Introduction to Symbolic AI
Coursework 1: Logic

Total possible: 72 marks.

1. [**10 marks**] Formalize each of the following in propositional logic, including all brackets required by the *strict* definition of a propositional formula (remember to give the correspondence between the basic sentences of the original and the propositional atoms):

    i. If Michel isn't either fulfilled or rich, he won't live another five years.

    ii. Unless the snowstorm doesn't arrive, Raheem will wear his boots; but I'm sure it *will* arrive.

    iii. If Akira and Toshiro are on set, then filming will begin if and only if the caterers have cleared out.

    iv. Either Irad arrived, or Sarah didn't: but not both!

    v. It's not the case both that Herbert heard the performance and Anne-Sophie did, if the latter didn't answer her phone calls.

    Solution

    [**2 marks per part: basically, 1 for the atoms, 1 for the form**.]

    i. $((\neg(p \vee q)) \rightarrow (\neg r))$
    p: Michel is fulfilled.
    q: Michel is rich.
    r: Michel will live for another five years.

    ii. $(((\neg p) \vee q) \wedge r)$
    p: The snowstorm will arrive.
    q: Raheem will wear his boots.
    r: I'm sure the snowstorm will arrive.

    iii. $((p \wedge q) \rightarrow (r \leftrightarrow s))$
    p: Akira is on set.
    q: Toshiro is on set.
    r: Filming will begin.
    s: The caterers have cleared out.

    iv. $((p \vee (\neg q)) \wedge \neg(p \wedge (\neg q)))$
    p: Irad arrived.
    q: Sarah arrived.

    v. $((\neg r) \rightarrow (\neg(p \wedge q)))$
    p: Herbert heard the performance.
    q: Anne-Sophie heard the performance.
    r: Anne-Sophie answered her phone calls.

2. [**8 marks**]

    i. What is the definition of the *satisfiability* of a propositional formula, A?

    ii. What is the definition of the *logical equivalence* of two propositional formulas A and B?

    iii. Prove that a propositional formula $\neg A$ is satisfiable iff $\neg\neg A \not\equiv \top$ (i.e., iff it's not the case that $\neg\neg A \equiv \top$).

    Solution

    [**2 marks each for (i) and (ii); 4 marks for (iii)**]

i. A propositional formula A is *satisfiable* iff there is an atomic evaluation function v such that $h_v(A) = \mathbf{t}$.

[Note: this could be stated with a bit more prose and a bit less 'symbolism', but the basic idea of atomic evaluation functions, etc., has to be present.]

ii. Two propositional formulas A and B are logically equivalent iff $h_v(A) = h_v(B)$, for all atomic evaluation functions v.

[Same point about symbolism as for (i).]

iii. For propositional A, then $\neg A$ is satisfiable

iff there is an atomic evaluation function v such that $h_v(\neg A) = \mathbf{t}$
[by the definition of satisfiability]

iff there is an atomic evaluation function v such that $h_v(\neg\neg A) = \mathbf{f}$
[by the clause for $\neg$ in Def. 1.5]

iff there is an atomic evaluation function v such that $h_v(\neg\neg A) \neq \mathbf{t}$
[obviously, since $\mathbf{t} \neq \mathbf{f}$!]

iff there is an atomic evaluation function v such that $h_v(\neg\neg A) \neq h_v(\top)$
[since $h_v(\top) = \mathbf{t}$, by Def. 1.5]

iff $\neg\neg A \not\equiv \top$
[by the definition of logical equivalence]

3. [**8 marks**] Use truth-tables to determine whether the following is valid or not:

$(p \wedge \neg q \leftrightarrow \neg(\neg r \vee \neg p)) \to (\neg\neg q \to r)$.

Solution

[**Roughly 1 mark per row; but also bearing in mind its general construction. If students don't construct the whole table, that is fine: full marks should be awarded if only a relevant row is shown.**]

p	q	r	$(p$	$\wedge$	$\neg$	q	$\leftrightarrow$	$\neg($	$\neg$	r	$\vee$	$\neg$	$p))$	$\longrightarrow$	$(\neg$	$\neg$	q	$\longrightarrow$	$r)$
t	t	t	t	f	f	t	f	t	f	t	f	f	t	t	t	f	t	t	t
t	t	f	t	f	f	t	t	f	t	f	t	f	t	f	t	f	t	f	f
t	f	t	t	t	t	f	t	t	f	t	f	f	t	t	f	t	f	t	t
t	f	f	t	t	t	f	f	f	t	f	t	f	t	t	f	t	f	t	f
f	t	t	f	f	f	t	t	f	f	t	t	t	f	t	t	f	t	t	t
f	t	f	f	f	f	t	t	f	t	f	t	t	f	f	t	f	t	f	f
f	f	t	f	f	t	f	t	f	f	t	t	t	f	t	f	t	f	t	t
f	f	f	f	f	t	f	t	f	t	f	t	t	f	t	f	t	f	t	f

The formula is clearly *invalid*.

4. [**14 marks**]

i. Which of the following are in CNF? Which are in DNF?

a. $p \wedge (\neg q \vee r)$

b. $\neg p$

c. $p \wedge (q \vee (p \wedge r))$

d. $\top$

e. $(p \wedge q) \vee (p \wedge q)$

f. $\neg\neg p \wedge (q \vee p)$

g. $p \wedge q$

h. $p \vee q$

ii. Define the property of the *refutation-soundness and -completeness* of a resolution derivation. Why is this property important?

iii. Apply unit propagation and the pure rule repeatedly, in order to reduce the following to their simplest forms (stating which rule you're applying, and indicate the literal involved):

a. $\{\{p, s\}, \{q, r\}, \{\neg s, q\}, \{\neg p, \neg r, \neg s\}\}$

b. $\{\{\neg p, q, r\}, \{\neg q\}, \{p, r, q\}, \{\neg r, q\}\}$

Solution

[**1 mark for each part of (i); 2 marks for (ii); 2 marks for each part of (iii)**]

i. a. CNF.
   b. CNF and DNF.
   c. Neither CNF nor DNF.
   d. CNF and DNF.
   e. DNF.
   f. Neither CNF nor DNF.
   g. CNF and DNF.
   h. CNF and DNF.

ii. Let $S \vdash_{\text{res(PL)}} A$ represent that there is a propositional resolution-derivation of A from S. Then the refutation-soundness and -completeness of propositional resolution is just the fact that $S \vdash_{\text{res(PL)}} \emptyset$ iff $S \models \bot$.

This is important, because in SAT-solving we are checking the satisfiability of some CNF S, and the refutation-soundness and -completeness means that S is satisfiable iff there is no resolution-derivation of $\emptyset$ from S. So if we can show there is no such derivation, we have demonstrated satisfiability.

iii. Note that there may be other ways to apply these. (It does not matter which rule is applied first, or which literal is chosen, etc.)

a. $\{\{p, s\}, \{q, r\}, \{\neg s, q\}, \{\neg p, \neg r, \neg s\}\}$
$\Rightarrow \{\{p, s\}, \{\neg p, \neg r, \neg s\}\}$ [by purity of q]
$\Rightarrow \{\{p, s\}\}$ [by purity of $\neg r$]
$\Rightarrow \{\}$ [by purity of p (or of s)]

b. $\{\{\neg p, q, r\}, \{\neg q\}, \{p, r, q\}, \{\neg r, q\}\}$
$\Rightarrow \{\{\neg p, r\}, \{p, r\}, \{\neg r\}\}$ [unit propagation on $\{\neg q\}$]
$\Rightarrow \{\{\neg p\}, \{p\}\}$ [unit propagation on $\{\neg r\}$]
$\Rightarrow \{\{\}\}$ [unit propagation on $\{\neg p\}$ (or on $\{p\}$)]

5. [**8 marks**] Use DP to determine whether the following argument is valid or not:

If I'm going, then you aren't.
If you're not going, then neither is Tara.
Either Tara's going or I'm not.
Tara's going unless I am.
So, you're going.

Solution

[**2 for the formalization; 2 for the initial argumentation; 3 for the application of DP; 1 for the conclusion**.]

We first formalize the argument in propositional logic.

Use:
p: I'm going.
q: You're going.
r: Tara's going.

The formalization of the argument is then:
$p \to \neg q$,
$\neg q \to \neg r$,
$r \vee \neg p$,

$r \vee p$.
So, q.

An argument $A_1, \ldots, A_n \models B$ is propositionally valid iff $(\bigwedge A_i) \wedge \neg B$ is unsatisfiable.

In the current case, this means checking whether $(p \to \neg q) \wedge (\neg q \to \neg r) \wedge (r \vee \neg p) \wedge (r \vee p) \wedge \neg q$ is unsatisfiable.

Since $(A \to B) \equiv (\neg A \vee B)$, then we must check the satisfiability of the CNF:

$$(\neg p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p) \wedge (r \vee p) \wedge \neg q$$

We then apply DP, using clausal-form notation:

$\{\{\neg p, \neg q\}, \{q, \neg r\}, \{r, \neg p\}, \{r, p\}, \{\neg q\}\}$
$\Rightarrow \{\{\neg r\}, \{r, \neg p\}, \{r, p\}\}$ [unit propagation on $\{\neg q\}$]
$\Rightarrow \{\{\neg p\}, \{p\}\}$ [unit propagation on $\{\neg r\}$]
$\Rightarrow \{\{\}\}$ [unit propagation on $\{p\}$ (or on $\{\neg p\}$)]
$\Rightarrow$ UNSATISFIABLE

So, the argument is *valid*.

6. [**12 marks**] Translate into first-order logic, giving as much logical structure as possible. Be sure to specify the signature for each part.

   i. All of Andrea's aunts' aunts gave a cupcake to someone other than Andrea.
   ii. There's a computer connected to every computer which isn't connected to itself.
   iii. Any painting by Paul Klee in a British gallery hangs in a room where all Kandinsky paintings in that gallery hang.
   iv. If there's somebody who loves nobody, then it's false that everybody loves somebody.

Solution

[**3 marks each: one for the signature, 2 for the formalization**.] For translations into logic, 1 mark for the signature, 1 for the logical form.]

i. Signature:

$$\begin{aligned} \mathcal{C} &= \{andrea\} \\ \mathcal{P}_1 &= \{c\} \\ \mathcal{P}_2 &= \{e\} \\ \mathcal{P}_3 &= \{g\} \end{aligned}$$

All other sets empty.
$c(X)$: X is a cupcake.
$a(X, Y)$: X is an aunt of Y.
$g(X, Y, Z)$: X gave Y to Z.
$\forall X(\exists Y(a(X, Y) \wedge a(Y, andrea)) \to \exists Z \exists W(c(Z) \wedge g(X, Z, W) \wedge W \neq andrea))$

ii. Signature:

$$\begin{aligned} \mathcal{C} &= \{\} \\ \mathcal{P}_1 &= \{c\} \\ \mathcal{P}_2 &= \{e\} \end{aligned}$$

All other sets empty.
$c(X)$: X is a computer.
$e(X, Y)$: X is connected to Y.
$\exists X(c(X) \wedge \forall Y(c(Y) \wedge \neg e(Y, Y) \to e(X, Y)))$

iii. Signature:

$$\begin{aligned} \mathcal{C} &= \{klee, kandinsky\} \\ \mathcal{P}_1 &= \{p, g, b, r\} \\ \mathcal{P}_2 &= \{by, h, in\} \end{aligned}$$

All other sets empty.
$p(X)$: X is a painting.
$g(X)$: X is a gallery.
$b(X)$: X is British.
$r(X)$: X is a room.
$by(X,Y)$: X is by Y.
$h(X,Y)$: X hangs in Y.
$in(X,Y)$ X is in Y.

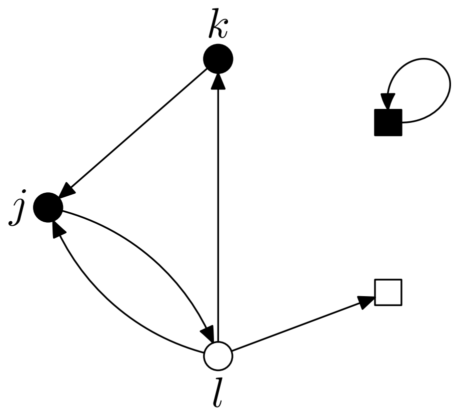
$$\begin{aligned}&\forall X \forall Y (p(X) \land by(X, klee) \land in(X,Y) \land g(Y) \land b(Y)\\ &\qquad \to \forall Z(p(Z) \land by(Z, kandinsky) \land in(Z,Y)\\ &\qquad\qquad \to \exists W(room(W) \land h(X,W) \land h(Z,W))))\end{aligned}$$

iv. Signature:

$$\mathcal{C} = \{\}$$
$$\mathcal{P}_1 = \{\}$$
$$\mathcal{P}_2 = \{l\}$$

$l(X,Y)$: X loves Y.
$\exists X \forall Y \neg l(X,Y) \to \neg \forall X \exists Y\, l(X,Y)$

7. [**12 marks**] Let $\mathcal{L}$ be a signature containing just four unary predicate symbols b, w, s and c, and a single binary relation symbol a; and three constants j, k and l. Consider the following $\mathcal{L}$-structure (D, φ), containing five objects:



The objects $\varphi(j)$, $\varphi(k)$ and $\varphi(l)$ are shown (indicated by the relevant letters). Further:

$\varphi(b)$ is the set of filled ('black') objects
$\varphi(w)$ is the set of unfilled ('white') objects
$\varphi(s)$ is the set of square objects
$\varphi(c)$ is the set of circular objects.
$\varphi(a)$ is the set of pairs (x,y) such that there is a directed arrow from x to y

For example, the object $\varphi(k)$, to the top in the diagram, is in $\varphi(b)$ and $\varphi(c)$, since it is drawn filled and circular.

Determine, for each of the following, whether it is true or false, and provide a justification in each case.

i. $\forall X(a(k,X) \to \neg(X = j))$
ii. $c(l) \to \exists X(b(X) \land c(X) \land a(l,X))$
iii. $\exists X \neg \exists Y(\neg(X = Y) \land a(X,Y))$
iv. $\forall X(\neg s(X) \to \exists Y(c(Y) \land b(Y) \land a(X,Y)))$
v. $\forall X(\exists Y(\neg(X = Y) \land a(X,Y)) \to \exists Y(a(X,Y) \land a(Y,X)))$
vi. $\forall X \forall Y(a(X,j) \land a(Y,j) \to (a(X,Y) \lor a(Y,X)))$

Solution

[**2 marks each. 1 mark for the true/false; 1 mark for the justification.**]

i. False. This just states that everything an arrow from k points to is not identical with j; whereas, in fact everything an arrow from k points to *is* identical with j.

   In more detail: if we take X to be j, then the antecedent $a(k, X)$ is true, but then the consequent states that $\neg(X = j)$ which is clearly false where X is j; a conditional where the antecedent is true and the consequent false is clearly false. (There is no need to check the case for the other possibilities for X—one is enough to show the claim is false, given the $\forall X$.)

ii. True. The antecedent states that l is a circle: clearly the case. So for the truth of the conditional, the consequent must be true. The consequent states that there is something black and circular which an arrow from l points to. There clearly is: k.

iii. True. This states that there is something, X, such that there is nothing X 'points to' which isn't X. In other words: there is something which doesn't point to anything other than itself. The black square at top right is such an object.

iv. False. This states that everything which isn't a square 'points to' a black circle. In other words, it states that every circle points to a black circle. This is true of l and k—so, if we take X to be l or k, the inner conditional is true. But if we take X to be j, the antecedent is true (since $\neg s(X)$ is true of j, because j is not a square), but the consequent is false, since j has no outgoing arrow pointing to a *black* circle.

v. False. This states that for any X, if X 'points to' something different from itself, then there is a 'two step' path from X to itself. For the formula to be true, then any X which makes the inner conditional's antecedent true must make its consequent true. So we just need to consider each X which points to something different from itself. Those are j, k and l—since the unfilled square at bottom-right doesn't point to *anything*, and the filled square at top-right doesn't point to anything different from itself. Now, there is a two-step path from l to itself—the path (l, j, l). And there is a two-step path from j to itself—the path (j, l, j). But there is no two-step path from k to itself. So if we take X to be k, the inner conditional is false, so the entire formula is false.

vi. False. This states that for any X which has an edge to j, and any Y which has an edge to j, then there is an edge in at least one direction between X and Y. If we take X and Y to be k and l (or vice versa), then the inner conditional is true. However, if we take X to be k and Y to be k, then the antecedent is true, but since there is no 'self-edge' from k to itsel, the consequent is false.