

499 fbelard 6  
c4 bb816 v1



Electronic submission

Wed - 19 Feb 2020 22:36:00

**bb816**

### Exercise Information

**Module:** 499 Modal Logic for Strategic Reasoning in AI

**Issued:** Wed - 05 Feb 2020

**Exercise:** 6 (CW)

**Due:** Wed - 19 Feb 2020

**Title:** Coursework2

**Assessment:** Individual

**FAO:** Belardinelli, Francesco (fbelard)

**Submission:** Electronic

### Student Declaration - Version 1

- I declare that this final submitted version is my unaided work.

Signed: (electronic signature) Date: 2020-02-11 15:58:26

**For Markers only:** (circle appropriate grade)

BARATH, Boris (bb816)	01187289	c4	2020-02-11 15:58:26	A*	A	B	C	D	E	F
-----------------------	----------	----	---------------------	----	---	---	---	---	---	---

1) a)  $(M, \pi) \models \varphi R \psi$

,  $\varphi$  releases  $\psi$   
iff  $\left[ \begin{array}{l} (\pi[i..j] \models \psi \text{ and } \pi[j] \models \varphi \text{ for some } i \geq 0 \text{ for } j \geq i) \\ \text{OR } (\pi[i.. \infty] \models \psi \text{ for all } i \geq 0) \end{array} \right]$  (1)  
OR  $(\pi[i.. \infty] \models \psi \text{ for all } i \geq 0)$  (2)  
 $\downarrow \psi \text{ true forever}$

b)  $\varphi R \psi \equiv (\varphi \wedge \psi) \vee (\psi \wedge X(\psi \vee (\varphi \wedge \psi)))$

c) the truth conditions match the LTL formula as:

1st part of the disjunction in the LTL formula satisfies  
the case if  $i = j$  in (1)

2nd part of the disjunction says that either  $\psi$  will  
remain true indefinitely, which satisfies (2)  
or that  $\psi$  will remain true until  $(\psi \wedge \varphi)$  is true  
which satisfies the  $j > i$  case from (1)

d)  $G \psi$  says that  $\psi$  will always be true.  
since  $p R q$  says  $q$  will hold until and once  
 $p$  becomes true, using the fact that  $\perp$  will  
never be true, we get that  $q$  will always be true  
 $\equiv Gq$  where  $p = \perp, q = \psi$

2) for conciseness, I will use  $\exists$  instead 'for some'  
and  $\forall$  instead 'for all'

$(M, q) \models EF\phi$  iff  $\exists \lambda \text{ from } q, \exists j \geq 0, (M, \lambda[j]) \models \phi$   
iff  $\exists \lambda \text{ from } q, \exists j \geq 0, (M, \lambda[j]) \models \phi$   
always true  $\rightarrow$  and  $\forall i, 0 \leq i < j, (M, \lambda[i]) \models \text{true}$   
iff  $(M, q) \models E(\text{true} \cup \phi)$

$(M, q) \models AF\phi$  iff  $\forall \lambda \text{ from } q, \exists j \geq 0, (M, \lambda[j]) \models \phi$   
iff  $\forall \lambda \text{ from } q, \exists j \geq 0, (M, \lambda[j]) \models \phi$   
always true  $\rightarrow$  and  $\forall i, 0 \leq i < j, (M, \lambda[i]) \models \text{true}$   
iff  $(M, q) \models A(\text{true} \cup \phi)$

$(M, q) \models EG\phi$  iff  $\exists \lambda \text{ from } q, \forall j \geq 0, (M, \lambda[j]) \models \phi$   
iff  $\neg \exists \lambda \text{ from } q, \forall j \geq 0, (M, \lambda[j]) \models \phi$   
iff  $\neg \forall \lambda \text{ from } q, \forall j \geq 0, (M, \lambda[j]) \models \phi$   
iff  $\neg \forall \lambda \text{ from } q, \exists j \geq 0, (M, \lambda[j]) \models \neg \phi$   
iff  $(M, q) \models \neg AF \neg \phi$

$(M, q) \models AG\phi$  iff  $\forall \lambda \text{ from } q, \forall j \geq 0, (M, \lambda[j]) \models \phi$   
iff  $\neg \neg \forall \lambda \text{ from } q, \forall j \geq 0, (M, \lambda[j]) \models \phi$   
iff  $\neg \exists \lambda \text{ from } q, \neg \forall j \geq 0, (M, \lambda[j]) \models \phi$   
iff  $\neg \exists \lambda \text{ from } q, \exists j \geq 0, (M, \lambda[j]) \models \neg \phi$   
iff  $(M, q) \models \neg EF \neg \phi$

3) Show that CTL is synt. frag. of CTL\*

a) proof by comparing syntax of CTL and CTL\*

let  $\phi, \psi$  be arbitrary state formulas,  $\sigma, \pi$  arbitrary path forml.

state formulas:	CTL	CTL*	path formulas:	CTL	CTL*
	$p$	$p$			$\phi$ (path form.)
	$\neg\phi$	$\neg\phi$			$\neg\pi$
	$\phi \wedge \psi$	$\phi \wedge \psi$			$\pi \wedge \sigma$
	$E\phi$	$E\pi$		$X\phi$	$X\pi$
	$A\pi$	$A\pi$		$\phi \vee \psi$	$\pi \vee \sigma$

from this comparison, it becomes apparent that CTL syntax is a subset of CTL\*, since CTL only supports state formulas after  $X$  and in  $\cup$ , while CTL\* supports both state and path formulas after  $X$  and in  $\cup$ .

b) can be expressed in CTL\* but not CTL:  $AFG\phi$   
 $EXX\phi$

4) Let  $M$  be a model,  $s$  a state,  $\pi$  a path,  
 $\phi, \phi'$  state formulas and  $\psi, \psi'$  path formulas

if we restrict semantics of  $CTL^*$  to  $CTL$ , we get:

$$(M, s) \models p \text{ iff } s \in V(p)$$

$$(M, s) \models \neg \phi \text{ iff } (M, s) \not\models \phi$$

$$(M, s) \models \phi \wedge \phi' \text{ iff } (M, s) \models \phi \text{ and } (M, s) \models \phi'$$

$$(M, s) \models E \psi \text{ iff for some path } \pi \text{ from } s, (M, \pi) \models \psi$$

$$(M, s) \models A \psi \text{ iff for all paths } \pi \text{ from } s, (M, \pi) \models \psi$$

|

these semantic rules make up definition 1.7

To show definition 1.8 take:

$(M, \pi) \models X\phi$  iff  $(M, \pi[1.. \infty]) \models \phi$  which we get from  
 $CTL^*$   $(M, \pi) \models X\psi$  and the fact that in  $CTL^*$  syntax  
(in def.1), a path formula  $\psi$  can consist of a state  
formula  $\phi$  \*

$(M, \pi) \models \phi \vee \phi'$  can be derived similarly from  $CTL^* (M, \pi) \models \psi \vee \psi'$   
and the same \* from def.1

these two rules make definition 1.8

5) a) In exercise 3 we showed that all expressions of CTL can be expressed in CTL\* as CTL is a syntactic fragment of CTL\*

furthermore, we showed that semantics of CTL are a subset of the semantics of CTL\*.

From these facts, it follows that, given an arbitrary formula  $\phi$  in CTL, we must have a syntactically and semantically equivalent formula  $\phi'$  in CTL\*

b) Example formulas which can be expressed in CTL\* but not CTL are formulas of the kind  $X\psi$  or  $\varphi U \pi$  where  $\psi, \varphi$  and  $\pi$  are path formulas. In CTL, only state formulas can occur before U and after X and V.

Eg:  $Exx_p$

6)

assume  $(M, t)$  and  $(M', t')$  are bisimilar

assume  $(M, \pi)$  and  $(M', \pi')$  are bisimilar

proof of  $(M, t) \models \phi$  iff  $(M', t') \models \phi$

by induction on structure of  $\phi$

- i)  $\Phi = p$  trivial, follows from definition 3a) and definition of bisimulation
- ii)  $\neg\phi$  follows from the negation of 3a), def. bisim. and the semantic rule  $(M, s) \models \phi \wedge \phi' \text{ iff } (M, s) \not\models \phi'$
- iii)  $\phi \wedge \phi'$  follows from definition of 3a) and the semantic rule  $(M, s) \models \phi \wedge \phi' \text{ iff } (M, s) \models \phi \wedge (M, s) \models \phi'$
- iv)  $E\Psi$  says there is a path where  $\Psi$  holds.  
 from the semantic rule, we have that  $(M, t) \models E\Psi$   
 iff there exists a path  $\pi$  starting from  $t$ , s.t.  $(M, \pi) \models \Psi$   
 - since  $(M, t)$  and  $(M', t')$  and also  $(M, \pi)$  and  $(M', \pi')$  are bisimilar, there must also be a path  $\pi'$  from  $t'$  such that  $(M', \pi') \models \Psi$ . proof on structure of  $\Psi$  is below.

6) v)  $\Box \Psi$  says  $\psi$  holds on all paths.

from the semantic rule for  $\Box \Psi$  we have

$(M, t) \models \Box \Psi$  iff for all paths  $\pi$  starting from  $t$ ,  
 $(M, \pi) \models \psi$

since  $(M, t)$  and  $(M', t')$  and also  $(M, \pi)$  and  $(M', \pi')$  are bisimilar, for every path  $\pi$  from  $t$  in  $M$ , there exists a path  $\pi'$  from  $t'$  in  $M'$  s.t.

$(M', t') \models \psi$ . (induction on  $\Psi$  below)

proof of  $(M, \pi) \models \psi$  iff  $(M', \pi') \models \psi'$

by induction on the structure of  $\Psi$

i)  $(M, \pi) \models \phi$  iff  $(M, \pi[\circ]) \models \phi$

by the definition of bisimulation on  $(M, \pi)$

and  $(M', \pi')$ , we have that  $(M, \pi[\circ])$  and  $(M', \pi'[\circ])$  are bisimilar. Furthermore, by the induction on  $\phi$  above, we have  $(M, \pi) \models \phi$  iff  $(M', \pi') \models \phi$

ii)  $(M, \pi) \models \neg \psi$  iff  $(M, \pi) \not\models \psi$  iff  $(M', \pi') \models \psi$

follows from def. 3a)

iii)  $(M, \pi) \models \psi \wedge \psi'$  iff  $(M, \pi) \models \psi$  and  $(M, \pi) \models \psi'$   
iff  $(M', \pi') \models \psi$  and  $(M', \pi') \models \psi'$

Follows from i) ii) and def. bisimulation

6) iv)  $(M, \pi) \models X\psi$  iff  $(M, \pi[1.. \infty]) \models \psi$

by the definition of bisimulation we have that

$(M, \pi[i])$  and  $(M', \pi'[i])$  are bisimilar  $\forall i \geq 0$

from def. bisimulation we have that

$(M, \pi) \models \psi$  iff  $(M', \pi') \models \psi$

Since  $\psi$  can be  $\phi$ , which we have proven

by induction - we have  $(M, \pi) \models X\psi$  iff

$(M', \pi') \models X\psi$

v)  $(M, \pi) \models \psi \cup \psi'$  iff  $(M, \pi[i.. \infty]) \models \psi'$  for some  $i \geq 0$

and  $(M, \pi[j.. \infty]) \models \psi$  for all  $0 \leq j < i$

similarly to iv) above ,  $\forall i \geq 0$  ,  $(M, \pi[i])$  and  $(M', \pi'[i])$  are bisimilar .

we now have  $(M, \pi[i.. \infty]) \models \psi'$  iff

$(M', \pi'[i.. \infty]) \models \psi'$

and  $(M, \pi[j.. \infty]) \models \psi$  iff

$(M', \pi'[j.. \infty]) \models \psi$

which can be any of the cases proven above

7) assume  $s_t, s_{t'}$  are finite states of models  $M, M'$  respectively

Proof of Hennessy-Milner :

$\Rightarrow : \Rightarrow$  follows from the fact that since  $(M, t)$  and  $(M', t')$  are bisimilar, they satisfy the same formulas and hence are modally equivalent (by def. of modal equivalence)

$\Leftarrow :$  assume  $(M, t)$  and  $(M', t')$  are modally equivalent. then, by def. of modal equivalence we know that they have the same theories (satisfy the same set of formulas.)

assume for a contradiction that there is a path  $\pi$  from  $t$  and there is no modally equivalent path  $\pi'$  from  $t'$ . But then,  $(M', t')$  and  $(M, t)$  do not necessarily satisfy the same set of formulas - contradiction to the modal equivalence of  $t, t'$

8)

proof:

$\Rightarrow$  assume  $(M, t)$  and  $(M', t')$  satisfy the same formulas of CTL. By exercise 7,  $(M, t)$  and  $(M', t')$  are bisimilar. From exercise 6 it follows that bisimulations preserve truth of modal formulas, so clearly  $(M, t)$  and  $(M', t')$  must satisfy the same formulas in  $CTL^*$

$\Leftarrow$  assume  $(M, t)$  and  $(M', t')$  satisfy the same formulas of  $CTL^*$

also, assume towards a contradiction that they do not satisfy the same formulas in CTL

- since CTL is a subset of  $CTL^*$ , then they cannot satisfy the same formulas in  $CTL^*$  which is a contradiction