BUGNET, Theodore (tvb15)

Imperial College London

Department of Computing
Academic Year 2019-2020
Page created Thu Feb 20 02:15:19 GMT 2020

CATE

499 fbelard 6 s5 tvb15 v1 0011900035756	Electronic submission **Wed - 19 Feb 2020 21:34:57** tvb15

Exercise Information

Module:	499 Modal Logic for Strategic Reasoning in AI	**Issued:**	Wed - 05 Feb 2020
Exercise:	6 (CW)	**Due:**	Wed - 19 Feb 2020
Title:	Coursework2	**Assessment:**	Individual
FAO:	Belardinelli, Francesco (fbelard)	**Submission:**	Electronic

Student Declaration - Version 1

- I declare that this final submitted version is my unaided work.

Signed: (electronic signature) Date: 2020-02-17 12:10:13

For Markers only: (circle appropriate grade)

BUGNET, Theodore (tvb15)	01048938	s5	2020-02-17 12:10:13	**A* A B C D E F**

499 Modal Logic CW2

Theodore Bugnet

February 19, 2020

Question 1

(a)

$\pi \models \phi R \psi$ iff, for all $i \geq 0$, either $\pi[i..\infty] \models \psi$ or there exists $j, 0 \leq j < i$, s.t. $\pi[j..\infty] \models \phi$.

(b)

$\phi R \psi \equiv \neg(\neg\phi U \neg\psi)$.

(c)

Using the definition of U, $\pi \models (\neg\phi U \neg\psi)$ iff $\exists i \geq 0$ s.t. $\pi[i..\infty] \models \neg\psi$ and $\forall j, 0 \leq j < i, \pi[j..\infty] \models \neg\phi$.

Therefore, $\pi \models \neg(\neg\phi U \neg\psi)$ iff $\forall i \geq 0$, either $\pi[i..\infty] \not\models \neg\psi$ or $\exists j, 0 \leq j < i, \pi[j..\infty] \not\models \neg\phi$.

Observing that $(\lambda \not\models \neg\alpha) \equiv (\lambda \models \alpha)$, it is clear that this matches the definition of $\phi R \psi$ presented in **(a)**.

(d)

$G\psi \equiv \neg F \neg\psi \equiv \neg(\top U \neg\psi) \equiv \neg(\neg\bot U \neg\psi)$, which, from the result in **(b)**, is equivalent to $\bot R \psi$.

Alternatively, using the result in **(a)**: $\phi = \bot$, so there is no j such that $\pi[j..\infty] \models \phi$, hence $\pi \models \bot R \psi$ iff $\pi[i..\infty] \models \psi$ for all $i \geq 0$ — which is exactly the definition of $\pi \models G\psi$.

Question 2

(a)

$(M, q) \models EF\Phi$ iff $(M, q) \models E(\top U \Phi)$. Using the definition of satisfaction of E, this is true iff, for some path λ starting from q, $(M, \lambda) \models \top U \Phi$. Using the definition of satisfaction for U, this is satisfied iff $(M, \lambda[i]) \models \Phi$ for some $i \geq 0$ and $(M, \lambda[j]) \models \top$ for all $0 \leq j < i$ — but the second conjunct is true by definition of $\top$.

Therefore, $(M, q) \models EF\Phi$ iff, for some path λ starting from q, $(M, \lambda[i]) \models \Phi$ for some $i \geq 0$.

(b)

$(M, q) \models AF\Phi$ iff $(M, q) \models A(\top U \Phi)$. Using the definition of satisfaction for A, this is true iff, for all paths λ starting from q, $(M, \lambda) \models \top U \Phi$. Using the same argument as in part **(a)**, this means that it is true iff, for all paths λ starting from q, $(M, \lambda[i]) \models \Phi$ for some $i \geq 0$.

(c)

$(M, q) \models EG\Phi$ iff $(M, q) \models \neg AF\neg\Phi$. Using the result from **(b)**, this is true iff $\neg\forall\lambda$ starting from q, $\exists i$ s.t. $(M, \lambda[i]) \not\models \Phi$. Simplifying yields: $\exists\lambda$ starting from q s.t. $\neg\exists i$ s.t. $(M, \lambda[i]) \not\models \Phi$, and hence, $\exists\lambda$ starting at q s.t. $\forall i \geq 0$ $(M, \lambda[i]) \models \Phi$.

(d)

$(M, q) \models AG\Phi$ iff $(M, q) \models \neg EF\neg\Phi$. Using the result from **(a)**, this is true iff $\neg\exists\lambda$ starting from q s.t. $\exists i \geq 0$ s.t. $(M, \lambda[i])\neg \models \Phi$. By the same argument as in **(c)**, this is equivalent to $\forall\lambda$ starting at q, $\forall i \geq 0$, $(M, \lambda[i]) \models \Phi$.

Question 3

(a)

The definition of state formulas, Φ, is identical between CTL and CTL*.

The path formulas, Ψ, are defined in CTL as $\Psi ::= X\Phi \mid \Phi U \Phi$. The definition of path formulas in CTL* includes $\Psi ::= \Phi \mid X\Psi \mid \Psi U \Psi$, alongside other forms. Therefore, using the rule $\Psi ::= \Phi$, from the next two rules it follows that $X\Phi$ and $\Phi U \Phi$ are valid CTL* path formulas - therefore, any CTL path formula is a CTL* path formula.

Therefore, every CTL formula is also a CTL* formula.

(b)

Take $\Phi = EXXp$. This formula satisfies the CTL* syntax definition, but not the CTL definition.

Question 4

For convenience, the rules presented in the coursework will be referred to by number in order, from 1 to 10.

The truth conditions for the state formulas in CTL* are identical to those in CTL, covered by rules 1-5.

Path formulas in CTL are of the shape $X\Phi$ or $\Phi U \Phi'$, where Φ and Φ' represent arbitrary state formulas. In CTL*, such formulas would be covered by rules 9 and 10. Considering each case in turn:

- $(M, \pi) \models X\Phi$ iff $(M, \pi[1 \ldots \infty]) \models \Phi$, according to rule 9. According to rule 6, the latter is the case iff $(M, \pi[1]) \models \Phi$. This is identical to the CTL truth condition for $(M, \lambda) \models \Phi$.

- $(M, \pi) \models \Phi U \Phi'$ iff $(M, \pi[i \ldots \infty]) \models \Phi'$ for some $i \geq 0$, and $(M, \pi[j \ldots \infty]) \models \Phi$ for all $0 \leq j < i$, according to rule 10. Again, using rule 6, this can be reduced to $(M, \pi[i]) \models \Phi$ for some $i \geq 0$, and $(M, \pi[j]) \models \Phi$ for all $0 \leq j < i$. This is, again, identical to the corresponding truth condition in CTL.

Therefore, all formulas using CTL syntax, when considered under CTL* semantics, have identical truth conditions to those defined by the CTL semantics.

Question 5

(a)

This follows trivially from question **3** and **4**. In **3**, it was shown that every formula Φ of CTL is also a formula of CTL* — in other words, $\Phi' = \Phi$. In **4**, it was shown that the fragment of CTL* syntax which corresponds exactly to CTL has identical semantics and truth conditions to the equivalent CTL syntax. Therefore, the CTL* formula Φ', which uses only CTL syntax, will have the exact same truth condition as the identical CTL formula Φ, and this holds any general CTL formula Φ.

(b)

Consider the CTL* formula $AFGa$. This has identical semantics to the LTL formula FGa. *TODO given time: prove that this has no CTL equivalent. Clarke-Draghicescu 1989 paper gives longer general proof using fairness, find something more elegant using only course material*

Question 6

In this answer, $M, M', \rightarrow, \rightarrow'$, etc. are as described in the question (i.e. respectively, two models, the relations corresponding to each model, etc.). Furthermore, the convention is adopted that any states marked with a prime (e.g. t', s') are elements of St', and those not marked such (e.g. t, s) are elements of St.

Lemma 1

Given two bisimilar states, such as t and t', for all paths λ starting at t there will be a bisimilar path λ' starting at t', and vice versa. In other words, without loss of generality, the paths π and π' given in the question can be assumed to start at t and t' respectively.

Proof.

First, consider state t, such that $(M, t) \approx (M', t')$. Call B the bisimulation responsible for this bisimilarity. By condition *(b)* of the bisimulation definition, for any states s s.t. $t \rightarrow s$, there must exist a state s' such that $t' \rightarrow' s'$ and B(s, s') — i.e. $(M, s) \approx (M', s')$. But this definition applies generally to any state, so by the same argument for any state q s.t. $s \rightarrow q$ for any s, in turn there must exist a state q such that $(M, q) \approx (M', q')$ and $s' \rightarrow' q'$, and so on and so forth.

By the definition of a path as a sequence of states related by the transition relation, and the definition of bisimilarity for paths, it follows that if $(M, t) \approx (M', t')$, then all paths π starting at t there will be a path π' starting at t' such that $(M, \pi) \approx (M', \pi')$.

An identical argument from M' to M can be constructed, using condition *(c)* of the bisimulation definition, to show that for any path π' starting at t', there will be a path π starting at t such that \$(M, π) $\approx (M', \pi')$. The reasoning is the exact same as above, except considering all possible s' s.t. $t' \to' s'$, observing that a corresponding s (i.e. bisimilar to s' and having the property that $t \to s$) must exist, and so forth, mirroring the rest of the argument.

Proof of CTL* preservation by bisimulations.

Consider the state formula Φ and bisimilar states (M, t) and (M', t'). Then, by induction on Φ, and using the definition of CTL* truth conditions:

- If $\Phi = p$, p being an atomic proposition, then by condition *(a)* of the bisimulation definition, $(M, t) \models p$ iff $(M', t') \models p$, and hence $(M, t) \models \Phi$ iff $(M', t') \models \Phi$.

- If $\Phi = \neg\phi$ for some state formula ϕ, then, given the inductive assumption that $(M, t) \models \phi$ iff $(M', t') \models \phi$, the converse is also true — i.e. $(M, t) \not\models \phi$ iff $(M', t') \not\models \phi$, and hence $(M, t) \models \Phi$ iff $(M', t') \models \Phi$.

- If $\Phi = \phi \wedge \phi'$ for some state formulas ϕ and ϕ', then, given the inductive assumption, we have $(M, t) \models \phi$ iff $(M', t') \models \phi$ and $(M, t) \models \phi'$ iff $(M', t') \models \phi'$, and therefore $(M, t) \models \phi \wedge \phi'$ iff $(M', t') \models \phi \wedge \phi'$, and hence $(M, t) \models \Phi$ iff $(M', t') \models \Phi$.

- If $\Phi = E\psi$ for some path formula ψ, then there exists a path (M, π) starting from t such that $(M, \pi) \models \psi$, and according to Lemma 1 there will also exist path (M', π') starting from t' bisimilar to (M, π). Then the truth is preserved if and only if we have that $(M, \pi) \models \psi$ iff $(M', \pi') \models \psi$ given these bisimilar (M, π) and (M', π'). This can be assumed by mutual induction on the path formulas.

- If $\Phi = A\psi$ for some path formula ψ, then since, according to Lemma 1, for every path (M, π) starting at t there will be a bisimilar path (M', π') starting at t' and vice versa, the truth is preserved if and only if, for every such pair of bisimilar paths (M, π) and (M', π'), we have that $(M, \pi) \models \psi$ iff $(M', \pi') \models \psi$. This can be assumed by mutual induction on the path formulas.

Now consider the path formula Ψ and bisimilar paths (M, π) and (M', π'). Then, by induction on Ψ, and using the definition of CTL* truth conditions:

- If $\Psi = \phi$ for some state formula ϕ, then by mutual induction on state formulas it can be assumed that $(M, \pi[0]) \models \phi$ iff $(M', \pi'[0]) \models \phi$, and hence $(M, \pi) \models \Psi$ iff $(M', \pi') \models \Psi$.

- If $\Psi = \neg\psi$ for some path formula ψ, then by an identical argument of that used for state formulas, we observe that the converse of the inductive assumption must be true, and $(M, \pi) \not\models \psi$ iff $(M', \pi') \not\models \psi$, and hence $(M, \pi) \models \Psi$ iff $(M', \pi') \models \Psi$.

- If $\Psi = \psi \wedge \psi'$, again, an argument identical to that for the similar case for state formulas is used, concluding that $(M, \pi) \models \psi \wedge \psi'$ iff $(M', \pi') \models \psi \wedge \psi'$, and hence $(M, \pi) \models \Psi$ iff $(M', \pi') \models \Psi$.

- If $\Psi = X\psi$, first observe that if paths (M, π) and (M', π') are bisimilar, then all paths $(M, \pi[1 \ldots \infty])$ will have bisimilar $(M', \pi'[1 \ldots \infty])$ and vice versa, directly following from the definition of path bisimilarity and Lemma 1. Therefore, by the inductive assumption, we have that $(M, \pi[1 \ldots \infty]) \models \psi$ iff $(M', \pi'[1 \ldots \infty]) \models \psi$, and hence $(M, \pi) \models \Psi$ iff $(M', \pi') \models \Psi$.

- If $\Psi = \psi U \psi'$, then note that the observation from above can be generalised for any i, i.e. given bisimilar (M, π) and (M', π'), all $(M, \pi[i \ldots \infty])$ will have bisimilar $(M', \pi'[i \ldots \infty])$ and vice versa. Therefore, using the inductive assumption, for some $i \geq 0$, we have $(M, \pi[i \ldots \infty]) \models \psi'$ iff $(M', \pi'[i \ldots \infty]) \models \psi'$. Similarly, for all j s.t. $0 \leq j < i$, $(M, \pi[j \ldots \infty]) \models \psi$ iff $(M', \pi'[j \ldots \infty]) \models \psi$. Hence, $(M, \pi) \models \Psi$ iff $(M', \pi') \models \Psi$.

Therefore, by mutual induction on path and state formulas, we have shown that for an arbitrary CTL* formula Φ and bisimilar states (M, t) and (M', t'), $(M, t) \models \Phi$ iff $(M', t') \models \Phi$. In other words, the truth of CTL* formulas is preserved by bisimulations.

Question 7

Let $St, St', \rightarrow, \rightarrow', V, V'$ be such that $M = (St, \rightarrow, V)$ and $M' = (St', \rightarrow', V')$ (M and M' as used in the question). Furthermore, let $\leftrightsquigarrow$ denote the relation between two states of being CTL-equivalent.

Consider the three conditions of the definition of a bisimulation, and whether $\leftrightsquigarrow$ satisfies them.

Condition *(a)* holds for $\leftrightsquigarrow$: if it didn't, i.e. some atomic proposition (call it p) were satisfied in one but not the other (w/o loss of generality, assume

$(M,t) \models p$ and $(M',t') \not\models p$), then take CTL formula p: it is satisfied in (M,t) but not (M',t'), contradicting the assumption of CTL-equivalency.

For condition *(b)*: Take some $s \in St$ s.t. $t \to s$ holds. Assume there exists no $s' \in St'$ such that the properties $t' \to' s'$ and $s \leftrightsquigarrow s'$ both hold. Let $S = q \in St' | t' \to q'$. Since St' is assumed finite, and $\to'$ is assumed serial, S is non-empty and finite. Furthermore, for every $s'_i \in S$, by assumption $s \not\leftrightsquigarrow s'$. Therefore, for every s'_i, there must exist a formula ϕ_i such that $(M,s) \models \phi_i$ and $(M', s'_i) \not\models \phi_i$.

Now, consider the CTL formula $\Phi = EX(\phi_1 \wedge \phi_2 \wedge \cdots \wedge \phi_n)$. By our assumption, every ϕ_i holds in (M,s), therefore $(M,t) \models \Phi$. But for every successor s'_i of t', there is a corresponding phi_i such that $(M', s'_i) \not\models \phi_i$, therefore $(M',t') \not\models \Phi$ — therefore, in this case, $(M,t) \not\leftrightsquigarrow (M',t')$.

Therefore, if we have that $(M,t) \leftrightsquigarrow (M',t')$, for every $s \in St$ s.t. $t \to s$, there must exist some $s' \in St'$ s.t. $t' \to' s' \wedge s \leftrightsquigarrow s'$. This makes $\leftrightsquigarrow$ satisfy condition *(b)* of the definition of a bisimulation.

Finally, consider condition *(c)*. The argument here is nearly identical to that for condition *(b)* — simply, rather than considering some $s \in St$ s.t. $t \to s$ and concluding that there must exist an $s' \in St'$ s.t. $t' \to' s' \wedge s \leftrightsquigarrow s'$, for this argument we instead consider some $s' \in St'$ s.t. $t' \to' s'$, and then use reasoning identical to that above to conclude that there must exist an $s \in St$ s.t. $t \to s \wedge s' \leftrightsquigarrow s$ for all such s'. This satisfies condition *(c)* of the definition of a bisimulation.

Therefore, we have determined that the relation $\leftrightsquigarrow$ — defined as the relation of being CTL-equivalent — satisfies the definition of being a bisimulation. In other words, if $t \in St$ is CTL-equivalent to $t' \in St'$, then (M,t) and (M',t') are bisimilar.

Question 8

From the result in Question 7, if (M,t) and (M',t') satisfy the same CTL formulas, then they are bisimilar. On the other hand, if they are bisimilar, they satisfy the same CTL* formulas, according to the result in Question 6. Therefore: if they satisfy the same CTL formulas, then they will satisfy the same CTL* formulas.

The converse — if they satisfy the same CTL* formulas, then they must satisfy the same CTL formulas — follows from the fact that CTL is a subset of CTL*, in other words, any CTL formula is also a CTL* formula (established in Questions 3 and 4).

It does not follow from this that, given a CTL* formula and the set of $(model, state)$ pairs that satisfy it, there will be a corresponding CTL formula satisfied by the same set - the notions are orthogonal. In fact, in Question 5 it was shown that there exists some CTL* formula for which no such corresponding CTL formula exists. Therefore, there is no contradiction: for such a CTL* formula and its set of $(model, state)$ pairs that satisfy it, the property defined as "being an element in this set" can be expressed in CTL*, but not in CTL. Thus, CTL* is more expressive than CTL.