Continuous-time fully distributed generalized Nash equilibrium seeking for multi-integrator agents

Mattia Bianchi [a], Sergio Grammatico [a]

[a] *Delft Center for Systems and Control, Delft University of Technology, The Netherlands*

Abstract

We consider a group of (multi)-integrator agents playing games on a network, in a partial-decision information scenario. We design fully distributed continuous-time controllers, based on consensus and primal-dual gradient dynamics, to drive the agents to a generalized Nash equilibrium. Our first solution adopts fixed gains, whose choice requires the knowledge of some global parameters of the game. Therefore, to adapt the procedure to setups where the agents do not have any global information, we introduce a controller that can be tuned in a completely decentralized fashion, thanks to the use of integral adaptive weights. We further introduce algorithms, both with constant and dynamic gains, specifically devised for generalized aggregative games. For all the proposed control schemes, we show convergence to a variational equilibrium, under Lipschitz continuity and strong monotonicity of the game mapping, by leveraging monotonicity properties and stability theory for projected dynamical systems.

1 Introduction

Generalized Nash equilibrium (GNE) problems arise in several engineering applications, including demand-side management in the smart grid [25], charging/discharging of electric vehicles [18], formation control [21], communication networks [14]. In these examples, multiple selfish decision makers, or agents, aim at optimizing their individual, yet inter-dependent, objective functions, subject to shared constraints. From a game-theoretic perspective, the goal is to design distributed GNE seeking algorithms, using the local information available to each agent. Moreover, in the cyber-physical sytems framework, games are often played by agents with their own dynamics [24], [28]. In this case, the "strategy" of each agent consists of the output of a dynamical system, and controllers have to be conceived to steer the physical processes to a Nash equilibrium, while ensuring closed-loop stability. Therefore, it is advantageous to consider continuous-time schemes, for which control-theoretic properties are more easily unraveled.

Literature review: A variety of different algorithms have been proposed to seek GNE in a distributed way [32], [33], [6]. A recent part of the literature focuses on aggregative games, for which the cost of each agent depends on the others agents' strategy only via an aggregative function [2], [4], [7]. These works refers to (aggregative) games played in a full-information setting, where each agent can access the decision of all the competitors (aggregate value), for example in the presence of a central coordinator that broadcasts the data to the network. Nevertheless, in many applications, the existence of a node with bidirectional communication with all the agents may be impractical, and the agents can only rely on local information. One solution is offered by pay-off based methods [15], [28], that are decentralized, but require the agents to measure their cost functions. Alternatively, in this paper, we assume that the agents agree on sharing some information with their neighbors. Each agent keeps an estimate of all the competitors' action and asymptotically reconstruct the true value, exploiting the data exchanged over the network. Such a partial-decision information scenario has been investigated for games without coupling constraints, resorting to (projected) gradient and consensus dynamics, both in discrete-time [26], [20], and continuous-time [17], [30], [8]. Of major interest for this paper is the method in [8], where a nonlinear averaging integral controller is used to tune on-line the weights of the communication. The advantage is to guarantee convergence to a Nash equilibrium (NE) without requiring the knowledge of any global parameter or the use of a constant, high-enough, gain, which is the solution proposed in [17]. Fewer works deal with generalized games. A double-layer algorithm was presented in [31]. Remarkably, Pavel in [23] derived a single-timescale, fixed step-size GNE learning algorithm, by leveraging an elegant operator splitting approach. The authors of [11] addressed aggregative games with equality constraints, via continuous-time design. Moreover, all the results mentioned above consider static or single-integrator agents only. Distributively driving a network of more complex physical systems to game theoretic solutions is still a relatively unexplored problem. With regard to aggregative games, a proportional integral feedback algorithm was developed in [9] to seek a NE in networks of passive nonlinear second-order systems; in [10], [34], continuous-time gradient-based controllers were introduced for some classes of nonlinear dynamics with uncertainties. The authors of [28] addressed generally coupled cost games played by linear agents, via an extremum seeking approach. NE problems arising in systems of multi-integrator agents were studied in [24]. Moreover, all the references cited do not consider generalized games. Despite the scarcity of results, the presence of coupling constraints is a significant extension, that arises naturally in a variety of fields, when the agents share some common resource or limitation [12, 2].

Contributions: Motivated by the above, in this paper we investigate continuous-time GNE seeking for networks of multi-integrator agents. We consider games with affine coupling constraints, played under partial-decision information. Specifically:

- We introduce two primal-dual projected-gradient controllers, for the case of single-integrator agents. The first is a continuous-time version of the algorithm in [23]. It employs a constant gain, whose choice requires the knowledge of the algebraic connectivity of the communication graph and of the Lipschitz and strong monotonicity constants of the game mapping. To relax this condition, we present a novel distributed averaging integral controller, that extends the solution of [8] to generalized games. In particular, the adaptive weights in place of the fixed global gain allows for a fully-decentralized tuning, that does not need any non-local information. For both algorithms, we prove convergence of primal and dual variables, under strong monotonicity and Lipschitz continuity of the game mapping. We are not aware of any other continuous-time GNE seeking scheme, for generally coupled costs games, whose convergence is guaranteed under such mild assumptions. (3)
- We propose a controller, with dynamic gains, specifically designed for generalized aggregative games. The agents keep and exchange an estimate of the aggregate value only, thus reducing communication and computation cost. With respect to [11], we can also handle inequality constraints. Furthermore, our algorithm requires no knowledge of global parameters and virtually no tuning. (4)
- We show how all of our controllers can be adapted to learning GNE in games with shared constraints played by multi-integrator agents. To the best of our knowledge, we are the first to address *generalized* games with higher-order dynamical agents. Besides, the use of adaptive weights still ensures convergence without any a priori information on the game. (5)

Some preliminary results of this paper have been submitted in [5], where algorithms with adaptive gains and aggregative games are not considered.

Basic notation: $\mathbb{R}$ ($\mathbb{R}_{\geq 0}$) denotes the set of (nonnegative) real numbers. For a differentiable function $g : \mathbb{R}^n \to \mathbb{R}$, $\nabla_x g(x)$ is its gradient. $\mathbf{0}$ ($\mathbf{1}$) denotes a matrix/vector with all elements equal to 0 (1); to improve clarity, we may add the dimension of these matrices/vectors as subscript. $I_n \in \mathbb{R}^{n\times n}$ denotes the identity matrix of dimension n. $A^\top$ and $\|A\|$ denote the transpose and the largest singular value of a matrix A, respectively. If A is symmetric, $\lambda_{\min}(A) := \lambda_1(A) \leq \cdots \leq \lambda_n(A) =: \lambda_{\max}(A)$ denote its eigenvalues. $A \succ 0$ stands for a symmetric positive definite matrix. $\mathrm{diag}(A_1, \dots, A_N)$ denotes the block diagonal matrix with the matrices $A_1, \dots, A_N$ on its diagonal. $A \otimes B$ denotes the Kronecker product of the matrices A and B. For $x, y \in \mathbb{R}^n$, let $x^\top y$ and $\|x\|$ denote the Euclidean inner product and norm, respectively. Given N vectors $y_1, \dots, y_N$, we may denote $\boldsymbol{y} := \mathrm{col}(y_1, \dots, y_N) = [y_1^\top \dots y_N^\top]^\top$, and, for each $i = 1, \dots, N$, $\boldsymbol{y}_{-i} := \mathrm{col}(y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_N)$.

Operator-theoretic definitions: A mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ is monotone (μ-strongly monotone, with $\mu > 0$) if, for all $x, y \in \mathbb{R}^n$, $(F(x) - F(y))^\top (x - y) \geq 0$ ($\geq \mu\|x - y\|^2$). A mapping $F : \mathbb{R}^m \to \mathbb{R}^n$ is θ-Lipschitz continuous, with $\theta > 0$, if, for all $x, y \in \mathbb{R}^m$, $\|F(x) - F(y)\| \leq \theta\|x - y\|$. $\overline{E}$ denotes the closure of a set E. Given a closed convex set $S \subseteq \mathbb{R}^n$, the mapping $\mathrm{proj}_S : \mathbb{R}^n \to S$ denotes the projection onto S , i.e., $\mathrm{proj}_S(v) := \mathrm{argmin}_{y\in S} \|y - v\|$. The set-valued mapping $\mathrm{N}_S : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ denotes the normal cone operator for the the set S, i.e., $\mathrm{N}_S(x) = \varnothing$ if $x \notin S$, $\left\{v \in \mathbb{R}^n \mid \sup_{z\in S}\, v^\top (z - x) \leq 0\right\}$ otherwise. The tangent cone operator of S is defined as $\mathrm{T}_S : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, $\mathrm{T}_S(x) = \overline{\bigcup_{\delta>0} \frac{1}{\delta}(S - x)}$. $\Pi_S(x, v) := \mathrm{proj}_{\mathrm{T}_S(x)}(v)$ denotes the projection on the tangent cone of S at x. By Moreau's Decomposition Theorem [1, Th. 6.30], it holds that $v = \mathrm{proj}_{\mathrm{T}_S(x)}(v) + \mathrm{proj}_{\mathrm{N}_S(x)}(v)$ and $\mathrm{proj}_{\mathrm{T}_S(x)}(v)^\top \mathrm{proj}_{\mathrm{N}_S(x)}(v) = 0$, for any $v \in \mathbb{R}^n$.

Lemma 1 *For any nonempty closed convex set $S \subseteq \mathbb{R}^q$, any $y, y' \in S$ and any $\xi \in \mathbb{R}^q$, it holds that*

$$(y - y')^\top \Pi_S(y, \xi) \leq (y - y')^\top \xi.$$

Thus, if $\Pi_S(y, \xi) = 0$, then $(y - y')^\top \xi \geq 0$. □

Proof. By Moreau's theorem, $(\xi - \Pi_C(y, \xi)) \in \mathrm{N}_S(y)$, hence for any $y, y' \in C$, $(y' - y)^\top (\xi - \Pi_C(y, \xi)) \leq 0$. ■

2 Mathematical Background

We consider a group of noncooperative agents $\mathcal{I} := \{1, \dots, N\}$, where each agent $i \in \mathcal{I}$ shall choose its decision variable (i.e., strategy) x_i from its local decision set $\Omega_i \subseteq \mathbb{R}^{n_i}$. Let $x = \mathrm{col}((x_i)_{i\in\mathcal{I}}) \in \Omega$ denote the stacked vector of all the agents' decisions, $\Omega := \times_{i\in\mathcal{I}} \Omega_i \subseteq \mathbb{R}^n$ the overall action space and $n := \sum_{i=1}^N n_i$. Moreover, let $x_{-i} = \mathrm{col}((x_j)_{j\in\mathcal{I}\setminus\{i\}})$ denote the collective strategy of all the agents, except that of agent i. The goal of each agent $i \in \mathcal{I}$ is to minimize its objective function $J_i(x_i, x_{-i})$, which depends on both the local strategy x_i and on the decision variables of the other agents x_{-i}. Furthermore, we address *generalized* games, where the coupling among the agents arises also via their feasible decision sets. In particular, we consider affine coupling constraints; thus the overall feasible set is

$$\mathcal{X} := \Omega \cap \{x \in \mathbb{R}^n \mid Ax \leq b\}, \tag{1}$$

where $A := [A_1, \ldots, A_N]$ and $b := \sum_{i=1}^N b_i$, with $A_i \in \mathbb{R}^{m\times n_i}$ and $b_i \in \mathbb{R}^m$ being local data. The game then is represented by N inter-dependent optimization problems:

$$\forall i \in \mathcal{I}: \begin{cases} \underset{y_i\in\mathbb{R}^{n_i}}{\mathrm{argmin}}\; J_i(y_i, x_{-i}) \\ \quad \text{s.t.} \quad (y_i, x_{-i}) \in \mathcal{X}. \end{cases} \tag{2}$$

In this paper, we consider the problem to compute a GNE, as formalized next.

Definition 1 *A collective strategy $x^* = \mathrm{col}((x_i^*)_{i\in\mathcal{I}})$ is a generalized Nash equilibrium if, for all $i \in \mathcal{I}$,*

$$x_i^* \in \underset{y_i}{\mathrm{argmin}}\, J_i\left(y_i, x_{-i}^*\right) \text{ s.t. } (y_i, x_{-i}^*) \in \mathcal{X}. \qquad \square$$

Next, we formulate standard convexity and regularity assumptions for the constraint sets and cost functions.

Standing Assumption 1 *For each $i \in \mathcal{I}$, the set Ω_i is non-empty, closed and convex; $\mathcal{X}$ is non-empty and satisfies Slater's constraint qualification; J_i is continuously differentiable and the function $J_i(\cdot, x_{-i})$ is convex for every x_{-i}.* $\square$

Moreover, among all the possible GNE, we focus on the important subclass of variational GNE (v-GNE) [12, Def. 3.11]. Under the previous assumption, x^* is a v-GNE of the game in (2) if and only if there exist a dual variable $\lambda^* \in \mathbb{R}^m$ such that the following KKT conditions are satisfied [12, Th. 4.8]:

$$\begin{aligned} \mathbf{0}_n &\in F(x^*) + A^\top \lambda^* + \mathrm{N}_\Omega(x^*) \\ \mathbf{0}_m &\in -(Ax^* - b) + \mathrm{N}^m_{\mathbb{R}_{\geq 0}}(\lambda^*), \end{aligned} \tag{3}$$

where F is the *pseudo-gradient* mapping of the game:

$$F(x) := \mathrm{col}\left(\nabla_{x_i} J_i(x_i, x_{-i})\right)_{i\in\mathcal{I}}. \tag{4}$$

A sufficient condition for the existence of a unique v-GNE is the strong monotonicity of the pseudo-gradient [13, Th. 2.3.3], as postulated next. This assumption was used, e.g., in [17, Ass. 2], [4, Ass. 3], [8, Ass. 4].

Standing Assumption 2 *The pseudo-gradient mapping F in (4) is μ-strongly monotone and θ_0-Lipschitz continuous, for some $\mu > 0$, $\theta_0 > 0$.* $\square$

3 Distributed generalized Nash equilibrium seeking

In this section,we consider the game in (2), where each agent is associated with the following dynamical system:

$$\forall i \in \mathcal{I}: \quad \dot{x}_i = \Pi_{\Omega_i}(x_i, u_i), \; x_i(0) \in \Omega_i. \tag{5}$$

Our aim is to design the inputs u_i to seek a v-GNE in a fully distributed way. Specifically, each agent only knows its own cost function J_i and feasible set Ω_i. Besides, agent i does not have full knowledge of x_{-i}, and only relies on the information exchanged locally with neighbors over a communication network $\mathcal{G}(\mathcal{I}, \mathcal{E})$. The unordered pair (i, j) belongs to the set of edges $\mathcal{E}$ if and only if agent j and i can exchange information. We denote $W = [w_{ij}]_{i,j\in\mathcal{I}} \in \mathbb{R}^{N\times N}$ the symmetric adjacency matrix of $\mathcal{G}$, with $w_{ij} > 0$ if $(i, j) \in \mathcal{E}$, $w_{ij} = 0$ otherwise; L the symmetric Laplacian matrix of $\mathcal{G}$; $\mathcal{N}_i := \{j \mid (i, j) \in \mathcal{E}\}$ the set of neighbors of agent i

Standing Assumption 3 *The communication graph $\mathcal{G}(\mathcal{I}, \mathcal{E})$ is undirected and connected.* $\square$

In the remainder of the section, we present two dynamic controllers to asymptotically drive the system in (5) towards a v-GNE, in a fully-distributed fashion.

3.1 Distributed generalized Nash equilibrium seeking algorithm with constant gain

Our first algorithm is the continuous-time counterpart of [23, Alg. 1]. To cope with partial-decision information, each agent keeps an estimate of all other agents' action. We denote $\boldsymbol{x}^i = \mathrm{col}((\boldsymbol{x}^i_j)_{j\in\mathcal{I}}) \in \mathbb{R}^{Nn}$, where $\boldsymbol{x}^i_i := x_i$ and $\boldsymbol{x}^i_j$ is agent i's estimate of agent j's action, for all $j \neq i$. Moreover, each agent keeps an estimate $\lambda_i \in \mathbb{R}^m_{\geq 0}$ of the dual variable, and an auxiliary variable $z_i \in \mathbb{R}^m$ to allow for distributed consensus of the multipliers estimates. Our proposed dynamics are summarized in Algorithm 1, where $c > 0$ is a *global* constant parameter and the initial conditions $\boldsymbol{x}^i_{-i}(0) \in \mathbb{R}^{n-n_i}, \lambda_i(0) \in \mathbb{R}^m_{\geq 0}, z_i(0) \in \mathbb{R}^m$ can be chosen arbitrarily.

We note that the agents exchange $\{\boldsymbol{x}^i, z_i, \lambda_i\}$ with their neighbors only, therefore the controller can be implemented distributedly. In steady state, agents should agree on their estimates, i.e., $\boldsymbol{x}^i = \boldsymbol{x}^j$, $\lambda_i = \lambda_j$, for all $i, j \in \mathcal{I}$. This motivates the presence of consensual terms for both primal and dual variables. We denote $\boldsymbol{E}_q := \{\boldsymbol{y} \in \mathbb{R}^{Nq} : \boldsymbol{y} = \mathbf{1}_N \otimes y, y \in \mathbb{R}^q\}$ the consensual space of dimension q, for an integer $q > 0$, and $\boldsymbol{E}_q^\perp$ its orthogonal complement. Specifically, $\boldsymbol{E}_n$ is the estimate

Algorithm 1 Distributed GNE seeking (constant gain)

For all $i \in \mathcal{I}$:

$$\begin{aligned} \dot{x}_i &= \Pi_{\Omega_i}\left(x_i, u_i\right) \\ u_i &= -\Big(\nabla_{x_i} J_i(x_i, \boldsymbol{x}^i_{-i}) + A_i^\top \lambda_i + c \sum_{j\in\mathcal{N}_i} w_{ij}(x_i - \boldsymbol{x}^j_i)\Big) \\ \dot{\boldsymbol{x}}^i_{-i} &= -c \textstyle\sum_{j\in\mathcal{N}_i} w_{ij}(\boldsymbol{x}^i_{-i} - \boldsymbol{x}^j_{-i}) \\ \dot{z}_i &= \textstyle\sum_{j\in\mathcal{N}_i} w_{ij}(\lambda_i - \lambda_j) \\ \dot{\lambda}_i &= \Pi_{\mathbb{R}^m_{\geq 0}}\Big(\lambda_i, A_i x_i - b_i - \sum_{j\in\mathcal{N}_i} w_{ij}(z_i - z_j + \lambda_i - \lambda_j)\Big) \end{aligned}$$

consensus subspace and $\boldsymbol{E}_m$ is the multiplier consensus subspace.

To write the closed-loop system in compact form, let us define, as in [23, Eq. 13-14], for all $i \in \mathcal{I}$,

$$\mathcal{R}_i := \left[\, \mathbf{0}_{n_i\times n_{<i}} \;\; I_{n_i} \;\; \mathbf{0}_{n_i \times n_{>i}} \,\right], \tag{6a}$$

$$\mathcal{S}_i := \begin{bmatrix} I_{n<i} & \mathbf{0}_{n<i\times n_i} & \mathbf{0}_{n<i\times n>i} \\ \mathbf{0}_{n>i\times n<i} & \mathbf{0}_{n>i\times n_i} & I_{n>i} \end{bmatrix}, \tag{6b}$$

where $n_{<i} := \sum_{j<i, j\in\mathcal{I}} n_j$, $n_{>i} := \sum_{j>i, j\in\mathcal{I}} n_j$. We note that $\mathcal{R}_i$ selects the i-th n_i dimensional component from an n-dimensional vector, while $\mathcal{S}_i$ removes it. Thus, $\mathcal{R}_i \boldsymbol{x}^i = \boldsymbol{x}^i_i = x_i$ and $\mathcal{S}_i \boldsymbol{x}^i = \boldsymbol{x}^i_{-i}$. We define $\mathcal{R} := \operatorname{diag}\left((\mathcal{R}_i)_{i\in\mathcal{I}}\right)$, $\mathcal{S} := \operatorname{diag}\left((\mathcal{S}_i)_{i\in\mathcal{I}}\right)$. It follows that $x = \mathcal{R}\boldsymbol{x}$ and $\operatorname{col}((\boldsymbol{x}^i_{-i})_{i\in\mathcal{I}}) = \mathcal{S}\boldsymbol{x} \in \mathbb{R}^{(N-1)n}$. Moreover, we have that

$$\mathcal{R}^\top \mathcal{R} + \mathcal{S}^\top \mathcal{S} = I_{Nn}. \tag{7}$$

Let $\boldsymbol{\lambda} := \operatorname{col}((\lambda_i)_{i\in\mathcal{I}})$, $\boldsymbol{\Lambda} := \operatorname{diag}\left((A_i)_{i\in\mathcal{I}}\right)$, $\boldsymbol{b} := \operatorname{col}((b_i)_{i\in\mathcal{I}})$, and, for any integer $q > 0$, $\boldsymbol{L}_q := L \otimes I_q$. Furthermore, we define the *extended pseudo-gradient* mapping $\boldsymbol{F}$ as:

$$\boldsymbol{F}(\boldsymbol{x}) := \operatorname{col}\big(\nabla_{x_i} J_i\left(x_i, \boldsymbol{x}^i_{-i}\right)_{i\in\mathcal{I}}\big). \tag{8}$$

Then, the closed-loop system, in compact form, reads as

$$\begin{aligned} \dot{\boldsymbol{x}} &= \mathcal{R}^\top \Pi_\Omega\big(\mathcal{R}\boldsymbol{x}, -\big(\boldsymbol{F}(\boldsymbol{x}) + \boldsymbol{\Lambda}^\top \boldsymbol{\lambda} + c\mathcal{R}\boldsymbol{L}_n \boldsymbol{x}\big)\big) \\ &\quad + \mathcal{S}^\top\left(-c\mathcal{S}\boldsymbol{L}_n \boldsymbol{x}\right) &\text{(9a)} \\ \dot{\boldsymbol{z}} &= \boldsymbol{L}_m \boldsymbol{\lambda} &\text{(9b)} \\ \dot{\boldsymbol{\lambda}} &= \Pi_{\mathbb{R}^{Nm}_{\geq 0}}\left(\boldsymbol{\lambda}, (\boldsymbol{\Lambda}\mathcal{R}\boldsymbol{x} - \boldsymbol{b} - \boldsymbol{L}_m \boldsymbol{\lambda} - \boldsymbol{L}_m \boldsymbol{z})\right). &\text{(9c)} \end{aligned}$$

The following lemma relates the equilibria of the system in (9) to the v-GNE of the game in (2). The proof is analogous to [23, Th. 1], hence it is omitted.

Lemma 2 *The following statements hold:*

i) Any equilibrium point $\bar{\boldsymbol{\omega}} = \operatorname{col}\left(\bar{\boldsymbol{x}}, \bar{\boldsymbol{z}}, \bar{\boldsymbol{\lambda}}\right)$ *of the dynamics in* (9) *is such that* $\bar{\boldsymbol{x}} = \mathbf{1}_N \otimes x^*$, $\bar{\boldsymbol{\lambda}} = \mathbf{1}_N \otimes \lambda^*$, *where the pair* (x^*, λ^*) *satisfies the KKT conditions in* (3)*, hence* x^* *is the v-GNE of the game in* (2).

ii) The set of equilibrium points of (9) *is nonempty.* □

We remark that, in Algorithm 1, each agent evaluates the gradient of its cost function in its local estimate, not on the actual collective strategy. In fact, only when the estimates belong to the consensus space, i.e., $\boldsymbol{x} = \mathbf{1} \otimes x$ (in the case of full information, for example), we have that $\boldsymbol{F}(\boldsymbol{x}) = F(x)$. It follows that the operator $\mathcal{R}^\top \boldsymbol{F}$ is not necessarily monotone, not even if the pseudo gradient F in (4) is strongly monotone (Standing Assumption 2). This is the main technical difficulty that arises when studying NE seeking under partial-information. To deal with this complication, cocoercivity of the extended pseudo-gradient (on the augmented estimate space) is sometimes postulated [30, Ass. 4], [27, Ass. 5]. However, this is a limiting assumption, which does not hold in general [23, Rem. 6]. Instead, our analysis is based on a weaker restricted monotonicity property, which can be guaranteed for any game satisfying Standing Assumptions 1-3, without additional hypotheses, as formalized in the next two statements.

Lemma 3 *The extended pseudo-gradient mapping* $\boldsymbol{F}$ *in* (8) *is* θ*-Lipschitz continuous, for some* $\mu \leq \theta \leq \theta_0$*: for any* $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^{Nn}$, $\|\boldsymbol{F}(\boldsymbol{x}) - \boldsymbol{F}(\boldsymbol{y})\| \leq \theta \|\boldsymbol{x} - \boldsymbol{y}\|$. □

Proof. See Appendix A ■

Lemma 4 ([23, Lem. 3]) *Let*

$$M_1 := \begin{bmatrix} \frac{\mu}{N} & -\frac{\theta_0+\theta}{2\sqrt{N}} \\ -\frac{\theta_0+\theta}{2\sqrt{N}} & c\lambda_2(L) - \theta \end{bmatrix}, \quad \underline{c} := \tfrac{(\theta_0+\theta)^2 + 4\mu\theta}{4\mu\lambda_2(L)}. \tag{10}$$

For any $c > \underline{c}$*, for any* $\boldsymbol{x}$ *and any* $\boldsymbol{y} \in \boldsymbol{E}_n$*, it holds that* $M_1 \succ 0$ *and also that*

$$\begin{aligned} (\boldsymbol{x} - \boldsymbol{y})^\top \left(\mathcal{R}^\top \left(\boldsymbol{F}(\boldsymbol{x}) - \boldsymbol{F}\left(\boldsymbol{y}\right)\right) + c\boldsymbol{L}_n \left(\boldsymbol{x} - \boldsymbol{y}\right)\right) \\ \geq \lambda_{\min}(M_1) \left\|\boldsymbol{x} - \boldsymbol{y}\right\|^2. \end{aligned}$$ □

Remark 1 *Lipschitz continuity of the extended-pseudo gradient is often postulated, e.g. [23, Ass. 4], [8, Ass. 5]. Lemma 3 shows that this condition can be inferred by Standing Assumptions 1-2, similarly to [29, Lem. 1].*

The restricted strong monotonicity property of the previous statement is not new in the context of of games played under partial-information, see, e.g., [23], [17], [26]. By leveraging Lemma 4, we next show the convergence of the dynamics in (9) to a v-GNE. For brevity of notation, let us define, in the remainder of the paper, the set

$$\boldsymbol{\Omega} := \{\boldsymbol{x} \in \mathbb{R}^{Nn} \mid \mathcal{R}\boldsymbol{x} \in \Omega\}. \tag{11}$$

Theorem 1 *Let $c > \underline{c}$, with $\underline{c}$ as in* (10). *For any initial condition in $\Xi = \boldsymbol{\Omega} \times \mathbb{R}^{mN} \times \mathbb{R}^{mN}_{\geq 0}$, the system in* (9) *has a unique Carathodory solution, which belongs to Ξ for all $t \geq 0$. The solution converges to an equilibrium* $\mathrm{col}\left(\bar{\boldsymbol{x}}, \bar{\boldsymbol{z}}, \bar{\boldsymbol{\lambda}}\right)$*, with $\bar{\boldsymbol{x}} = \mathbf{1}_N \otimes x^*$, $\bar{\boldsymbol{\lambda}} = \mathbf{1}_N \otimes \lambda^*$, where the pair (x^*, λ^*) satisfies the KKT conditions in* (3)*, hence x^* is the v-GNE of the game in* (2). □

Proof. See Appendix E ■

3.2 Distributed generalized Nash equilibrium seeking algorithm with adaptive gains

The dynamic controller proposed in the last subsection allows to seek a v-GNE in a fully distributed way, provided that the global fixed gain c is chosen high-enough, as in Theorem 1. However, selecting a gain that ensures convergence requires global knowledge about the graph $\mathcal{G}$, i.e., the algebraic connectivity, and about the game mapping, i.e., the strong monotonicity and Lipschitz constants. These parameters are unlikely to be available locally in a network system, when the cost function of each agent is private. To overcome this limitation and enhance the scalability of the design, the authors of [8] proposed a controller for the integrator systems in (5), where the gain is tuned online, thus relaxing the need for global information, for games without coupling constraints. In this section we extend their result to the GNE problem, i.e., to games with shared constraints.

Our proposed controller is given in Algorithm 2. For all $i \in \mathcal{I}$, k_i is the adaptive gain of agent i, $\gamma_i > 0$ is a constant local parameter, $\rho^i = \mathrm{col}((\rho^i_j)_{j\in\mathcal{I}})$, and the initial conditions $\boldsymbol{x}^i_{-i}(0) \in \mathbb{R}^{n-n_i}, k_i(0) \in \mathbb{R}, \lambda_i(0) \in \mathbb{R}^m_{\geq 0}, z_i(0) \in \mathbb{R}^m$ can be chosen arbitrarily. We can

Algorithm 2 Distributed GNE seeking (adaptive gains)

For all $i \in \mathcal{I}$:

$$\begin{aligned}
\dot{x}_i &= \Pi_{\Omega_i}\left(x_i, u_i\right)\\
u_i &= -\nabla_{x_i} J_i(x_i, \boldsymbol{x}^i_{-i}) - A_i^\top \lambda_i - \sum_{j\in\mathcal{N}_i} w_{ij}(k_j \rho^j_i - k_i \rho^i_i)\\
\dot{\boldsymbol{x}}^i_{-i} &= -\textstyle\sum_{j\in\mathcal{N}_i} w_{ij}(k_j \rho^j_{-i} - k_i \rho^i_{-i})\\
\dot{k}_i &= \gamma_i \|\rho^i\|^2, \qquad \rho^i = \textstyle\sum_{j\in\mathcal{N}_i} w_{ij}\left(\boldsymbol{x}^j - \boldsymbol{x}^i\right)\\
\dot{z}_i &= \textstyle\sum_{j\in\mathcal{N}_i} w_{ij}\left(\lambda_i - \lambda_j\right)\\
\dot{\lambda}_i &= \Pi_{\mathbb{R}^m_{\geq 0}}\Big(\lambda_i, A_i x_i - b_i - \sum_{j\in\mathcal{N}_i} w_{ij}(z_i - z_j + \lambda_i - \lambda_j)\Big)
\end{aligned}$$

rewrite the overall closed-loop, in compact form, as

$$\begin{aligned}
\dot{\boldsymbol{x}} &= \mathcal{R}^\top \Pi_{\Omega}\left(\mathcal{R}\boldsymbol{x}, -\left(\boldsymbol{F}(\boldsymbol{x}) + \boldsymbol{\Lambda}^\top\boldsymbol{\lambda} + \mathcal{R}\left(LKL\otimes I_n\right)\boldsymbol{x}\right)\right)\\
&\quad + \mathcal{S}^\top\left(-\mathcal{S}\left(LKL\otimes I_n\right)\boldsymbol{x}\right) && \text{(12a)}\\
\dot{\boldsymbol{k}} &= D\left(\boldsymbol{\rho}\right)^\top\left(\Gamma\otimes I_n\right)\boldsymbol{\rho}, \qquad \boldsymbol{\rho} = -(L\otimes I_n)\boldsymbol{x} && \text{(12b)}\\
\dot{\boldsymbol{z}} &= \boldsymbol{L}_m\boldsymbol{\lambda} && \text{(12c)}\\
\dot{\boldsymbol{\lambda}} &= \Pi_{\mathbb{R}^{Nm}_{\geq 0}}\left(\boldsymbol{\lambda}, (\boldsymbol{\Lambda}\mathcal{R}\boldsymbol{x} - \boldsymbol{b} - \boldsymbol{L}_m\boldsymbol{\lambda} - \boldsymbol{L}_m\boldsymbol{z})\right). && \text{(12d)}
\end{aligned}$$

where $\boldsymbol{k} := \mathrm{col}\left(k_1, \ldots, k_N\right)$, $\boldsymbol{\rho} := \mathrm{col}\left(\rho^1, \ldots, \rho^N\right)$, $\Gamma := \mathrm{diag}\left(\gamma_1, \ldots, \gamma_N\right)$, $K := \mathrm{diag}\left(k_1, \ldots, k_N\right)$, $D(\boldsymbol{\rho}) := \mathrm{diag}\left(\rho^1, \ldots, \rho^N\right)$.

Lemma 5 *The following statements hold:*

i) Any equilibrium point $\mathrm{col}\left(\bar{\boldsymbol{x}}, \bar{\boldsymbol{k}}, \bar{\boldsymbol{z}}, \bar{\boldsymbol{\lambda}}\right)$ *of* (12) *is such that $\bar{\boldsymbol{x}} = \mathbf{1}_N \otimes x^*$, $\bar{\boldsymbol{\lambda}} = \mathbf{1}_N \otimes \lambda^*$, where the pair (x^*, λ^*) satisfies the KKT conditions in* (3)*, hence x^* is the v-GNE of the game in* (2).

ii) The set of equilibrium points of (12) *is nonempty.* □

Proof. See Appendix B ■

The following result is analogous to Lemma 4. The proof relies on the decomposition of $\boldsymbol{x}$ along the consensus subspace $\boldsymbol{E}_n$, where $\boldsymbol{F}$ is strongly monotone, and the disagreement subspace $\boldsymbol{E}_n^\perp$, where $LKL\otimes I_n$ is strongly monotone.

Lemma 6 *Let*

$$M_2 := \begin{bmatrix} \frac{\mu}{N} & -\frac{\theta_0+\theta}{2\sqrt{N}} \\ -\frac{\theta_0+\theta}{2\sqrt{N}} & k^*\lambda_2(L)^2 - \theta \end{bmatrix}, \ \underline{k} := \frac{(\theta_0+\theta)^2 + 4\mu\theta}{4\mu\lambda_2(L)^2}. \quad (13)$$

For any $k^* > \underline{k}$ and $K^* = I_N k^*$, for any $\boldsymbol{x} \in R^{Nn}$ and any $\boldsymbol{y} \in \boldsymbol{E}_n$, it holds that $M_2 \succ 0$ and also that

$$\begin{aligned}
&(\boldsymbol{x} - \boldsymbol{y})^\top \mathcal{R}^\top\left(\boldsymbol{F}(\boldsymbol{x}) - \boldsymbol{F}\left(\boldsymbol{y}\right)\right)\\
&\quad + \left(LK^*L\otimes I_M\right)(\boldsymbol{x} - \boldsymbol{y})) \geq \lambda_{\min}(M_2)\left\|\boldsymbol{x} - \boldsymbol{y}\right\|^2. \quad \square
\end{aligned}$$

Proof. See Appendix C ■

Building on this property, we can now present the main result of this section.

Theorem 2 *For any initial condition in $\Xi = \boldsymbol{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{mN} \times \mathbb{R}^{mN}_{\geq 0}$, the system in* (12) *has a unique Carathodory solution, which belongs to Ξ for all $t \geq 0$. The solution converges to an equilibrium* $\mathrm{col}\left(\bar{\boldsymbol{x}}, \bar{\boldsymbol{k}}, \bar{\boldsymbol{z}}, \bar{\boldsymbol{\lambda}}\right)$*, with $\bar{\boldsymbol{x}} = \mathbf{1}_N \otimes x^*$, $\bar{\boldsymbol{\lambda}} = \mathbf{1}_N \otimes \lambda^*$ and the pair (x^*, λ^*) satisfies the KKT conditions in* (3)*, hence x^* is the v-GNE of the game in* (2). □

Proof. See Appendix D ■

Remark 2 *Algorithm 2 allows for a fully uncoupled tuning. Specifically, each agent i can choose locally the initial conditions and the rate γ_i, independently of the other agents and without any need for communication or knowledge of global parameters. Compared to Algorithm 1, the agents exchange some extra information, namely the variables $(k_i\rho^i)_{i\in\mathcal{I}}$.*

4 Distributed generalized Nash equilibrium seeking for aggregative games

In this section, we focus on aggregative games. We assume that $n_i = \bar{n} > 0$ for all $i \in \mathcal{I}$ (hence $n = N\bar{n}$). In (average) aggregative games, the cost function of each agent depends on the local decision and on the value of the average strategy, i.e., $\operatorname{avg}(x) := \frac{1}{N}\sum_{i\in\mathcal{I}} x_i$. It follows that, for each $i \in \mathcal{I}$, there is a function $f_i : \mathbb{R}^{\bar{n}} \times \mathbb{R}^{\bar{n}} \to \mathbb{R}$ such that the original cost function J_i in (2) can be written as

$$J_i(x_i, x_{-i}) =: f_i(x_i, \operatorname{avg}(x)). \tag{14}$$

Since an aggregative game is only a particular instance of the game in (2), all the considerations on the existence and uniqueness of a v-GNE and equivalence with the KKT conditions in (3) are still valid.

Moreover, Algorithms 1-2 could still be used to drive a system of single integrators (5) towards a v-GNE. This would require each agent to keep (and exchange) an estimate of all other agents' action, i.e., a vector of $(N-1)\bar{n}$ components. In practice, however, the cost of each agent is only a function of the aggregative value $\operatorname{avg}(x)$, whose dimension $\bar{n}$ is independent of the number N of agents. To reduce the communication and computation burden, in this section, we introduce two distributed controllers, that are scalable with the number of agents, specifically designed to seek a v-GNE in aggregative games.

Our proposed dynamics are obtained by adapting Algorithms 1, 2 to take into account the aggregative structure of the game, and are illustrated in Algorithms 3, 4, respectively. Since the agents rely on local information only, they don't have access to the actual value of the average strategy. Therefore, we embed each agent with an auxiliary error variable $\varsigma_i \in \mathbb{R}^{\bar{n}}$, that is an estimate of the quantity $\operatorname{avg}(x) - x_i$. Each agent aims at asymptotically reconstructing the true aggregate value, based on the information received from its neighbors. We used the notation

$$\nabla_{x_i} f_i(x_i, \sigma^i) = \nabla_y f_i(y, \sigma^i)|_{y=x_i} + \frac{1}{N}\nabla_y f_i(x_i, y)|_{y=\sigma^i}.$$

Algorithm 3 Distributed GNE seeking in aggregative games (constant gain)

For all $i \in \mathcal{I}$:
Initialize $\varsigma_i = \mathbf{0}_{\bar{n}}$;

$$\begin{aligned}
\dot{x}_i &= \Pi_{\Omega_i}(x_i, u_i)\\
u_i &= -\nabla_{x_i} f_i(x_i, \sigma^i) - A_i^\top \lambda_i - c\textstyle\sum_{j\in\mathcal{N}_i} w_{ij}(\sigma^i - \sigma^j)\\
\dot{\varsigma}_i &= -c\textstyle\sum_{j\in\mathcal{N}_i} w_{ij}(\sigma^i - \sigma^j), \qquad \sigma^i = x_i + \varsigma_i\\
\dot{z}_i &= \textstyle\sum_{j\in\mathcal{N}_i} w_{ij}(\lambda_i - \lambda_j)\\
\dot{\lambda}_i &= \Pi_{\mathbb{R}^m_{\geq 0}}\Big(\lambda_i, A_i x_i - b_i - \sum_{j\in\mathcal{N}_i} w_{ij}(z_i - z_j + \lambda_i - \lambda_j)\Big)
\end{aligned}$$

Algorithm 4 Distributed GNE seeking in aggregative games (adaptive gains)

For all $i \in \mathcal{I}$:
Initialize $\varsigma_i = \mathbf{0}_{\bar{n}}$;

$$\begin{aligned}
\dot{x}_i &= \Pi_{\Omega_i}(x_i, u_i)\\
u_i &= -\nabla_{x_i} f_i(x_i, \sigma^i) - A_i^\top \lambda_i - \textstyle\sum_{j\in\mathcal{N}_i} w_{ij}(k_j\rho^j - k_i\rho^i)\\
\dot{\varsigma}_i &= -\textstyle\sum_{j\in\mathcal{N}_i} w_{ij}(k_j\rho^j - k_i\rho^i) \qquad \sigma^i = x_i + \varsigma_i\\
\dot{k}_i &= \gamma_i\|\rho^i\|^2 \qquad \rho^i = \textstyle\sum_{j\in\mathcal{N}_i} w_{ij}\left(\sigma^j - \sigma^i\right)\\
\dot{z}_i &= \textstyle\sum_{j\in\mathcal{N}_i} w_{ij}(\lambda_i - \lambda_j)\\
\dot{\lambda}_i &= \Pi_{\mathbb{R}^m_{\geq 0}}\Big(\lambda_i, A_i x_i - b_i - \sum_{j\in\mathcal{N}_i} w_{ij}(z_i - z_j + \lambda_i - \lambda_j)\Big)
\end{aligned}$$

We note that, in Algorithms 3, 4, the agents send and receive the quantities $\sigma_i \in \mathbb{R}^{\bar{n}}$, instead of exchanging the variables $\boldsymbol{x}^i \in \mathbb{R}^{N\bar{n}}$, like in Algorithms 1, 2.

Let $\boldsymbol{\varsigma} := \operatorname{col}((\varsigma_i)_{i\in\mathcal{I}})$, $\boldsymbol{\sigma} := \operatorname{col}((\sigma^i)_{i\in\mathcal{I}})$. Furthermore, let us define the *extended pseudo-gradient* mapping $\tilde{\boldsymbol{F}}$ as

$$\tilde{\boldsymbol{F}}(x, \boldsymbol{\sigma}) := \operatorname{col}\left(\left(\nabla_{x_i} f_i(x_i, \sigma^i)\right)_{i\in\mathcal{I}}\right). \tag{15}$$

Lemma 7 *The mapping $\tilde{\boldsymbol{F}}$ in (15) is $\tilde{\theta}$-Lipschitz continuous, for some $\tilde{\theta} > 0$: for any $(x, \boldsymbol{\sigma}), (x', \boldsymbol{\sigma}') \in \mathbb{R}^{2n}$, $\|\tilde{\boldsymbol{F}}(x, \boldsymbol{\sigma}) - \tilde{\boldsymbol{F}}(x', \boldsymbol{\sigma}')\| \leq \tilde{\theta}\,\|\operatorname{col}(x - x', \boldsymbol{\sigma} - \boldsymbol{\sigma}')\|$. Therefore, the mapping $\tilde{\boldsymbol{F}}(x, \cdot)$ is $\tilde{\theta}_\sigma$-Lipschitz continuous, for some $0 < \tilde{\theta}_\sigma \leq \tilde{\theta}$, for all $x \in \mathbb{R}^n$.* □

Proof. It follows from Lemma 3, by observing that $\tilde{\boldsymbol{F}}(x, \boldsymbol{\sigma}) = \boldsymbol{F}((x, (I_N \otimes \mathbf{1}_{N-1} \otimes I_{\bar{n}})(\frac{N}{N-1}\boldsymbol{\sigma} - \frac{1}{N-1}x)))$. ■

We note that, in Algorithms 3, 4, each agent evaluates the gradient of its cost function in its local estimate of the average strategy. Only if all the estimates coincide with the actual value, i.e., $\boldsymbol{\sigma} = \mathbf{1}_N \otimes \operatorname{avg}(x)$, we can conclude that $\tilde{\boldsymbol{F}}(x, \boldsymbol{\sigma}) = F(x)$, F as in (4).

The dynamics in Algorithms 3, 4 can be rewritten in compact form, as

$$\dot{x} = \Pi_\Omega\big(x, -\tilde{\boldsymbol{F}}(x,\boldsymbol{\sigma}) - \boldsymbol{\Lambda}^\top\boldsymbol{\lambda} - c\boldsymbol{L}_{\bar{n}}\boldsymbol{\sigma}\big) \tag{16a}$$
$$\dot{\boldsymbol{\varsigma}} = -c\boldsymbol{L}_{\bar{n}}(\boldsymbol{\sigma}), \qquad \boldsymbol{\sigma} = x + \boldsymbol{\varsigma} \tag{16b}$$
$$\dot{\boldsymbol{z}} = \boldsymbol{L}_m\boldsymbol{\lambda} \tag{16c}$$
$$\dot{\boldsymbol{\lambda}} = \Pi_{\mathbb{R}^{Nm}_{\geq 0}}\left(\boldsymbol{\lambda}, (\boldsymbol{\Lambda}x - \boldsymbol{b} - \boldsymbol{L}_m\boldsymbol{\lambda} - \boldsymbol{L}_m\boldsymbol{z})\right), \tag{16d}$$

and

$$\dot{x} = \Pi_\Omega\big(x, -\tilde{\boldsymbol{F}}(x,\boldsymbol{\sigma}) - \boldsymbol{\Lambda}^\top\boldsymbol{\lambda} - (LKL\otimes I_{\bar{n}})\boldsymbol{\sigma}\big) \tag{17a}$$
$$\dot{\boldsymbol{\varsigma}} = -(LKL\otimes I_{\bar{n}})\boldsymbol{\sigma}, \qquad \boldsymbol{\sigma} = x + \varsigma \tag{17b}$$
$$\dot{\boldsymbol{k}} = D(\boldsymbol{\rho})^\top(\Gamma\otimes I_{\bar{n}})\boldsymbol{\rho}, \qquad \boldsymbol{\rho} = -(L\otimes I_{\bar{n}})\boldsymbol{\sigma} \tag{17c}$$
$$\dot{\boldsymbol{z}} = \boldsymbol{L}_m\boldsymbol{\lambda} \tag{17d}$$
$$\dot{\boldsymbol{\lambda}} = \Pi_{\mathbb{R}^{Nm}_{\geq 0}}\left(\boldsymbol{\lambda}, (\boldsymbol{\Lambda}x - \boldsymbol{b} - \boldsymbol{L}_m\boldsymbol{\lambda} - \boldsymbol{L}_m\boldsymbol{z})\right), \tag{17e}$$

respectively.

The convergence analysis of the dynamics in (16), (17), to a v-GNE makes use of an invariance property of the systems, namely that $\mathrm{avg}(x) = \mathrm{avg}(\boldsymbol{\sigma})$ along any trajectory, if the initial conditions are chosen opportunely. In fact, it is crucial to set $\varsigma_i(0) = \mathbf{0}_{\bar{n}}$, that ensures $\mathrm{avg}(x(0)) = \mathrm{avg}(\boldsymbol{\sigma}(0))$. Indeed, the dynamics in (16b) or (17b) can be regarded as a continuous time dynamic tracking [20] for the time-varying quantity $\mathrm{avg}(x) - x$. By leveraging this invariance property, we obtain a refinement of the restricted strong monotonicity properties in Lemmas 4, 6, as demonstrated next.

Lemma 8 ([16, Lemma 4]) *Let*

$$M_3 := \begin{bmatrix} \mu & -\frac{\tilde{\theta}_\sigma}{2} \\ -\frac{\tilde{\theta}_\sigma}{2} & c\lambda_2(L) \end{bmatrix}, \quad \underline{c} := \tfrac{\tilde{\theta}_\sigma^2}{4\mu\lambda_2(L)} \tag{18}$$

For any $c > \underline{c}$, for any $(x,\boldsymbol{\sigma})$ such that $\mathrm{avg}(x) = \mathrm{avg}(\boldsymbol{\sigma})$ and any $(x',\boldsymbol{\sigma}')$ such that $\boldsymbol{\sigma}' = \mathbf{1}_N\otimes\mathrm{avg}(x')$, it holds that $M_3 \succ 0$, and also that

$$\begin{aligned}&(x-x')^\top(\tilde{\boldsymbol{F}}(x,\boldsymbol{\sigma}) - \tilde{\boldsymbol{F}}(x',\boldsymbol{\sigma}')) \\ &+(\boldsymbol{\sigma}-\boldsymbol{\sigma}')^\top(c\boldsymbol{L}_{\bar{n}}(\boldsymbol{\sigma}-\boldsymbol{\sigma}')) \\ &\geq \lambda_{\min}(M_3)\left\|\mathrm{col}\left(x-x', \boldsymbol{\sigma} - \mathbf{1}_N\otimes\mathrm{avg}(x)\right)\right\|^2. \quad \square\end{aligned}$$

Lemma 9 *Let*

$$M_4 = \begin{bmatrix} \mu & -\frac{\tilde{\theta}_\sigma}{2} \\ -\frac{\tilde{\theta}_\sigma}{2} & k^*\lambda_2(L)^2 \end{bmatrix}, \quad \underline{k} = \tfrac{\tilde{\theta}_\sigma^2}{4\mu\lambda_2(L)^2} \tag{19}$$

For any $k^* > \underline{k}$ and $K^* = I_N K^*$, for any $(x,\boldsymbol{\sigma})$ such that $\mathrm{avg}(x) = \mathrm{avg}(\boldsymbol{\sigma})$ and any $(x',\boldsymbol{\sigma}')$ such that $\boldsymbol{\sigma}' = \mathbf{1}_N\otimes\mathrm{avg}(x')$, it holds that $M_4 \succ 0$, and also that

$$\begin{aligned}&(x-x')^\top(\tilde{\boldsymbol{F}}(x,\boldsymbol{\sigma}) - \tilde{\boldsymbol{F}}(x',\boldsymbol{\sigma}')) \\ &+(\boldsymbol{\sigma}-\boldsymbol{\sigma}')^\top((LK^*L\otimes I_{\bar{n}})(\boldsymbol{\sigma}-\boldsymbol{\sigma}')) \\ &\geq \lambda_{\min}(M_4)\left\|\mathrm{col}\left(x-x', \boldsymbol{\sigma} - \mathbf{1}_N\otimes\mathrm{avg}(x)\right)\right\|^2. \quad \square\end{aligned}$$

Proof. See Appendix F ■

We are now ready to prove the main results of this section.

Theorem 3 *Let $c > \underline{c}$, with $\underline{c}$ as in (18). For any initial condition in $\Xi = \Omega\times\mathbb{R}^n\times\mathbb{R}^{mN}\times\mathbb{R}^{mN}_{\geq 0}$ such that $\boldsymbol{\varsigma}(0) = \mathbf{0}_n$, the system in (16) has a unique Carathodory solution that belongs to Ξ for all $t\geq 0$. The solution converges to an equilibrium $\mathrm{col}(\bar{x},\bar{\boldsymbol{\varsigma}},\bar{\boldsymbol{z}},\bar{\boldsymbol{\lambda}})$, with $\bar{x}+\bar{\boldsymbol{\varsigma}} = \mathbf{1}_N\otimes\mathrm{avg}(\bar{x})$, $\bar{\boldsymbol{\lambda}} = \mathbf{1}_N\otimes\lambda^*$, where the pair $(\bar{x},\lambda^*)$ satisfies the KKT conditions in (3), hence $\bar{x}$ is the v-GNE of the game in (2).* □

Proof. See Appendix H. ■

Theorem 4 *For any initial condition in $\Xi = \Omega\times\mathbb{R}^n\times\mathbb{R}^N\times\mathbb{R}^{mN}\times\mathbb{R}^{mN}_{\geq 0}$ such that $\boldsymbol{\varsigma}(0) = \mathbf{0}_n$, the system in (17) has a unique Carathodory solution, which belongs to Ξ for all $t\geq 0$. The solution converges to an equilibrium $\mathrm{col}\left(\bar{x},\bar{\boldsymbol{\varsigma}},\bar{\boldsymbol{k}},\bar{\boldsymbol{z}},\bar{\boldsymbol{\lambda}}\right)$, with $\bar{x}+\bar{\boldsymbol{\varsigma}} = \mathbf{1}_N\otimes\mathrm{avg}(\bar{x})$, $\bar{\boldsymbol{\lambda}} = \mathbf{1}_N\otimes\lambda^*$, where the pair $(\bar{x},\lambda^*)$ satisfies the KKT conditions in (3), hence $\bar{x}$ is the v-GNE of the game in (2).* □

Proof. See Appendix G ■

5 Distributed generalized Nash equilibrium seeking for multi-integrator systems

In this section, we consider a game as in (2) under the following additional assumption, which, to the best of our knowledge, has been always used in NE problems in the presence of higher-order dynamical agents [24, Ass. 1], [10, Def. 1].

Assumption 1 $\Omega = \mathbb{R}^n$. □

Besides, we assume that, for all $i\in\mathcal{I}$, agent is modeled as an integrator of order r_i, $r_i > 1$, i.e., of the form $x_i^{(r_i)} = u_i$ (single-integrators are not included for ease of presentation). Equivalently, using a notation that will be useful in the following (see [24]), we have

$$\forall i\in\mathcal{I}: \qquad \begin{cases}\dot{x}_i = C_i v_i & \text{(20a)} \\ \dot{v}_i = E_i v_i + G_i u_i, & \text{(20b)}\end{cases}$$

where, for all $i \in \mathcal{I}$, $x_i \in \mathbb{R}^{n_i}$, $v_i \in \mathbb{R}^{(r_i-1)n_i}$,

$$E_i = \begin{bmatrix} \mathbf{0}_{n_i(r_i-2)\times n_i} & I_{n_i(r_i-2)} \\ \mathbf{0}_{n_i\times n_i} & \mathbf{0}_{n_i\times n_i(r_i-2)} \end{bmatrix},\; G_i = \begin{bmatrix} \mathbf{0}_{n_i(r_i-2)\times n_i} \\ I_{n_i} \end{bmatrix},$$
$$C_i = \begin{bmatrix} I_{n_i} \; \mathbf{0}_{n_i\times n_i(r_i-2)} \end{bmatrix},\quad v_i = \mathrm{col}\big(x_i^{(1)}, \dots, x_i^{(r_i-1)}\big).$$

Our aim is to drive the agents' actions (i.e., the x_i coordinates of each agent state (x_i, v_i)) to a v-GNE of the game in (2). Moreover, at steady state, the generalized velocities v_i of all the agents must be zero. We emphasize that we are not able to directly control the agent strategy x_i for the system in (20).

We assume that each agent is able to measure its own generalized velocity v_i. As in [24], we consider the input

$$u = \tilde{u}_i - [I_{n_i} \quad c_i^\top \otimes I_{n_i}]v_i = \tilde{u}_i - \sum_{k=0}^{r_i-2} c_{i,k} x_i^{(k+1)}, \quad (21)$$

where $c_i^\top = [c_{i,1} \;\dots\; c_{i,(r_i-2)}]$ and $(c_{i,0} = 1, c_{i,1}, \dots, c_{i,(r_i-2)}, c_{i,(r_i-1)} = 1)$ are the ascending coefficients of any Hurwitz polynomial of order $(r_i - 1)$. Besides, we define, for all $i \in \mathcal{I}$, the transformation $(x_i, v_i) \mapsto (\zeta_i, v_i)$, where

$$\zeta_i := x_i + [\, c_i^\top \otimes I_{n_i} \;\; I_{n_i}\,]v_i. \quad (22)$$

Here, ζ_i can be interpreted as a prediction of the position of agent i, given its current state. Each closed-loop system, in the new coordinates, reads as

$$\forall i \in \mathcal{I}: \quad \begin{cases} \dot{\zeta}_i = \tilde{u}_i & (23a) \\ \dot{v}_i = \tilde{E}_i v_i + G_i \tilde{u}_i, & (23b) \end{cases}$$

where

$$\tilde{E}_i = \begin{bmatrix} \mathbf{0}_{n_i(r_i-2)\times n_i} & I_{n_i(r_i-2)} \\ -I_{n_i} & -c_i^\top \otimes I_{n_i} \end{bmatrix}.$$

We note that the dynamics of the new variable ζ_i in (23a), under Assumption 1, are identical to the single-integrator in (5), with translated input $\tilde{u}_i$. As such, we are in a position to design the inputs $\tilde{u}_i$ according to Algorithm 2 (or 1, or 3 or 4 for aggregative games), to drive the variable $\zeta := \mathrm{col}((\zeta_i)_{i\in\mathcal{I}})$ to an equilibrium $\bar{\zeta} = x^*$, where x^* is the v-GNE for the game in (2). Moreover, we note that $\tilde{E}_i$ is a Hurwitz matrix, as it is in canonical controllable form (modulo Kronecker multiplication by identity), and the coefficients of the last row are by assumption the coefficients of an Hurwitz polynomial. Therefore, the velocity dynamics (23b) are Input-to-state-stable (ISS) with respect to the input u_i [19, Lemma 4.6]. Finally, we remark that, at any equilibrium of (23), $v_i = \mathbf{0}_{n_i(r_i-1)}$, hence $\zeta_i = x_i$, for all $i \in \mathcal{I}$. Building on this considerations, we propose Algorithm 5 to drive the multi-integrator agents (20) towards a v-GNE.

Algorithm 5 Distributed GNE seeking for multi-integrator agents (adaptive gain)

For all $i \in \mathcal{I}$:

$$\begin{aligned}
\dot{x}_i &= C_i v_i \\
\dot{v}_i &= E_i v_i + G_i u_i \\
u_i &= \tilde{u}_i - [I_{n_i} \quad c_i^\top \otimes I_{n_i}]v_i \\
\tilde{u}_i &= -\nabla_i J_i(\boldsymbol{\zeta}_i^i, \boldsymbol{\zeta}_{-i}^i) - A_i^\top \lambda_i - \sum_{j\in\mathcal{N}_i} w_{ij}(k_j \rho_i^j - k_i \rho_i^i) \\
\dot{\boldsymbol{\zeta}}_{-i}^i &= -\textstyle\sum_{j\in\mathcal{N}_i} w_{ij}(k_j \rho_{-i}^j - k_i \rho_{-i}^i) \\
\boldsymbol{\zeta}_i^i &= x_i + [\, c_i^\top \otimes I_{n_i} \;\; I_{n_i}\,]v_i \\
\dot{k}_i &= \gamma_i \|\rho^i\|^2 \qquad\qquad \rho^i = \textstyle\sum_{j\in\mathcal{N}_i} w_{ij}\left(\boldsymbol{\zeta}^j - \boldsymbol{\zeta}^i\right) \\
\dot{z}_i &= \textstyle\sum_{j\in\mathcal{N}_i} w_{ij}\,(\lambda_i - \lambda_j) \\
\dot{\lambda}_i &= \Pi_{\mathbb{R}^m_{\geq 0}}\Big(\lambda_i, A_i \boldsymbol{\zeta}_i^i - b_i - \sum_{j\in\mathcal{N}_i} w_{ij}(z_i - z_j + \lambda_i - \lambda_j)\Big)
\end{aligned}$$

Differently from Algorithm 2, the agents are not keeping an estimate of other agents action, but of other agents *prediction*. Here, $\boldsymbol{\zeta}^i = (\mathrm{col}(\boldsymbol{\zeta}_j^i)_{j\in\mathcal{I}})$, and $\boldsymbol{\zeta}_j^i$ represents agent i's estimation of the quantity ζ_j for $j \neq i$, while $\boldsymbol{\zeta}_i^i = \zeta_i$. By defining $C = \mathrm{diag}((C_i)_{i\in\mathcal{I}})$, $\tilde{E} = \mathrm{diag}((\tilde{E})_{i\in\mathcal{I}})$, $G = \mathrm{diag}((G_i)_{i\in\mathcal{I}})$, $B = \mathrm{diag}(([\, c_i^\top \otimes I_{n_i} \;\; I_{n_i}\,])_{i\in\mathcal{I}})$, we can rewrite the closed-loop system in compact form as

$$\begin{aligned}
\dot{x} &= Cv && (24a)\\
\dot{v} &= \tilde{E}v - G\big(\boldsymbol{F}(\boldsymbol{\zeta}) + \boldsymbol{\Lambda}^\top \boldsymbol{\lambda} + \mathcal{R}\,(LKL \otimes I_n)\,\boldsymbol{\zeta}\big) && (24b)\\
\mathcal{S}\dot{\boldsymbol{\zeta}} &= -\mathcal{S}\,(LKL \otimes I_n)\,\boldsymbol{\zeta}, \qquad \mathcal{R}\boldsymbol{\zeta} = x + Bv && (24c)\\
\dot{\boldsymbol{k}} &= D\,(\boldsymbol{\rho})^\top\,(\Gamma \otimes I_n)\,\boldsymbol{\rho}, \qquad \boldsymbol{\rho} = -(L \otimes I_n)\boldsymbol{\zeta} && (24d)\\
\dot{\boldsymbol{z}} &= \boldsymbol{L}_m \boldsymbol{\lambda} && (24e)\\
\dot{\boldsymbol{\lambda}} &= \Pi_{\mathbb{R}^{Nm}_{\geq 0}}\left(\boldsymbol{\lambda}, (\boldsymbol{\Lambda}\mathcal{R}\boldsymbol{\zeta} - \boldsymbol{b} - \boldsymbol{L}_m\boldsymbol{\lambda} - \boldsymbol{L}_m\boldsymbol{z})\right) && (24f)
\end{aligned}$$

Theorem 5 *Let Assumption 1 hold. For any initial condition with $\boldsymbol{\lambda}(0) \in \mathbb{R}^m_{\geq 0}$, the system in (24) has a unique Carathodory solution, such that $\boldsymbol{\lambda}(t) \in \mathbb{R}^{Nm}_{\geq 0}$, for every $t \geq 0$. The solution converges to an equilibrium $\mathrm{col}\left(\bar{x}, \bar{v}, \bar{\boldsymbol{\zeta}}, \bar{\boldsymbol{k}}, \bar{\boldsymbol{z}}, \bar{\boldsymbol{\lambda}}\right)$, with $\bar{x} = x^*, \bar{v} = \mathbf{0}_r$ and $r = \sum_{i\in\mathcal{I}}(r_i - 1)n_i$, $\bar{\boldsymbol{\zeta}} = \mathbf{1}_N \otimes x^*$, $\bar{\boldsymbol{\lambda}} = \mathbf{1}_N \otimes \lambda^*$, where the pair (x^*, λ^*) satisfies the KKT conditions in (3), hence x^* is the v-GNE for the game in (2).* □

Proof. See Appendix I ■

Remark 3 *To deal with single integrator agents, i.e., if $r_i = 1$ for some $i \in \mathcal{I}$, it is enough to negect v_i in Algorithm 5, i.e., $\dot{x}_i = u_i = \tilde{u}_i$, $\boldsymbol{\zeta}_i^i = x_i$ essentially, we retrieve the controller in Algorithm 2.* □

Algorithm 5 can be tuned in a fully decentralized way and without the need for any global information. We remark that Algorithm 5 is derived by generating $\tilde{u}_i$ in (23) according to Algorithm 2. However, the proof of Theorem 5 is not based on the specific structure of Algorithm 2, but only on its convergence properties. Hence the result still holds if another controller with similar features is employed in place of Algorithm 2, allowing to select the one that best suits the problem at hand. For example, by picking $\tilde{u}_i$ according to Algorithm 4, we obtain a controller to drive a group of multi-integrator agents toward a v-GNE of an aggregative game, that can be tuned in a fully-uncoupled manner; by exploiting the controller in [11, Eq. 11], we could address aggregative games with equality constraints played over balanced digraphs. In [24], the authors considered NE problems (without shared constraints) and chose the inputs $\tilde{u}_i$ based on the algorithm presented in [17, Eq. 47] (we retrieve this controller when considering Algorithm 1 in the absence of coupling constraints). The controller in [17] achieves exponential convergence to a NE, hence ISS with respect to possible additive disturbances [19, Lemma 4.6]. Therefore, in [24], the authors were able to handle the presence of deterministic disturbances, via an asymptotic observer and by leveraging ISS arguments. We have not guaranteed this robustness, i.e., exponential convergence, for the primal-dual dynamics in (9). However, the controller in [24] is designed for games without any local or shared constraints (in this case, the NE problem reduces to finding a zero of the game mapping). On the contrary, the controller in Algorithm 5 drives the system in (20) to a v-GNE of a generalized game, and ensures for the coupling constraints to be satisfied asymptotically. We also remark that, like in [24], we assume the absence of constraints on the local feasible set of each agent (Assumption 1). Nevertheless, if some are present, they can be dualized as the coupling constraints, and hence satisfied asymptotically.

6 Illustrative applications

6.1 Mobile sensor network

We consider a group of five robots moving in a plane as in [24]. Each agent $i \in \mathcal{I} = \{1, \dots, 5\}$ has a cost function $J_i(p_i, p_{-i}) := p_i^T p_i + d_i^\top p_i + \sum_{j\in\mathcal{I}} \|p_i - p_j\|^2$, with $p_i = \mathrm{col}(x_i, y_i)$ its cartesian coordinates, $d_i \in \mathbb{R}^2$ random local parameters. We impose the local constraints $0.1 \leq y_i \leq 0.5$, $\forall i \in \mathcal{I}$. The robots can communicate over a random undirected connected graph $\mathcal{G}(\mathcal{I}, \mathcal{E})$. In order for all the robots to maintain communication with their neighbors, we impose the Chebyschev distance between any two neighboring robots to be smaller than 0.2. Hence, the (affine) coupling constraints are represented by $\max\{|x_i - x_j|, |y_i - y_j|\} \leq 0.2, \forall (i,j) \in \mathcal{E}$. We consider both velocity-actuated and force-actuated robots. We set $c = 30$ to satisfy the condition in Theorem 1; $\gamma_i = 1, \forall i \in \mathcal{I}$; initial conditions are chosen randomly.

Velocity-actuated robots: Each agent has a dynamic as in (5). Figure 1 compares the results for Algorithms 1 and 2 and shows convergence of both to the unique v-GNE and asymptotic satisfaction of the coupling constraints.

Force-actuated robots: Each agent is modeled as a double-integrator, i.e., as in (20) with $r_i = 2$. The local constraints are considered as part of the coupling constraints, hence they are dualized and satisfied only asymptotically (see 5). We simulate Algorithm 5 and the analogous algorithm with constant gain (obtained by choosing $\tilde{u}_i$ in (23a) according to Algorithm 1). The results are illustrated in Figure 2. Finally, in Figure 3, we compare the trajectories of the five robots in the velocity- and force-actuated scenario, under adaptive gain algorithms. In both cases, the agent are converging to the unique v-GNE. However, the local constraints are satisfied along the whole trajectory for single-integrator agents, only asymptotically for the double-integrator agents.

6.2 Resource allocation

We consider a resource allocation problem, modeled as an aggregative game [3]. Each of N agents has to complete a task in $\bar{n}$ time slots. We denote $x_i(h) \in [0,1]$, the ratio of the task that agent i allocates at time slot h, $h = 1, \dots, \bar{n}$. The local feasible sets are given by $\Omega_i := \{x_i \in \mathbb{R}^{\bar{n}} | \mathbf{0}_n \leq x_i \leq u_i, \mathbf{1}_{\bar{n}}^\top x_i = 1\}, i \in \mathcal{I}$, where u_i is the vector of maximum task allocations of agent i. The agents share the coupling constraints $\sum_{i=1}^N x_i(h) := Ax \leq b(h), h = 1, \dots, \bar{n}$, representing the maximal collective weighted allocation possible per time slot. The objective of agent i is to follow as much as possible an ideal strategy $\bar{x}_i$, while avoiding to allocate its task in time slots congested by other agents. This is represented by the cost function $J_i(x_i, \mathrm{avg}(x)) := \frac{1}{2} a_i \|x_i - \bar{x}_i\|^2 + x_i^\top Q_i \,\mathrm{avg}(x)$, where $a_i > 0$ and $Q_i \succeq 0$. In our simulations, we randomly generate the parameters as $N = 5$, $\bar{n} = 5$, $a_i \sim [1,2]$, Q_i to ensure Standing Assumption 2. To ensure feasibility, we set $u_i = \frac{2}{\bar{n}} \mathbf{1}_{\bar{n}}$ and $b = \frac{1}{2} Au$, $b_i = \frac{b}{N}$, $i \in \mathcal{I}$. We assume that $\bar{x}_i = \mathrm{col}(1, 0, \dots, 0)$, $i \in \mathcal{I}$, i.e., each agent has an incentive to complete its task in the first time slot. Figure 4 shows a comparison between Algorithm 3 and 4, assuming single integrator dynamics (5) for the agents, randomly generated initial conditions and communication graph, $c = 25$ to satisfy the condition in Theorem 3 and $\gamma_i = 1$ for all $i \in \mathcal{I}$.

7 Conclusion and outlook

Generalized games played by multi-integrator systems can be solved via continuous-time fully-distributed primal-dual pseudogradient controllers, provided that the game mapping is strongly monotone and Lipschitz continuous. Convergence can be ensured even without

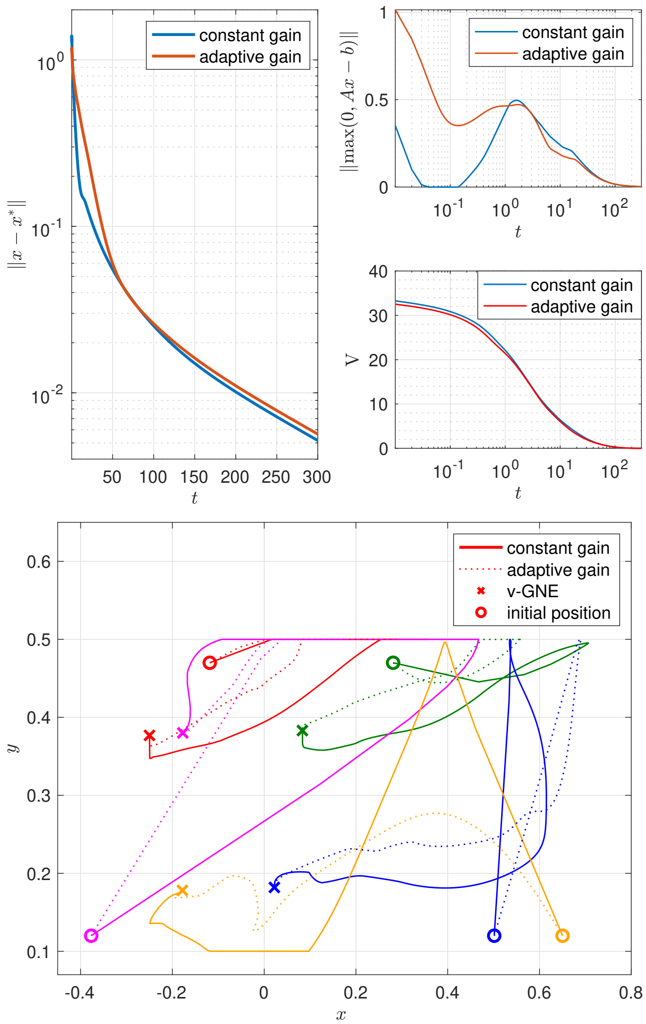

Fig. 1. Results of Algorithms 1- 2 for velocity-actuated robots.

a-priori knowledge on the game parameters, via integral consensus. Seeking an equilibrium when the agents are characterized by constrained dynamics is currently an unexplored problem. The extension of our results to networks of heterogeneous dynamical systems is left as future research.

8 Appendices

Appendix A Proof of Lemma 3

Let us define $\boldsymbol{x} = \operatorname{col}((\boldsymbol{x}^i)_{i\in\mathcal{I}})$, $\boldsymbol{y} = \operatorname{col}((\boldsymbol{y}^i)_{i\in\mathcal{I}})$. By Standing Assumption 2, we have, for all $i \in \mathcal{I}$,

$$\|\nabla_i J_i(\boldsymbol{x}^i) - \nabla_i J_i(\boldsymbol{y}^i)\| \leq \|F(\boldsymbol{x}^i) - F(\boldsymbol{y}^i)\| \leq \theta_0 \|\boldsymbol{x}^i - \boldsymbol{y}^i\|.$$

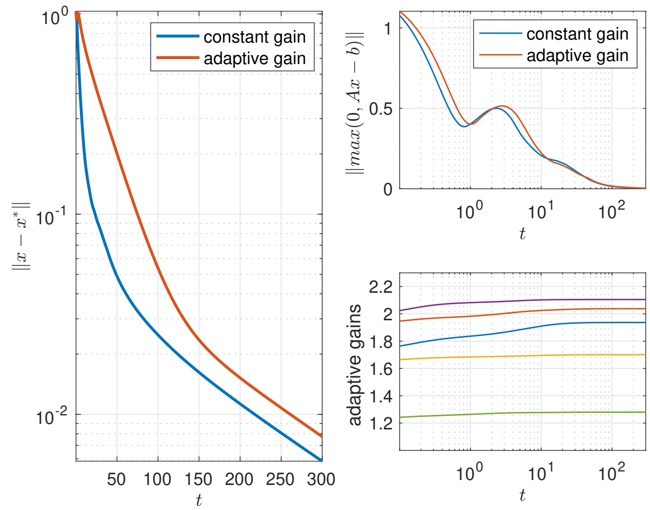

Fig. 2. Results of Algorithm 5 for force-actuated robots.

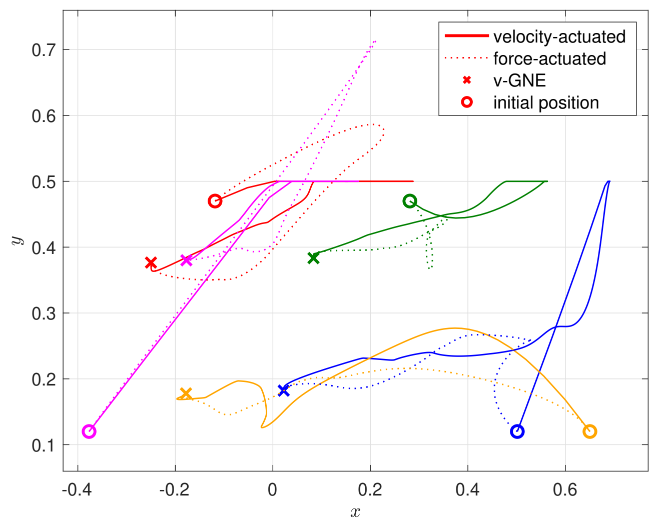

Fig. 3. Velocity- and force-actuated robots trajectories, with adaptive gains.

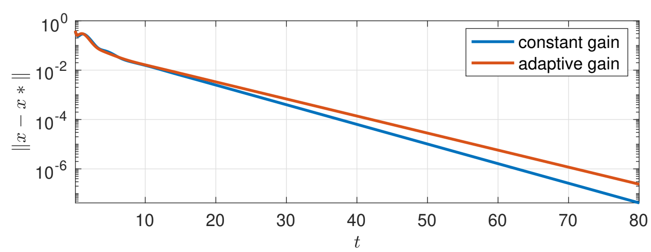

Fig. 4. Results of Algorithms 3-4, for single-integrator agents.

Hence

$$\begin{aligned}\|\boldsymbol{F}(\boldsymbol{x}) - \boldsymbol{F}(\boldsymbol{y})\|^2 &= \textstyle\sum_{i\in\mathcal{I}} \|\nabla_i J_i(\boldsymbol{x}^i) - \nabla_i J_i(\boldsymbol{y}^i)\|^2 \\ &\leq \theta_0^2 \textstyle\sum_{i\in\mathcal{I}} \|\boldsymbol{x}^i - \boldsymbol{y}^i\|^2 = \theta_0^2 \|\boldsymbol{x} - \boldsymbol{y}\|^2.\end{aligned}$$

Then $\theta \geq \mu$ follows by choosing $\mathcal{S}\boldsymbol{x} = \mathcal{S}\boldsymbol{y}$, $\boldsymbol{x} \neq \boldsymbol{y}$. ∎

Appendix B Proof of Lemma 5

We follow the arguments of [23, Th. 1]. We first notice that, under Standing Assumption 3, we have, for any $q > 0$,

$$\operatorname{Range}(L \otimes I_q) = \operatorname{Null}\left(\mathbf{1}_N^\top \otimes I_q\right) = \boldsymbol{E}_q^\perp, \tag{25}$$
$$\operatorname{Null}(L \otimes I_q) = \operatorname{Range}(\mathbf{1}_N \otimes I_q) = \boldsymbol{E}_q. \tag{26}$$

i) For any equilibrium $\operatorname{col}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{k}}, \bar{\boldsymbol{z}}, \bar{\boldsymbol{\lambda}})$ of (12), we have:

$$\begin{aligned}
\mathbf{0}_{Nn} &\in \mathcal{R}^\top \boldsymbol{F}(\bar{\boldsymbol{x}}) + \mathcal{R}^\top \boldsymbol{\Lambda}^\top \bar{\boldsymbol{\lambda}} + \left(L\bar{K}L \otimes I_n\right)\bar{\boldsymbol{x}} \\
&\quad + \mathcal{R}^\top \mathrm{N}_\Omega(\mathcal{R}\bar{\boldsymbol{x}}) && (27)\\
\mathbf{0}_N &= D(\bar{\boldsymbol{\rho}})^\top (\Gamma \otimes I_n)\bar{\boldsymbol{\rho}}, \quad \bar{\boldsymbol{\rho}} = -(L \otimes I_n)\bar{\boldsymbol{x}} && (28)\\
\mathbf{0}_{Nm} &= -\boldsymbol{L}_m \bar{\boldsymbol{\lambda}} && (29)\\
\mathbf{0}_{Nm} &\in -\boldsymbol{\Lambda}\mathcal{R}\bar{\boldsymbol{x}} + \boldsymbol{b} + \boldsymbol{L}_m(\bar{\boldsymbol{\lambda}} + \bar{\boldsymbol{z}}) + \mathrm{N}_{\mathbb{R}^{Nm}_{\geq 0}}\left(\bar{\boldsymbol{\lambda}}\right), && (30)
\end{aligned}$$

where $\bar{K} = \operatorname{diag}\left(\bar{k}_1, \ldots, \bar{k}_N\right)$, and we have used (7). By (28) we have $\bar{\boldsymbol{\rho}} = \mathbf{0}_{Nn}$, i.e., $\bar{\boldsymbol{x}} \in \boldsymbol{E}_n$ by (26), and by (29) and (26), we have $\bar{\boldsymbol{\lambda}} \in \boldsymbol{E}_m$. Therefore $\bar{\boldsymbol{x}} = \mathbf{1}_N \otimes x^*$ and $\bar{\boldsymbol{\lambda}} = \mathbf{1}_N \otimes \lambda^*$, for some $x^* \in \mathbb{R}^n$ and $\lambda^* \in \mathbb{R}^m$. By premultiplying (27) by $(\mathbf{1}_N^\top \otimes I_n)$, by (26) and since $(\mathbf{1}_N^\top \otimes I_n)\mathcal{R}^\top = I_n$, $\boldsymbol{F}(\mathbf{1}_N \otimes x^*) = F(x^*)$, $\mathcal{R}\bar{\boldsymbol{x}} = x^*$, and $\boldsymbol{\Lambda}^\top(\mathbf{1}_N \otimes \lambda^*) = A^\top \lambda^*$, we retrieve the first KKT condition in (3). We obtain the second condition in (3) by premultiplying (30) by $(\mathbf{1}_N^\top \otimes I_m)$ and using that $(\mathbf{1}_N^\top \otimes I_m)\boldsymbol{b} = b$, $(\mathbf{1}_N^\top \otimes I_m)\boldsymbol{L}_m = 0$ by (26) and symmetry of L, $(\mathbf{1}_N^\top \otimes I_m)\boldsymbol{\Lambda} = A$ and $(\mathbf{1}_N^\top \otimes I_m)N_{\mathbb{R}^{Nm}_{\geq 0}}(\mathbf{1}_N \otimes \lambda^*) = N\mathrm{N}_{\mathbb{R}^m_{\geq 0}}(\lambda^*) = \mathrm{N}_{\mathbb{R}^m_{\geq 0}}(\lambda^*)$.
ii) Let (x^*, λ^*) be any pair that satisfies the KKT conditions in (3). By taking $\bar{\boldsymbol{x}} = \mathbf{1}_N \otimes x^*$, $\bar{\boldsymbol{\lambda}} = \mathbf{1}_N \otimes \lambda^*$ and any $\bar{\boldsymbol{k}}$, (27)-(29) are satisfied as above. It suffices to show that there exists $\bar{\boldsymbol{z}}$ such that (30) is satisfied, or equivalently that $(-\boldsymbol{\Lambda}\mathcal{R}\bar{\boldsymbol{x}} + \boldsymbol{b} + \boldsymbol{L}_m\bar{\boldsymbol{\lambda}} + \bar{v}) \in \operatorname{Range}(\boldsymbol{L}_m)$, for some $\bar{v} \in \mathrm{N}_{\mathbb{R}^{Nm}_{\geq 0}}\left(\bar{\boldsymbol{\lambda}}\right)$. By (3), there exists $v^* \in \mathrm{N}_{\mathbb{R}^m_{\geq 0}}(\lambda^*)$ such that $Ax^* - b - v^* = \mathbf{0}_n$. Since $\bar{\boldsymbol{\lambda}} = \mathbf{1}_N \otimes \lambda^*$, $\mathrm{N}_{\mathbb{R}^{Nm}_{\geq 0}}(\bar{\boldsymbol{\lambda}}) = \times_{i \in \mathcal{I}} \mathrm{N}_{\mathbb{R}^m_{\geq 0}}(\lambda^*)$, and it follows by properties of cones that $\operatorname{col}(v_1^*, \ldots, v_N^*) \in \mathrm{N}_{\mathbb{R}^{Nm}_{\geq 0}}(\bar{\boldsymbol{\lambda}})$ with $v_1^* = \cdots = v_N^* = \frac{1}{N}v^*$. Therefore $(\mathbf{1}_N^\top \otimes I_m)\left(-\boldsymbol{\Lambda}\mathcal{R}\bar{\boldsymbol{x}} + \boldsymbol{b} + \boldsymbol{L}_m\bar{\boldsymbol{\lambda}} + \operatorname{col}(v_1^*, \ldots, v_N^*)\right) = b - Ax^* + v^* = \mathbf{0}_m$, and the conclusion follows since $\operatorname{Null}\left(\mathbf{1}_N^\top \otimes I_m\right) = \operatorname{Range}(\boldsymbol{L}_m)$ by (25). ∎

Appendix C Proof of Lemma 6

Let $\boldsymbol{y} = \mathbf{1}_n \otimes y$, for some $y \in \mathbb{R}^n$. We decompose $\boldsymbol{x} = \boldsymbol{x}^\perp + \boldsymbol{x}^\parallel$, where $\boldsymbol{x}^\parallel$ and $\boldsymbol{x}^\perp$ are the projection of $\boldsymbol{x}$ on the consensus subspace $\boldsymbol{E}_n$ and on the orthogonal subspace $\boldsymbol{E}_n^\perp$, respectively. Therefore $\boldsymbol{x}^\parallel = \mathbf{1}_N \otimes \hat{x}$, for some $\hat{x} \in \mathbb{R}^n$. By [23, Eq. 50],

$$\begin{aligned}
(\boldsymbol{x} - \boldsymbol{y})^\top \mathcal{R}^\top(\boldsymbol{F}(\boldsymbol{x}) - \boldsymbol{F}(\boldsymbol{y})) &\geq -\theta\|\hat{x} - y\|\|\boldsymbol{x}^\perp\| \\
&+ \mu\|\hat{x} - y\|^2 - \theta\|\boldsymbol{x}^\perp\|^2 - \theta_0\|\boldsymbol{x}^\perp\|\|\hat{x} - y\|.
\end{aligned}$$

For any $k^* > \underline{k} > 0$, we have $K^* \succ 0$ and, by (26), $\operatorname{Null}(LK^*L \otimes I_n) = \boldsymbol{E}_n$. Therefore it holds that

$$(\boldsymbol{x} - \boldsymbol{x}')^\top (LK^*L \otimes I_n)(\boldsymbol{x} - \boldsymbol{x}') \geq -k^*\lambda_2(L)^2\|\boldsymbol{x}^\perp\|^2.$$

By $\|\hat{x} - x'\| = \frac{1}{\sqrt{N}}\left\|\boldsymbol{x}^\parallel - \boldsymbol{y}\right\|$, we conclude that

$$\begin{aligned}
&(\boldsymbol{x} - \boldsymbol{y})^\top \left(\mathcal{R}^\top (\boldsymbol{F}(\boldsymbol{x}) - \boldsymbol{F}(\boldsymbol{y})) + (LK^*L \otimes I_n)(\boldsymbol{x} - \boldsymbol{y})\right) \\
&\qquad \geq \begin{bmatrix} \|\boldsymbol{x}^\perp\| \\ \|\boldsymbol{x}^\parallel - \boldsymbol{y}\| \end{bmatrix}^\top M_2 \begin{bmatrix} \|\boldsymbol{x}^\perp\| \\ \|\boldsymbol{x}^\parallel - \boldsymbol{y}\| \end{bmatrix},
\end{aligned}$$

with M_2 as in (13) and, for $k^* > \underline{k}$, $M_2 \succ 0$ by Silvester's criterion. The conclusion follows since, by orthogonality, $\|\boldsymbol{x}^\parallel - \boldsymbol{y}\|^2 + \|\boldsymbol{x}^\perp\|^2 = \|\boldsymbol{x} - \boldsymbol{y}\|^2$. ∎

Appendix D Proof of Theorem 2

We first rewrite the dynamics in (12) as

$$\dot{\boldsymbol{\omega}} = \Pi_\Xi(\boldsymbol{\omega}, -\mathcal{B}(\boldsymbol{\omega}) - \Phi\boldsymbol{\omega}), \tag{31}$$

where $\boldsymbol{\omega} := \operatorname{col}(\boldsymbol{x}, \boldsymbol{k}, \boldsymbol{z}, \boldsymbol{\lambda})$,

$$\mathcal{B}(\boldsymbol{\omega}) = \begin{bmatrix} \mathcal{R}^\top \boldsymbol{F}(\boldsymbol{x}) + (LKL \otimes I_n)\boldsymbol{x} \\ -D(\boldsymbol{\rho})^\top(\Gamma \otimes I_n)\boldsymbol{\rho} \\ \mathbf{0}_{Nm} \\ \boldsymbol{L}_m\boldsymbol{\lambda} + \boldsymbol{b} \end{bmatrix}, \Phi = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathcal{R}^\top\boldsymbol{\Lambda}^\top \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\boldsymbol{L}_m \\ -\boldsymbol{\Lambda}\mathcal{R} & \mathbf{0} & \boldsymbol{L}_m & \mathbf{0} \end{bmatrix}.$$

Let us define the quadratic Lyapunov function

$$\begin{aligned}
V &= \tfrac{1}{2}\|\boldsymbol{\omega} - \bar{\boldsymbol{\omega}}\|_Q^2 \\
&= \tfrac{1}{2}(\|\boldsymbol{x} - \bar{\boldsymbol{x}}\|^2 + \|\boldsymbol{k} - \bar{\boldsymbol{k}}\|_{\Gamma^{-1}}^2 + \|\boldsymbol{z} - \bar{\boldsymbol{z}}\|^2 + \|\boldsymbol{\lambda} - \bar{\boldsymbol{\lambda}}\|^2),
\end{aligned}$$

where $Q =: \operatorname{diag}(I_{Nn}, \Gamma^{-1}, I_{Nm}, I_{Nm})$ and $\bar{\boldsymbol{x}} = \mathbf{1}_N \otimes x^*$, $\bar{\boldsymbol{\lambda}} = \mathbf{1}_N \otimes \lambda^*$, where the pair (x^*, λ^*) satisfies the KKT conditions in (3), $\bar{\boldsymbol{k}}$ such that $k^* := \min(\bar{\boldsymbol{k}}) \geq \underline{k}$, with $\underline{k}$ as in (13), $\bar{\boldsymbol{z}}$ chosen such that $\bar{\boldsymbol{\omega}} := \operatorname{col}\left(\bar{\boldsymbol{x}}, \bar{\boldsymbol{k}}, \bar{\boldsymbol{z}}, \bar{\boldsymbol{\lambda}}\right)$ is an equilibrium for (12), and such a $\bar{\boldsymbol{z}}$ exists by the proof of Lemma 5. We notice that $Q\bar{\boldsymbol{\omega}}$ is an equilibrium of (31) if $\bar{\boldsymbol{\omega}}$ is, and $Q\boldsymbol{\omega} \in \Xi$ if $\boldsymbol{\omega} \in \Xi$, because of the specific structure of the matrix Q, of the set Ξ and of the dynamics in (31). Therefore, we can apply Lemma 1 to obtain

$$\begin{aligned}
\dot{V}(\boldsymbol{\omega}) :&= \nabla V(\boldsymbol{\omega})\dot{\boldsymbol{\omega}} = (\boldsymbol{\omega} - \bar{\boldsymbol{\omega}})^\top Q\dot{\boldsymbol{\omega}} = \\
&= (\boldsymbol{\omega} - \bar{\boldsymbol{\omega}})^\top Q\Pi_\Xi(\boldsymbol{\omega}, -\mathcal{B}(\boldsymbol{\omega}) - \Phi\boldsymbol{\omega})
\end{aligned}$$

$$\leq (\boldsymbol{\omega} - \bar{\boldsymbol{\omega}})^\top Q(-\mathcal{B}(\boldsymbol{\omega}) - \Phi\boldsymbol{\omega}). \tag{32}$$

We remark that the inequality above has to be intended to hold point-wise, since we still have to prove the existence of a solution to (31). By Lemma 1, it also holds that $(\boldsymbol{\omega} - \bar{\boldsymbol{\omega}})^\top Q(-\mathcal{B}(\bar{\boldsymbol{\omega}}) - \Phi\bar{\boldsymbol{\omega}}) \leq 0$. By subtracting this term from (32), we obtain

$$\begin{aligned} \dot{V}(\boldsymbol{\omega}) &\leq -(\boldsymbol{\omega} - \bar{\boldsymbol{\omega}})^\top Q \left(\mathcal{B}(\boldsymbol{\omega}) - \mathcal{B}(\bar{\boldsymbol{\omega}}) + \Phi(\boldsymbol{\omega} - \bar{\boldsymbol{\omega}})\right) \\ &= -(\boldsymbol{x} - \bar{\boldsymbol{x}})^\top \mathcal{R}^\top \left(\boldsymbol{F}(\boldsymbol{x}) - \boldsymbol{F}(\bar{\boldsymbol{x}})\right) \\ &\quad - (\boldsymbol{x} - \bar{\boldsymbol{x}})^\top \left(LKL \otimes I_n\right)(\boldsymbol{x} - \bar{\boldsymbol{x}}) \\ &\quad + (\boldsymbol{k} - \bar{\boldsymbol{k}})^\top \Gamma^{-1} D(\boldsymbol{\rho})^\top \left(\Gamma \otimes I_n\right) \boldsymbol{\rho} \\ &\quad - (\boldsymbol{\lambda} - \bar{\boldsymbol{\lambda}})^\top \boldsymbol{L}_m (\boldsymbol{\lambda} - \bar{\boldsymbol{\lambda}}), \end{aligned} \tag{33}$$

where, in the last equality, we used that $Q\Phi = \Phi$, $\Phi^\top = -\Phi$, and $\bar{\boldsymbol{\rho}} := -\boldsymbol{L}_n \bar{\boldsymbol{x}} = 0$. The third addend in (33) can be rewritten as

$$\begin{aligned} &(\boldsymbol{k} - \bar{\boldsymbol{k}})^\top \Gamma^{-1} D(\boldsymbol{\rho})^\top \left(\Gamma \otimes I_n\right) \boldsymbol{\rho} = \textstyle\sum_{i=1}^N \left(k_i - \bar{k}_i\right) \boldsymbol{\rho}^{i\top} \boldsymbol{\rho}^i \\ &= \boldsymbol{\rho}^\top \left(\left(K - \bar{K}\right) \otimes I_n\right) \boldsymbol{\rho} = \boldsymbol{x}^\top \left(L\left(K - \bar{K}\right) L \otimes I_n\right) \boldsymbol{x} \\ &= (\boldsymbol{x} - \bar{\boldsymbol{x}})^\top \left(L\left(K - \bar{K}\right) L \otimes I_n\right)(\boldsymbol{x} - \bar{\boldsymbol{x}}), \end{aligned}$$

where $\bar{K} := \operatorname{diag}(\bar{\boldsymbol{k}})$. Therefore the sum of the second and third term in (33) is $-(\boldsymbol{x} - \bar{\boldsymbol{x}})^\top \left(L\bar{K}L \otimes I_n\right)(\boldsymbol{x} - \bar{\boldsymbol{x}}) \leq -(\boldsymbol{x} - \bar{\boldsymbol{x}})^\top \left(LK^*L \otimes I_n\right)(\boldsymbol{x} - \bar{\boldsymbol{x}})$, where $K^* := k^* I_n$. For the last addend in (33), we can write $(\boldsymbol{\lambda} - \bar{\boldsymbol{\lambda}})^\top \boldsymbol{L}_m (\boldsymbol{\lambda} - \bar{\boldsymbol{\lambda}}) = \boldsymbol{\lambda}^\top \boldsymbol{L}_m \boldsymbol{\lambda}$ by (26) and, by [1, Th. 18.15], $\boldsymbol{\lambda}^\top \boldsymbol{L}_m \boldsymbol{\lambda} \geq \frac{1}{2\lambda_{\max}(L)} \|\boldsymbol{L}_m \boldsymbol{\lambda}\|^2$. By Lemma 6, we finally get

$$\dot{V} \leq -\lambda_{\min}(M_1)\|\boldsymbol{x} - \bar{\boldsymbol{x}}\|^2 - \tfrac{1}{2\lambda_{\max}(L)} \|\boldsymbol{L}_m \boldsymbol{\lambda}\|^2, \tag{34}$$

with $M_2 \succ 0$ as in Lemma 6.

Let $\mathcal{P}$ be any compact sublevel set of V (notice that V is radially unbounded) containing the initial condition $\boldsymbol{\omega}(0) \in \Xi$. $\mathcal{P}$ is invariant for the dynamics, since $\dot{V}(\boldsymbol{\omega}) \leq 0$ by (34). The set $\mathcal{P} \cap \Xi$ is compact and convex, therefore, by continuity, $\mathcal{B} + \Phi$ is Lipschitz continuous on it. We conclude that, for any initial condition, there exists a unique Carathodory solution to (9), that belongs to $\mathcal{P} \cap \Xi$ (and therefore is bounded) for every t [22, Th. 2.5]. Moreover, by [8, Th. 2], the solution converges to the largest invariant set $\mathcal{O}$ contained in the set $\mathcal{Z} := \{\boldsymbol{\omega} \text{ s.t. } \dot{V}(\boldsymbol{\omega}) = 0\}$.

We can already conclude boundedness of the trajectories of (31) and that $\boldsymbol{x}$ converges to the point $\mathbf{1}_N \otimes x^*$, with x^* the unique v-GNE of the game in (2). We will now show convergence of the other variables.

We first characterize the points $\operatorname{col}(\hat{\boldsymbol{x}}, \hat{\boldsymbol{k}}, \hat{\boldsymbol{z}}, \hat{\boldsymbol{\lambda}}) \in \mathcal{Z}$, for which the quantities in (32)-(34) must be zero. By (34), $\hat{\boldsymbol{x}} = \bar{\boldsymbol{x}} = \mathbf{1}_N \otimes x^*$, and $\hat{\boldsymbol{\lambda}} \in \boldsymbol{E}_m$, i.e. $\hat{\boldsymbol{\lambda}} = \mathbf{1}_N \otimes \hat{\lambda}$, for some $\hat{\lambda} \in \mathbb{R}^m_{\geq 0}$. Also, by expanding (32), and, by using that $\hat{\boldsymbol{x}} = \bar{\boldsymbol{x}}$, $\hat{\boldsymbol{\rho}} := -\boldsymbol{L}_n \hat{\boldsymbol{x}}$, and by (26), we have

$$\begin{aligned} 0 &= (\hat{\boldsymbol{\lambda}} - \bar{\boldsymbol{\lambda}})^\top (\boldsymbol{\Lambda}\mathcal{R}\bar{\boldsymbol{x}} - \boldsymbol{b} - \boldsymbol{L}_m \hat{\boldsymbol{z}}) = (\hat{\lambda} - \lambda^*)^\top (Ax^* - b) \\ &= \hat{\lambda}^\top (Ax^* - b) = \hat{\boldsymbol{\lambda}}^\top (\boldsymbol{\Lambda}\mathcal{R}\bar{\boldsymbol{x}} - \boldsymbol{b} - \boldsymbol{L}_m(\hat{\boldsymbol{\lambda}} + \hat{\boldsymbol{z}})), \end{aligned} \tag{35}$$

where in the second equality we have again used (26) and the third equality follows from the KKT conditions in (3). This concludes the characterization of the set $\mathcal{Z}$.

Then, let $\underline{\boldsymbol{\omega}}(t) = \operatorname{col}(\underline{\boldsymbol{x}}(t), \underline{\boldsymbol{k}}(t), \underline{\boldsymbol{z}}(t), \underline{\boldsymbol{\lambda}}(t))$ be the trajectory starting at any $\operatorname{col}(\underline{\boldsymbol{x}}, \underline{\boldsymbol{k}}, \underline{\boldsymbol{z}}, \underline{\boldsymbol{\lambda}}) \in \mathcal{O}$. By invariance of $\mathcal{O}$, $\underline{\boldsymbol{\omega}}(t)$ must lie in $\mathcal{Z} \supseteq \mathcal{O}$ for all $t \geq 0$. Hence, $\underline{\boldsymbol{x}}(t) \equiv \bar{\boldsymbol{x}}$. Therefore, by (12b), $\dot{\underline{\boldsymbol{k}}}(t) = 0$ for all t, or $\underline{\boldsymbol{k}}(t) \equiv \underline{\boldsymbol{k}}$. Since $\underline{\boldsymbol{\lambda}}(t) \in \boldsymbol{E}_m$ for all t, $\dot{\underline{\boldsymbol{z}}}(t) = 0$, for all t, by (12c), or $\underline{\boldsymbol{z}}(t) \equiv \underline{\boldsymbol{z}}$. Hence the quantity $v := (\boldsymbol{\Lambda}\mathcal{R}\underline{\boldsymbol{x}}(t) - \boldsymbol{b} - \boldsymbol{L}_m \underline{\boldsymbol{\lambda}}(t) - \boldsymbol{L}_m \underline{\boldsymbol{z}}(t))$ is a constant along the trajectory $\underline{\boldsymbol{\omega}}$. Suppose by contradiction that $v_k > 0$, where v_k denotes the k-th component of v. Then, by (12d) and properties of the normal cone, $\dot{\underline{\boldsymbol{\lambda}}}(t)_k = v_k$ for all t, and $\underline{\boldsymbol{\lambda}}(t)$ grows indefinitely. Since all the solutions of (12) are bounded, this is a contradiction. Therefore, $v \leq 0$, and $\underline{\boldsymbol{\lambda}}(t)^\top v = 0$ by (35). Equivalently, $v \in N_{\mathbb{R}^{Nm}_{\geq 0}}(\underline{\boldsymbol{\lambda}}(t))$, hence $\dot{\underline{\boldsymbol{\lambda}}}(t) = 0$, for all t. We conclude that all the points in the set $\mathcal{O}$ are equilibria.

The set $\Lambda(\boldsymbol{\omega}_0)$ of ω-limit points [1] of the solution to (12) starting from any $\boldsymbol{\omega}_0 \in \Xi$ is nonempty (by Bolzano-Weierstrass theorem, since all the trajectories of (12) are bounded) and invariant (as in proof of [8, Lemma 5]). By $\dot{V} \leq 0$ it follows that V must be constant on $\Lambda(\boldsymbol{\omega}_0)$, hence $\Lambda(\boldsymbol{\omega}_0) \subseteq \mathcal{Z}$ (see proof of [8, Th.2]). Also $\Lambda(\boldsymbol{\omega}_0)$ is invariant, so $\Lambda(\boldsymbol{\omega}_0) \subseteq \mathcal{O}$.

We want now to show that, for any for any $\boldsymbol{\omega}_0 \in \Xi$, $\Lambda(\boldsymbol{\omega}_0)$ is a singleton. If $\Lambda(\boldsymbol{\omega}_0)$ is a singleton, the solution converges to that point [1, Lemma 1.14], which is an equilibrium, since $\Lambda(\boldsymbol{\omega}_0) \subseteq \mathcal{O}$. For the sake of contradiction, we assume that there are two ω-limit points $\boldsymbol{\omega}_1 = \operatorname{col}(\bar{\boldsymbol{x}}, \underline{\boldsymbol{k}}, \boldsymbol{z}_1, \boldsymbol{\lambda}_1) \in \Lambda(\boldsymbol{\omega}_0)$, $\boldsymbol{\omega}_2 = \operatorname{col}(\bar{\boldsymbol{x}}_2, \underline{\boldsymbol{k}}, \boldsymbol{z}_2, \boldsymbol{\lambda}_2) \in \Lambda(\boldsymbol{\omega}_0)$, with $\boldsymbol{\omega}_1 \neq \boldsymbol{\omega}_2$. We note that $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ must have the same vector of adaptive gains $\underline{\boldsymbol{k}}$ by definition of ω-limit point, since the k_i in Algorithm 2 are nonincreasing. Let us define $\boldsymbol{\omega}_3 = \operatorname{col}(\bar{\boldsymbol{x}}, \underline{\boldsymbol{k}} + \mathbf{1}\alpha, \boldsymbol{z}_1, \boldsymbol{\lambda}_1)$, $\alpha \in \mathbb{R}$ chosen such that $\min(\underline{\boldsymbol{k}} + \mathbf{1}\alpha) > \underline{k}$, $\underline{k}$ as in (13). By (34), the quantity $\|\boldsymbol{\omega}(t) - \boldsymbol{\omega}_3\|_Q$ is nonincreasing along any trajectory $\boldsymbol{\omega}(t)$ of (31). Therefore, by definition of ω-limit point, it must hold that $\|\boldsymbol{\omega}_1 - \boldsymbol{\omega}_3\|_Q = \|\boldsymbol{\omega}_2 - \boldsymbol{\omega}_3\|_Q$, or $\|\operatorname{col}(\mathbf{0}_{Nn}, \alpha\mathbf{1}_N, \mathbf{0}_{Nm}, \mathbf{0}_{Nm})\|_Q = \|\operatorname{col}(\mathbf{0}_{Nn}, \alpha\mathbf{1}_N, \boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2, \boldsymbol{z}_2 - \boldsymbol{z}_1)\|_Q$. Equivalently, $\boldsymbol{\omega}_1 = \boldsymbol{\omega}_2$, that is a contradiction. ■

[1] $z : [0, \infty) \to \mathbb{R}^n$ has an ω-limit point at $\bar{z}$ if there exists a nonnegative diverging sequence $\{t_k\}_{k \in \mathbb{N}}$ such that $z(t_k) \to \bar{z}$

Appendix E Proof of Theorem 1

The proof is analogous to the proof of Theorem 2 and hence it is only sketched here. Consider the Lyapunov function $V = \frac{1}{2}\|\boldsymbol{\omega} - \bar{\boldsymbol{\omega}}\|^2$, where $\boldsymbol{\omega} = \mathrm{col}(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{\lambda})$, and $\bar{\boldsymbol{\omega}} = \mathrm{col}\left(\bar{\boldsymbol{x}}, \bar{\boldsymbol{z}}, \bar{\boldsymbol{\lambda}}\right)$ is any equilibrium point of (9). By proceeding like in the proof of Theorem 2 and by defining

$$\Phi = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathcal{R}^\top \boldsymbol{\Lambda}^\top \\ \mathbf{0} & \mathbf{0} & -\boldsymbol{L}_m \\ -\boldsymbol{\Lambda}\mathcal{R} & \boldsymbol{L}_m & \mathbf{0} \end{bmatrix}, \ \mathcal{B}(\boldsymbol{\omega}) = \begin{bmatrix} \mathcal{R}^\top \boldsymbol{F}(\boldsymbol{x}) + c\boldsymbol{L}_n \boldsymbol{x} \\ \mathbf{0}_{Nm} \\ \boldsymbol{L}_m \boldsymbol{\lambda} + \boldsymbol{b} \end{bmatrix},$$

we leverage Lemma 4 to obtain

$$\dot{V}(\boldsymbol{\omega}) \leq -\lambda_{\min}(M_1)\|\boldsymbol{x} - \bar{\boldsymbol{x}}\|^2 - \tfrac{1}{2\lambda_{\max}(L)}\|\boldsymbol{L}_m\boldsymbol{\lambda}\|^2 \quad (36)$$

with M_1 as in Lemma 4. As in Theorem 2, we can show the convergence to the set $\mathcal{O}$ of equilibria and that the ω-limit set $\Lambda(\boldsymbol{\omega}_0) \subseteq \mathcal{O}$ and it is nonempty, for any $\boldsymbol{\omega}_0$. Finally, since the distance to any equilibrium point along any trajectory of (9) is non-increasing by (36), and since the solution of (12) has an ω-limit point at an equilibrium, it follows that the solution converges to that equilibrium. The last claim follows by Lemma 2. ∎

Appendix F Proof of Lemma 9

By Standing Assumption 2 and Lemma 7, we have

$$\begin{aligned} &(x - x')^\top(\tilde{\boldsymbol{F}}(x, \boldsymbol{\sigma}) - \tilde{\boldsymbol{F}}(x', \mathbf{1}_N \otimes \mathrm{avg}(x'))) \\ &= (x - x')^\top\left(\tilde{\boldsymbol{F}}(x, \boldsymbol{\sigma}) - \tilde{\boldsymbol{F}}(x, \mathbf{1}_N \otimes \mathrm{avg}(x))\right. \\ &\quad \left.\tilde{\boldsymbol{F}}(x, \mathbf{1}_N \otimes \mathrm{avg}(x) - \tilde{\boldsymbol{F}}(x', \mathbf{1}_N \otimes \mathrm{avg}(x')\right) \\ &\geq \mu\|x - x'\|^2 - \tilde{\theta}_\sigma\|x - x'\|\|\boldsymbol{\sigma} - \mathbf{1}_N \otimes \mathrm{avg}(x)\|. \end{aligned}$$

Moreover, we note that $(\boldsymbol{\sigma} - \mathbf{1}_N \otimes \mathrm{avg}(x)) \in \boldsymbol{E}_{\bar{n}}^\perp$, since $\mathrm{avg}(\boldsymbol{\sigma}) = \mathrm{avg}(x)$, and $\boldsymbol{\sigma'} \in \boldsymbol{E}_{\bar{n}}$. Hence, by (26), we have

$$\begin{aligned} &(\boldsymbol{\sigma} - \boldsymbol{\sigma'})^\top(LKL \otimes I_{\bar{n}})(\boldsymbol{\sigma} - \boldsymbol{\sigma'}) \\ &\quad \geq k^*\lambda_2(L)^2\|\boldsymbol{\sigma} - \mathbf{1}_N \otimes \mathrm{avg}(x)\|^2, \end{aligned}$$

and the conclusion follows readily. ∎

Appendix G Proof of Theorem 4

Let $\boldsymbol{\omega} = \mathrm{col}(x, \boldsymbol{\varsigma}, \boldsymbol{k}, \boldsymbol{z}, \boldsymbol{\lambda})$, $\Psi := \{\boldsymbol{\omega} \in \Xi \mid \mathrm{avg}(\boldsymbol{\varsigma}) = \mathbf{0}_{\bar{n}}\}$. First, we show that the set Ψ is invariant for the system in (17). It is enough to note that, for all $\boldsymbol{\omega} \in \Xi$,

$$\nabla_{\boldsymbol{\omega}}(\mathrm{avg}(\boldsymbol{\varsigma}))^\top \dot{\boldsymbol{\omega}} = -\frac{1}{N}(\mathbf{1}_N^\top \otimes I_{\bar{n}})(LKL \otimes I_{\bar{n}})\boldsymbol{\sigma} = \mathbf{0}_{\bar{n}}.$$

Next, analogously to the proof of Lemma 5, it can be shown that any equilibrium point $\bar{\boldsymbol{\omega}} := \mathrm{col}\left(\bar{x}, \bar{\boldsymbol{\varsigma}}, \bar{\boldsymbol{k}}, \bar{\boldsymbol{z}}, \bar{\boldsymbol{\lambda}}\right) \in \Psi$ of (17) is such that $\bar{\boldsymbol{\lambda}} = \mathbf{1}_N \otimes \lambda^*$, the pair $(\bar{x}, \lambda^*)$ satisfies the KKT conditions in (3), and $\bar{\boldsymbol{\sigma}} := \bar{x} + \bar{\boldsymbol{\varsigma}} = \mathbf{1} \otimes \mathrm{avg}(\bar{x})$. Moreover, for any pair (x^*, λ^*) satisfying the KKT conditions in (3), there exists $\bar{\boldsymbol{z}} \in \mathbb{R}^{mN}$ such that $\mathrm{col}(x^*, \mathbf{1}_N \otimes \mathrm{avg}(x^*) - x^*, \boldsymbol{k}, \bar{\boldsymbol{z}}, \mathbf{1}_N \otimes \lambda^*) \in \Psi$ is an equilibrium for (17), for any $\boldsymbol{k} \in \mathbb{R}^N$. Therefore, the set of equilibria in Ψ is nonempty. The proof is omitted because of space limitations.

Let $\bar{\boldsymbol{\omega}} = \mathrm{col}\left(\bar{x}, \bar{\boldsymbol{\varsigma}}, \bar{\boldsymbol{k}}, \bar{\boldsymbol{z}}, \bar{\boldsymbol{\lambda}}\right) \in \Psi$ be an equilibrium of (17) such that $k^* = \min(\bar{\boldsymbol{k}}) > \underline{k}$, $\underline{k}$ as in (19), and consider the quadratic Lyapunov function $V = \frac{1}{2}\|\boldsymbol{\omega} - \bar{\boldsymbol{\omega}}\|_Q^2$, where $Q = \mathrm{diag}(I_n, I_n, \Gamma^{-1}, I_{Nm}, I_{Nm})$. By proceeding like in the proof of Theorem 2 and by defining

$$\mathcal{B}(\boldsymbol{\omega}) = \begin{bmatrix} \tilde{\boldsymbol{F}}(x, \boldsymbol{\sigma}) + (LKL \otimes I_{\bar{n}})\boldsymbol{\sigma} \\ (LKL \otimes I_{\bar{n}})\boldsymbol{\sigma} \\ -D(\boldsymbol{\rho})^\top(\Gamma \otimes I_{\bar{n}})\boldsymbol{\rho} \\ \mathbf{0}_{Nm} \\ \boldsymbol{L}_m\boldsymbol{\lambda} + \boldsymbol{b} \end{bmatrix}, \ \Phi = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{\Lambda}^\top \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\boldsymbol{L}_m \\ -\boldsymbol{\Lambda} & 0 & 0 & \boldsymbol{L}_m & \mathbf{0} \end{bmatrix},$$

we obtain, by Lemma 9, that for all $\boldsymbol{\omega} \in \Psi$,

$$\begin{aligned} \dot{V}(\boldsymbol{\omega}) \leq &-\lambda_{\min}(M_4)(\|x - \bar{x}\|^2 + \|\boldsymbol{\sigma} - \mathbf{1}_N \otimes x\|^2) \\ &- \tfrac{1}{2\lambda_{\max}(L)}\|\boldsymbol{L}_m\boldsymbol{\lambda}\|^2, \end{aligned}$$

with $M_4 \succ 0$ as in (19).

Let $\mathcal{P} = \bar{\mathcal{P}} \cap \Psi$, where $\bar{\mathcal{P}}$ is any compact sublevel set of V containing the initial condition. Since $\boldsymbol{\varsigma}(0) = \mathbf{0}_n$, we have that $\boldsymbol{\omega}(0) \in \mathcal{P}$. $\mathcal{P}$ is compact, convex and invariant. Then, the existence of a unique solution of the system in (17) and convergence to an equilibrium point in $\mathcal{P}$ (hence, in Ψ) follows as for Theorem 2. ∎

Appendix H Proof of Theorem 3

The proof follows as for Theorem 4, by defining $V = \frac{1}{2}\|\boldsymbol{\omega} - \bar{\boldsymbol{\omega}}\|^2$, where $\bar{\boldsymbol{\omega}} := \mathrm{col}\left(\bar{x}, \bar{\boldsymbol{\varsigma}}, \bar{\boldsymbol{z}}, \bar{\boldsymbol{\lambda}}\right) \in \Psi$ is any equilibrium of (16) and it can be shown that such an equilibrium exists,

$$\mathcal{B}(\boldsymbol{\omega}) := \begin{bmatrix} \tilde{\boldsymbol{F}}(x, \boldsymbol{\sigma}) + c\boldsymbol{L}_{\bar{n}}\boldsymbol{\sigma} \\ c\boldsymbol{L}_{\bar{n}}\boldsymbol{\sigma} \\ \mathbf{0}_{Nm} \\ \boldsymbol{L}_m\boldsymbol{\lambda} + \boldsymbol{b} \end{bmatrix}, \ \Phi := \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{\Lambda}^\top \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\boldsymbol{L}_m \\ -\boldsymbol{\Lambda} & \mathbf{0} & \boldsymbol{L}_m & \mathbf{0} \end{bmatrix},$$

and by exploiting Lemma 8.

Appendix I Proof of Theorem 5

By applying the transformation $x \mapsto \mathcal{R}\boldsymbol{\zeta} = x + Bv$ to the system in (24), we obtain:

$$\dot{v} = \tilde{E}v - G(\boldsymbol{F}(\boldsymbol{\zeta}) + \boldsymbol{\Lambda}^\top\boldsymbol{\lambda} + \mathcal{R}\left(LKL \otimes I_n\right)\boldsymbol{\zeta}) \quad \text{(37a)}$$

$$
\begin{aligned}
\dot{\boldsymbol{\zeta}} &= -\mathcal{R}^\top (\boldsymbol{F}(\boldsymbol{\zeta}) + \boldsymbol{\Lambda}^\top \boldsymbol{\lambda} + \mathcal{R}\left(LKL \otimes I_n\right) \boldsymbol{\zeta}) \\
&\quad - \mathcal{S}^\top \mathcal{S}\left(LKL \otimes I_n\right) \boldsymbol{\zeta} && \text{(37b)} \\
\dot{\boldsymbol{k}} &= D\left(\boldsymbol{\rho}\right)^\top \left(\Gamma \otimes I_n\right) \boldsymbol{\rho}, \qquad \boldsymbol{\rho} = -(L \otimes I_n)\boldsymbol{\zeta} && \text{(37c)} \\
\dot{\boldsymbol{z}} &= \boldsymbol{L}_m \boldsymbol{\lambda} && \text{(37d)} \\
\dot{\boldsymbol{\lambda}} &= \Pi_{\mathbb{R}^{Nm}_{\geq 0}} \left(\boldsymbol{\lambda}, (\boldsymbol{\Lambda}\mathcal{R}\boldsymbol{\zeta} - \boldsymbol{b} - \boldsymbol{L}_m\boldsymbol{\lambda} - \boldsymbol{L}_m\boldsymbol{z})\right). && \text{(37e)}
\end{aligned}
$$

The system (37) is in cascade form for (37a) with respect to (37b)-(37e). We note also that, under Assumption 1, the subsystem (37b)-(37e) is exactly (9). Hence, , by Theorem 2, there exists a unique solution to (37b)-(37e), that is bounded and converges to an equilibrium point col $\left(\bar{\boldsymbol{\zeta}}, \bar{\boldsymbol{k}}, \bar{z}, \bar{\boldsymbol{\lambda}}\right)$, with $\bar{\boldsymbol{\zeta}} = \mathbf{1}_N \otimes x^*$, $\bar{\boldsymbol{\lambda}} = \mathbf{1}_N \otimes \lambda^*$, where the pair (x^*, λ^*) satisfies KKT conditions in (3). On the other hand, the dynamics in (37a) are ISS with respect to the input $\tilde{u} := -(\boldsymbol{F}(\boldsymbol{\zeta}) + \boldsymbol{\Lambda}^\top \boldsymbol{\lambda} + \mathcal{R}\left(LKL \otimes I_n\right) \boldsymbol{\zeta})$ [19, Lemma 4.6], and this input is bounded, by boundedness of the trajectory $(\boldsymbol{\zeta}, \boldsymbol{k}, \boldsymbol{z}, \boldsymbol{\lambda})$ and Lipschitz continuity in Lemma 3. Moreover, since $\bar{\boldsymbol{\zeta}} = \mathbf{1}_N \otimes x^*$, $\bar{\boldsymbol{\lambda}} = \mathbf{1}_N \otimes \lambda^*$, by the KKT conditions in (3) and by continuity, we have $\tilde{u} \to \mathbf{0}_n$ for $t \to \infty$. Therefore, $v(t) \to \mathbf{0}_r$ for $t \to \infty$ (this follows from definition of ISS, see [19, Ex. 4.58]). By definition of $\zeta_i = \mathcal{R}_i \boldsymbol{\zeta}^i$ in (22), it also follows that $x \to x^*$. ∎

References

[1] H. H. Bauschke and P. L. Combettes. *Convex analysis and monotone operator theory in Hilbert spaces*, volume 2011. Springer, 2017.

[2] G. Belgioioso and S. Grammatico. Semi-decentralized Nash equilibrium seeking in aggregative games with separable coupling constraints and non-differentiable cost functions. *IEEE Control Systems Letters*, 1(2):400–405, 2017.

[3] G. Belgioioso and S. Grammatico. A Douglas-Rachford splitting for semi-decentralized equilibrium seeking in generalized aggregative games. In *2018 IEEE Conference on Decision and Control (CDC)*, pages 3541–3546, 2018.

[4] G. Belgioioso and S. Grammatico. Projected-gradient algorithms for generalized equilibrium seeking in aggregative games are preconditioned forward-backward methods. *2018 European Control Conference (ECC)*, pages 2188–2193, 2018.

[5] M. Bianchi and S. Grammatico. A continuous-time distributed generalized Nash equilibrium seeking algorithm over networks for double-integrator agents. *arXiv preprint arXiv:1910.11608*, 2019, (submitted to the European Control Conference 2020).

[6] E. Brgens and C. Kanzow. A distributed regularized jacobi-type admm-method for generalized Nash equilibrium problems in Hilbert spaces. *Numerical Functional Analysis and Optimization*, 39(12):1316–1349, 2018.

[7] C. De Persis and S. Grammatico. Continuous-time integral dynamics for aggregative game equilibrium seeking. *2018 European Control Conference (ECC)*, pages 1042–1047, 2018.

[8] C. De Persis and S. Grammatico. Distributed averaging integral Nash equilibrium seeking on networks. *Automatica, DOI: 10.1016/j.automatica.2019.108548*, 2019.

[9] C. De Persis and N. Monshizadeh. A feedback control algorithm to steer networks to a Cournot-Nash equilibrium. *IEEE Transactions on Control of Network Systems, DOI: 10.1109/TCNS.2019.2897907*, 2019.

[10] Z. Deng and S. Liang. Distributed algorithms for aggregative games of multiple heterogeneous EulerLagrange systems. *Automatica*, 99:246 – 252, 2019.

[11] Z. Deng and X. Nian. Distributed generalized Nash equilibrium seeking algorithm design for aggregative games over weight-balanced digraphs. *IEEE Transactions on Neural Networks and Learning Systems*, 30(3):695–706, 2019.

[12] F. Facchinei and C. Kanzow. Generalized Nash equilibrium problems. *Annals of Operations Research*, 175(1):177–211, 2010.

[13] F. Facchinei and J. Pang. *Finite-dimensional variational inequalities and complementarity problems*. Springer Science & Business Media, 2007.

[14] F. Facchinei and J. Pang. Nash equilibria: the variational approach. In Daniel P. Palomar and Yonina C.Editors Eldar, editors, *Convex Optimization in Signal Processing and Communications*, page 443493. Cambridge University Press, 2009.

[15] P. Frihauf, M. Krstic, and T. Basar. Nash equilibrium seeking in noncooperative games. *IEEE Transactions on Automatic Control*, 57(5):1192–1207, 2012.

[16] D. Gadjov and L. Pavel. Single-timescale distributed GNE seeking for aggregative games over networks via forward-backward operator splitting. *arXiv preprint arXiv:1908.00107*.

[17] D. Gadjov and L. Pavel. A passivity-based approach to Nash equilibrium seeking over networks. *IEEE Transactions on Automatic Control*, 64(3):1077–1092, 2019.

[18] S. Grammatico. Dynamic control of agents playing aggregative games with coupling constraints. *IEEE Transactions on Automatic Control*, 62(9):4537–4548, 2017.

[19] H. K. Khalil. *Nonlinear Systems*. Pearson Education. Prentice Hall, 2002.

[20] J. Koshal, A. Nedic, and U. V. Shanbhag. Distributed algorithms for aggregative games on graphs. *Operations Research*, 64:680–704, 2016.

[21] W. Lin, Z. Qu, and M. A. Simaan. Distributed game strategy design with application to multi-agent formation control. In *53rd IEEE Conference on Decision and Control*, pages 433–438, 2014.

[22] A. Nagurney and D. Zhang. *Projected Dynamical Systems and Variational Inequalities with Applications*. International Series in Operations Research & Management Science. Springer, 2012.

[23] L. Pavel. Distributed GNE seeking under partial-decision information over networks via a doubly-augmented operator splitting approach. *IEEE Transactions on Automatic Control, DOI: 10.1109/TAC.2019.2922953*, 2019.

[24] A. R. Romano and L. Pavel. Dynamic Nash equilibrium seeking for higher-order integrators in networks. In *2019 18th European Control Conference (ECC)*, pages 1029–1035, 2019.

[25] W. Saad, Z. Han, H. V. Poor, and T. Basar. Game-theoretic methods for the smart grid: An overview of microgrid systems, demand-side management, and smart grid communications. *IEEE Signal Processing Magazine*, 29(5):86–105, 2012.

[26] F. Salehisadaghiani, W. Shi, and L. Pavel. Distributed Nash equilibrium seeking under partial-decision information via the alternating direction method of multipliers. *Automatica*, 103:27 – 35, 2019.

[27] W. Shi and L. Pavel. Lana: An ADMM-like Nash equilibrium seeking algorithm in decentralized environment. In *2017 American Control Conference (ACC)*, pages 285–290, May 2017.

[28] M. S. Stankovic, K. H. Johansson, and D. M. Stipanovic. Distributed seeking of Nash equilibria with applications to mobile sensor networks. *IEEE Transactions on Automatic Control*, 57(4):904–919, 2012.

[29] T. Tatarenko, W. Shi, and A. Nedi. Accelerated gradient play algorithm for distributed Nash equilibrium seeking. In *2018 IEEE Conference on Decision and Control (CDC)*, pages 3561–3566, 2018.

[30] M. Ye and G. Hu. Distributed Nash equilibrium seeking by a consensus based approach. *IEEE Transactions on Automatic Control*, 62(9):4811–4818, 2017.

[31] P. Yi and L. Pavel. Distributed generalized Nash equilibria computation of monotone games via double-layer preconditioned proximal-point algorithms. *IEEE Transactions on Control of Network Systems*, 6(1):299–311, 2019.

[32] P. Yi and L. Pavel. An operator splitting approach for distributed generalized Nash equilibria computation. *Automatica*, 102:111 – 121, 2019.

[33] C. Yu, M. van der Schaar, and A. H. Sayed. Distributed learning for stochastic generalized Nash equilibrium problems. *IEEE Transactions on Signal Processing*, 65(15):3893–3908, 2017.

[34] Y. Zhang, S. Liang, X. Wang, and H. Ji. Distributed Nash equilibrium seeking for aggregative games with nonlinear dynamics under external disturbances. *IEEE Transactions on Cybernetics*, pages 1–10, 2019.