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Switching Surfaces in N-Person Differential Games[1]

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Abstract. Switching surfaces in N-person differential games are essentially similar to those encountered in optimal control and two-person, zero-sum differential games. The differences between the Nash noncooperative solution and the saddle-point solution are reflected in the dispersal surfaces. These are discussed through classification and construction procedures for switching surfaces. A simple example of a two-person, nonzero-sum game is considered. A complete solution of this game will be presented in a companion paper (Ref. 1).

1. Introduction

Differential games with more than two players have been receiving attention lately (Refs. 2–6). In an important class of these problems, in which the players are permitted to use discontinuous strategies, the discontinuities in optimal strategies of the players lie on certain surfaces in the *playing space* of the game. Furthermore, the solution of the game might often exhibit surfaces containing singular and abnormal solutions. All such surfaces are termed here as the switching surfaces, and this paper is devoted to their study.

Since the strategy of any player with perfect information is a feedback control law, the solution of the game requires (loosely speaking) the mandatory solution of the synthesis problem for all those players endowed with perfect information. Perhaps, one could also solve the synthesis problem for the remaining players,[4] for convenience in repre-

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[4] The misleading term *optimal open-loop feedback control* is used by some authors for this solution for players with no observations.

senting their control laws and the optimal paths in the playing space of the game. The construction of switching surfaces holds in either case with suitable interpretation.

The solution of the game between the switching surfaces is obtained in a routine fashion by integrating the canonical equations obtained by the appropriate necessary conditions (Refs. 2–6). This is referred to as the *solution in the small* (Ref. 1). In contrast to this, the construction of the switching surfaces itself constitutes the *solution in the large*. This latter aspect is on an uneasy terrain in the literature and mostly is example oriented.

In Section 2, we formulate the problem and discuss the solution aspects, restricting our attention to the noncooperative situation. This is followed in Section 3 with a general classification and construction of the switching surfaces in optimal control and differential game problems. In Section 4, we give simple examples illustrating the method of construction of the switching surfaces.

2. Formulation of *N*-Person Differential Games

The state of an N-person differential game, x of dimension n, satisfies the vector differential equation

$$\dot{x} = f(x, \mathbf{u}, t), \tag{1}$$

where

$$\mathbf{u} = (u^1, \ldots, u^p, \ldots, u^N). \tag{2}$$

In (2), u^p is the r^p-dimensional control action vector of the pth player. The region $\mathscr{R}$ of interest in the state–time space in which the game evolves is known as the playing space.

The state of the game is to be transferred by the control actions of the players from an initial state $x(t_0) = x_0$ to a final state contained in a terminal surface of dimension n given by

$$x_f = X(\sigma), \qquad t_f = T(\sigma), \tag{3}$$

where σ ranges over an n-dimensional cube. The terminal surface forms a part of the boundary of $\mathscr{R}$.

The pth player chooses his control action $u^p \in \Omega^p$ so as to minimize his payoff functional

$$J^p[x_0, t_0, \mathbf{u}] = \phi^p(x_f, t_f) + \int_{t_0}^{t_f} L^p(x, \mathbf{u}, t)\, dt. \tag{4}$$

The strategy of any player is a function of his own information of the state of the game. No notational difference will be made between the control action of a player and his strategy.

The functions f and L^p are assumed to belong to the class $C^{(1)}$. The functions T and X in (3) are assumed piecewise $C^{(1)}$, and ϕ^p is of class $C^{(1)}$ on each smooth section of the terminal surface. These are some of the usual assumptions made in the case of optimal control problems as well.

The noncooperative solution,[5] in which unrestricted communication and binding agreements are not allowed between the players, is given in terms of the equilibrium and minmax points of the game (Ref. 19). The game formulated in (1)–(4) with the initial condition $x(t_0) = x_0$ has an equilibrium point, if a joint strategy $\mathbf{u}^* = (u^{1*},..., u^{p*},..., u^{N*})$ exists such that, for $p = 1, 2,..., N$, we have

$$J^p[x_0, t_0, (\mathbf{u}^*; u^p)] \geqslant J^p[x_0, t_0, \mathbf{u}^*], \tag{5}$$

where

$$(\mathbf{u}^*; u^p) = (u^{1*},..., u^{p-1*}, u^p, u^{p+1*},..., u^{N*}). \tag{6}$$

Thus, no player can unilaterally deviate from such a strategy to improve his payoff. If, for any (x_0, t_0), there exists a unique equilibrium strategy, then it is the noncooperative solution. If not, additional concepts are introduced to define the solution.

Two equilibrium strategies $\mathbf{u}^*$ and $\hat{\mathbf{u}}$ are equivalent if, for $p = 1, 2,..., N$, we have

$$J^p[x_0, t_0, \mathbf{u}^*] = J^p[x_0, t_0, \hat{\mathbf{u}}]. \tag{7}$$

Two equilibrium strategies $\mathbf{u}^*$ and $\hat{\mathbf{u}}$ are said to be interchangeable if any recombined strategy $\tilde{\mathbf{u}}$ (every $\tilde{\mathbf{u}}^p$ is either u^{p*} or $\hat{u}^p$) is also an equilibrium strategy. If all the equilibrium strategies of a game are interchangeable, then they constitute the Nash noncooperative solution.[6] The above concepts are readily applicable to the null-information case, in which the players have no information of the state of the game, except the initial condition.

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[5] The solution concepts to be discussed are strictly applicable for the normal form of the game. The construction of the maximum nonvoid class of playable strategies representing the normal form for a perfect information differential game is given in Ref. 6.

[6] If not, the solution is defined through certain reduction procedures on the game. Minmax strategies also may have to be considered for solution in this case. These concepts are discussed with reference to differential games by means of an example in Ref. 1.

For the perfect information game, we have to consider the game situation for all the initial conditions in the playing space $\mathscr{R}$. Here, we consider a class of games having the Nash noncooperative solution. We further assume that the solution $\mathbf{u}^*$ induces a regular decomposition on $\mathscr{R}$ (Ref. 6). Thus, discontinuities in $\mathbf{u}^*$ lie on certain well-defined surfaces in $\mathscr{R}$. The corners of the optimal paths lie on the transition surfaces. Starting points from which multiple paths arise lie on the dispersal surfaces. Those multiple paths arising due to the discontinuities in u^{p*} should obviously yield the same payoff to the pth player. But this need not be true for players whose strategies are continuous at the dispersal surfaces. Such a situation has no parallel in two-person, zero-sum games, since saddle points are automatically both equivalent and interchangeable. We state below the necessary conditions to be satisfied for such a solution $\mathbf{u}^*$. These are stated as generalizations of the earlier results (Refs. 2–6).

Theorem 2.1. If $\mathbf{u}^*$ is a Nash noncooperative solution of the perfect-information game defined by (1)–(4), then there exist adjoint variables $(\lambda_0{}^p, \lambda^p)$, $\lambda_0{}^p \geqslant 0$ for $p = 1, 2,..., N$, one set for each player, such that each set is nonzero along the trajectory and satisfies the following conditions.

(i) *Euler–Lagrange Equations.* Between the corners of the optimal trajectory x^*, the variables x and λ^p satisfy the equations[7]

$$\dot{x} = f(x, \mathbf{u}^*, t), \tag{8}$$

$$\dot{\lambda}^p = -H_x^{p*} - \sum_j H_{u^{j*}}^{p*} u_x^{j*}, \tag{9}$$

where

$$H^p(x, \lambda^p, \mathbf{u}, t) = \lambda_0{}^p L(x, \mathbf{u}, t) + \langle \lambda^p, f(x, \mathbf{u}, t) \rangle. \tag{10}$$

(ii) *Corner Conditions.* At corners of x^*, corresponding to the discontinuities in the strategies of one or more players, the following condition is satisfied across the manifold of discontinuity (or switching surface):

$$(H^{p+} - H^{p-})\, dt - (\lambda^{p+} - \lambda^{p-})\, dx = 0, \tag{11}$$

where dt and dx are arbitrary displacements along the manifold at the corner point.

[7] Subscripts on H, u, W indicate partial derivatives.

Also, if the trajectories on either side of the switching surface are not tangential to it, and if all except one player, say the pth player, employ continuous strategies across the manifold, then his λ^p is continuous.

(iii) *Transversality Condition.* For $p = 1, 2, \ldots, N$, at the terminal time t_f, we have[8]

$$\lambda_0{}^p \phi_\sigma{}^p + H^p T_\sigma - \lambda^p X_\sigma = 0. \tag{12}$$

(iv) *Equilibrium Point Principle.* For every t, $u^*(t)$ corresponds to an equilibrium point in the Hamiltonians; that is, for $p = 1, 2, \ldots, N$,

$$H^p(x, \lambda^p, (\mathbf{u}^*; u^p), t) \geqslant H^p(x, \lambda^p, \mathbf{u}^*, t) = \min_{u^p \in \Omega^p} H^p(x, \lambda^p, (\mathbf{u}^*; u^p), t). \tag{13}$$

If u^p is interior to Ω^p, (13) implies that

$$\partial H^p / \partial u^p = 0, \tag{14}$$

$$\partial^2 H^p / \partial u^{p^2} \geqslant 0. \tag{15}$$

Inequality (15), is known as the Legendre–Clebsch condition.

(v) *Hamilton–Jacobi Equation.* The noncooperative value function vector $\mathbf{W} = (W^1, \ldots, W^p, \ldots, W^N)$ of the game satisfies a Hamilton–Jacobi equation; that is, for $p = 1, 2, \ldots, N$,

$$\begin{aligned} -W_t{}^p(x, t) &= \min_{u^p \in \Omega^p} \{\langle W_x{}^p(x, t), f(x, (\mathbf{u}^*; u^p), t)\rangle + \lambda_0{}^p L^p(x, (\mathbf{u}^*; u^p), t)\} \\ &= H^p(x, W_x{}^p, \mathbf{u}^*, t), \end{aligned} \tag{16}$$

where

$$W^p(x, t) = J^p[x, t, \mathbf{u}^*]. \tag{17}$$

Equation (16) has the final condition

$$W^p(x_f, t_f) = \phi^p(x_f, t_f). \tag{18}$$

The results follow from the one-sided optimal control problems of the players. The corresponding results for the null-information case appear in Refs. 2, 4, 5, and the essential differences between these two forms of information are discussed by Starr and Ho (Refs. 4–5). For the null-information case, the adjoint equation (9) is modified with $u_x^{p*} = 0$, and the Hamilton–Jacobi equations (16) need not be satisfied.

The optimal paths on which the theorem holds with $\lambda_0{}^p = 1$ are called the normal paths. The canonical equations (8), (9) and the

[8] Equation (11) can also be expressed in this form if the switching surface has a parametric representation similar to Eq. (3).

minimum principle (13) are satisfied between the switching surfaces. So, for obtaining the solution, one should construct the Hamiltonians for the players and find the equilibrium point in them. If, on an optimal trajectory, the minimum principle fails to determine u^*, then we call such paths singular paths.

In the next section, we classify the switching surfaces encountered in differential games and optimal control problems.

3. Classification and Construction of Switching Surfaces

Switching surfaces can be exhaustively classified by considering the nature of the optimal paths on the surface and its neighborhood, i.e., whether the paths enter, leave, or are parallel to the surface (not necessarily of the same nature on either side). The universal and dispersal surfaces of Isaacs (Ref. 7) derive their names on this basis. He also showed that some of the surfaces under this classification are unrealizable.

A different method is to label a switching surface as belonging to the player or players whose strategies are discontinuous across the surface. Yet another classification is based on the method of construction or the condition to be satisfied on the switching surface. Thus, transition surfaces are constructed by applying the corner conditions (11). The other candidates in this classification are the singular, dispersal, and abnormal surfaces. Before going into the construction of these surfaces, we present a general discussion on the construction of the switching surfaces in optimal control problems and differential games.

The construction of switching surfaces in optimal control problems appears in contemporary literature. The familiar problems have the Hamiltonians linear or sectionally linear in the control variables on which bounds are specified. The resulting bang–bang, three-level, and other controls are given in terms of signum, dead zone, or similar functions of a suitable switching function with the state and adjoint variables (x, λ) as its arguments. Under the usual smoothness assumptions on the formulation functions [f and L are assumed of class $C^{(1)}$], the switching function as well as λ are continuous functions of their respective arguments. In these problems, the construction of switching surfaces, which are mainly of the transition type, is straight-forward.

On the other hand, in differential games, the switching function as well as the adjoint variables λ^p need not be continuous in spite of similar smoothness assumptions on f and L^p. This situation arises because any discontinuities in the optimal strategies of the other players are reflected in the dynamic equations of the one-sided optimal control problem of the remaining player p. The preceding result is true whatever be the information patterns to the players. The corner conditions stated by Berkovitz (Ref. 8) for such a problem are equivalent to the results stated as corner conditions (11) in Theorem 2.1.

The possible discontinuities in the adjoint variables, together with the simultaneity involved in obtaining the strategies of all the players, makes the construction of switching surfaces more difficult in differential games. This explains to a large extent the occurrence of certain unusual transition surfaces, termed jocularly as bang–bang–bang surfaces by Isaacs (Ref. 9). In what follows, we consider one-by-one the singular, dispersal, and abnormal surfaces.

3.1. Singular Surfaces. Singular surfaces contain singular optimal paths. Singular extremals in optimal control have been studied in the literature (Refs. 10–12). A definition of Robbins (Ref. 12) is generalized here. An extremal arc is singular if, at each point of the arc, there is some allowable first-order weak control variation for at least one player p, which leaves his Hamiltonian H^p unchanged to second order. If u^p is interior to the restraint set Ω^p, then this condition reduces to $H^p_{u^p u^p}$ being singular or that the Legendre–Clebsch condition (15) is satisfied in its weak form only. As in optimal control, the most common examples are the linear singular arcs. These arise when at least one of the Hamiltonians, say H^p, is linear or sectionally linear in some components of the corresponding u^p. Most universal surfaces of Isaacs (Ref. 7) fall in this category.

Of the various methods of obtaining the singular arcs, we present here the Robbins' second variation method. The linear singular control variables cannot be determined from (14) of the minimum principle. However, differentiation of $H^p_{u^p}$ a sufficient number of times equal to the order of singularity will determine these variables after suitable manipulation. This technique consists in substituting the canonical equations and the expressions for the nonsingular control variables after each differentiation.

The generalized Legendre–Clebsch condition is stated as a test for the optimality of singular extremals. For player p, $\Phi_k{}^p$ is formed for different values of k, where

$$\Phi_k{}^p = \{(d/dt)^k H^p_{u^p}\}_{u^p}. \tag{19}$$

The first time $\Phi_k{}^p$ is not equal to a null matrix, k should be even, that is, $k = 2l$. The matrix $(-1)^l \Phi^p_{2l}$ must be positive semidefinite. If this condition fails, the singular extremal is not optimal. The two-person, zero-sum version of this condition is given by Anderson (Ref. 13) along with a few junction conditions. Such junction conditions are applicable only when all players, except one, use continuous strategies across the junction.

The analytical difficulties in the actual construction mount with several players having singular control variables and with different control variables having different orders of singularity.

3.2. Dispersal Surfaces. Dispersal surfaces contain starting points in $\mathscr{R}$ where multiple optimal paths arise. Multiple optimal paths result on account of the discontinuities in the strategies of one or more players. These discontinuities can be interpreted as different strategy choices for the concerned player corresponding to each different path at the starting point. If we freeze the strategies of the other players at this point, then the payoffs to the player on the different paths, resulting from his different strategy choices, should be equal. In a general N-person differential game, the payoff to the other players on these paths need not be equal.

In optimal control problems with the usual smoothness assumptions on f and L and the terminal surface, dispersal surfaces are rarely met with. Isaacs discusses thoroughly the construction of these surfaces in two-person, zero-sum differential games. Dispersal surfaces of any player are constructed based on the property that the payoff to this player is independent of the multiple optimal paths when the strategies of the others are frozen at the surface, as described earlier. However, on the dispersal surface $N^p_{i_1,i_2,\ldots,i_k}$ of player p, the following condition is satisfied for any variations dx, dt on the surface (see Ref. 14 for a similar result):

$$\begin{aligned} H^p(x, \lambda^p_{i_1}, (\mathbf{u}^*; u^*_{i_1}), t) - \lambda^p_{i_1}\, dx &= H^p(x, \lambda^p_{i_2}, (\mathbf{u}^*; u^*_{i_2}), t) - \lambda^p_{i_2}\, dx \\ &\;\vdots \\ &= H^p(x, \lambda^p_{i_k}, (\mathbf{u}^*; u^*_{i_k}) - \lambda^p_{i_k}\, dx. \qquad (20) \end{aligned}$$

The subscripts $i_1, i_2, \ldots, i_k$ denote the different optimal paths corresponding to the different strategy choices.

3.3. Abnormal Surfaces. Abnormal surfaces are surfaces containing abnormal solutions. On the abnormal paths, the minimum principle of Theorem 21 is satisfied with $\lambda_0{}^p = 0$. Thus, $\lambda_0{}^p = 0$ in the expression (10) for the Hamiltonian and the transversality and corner conditions (11)–(12). The semipermeable surfaces of Isaacs (Ref. 7) follow the same construction and constitute examples of abnormal surfaces.

Results on the optimality of abnormal solutions appear in optimal control literature (see for example, Ref. 15, Theorem 12, p. 364, and Ref. 16, pp. 37f and 53). Similar considerations arise in differential games as well. The abnormal surfaces in problems with time as payoff have a special significance in two-person, zero-sum games as is evidenced by the concept of barrier (Ref. 7). This surface does not exhibit time optimality but corresponds to the neutral outcome associated with the game of kind. In a general problem, the roles of the players with regard to the termination of the game have to be specified to determine the optimality of abnormal solutions. Thus, Isaacs specifies a drastic penalty for nontermination to both the players in a game (see Appendix of Ref. 9) and, under this assumption, the abnormal solutions of this game were shown to be optimal.

The construction of switching surfaces across which more than one player employ discontinuous strategies is much more difficult than the construction of one-player switching surfaces. Examples are the bang-bang–bang surfaces (Ref. 9) and the equivocal surfaces (Ref. 7) of Isaacs and the switch envelope of Breakwell and Merz (Ref. 17). It is in particular the construction of these surfaces that is highly example oriented.

In the next section, we present a few simple examples. Included also is an example of a new type of dispersal surface for nonzero-sum games which has no counterpart in two-person, zero-sum differential games.

4. Examples of Switching Surfaces

In this section, we consider the game described by a double integral plant defined by

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = u^1 + cu^2, \tag{21}$$

with the control variables constrained as follows:

$$|u^1| \leqslant 1, \qquad |u^2| \leqslant 1. \tag{22}$$

The terminal state is specified as the origin at the free terminal time t_f. The payoffs to the players are given by

$$\begin{aligned} J^1[x_0, u^1, u^2] &= \int_0^{t_f} dt, \\ J^2[x_0, u^1, u^2] &= \int_0^{t_f} \{|u^1| + b\,|u^2|\}\, dt. \end{aligned} \tag{23}$$

This problem originally arose in a bicriterion optimal control problem: a satellite attitude controller represented by a double-integral plant with time and fuel minimization as the twin objectives. This problem was solved fully for the case $c > b$ (Ref. 18). Here, we obtain the singular and abnormal solutions of the noncooperative solution. We also obtain the complete noncooperative solution of the game for the case $2 < c < b$ under a modified playing space and terminal specification. This exhibits a dispersal surface for the second player.

The Hamiltonians for the players are given by

$$\begin{aligned} H^1 &= 1 + \lambda_1{}^1 x_2 + \lambda_2{}^1(u^1 + cu^2), \\ H^2 &= | u^1 | + b \, | u^2 | + \lambda_1{}^2 x_2 + \lambda_2{}^2(u^1 + cu^2). \end{aligned} \tag{24}$$

For obtaining the equilibrium strategies, H^1 and H^2 are to be minimized with respect to u^1 and u^2, respectively. Thus, we have

$$u^{1*} = -\text{sign}(\lambda_2{}^1), \qquad u^{2*} = -\text{dez}(\lambda_2{}^2 c/b), \tag{25}$$

where we define

$$\text{sign}\, z = \begin{cases} +1, & z > 0, \\ -1, & z < 0, \end{cases} \tag{26}$$

$$\text{dez}(z) = \begin{cases} +1, & z > 1, \\ 0, & -1 < z < 1, \\ -1, & z < -1. \end{cases} \tag{27}$$

When neither player has any observations, the adjoint equations are given for $p = 1, 2$ as

$$\dot{\lambda}_1{}^p = 0, \qquad \dot{\lambda}_2{}^p = -\lambda_1{}^p. \tag{28}$$

However, since u^{1*} and u^{2*} assume only the values ± 1 and 0, in view of (25)–(27), the expressions $u^{p*}_{x_l}$ for p, $l = 1, 2$ will be zero. Hence, (28) is valid when the players have perfect information as well.

Since the terminal specification violates the dimensionality requirement, we choose the terminal surface as $x_1 = a \cos \theta$ and $x_2 = a \sin \theta$, and then apply the transversality condition (12) with $\sigma = (t_f, \theta)$. We have

$$[\lambda_1{}^1(t_f), \lambda_2{}^1(t_f)] \begin{bmatrix} a \sin \theta, & -a \sin \theta \\ u^1 + cu^2, & a \cos \theta \end{bmatrix} = [-1, 0], \tag{29}$$

$$[\lambda_1{}^2(t_f), \lambda_2{}^2(t_f)] \begin{bmatrix} a \sin \theta, & -a \sin \theta \\ u^1 + cu^2, & a \cos \theta \end{bmatrix} = [-\{| u^1 | + b \, | u^2 |\}, 0]. \tag{30}$$

Solving (29)–(30) and letting $a \to 0$, we have

$$\lambda_1^p(t_f) = \lambda_2^p(t_f) \cot \theta, \qquad p = 1, 2, \tag{31}$$

$$\lambda_2^2(t_f) = \{| u^1 | + b | u^2 |\} \lambda_2^1(t_f) = -(| u^1 | + b | u^2 |)/(u^1 + cu^2). \tag{32}$$

4.1. Singular Solutions. A singular u^1 requires that $\lambda_2^1(t) = 0$ and $\dot{\lambda}_2^1(t) = -\lambda_1^1(t) = 0$; this violates the condition $H^1 = 0$ on the trajectory and, hence, does not arise. On the other hand, a singular u^2 requires that

$$\lambda_2^2(t)c/b = \pm 1 \qquad \text{or} \qquad \lambda_2^2(t) = \pm b/c, \tag{33}$$

and

$$\dot{\lambda}_2^2(t) = -\lambda_1^2(t) = 0. \tag{34}$$

Thus, for the terminal sequence $\left[\begin{smallmatrix}-1\\-\epsilon\end{smallmatrix}\right]$, $0 \leqslant \epsilon \leqslant 1$, to be optimal, we should have from the transversality condition (12) and (33) (considering the upper values) and (34)

$$\lambda_1^1(t) = \lambda_1^2(t) = 0, \tag{35}$$

$$\lambda_2^2(t) = (1 + b\epsilon) \lambda_2^1(t) = -(1 + b\epsilon)/(1 + c\epsilon) = b/c. \tag{36}$$

Equation (36) requires that $b = c$ and that ϵ is a constant on the trajectory. It can be seen that this sequence does not violate the generalized Legendre–Clebsch condition. Considering the lower values of (33), we can show a similar result for the terminal sequence $\left[\begin{smallmatrix}+1\\+\epsilon\end{smallmatrix}\right]$. The resulting trajectories for $0 \leqslant \epsilon \leqslant 1$ are shown in regions G_5 and G_6 of Fig. 1. The equation of the trajectory $\gamma_{1\epsilon}$ along which the state reaches the origin with the control law $\left[\begin{smallmatrix}\pm 1\\\pm\epsilon\end{smallmatrix}\right]$, $0 \leqslant \epsilon \leqslant 1$, is obtained as follows.

On integrating (21) with the initial condition $x_1(0) = s_1$ and $x_2(0) = s_2$, we have

$$x_2 = s_2 + (u^1 + cu^2)t, \qquad x_1 = s_1 + s_2 t + \tfrac{1}{2}(u^1 + cu^2)t^2. \tag{37}$$

For the system to reach the origin, there must exist some $t_f > 0$ such that $x_1(t_f) = x_2(t_f) = 0$ in (37). Hence, we get

$$t_f = -s_2/(u^1 + cu^2), \qquad \operatorname{sign} s_2 = -\operatorname{sign}(u^1 + cu^2), \tag{38}$$

and

$$s_1 = -s_2^2/2(u^1 + cu^2). \tag{39}$$

Thus, $\gamma_{1\epsilon}$ is given by

$$\gamma_{1\epsilon} = \{(x_1, x_2) : x_1 = -x_2^2 \operatorname{sign} x_2/2(1 + c\epsilon)\}. \tag{40}$$

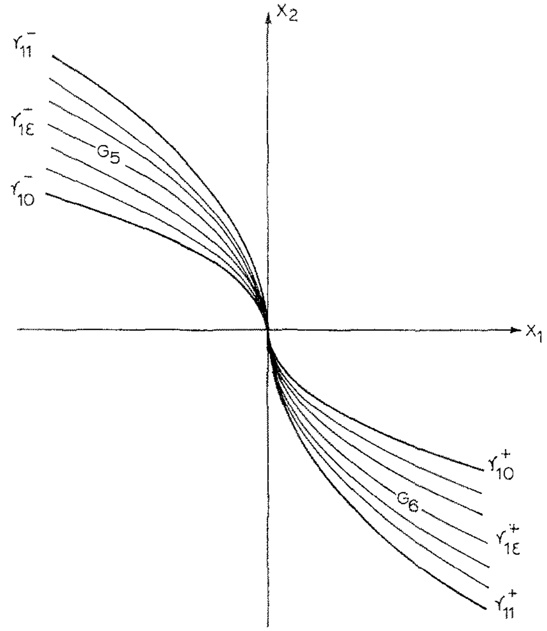

Fig. 1. Singular solutions for the example of the double integral plant.

Now, we can determine whether these trajectories are optimal for the perfect-information case by verifying the Hamilton–Jacobi equation (12). Expressing the control laws as feedback policies in the region G_5, we have, in view of (38)–(40),

$$u^1 = -1, \qquad u^2 = -\epsilon = (x_2{}^2 + 2x_1)/2x_1c, \tag{41}$$

$$W^1(x_1, x_2) = x_2/(1 + c\epsilon) = -2x_1/x_2, \qquad W^2(x_1, x_2) = x_2. \tag{42}$$

Now, for Player 1, we have the Hamilton–Jacobi equation

$$0 = \min_{|u^1|\leqslant 1} \{1 + (-2/x_2)\, x_2 + (2x_1/x_2{}^2)(u^1 + cu^2)\}$$

or

$$u^{1*} = -\text{sign}(2x_1/x_2{}^2) = +1. \tag{43}$$

Since (43) contradicts (41), the cluster of singular solutions in G_5 (and similarly in G_6) are not optimal for Player 1.

However, by writing the Hamilton–Jacobi equation for Player 2, it can be easily seen that the cluster of singular solutions in G_5 and G_6 are optimal for this player. Which of the singular solutions in (G_5, G_6) are optimal under the null and perfect information cases can only be answered by a complete solution of the problem, which will be given in Ref. 1.

We obtain the abnormal solutions of the same problem now. Since $\lambda_0^1 = \lambda_0^2 = 0$ in this case, (24)–(25) are modified as follows:

$$H^1 = \lambda_1^1 x_2 + \lambda_2^1(u^1 + cu^2), \qquad H^2 = \lambda_1^2 x_2 + \lambda_2^2(u^1 + cu^2), \tag{44}$$

and

$$u^{1*} = -\operatorname{sign}(\lambda_2^1), \qquad u^{2*} = -\operatorname{sign}(\lambda_2^2). \tag{45}$$

The adjoint equation (28) are not modified, obviously.

The curves γ_{11}^+, γ_{11}^-, $\gamma_{1,-1}$, $\gamma_{-1,1}$ are the abnormal curves with the corresponding control sequences

$$\begin{bmatrix}+1\\+1\end{bmatrix}, \quad \begin{bmatrix}-1\\-1\end{bmatrix}, \quad \begin{bmatrix}+1\\-1\end{bmatrix}, \quad \begin{bmatrix}-1\\+1\end{bmatrix}.$$

We show this below for γ_{11}^- by showing that it satisfies the required necessary conditions. The results for the other curves follow similarly.

From the transversality conditions (14),[9] we have λ_1^1 and λ_1^2 arbitrary and

$$\lambda_2^1(t_f) = \lambda_2^2(t_f) = 0. \tag{46}$$

By (38), we have for any initial state (x_1, x_2) on γ_{11}^-

$$t_f = x_2/(1 + c). \tag{47}$$

From (28), (46), (47), on integration, we have

$$\lambda_2^p(0) = \lambda_1^p\, x_2/(1 + c). \tag{48}$$

Equations (45) and (48) yield

$$u^1 = u^2 = -1, \tag{49}$$

with λ_1^p being any negative number for $p = 1, 2$.

The optimality of the different abnormal solutions can be decided in the context of the full solution of the game.

Now, we modify the playing space, the terminal specification, and the payoff functions of the game formulated in (21)–(23). The terminal surface $T_1 U T_2$ is shown in Fig. 2. The space between T_1 and T_2 is the playing space. We have

$$\begin{aligned} T_1 &= \{(x_1, x_2) : x_1 = -x_2^2/2(1 + c)\}, \\ T_2 &= \{(x_1, x_2) : x_1 = -\tfrac{1}{2}x_2^2\}. \end{aligned} \tag{50}$$

[9] Equations (29)–(30) are modified with null vectors on the right-hand side in this case.

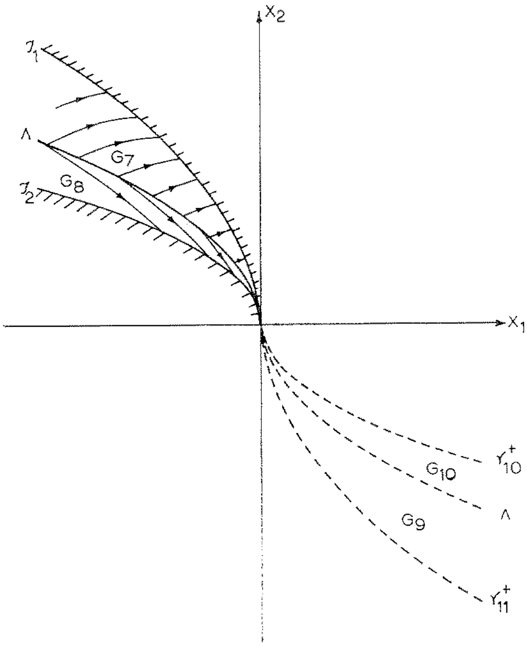

Fig. 2. An example of a game with a dispersal surface.

Comparing (40) and (50), we know that T_1 and T_2 are identical with γ_{11}^- and γ_{10}^-, respectively.

The payoff functionals are of the composite type given by

$$\begin{aligned} J^1[x_0, \mathbf{u}] &= \phi^1(x_f) + \int_0^{t_f} dt, \\ J^2[x_0, \mathbf{u}] &= \phi^2(x_f) + \int_0^{t_f} \{|u^1| + b\,|u^2|\}\, dt, \end{aligned} \tag{51}$$

where x_f is the terminal state at the free terminal time t_f. The functions ϕ^1 and ϕ^2 are given below as

$$\phi^1(x_f) = \begin{cases} x_{2f}/(1+c), & x_f \in T_1, \\ x_{2f}, & x_f \in T_2, \end{cases} \tag{52}$$

$$\phi^2(x_f) = \begin{cases} x_{2_f}(1+b)/(1+c), & x_f \in T_1, \\ x_{2_f}, & x_f \in T_2. \end{cases} \tag{53}$$

The Hamiltonians for the players, the adjoint equations, and the optimal control actions are given again as in (24), (28), (25), respectively.

Applying the transversality condition (12) at T_1, with $\sigma = (x_{2_f}, t_f)$, we have

$$\begin{aligned} 1/(1+c) - \lambda_1^1[-x_{2_f}/(1+c)] - \lambda_2^1 &= 0, \\ 1 + \lambda_1^1 x_{2_f} + \lambda_2^1(u^1 + cu^2) &= 0, \\ (1+b)/(1+c) - \lambda_1^2[-x_{2_f}/(1+c)] - \lambda_2^2 &= 0, \\ |u^1| + b\,|u^2| + \lambda_1^2 x_{2_f} + \lambda_2^2(u^1 + cu^2) &= 0. \end{aligned} \tag{54}$$

The quantities in (54) refer to time t_f. Solving (54) to be consistent with (25) and (28) yields

$$\begin{aligned} &\lambda_1^1(t_f) = -1/x_{2_f}, \qquad \lambda_2^1(t_f) = 0, \\ &\lambda_1^2(t_f) = -(2+c+b)/(2+c)\, x_{2f}, \qquad \lambda_2^2(t_f) = b/(2+c), \end{aligned} \tag{55}$$

and

$$u^1(t) = 1, \qquad u^2(t) = 0 \qquad \text{for} \quad t < t_f. \tag{56}$$

By a similar application of T_2, we have

$$\begin{aligned} 1 - \lambda_1^1(-x_{2_f}) - \lambda_2^1 &= 0, \\ 1 + \lambda_1^1 x_{2_f} + \lambda_2^1(u^1 + cu^2) &= 0, \\ 1 - \lambda_1^2(-x_{2_f}) - \lambda_2^2 &= 0, \\ |u^1| + b\,|u^2| + \lambda_1^2 x_{2_f} + \lambda_2^2(u^1 + cu^2) &= 0. \end{aligned} \tag{57}$$

Solving (57), (25), (28) consistently, we have

$$\begin{aligned} &\lambda_1^1(t_f) = -1/x_{2_f}, \qquad \lambda_2^1(t_f) = 0, \\ &\lambda_1^2(t_f) = [-1 - b - b(1-c)/(c-2)]/x_{2_f}, \qquad \lambda_2^2(t_f) = b/(c-2), \end{aligned} \tag{58}$$

and

$$u^1(t) = 1, \qquad u^2(t) = -1 \qquad \text{for} \qquad t < t_f. \tag{59}$$

The optimal paths resulting from the control laws (56) and (59) are made to flood the playing space. From (56) and (59), it is clear that the strategy of Player 1 is continuous in the playing space and that of the second player is discontinuous. Hence, there arises a switching surface which, in this case, is the second player's dispersal surface Λ. For starting points on Λ, W^2 must be the same whether the optimal paths reach T_1 or T_2. By actually equating the values, it is straightforward to show that Λ is given by

$$\Lambda = \{(x_1, x_2) : x_1 = -\tfrac{1}{2}\eta x_2^2,\ x_2 \geqslant 0\}, \tag{60}$$

where η is given by

$$(b+1)/(c-1) - [(2+b-c)/(c-1)]\sqrt{[(1+(c-1)\eta)/(c-2)]}$$
$$= -1 + (2+c+b)\sqrt{(1+\eta)}/\sqrt{[(1+c)(2+c)]}. \tag{61}$$

The optimality of this solution is conclusively shown by verifying the Hamilton–Jacobi equation. A similar construction between γ_{11}^{+} and γ_{10}^{+} shown in broken lines in Fig. 2 will be used in Ref. 1.

From (52) and (38), one can easily see that $\phi^1(x_f)$ is the time taken for the system state x_f to reach the origin along either T_1 or T_2 as the case may be. Hence, W^1 can be interpreted as the total time taken for the system state to reach the origin along the optimal path and the terminal surface. Thus, for any point A on Λ, the value of W^1 for either optimal path can be written as the line integral

$$W^1 = \int_A^0 dt = \int_A^0 dx_1/x_2\,. \tag{62}$$

As the trajectory reaching T_2 is below that reaching T_1, it follows from (62) that the time taken for the former path is larger than that of the latter. Hence, the two equilibrium points are nonequivalent for Player 1. There is no counterpart of this result in two-person, zero-sum games, for obvious reasons.

5. Conclusions

The Nash noncooperative solution of an N-person differential game is described for a class of games with null and perfect information to the players. Switching surfaces encountered in optimal control and differential games are classified, and the method of construction of these surfaces is reviewed with reference to the N-person differential games. An example brings out the distinct nature of dispersal surfaces in the Nash noncooperative solution. The optimality of singular and abnormal solutions of the example can be established by the complete solution of the game (Ref. 1).

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