(2\,\underline{\mathbf{\Omega}}\times\underline{\mathbf{u}})_z &= (2\,\Omega\, cos\theta)\, u_y
\end{aligned} \tag{5.15}$$

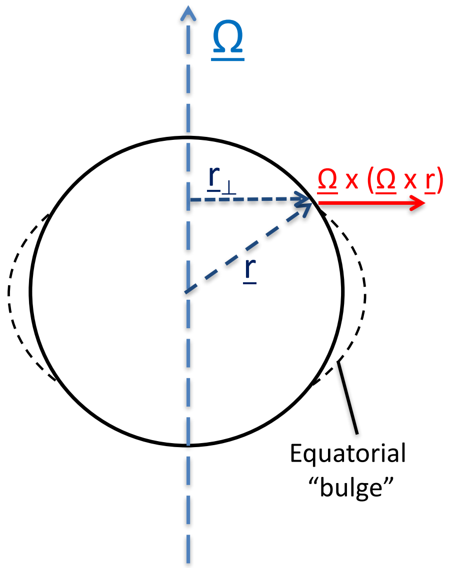

Figure 5.6: The centrifugal acceleration term.

where θ denotes the latitude ($\theta = 0$ at equator, $\theta = \pm 90°$ at north / south poles, respectively) and we defined the variable f (called the *Coriolis parameter*) as

$$f = 2\,\Omega\, sin\,\theta \tag{5.16}$$

The Coriolis parameter is often used in Geophysical Fluid Dynamics.

5.2.3 The centrifugal acceleration term

Often it is useful to find approximate values for the terms in the momentum equation (5.13) in order to carry out simple analyses and understand the flow of atmospheric or ocean particles without fully solving the equation of motion. Here, we will investigate the magnitude of the centrifugal term $\underline{\mathbf{\Omega}} \times (\underline{\mathbf{\Omega}} \times \underline{\mathbf{r}})$ (or $-\Omega^2\, \underline{\mathbf{r}}_\perp$, the two expressions are mathematically equivalent ($\underline{\mathbf{r}}_\perp$ being the component of $\underline{\mathbf{r}}$ perpendicular to $\underline{\mathbf{\Omega}}$)) in the horizontal and radial directions. The centrifugal term is illustrated in Figure 5.6.

We have for Earth a rotation frequency of $\Omega \approx 7.27{\cdot}10^{-5}\ \mathrm{s}^{-1}$ and radius of $r \approx 6.4{\cdot}10^{6}$ m. Gravity acceleration is on average $g \approx 9.81\ \mathrm{m/s}^2$.

1. *Radial direction*: Here we can compare gravity to the centrifugal acceleration term at the equator (θ=0):

$$\alpha = \frac{|\text{radial centrif. term}|}{|\text{gravity}|} = \frac{\Omega^2\, r\cos^2(\theta)}{g} = \frac{(7.27\cdot 10^{-5})^2\cdot 6.4\cdot 10^6}{9.81} \approx 3\cdot 10^{-3} \tag{5.17}$$

This result illustrates that Newtonian gravity is much larger than the centrifugal accelerations caused by Earth's rotation. On other planets, in particular Saturn, this ratio is considerably larger, $\alpha_{Saturn} \approx 0.16$: Saturn is more oblate and has an equatorial "bulge", see Figure 5.6!

2. *Horizontal direction*: The centrifugal acceleration also has a horizontal component which can be much larger when compared to other relevant quantities, in particular Coriolis acceleration. Their ratio is for mid-latitudes ($\theta = 45°$), assuming an average wind speed of $u \approx 10$ m/s:

$$\beta = \frac{\text{horiz. centrif. term}}{\text{Coriolis term}} \approx \frac{\Omega^2\, r \cos(\theta)\sin(\theta)}{2\,\Omega\, u \sin(\theta)} = \frac{\Omega\, r\cos(\theta)}{2\,u} \approx 30 \tag{5.18}$$

This shows that the horizontal centrifugal term can dominate over the Coriolis term. It is in fact often largely balanced by the pressure gradient term. This may confuse you: in Section 5.3 we introduce the geostrophic balance where pressure gradients are balanced by Coriolis forces. What about the centrifugal forces in that case? Importantly, the centrifugal acceleration always points in the same direction, independent of wind velocity, while the Coriolis acceleration changes with wind velocity. So, the centrifugal acceleration can be regarded as a quasi-uniform background force which is super-imposed by Coriolis and pressure forces. On a global scale, background pressure gradients are balanced by centrifugal forces, but when it comes to smaller–scale weather systems, those are dominated by the balance between perturbations in pressure and Coriolis. So, the centrifugal forces are something to keep in the back of your mind but they don't affect the weather systems that are dominated by geostrophic flow.

5.2.4 Rossby number

Often it is valid to neglect the nonlinear acceleration terms of the momentum equation (specifically, the advection/inertia term – recall that when we went from equation (3.6) to (3.10) we converted the advection term into an inertia term). These are of the order of u^2/L (u being the wind speed and L a horizontal length scale). The Coriolis parameter is of the order of $f\,u$. The ratio of nonlinear terms (or, inertia) to the Coriolis term is called the *Rossby number:*

$$Ro = \frac{\text{Inertia}}{\text{Coriolis acceleration}} \approx \frac{u^2/L}{fu} = \frac{u}{fL} \tag{5.19}$$

In the atmosphere, $u \approx 10$ m/s, $f \approx 10^{-4}$ s^{-1} and $L \approx 1000$ km, giving $Ro \approx 0.1$. The value can be even smaller in the ocean and justifies our neglect of the nonlinear (inertial) terms. However, when looking at phenomena in the atmosphere on smaller scale (small L) then Ro can exceed 1.

- If $Ro \ll 1$ then Coriolis acceleration dominates: this applies to large–scale flows. Often, the *geostrophic balance* (see Section 5.3) holds in those regimes.

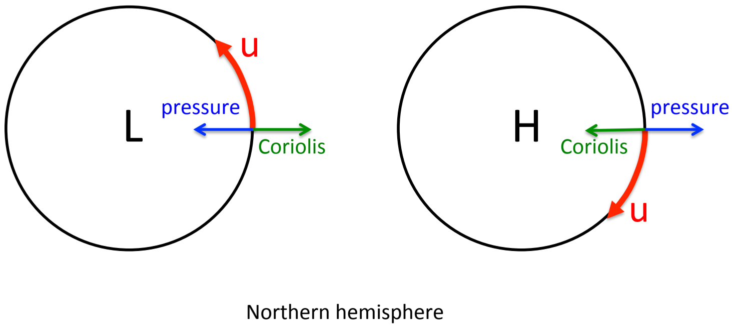

Figure 5.7: Geostrophic flow around low and high pressure centres. The pressure acceleration ($-\nabla p$) is indicated as a blue arrow, the Coriolis acceleration as a green arrow. The thick red arrow denotes the geostrophic wind direction. The geostrophic wind direction is opposite in the southern hemisphere.

- If $Ro \gg 1$ then inertia dominates: this applies to smaller–scale problems (e.g., shooting a soccer ball over the field. Here, Coriolis is irrelevant and the ball flies into the direction in which it was shot, without any diversion due to rotational forces). The eye of a cyclone also is a regime of large Ro, where Coriolis forces play a minor role in the flow.

5.3 Geostrophic flow

Lecture 9

Large-scale flows (away from boundaries and in steady state) are often well approximated by the balance between Coriolis acceleration and pressure gradient, called the *geostrophic approximation*. The geostrophic approximation consists in equating the Coriolis and pressure accelerations, which in cartesian coordinates is given by:

$$\begin{aligned} f\, u_y &= \frac{1}{\rho}\frac{\partial p}{\partial x} \quad \Rightarrow \quad u_y = \frac{1}{f\rho}\frac{\partial p}{\partial x} \\ f\, u_x &= -\frac{1}{\rho}\frac{\partial p}{\partial y} \quad \Rightarrow \quad u_x = -\frac{1}{f\rho}\frac{\partial p}{\partial y} \end{aligned} \tag{5.20}$$

The geostrophic balance of Eq.(5.20) is useful in that it allows to calculate winds from pressure measurements alone. As illustrated in Figures 5.7 and 5.8, geostrophic winds flow roughly along isobars (i.e., not across them), clockwise around high pressure regions and anti-clockwise around low pressure regions (in the northern hemisphere).

See also: Problem Sheet 3, Question 3

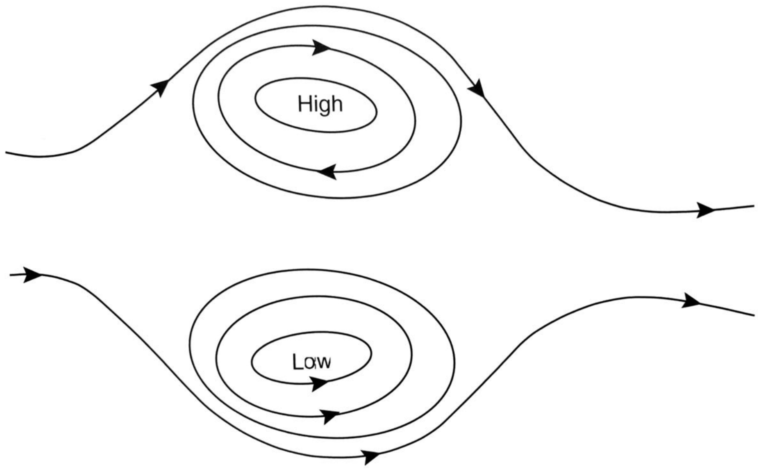

Figure 5.8: Schematic of geostrophic flow with a positive value of the Coriolis parameter f (i.e., northern hemisphere conditions). For negative f (southern hemisphere) these directions are opposite (Credit: Vallis, 2006).

5.4 Thermal wind

The thermal wind equation relates vertical changes of wind (or fluid-) velocity to horizontal density gradients. This is useful for situations where horizontal profiles of density measurements are present to infer wind velocities. This method is popular with oceanographers and also in atmospheric physics.

The thermal wind equation is obtained by using the geostrophic wind equations (5.20) in combination with the hydrostatic balance equation given by

$$dp = -\rho\, g\, dz \tag{5.21}$$

where dp is a change in pressure, ρ the mass density, g gravitational acceleration and dz a change in height. The relation states that any vertical change in pressure is equal to the change in weight of the atmosphere (or ocean) above. This is related to the fundamental concept that the pressure is equal to the weight of the atmosphere/ocean above (divided by a column area).

We may use equation (5.21) to replace the pressure in equations (5.20) with densities and obtain the thermal wind equations for the two horizontal components as

$$\begin{aligned} \frac{\partial u_y}{\partial z} &= -\frac{g}{\rho\, f}\,\frac{\partial \rho}{\partial x} \\ \frac{\partial u_x}{\partial z} &= \frac{g}{\rho\, f}\,\frac{\partial \rho}{\partial y} \end{aligned} \tag{5.22}$$

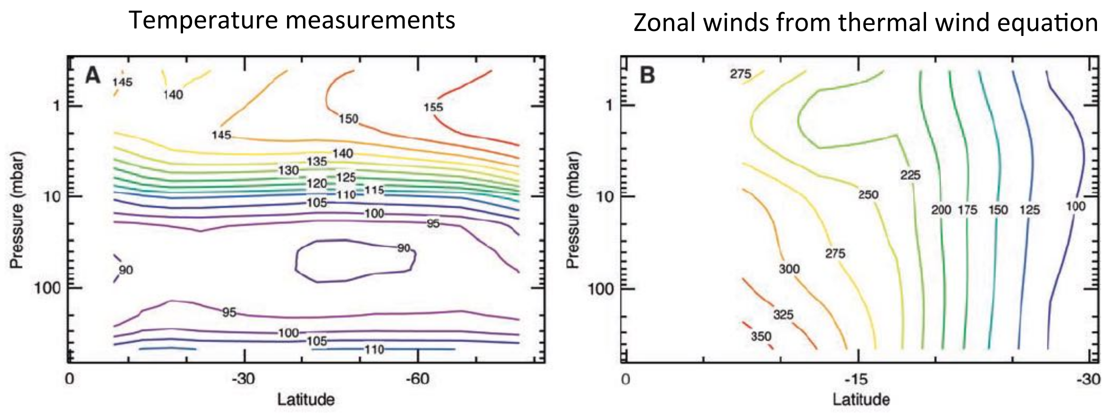

Figure 5.9: Zonally averaged temperatures and zonal winds above Saturn's clouds. The left panel shows temperatures, the right panel shows zonal winds obtained from the thermal wind equation. Temperatures were observed by the Cassini Composite Infrared Spectrometer (CIRS) in late 2002 during Cassini's approach towards Saturn. (from Flasar et al., Science, 307, 2005)

If the velocity is known at some height level then the horizontal velocities can be estimated by integrating up from the level where the velocity is known by using the thermal wind equations (5.22).

These equations can be further converted to allow calculation of the wind field from temperatures (instead of densities). For a thin atmosphere the zonally (longitudinally) averaged thermal wind equation in terms of latitudinal temperature gradients is in spherical coordinates (r, θ, ϕ) given by

$$\frac{\partial}{\partial r}\left(f\,u_\phi + \frac{u_\phi^2\,tan(\theta)}{r}\right) = -\frac{g}{T}\,\frac{1}{r}\left(\frac{\partial T}{\partial \theta}\right)_p \tag{5.23}$$

This relates zonally averaged vertical (radial) gradients of *zonal* (east-west) winds u_ϕ at latitude θ to latitudinal gradients of temperature T on a level of constant pressure p. Gravity acceleration is g and r is the planet radius. Since we averaged longitudinally, the other horizontal wind component (the north-south, or *meridional* wind) is zero.

Equation (5.23) is useful for inferring winds from temperature measurement and is widely used on Earth and to study other planets. An example for Saturn is shown in Figure 5.9. Since winds cannot easily be measured remotely the thermal wind equation is invaluable since it at least allows scientists to calculate them from temperature or density measurements.

– END –