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Looking Forward Approach in Cooperative Differential Games

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New approach to the definition of solution in cooperative differential games is considered. The approach is based on artificially truncated information about the game. It assumed that at each time, instant players have information about the structure of the game (payoff functions, motion equations) only for the next fixed time interval. Based on this information they make the decision. Looking Forward Approach is applied to the cases when the players are not sure about the dynamics of the game on the whole time interval $[0, T]$ and orient themselves on the game dynamics defined on the smaller time interval $\overline{T}$ ($0 < \overline{T} < T$), on which they surely know that the game dynamics is not changing.

Keywords: Differential game; time consistency; imputation distribution procedure; looking forward approach; truncated subgame.

Subject Classification: 49N70, 91A12, 90C39

1. Introduction

The n-person differential cooperative game $\hat{\Gamma}_j(x_0, 0, T)$ with prescribed duration T, starting from initial state $x_0 \in R^n$ and initial time instant $t = 0$ is given. The structure of the game is based on motion equations:

$$\dot{x} = g(t, x, u) \tag{1}$$

with initial condition $x(0) = x_0$, where $x \in R^n$, $u_i \in U_i \subset \operatorname{Comp} R^k$ (k-dimensional compact space), $i = 1, \ldots, n$. Suppose that function $g(t, x, u)$ has all necessary properties for existence, uniqueness and continuability of the solution of the system (1) for any initial conditions and any feedback strategies (see Basar and Olsder [1987]).

Let N be the set of players. The payoff function of player $i \in N$ is defined on the fixed time interval $[0, T]$:

$$K_i(x, 0, T; u) = \int_0^T h_i(x(\tau), u(\tau))d\tau, \tag{2}$$

where $h_i(x(\tau), u(\tau))$ is a continuous function, $x(\tau)$ is the trajectory (solution) of the system (1) with initial condition $x(0) = x_0$, which is realized on the time interval $[0, T]$ when the controls $u = (u_1 \dots u_n)$ are used. In this game model players use an artificially truncated information about the game. At each time instant $t \in [j\Delta t, (j+1)\Delta t]$ they have the information about the structure (motion equations) of the game for the next fixed period of time $[j\Delta t, j\Delta t + \overline{T}]$, $j = 0, \dots, l$, where $l = \frac{T}{\Delta t} - \frac{\overline{T}}{\Delta t}$ is a number of time instants in the game $\Gamma(x_0, 0, T)$ when information about the game updates (suppose that $\frac{T}{\Delta t}$ and $\frac{\overline{T}}{\Delta t}$ are integers). Based on this information they make their decisions. When information about the game updates players recalculate their decisions using the new updated information. The Looking Forward Approach for cooperative differential games is used to consider such cases when players are not sure about the dynamics of the game on the time interval $[0, T]$ on which the game $\Gamma(x_0, 0, T)$ is defined and orient themselves on the game dynamics defined on the smaller time interval with length $\overline{T}$, on which they surely know that the dynamics of the game $\Gamma(x_0, 0, T)$ is not changing (Fig. 1). The problems of this type often occur in real-life situations since it is not easy to forecast the dynamics or structure of the conflicting process on the large time interval. In the same time one can assume that on small time intervals $\Delta t > 0$ the game structure is not changing and on each next time interval it does not change essentially compared to the previous one. This last comment brings us to the idea to use for such kind of games Looking Forward Approach, considering at the time instant $t = j\Delta t$ the structure of the game as if it is not changing on the next time interval $[j\Delta t, j\Delta t + \overline{T}]$.

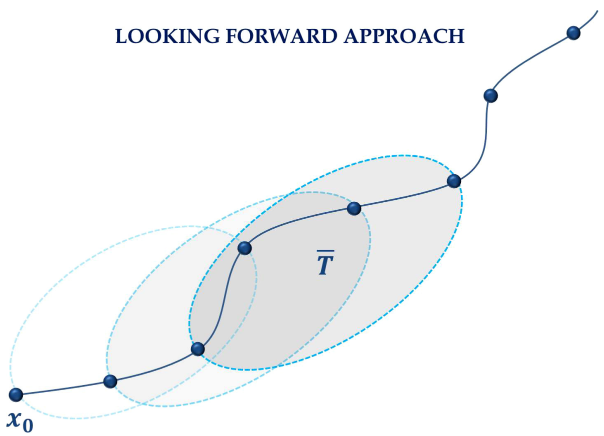

Fig. 1. Suppose that the game evolves along the trajectory (blue line). Blue ovals denote information which players use during the game.

2. Definition of the Game

During the first time interval $[0, \Delta t]$ players in the game have information about the structure (motion equations) of the game on the time interval $[0, \overline{T}]$ and expert evaluation of the terminal payoff depending on the position of players at the time instant $\overline{T}$. At the time instant Δt information about the game updates and on the second time interval $(\Delta t, 2\Delta t]$ players have information about the structure of the game on the time interval $(\Delta t, \Delta t + \overline{T}]$ and expert evaluation of the terminal payoff at the time instant $\Delta t + \overline{T}$ and so on. Expert evaluation gives players the opportunity to estimate the payoff which can be received by players on the time interval $[j\Delta t + \overline{T}, T]$ (time interval with the unknown structure) using specific controls on the time interval $[j\Delta t, j\Delta t + \overline{T}]$. Therefore expert evaluation contains some information about the structure of the game $\Gamma(x_0, 0, T)$. To include this fact in the model introduce the following definition (Fig. 2).

Definition 1. Consider the family of truncated subgames $\hat{\Gamma}_j(x, j\Delta t, j\Delta t + \overline{T})$ for $j = 0, \ldots, l$, where $l = \frac{T}{\Delta t} - \frac{\overline{T}}{\Delta t}$. The game starts at the time instant $j\Delta t$ from the initial state $x_j = x(j\Delta t)$. The truncated subgame terminates at the time instant $j\Delta t + \overline{T}$. The payoff function of player i is equal to

$$\hat{K}_i^j(x, j\Delta t, j\Delta t + \overline{T}; u_j) = \int_{j\Delta t}^{j\Delta t + \overline{T}} h_i(x(\tau), u(\tau))d\tau + \hat{q}_i(j\Delta t + \overline{T}, x) \tag{3}$$

subject to

$$\begin{aligned} \dot{x} &= g(t, x, u), \\ x_j &= x(j\Delta t), \end{aligned} \tag{4}$$

where $\hat{q}_i(j\Delta t + \overline{T}, x)$ is the expected terminal payoff of player $i \in N$.

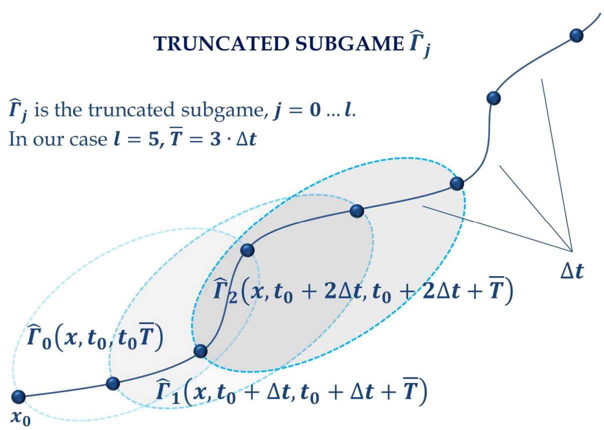

Fig. 2. Truncated information in each oval can be modelled by corresponding truncated subgame.

The notion of expected terminal payoff gives players the opportunity to use simple forecast of situation in the future, such as expert evaluation. In a real-life applications expert evaluation can be calculated using autoregressive models for trajectory and corresponding payoff for any initial conditions of the system (1) and feedback strategies on the time interval where the structure of the game is not certain. In any case expert evaluation can be given as an expected value of payoff of player i for each truncated subgame. The structure of the truncated subgame $\hat{\Gamma}_j(x, j\Delta t, j\Delta t + \overline{T})$ coincides with the structure of the game $\Gamma(x_0, 0, T)$ on the time interval $[j\Delta t, j\Delta t + \overline{T}]$ except that in each truncated subgame players also use the expert evaluation to get better result.

Suppose all players agreed to cooperate, then in each truncated subgame $\hat{\Gamma}_j(x, j\Delta t, j\Delta t + \overline{T})$, $j = 0, \ldots, l$ players seek to maximize sum of their expected payoff:

$$\max_{u_1^j \ldots u_n^j} \sum_{i=1}^{n} \hat{K}_i^j(x, j\Delta t, j\Delta t + \overline{T}; u_j)$$
$$= \max_{u_1^j \ldots u_n^j} \sum_{i=1}^{n} \int_{j\Delta t}^{j\Delta t + \overline{T}} h_i(x(\tau), u(\tau)) d\tau + \hat{q}_i(j\Delta t + \overline{T}, x). \quad (5)$$

The solution to the control problem (4)–(5) can be characterized as follows:

Optimal control $u_j^*(t) = (u_1^{*j}(t) \ldots u_n^{*j}(t))$ for each cooperative truncated subgame $\hat{\Gamma}_j(x, j\Delta t, j\Delta t + \overline{T})$ can be found using the following theorem [Yeung and Petrosyan, 2012].

Theorem 1. *A set of controls* $\{u_i^{*j}(t) = \psi_i^{*j}(t, x), i \in N\}$ *provides an optimal solution to the control problem* (4)–(5) *if there exist continuously differentiable functions* $W^{(j\Delta t)}(t, x) : [j\Delta t, j\Delta t + \overline{T}] \times R^m \longrightarrow R$, *satisfying the following set of partial differential equations*:

$$-W_t^{(j\Delta t)}(t, x) = \max_{u_1 \ldots u_n} \left\{ \sum_{i=1}^{n} h_i(x(\tau), u(\tau)) + W_x^{(j\Delta t)}(t, x) g(t, x, u) \right\},$$
$$W^{(j\Delta t)}(j\Delta t + \overline{T}, x) = \hat{Q}_i(j\Delta t + \overline{T}, x), \quad (6)$$

where $\hat{Q}(j\Delta t + \overline{T}, x) = \sum_{i=1}^{n} \hat{q}_i(j\Delta t + \overline{T}, x)$ *is a sum of expert evaluations for truncated subgame* $\hat{\Gamma}_j(x, j\Delta t, j\Delta t + \overline{T})$.

Since Looking Forward Approach implies that players in the game $\Gamma(x, 0, T)$ use truncated information about the structure (motion equations) of the game, it is impossible to construct a cooperative trajectory for the whole game $\Gamma(x, 0, T)$. Instead of cooperative trajectory we build a conditionally cooperative trajectory

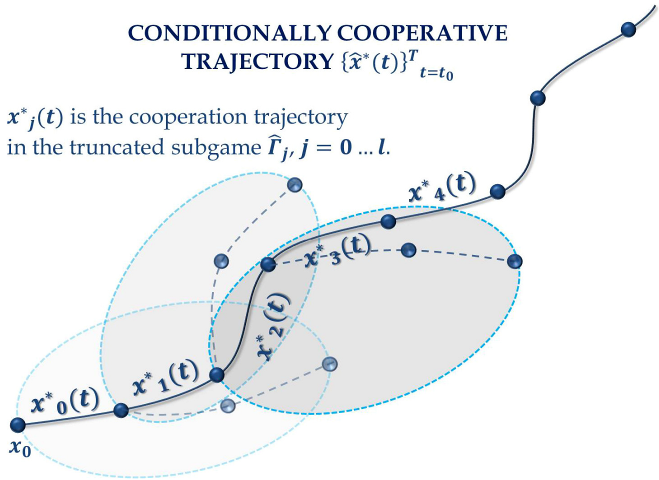

Fig. 3. Conditionally cooperative trajectory is defined as composition of cooperative trajectories for each truncated subgame. Each of them is represented by solid and dotted line. Solid line is a part of cooperative trajectory which contains in conditionally cooperative trajectory, dotted line represents part of cooperative trajectory which is not used by players.

$\{\hat{x}^*(t)\}^T_{t=0}$ as follows:

$$\{\hat{x}^*(t)\}^T_{t=0} = \begin{cases} x_0^*(t), & t \in [0, \Delta t], \\ x_1^*(t), & t \in (\Delta t, 2\Delta t], \\ \dots \\ x_j^*(t), & t \in (j\Delta t, (j+1)\Delta t], \\ \dots \\ x_l^*(t), & t \in (l\Delta t, (l+1)\Delta t], \end{cases} \tag{7}$$

where $x_j^*(t)$ is a cooperative trajectory, which corresponds to the optimal control $\psi_j^*(t,x) = (\psi_1^{*j}(t,x) \dots \psi_n^{*j}(t,x))$ in the truncated subgame $\hat{\Gamma}_j(x_{j-1}^*, j\Delta t, j\Delta t + \overline{T})$.

As it follows from (4), on the time interval $[0, \Delta t]$ we use part of cooperative trajectory $x_0^*(t)$ in the truncated subgame $\hat{\Gamma}_j(x_0, 0, \overline{T})$. At the time instant Δt information about the game updates and position of players is $x_0^*(\Delta t)$. On the time interval $(\Delta t, 2\Delta t]$ we use part of cooperative trajectory $x_1^*(t)$ in the truncated subgame $\hat{\Gamma}_j(x_0^*(\Delta t), \Delta t, \Delta t + \overline{T})$ which starts at the time instant Δt, in the position $x_0^*(\Delta t)$. At the time instant $j\Delta t$ information about the game updates and position of the players is $x_{j-1}^*(j\Delta t)$. On the time interval $(j\Delta t, (j+1)\Delta t]$ we use part of cooperative trajectory $x_j^*(t)$ in the truncated subgame $\hat{\Gamma}_j(x_{j-1}^*(j\Delta t), j\Delta t, j\Delta t + \overline{T})$ which starts at the time instant $j\Delta t$, in the position $x_{j-1}^*(j\Delta t)$. Therefore conditionally cooperative trajectory $\{\hat{x}^*(t)\}^T_{t=0}$ is defined as composition of cooperative trajectories $x_j^*(t)$ for each truncated subgame $\hat{\Gamma}_j(x_{j-1}^*, j\Delta t, j\Delta t + \overline{T})$ defined on time intervals $[j\Delta t, (j+1)\Delta t]$ (Fig. 3).

3. Truncated Cooperative Subgame

Consider the truncated cooperative subgame $\hat{\Gamma}_j(x^*_{j-1}, j\Delta t, j\Delta t + \overline{T})$ with initial condition x^*_{j-1} defined at time instant $j\Delta t$ and duration $\overline{T}$. For each coalition $S \subset N$ defines the values of characteristic function as

$$V(S, x^*_j, j\Delta t, j\Delta t + \overline{T}) = \begin{cases} \sum_{i=1}^{n} \hat{K}^j_i(x^*_{j-1}, j\Delta t, j\Delta t + \overline{T}; u^{*j}_1 \ldots u^{*j}_n), & S = N, \\ V_j(S), & S \subset N, \\ 0, & S = \emptyset, \end{cases} \tag{8}$$

where $V_j(S)$ is the value of zero-sum truncated subgame $\hat{\Gamma}_j(x^*_{j-1}, j\Delta t, j\Delta t + \overline{T})$ between coalition S and $N\backslash S$.

Using the characteristic function in each truncated subgame $\hat{\Gamma}_j(x^*_{j-1}, j\Delta t, j\Delta t + \overline{T})$ define an imputation $\xi^j(x^*_j, j\Delta t, j\Delta t + \overline{T})$ for each truncated subgame as follows:

$$\begin{gathered} \xi^j_i(x^*_j, j\Delta t, j\Delta t + \overline{T}) \geq V(\{i\}, x^*_j, j\Delta t, j\Delta t + \overline{T}), \quad i \in N, \\ \sum_{i=1}^{n} \xi^j_i(x^*_j, j\Delta t, j\Delta t + \overline{T}) = V(\{i\}, x^*_j, j\Delta t, j\Delta t + \overline{T}). \end{gathered} \tag{9}$$

The set of all possible imputations for each truncated subgame is denoted as $E_j(x^*_{j-1}, j\Delta t, j\Delta t + \overline{T})$. Suppose that in each truncated subgame $\hat{\Gamma}_j(x^*_{j-1}, j\Delta t, j\Delta t + \overline{T})$ optimal solution $M_j(x^*_{j-1}, j\Delta t, j\Delta t + \overline{T}) \subset E_j(x^*_{j-1}, j\Delta t, j\Delta t + \overline{T})$ is chosen, $M_j(x^*_{j-1}, j\Delta t, j\Delta t + \overline{T}) \neq \emptyset$ (it can be Core, NM solution, nucleolus, Shapley value).

It is easy to suggest that distribution of the total payoff of players in the game $\Gamma(x_0, 0, T)$ along the conditionally cooperative trajectory $\{\hat{x}^*(t)\}^T_{t=0}$ must be organized as composition of imputations of total payoff for each time interval $[j\Delta t, (j+1)\Delta t]$, $j = 0, \ldots, l$, because of the structure of the game $\Gamma(x_0, 0, T)$. In what follows we shall try to construct a new optimality principle (solution concept). But it is easy to see that using optimal solutions for each truncated subgame $\hat{\Gamma}_j(x^*_{j-1}, j\Delta t, j\Delta t + \overline{T})$ described above it is impossible to construct such composition. Optimal solution for each truncated subgame is defined for the same time interval as the truncated subgame, $[j\Delta t, j\Delta t + \overline{T}]$. We need the solution, which will be defined just for the time interval $[j\Delta t, (j+1)\Delta t]$, because the information about the structure of the game updates after every Δt time interval. Suppose we use optimal solution for truncated subgame $\hat{\Gamma}_j(x^*_{j-1}, j\Delta t, j\Delta t + \overline{T})$ during the whole time interval $[(j\Delta t, j\Delta t + \overline{T}]$, but at the time instant $t = (j+1)\Delta t$ information about the game structure updates and the new truncated subgame $\hat{\Gamma}_j(x^*_{j-1}, j\Delta t, j\Delta t + \overline{T})$ starts, but the chosen solution is no longer optimal, because it is based upon the outdated information. To get along with this problem we will use IDP [Petrosian *et al.* 2012] for each truncated subgame. IDP also gives us time consistency of our newly built solution concept and opportunity to define the cooperative solution in each time instant from $[j\Delta t, j\Delta t + \overline{T}]$.

4. Construction of Solution Concept

In order to construct the solution concept for the game $\Gamma(x_0, 0, T)$ we define a family of subgames of truncated subgame $\hat{\Gamma}_j(x^*_{j-1}, j\Delta t, j\Delta t + \overline{T})$ along its cooperative trajectory $x^*_j(t)$ as

$$\hat{\Gamma}_j(x^*_j, t, j\Delta t + \overline{T}), \tag{10}$$

where $t \in (j\Delta t, j\Delta t + \overline{T}]$ is a starting time of the subgame.

Characteristic function along $x^*_j(t)$ in the family of subgames of truncated subgame $\hat{\Gamma}_j(x^*_j, t, j\Delta t + \overline{T})$ on the time interval $(j\Delta t, j\Delta t + \overline{T}]$, $j = 0, \ldots, l$ is defined as in (9). Define the set of imputations in $\hat{\Gamma}_j(x^*_j, t, j\Delta t + \overline{T})$ along x^*_j as $E_j(x^*_j, t, j\Delta t + \overline{T})$, where $t \in (j\Delta t, j\Delta t + \overline{T}]$, $j = 0, \ldots, l$.

Suppose that for each subgame of previously defined truncated subgame $\hat{\Gamma}_j(x^*_j, j\Delta t, j\Delta t + \overline{T})$ optimal solution $M_j(x^*_j, t, j\Delta t + \overline{T}) \neq \emptyset$ along the cooperative trajectory x^*_j is chosen.

Suppose that for each truncated subgame $\hat{\Gamma}_j(x^*_j, j\Delta t, j\Delta t + \overline{T})$ in the starting positions $x^*_j(j\Delta t)$ players agreed to choose an imputation

$$\xi_j(x^*_j, j\Delta t, j\Delta t + \overline{T}) \in M_j(x^*_j, j\Delta t, j\Delta t + \overline{T}) \tag{11}$$

and corresponding IDP

$$B_j(t) = [B^j_1(t) \ldots B^j_n(t)], \tag{12}$$

where $t \in (j\Delta t, j\Delta t + \overline{T}]$, which guaranties the time consistency of this imputation [Petrosian *et al.* 2012]:

$$\xi_j(x^*_j, j\Delta t, j\Delta t + \overline{T}) = \int_{j\Delta t}^{j\Delta t + \overline{T}} B_j(t, x^*_j)dt. \tag{13}$$

First consider the following condition concerning the imputation $\xi_j(x^*_j, t, j\Delta t + \overline{T})$ that this function is twice continuously differentiable in t and x^*_j. Theorem characterizing a formula for $B_j(t, x^*_j)$ can be provided as follows [Yeung and Petrosyan, 2012]:

Theorem 2. *If previous condition is satisfied, then IDP with an instantaneous payment at time* $\tau \in [t, T]$ *is given by*:

$$\begin{aligned} B_j(\tau, x^*_j) = &-[\xi^j_t(x^*_j, t, j\Delta t + \overline{T})|_{t=\tau}] \\ &- [\xi^j_{x^*_j}(x^*_j, t, j\Delta t + \overline{T})|_{t=\tau}]g[\tau, x^*_j, \psi^{*j}_1(\tau, x) \ldots \psi^{*j}_n(\tau, x)]. \end{aligned} \tag{14}$$

Consider a new concept of solution in the differential game with Looking Forward Approach in the game $\Gamma(x, 0, T)$ using IDP $B_j(t)$ for each chosen imputation $\xi_j(x^*_j, j\Delta t, j\Delta t + \overline{T}) \in M_j(x^*_j, j\Delta t, j\Delta t + \overline{T})$ in the truncated subgame

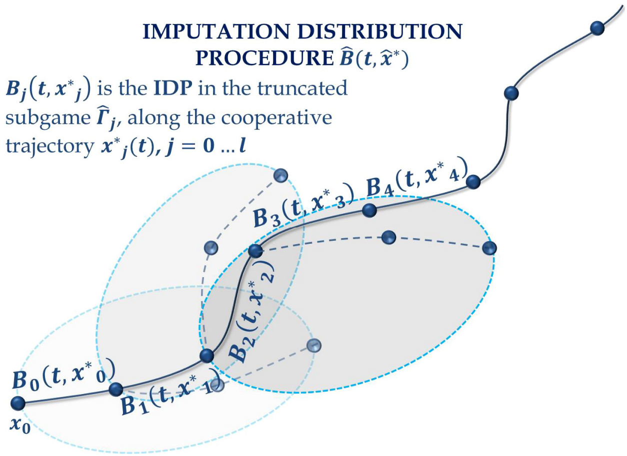

Fig. 4. IDP in the game is defined as composition of IDP's for each truncated subgame.

$\hat{\Gamma}_j(x_j^*, j\Delta t, j\Delta t + \overline{T})$ (Fig. 4):

$$\hat{B}(t) = \begin{cases} B_0(t), & t \in [0, \Delta t], \\ \ldots \\ B_j(t), & t \in (j\Delta t, (j+1)\Delta t], \\ \ldots \\ B_l(t), & t \in (l\Delta t, (l+1)\Delta t]. \end{cases} \tag{15}$$

Introduce the following vector:

$$\hat{\xi}(x_0, 0, T) = \int_0^T \hat{B}(\tau)d\tau = \sum_{j=0}^{l} \int_{j\Delta t}^{(j+1)\Delta t} B_j(\tau)d\tau. \tag{16}$$

Define the set $\hat{M}$ or $\hat{M}(x_0, 0, T)$ as set of all possible $\hat{\xi}(x_0, 0, T)$ for different imputations $\xi_j(x_j^*, j\Delta t, j\Delta t+\overline{T}) \in M_j(x_j^*, j\Delta t, j\Delta t+\overline{T})$ and corresponding IDPs $B_j(t)$, $j = 0, \ldots, l$.

It is easy to see that solution concept $\hat{M}$ is time consistent. But there is another surprising property of $\hat{M}$. To understand this, we introduce the definition.

Definition 2. Solution $M(x, 0, T)$ is called strongly Δt-time-consistent if for all $\xi(x, 0, T) \in M(x, 0, T)$ there exists IDP $B(t)$, which satisfies

$$\omega[\xi(x, 0, T); x^*(j\Delta t), 0, j\Delta t] \oplus M(x^*, j\Delta t, T) \subset M(x, 0, T), \tag{17}$$

where $\omega[\xi(x, 0, T); x^*(j\Delta t), 0, j\Delta t] = \int_0^{j\Delta t} \hat{B}(\tau)d\tau$, $j = 0, \ldots, l$, $a \oplus A = \{a + a' : a' \in A, a \in R^n, A \subset R^n\}$.

Theorem 3. *Solution concept $\hat{M}$ is strongly Δt-time-consistent in the game $\Gamma(x_0, 0, T)$.*

Proof. Consider

$$\bar{\xi} \in \omega[\hat{\xi}(x_0, 0, T); x^*(j\Delta t), 0, j\Delta t] \oplus \hat{M}(x^*, j\Delta t, T), \tag{18}$$

then $\bar{\xi}$ can be represented in the following form

$$\begin{aligned} \bar{\xi} &= \int_0^{j\Delta t} \hat{B}'(\tau)d\tau + \hat{\xi}''(x^*, j\Delta t, T) \\ &= \sum_{k=0}^{j-1} \int_{k\Delta t}^{(k+1)\Delta t} B_k'(\tau)d\tau + \hat{\xi}''(x^*, j\Delta t, T), \end{aligned} \tag{19}$$

where $\hat{\xi}''(x^*, j\Delta t, T) \in \hat{M}(x^*, j\Delta t, T)$, $\sum_{k=0}^{j-1} \int_{k\Delta t}^{(k+1)\Delta t} B_k'(\tau)d\tau$ is defined for a fixed

$$\xi_k'(x_k^*, k\Delta t, k\Delta t + \overline{T}) \in M_k(x_k^*, k\Delta t, k\Delta t + \overline{T}), \quad k = 0 \cdots j-1. \tag{20}$$

But $\hat{\xi}''(x^*, j\Delta t, T) = \int_{j\Delta t}^{T} \hat{B}''(\tau)d\tau = \sum_{k=j}^{l} \int_{k\Delta t}^{(k+1)\Delta t} B_k''(\tau)d\tau$ is defined for some

$$\xi_k''(x_k^*, k\Delta t, k\Delta t + \overline{T}) \in M_k(x_k^*, k\Delta t, k\Delta t + \overline{T}), \quad k = j, \ldots, l. \tag{21}$$

In the same time any element of $\hat{M}(x_0, 0, T)$ has the form (12). And if we take

$$\hat{B}(t) = \begin{cases} \hat{B}'(t), & t \in [0, j\Delta t), \\ \hat{B}''(t), & t \in [j\Delta t, T], \end{cases} \tag{22}$$

we get that our solution can be represented as

$$\bar{\xi} = \sum_{k=0}^{j-1} \int_{k\Delta t}^{(k+1)\Delta t} B_k'(\tau)d\tau + \sum_{k=j}^{l} \int_{k\Delta t}^{(k+1)\Delta t} B_k''(\tau)d\tau = \int_0^T \hat{B}(\tau)d\tau \tag{23}$$

which shows that according to the definition (12) $\bar{\xi} \in \hat{M}(x_0, 0, T)$. □

5. Looking Forward Approach in Cooperative Extraction

The following example for each truncated cooperative subgame was considered in Jorgensen and Yeung [1999], the problem of time consistency in this example was considered in Yeung and Petrosyan [2012]. We applied Looking Forward Approach to the example.

Consider the differential resource extracting game with two asymmetric extractors. Motion equation for the resource stock $x(t) \in X \subset R$ has the following form:

$$\begin{aligned} \dot{x} &= a\sqrt{x(t)} - bx(t) - u_1 - u_2, \\ x(0) &= x_0, \end{aligned} \tag{24}$$

where u_i is a harvest rate of extractor, $i = 1, 2$.

The payoff function of extractor i is equal to

$$K_i(x,0,T;u_1,u_2) = \int_0^T \left[\sqrt{u_i(t)} - \frac{c_i}{\sqrt{x(t)}} u_i(t)\right] dt, \tag{25}$$

where c_i is a constant $(c_1 \neq c_2)$.

The game is defined on the time interval $[0,T]$. Suppose that at each time instant $t \in [j\Delta t,(j+1)\Delta t]$, $j = 0,\ldots,l$ players have just truncated information about the structure (motion equations) of the game on the time interval $[j\Delta t, j\Delta t + \overline{T}]$ and expert evaluation of the expected terminal payoff at the time instant $j\Delta t + \overline{T}$. To include this fact in the model, we introduce the truncated subgame $\hat{\Gamma}_j(x, j\Delta t, j\Delta t + \overline{T})$, $j = 0,\ldots,l$, where $l = \frac{T}{\Delta t} - \frac{\overline{T}}{\Delta t}$. Motion equations and initial conditions for the truncated subgame have the following form:

$$\begin{aligned} \dot{x} &= a\sqrt{x(t)} - bx(t) - u_1 - u_2, \\ x(j\Delta t) &= x_j, \end{aligned} \tag{26}$$

and payoff function

$$\begin{aligned} &\hat{K}_i^j(x, j\Delta t, j\Delta t + \overline{T}; u_j) \\ &\quad = \int_{j\Delta t}^{j\Delta t + \overline{T}} \left[\sqrt{u_i(t)} - \frac{c_i}{\sqrt{x(t)}} u_i(t)\right] dt + \hat{q}_i(j\Delta t + \overline{T}, x), \end{aligned} \tag{27}$$

where $\hat{q}_i(j\Delta t + \overline{T}, x)$ is an expert evaluation of payoff of player i for truncated subgame $\hat{\Gamma}_j(x, j\Delta t, j\Delta t + \overline{T})$.

Value function of extractor $i = 1,2$ in the game (26)–(27) in cooperative setting (the payoff of player i in fixed Nash equilibria as function of initial conditions) is [Jorgensen and Yeung, 1999]

$$V^{(j\Delta t)i}(t,x) = A_i^j(t)\sqrt{x} + C_i^j(t), \tag{28}$$

where $i,k = 1,2$, $i \neq k$, $j = 0,l$, $A_i^j(t)$, $B_i^j(t)$, $A_k^j(t)$, $B_k^j(t)$ satisfy

$$\begin{aligned} \dot{A}_i^j(t) &= \left[\frac{b}{2}\right] A_i^j(t) - \frac{1}{2\left[c_i + \frac{A_i^j(t)}{2}\right]} + \frac{c_i}{4\left[c_i + \frac{A_i^j(t)}{2}\right]^2} \\ &\quad + \frac{A_i^j(t)}{8\left[c_i + \frac{A_i^j(t)}{2}\right]^2} + \frac{A_i^j(t)}{8\left[c_k + \frac{A_k^j(t)}{2}\right]^2}, \\ \dot{C}_i^j(t) &= -\frac{a}{2} A_i^j(t), \\ A_i^j(j\Delta t + \overline{T}) &= \hat{q}_i(j\Delta t + \overline{T}, x), \quad C_i^j(j\Delta t + \overline{T}) = 0. \end{aligned}$$

Consider the case when in each truncated subgame $\hat{\Gamma}_j(x, j\Delta t, j\Delta t + \overline{T})$, $j = 0, \ldots, l$ the resources extractors agree to act cooperatively and follow the optimality principle under which they would maximize their joint payoffs and share the excess of the total cooperative payoff over the sum of individual noncooperative payoffs proportional to the agents noncooperative payoffs.

Maximized expected joint payoff can be derived as follows [Yeung and Petrosyan, 2012; Jorgensen and Yeung, 1999]:

$$W^{(j\Delta t)}(t, x) = A^j(t)\sqrt{x} + C^j(t), \tag{29}$$

where $j = 0, \ldots, l$, $A^j(t)$, $B^j(t)$, $A^j(t)$, $B^j(t)$ satisfy

$$\begin{aligned}
\dot{A}^j(t) &= \left[\frac{b}{2}\right] A^j(t) - \frac{1}{2\left[c_1 + \frac{A^j(t)}{2}\right]} + \frac{1}{4\left[c_i + \frac{A^j(t)}{2}\right]^2} \\
&\quad + \frac{c_1}{4\left[c_1 + \frac{A^j(t)}{2}\right]^2} + \frac{c_2}{4\left[c_2 + \frac{A^j(t)}{2}\right]^2} \\
&\quad + \frac{A^j(t)}{8\left[c_1 + \frac{A^j(t)}{2}\right]^2} + \frac{A^j(t)}{8\left[c_2 + \frac{A^j(t)}{2}\right]^2}, \\
\dot{C}^j(t) &= -\frac{a}{2} A^j(t), \\
A^j(j\Delta t + \overline{T}) &= \hat{Q}(j\Delta t + \overline{T}, x), \quad C^j(j\Delta t + \overline{T}) = 0.
\end{aligned}$$

Optimal cooperative trajectory for each truncated subgame has the following form [Yeung and Petrosyan, 2012; Jorgensen and Yeung, 1999]:

$$x_j^*(t) = \varpi_j^2(j\Delta t, t)\left[\sqrt{x(j\Delta t)} + \int_{j\Delta t}^{t} \varpi_j(j\Delta t, \tau)^{-1} H_1^j \, d\tau\right]^2, \quad t \in [j\Delta t, j\Delta t + \overline{T}], \tag{30}$$

where

$$\begin{aligned}
\varpi_j(j\Delta t, t) &= \exp\left[\int_{j\Delta t}^{t} [H_2^j(\tau)] d\tau\right] \\
H_1^j &= \frac{1}{2} a, \\
H_2^j(t) &= -\left[\frac{1}{2} b + \frac{1}{8\left[c_1 + \frac{A^j(t)}{2}\right]^2} + \frac{A^j(t)}{8\left[c_2 + \frac{A^j(t)}{2}\right]^2}\right].
\end{aligned}$$

According to (7) and (30) conditionally cooperative trajectory $\{\hat{x}^*\}_{t=0}^T$ in the game has the following form:

$$\{\hat{x}^*(t)\}_{t=0}^T = \begin{cases} \varpi_0^2(0,t)\left[\sqrt{x_0} + \int_0^t \varpi_0(0,t)^{-1} H_1^0 dt\right]^2, & t \in [0, \Delta t], \\ \cdots \\ \varpi_j^2(j\Delta t, t)\left[\sqrt{x_{j-1}^*(j\Delta t)} + \int_{j\Delta t}^t \varpi_j(j\Delta t, t)^{-1} H_1^j dt\right]^2, & t \in (j\Delta t, (j+1)\Delta t], \\ \cdots \\ \varpi_l^2(l\Delta t, t)\left[\sqrt{x_{j-1}^*(l\Delta t)} + \int_{l\Delta t}^t \varpi_l(l\Delta t, t)^{-1} H_1^l dt\right]^2, & t \in (l\Delta t, (l+1)\Delta t]. \end{cases} \quad (31)$$

5.1. *Subgame consistent cooperative extraction*

As optimality principle (proportional solution) in each truncated cooperative subgame we use:

$$\begin{aligned} \xi_j^i(x_j^*, j\Delta t, j\Delta t + \overline{T}) &= \frac{V^{(j\Delta t)i}(t, x_j^*)}{\sum_{k=1}^2 V^{(j\Delta t)k}(t, x_j^*)} W^{(j\Delta t)}(t, x_j^*) \\ &= \frac{[A_i^j(t)\sqrt{x_j^*} + C_i^j(t)]}{\sum_{k=1}^2 [A_k^j(t)\sqrt{x_j^*} + C_k^j(t)]} \\ &\quad \times [A^j(t)\sqrt{x_j^*} + C^j(t)]. \end{aligned} \quad (32)$$

Denote the corresponding IDP by

$$B_j(t, x_j^*) = [B_1^j(t, x_j^*), B_2^j(t, x_j^*)]. \quad (33)$$

As it was done by Yeung and Petrosyan [2012] one can derive the following formula for IDP $B_j(t, x_j^*)$:

$$\begin{aligned} B_j(\tau, x_j^*) = &-[\xi_t^{(j)}(x_j^*, t, j\Delta t + \overline{T})|_{t=\tau}] \\ &- [\xi_{x_j^*}^{(j)}(x_j^*, t, j\Delta t + \overline{T})|_{t=\tau}] \\ &\times \left[a\sqrt{x_j^*} - bx_j^* - \frac{x_j^*}{4[c_1 + A^j(t)]^2} - \frac{x_j^*}{4[c_2 + A^j(t)]^2}\right]. \end{aligned} \quad (34)$$

Using (34) and (15) we can construct IDP $\hat{B}(t)$ for the Cooperative Extraction game with Looking Forward Approach:

$$\hat{B}(t) = \begin{cases} -[\xi_t^{(0)}(x_0^*, t, \overline{T})|_{t=\tau}] - [\xi_{x_0^*}^{(0)}(x_0^*, t, \overline{T})|_{t=\tau}] \\ \quad \times \left[a\sqrt{x_0^*} - bx_0^* - \dfrac{x_0^*}{4[c_1 + A^0(t)]^2} - \dfrac{x_0^*}{4[c_2 + A^0(t)]^2}\right], \\ \quad t \in [0, \Delta t], \\ -[\xi_t^{(j)}(x_j^*, t, j\Delta t + \overline{T})|_{t=\tau}] - [\xi_{x_j^*}^{(j)}(x_j^*, t, j\Delta t + \overline{T})|_{t=\tau}] \\ \quad \times \left[a\sqrt{x_j^*} - bx_j^* - \dfrac{x_j^*}{4[c_1 + A^j(t)]^2} - \dfrac{x_j^*}{4[c_2 + A^j(t)]^2}\right], \\ \quad t \in (j\Delta t, (j+1)\Delta t], \\ -[\xi_t^{(l)}(x_l^*, t, l\Delta t + \overline{T})|_{t=\tau}] - [\xi_{x_l^*}^{(l)}(x_l^*, t, l\Delta t + \overline{T})|_{t=\tau}] \\ \quad \times \left[a\sqrt{x_l^*} - bx_l^* - \dfrac{x_l^*}{4[c_1 + A^l(t)]^2} - \dfrac{x_l^*}{4[c_2 + A^l(t)]^2}\right], \\ \quad t \in (l\Delta t, (l+1)\Delta t]. \end{cases} \tag{35}$$

Substituting $\hat{B}(t)$ from (35) in (16) we get the analytic expression for the optimal solution $\hat{\xi}(x_0, 0, T)$ for the Cooperative Extraction game with Looking Forward Approach.

6. Conclusion

In a real-life game, when there is no certain information about the future after a few stages and it reviles step by step, where it is impossible to construct optimal solution for the whole game the new solution concept is presented. This solution is based upon the payoff distribution mechanism [Petrosian *et al.*, 2012].

It is proved that this newly built solution is not only time consistent (which is a very rare event in cooperative differential games), but also strongly Δt-time-consistent.

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