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Student Declaration - Version 1

- I declare that this final submitted version is my unaided work.

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Modal Logic CW 2

iv915

January 2020

1. (a) For a path π , then $\pi \models \phi R \psi$ iff either

- $\pi[i..\infty] \models \phi$ for some $i \geq 0$ and $\pi[j..\infty] \models \psi$ for all $0 \leq j \leq i$, or
- $\pi[i..\infty] \models \psi$ for all $i \geq 0$

2

- (b)

$$(\psi U (\psi \wedge \phi)) \vee \neg(\top U \neg\psi)$$

1

Solution could have been simplified further

- (c) $\pi \models (\psi U (\psi \wedge \phi)) \vee \neg(\top U \neg\psi)$ if and only if $\pi \models \psi U (\psi \wedge \phi)$ or $\pi \models \neg(\top U \neg\psi)$

$\pi \models \psi U (\psi \wedge \phi)$, iff (from lecture) $\pi[i..\infty] \models \psi \wedge \phi$ for some $i \geq 0$, and $\pi[j..\infty] \models \psi$ for all $0 \leq j < i$. Then $\pi[i..\infty] \models \psi \wedge \phi$ iff $\pi[i..\infty] \models \psi$ and $\pi[i..\infty] \models \phi$. Clearly, this is true iff $\pi[i..\infty] \models \phi$, and $\pi[j..\infty] \models \psi$ for $0 \leq j \leq i$ - which is precisely the first condition.

$\pi \models \neg(\top U \neg\psi)$ iff $\pi \not\models \top U \neg\psi$. $\pi \not\models \top U \neg\psi$ if, for all $i \geq 0$ such that $\pi[i..\infty] \models \neg\psi$, then there exists $0 \leq j < i$ such that $\pi[j..\infty] \not\models \top$. But by the semantics of \top , this cannot be the case - so in fact there exists no such $i \geq 0$ such that $\pi[i..\infty] \models \neg\psi$ - or equivalently $\pi[i..\infty] \models \psi$. So this is true iff $\forall i \geq 0$, then $\pi[i..\infty] \models \psi$ - which is precisely the second condition.

3

Solution correct and very well explained

- (d) We already know that $G\psi \equiv \neg(\top U \neg\psi)$ (from lecture) - so it would suffice to show that $\psi U (\psi \wedge \perp) \equiv \perp$. Using $\psi \wedge \perp \equiv \perp$, this can be simplified to showing that $\psi U \perp \equiv \perp$.

Let λ be any path such that $\lambda \models \psi U \perp$. Then for some $i \geq 0$, $\lambda[i..\infty] \models \perp$ and for all $0 \leq j < i$, $\lambda[j..\infty] \models \psi$ - but this is a contradiction, so no such λ exists. Therefore, the paths satisfying $\psi U \perp$ and \perp in any model are the same - so $\psi U \perp \equiv \perp$.

2

Solution is correct and explained but could have been presented with more clarity. Please clearly separate steps in solutions

2. We first show that $(M, \pi) \models (\top U \Phi)$ iff for some $j \geq 0$, $(M, \pi[j]) \models \Phi$.

By definition of U , then $(M, \pi) \models (\top U \Phi)$ iff for some $j \geq 0$, $(M, \pi[j]) \models \Phi$ and $\forall 0 \leq i < j$, $(M, \pi[i]) \models \top$. But by the definition of \top , then this

second condition always holds - so $(M, \pi) \models (\top U \Phi)$ iff for some $j \geq 0$, $(M, \pi[j]) \models \Phi$.

- $(M, q) \models EF\Phi$ iff for some path λ from q , then $(M, \lambda) \models \top U \Phi$ - but this is iff for some $j \geq 0$, $(M, \lambda[j]) \models \Phi$. 2
 So $(M, q) \models EF\Phi$ iff for some path λ from q , for some $j \geq 0$, then $(M, \lambda[j]) \models \Phi$.
- $(M, q) \models AF\Phi$ iff for every path λ from q , then $(M, \lambda) \models \top U \Phi$ - but this is iff for some $j \geq 0$, $(M, \lambda[j]) \models \Phi$. 2
 So $(M, q) \models AF\Phi$ iff for every path λ from q , for some $j \geq 0$, then $(M, \lambda[j]) \models \Phi$.
- $(M, q) \models EG\Phi$ iff $(M, q) \not\models AF\neg\Phi$ - so iff it is not true that for every path λ from q , for some $j \geq 0$, $(M, \lambda[j]) \models \neg\Phi$. 2
 Then this is not true iff for some path λ from q , every $j \geq 0$, $(M, \lambda[j]) \not\models \neg\Phi$ - so $(M, \lambda[j]) \models \Phi$.
 So $(M, q) \models EG\Phi$ iff for some path λ from q , for every $j \geq 0$, then $(M, \lambda[j]) \models \Phi$.
- $(M, q) \models AG\Phi$ iff $(M, q) \not\models EF\neg\Phi$ - so iff it is not true that for some path λ from q , for some $j \geq 0$, $(M, \lambda[j]) \models \neg\Phi$. 2
 Then this is not true iff for every path λ from q , every $j \geq 0$, $(M, \lambda[j]) \not\models \neg\Phi$ - so $(M, \lambda[j]) \models \Phi$.
 So $(M, q) \models AG\Phi$ iff for every path λ from q , for every $j \geq 0$, then $(M, \lambda[j]) \models \Phi$.

3. (a) We show this by structural induction on the sufficient connectives \neg, \wedge, A, E, X, U , over both path and state formulae.

If $p \in \text{AP}$ is a state formula of CTL, then by construction p is a state formula of CTL*.

Assume $\neg\Phi$ is a state formula of CTL. Then Φ is a state formula of CTL, so Φ is a state formula of CTL*. So by construction $\neg\Phi$ is a state formula of CTL*.

Assume $\Phi \wedge \Psi$ is a state formula of CTL. Then Φ, Ψ are state formulae of CTL - so they are state formulae of CTL*. So by construction, $\Phi \wedge \Psi$ is a state formula of CTL*.

Assume $E\phi$ is a state formula of CTL. Then ϕ is a path formula of CTL - so it is a path formula of CTL*. So by construction $E\phi$ is a path formula of CTL*.

Assume $A\phi$ is a state formula of CTL. Then ϕ is a path formula of CTL - so it is a path formula of CTL*. So by construction $A\phi$ is a path formula of CTL*.

Assume $X\Phi$ is a path formula of CTL. Then Φ is a state formula of CTL, and so Φ is a state formula of CTL*. So by construction $X\Phi$ is a path formula of CTL*.

Assume $\Phi U \Psi$ is a path formula of CTL. Then Φ, Ψ are state formulae of CTL - so they are state formulae of CTL*. So by construction $\Phi U \Psi$ is a path formula of CTL*.

So by induction, all state and path formulae of CTL are state and path formulae of CTL*, respectively.

Since the formulae of CTL are all and exactly the state formulae, and all state formulae of CTL are state formulae of CTL*, and the formulae of CTL* are all and exactly the state formulae - then CTL is a syntactic fragment of CTL*.

3

- (b) $E((\neg Xp) \wedge q)$ - this does not belong to CTL since the contents of the E is a path formula containing conjunction and propositional atoms, both not allowed in CTL.

2

4. Denote the semantic judgement of CTL as \models_{CTL} . We prove by induction that if $\Phi, \psi \in \text{CTL}$, then $(M, s) \models \Phi \iff (M, s) \models_{\text{CTL}} \Phi$, and $(M, \pi) \models \psi \iff (M, \pi) \models_{\text{CTL}} \psi$.

For the state formula p , then the result is trivial, since $(M, s) \models p \iff s \in V(p) \iff (M, s) \models_{\text{CTL}} p$.

For the state formula $\neg\Phi$, then $(M, s) \models \neg\Phi \iff (M, s) \not\models \Phi$ - and by the inductive hypothesis - $\iff (M, s) \not\models_{\text{CTL}} \Phi \iff (M, s) \models_{\text{CTL}} \neg\Phi$.

For the state formula $\Phi \wedge \Phi'$, then $(M, s) \models \Phi \wedge \Phi'$ iff $(M, s) \models \Phi$ and $(M, s) \models \Phi'$. By the inductive hypothesis, this is iff $(M, s) \models_{\text{CTL}} \Phi$ and $(M, s) \models_{\text{CTL}} \Phi'$, so iff $(M, s) \models_{\text{CTL}} \Phi \wedge \Phi'$.

For the state formula $E\psi$, then $(M, s) \models E\psi$ iff for some path π starting from s , $(M, \pi) \models \psi$. By the inductive hypothesis, this is iff for some path π starting from s , $(M, \pi) \models_{\text{CTL}} \psi$, and so iff $(M, s) \models_{\text{CTL}} E\psi$.

For the state formula $A\psi$, then $(M, s) \models A\psi$ iff for every path π starting from s , $(M, \pi) \models \psi$. By the inductive hypothesis, this is iff for every path π starting from s , $(M, \pi) \models_{\text{CTL}} \psi$, and so iff $(M, s) \models_{\text{CTL}} A\psi$.

For the path formula $X\psi$ then for $X\psi$ to be a formula of the CTL fragment, then $\psi = \Phi$. Then $(M, \pi) \models X\Phi$ iff $(M, \pi[1..]) \models \Phi$ - and by definition, this is iff $(M, \pi[1]) \models \Phi$. By the inductive hypothesis, then this is iff $(M, \pi[1]) \models_{\text{CTL}} \Phi$ - which is iff $(M, \pi) \models_{\text{CTL}} X\Phi$.

For the path formula $\psi U \psi'$, the for this to be a formula of the CTL fragment, then $\psi = \Phi, \psi' = \Phi'$. Then $(M, \pi) \models \Phi U \Phi'$ iff $(M, \pi[i..\infty]) \models \Phi'$ for some $i \geq 0$, and for all $0 \leq j < i$, then $(M, \pi[j..\infty]) \models \Phi$. By definition this is iff $(M, \pi[i]) \models \Phi'$ for some $i \geq 0$, and for all $0 \leq j < i$, then $(M, \pi[j]) \models \Phi$. By the inductive definition, then this is iff $(M, \pi[i]) \models_{\text{CTL}} \Phi'$ for some $i \geq 0$, and for all $0 \leq j < i$, then $(M, \pi[j]) \models_{\text{CTL}} \Phi$ - which is iff $(M, \pi) \models_{\text{CTL}} \Phi U \Phi'$.

5

5. (a) Take $\Phi' = \Phi$. By Question 4, the semantics of formulae of CTL* which lie inside the CTL fragment are the same as for the corresponding CTL formula. Therefore,

$$(M, s) \models \Phi \text{ (in CTL*)} \iff (M, s) \models \Phi \text{ (in CTL)}$$

2

- (b) We consider the LTL formula $F(p \wedge Xp)$, which from lectures is not expressible in CTL.

1

No attempt to justify the example

This formula is equivalent to the state formula $AF(p \wedge Xp)$ in CTL*.

6. We use the fact that if there exists a bisimulation between M and M' , then for every path $\pi \in M$, there exists a bisimilar path $\pi' \in M'$ such that $\pi[0]$ is bisimilar to $\pi'[0]$.

We can construct a finite prefix of such a path of length n for any choice of n , simply taking $\pi'[0]$ to be any world bisimilar to $\pi[0]$ and repeatedly applying the forth condition.

Using this fact, we can perform the proof by mutual induction.

For the state formula p , as (M, t) and (M', t') are bisimilar, then

$$(M, t) \models p \iff t \in V(p) \iff t' \in V'(p) \iff (M', t') \models p$$

For the state formula $\neg\Phi$, then $(M, t) \models \neg\Phi$ iff $(M, t) \not\models \Phi$. By the inductive hypothesis, this is iff $(M', t') \not\models \Phi$, so iff $(M', t') \models \neg\Phi$.

For the state formula $\Phi \wedge \Phi'$, then $(M, t) \models \Phi \wedge \Phi'$ iff $(M, t) \models \Phi$ and $(M, t) \models \Phi'$. By the inductive hypothesis, this is iff $(M', t') \models \Phi$ and $(M', t') \models \Phi'$ - so iff $(M', t') \models \Phi \wedge \Phi'$.

For the state formula $E\psi$, then $(M, t) \models E\psi$ only if for some π starting from t , then $(M, \pi) \models \psi$. Then by the above result, there exists $\pi' \in M'$ starting from t' such that π and π' are bisimilar - so by the inductive hypothesis $(M', \pi') \models \psi$ - and as $\pi'[0] = t'$, then $(M', t') \models E\psi$. Since this argument is symmetric, the result is an iff.

For the state formula $A\psi$, then $(M, t) \models A\psi$ only if for every π starting from t , then $(M, \pi) \models \psi$. Let $\pi' \in M'$ be a path starting from t' - then by the above result there exists $\pi \in M$ starting from t such that π is bisimilar to π' . But as π starts from t , then $(M, \pi) \models \psi$ - so by the inductive hypothesis, $(M', \pi') \models \psi$. Since π' was arbitrary, then $(M', t') \models A\psi$. Since this argument is symmetric, the result is an iff.

For the path formula Φ , then $(M, \pi) \models \Phi$ iff $(M, \pi[0]) \models \Phi$. As π' is bisimilar to π , then $\pi'[0]$ is bisimilar to $\pi[0]$ - so by the inductive hypothesis, this is iff $(M', \pi'[0]) \models \Phi$, and so iff $(M', \pi') \models \Phi$.

For the path formulae $\neg\psi, \psi \wedge \psi'$, the proof is the same as for the state formulae version.

For the path formula $X\psi$, then $(M, \pi) \models X\psi$ iff $(M, \pi[1..\infty]) \models \psi$. As π is bisimilar to π' , then $\pi[1..\infty]$ is bisimilar to $\pi'[1..\infty]$ - so by the inductive hypothesis, this is iff $(M', \pi'[1..\infty]) \models \psi$, and so iff $(M', \pi') \models \Phi$.

For the path formula $\psi U \psi'$, then $(M, \pi) \models \psi U \psi'$ only if for some $i \geq 0$, $(M, \pi[i..\infty]) \models \psi'$ and for all $0 \leq j < i$, $(M, \pi[j..\infty]) \models \psi$. As π is bisimilar to π' , then all of these subpaths are bisimilar to their corresponding subpaths in π' - so by the inductive hypothesis, this is only if $(M', \pi'[i..\infty]) \models \psi'$ and for all $0 \leq j < i$, $(M', \pi'[j..\infty]) \models \psi$. So $(M', \pi') \models \psi U \psi'$. Since this argument is symmetric, the result is an iff.

6

All works well justified and written very well. Well Done!

7. (\implies) - as in the proof of Theorem 35, it suffices to show that \rightsquigarrow is a bisimulation. Assume $u \in M, u' \in M'$ such that $u \rightsquigarrow u'$.

Then

$$u \in V(p) \iff (M, u) \models p \iff (M', u') \models p \iff u' \in V'(p)$$

Assume $v \in M$ such that $u \rightarrow v$. Assume for a contradiction that there exists no $v' \in M'$ such that $u' \rightarrow v'$ and $v \rightsquigarrow v'$. By the assumption that the set of states of M' is finite, then the set

$$S = \{v' \in M \mid u' \rightarrow v'\}$$

is finite. As we only consider infinite paths, it must also be nonempty. Therefore, for each $w \in S$, we can take Φ_w some state formula in CTL such that $(M, v) \models \Phi_w$ and $(M', w) \not\models \Phi_w$. Then the assertion $\bigwedge_{w \in S} \Phi_w$ has the property that $(M, v) \models \bigwedge_{w \in S} \Phi_w$.

Considering some path starting at u for which the next state is v , we then have that $(M, u) \models EX \bigwedge_{w \in S} \Phi_w$. As this is a formula of CTL, then by the assumption that $u \rightsquigarrow u'$, then $(M', u') \models EX \bigwedge_{w \in S} \Phi_w$ - so for some path π starting from u' , then $(M', \pi[1]) \models \bigwedge_{w \in S} \Phi_w$. But then $\pi[1] \in S$,

so in particular $(M', \pi[1]) \not\models \Phi_{\pi[1]}$, so $(M', \pi[1]) \not\models \bigwedge_{w \in S} \Phi_w$ - so we have a contradiction. So in fact there exists some $v' \in M'$ such that $u' \rightarrow v'$ and $v \rightsquigarrow v'$. So the forth condition is satisfied.

Since the above argument is symmetric, the back condition is also satisfied.

So \rightsquigarrow is a bisimulation, and by assumption $t \rightsquigarrow t'$, so (M, t) and (M', t') are bisimilar.

(\Leftarrow) - follows from 4 and 6. If (M, t) and (M', t') are bisimilar, then they satisfy the same CTL* formulae - and so they satisfy the same CTL formulae in the CTL* fragment, and so they satisfy the same CTL formulae. 6

8. (\Rightarrow) - Assume (M, t) and (M', t') satisfy the same formulae of CTL. Then by 7, they are bisimilar. Then by 6, $(M, t) \models \Phi \iff (M', t') \models \Phi$, for $\Phi \in \text{CTL}^*$ - so they satisfy the same formulae of CTL*.

(\Leftarrow) - Assume (M, t) and (M', t') satisfy the same formulae of CTL*.

Then $(M, t) \models_{\text{CTL}} \Phi$, for $\Phi \in \text{CTL}$ if and only if $(M, t) \models \Phi$ as a formula in the syntactic fragment of CTL* - if and only if $(M', t') \models \Phi$ as a formula in the syntactic fragment, if and only if $(M', t') \models_{\text{CTL}} \Phi$.

While more formulae could be expressible in CTL* than in CTL, those formulae are not more expressive in a way that can distinguish between bisimilar states - and can only distinguish between states which are not bisimilar, perhaps for the reason that they are already not bisimilar. 5

Out of 49

1			
a/2	b/2	c/3	d/3
2	Solution correct and in fully simplified form 1	Solution correct and very well explained 3	Required condition shown but relation between the required conditions not made explicit 2

2			
a/2	b/2	c/2	d/2
2	2	2	2

3			
a/3	b/2		
3	2		

4			
/5			
5			

5			
a/2	b/2		
2	No attempt to justify the example 1		

6	7	8
/6	/6	/5
All works well justified and written very well. Well Done!		
6	6	5