Mesh Smoothing

0 Iterations 5 Iterations 20 Iterations

Outline

- Differential Geometry
- **Discrete Differential Geometry**
- Mesh Quality Measures

Laplace Operator

- Laplace operator

Laplace operator

gradient operator

2nd partial derivatives

$$\Delta f = \mathrm{div} \nabla f = \sum_i \frac{\partial^2 f}{\partial x_i^2}$$

function in Euclidean space

divergence operator

Cartesian coordinates

Laplace-Beltrami Operator

Extension of Laplace to **functions on manifolds**

Laplace-Beltrami

gradient operator

$$\Delta_{\mathcal{S}} f = \mathrm{div}_{\mathcal{S}} \nabla_{\mathcal{S}} f$$

function on manifold $\mathcal{S}$

divergence operator

Laplace-Beltrami Operator

Extension of Laplace to **functions on manifolds**

Laplace-Beltrami

gradient operator

mean curvature

$$\Delta_{\mathcal{S}}\mathbf{x} = \mathrm{div}_{\mathcal{S}}\ \nabla_{\mathcal{S}}\mathbf{x} = -2H\mathbf{n}$$

coordinate function

divergence operator

surface normal

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Discrete Curvatures

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- How to discretize curvatures on a mesh?

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$$K(v) \approx \frac{1}{A(v)} \int_{A(v)} K(\mathbf{x}) \, \mathrm{d}A$$

Discrete Curvatures

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  - Laplace-Beltrami gives us mean curvature H
  - Discretize Gaussian curvature K

Laplace Operator on Meshes?

- Extend finite differences to meshes?
  - What weights per vertex / edge?

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1D grid

Laplace Operator on Meshes?

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1D grid

2D grid

Laplace Operator on Meshes?

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1D grid 2D grid 2D/3D mesh

Uniform Laplacian Matrix

Uniform Laplacian Matrix

- Uniform discretization

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- Uniform discretization

$$\Delta_{\text{uni}} f(v_i) \;:=\; \frac{1}{|\mathcal{N}_1(v_i)|} \sum_{v_j \in \mathcal{N}_1(v_i)} (f(v_j) - f(v_i))$$

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- Properties
  - depends only on connectivity
  - simple and efficient
  - bad approximation for irregular triangulations
    - can give non-zero *H* for planar meshes
    - tangential drift for mesh smoothing

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Discrete Laplace-Beltrami

- Cotangent discretization

$$\Delta_{\mathcal{S}} f(v_i) \;:=\; \frac{1}{2A(v_i)} \sum_{v_j \in \mathcal{N}_1(v_i)} \left(\cot \alpha_{ij} + \cot \beta_{ij}\right) \left(f(v_j) - f(v_i)\right)$$

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#1: Barycentric Cells

- Connect edge midpoints and triangle barycenters
  - Simple to compute
  - Area is 1/3 of triangle areas

#2: Voronoi Cells

- Connect edge midpoints and
  - Circumcenters for all triangles
  - What about obtuse triangles?

#3: Mixed Cells

- Connect edge midpoints and
  - Circumcenters for non-obtuse triangles
  - Midpoint of opposite edge for obtuse triangles
  - Better approximation, more complex to compute...

Discrete Laplace-Beltrami

- Cotangent discretization

$$\Delta_{\mathcal{S}} f(v) := \frac{1}{2A(v)} \sum_{v_i \in \mathcal{N}_1(v)} (\cot \alpha_i + \cot \beta_i)\,(f(v_i) - f(v))$$

- Problems
  - weights can become negative (when?)
  - depends on triangulation
- Still the most widely used discretization

Discrete Curvatures

- Mean curvature (absolute value)

$$H = \frac{1}{2} \|\Delta_{\mathcal{S}} \mathbf{x}\|$$

- Gaussian curvature

$$K = (2\pi - \sum_j \theta_j)/A$$

Literature

- Taubin: *A signal processing approach to fair surface design*, SIGGRAPH 1996.
- Desbrun et al.: *Implicit Fairing of Irregular Meshes using Diffusion and Curvature Flow*, SIGGRAPH 1999.
- Meyer et al.: *Discrete Differential-Geometry Operators for Triangulated 2-Manifolds*, VisMath 2002.
- Wardetzky, Mathur, Kaelberer, Grinspun: Discrete Laplace Operators: No free lunch, SGP 2007

Outline

- Differential Geometry
- Discrete Differential Geometry
- **Mesh Quality Measures**

Mesh Quality

- Visual inspection of “sensitive” attributes
  - Specular shading

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  - Specular shading
  - Reflection lines

Mesh Quality

- Visual inspection of “sensitive” attributes
  - Specular shading
  - Reflection lines
    - differentiability one order lower than surface
    - can be efficiently computed using graphics hardware

C^0 C^1 C^2

Mesh Quality

- Visual inspection of “sensitive” attributes
  - Specular shading
  - Reflection lines
  - Curvature
    - Mean curvature

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Mesh Quality Criteria

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Mesh Quality Criteria

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- Fairness
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- Adaptive tessellation
  - Low complexity
- Triangle shape
  - Numerical robustness

Mesh Optimization

- Smoothness
  - ➡ Smoothing
- Fairness
  - ➡ Fairing
- Adaptive tessellation
  - ➡ Decimation
- Triangle shape
  - ➡ Remeshing

#1: Filter Scanning Noise

- Filter out high frequency noise

Desbrun, Meyer, Schroeder, Barr
Implicit Fairing of Irregular Meshes using Diffusion and Curvature Flow
SIGGRAPH 99

Mesh Smoothing

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#2: Feature Modification

- Advanced Filtering

Original | Low-Pass | Exaggerate

Kim, Rosignac: *Geofilter: Geometric Selection of Mesh Filter Parameters*, Eurographics 05

#3: Fair Surface Design

- Fair Surface Design

Schneider, Kobbelt: *Geometric fairing of irregular meshes for free-form surface design*, CAGD 18(4), 2001

#4: Hole Filling

- Hole-filling with energy-minimizing patches

#5: Mesh Parameterization

- Mesh parameterization

Levy et al.: Least squares conformal maps for automatic texture atlas generation, SIGGRAPH 2002.

#6: Surface Deformation

- Mesh deformation

Kobbelt, Campagna, Vorsatz, Seidel: *Interactive Multi-Resolution Modeling on Arbitrary Meshes*, SIGGRAPH 98

Botsch, Kobbelt: *An intuitive framework for real-time freeform modeling*, SIGGRAPH 04

Laplace Operator on Surfaces Δ

- Uniform Laplace operator
  - depends on mesh connectivity only
  - approximates second derivative on surfaces

- Cotan operator (Laplace Beltrami)

$$\Delta = M^{-1}C$$

$$M = M^T \qquad C = C^T$$

$$\Delta^T \neq \Delta$$

$$c_{ij} = \frac{1}{2}\left(\cot(\alpha_{ij}) + \cot(\beta_{ij})\right)$$

$$c_{ii} = -\sum_{j \in N_1(i)} c_{ij}$$

Common Problem Types

- Laplace equation (special case of Poisson equation)

$$\Delta f = 0$$

$$M^{-1}Cf = 0 \rightarrow Cf = 0$$

subject to boundary conditions

$$f(x) = f_0 \quad x \in \partial B$$

- Poisson equation

$$\Delta f = g$$

$$M^{-1}Cf = g \rightarrow Cf = Mg$$

- Differential equation

$$f_t = \frac{\partial f}{\partial t} = \Delta f$$

- Eigen/spectral analysis

$$\Delta \phi_i = \lambda_i \phi_i$$

$$\left(M^{-\frac{1}{2}} C M^{-\frac{1}{2}}\right) M^{\frac{1}{2}} \phi_i = \lambda_i M^{\frac{1}{2}} \phi_i$$

$$\left(M^{-\frac{1}{2}} C M^{-\frac{1}{2}}\right) \xi_i = \lambda_i \xi_i \qquad \rightarrow \phi_i = M^{-\frac{1}{2}} \xi_i$$

Outline

- **Spectral Analysis**
- Diffusion Flow
- Energy Minimization

Fourier Transform

- Represent a function as a weighted sum of sines and cosines

Joseph Fourier 1768 - 1830

$$f(x) = a_0 + a_1 \cos(x)$$

Fourier Transform

- Represent a function as a weighted sum of sines and cosines

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$$f(x) = a_0 + a_1 \cos(x) + a_2 \cos(3x)$$

Fourier Transform

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$$f(x) = a_0 + a_1 \cos(x) + a_2 \cos(3x) + a_3 \cos(5x)$$

Fourier Transform

- Represent a function as a weighted sum of sines and cosines

Joseph Fourier 1768 - 1830

$$f(x) = a_0 + a_1 \cos(x) + a_2 \cos(3x) + a_3 \cos(5x) + a_4 \cos(7x) + \ldots$$

Fourier Transform

Spatial Domain

Frequency Domain

Fourier Transform

$$F(\omega) = \int_{-\infty}^{\infty} f(x)\, \mathrm{e}^{-2\pi \mathrm{i} \omega x} \mathrm{d}x$$

Fourier Transform

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$$f(x) = \int_{-\infty}^{\infty} F(\omega)\, \mathrm{e}^{2\pi \mathrm{i} \omega x}\, \mathrm{d}\omega$$

Convolution

- Smooth signal by convolution with a kernel function

$$h(x) \;=\; f * g \;:=\; \int f(y) \cdot g(x-y)\,\mathrm{d}y$$

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- Example: Gaussian blurring

Convolution

- Smooth signal by convolution with a kernel function

$$h(x) \;=\; f * g \;:=\; \int f(y) \cdot g(x-y)\,\mathrm{d}y$$

- Convolution in spatial domain ⇔

  Multiplication in frequency domain

$$H(\omega) \;=\; F(\omega) \cdot G(\omega)$$

Fourier Analysis

- Low-pass filter discards high frequencies

spatial domain

frequency domain

Fourier Analysis

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Fourier Transform

- Spatial domain $f(x)$ → Frequency domain $F(\omega)$

$$F(\omega) = \int_{-\infty}^{\infty} f(x)\, \mathrm{e}^{-2\pi \mathrm{i} \omega x} \mathrm{d}x$$

Fourier Transform

- Spatial domain $f(x)$ → Frequency domain $F(\omega)$

$$F(\omega) = \int_{-\infty}^{\infty} f(x)\, \mathrm{e}^{-2\pi \mathrm{i} \omega x} \mathrm{d}x$$

- Multiply by low-pass filter $G(\omega)$

$$F(\omega) \leftarrow F(\omega) \cdot G(\omega)$$

Fourier Transform

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- Frequency domain $F(\omega)$ → Spatial domain $f(x)$

$$f(x) = \int_{-\infty}^{\infty} F(\omega)\, \mathrm{e}^{2\pi \mathrm{i} \omega x}\, \mathrm{d}\omega$$

Fourier Transform

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- Consider L^2 function space with inner product

$$\langle f, g \rangle := \int_{-\infty}^{\infty} f(x)\, \overline{g(x)}\, \mathrm{d}x$$

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- Complex "waves" build an orthonormal basis

$$e_\omega(x) \;:=\; \mathrm{e}^{-2\pi \mathrm{i}\omega x} \;=\; \cos(2\pi\omega x) - \mathrm{i}\sin(2\pi\omega x)$$

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- Fourier transform is a change of basis

$$f(x) = \int_{-\infty}^{\infty} \langle f, e_\omega \rangle\; e_\omega\; \mathrm{d}\omega \qquad f(x) = \sum_{\omega=-\infty}^{\infty} \langle f, e_\omega \rangle\; e_\omega\; \mathrm{d}\omega$$

Fourier Analysis on Meshes?

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$$\Delta\left(e^{2\pi \mathrm{i}\omega x}\right) = \frac{\mathrm{d}^2}{\mathrm{d}x^2} e^{2\pi \mathrm{i}\omega x} = -\left(2\pi\omega\right)^2 e^{2\pi \mathrm{i}\omega x}$$

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➡ **Use Eigenfunctions of discrete Laplace-Beltrami**

Discrete Laplace-Beltrami

- Function values sampled at mesh vertices

$$\mathbf{f} = [f_1, f_2, \ldots, f_n] \in \mathbb{R}^n$$

- Discrete Laplace-Beltrami (per vertex)

$$\Delta_{\mathcal{S}} f(v_i) := \frac{1}{2A_i} \sum_{v_j \in \mathcal{N}_1(v_i)} (\cot\alpha_{ij} + \cot\beta_{ij}) \left(f(v_j) - f(v_i)\right)$$

Discrete Laplace-Beltrami

- Discrete Laplace *operator* (per mesh)
  - Sparse matrix

$$\mathbf{L} = \mathbf{M}^{-1}\mathbf{C} \in \mathbb{R}^{n \times n}$$

$$\begin{pmatrix} \vdots \\ \Delta_{\mathcal{S}} f(v_i) \\ \vdots \end{pmatrix} = \mathbf{L} \cdot \begin{pmatrix} \vdots \\ f(v_i) \\ \vdots \end{pmatrix}$$

Discrete Laplace-Beltrami

- Discrete Laplace *operator* (per mesh)

$$\mathbf{L} = \mathbf{M}^{-1}\mathbf{C} \in \mathbb{R}^{n \times n}$$

  – Sparse matrix

  – Note fraction 1/2 can either be in C or in M matrix.

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- Discrete Laplace *operator* (per mesh)

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  – Sparse matrix

$$\mathbf{C}_{ij} = \begin{cases} \cot\alpha_{ij} + \cot\beta_{ij}, & i \neq j\,,\ j \in \mathcal{N}_1(v_i) \\ -\sum_{v_j \in \mathcal{N}_1(v_i)} (\cot\alpha_{ij} + \cot\beta_{ij}) & i = j \\ 0 & \text{otherwise} \end{cases}$$

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$$\mathbf{M}^{-1} = \operatorname{diag}\left(\ldots, \frac{1}{2A_i}, \ldots\right)$$

  - Note fraction 1/2 can either be in C or in M matrix.

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- Discrete Laplace-Beltrami matrix $\mathbf{L}$
  - Eigenvectors are “natural vibrations”
  - Eigenvalues are “natural frequencies”

Discrete Laplace-Beltrami

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  - Eigenvalues are “natural frequencies”

Spectral Analysis

1. Setup Laplace-Beltrami matrix $\mathbf{L}$
2. Compute k smallest eigenvectors $\{\mathbf{e}_1, \ldots, \mathbf{e}_k\}$
3. Reconstruct mesh from those (component-wise)

$$\mathbf{x} := [x_1, \ldots, x_n] \qquad \mathbf{y} := [y_1, \ldots, y_n] \qquad \mathbf{z} := [z_1, \ldots, z_n]$$

$$\mathbf{x} \leftarrow \sum_{i=1}^{k} \left(\mathbf{x}^T \mathbf{e}_i\right) \mathbf{e}_i \qquad \mathbf{y} \leftarrow \sum_{i=1}^{k} \left(\mathbf{y}^T \mathbf{e}_i\right) \mathbf{e}_i \qquad \mathbf{z} \leftarrow \sum_{i=1}^{k} \left(\mathbf{z}^T \mathbf{e}_i\right) \mathbf{e}_i$$

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Bruno Levy: *Laplace-Beltrami Eigenfunctions: Towards an algorithm that understands geometry*, Shape Modeling and Applications, 2006

Spectral Analysis

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Too complex for large meshes!

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Review: Curves

Differential Geometry of Curves

- chord length and arc length
- arc-length parameterization
- Frenet frame

$$s = s(t) = \int_a^t \|\mathbf{x}_t\| \, \mathrm{d}t$$

$$\mathbf{t} = \frac{\mathbf{x}_t}{\|\mathbf{x}_t\|}$$

$$\mathbf{n} = \mathbf{b} \times \mathbf{t}$$

$$\mathbf{b} = \frac{\mathbf{x}_t \times \mathbf{x}_{tt}}{\|\mathbf{x}_t \times \mathbf{x}_{tt}\|}$$

Frenet frame

Review: Curves

Differential Geometry of Curves

– ***Frenet-Serret***: Derivatives w.r.t. arc length s

$$\begin{aligned} \mathbf{t}_s &= & +\kappa\mathbf{n} & \\ \mathbf{n}_s &= -\kappa\mathbf{t} & & +\tau\mathbf{b} \\ \mathbf{b}_s &= & -\tau\mathbf{n} & \end{aligned}$$

– **Curvature** (*deviation from straight line*)

$$\kappa = \|\mathbf{x}_{ss}\|$$

– **Torsion** (*deviation from planarity*)

$$\tau = \frac{1}{\kappa^2}\det([\mathbf{x}_s, \mathbf{x}_{ss}, \mathbf{x}_{sss}])$$

Review: Surface

Differential Geometry of Surfaces

– **First fundamental form**

$$\mathbf{I} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} := \begin{pmatrix} \mathbf{x}_u^T \mathbf{x}_u & \mathbf{x}_u^T \mathbf{x}_v \\ \mathbf{x}_u^T \mathbf{x}_v & \mathbf{x}_v^T \mathbf{x}_v \end{pmatrix} \qquad \left\langle \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}, \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} \right\rangle := \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}^T \mathbf{I} \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}$$

- **Angle**

$$\mathbf{t}_1^T \mathbf{t}_2 = \langle (\alpha_1, \beta_1), (\alpha_1, \beta_1) \rangle$$

- **Length**

$$\mathrm{d}s^2 = E\mathrm{d}u^2 + 2F\mathrm{d}u\mathrm{d}v + G\mathrm{d}v^2$$

- **Area**

$$\mathrm{d}A = \sqrt{EG - F^2}\mathrm{d}u\,\mathrm{d}v$$

Review: Surface

Differential Geometry of Surfaces

– **Second fundamental form**

$$\mathbf{II} = \begin{pmatrix} e & f \\ f & g \end{pmatrix} := \begin{pmatrix} \mathbf{x}_{uu}^T \mathbf{n} & \mathbf{x}_{uv}^T \mathbf{n} \\ \mathbf{x}_{uv}^T \mathbf{n} & \mathbf{x}_{vv}^T \mathbf{n} \end{pmatrix}$$

- **normal curvature**

$$\kappa_n(\bar{\mathbf{t}}) = \frac{\bar{\mathbf{t}}^T \, \mathbf{II} \, \bar{\mathbf{t}}}{\bar{\mathbf{t}}^T \, \mathbf{I} \, \bar{\mathbf{t}}} = \frac{ea^2 + 2fab + gb^2}{Ea^2 + 2Fab + Gb^2}$$

- **principal curvatures**

$$\kappa_1 = \max_{\phi} \kappa_n(\phi) \qquad \kappa_2 = \min_{\phi} \kappa_n(\phi)$$

- **mean and Gaussian curvature**

$$H = \frac{\kappa_1 + \kappa_2}{2} \qquad K = \kappa_1 \cdot \kappa_2$$

Outline

- Spectral Analysis
- **Diffusion Flow**
- Energy Minimization

Diffusion Flow on Height Fields

- Diffusion equation

diffusion constant

$$\frac{\partial f}{\partial t} = \lambda \Delta f$$

Laplace operator

Diffusion Flow on Meshes

- Iterate $\mathbf{p}_i \leftarrow \mathbf{p}_i + \lambda \Delta \mathbf{p}_i$

0 Iterations

5 Iterations

20 Iterations

Uniform Laplace Discretization

- Smoothes geometry and triangulation
- Can be non-zero even for planar triangulations
- Vertex drift can lead to distortions
- Might be desired for mesh regularization

Desbrun et al., Siggraph 1999

Curvature Flow

- Use diffusion flow with Laplace-Beltrami

$$\frac{\partial \mathbf{p}}{\partial t} = \lambda \Delta_{\mathcal{S}} \mathbf{p}$$

- Laplace-Beltrami is parallel to surface normal

$$\frac{\partial \mathbf{p}}{\partial t} = -2\lambda H \mathbf{n}$$

➡ Avoids vertex drift on surface

Comparison

Original | Uniform Laplace | Laplace-Beltrami

Numerical Integration

Numerical Integration

- Write update $\mathbf{p}_i^{(t+1)} = \mathbf{p}_i^{(t)} + \lambda \Delta \mathbf{p}_i^{(t)}$ in matrix notation

$$\mathbf{P}^{(t)} = \left(\mathbf{p}_1^{(t)}, \ldots, \mathbf{p}_n^{(t)}\right)^T \in \mathbb{R}^{n \times 3}$$

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- **Implicit** integration is unconditionally stable

$$(\mathbf{I} - \lambda \mathbf{L})\, \mathbf{P}^{(t+1)} = \mathbf{P}^{(t)}$$

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➡ Symmetrize by multiplying $\mathbf{M}$ from left

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- Solve sparse symmetric positive definite system
  - ➡ Iterative conjugate gradients, sparse Cholesky

Outline

- Spectral Analysis
- Diffusion Flow
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Fairness

- Idea: Penalize “unaesthetic behavior”
- **Measure** fairness
  - Principle of the simplest shape
  - Physical interpretation
- **Minimize** some fairness functional
  - Surface area, curvature
  - Membrane energy, thin plate energy

Non-Linear Energies

- Membrane energy (surface area)

$$\int_{\mathcal{S}} \mathrm{d}A \;\rightarrow\; \min \quad \text{with} \quad \delta\mathcal{S} = \mathbf{c}$$

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$$\int_{\mathcal{S}} \kappa_1^2 + \kappa_2^2 \,\mathrm{d}A \;\rightarrow\; \min \quad \text{with} \quad \delta\mathcal{S} = \mathbf{c}, \;\; \mathbf{n}(\delta\mathcal{S}) = \mathbf{d}$$

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- Too complex ... simplify energies

Membrane Surfaces

- Surface parameterization

$$\mathbf{x} : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

- Membrane energy (surface area)

$$\min_{\mathbf{x}} \int_{\Omega} \|\mathbf{x}_u\|^2 + \|\mathbf{x}_v\|^2 \mathrm{d}u\mathrm{d}v$$

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$$L(f + \lambda u) = \int_a^b (f' + \lambda u')^2 = \int_a^b f'^2 + 2\lambda f' u' + \lambda^2 u'^2$$

- If f minimizes L, the following has to vanish

$$\left.\frac{\partial L(f + \lambda u)}{\partial \lambda}\right|_{\lambda=0} = \int_a^b 2 f' u' \ \stackrel{!}{=}\ 0$$

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➡ ***Euler-Lagrange equation***

Bivariate Variational Calculus

- Find minimum of functional

$$\underset{f}{\operatorname{argmin}} \int_{\Omega} L\left(f_{uu}, f_{vv}, f_u, f_v, f, u, v\right)$$

- Euler-Lagrange PDE defines the minimizer f

$$\frac{\partial L}{\partial f} - \frac{\partial}{\partial u}\frac{\partial L}{\partial f_u} - \frac{\partial}{\partial v}\frac{\partial L}{\partial f_v} + \frac{\partial^2}{\partial u^2}\frac{\partial L}{\partial f_{uu}} + \frac{\partial^2}{\partial u \partial v}\frac{\partial L}{\partial f_{uv}} + \frac{\partial^2}{\partial v^2}\frac{\partial L}{\partial f_{vv}} = 0$$

  – Again, subject to suitable boundary constraints

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- Variational calculus

$$\Delta \mathbf{x} = 0$$

Laplace-Beltrami Operator

- Extension of Laplace to functions on manifolds

Laplace-Beltrami

gradient operator

mean curvature

$$\Delta_{\mathcal{S}}\mathbf{x} = \mathrm{div}_{\mathcal{S}}\, \nabla_{\mathcal{S}}\mathbf{x} = -2H\mathbf{n}$$

$$\nabla\mathbf{x} = [\partial\mathbf{x}/\partial u \quad \partial\mathbf{x}/\partial v]$$

$$\mathrm{div}\,\mathbf{y} = \nabla\cdot\mathbf{y} = \partial\mathbf{y}/\partial u + \partial\mathbf{y}/\partial v$$

$$\mathrm{div}(\nabla\mathbf{x}) = \Delta\mathbf{x}$$

Thin-Plate Surfaces

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$$\int_\Omega \|\mathbf{p}_{uu}\|^2 + 2\,\|\mathbf{p}_{uv}\|^2 + \|\mathbf{p}_{vv}\|^2 \,\mathrm{d}u\mathrm{d}v \;\rightarrow\; \min$$

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- Variational calculus

$$\Delta^2 \mathbf{p} = 0$$

Energy Functionals

Membrane

$$\Delta_{\mathcal{S}}\mathbf{p} = 0$$

Thin Plate

$$\Delta_{\mathcal{S}}^{2}\mathbf{p} = 0$$

$$\Delta_{\mathcal{S}}^{3}\mathbf{p} = 0$$

Analysis

- Minimizer surfaces satisfy Euler-Lagrange PDE

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- They are ***stationary surfaces of Laplacian flows***

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Analysis

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$$\frac{\partial \mathbf{p}}{\partial t} = \Delta_{\mathcal{S}}^{k}\mathbf{p}$$

- Explicit flow integration corresponds to iterative solution of linear system

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