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CO499: Modal Logic
Course Work 2

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February 19, 2020

1 Question 1

"φ releases ψ" φRψ: ψ remains true until and including once φ becomes true. If φ never become true, then ψ must remain true forever.
(a) **Make the definition of the informally explained operator R precise by providing truth conditions for formulas φRψ in terms of a model M, path φ, and the truth of φ and ψ , similarly to Def. 1.4 in Lecture 5.**

$\pi \models \varphi R\psi$ iff one of the two following conditions holds true
(1) for all $i \geq 0$, $\pi[i,\infty] \not\models \varphi$ and $\pi[i,\infty] \models \psi$
(2) for some $i \geq 0$,$\pi[i,\infty] \models \varphi$ and for all $0 \leq j \leq i$ $\pi[j,\infty] \models \psi$
These conditions can be simplified as:
(1) for all $i \geq 0$, $\pi[i,\infty] \models \psi$
Otherwise,
(2) for some $i \geq 0$,$\pi[i,\infty] \models \varphi$ and for all $0 \leq j \leq i$ $\pi[j,\infty] \models \psi$

$(M,q) \models \varphi R\psi$ iff $\pi \models \varphi R\psi$ for every path π in M starting from q.
$M \models \varphi R\psi$ iff $(M,q0) \models \varphi R\psi$ for every initial state q0 in M.

R means that only when φ holds, can $\neg\psi$ holds in the future.

(b)**Now provide an LTL formula (by using atoms, Boolean connectives, and operators next X and until U only) that formalizes the meaning of the release operator R at point (a).**
According to (a), operator R means only when φ holds, can $\neg\psi$ holds in the future.
That is to say that it is always the case that there exist φ before $\neg\psi$.
In other words, it is not the case that $\neg\varphi$ always holds until $\neg\psi$.

The meaning of operator R can be formalized as:
$\neg(\neg\varphi U \neg\psi)$

(c) **Check that the truth conditions provided in (a) match the LTL formula in (b). That is, the LTL formula is true iff the corresponding condition is satisfied.**

$\pi \models \neg(\neg\varphi U \neg\psi)$ iff $\pi \not\models (\neg\varphi U \neg\psi)$ (by definition of $\neg$)
iff it is not the case that there exists $i \geq 0$ such that $\pi[i..\infty] \models \neg\psi$ and for all $0 \leq j < i$, $\pi[j..\infty] \models \neg\varphi$ (by definition of $\phi U \psi$)
iff it is not the case that there exists $i \geq 0$ such that $\pi[i..\infty] \not\models \psi$ and for all $0 \leq j < i$ $\pi[j..\infty] \not\models \varphi$ (by definition of $\neg$)
iff (1) for all $i \geq 0$, $\pi[i..\infty] \models \psi$, or (2) if there exist $i \geq 0$ such that $\pi[i..\infty] \not\models \psi$ then we can always find $j < i$ such that $\pi[j..\infty] \models \varphi$
iff (1) for all $i \geq 0$, $\pi[i..\infty] \models \psi$, or (2) for some $i \geq 0$,$\pi[i,\infty] \models \varphi$ and $\pi[j,\infty] \models \psi$ for all $0 \leq j \leq i$(by definition of first-order reasoning)

This satisfies the truth condition provided in (a).

(d) **By using your answers to points (a)-(c), check that the always operator $G\psi$ can be expressed as $\bot R\psi$.**
$G\psi$ can be written as $= \neg F \neg\psi$
$\neg F \neg\psi$
$= \neg(trueU\neg\psi)$ (replace true with not false)
$= \neg(\neg\bot U \neg\psi)$ [1]
According to (b), we can rewrite formula [1] as $\bot R\psi$

2 Question 2

Prove the following equivalences by using the definition of satisfaction $\models$ for CTL

(1)$(M,q) \models EF\Phi$ iff for some path λ from q, for some j ≥ 0 $(M,\lambda[j]) \models \Phi$
(2)$(M,q) \models AF\Phi$ iff for every path λ from q, for some j ≥ 0 $(M,\lambda[j]) \models \Phi$
(3)$(M,q) \models EG\Phi$ iff for some path λ from q, for all j ≥ 0 $(M,\lambda[j]) \models \Phi$
(4)$(M,q) \models AG\Phi$ iff for every path λ from q, for all j ≥ 0 $(M,\lambda[j]) \models \Phi$

(1) $(M,q) \models EF\Phi$ iff for some path λ starting from q, $(M,\lambda) \models trueU\Phi$
iff $(M,\lambda[i]) \models \Phi$ for some $i \geq 0$ and $(M,\lambda[j]) \models True$ for all $0 \leq j < i$
Because $(M,\lambda[j]) \models True$ holds.
iff for some path λ from q, for some $i \geq 0$,$(M,\lambda[i]) \models \Phi$
iff for some path λ from q, for some $j \geq 0$,$(M,\lambda[j]) \models \Phi$

(2) $(M,q) \models AF\Phi$ iff for all path λ starting from q, $(M,\lambda) \models trueU\Phi$
iff $(M,\lambda[i]) \models \Phi$ for some $i \geq 0$ and $(M,\lambda[j]) \models True$ for all $0 \leq j < i$
Because $(M,\lambda[j]) \models True$ holds.
iff for every path λ from q, for some $i \geq 0$,$(M,\lambda[i]) \models \Phi$
iff for every path λ from q, for some $j \geq 0$,$(M,\lambda[j]) \models \Phi$

(3)$(M,q) \models EG\Phi$ iff $(M,q) \models \neg AF\neg\Phi$
iff it is not the case that $(M,q) \models AF\neg\Phi$
iff it is not the case that for every path λ from q, for some j ≥ 0

$(M, \lambda[j]) \models \neg\Phi$ (As proved in (2))
iff there exists some path λ from q,for all j ≥ 0, $(M, \lambda[j]) \models \Phi$
iff for some path λ from q, for all j ≥ 0 $(M, \lambda[j]) \models \Phi$

(4)$(M, q) \models AG\Phi$ iff $(M, q) \models \neg EF\neg\Phi$
iff it is not the case that $(M, q) \models EF\neg\Phi$
iff it is not the case that for some path λ from q, for some j ≥ 0 $(M, \lambda[j]) \models \neg\Phi$ (As proved in (1))
iff for every path λ from q, for all j ≥ 0 $(M, \lambda[j]) \models \Phi$

3 Question 3

(a)Show that every formula Φ in CTL is also a formula in CTL* by definition.

Let Φ be a state formula and ψ a path formula.
The definition of syntax of CTL is :
For state formula $\Phi ::= p|\neg\Phi|\Phi \wedge \Phi|E\psi|A\psi$, which is completely the same as that of CTL*
So a state formula in CTL is also a state formula in CTL*
For path formula in CTL $\psi ::= X\Phi|\Phi U \Phi$
(1)$\psi = X\Phi$. Because in CTL*, ψ can be derived from $X\psi$, and $X\psi$ can be derived from $X\Phi$. So it is also a path formula in CTL* by difinition.
(2)$\psi = \Phi U \Phi$. Because in CTL*, ψ can be derived from $\psi U \psi$, and $\psi U \psi$ can be derived from $\Phi U \Phi$. So it is also a path formula in CTL* by difinition.

(b)We now show that there exists formula Φ in CTL* that does not belong to CTL by providing examples.

Given AP $= \{a = 1, b = 2\}$ and a formula $\Phi = A(X(a = 1) \wedge X(b = 2))$. Φ is a formula in CTL* because $X(a = 1) \wedge X(b = 2)$ is a path formula and Φ is a state formula by definition in CTL*.
But in CTL, $X(a = 1) \wedge X(b = 2)$ is not a valid path formula. So $\Phi = A(X(a = 1) \wedge X(b = 2))$ is not a valid formula in CTL.

4 Question 4

Show that if we restrict Def. 2 to formulas in CTL (which we can do, as CTL is a fragment of CTL),then we obtain the same truth conditions as in Def. 1.7 and 1.8 in Lecture 5..

We only need to prove it is true in the follow formulas, as the CTL* defines the state formula in the same way as CTL.
(1)$(M, \pi) \models \Phi$

$(2)(M,\pi) \models X\psi$
$(3)(M,\pi) \models \psi U \psi'$

$(1)(M,\pi) \models \Phi$ iff $(M,\pi[0]) \models \Phi$ where $\pi[0]$ is the initial state in path π.
iff $(M,s) \models \psi$ (Here let s equal to $\pi[0]$)
iff $(M,s) \models \psi$ (The same truth conditions in CTL.)
$(2)(M,\pi) \models X\psi$ iff $(M,\pi[1...\infty]) \models \psi$ (by definition of X)
iff $(M,\pi[1]) \models \psi$ where $\pi[1]$ is the initial state in path π (by definition of $(M,\pi) \models \Phi$)
iff $(M,\pi[1]) \models \psi$ (The same truth conditions in CTL. Here ψ is a valid state formula and $X\psi$ is a valid path formula in CTL)

$(3)(M,\pi) \models \psi U \psi'$ iff $(M,\pi[i...\infty]) \models \psi'$ for some i ≥ 0, and $(M,\pi[j...\infty]) \models \psi$ for all $0 \leq j < i$(by definition of U)
iff $(M,\pi[i]) \models \psi'$ for some i ≥ 0, and $(M,\pi[j]) \models \psi$ for all $0 \leq j < i$ where $\pi[j]$ is the initial state in path $\pi[j...\infty]$ and $\pi[i]$ is the initial state in path $\pi[i...\infty]$ (by definition of $(M,\pi) \models \Phi$)
iff $(M,\pi[i]) \models \psi'$ for some i ≥ 0, and $(M,\pi[j]) \models \psi$ for all $0 \leq j < i$ (The same truth conditions in CTL. Here ψ andψ' are valid state formulas and $\psi U \psi'$ is a valid path formula in CTL)

5 Question 5

Show that CTL is strictly more expressive than CTL

(a)We need to first prove that CTL* is more expressive than CTL.
According to question(3), CTL is a strict syntactic fragment of CTL*. So every formula which is valid in CTL is also valid in CTL*.
This means that for every formula Φ of CTL, Φ is also a formula in CTL*. For every model M, and initial state s, for every formula Φ in CTL, thre exists some formula Φ', s.t $(M,s) \models \Phi$ iff $(M,s) \models \Phi'$. Here we can choose $\Phi' = \Phi$. It statisfies the contidion that Φ' and Φ are equivalent.

(b)Next, we need to prove that there exists some formula Φ in CTL* for which ther exists no equivalent formula Φ' in CTL.
Define $\Phi = FGa, a \in AP$
It is a valid formula in CTL*.But there is no equivalent formula in CTL which can express the same meaning.

According to Clarke Draghicescu's Theorem, let Φ be a CTL formula and let φ be the LTL formula obtained by deleting all path quantifiers in Φ. Either $\Phi \equiv \varphi$ or there is no LTL formula equivalent to Φ.Next we will try adding all combinations of path quantifiers to FGa and prove that none of them is equvalent to FGa. So there is no formula in CTL equivalent to FGa. Noted, LTL formula is also CTL* formula,

We now prove that AFAGa, AFEGa, EFEGa and EFAGa are all not equivalent formula to FGa.
1) From Figure 1, $(M,s0) \models FGa$ but $(M,s0) \not\models AFAGa$.
Consider the path $(s0)^w$. Because $(M,s1) \not\models a$, $(M, path[s0,s1,s2...]) \not\models Ga$, $(M,s0) \not\models AGa$. So $(M,s0) \not\models AFAGa$. AFAGa is not equivalent to FGa.

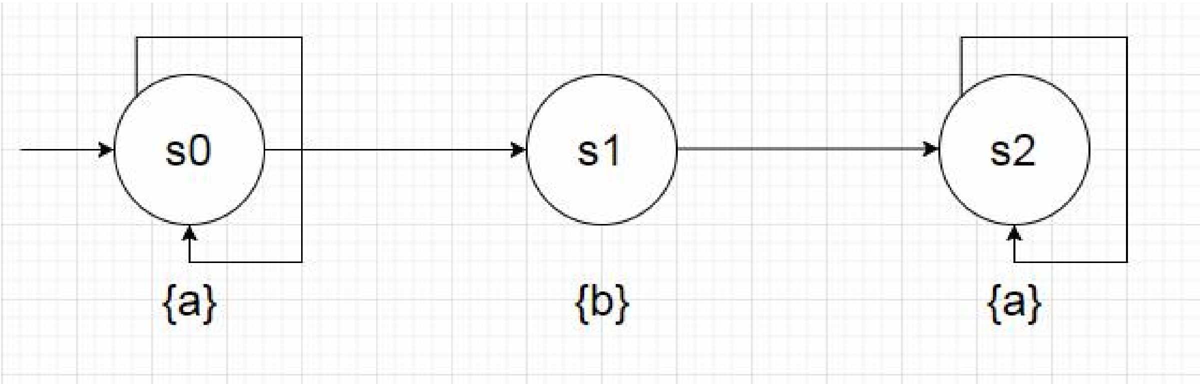

Figure 1: Model M

2) From Figure 2, Consider the path $\lambda = (s2)^w$. Because $(M,s2) \models a$, $(M,\lambda) \models Ga$, $(M,s2) \models AGa$ and $(M,s2) \models EGa$. So $(M,s0) \models EFAGa$ and $(M,s0) \models EFEGa$.
But $(M,s0) \not\models FGa$, considering path $(s0)^w$. So EFAGa and EFEGa are not equivalent to FGa.

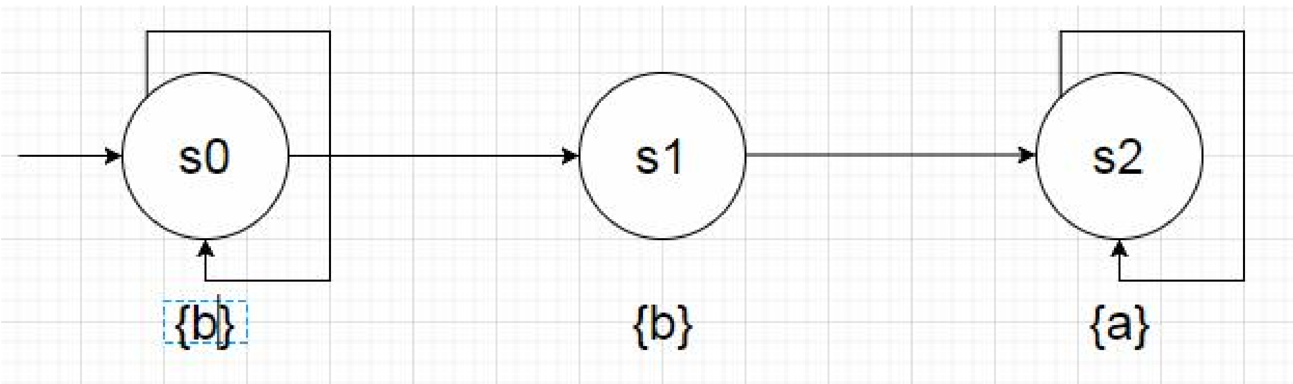

Figure 2: Model M

3) From Figure 3, Consider the path $\lambda = s0,(s1,s2)^w$. Because $(M,s1) \not\models a$, $(M,\lambda) \not\models FGa$, $(M,s0) \not\models FGa$
Considering all the paths starting from s0. Let $\lambda 1 = (s0)^w$,$\lambda 2 = s0,s1,(s2)^w$, $\lambda 3 = s0,(s1,s2)^w$. Because $(M,s2) \models a$, $(M,(s2)^w) \models Ga$. So $(M,s2) \models EGa$. Also, $(M,s0) \models EGa$.As a result, $(M,\lambda 1) \models FEGa$, $(M,\lambda 2) \models FEGa$ and $(M,\lambda 3) \models FEGa$.
So $(M,s0) \models AFEGa$. But$(M,s0) \not\models FGa$. As a result, AFEGa is not equivalent to FGa.

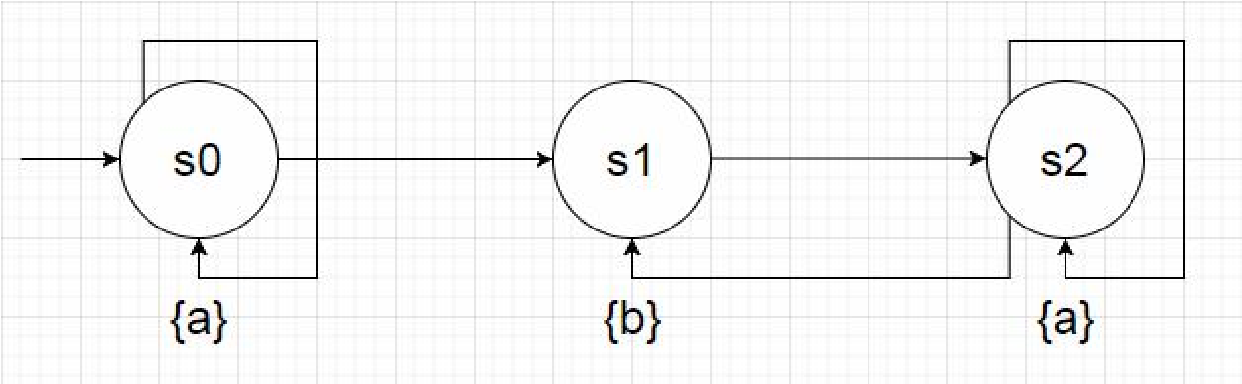

Figure 3: Model M

Now we have shown that CTL* is strictly more expressive than CTL.

6 Question 6

Conclude that the truth of CTL formulas is preserved by bisimulations.
Assume that (M, t) and (M', t') are bisimilar, (M, π) and (M', π') are also bisimilar, Φ is a state formula, and ψ is a path formula. Then show that
$(M,t) \models \Phi$ iff $(M',t') \models \Phi$
$(M,\pi) \models \psi$ iff $(M',\pi') \models \psi$

Proof:
We prove by induction on Φ and ψ.
For state formula:
(1) When Φ is in the form of atom p. Because for all atoms p, $u \in V(p)$ iff $u' \in V'(p)$ (by definition of bisimulation (a)), $(M,t) \models p$ iff $(M',t') \models p$

(2) When Φ is in the form of $E\psi$.
$(M,t) \models E\psi$ iff for some path $\pi[0,...,i]$ starting from t,$(M,\pi) \models \psi$ (by definition of path quantifier E)
Suppose $\pi[0] = t$,$\pi[1] = u^1$,...,$\pi[i] = u^k$. Because (M, t), (M', t') and (M, π) , (M', π') are bisimilar, B(t,t') holds.
If $u^1 \in St$ and $t \rightarrow u^1$,there is $v^1 \in St'$ such that $t' \rightarrow v^1$ and $B(u^1,v^1)$ (by definition of bisimulation forth)
Similarly, we can continue to apply the rule of **forth**. In this way, we can find $v^2,...,v^k$ such that $v^{i-1} \rightarrow v^i,(i = 2,..,k)$ and $B(u^i,v^i),(i = 2,..,k)$.
So we find path π' ($\pi'[0] = t'$,$\pi'[1] = v^1$,...,$\pi'[i] = v^k$.) starting from t', satisfying $(M',\pi') \models \psi$
So $(M',t') \models E\psi$
Similarly, by appling the rule of **back**, we can prove that $(M',t') \models E\psi \rightarrow (M,t) \models E\psi$
As a result,$(M,t) \models E\psi$ iff $(M',t') \models E\psi$.

(3) When Φ is in the form of $A\psi$
$(M,t) \models A\psi$ iff for all path $\pi[0,...,i]$ starting from t,$(M,\pi) \models \psi$ (by definition of path quantifier A)
Let us choose an arbitrary path such that $\pi[0] = t$,$\pi[1] = u^1$,...,$\pi[i] = u^k$. Because (M, t), (M', t') and (M, π) , (M', π') are bisimilar, B(t,t') holds.
If $u^1 \in St$ and $t \to u^1$,there is $v^1 \in St'$ such that $t' \to v^1$ and $B(u^1, v^1)$ (by definition of bisimulation forth)
Similarly, we can continue to apply the rule of **forth**. In this way, we can find $v^2,...,v^k$ such that $v^{i-1} \to v^i, (i = 2,..,k)$ and $B(u^i, v^i), (i = 2,..,k)$.
So we find path π' ($\pi'[0] = t'$,$\pi'[1] = v^1$,...,$\pi'[i] = v^k$.) starting from t', satisfying $(M', \pi') \models \psi$
Because path π' is also arbitrary, it holds true for all the path π'. So $(M', t') \models A\psi$
Similarly, by appling the rule of **back**, we can prove that $(M', t') \models A\psi \to (M, t) \models A\psi$
As a result, $(M, t) \models A\psi$ iff $(M', t') \models A\psi$.

(4) When Φ is in the form of $\neg\Phi$ or $\Phi \wedge \Phi'$,$(M,t) \models \Phi$ iff $(M',t') \models \Phi$ naturally holds true based on (1),(2) and (3)

For path formula :
(5) When ψ is in the form of $X\psi$
$(M,\pi) \models X\psi$ iff $(M,\pi[1,...\infty]) \models \psi$
Because (M,π) and (M',π') are bisimilar, we can find path π' for every $i > 0$, $(M,\pi[i])$and $(M',\pi'[i])$ are bisimilar.
As proved above in **(1)-(4)**, for each state formula $\pi[i]$ and $\pi'[i]$ the truth of CTL* formulas is preserved.
So $(M',\pi') \models X\psi$
Similarly,$(M',\pi') \models X\psi$ can get$(M,\pi) \models X\psi$
So,$(M,\pi) \models X\psi$ iff$(M',\pi') \models X\psi$

(6) When ψ is in the form of $\psi U \psi'$
$(M,\pi) \models \psi U \psi'$ iff $(M,\pi[i,...\infty]) \models \psi'$ for some $i \geq 0$, and $(M,\pi[j,...\infty]) \models \psi$ for all $0 \leq j < i$
Because (M,π) and (M',π') are bisimilar, we can find path π' for every $i > 0$, $(M,\pi[i])$and $(M',\pi'[i])$ are bisimilar.
As proved above in **(1)-(4)**, $E\psi$and $A\psi$are preserved by bisimulation. This means that path π' satisfies that $(M',\pi'[i,...\infty]) \models \psi'$ for some $i \geq 0$, and $(M',\pi'[j,...\infty]) \models \psi$ for all $0 \leq j < i$
So $(M',\pi') \models \psi U \psi'$
Similarly, $(M',\pi') \models \psi U \psi'$ can get $(M,\pi) \models \psi U \psi'$

(7) When ψ is in the form of Φ
$(M,\pi) \models \Phi$ iff $(M,\pi[0]) \models \Phi$, where $\pi[0]$ is the initial state in path π.
Because (M,π) and (M',π') are bisimilar, $(M',\pi'[0]) \models \Phi$, where $\pi'[0]$ is the initial state in path π'. So $(M',\pi') \models \Phi$
Similarly,$(M',\pi') \models \Phi$ can get $(M,\pi) \models \Phi$

So $(M, \pi) \models \Phi$ iff $(M', \pi') \models \Phi$

(8) When ψ is in the form of $\neg\psi$ or $\psi \wedge \psi'$, $(M, \pi) \models \psi$ iff $(M', \pi') \models \psi$ naturally holds true based on (5),(6) and (7)

7 Question 7

Prove a version of the Hennessy-Milner theorem for CTL (Theorem 35 in Lecture 2)

Proof: $\Longrightarrow$ If $t \in M$ and $t' \in M'$ are CTL-equivalent, we need to prove that the relation is bisimulation.
Here let $M = \{St, \rightarrow, V\}$,$M' = \{St', \rightarrow, V'\}$, B is a relation on St x St' such that for every $u \in St$ and $u' \in St'$, if B(u,u'),
(1) for all atoms, because equivalent worlds satisfy the same atoms, $u \in V(p)$ iff $u' \in V'(p)$.
(2) forth: $w, v \in St$, $w' \in St'$. Assume that B(w,w') and R(w,v). We point out that there is a contradiction by assuming that for no $v' \in St', R(w', v') and B(v, v')$
Let $St' = \{v[0]' \in M' | R(w', v[0]')\} \vee \{v[i]' \in M' | R(v[i-1]', v[i]')\}$. By assumption, there exists a formula Φ such that $(M, v[i]) \models \Phi$ but $(M', v'[i]) \not\models \Phi$. Here let path $\lambda[0] = w, \lambda'[0] = w', \lambda[i] = v[i], \lambda'[i] = v'[i]$. So $(M, \lambda[i-1]) \models X\Phi$ but$(M', \lambda[i-1]') \not\models X\Phi$, which contradicts CTL-equivalence .
(3) back can be check similarly.
(4) Further, for path π, if (M, π) and (M', π') are not bisimilar, we can find a contradiction that $(M, s) \models A\psi$ but $(M', s') \not\models A\psi$, contradicting the equivalence.
So B(w,w') is indeed a bisimulation.

This relationship is actually if and only if.
$\Longleftarrow$ follows from question 6 that if two models are bisimulate, the truth will be preserved in CTL* (also in CTL)

8 Question 8

By comparing the results at points (5), (6), and (7) show that, even though CTL* is strictly more expressive than CTL, the two logics have the same distinguishing power: (M, t) and (M', t') satisfy the same formulas of CTL iff they satisfy the same formulas of CTL* (prove this latter fact!).

Elaborate briefly on these apparently contradictory features of CTL and CTL* .

Expressive power and distinguishing power are actually two different notions.

For expressive power : we say **L** is at least as expressive as **L'** ($L \geq L'$) iff $\forall \varphi' \in L', \exists \varphi \in L$ such that $\varphi \equiv \varphi'$

For distinguishing power: we say **L** is at least as distinguishing as **L'**($L \geq L'$) iff for any **(M,t)** and **(M',t')**,
if $\forall \varphi \in L, (M,t) \models \varphi \Longleftrightarrow (M',t') \models \varphi$,
then $\forall \varphi \in L', (M,t) \models \varphi \Longleftrightarrow (M',t') \models \varphi$

So expressive power focuses on the syntax and semantics of the language. And distinguishing power actually focuses on the execution tree.
We say two Kripke structures satisfy the same CTL or CTL* formulas iff they have the same set of executions(ie they are trace-equivalent).

From (5), we know that CTL* is strictly more expressive than CTL. So it is obvious that if (M, t) and (M', t') satisfy the same formula of CTL, they also satisfy the same formula of CTL*.So CTL* is at least as distinguishing as CTL.
We need to further show that CTL is also as distinguishing as CTL*. (M, t) and (M', t') satisfy the same formulas of CTL if they satisfy the same formulas of CTL*.

From (7) we know that if $t \in M, t' \in M'$ are CTL-equivalent ,then (M, t) and (M', t') are bisimilar.
Similar to the result in (7), we can also prove that if $t \in M, t' \in M'$ are CTL*-equivalent ,then (M, t) and (M', t') are bisimilar.
This means that **two (finitely branching) Kripke structures satisfy the same CTL (or CTL*) formulas iff they are bisimilar.**
(Hennessy, 1980)

So if (M,t) and (M',t') are CTL*-equivalent
$\longrightarrow$ (M,t) and (M',t') are bisimilar.
$\longrightarrow$ (M,t) and (M',t') can satisfy the same CTL formulas(CTL - equivalent).

Also we notice that although CTL is not more expressive than LTL. Nor is LTL more expressive than CTL. But CTL has more distinguishing power than LTL. The same holds true CTL*. CTL* has more distinguishing power than LTL.