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### Exercise Information

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### Student Declaration - Version 1

- I declare that this final submitted version is my unaided work.

Signed: (electronic signature) Date: 2020-02-06 16:36:17

**For Markers only:** (circle appropriate grade)

BROWANG, (mgb19)	Matteo	01797461	s5	2020-02-06 16:36:17	A*	A	B	C	D	E	F
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## COURSEWORK 2 : TEMPORAL LOGIC

IMPERIAL COLLEGE LONDON

DEPARTMENT OF COMPUTING

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# Modal Logic

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## Question 1

(a) The truth condition that defines  $\varphi R \psi$  are:

$\pi \models \varphi R \psi$  iff  $\pi[i \dots \infty] \models \varphi$  for some  $i \geq 0$ , and  $\pi[j \dots \infty] \models \psi$  for all  $0 \leq j \leq i$

1

Got one solution but missed the other

(b) A LTL formula that formalizes the meaning of  $\varphi R \psi$  is:

$$(\psi \wedge X\psi)U\varphi$$

1

Correct except for the missing condition from part a

(c) We translate the formula found in (b) to the corresponding truth conditions according to the definition in Lecture 5:

$\pi \models (\psi \wedge X\psi)U\varphi$  iff  $\pi[i \dots \infty] \models \varphi$  for some  $i \geq 0$ , and  $\pi[j \dots \infty] \models (\psi \wedge X\psi)$  for all  $0 \leq j < i$   
iff  $\pi[i \dots \infty] \models \varphi$  for some  $i \geq 0$ , and  $\pi[j \dots \infty] \models \psi$  and  
 $\pi[j \dots \infty] \models X\psi$  for all  $0 \leq j < i$   
iff  $\pi[i \dots \infty] \models \varphi$  for some  $i \geq 0$ , and  $\pi[j \dots \infty] \models \psi$  and  
 $\pi[j+1 \dots \infty] \models \psi$  for all  $0 \leq j < i$   
iff  $\pi[i \dots \infty] \models \varphi$  for some  $i \geq 0$ , and  $\pi[j \dots \infty] \models \psi$  for all  $0 \leq j \leq i$

This last truth condition exactly corresponds to the truth condition that we found in (a).

2

Solution correct and very well explained. However, both conditions not satisfied due to error in a

(d) We write the truth condition (that we found (a)) corresponding to  $\perp R \psi$ :

$\pi \models \perp R \psi$  iff  $\pi[i \dots \infty] \models \perp$  for some  $i \geq 0$ , and  $\pi[j \dots \infty] \models \psi$  for all  $0 \leq j \leq i$

By definition  $\perp$  is never true, so this truth condition becomes:

$\pi \models \perp R \psi$  iff  $\pi[j \dots \infty] \models \psi$  for all  $j \geq 0$

This is the truth condition of  $G\psi$  and so  $G\psi$  can be expressed as  $\perp R \psi$ .

2

Solution well explained and correct, although missing the resolution of one solution due to the error in a

## Question 2

$(M, q) \models EF\phi$  iff  $(M, q) \models E(trueU\phi)$

iff for some path  $\lambda$  starting from  $q$ ,  $(M, \lambda) \models trueU\phi$

iff for some path  $\lambda$  starting from  $q$ , for some  $j \geq 0$ ,  $(M, \lambda[j]) \models \phi$

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and  $(M, \lambda[i]) \models \text{true}$  for all  $0 \leq i < j$   
iff for some path  $\lambda$  starting from  $q$ , for some  $j \geq 0$ ,  $(M, \lambda[j]) \models \phi$  2

$(M, q) \models EF\phi$  iff  $(M, q) \models E(\text{true}U\phi)$   
iff for every path  $\lambda$  starting from  $q$ ,  $(M, \lambda) \models \text{true}U\phi$   
iff for every path  $\lambda$  starting from  $q$ , for some  $j \geq 0$ ,  $(M, \lambda[j]) \models \phi$   
and  $(M, \lambda[i]) \models \text{true}$  for all  $0 \leq i < j$   
iff for every path  $\lambda$  starting from  $q$ , for some  $j \geq 0$ ,  $(M, \lambda[j]) \models \phi$  2

$(M, q) \models EG\phi$  iff  $(M, q) \models \neg AF\neg\phi$   
iif  $(M, q) \not\models AF\neg\phi$   
iff for some path  $\lambda$  starting from  $q$ , for all  $j \geq 0$ ,  $(M, \lambda[j]) \not\models \neg\phi$   
(this is the negation of the truth condition of  $AF\neg\phi$  according to the previous result)  
iff for some path  $\lambda$  starting from  $q$ , for all  $j \geq 0$ ,  $(M, \lambda[j]) \models \phi$  2

$(M, q) \models AG\phi$  iff  $(M, q) \models \neg EF\neg\phi$   
iif  $(M, q) \not\models EF\neg\phi$   
iff for every path  $\lambda$  starting from  $q$ , for all  $j \geq 0$ ,  $(M, \lambda[j]) \not\models \neg\phi$   
(this is the negation of the truth condition of  $EF\neg\phi$  according to the first result)  
iff for every path  $\lambda$  starting from  $q$ , for all  $j \geq 0$ ,  $(M, \lambda[j]) \models \phi$  2

### Question 3

(a) Let  $\psi$  be a path formula of CTL and  $\phi$  be a formula of CTL.  
We prove by mutual induction on their structure that  $\psi$  is a path formula of CTL\*  
and  $\phi$  is a state formula of CTL\*:

- $\psi = X\phi$  with  $\phi$  a path CTL\* formula: by Def. 1.,  $\psi$  is a CTL\* path formula.
- $\psi = \varphi U\phi$  with  $\varphi$  and  $\phi$  path CTL\* formulas: by Def. 1.,  $\psi$  is a CTL\* path formula.
- $\phi = a \in AP$ : by Def. 1.,  $\phi$  is a CTL\* formula.
- $\phi = \neg\psi$  with  $\psi$  a CTL\* formula: by Def. 1.,  $\phi$  is a CTL\* formula.
- $\phi = \varphi \wedge \psi$  with  $\varphi$  and  $\psi$  CTL\* formulas: by Def. 1.,  $\phi$  is a CTL\* formula.
- $\phi = E\psi$  with  $\psi$  a CTL\* path formula: by Def. 1.,  $\phi$  is a CTL\* formula.
- $\phi = A\psi$  with  $\psi$  a CTL\* path formula: by Def. 1.,  $\phi$  is a CTL\* formula.

So every formula of CTL is a formula of CTL\*. 3

(b) We consider the formula  $\phi = AX\psi$  with  $\psi$  an atom.  
Then, by Def. 1.,  $\phi$  is CTL\* formula but not a CTL formula (the X operator is followed by a path formula). 2

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## Question 4

The 5 state formulas of CTL\* exactly correspond to the 5 state formulas of CTL. Concerning the path formulas, if we restrict these formulas to CTL, then we can't have path formulas of the form  $\phi$  (with  $\phi$  a state formula),  $\neg\psi$  and  $\psi \wedge \psi'$  (with  $\psi$  and  $\psi'$  path formula), so we can't apply the rules  $(M, \pi) \models \phi$ ,  $(M, \pi) \models \neg\psi$  and  $(M, \pi) \models \psi \wedge \psi'$ .

Then it remains the path formulas  $(M, \pi) \models X\psi$  and  $(M, \pi) \models \psi U \psi'$  which correspond exactly to the path formulas of CTL.

So if we restrict these formulas to CTL, we obtain the same truth conditions as in CTL.

3

A direct proof of the equivalence between truth conditions is required.

## Question 5

(a) We show in **Question 3** that every CTL formula is also a CTL\* formula. Since every formula is equivalent to itself, then every formula of CTL has an equivalent formula (itself) in CTL\* so CTL\* is more expressive than CTL.

2

(b) Let's consider the formula  $\phi = F(a \wedge Xa)$  with  $a$  an atom and  $F$  is defined as in CTL ( $F\psi = true U \psi$ ).

By Def. 1.,  $\phi$  is a state formula of CTL\* but  $\phi$  is not a CTL formula.

We show in the lecture that  $\phi$  is also a LTL formula and that there is no CTL formula  $\phi'$  equivalent to  $\phi$ .

Then CTL\* is strictly more expressive than CTL.

1

Though the example is justified in the lectures, a proof should have been given

## Question 6

We prove this result by mutual induction on  $\phi$  and  $\psi$  ( $p$  is an atom,  $\varphi$  and  $\varphi'$  are state formulas,  $\theta$  and  $\theta'$  are path formulas):

- $\phi = p : (M, t) \models \phi \Leftrightarrow t \in V(p) \Leftrightarrow t' \in V'(p) \Leftrightarrow (M', t') \models \phi$
- $\phi = \neg\varphi : (M, t) \models \phi \Leftrightarrow (M, t) \not\models \varphi \Leftrightarrow (M', t') \not\models \varphi \Leftrightarrow (M', t') \models \phi$
- $\phi = \varphi \wedge \varphi' : (M, t) \models \phi \Leftrightarrow (M, t) \models \varphi \text{ and } (M, t) \models \varphi' \Leftrightarrow (M', t') \models \varphi \text{ and } (M', t') \models \varphi' \Leftrightarrow (M', t') \models \phi$
- $\psi = \varphi : \text{Since, by definition } (M, \pi[0]) \text{ and } (M', \pi'[0]) \text{ are bisimilar, we have :}$   
 $(M, \pi) \models \psi \Leftrightarrow (M, \pi[0]) \models \varphi \Leftrightarrow (M', \pi'[0]) \models \varphi \Leftrightarrow (M', \pi') \models \varphi$
- $\psi = \neg\theta : (M, \pi) \models \psi \Leftrightarrow (M, \pi) \not\models \theta \Leftrightarrow (M', \pi') \not\models \theta \Leftrightarrow (M', \pi') \models \psi$
- $\psi = \theta \wedge \theta' : (M, \pi) \models \psi \Leftrightarrow (M, \pi) \models \theta \text{ and } (M, \pi) \models \theta' \Leftrightarrow (M', \pi') \models \theta \text{ and } (M', \pi') \models \theta' \Leftrightarrow (M', \pi') \models \psi$

- $\psi = X\theta$  : By definition of bisimilarity between path, for every  $i \geq 0$ ,  $(M, \pi[i])$  and  $(M', \pi'[i])$  are bisimilar, so for every  $i \geq 0$ ,  $(M, \pi[i+1])$  and  $(M', \pi'[i+1])$  are bisimilar, so  $(M, \pi[1 \dots \infty])$  and  $(M', \pi'[1 \dots \infty])$  are also bisimilar, then:

$$(M, \pi) \models \psi \Leftrightarrow (M, \pi[1 \dots \infty]) \models \theta \Leftrightarrow (M', \pi'[1 \dots \infty]) \models \theta \Leftrightarrow (M', \pi') \models \psi$$

- $\psi = \theta U \theta'$ : By definition of bisimilarity between path, for every  $i \geq 0$ ,  $(M, \pi[i])$  and  $(M', \pi'[i])$  are bisimilar, so for every  $k$   $(M, \pi[k \dots \infty])$  and  $(M', \pi'[k \dots \infty])$  are also bisimilar, then:

$$(M, \pi) \models \psi \Leftrightarrow (M, \pi[i \dots \infty]) \models \theta' \text{ for some } i \geq 0 \text{ and } (M, \pi[j \dots \infty]) \models \theta \text{ for every } 0 \leq j < i \Leftrightarrow (M', \pi'[i \dots \infty]) \models \theta' \text{ and } (M, \pi[j \dots \infty]) \models \theta \text{ for every } 0 \leq j < i \Leftrightarrow (M', \pi') \models \psi$$

- $\phi = E\theta$  : By definition of the forth property of a bisimulation, for every path  $\pi = (t = t_0, t_1, t_2, \dots)$  starting from  $t$  and for every  $i \geq 0$ , since  $t_i \rightarrow t_{i+1}$ , there is  $t'_i$  and  $t'_{i+1}$  such that  $B(t_i, t'_i)$ ,  $B(t_{i+1}, t'_{i+1})$  and  $t'_i \rightarrow t'_{i+1}$ . So there exists a path  $\pi' = (t' = t'_0, t'_1, t'_2, \dots)$  starting from  $t'$  such that for every  $i \geq 0$ ,  $(M, t_i)$  and  $(M', t'_i)$  are bisimilar. By definition,  $\pi$  and  $\pi'$  are bisimilar. This is also true in the other way with the back property. Then:

$$(M, t) \models \phi \Leftrightarrow \text{For some path } \pi \text{ starting from } t, (M, \pi) \models \theta \Leftrightarrow \text{For some path } \pi' \text{ (bisimilar to } \pi) \text{ starting from } t', (M', \pi') \models \theta \Leftrightarrow (M', t') \models \theta$$

- $\phi = A\theta$ : With the previous result we can show that for every path  $\pi$  starting from  $t'$  there exists exactly one path  $\pi'$  starting from  $t$  such that  $\pi$  and  $\pi'$  are bisimilar and viceversa. Then:

$$(M, t) \models \phi \Leftrightarrow \text{For every path } \pi \text{ starting from } t, (M, \pi) \models \theta \Leftrightarrow \text{For every path } \pi' \text{ starting from } t', (M', \pi') \models \theta \Leftrightarrow (M', t') \models \phi$$

Hence,

$$(M, t) \models \phi \Leftrightarrow (M', t') \models \phi$$

$$(M, \pi) \models \psi \Leftrightarrow (M', \pi') \models \psi$$

This means that CTL\* formulas are invariant by bisimulation, so the truth of CTL\* formulas is preserved by bisimulation.

6

All cases considered and well thought out

## Question 7

We consider the relation  $B$  such that for every  $w, w' \in St \times St'$ ,  $B(w, w')$  iff  $w$  and  $w'$  are CTL-equivalent (then we have  $B(t, t')$ ).

Let's show that  $B$  is a bisimulation. We consider two states  $w, w' \in St \times St'$  such that  $B(w, w')$ .

- Since  $w$  and  $w'$  satisfy the same formulas, in particular they satisfy the same atomic formulas, so for all atoms  $p$ ,  $w \in V(p)$  iff  $w' \in V'(p)$ .

- 
- To prove the Forth property, we consider  $v \in St$  such that  $w \rightarrow v$ . Let's assume that there is no  $v' \in M'$  such that  $w' \rightarrow' v'$  and  $B(v, v')$ . We define  $S' = \{u' \in M' \mid w' \rightarrow' u'\}$ . Since we suppose that  $St$  and  $St'$  are finite, then  $S' = u'_1, u'_2, \dots, u'_k$  is finite. By assumption, for every  $u'_i \in S'$  there is a CTL formula  $\psi_i$  such that  $(M, v) \models \psi_i$  and  $(M', u'_i) \not\models \psi_i$ .

Then, we have  $(M, w) \models AX(\psi_1 \wedge \psi_2 \wedge \dots \wedge \psi_k)$  (we can just consider the paths of the form  $(w, v, \dots)$ ) but  $(M', w') \not\models AX(\psi_1 \wedge \psi_2 \wedge \dots \wedge \psi_k)$  since none of the successor states (items of  $S'$ ) of  $w'$  satisfies  $\psi_1 \wedge \psi_2 \wedge \dots \wedge \psi_k$ .

This result contradicts the equivalence of  $w$  and  $w'$ , so there exists  $v' \in M'$  such that  $w' \rightarrow' v'$  and  $B(v, v')$ .

- In the same way, we can prove the Back property.

Hence,  $B$  is a bisimulation between  $M$  and  $M'$  and  $B(t, t')$  so  $(M, t)$  and  $(M', t')$  are bisimilar.

5

Correct methodology but no actual attempt is seen to prove the back relation

## Question 8

We suppose that  $(M, t)$  and  $(M', t')$  satisfy the same formulas in CTL\*.

Then, since every CTL formula is equivalent to a CTL\* formula according to **Question 5**, every CTL formula  $\psi$  that  $(M, t)$  satisfies is equivalent to a CTL\* formula  $\psi'$ , and  $(M, t)$  also satisfies  $\psi'$ , so  $(M', t')$  also satisfies  $\psi'$ . Since  $\psi$  and  $\psi'$  are equivalent,  $(M', t')$  also satisfies  $\psi$ .

Then,  $(M, t)$  and  $(M', t')$  satisfy the same formulas in CTL.

We suppose that  $(M, t)$  and  $(M', t')$  satisfy the same formulas in CTL, so  $(M, t)$  and  $(M', t')$  are CTL-equivalent.

Then, according to **Question 7**,  $(M, t)$  and  $(M', t')$  are bisimilar.

According to **Question 6**, the truth of CTL\* are preserved by bisimulation, which means that  $(M, t)$  and  $(M', t')$  satisfy the same formulas in CTL\*.

4

Inductions are correct, but no comment is made regarding the apparent contradiction

1			
a/2	b/2	c/3	d/3
Got one solution but missed the other	Solution doesn't adhere to the statements made by 'Release'	Solution correct and very well explained. However, both conditions not satisfied due to error in a	Solution well explained and correct, although missing the resolution of one solution due to the error in a
1	0	2	2

2			
a/2	b/2	c/2	d/2
2	2	2	2

3	
a/3	b/2
3	2

4	
/5	
A direct proof of the equivalence between truth conditions is required.	
3	

5	
a/2	b/2
2	Though the example is justified in the lectures, a proof should have been given 1

6	7	8
/6	/6	/5
All cases considered and well thought out	Correct methodology but no actual attempt is seen to prove the back relation	Inductions are correct, but no comment is made regarding the apparent contradiction
6	5	4