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Brief paper

# Distributed convergence to Nash equilibria in network and average aggregative games ☆

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ABSTRACT

We consider network aggregative games where each player minimizes a cost function that depends on its own strategy and on a convex combination of the strategies of its neighbors. As a first contribution, we propose a class of distributed algorithms that can be used to steer the strategies of the rational agents to a Nash equilibrium configuration, with guaranteed convergence under different sufficient conditions depending on the cost functions and on the network. A distinctive feature of the proposed class of algorithms is that agents use optimal responses instead of gradient type of strategy updates. As a second contribution, we show that the algorithm suggested for network aggregative games can also be used to recover a Nash equilibrium of average aggregative games (i.e., games where each agent is affected by the average of the strategies of the whole population) in a distributed fashion, that is, without requiring a central coordinator. We apply our theoretical results to multi-dimensional, convex-constrained opinion dynamics and to demand-response schemes for energy management.



## 1. Introduction

Aggregative games (Jensen, 2010) have been suggested to model applications in which the well-being of an agent is not the result of its individual two-player interactions, as in traditional games, but rather depends on the *aggregate* behavior of a (possibly agent dependent) subset of the population. In this paper we consider two specific types of aggregative games: network aggregative games (NAGs) and average aggregative games (AAGs). The main difference between the two frameworks is that in NAGs the aggregate is agent dependent and is obtained as the convex combination of the strategies of each agent's neighbors as given by an underlying network while in AAGs the aggregate is the same for each agent and coincides with the average of the strategies of the whole population. In both cases, our objective is to devise distributed algorithms that steer the agents' strategies to an (almost) Nash equilibrium by using local communications. Note that we consider cases in which agents may be heterogeneous in their cost functions and constraint sets, and have no information regarding the settings of the rest of the agents. Consequently, coordination needs to be achieved by iterative (local) communications.

The topic of distributed coordination towards a Nash equilibrium has received a lot of attention in the past years. For example (Kannan & Shanbhag, 2012; Yi & Pavel, 2019b; Yin, Shanbhag, & Mehta, 2011) suggest algorithms where any specific agent is allowed to communicate only with the agents that affect its cost function. We note that in the AAGs considered here the cost function of each agent is affected by the average strategy of all the agents. Hence, these schemes can be applied to AAGs, but would require in this context communications among all the agents. Distributed algorithms that use local communications as in the setting of this paper, have been suggested for example in Pavel (2018, 2019), Salehisadaghiani and Pavel (2016), Salehisadaghiani, Shi, and Pavel (2019) and Yi and Pavel (2019a) for general games and in Koshal, Nedić, and Shanbhag (2016), Liang, Yi, and Hong (2017) and Ye and Hu (2017) for AAGs. The distributed algorithms suggested in these works assume that agents update their strategies by performing a gradient step. On the contrary, in this work we assume that agents update their strategies by using an "optimal response", that is, by choosing

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at each iteration the strategy that minimizes their cost function given the current neighbors' aggregate. This type of update is motivated by applications where the iterations are performed over time and consequently agents are interested in minimizing their cost at each iteration. The simplest iteration satisfying the type of updates considered in this work is the so called best response (BR) dynamics. It is known that if the game is potential then *sequential* BR dynamics converge under fairly general assumptions Monderer and Shapley (1996). This is however not the case in our setup. In fact we consider games that are *not necessarily potential* and moreover we consider *simultaneous* updates (since we study large population games). To address this scenario, we introduce a class of distributed algorithms[1] and we prove convergence for cases where the BR dynamics fail to converge. Our key idea is to extend our previous work (Grammatico, Parise, Colombino, & Lygeros, 2016), that is tailored to AAGs and relies on the presence of a central operator, in two separate directions:

(T1) we study coordination to the Nash equilibrium of a NAG instead of an AAG by allowing agents to communicate with their neighbors;
(T2) we study coordination to an $\varepsilon$-Nash equilibrium of an AAG by using local communications instead of relying on a central operator.

To achieve such tasks, we suggest a class of distributed algorithms that depend on an integer parameter $\nu \in \mathbb{N}$, which represents the number of communications performed by the agents between two strategy updates, and on an update mapping $\Phi_k$. Such class relates to the algorithm proposed in Grammatico et al. (2016), which is a decentralized method for AAGs with $N$ agents and $n$ dimensional strategy vectors. Convergence of the algorithm suggested in Grammatico et al. (2016) is proven by showing that a certain aggregation mapping (corresponding to an update of the *global aggregate* broadcast by the central operator) possesses suitable regularity properties in $\mathbb{R}^n$. We here prove convergence by showing regularity properties of an *extended aggregation mapping* - which lives in $\mathbb{R}^{Nn}$. The extension from $\mathbb{R}^n$ to $\mathbb{R}^{Nn}$ is due to the fact that, since we here consider a distributed instead of a decentralized problem, to address (T1) and (T2) each agent needs to maintain a *different local aggregate*. As main technical contribution we derive sufficient conditions in terms of $\nu$, $\Phi_k$, the cost functions and the network to guarantee that this *extended mapping* possesses suitable regularity properties, which in turn guarantee convergence of the suggested algorithm. We then show in Corollary 1 that task (T1) above can be achieved by using the proposed algorithm with $\nu = 1$ and in Corollary 2 that task (T2) can be solved by using the proposed algorithm with $\nu \gg 1$ (the main idea being that – under suitable assumptions – each agent can distributedly recover the average of the other agents' strategies by communicating a sufficiently large number of times as shown by standard consensus theory results). We employ these two results in Section 5.1 to study a new model of opinion dynamics that, differently from the literature, allows for interdependencies among different topics, and in Section 5.2 to derive a hierarchical scheme guaranteeing convergence in a widespread model of demand-response for smart energy management under less stringent assumptions than in the previous literature (Grammatico et al., 2016; Ma, Callaway, & Hiskens, 2013; Ma, Hu, & Spanos, 2014). The proofs are in the Appendix.

*Notation:* $\mathbb{Z}[a,b] := [a,b] \cap \mathbb{Z}$. $\mathbb{1}_n$ denotes the vector of all 1. For $Q \in \mathbb{R}^{n\times n}$, $Q \succcurlyeq (\succ) 0$ if $Q = Q^\top$ and $x^\top Q x \geq (>) 0$ for all $x$. If $Q \succcurlyeq 0$, $Q^{\frac{1}{2}} \succcurlyeq 0$ denotes its principal square root. For $Q \succ 0$, $\mathcal{H}_Q$ is the Hilbert space with product $\langle x, y\rangle_Q := x^\top Q y$ and norm $\|\cdot\|_Q$; for simplicity we set $\|x\| := \|x\|_{I_n}$. $A \otimes B$ is the Kronecker product. For any $\nu \in \mathbb{N}$, $P^\nu_{ij}$ denotes the element $(i,j)$ of $P^\nu$. $P$ is row-stochastic if $\sum_{j=1}^N P_{ij} = 1\ \forall i$; doubly-stochastic if $P$ and $P^\top$ are row-stochastic; primitive if there exists $h \in \mathbb{N}$ such that $P^h$ is element-wise strictly positive. $\|P\|_Q$ is the matrix norm induced by $\|\cdot\|_Q$.

[1] Parametrized by the number of communications between two strategy updates and by the mapping used for the update of the local aggregates.

## 2. The game setup

### 2.1. Network and average aggregative games

We consider a population of $N \in \mathbb{N}$ heterogeneous agents, where each agent $i \in \mathbb{Z}[1,N]$ controls a decision variable $x^i$, taking values in the set $\mathcal{X}^i \subset \mathbb{R}^n$, and interacts with the other agents via a directed network. We specify the network by the weighted adjacency matrix $P \in \mathbb{R}^{N\times N}$, whose element $P_{ij} \in [0,1]$ denotes the strength (or relevance) of the communication from agent $j$ to agent $i$, $P_{ij} = 0$ implying no communication. The aim of agent $i$ is to minimize its individual deterministic cost $J^i\left(x^i, \sigma^i(x)\right)$ that depends on its own strategy $x^i$ and on an aggregate quantity $\sigma^i(x) := \sum_{j=1}^N P_{ij} x^j$. Formally, each agent $i \in \mathbb{Z}[1,N]$ aims at computing its *best response* (BR) to the neighbors' aggregate state $\sigma^i$

$$\begin{aligned} x^i_{\mathrm{br}}(\sigma^i) &:= \arg\min_{x^i \in \mathcal{X}^i} J^i\left(x^i, \sigma^i(x)\right) \\ &= \arg\min_{x^i \in \mathcal{X}^i} J^i\left(x^i, P_{ii}x^i + \textstyle\sum_{j\neq i}^N P_{ij} x^j\right). \end{aligned} \tag{1}$$

Since the neighbors' aggregate state $\sigma^i(x)$ is different for each agent and is specified by the network, we refer to this problem as a *network aggregative game* (NAG). By selecting as $P$ the complete network (i.e. $P = \frac{1}{N}\mathbb{1}_N\mathbb{1}_N^\top$) we obtain the special class of *average aggregative games* (AAGs) for which $\sigma^i(x) \equiv \frac{1}{N}\sum_{j=1}^N x^j =: \bar\sigma(x)$ and

$$x^i_{\mathrm{br}}(\bar\sigma) := \arg\min_{x^i \in \mathcal{X}^i} J^i\left(x^i, \tfrac{1}{N}x^i + \tfrac{1}{N}\textstyle\sum_{j\neq i}^N x^j\right). \tag{2}$$

A set of strategies in which every agent is playing the BR to the other players' strategies is a Nash equilibrium.

**Definition 1** (*Nash Equilibrium*). A set of strategies $\left\{\bar x^i\right\}_{i=1}^N$ is an $\varepsilon$-Nash equilibrium for (1) if, for all $i \in \mathbb{Z}[1,N]$, $\bar x^i \in \mathcal{X}^i$ and $J^i\left(\bar x^i, \sum_{j=1}^N P_{ij}\bar x^j\right) \leq \min_{x^i \in \mathcal{X}^i} J^i\left(x^i, P_{ii}x^i + \sum_{j\neq i}^N P_{ij}\bar x^j\right) + \varepsilon$. If this holds for $\varepsilon = 0$ then $\left\{\bar x^i\right\}_{i=1}^N$ is a Nash equilibrium.

We consider games satisfying the following.

**Assumption 1** (*Game Setting*). Each agent $i \in \mathbb{Z}[1,N]$ is subject to a personalized convex and compact constraint set $\mathcal{X}^i \subset \mathbb{R}^n$. The weighted adjacency matrix $P \in \mathbb{R}^{N\times N}$ is *row-stochastic*. The minimizer in (1) is unique. The cost functions $J^i\left(x^i, z^i\right)$ are uniformly Lipschitz in the second argument $z^i$ with constant $L_J$, that is, for all $i \in \mathbb{Z}[1,N]$ it holds $|J^i\left(x^i, z^i_A\right) - J^i\left(x^i, z^i_B\right)| \leq L_J \|z^i_A - z^i_B\|$ for all $z^i_A, z^i_B$.

The assumption above is satisfied e.g. if $J^i\left(x^i, z^i\right) = \upsilon^i(x^i) + p(z^i)^\top x^i$, for some $\upsilon^i$ strongly convex and $p$ Lipschitz in $z^i$, as typical of demand response games (see Remark 2).

### 2.2. Optimal responses and aggregation mappings

We define the *optimal response* of agent $i$ to a fixed reference $z^i \in \mathbb{R}^n$, according to the following formula

$$x^{i\star}(z^i) := \arg\min_{x^i \in \mathcal{X}^i} J^i(x^i, z^i). \tag{3}$$

**Remark 1.** Even if $z^i = \sigma^i(x)$ the optimal response differs from the BR since the optimization in (3) is performed only with respect to the first argument of $J^i(\cdot,\cdot)$. This difference vanishes if the influence of $x^i$ in $\sigma^i(x)$ is null (i.e. if $P_{ii} = 0$ for all $i$) in which case $x^{i\star}(\sigma^i) = x^i_{\mathrm{br}}(\sigma^i)$.

In the rest of the paper we make the following regularity assumption, as by Definition 2 in Appendix A.

**Assumption 2** (*Regularity of Optimal Responses*)**.** The minimizer in (3) is unique. The mappings $\{x^{i\star}\}_{i=1}^N$ are uniformly Lipschitz with constant $L_x$ (i.e., $\|x^{i\star}(z^i_A) - x^{i\star}(z^i_B)\| \le L_x\|z^i_A - z^i_B\|\ \forall i, z^i_A, z^i_B$). Moreover, there exists $S \in \mathbb{R}^{n\times n}$, $S \succ 0$, such that at least one of the following statements holds in $\mathcal{H}_S$:

(A2.a) $x^{i\star}$ is a *contraction* for all $i \in \mathbb{Z}[1,N]$;
(A2.b) $x^{i\star}$ is *non-expansive* for all $i \in \mathbb{Z}[1,N]$;
(A2.c) $x^{i\star}$ is *firmly non-expansive* for all $i \in \mathbb{Z}[1,N]$;
(A2.d) $-x^{i\star}$ is *monotone* for all $i \in \mathbb{Z}[1,N]$.

**Remark 2.** Sufficient conditions to satisfy Assumption 2 in the case of agents with quadratic cost functions are given in Grammatico et al. (2016). An example of an AAG satisfying (A2.a-b) with $S = I_n$ is given in Ma et al. (2013), in the context of electric vehicle charging with nonlinear price functions. Examples of NAGs satisfying (A2.a) with $S = I_n$ are discussed in Bramoullé and Kranton (2016). Therein it is shown that such games are potential if a condition related to the symmetry of the network is met.

Let $\boldsymbol{z} := \left[z^1; \dots; z^N\right] \in \mathbb{R}^{Nn}$ be a vector of (possibly different) reference vectors for each agent and define the mapping $\boldsymbol{x}^\star : \mathbb{R}^{Nn} \to \mathcal{X}_{1\times N}$ as

$$\boldsymbol{x}^\star(\boldsymbol{z}) := \left[x^{1\star}(z^1); \dots; x^{N\star}(z^N)\right] \in \mathbb{R}^{Nn}, \tag{4}$$

whose $i$th component is the optimal strategy computed by agent $i$ in response to $z^i$. The mapping $\boldsymbol{x}^\star$ in (4) can be used to define an *aggregation mapping* $\boldsymbol{\mathcal{A}}_1$ that, given a reference vector $\boldsymbol{z}$, returns the updated estimates of the neighbors' aggregate states, after one optimization and one communication step.[2] Formally, $\boldsymbol{\mathcal{A}}_1 : \mathbb{R}^{Nn} \to (P \otimes I_n)\mathcal{X}_{1\times N} \subset \mathbb{R}^{Nn}$ and

$$\begin{aligned}\boldsymbol{\mathcal{A}}_1(\boldsymbol{z}) &:= \begin{bmatrix} \mathcal{A}_1^1(\boldsymbol{z}) \\ \vdots \\ \mathcal{A}_1^N(\boldsymbol{z}) \end{bmatrix} := \begin{bmatrix} \sum_{j=1}^N P_{1j} x^{j\star}(z^j) \\ \vdots \\ \sum_{j=1}^N P_{Nj} x^{j\star}(z^j) \end{bmatrix} \\ &= (P \otimes I_n)\boldsymbol{x}^\star(\boldsymbol{z}) =: \boldsymbol{\mathcal{P}}_1 \boldsymbol{x}^\star(\boldsymbol{z}).\end{aligned} \tag{5}$$

In Section 4.1 we show that fixed points of $\mathcal{A}_1(\boldsymbol{z})$ are related to Nash equilibria of the corresponding NAG, thus justifying our interest in such mapping for solving task (T1). To develop distributed algorithms for AAGs and solve task (T2) we instead make use of a generalization of $\boldsymbol{\mathcal{A}}_1$, which we term the *multi-communication aggregation mapping* $\boldsymbol{\mathcal{A}}_\nu$. This map returns the updated estimates of the neighbors' aggregate states, after one optimization and $\nu$ (instead of one) communication steps, $\boldsymbol{\mathcal{A}}_\nu : \mathbb{R}^{Nn} \to (P^\nu \otimes I_n)\mathcal{X}_{1\times N} \subset \mathbb{R}^{Nn}$

$$\boldsymbol{\mathcal{A}}_\nu(\boldsymbol{z}) := (P^\nu \otimes I_n)\boldsymbol{x}^\star(\boldsymbol{z}) =: \boldsymbol{\mathcal{P}}_\nu \boldsymbol{x}^\star(\boldsymbol{z}). \tag{6}$$

We show in Section 4.2 that, for suitable choices of $P$, fixed points of $\mathcal{A}_\nu(\boldsymbol{z})$ are related to $\varepsilon$-Nash equilibria of the corresponding AAG (with $\varepsilon$ arbitrarily small for $N$, $\nu$ sufficiently large, as specified in the proof of Proposition 2). Before presenting the exact relation between the Nash equilibria of NAGs/AAGs and the fixed points of $\mathcal{A}_1(\boldsymbol{z})/\mathcal{A}_\nu(\boldsymbol{z})$, we present in the next section a class of distributed algorithms that guarantees convergence to such fixed points. In Section 4 we then show how such algorithms can be used to solve (T1) and (T2).

[2] Note that this aggregation mapping lives in $\mathbb{R}^{Nn}$ and can be seen as an extension of the aggregation mapping considered in Grammatico et al. (2016), which lives in $\mathbb{R}^n$, for cases where each agent may have a *different* local aggregate.

## 3. A parametric class of distributed algorithms

One of the main challenges in aggregative games, and games in general, is to characterize the evolution of the players' strategies when the game is repeated iteratively. For this problem to be well defined one has to specify the update rule used by each agent $i$, at iteration $k$, to select its updated strategy $x^i_{(k+1)}$ in response to the strategies of the other players. One of the most natural type of update is obtained when agents select as next strategy the optimal response to the previous neighbors' aggregate leading to the dynamics detailed in Algorithm 1.

**Algorithm 1**. Optimal response dynamics in NAGs

**Initialization**. Set $k = 0$, set $z^i_{(0)} = \sigma^i(x_{(0)}) \in \mathbb{R}^n\ \forall i$.

**Iterate until convergence**. Each agent $i$:

O) Computes its optimal strategy with respect to $z^i_{(k)}$
$\quad x^i_{(k+1)} \leftarrow x^{i\star}(z^i_{(k)}) = \operatorname{argmin}_{x^i \in \mathcal{X}^i} J^i(x^i, z^i_{(k)})$

C) Communicates once with the neighbors
$\quad \mathcal{A}^i_1 \leftarrow \sum_{j=1}^N P_{ij} x^j_{(k+1)}$

U) Updates the reference
$\quad z^i_{(k+1)} \leftarrow \mathcal{A}^i_1$

Note that if $P_{ii} = 0$ for all $i$ Algorithm 1 coincides with the BR dynamics, as noted in Remark 1. In this paper we suggest a new class of update rules, summarized in Algorithm 2, which generalizes Algorithm 1 by (i) allowing agents to communicate $\nu$ times instead of once per each iteration (more in detail, $\nu_1$ times before the strategy update leading to the local a-priori aggregate $\mathcal{A}^i_{\nu_1,0}(\boldsymbol{z}_{(k)})$ and $\nu_2$ times after the strategy update leading to the a-posteriori aggregate $\mathcal{A}^i_{\nu_1,\nu_2}(\boldsymbol{z}_{(k)})$) and (ii) introducing an "memory term" in the update of $z^i_{(k+1)}$ via the mapping $\Phi_k : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. Note that a single iteration of Algorithm 2 updates the reference $\boldsymbol{z}_{(k)} = [z^1_{(k)};\ \dots; z^N_{(k)}]$ as $\boldsymbol{z}_{(k+1)} = \boldsymbol{\Phi}_k\left(\boldsymbol{z}_{(k)}, \boldsymbol{\mathcal{A}}_{\nu_1,\nu_2}\left(\boldsymbol{z}_{(k)}\right)\right)$ where

$$\boldsymbol{\mathcal{A}}_{\nu_1,\nu_2}(\boldsymbol{z}) := \boldsymbol{\mathcal{P}}_{\nu_2}\boldsymbol{x}^\star(\boldsymbol{\mathcal{P}}_{\nu_1}\boldsymbol{z}) \in (P^{\nu_2} \otimes I_n)\mathcal{X}_{1\times N}, \tag{7}$$

and for $\boldsymbol{z} = [z^1;\ \dots; z^N]$, $\boldsymbol{a} = [a^1;\ \dots; a^N]$ we set $\boldsymbol{\Phi}_k(\boldsymbol{z}, \boldsymbol{a}) := [\Phi_k(z^1, a^1);\ \dots; \Phi_k(z^N, a^N)]$.

### 3.1. Settings for Algorithm 2

Algorithm 2 describes a class of learning dynamics, parameterized by $\nu_1, \nu_2$ and by the mappings $\Phi_k$. The simplest dynamics is obtained by using as mapping the Picard–Banach iteration (Berinde, 2007, Theorem 2.1), so that,

$$\boldsymbol{z}_{(k+1)} = \boldsymbol{\Phi}^{\mathrm{P-B}}\left(\boldsymbol{z}_{(k)}, \boldsymbol{\mathcal{A}}_{\nu_1,\nu_2}\left(\boldsymbol{z}_{(k)}\right)\right) := \boldsymbol{\mathcal{A}}_{\nu_1,\nu_2}\left(\boldsymbol{z}_{(k)}\right). \tag{8}$$

Note that if we set $(\nu_1, \nu_2) = (0, 1)$ then under the Picard–Banach iteration Algorithm 2 coincides with Algorithm 1. To ensure convergence under less stringent assumptions, we suggest the use of more general fixed point iterations such as the Krasnoselskij iteration

$$\boldsymbol{\Phi}^{\mathrm{K}}(\boldsymbol{z}_{(k)}, \boldsymbol{\mathcal{A}}_{\nu_1,\nu_2}\left(\boldsymbol{z}_{(k)}\right)) := (1-\lambda)\boldsymbol{z}_{(k)} + \lambda\boldsymbol{\mathcal{A}}_{\nu_1,\nu_2}\left(\boldsymbol{z}_{(k)}\right) \tag{9}$$

with $\lambda \in (0, 1)$ as detailed in (Berinde, 2007, Theorem 3.2), and the step-dependent Mann iteration (Berinde, 2007, Theorem 4.11)

$$\boldsymbol{\Phi}^{\mathrm{M}}_k(\boldsymbol{z}_{(k)}, \boldsymbol{\mathcal{A}}_{\nu_1,\nu_2}(\boldsymbol{z}_{(k)})) := (1-\alpha_k)\boldsymbol{z}_{(k)} + \alpha_k\boldsymbol{\mathcal{A}}_{\nu_1,\nu_2}(\boldsymbol{z}_{(k)}), \tag{10}$$

**Algorithm 2**. Distributed learning dynamics in NAGs

**Initialization**. Set $k = 0$, choose $z^i_{(0)} \in \mathbb{R}^n$ $\forall i$, the mappings $\{\Phi_k\}_{k=1}^{\infty}$ and $\nu_1, \nu_2 \in \mathbb{Z}_{\geq 0}$, s.t. $\nu_1 + \nu_2 = \nu$.

**Iterate until convergence**. Each agent $i$:

C$_1$) Communicates $\nu_1$ times

$\mathcal{A}^i_{0,0} \leftarrow z^i_{(k)}$

for $s = 0$ to $s = \nu_1 - 1$ do

$\mathcal{A}^i_{s+1,0} \leftarrow \sum_{j=1}^N P_{ij}\mathcal{A}^j_{s,0}$

end

O) Computes its optimal strategy with respect to $\mathcal{A}^i_{\nu_1,0}$

$x^i_{(k+1)} \leftarrow x^{i\star}(\mathcal{A}^i_{\nu_1,0}) = \operatorname{argmin}_{x^i\in\mathcal{X}^i} J^i(x^i, \mathcal{A}^i_{\nu_1,0})$

C$_2$) Communicates $\nu_2$ times

$\mathcal{A}^i_{\nu_1,0} \leftarrow x^i_{(k+1)}$

for $s = 0$ to $s = \nu_2 - 1$ do

$\mathcal{A}^i_{\nu_1,s+1} \leftarrow \sum_{j=1}^N P_{ij}\mathcal{A}^j_{\nu_1,s}$

end

U) Updates the reference

$z^i_{(k+1)} \leftarrow \Phi_k\left(z^i_{(k)}, \mathcal{A}^i_{\nu_1,\nu_2}\right)$

where the sequence $(\alpha_k)_{k=1}^{\infty}$ is such that $\alpha_k \in (0,1)$ $\forall k \geq 0$, $\lim_{k\to\infty}\alpha_k = 0$ and $\sum_{k=1}^{\infty}\alpha_k = \infty$ (e.g., $\alpha_k = 1/k$).

The main difference of $\Phi^{\mathrm{K}}$, $\Phi^{\mathrm{M}}_k$ from $\Phi^{\mathrm{P\text{-}B}}$ is the introduction of a memory term so that the update of the reference $z^i_{(k+1)}$ does not depend only on the current a-posteriori aggregate $\mathcal{A}^i_{\nu_1,\nu_2}(\boldsymbol{z}_{(k)})$, but also on the value $z^i_{(k)}$ of the reference at the previous iteration. From a mathematical perspective we show that this additional term facilitates convergence. From a descriptive perspective this memory term can be understood as agents trying to avoid sudden changes in their reference update.

### 3.2. Conditions for convergence of Algorithm 2

We next provide conditions under which the sequence of vectors used to compute the optimal responses, i.e.

$$\left(\left[\mathcal{A}^1_{\nu_1,0}(\boldsymbol{z}_{(k)});\ \ldots;\ \mathcal{A}^N_{\nu_1,0}(\boldsymbol{z}_{(k)})\right]\right)_{k=1}^{\infty}, \tag{11}$$

in Algorithm 2 converges, as $k$ tends to infinity, to a fixed point of the aggregation mapping $\boldsymbol{\mathcal{A}}_\nu$ in (6). This result is used in the next section to prove convergence of Algorithm 2 to a Nash equilibrium for NAGs (T1) and to an $\varepsilon$-Nash equilibrium for AAGs (T2).

**Theorem 1** (*Global Convergence of Algorithm 2*). *Under Assumption 1, the following conditions guarantee that the sequence in (11) converges, for any initial reference $\boldsymbol{z}_{(0)} \in \mathbb{R}^{Nn}$, to a fixed point of $\boldsymbol{\mathcal{A}}_\nu$ in (6).*

| | *Mapping* | | $(\nu_1, \nu_2)$ | *Regularity* | *Network* |
|---|---|---|---|---|---|
| *1.* | $\Phi^{P\text{-}B}$ | *In (8)* | $(0, \nu)$ | *(A2.a)* | $\|P\| \leq 1$ |
| *2.* | $\Phi^{K}$ | *In (9)* | $(0, \nu)$ | *(A2.b)* | $\|P\| \leq 1$ |
| *3.* | $\Phi^{P\text{-}B}$ | *In (8)* | $(\frac{\nu}{2}, \frac{\nu}{2})$ | *(A2.c)* | $P = P^\top$ |
| *4.* | $\Phi^M_k$, $\Phi^{K*}$ | *In (10)* | $(\frac{\nu}{2}, \frac{\nu}{2})$ | *(A2.d)* | $P = P^\top$ |

*$^*$with $\lambda \in \left(0, \frac{2}{1+\mu^2}\right)$, where $\mu$ is the Lipschitz constant of $\boldsymbol{\mathcal{A}}_{\frac{\nu}{2},\frac{\nu}{2}}(\boldsymbol{z})$.*

*The mapping $\boldsymbol{\mathcal{A}}_\nu$ has a unique fixed point in case 1.*[3] □

[3] The condition $\|P\| \leq 1$ is satisfied by any doubly stochastic matrix (by Hölder's inequality). Under (A2.a), using $\Phi^{\mathrm{P\text{-}B}}$, and (A2.d), using $\Phi^{\mathrm{K}}$, Algorithm 2 converges exponentially fast (Berinde, 2007, Theorems 2.1 and 3.6).

## 4. Distributed convergence to Nash equilibria

### 4.1. Network aggregative games

The next proposition shows that the fixed points of the aggregation mapping $\boldsymbol{\mathcal{A}}_1$ can be used to find a Nash equilibrium of a NAG game, for any population size $N$ and for any given network $P$ (with no self loops).

**Proposition 1** (*NAG Nash Equilibria*). *Under Assumptions 1 and 2 the mapping $\boldsymbol{\mathcal{A}}_1$ in (5) admits at least one fixed point. If $P_{ii} = 0$ for all $i \in \mathbb{Z}[1, N]$ and $\bar{\boldsymbol{z}} = \boldsymbol{\mathcal{A}}_1(\bar{\boldsymbol{z}})$ is a fixed point of $\boldsymbol{\mathcal{A}}_1$, then the set of strategies $\left\{x^{i\star}\left(\bar{z}^i\right)\right\}_{i=1}^N$, with $x^{i\star}$ as in (3) for all $i \in \mathbb{Z}[1, N]$, is a Nash equilibrium for (1).* □

Proposition 1 together with Theorem 1 immediately leads to the following result.

**Corollary 1.** *Suppose that Assumption 1 holds. Then under conditions 1 and 2 of Theorem 1, Algorithm 2 converges, from any initial configuration, to a Nash equilibrium for the NAG in (1) upon choosing $\nu = 1$.*

Note that Corollary 1 implicitly ensures the existence of at least one Nash equilibrium for all NAGs satisfying Assumptions 1 and 2. Moreover, by Remark 1, Corollary 1 guarantees convergence of the simultaneous BR dynamics for NAGs that satisfy condition 1. If the less restrictive condition 2 is met, convergence can be obtained by using the update with memory given in (9).

### 4.2. Average aggregative games

The result in the previous subsection cannot be directly applied to AAGs since the complete network: (i) requires communications among all the agents and (ii) has self loops of weight $\frac{1}{N}$. To overcome these issues we suggest the use of Algorithm 2 over a generic network $P$ (which is in this case a design parameter and will typically be sparse) instead of the complete network and we derive asymptotic results in terms of the population size $N$ (so that the weight of the self loops becomes negligible). To this end, we start by assuming that $\mathcal{X}^i$, $L_{\mathrm{J}}$ and $L_{\mathrm{x}}$ are uniformly bounded over $N$.

**Assumption 3** (*Uniform Compactness*). There exist a compact set $\mathcal{X} \subset \mathbb{R}^n$ and $\bar{L}_{\mathrm{J}}$, $\bar{L}_{\mathrm{x}}$ such that, $\mathcal{X}^i(N) \subseteq \mathcal{X}$, $L_J(N) \leq \bar{L}_J$ and $L_x(N) \leq \bar{L}_x$ for all population sizes $N$.

To compensate for the fact that we use a network $P$ instead of the complete network we allow agents to communicate a large number of times $\nu \gg 1$ (instead of once) and we require the following assumption on $P$.

**Assumption 4** (*Asymptotic Average Consensus*). For all population sizes $N$, the weighted adjacency matrix $P = P_N$ satisfies $\lim_{\nu\to\infty} P^\nu = \frac{1}{N}\mathbb{1}_N\mathbb{1}_N^\top$. Equivalently, $P$ is primitive and doubly stochastic (Olfati-Saber, Fax, & Murray, 2007).

In other words, the network $P$ should be chosen such that, by iteratively communicating, the agents asymptotically reach consensus on the population average. Under these assumptions, we guarantee convergence to an $\varepsilon$-Nash equilibrium (instead of an exact equilibrium) where $\varepsilon$ is a function of both $\nu$ and $N$.

**Proposition 2** (*AAG Nash Equilibria*). *Suppose that Assumptions 1–4 hold. The mapping $\boldsymbol{\mathcal{A}}_\nu$ in (6) admits at least one fixed point. Moreover, for all $\varepsilon > 0$ there exists $\bar{N}$ such that: for all $N > \bar{N}$, there exists $\bar{\nu} > 0$ such that, for all $\nu \geq \bar{\nu}$, if $\bar{\boldsymbol{z}}$ is a fixed point of $\boldsymbol{\mathcal{A}}_\nu$ in (6), then the set $\left\{x^{i\star}\left(\bar{z}^i\right)\right\}_{i=1}^N$, with $x^{i\star}$ as in (3), is an $\varepsilon$-Nash equilibrium for (2).* □

| | | | $(\nu_1, \nu_2)$ | *Cost* $(\forall i)$ | *Network* |
|---|---|---|---|---|---|
| *1.* | $\Phi^{P-B}$ | *In* (8) | $(0, 1)$ | $\theta_i > 0$ | $\|P\| \leq 1$ |
| *2.* | $\Phi^{K}$ | *In* (9) | $(0, 1)$ | | $\|P\| \leq 1$ □ |

In words, if $\theta_i > 0$ for all $i \in \mathbb{Z}[1, N]$ and $\|P\| \leq 1$, the BR dynamics converge to the unique Nash equilibrium. If instead there exists $i$ such that $\theta_i = 0$, then a Nash equilibrium can be reached by using the update rule with memory.

To investigate the performance of the two schemes we consider a case study where each agent $i$ has $n = 2$ opinions, regarding two different topics, taking values in $\mathcal{X}^i := \{[x_1, x_2]^\top \mid \|x_1 - x_2\|^2 \leq 0.3\}$ and is either a follower or stubborn with $\theta_i = 1$. Fig. 1 shows that the convergence speed depends only mildly on the population size, suggesting that our approach is scalable.

Proposition 2 together with Theorem 1 immediately leads to the following result.

**Corollary 2.** *Suppose that Assumptions 1–4 hold. Under the conditions of Theorem 1, Algorithm 2 converges to an $\varepsilon$-Nash equilibrium for the AAG in (2) for all $N > \bar{N}$ and $\nu \geq \bar{\nu}$, as defined in Proposition 2.* □

We note that for any desired $\varepsilon > 0$, the proof of Proposition 2 allows one to derive lower bounds on $N$ and $\nu$ ensuring that $\{x^{i\star}(\bar{z}^i)\}_{i=1}^N$ is an $\varepsilon$-Nash equilibrium. Moreover, the minimum number of required communications $\bar{\nu}$ can be computed in a distributed fashion and for symmetric networks the dependence of $\varepsilon$ on $\nu$ can be further specified in terms of the spectral properties of $P$, as detailed in Parise, Grammatico, Gentile, and Lygeros (2015). Finally, conditions 3 and 4 of Theorem 1 require $\nu$ to be even. This is not a restriction since in task (T2) the network $P$ and the number of communications $\nu$ are design parameters.

## 5. Applications

### *5.1. NAGs: multidimensional opinion dynamics*

Assume that each agent $i \in \mathbb{Z}[1, N]$ has a vector $x^i \in [0, 1]^n$ of opinions regarding $n \in \mathbb{N}$ topics. Each component $x^i_s \in [0, 1]$ represents the opinion of agent $i$ about topic $s \in \mathbb{Z}[1, n]$. We denote by $x^i_{(0)} \in [0, 1]^n$ the initial opinion of agent $i$ and assume that at every iteration $k$ each agent $i$ communicates once ($\nu = 1$) with its neighbors and updates its opinion by computing the BR

$$\begin{aligned} x^i_{\mathrm{br}}\left(\sigma^i\right) &:= \operatorname*{arg\,min}_{x^i \in \mathcal{X}^i} \sum_{j \neq i}^{N} (P_{ij}\|x^i - x^j\|^2) + \theta_i \|x^i - x^i_{(0)}\|^2 \\ &= \operatorname*{arg\,min}_{x^i \in \mathcal{X}^i} \|x^i\|^2 - 2\sigma^i(x)^\top x^i + \theta_i \|x^i - x^i_{(0)}\|^2, \end{aligned} \tag{12}$$

where w.l.o.g. $P_{ii} = 0 \; \forall i$. The cost function in (12) comprises two terms: the first one models the influence of the neighbors to the new opinion of agent $i$, the second one models the "stubbornness" of agent $i$ about its initial opinion. Additional constraints on the agents' opinions across the $n$ topics, as for example the fact that the opinions regarding two topics should not differ more than a given threshold, as well as hard constraints on single topics can be encoded via the constraint set $\mathcal{X}^i \subseteq [0, 1]^n$. The agents are assumed to be heterogeneous in the sense that $\theta_i \geq 0$, $\mathcal{X}^i$ and $\{P_{ij}\}_{j \neq i}^N$ may be different for every agent. We refer to agents for which $\theta_i = 0$ and $\mathcal{X}^i = [0, 1]^n$ as *followers* and to all the remaining ones as *stubborn*.

In the absence of constraints, the solution to (12) for each topic decouples, leading to the BR dynamics

$$x^i_{(k+1)} = x^i_{\mathrm{br}}(\sigma^i_{(k)}) = \tfrac{1}{1+\theta_i} \textstyle\sum_{j \neq i} P_{ij} x^j_{(k)} + \tfrac{\theta_i}{1+\theta_i} x^i_{(0)},$$

which are a particular case of the Friedkin and Johnsen model (Friedkin & Johnsen, 1999), with parameters $\Lambda := \mathrm{diag}\left(\frac{1}{1+\theta_1}, \ldots, \frac{1}{1+\theta_N}\right)$ and $W := P$, and coincide with the DeGroot model if all agents are followers. It is shown in Ghaderi and Srikant (2014) that such decoupled BR dynamics converge to a Nash equilibrium of the NAG with costs as in (12). Proposition 1 and Corollary 1 allow us to extend the analysis to the *multi-dimensional* case with *stubborn agents* and *generic convex constraints*.

**Corollary 3.** *Suppose that Assumption 1 holds. The following iterations and conditions guarantee convergence of the opinions computed according to Algorithm 2, from any initial configuration, to a Nash equilibrium for (12).*

### *5.2. AAGs: a demand-response scheme*

Consider a population of $N \in \mathbb{N}$ loads whose electricity consumption $u^i = \left[u^i_1, \ldots, u^i_T\right] \in \mathbb{R}^T$ over the horizon $\mathcal{T} = \mathbb{Z}[1, T]$ is scheduled according to the following demand-response scheme

$$\begin{aligned} u^{i\star}(\bar{\sigma}) := \operatorname*{arg\,min}_{u^i \in \mathbb{R}^T} \quad & \textstyle\sum_{t \in \mathcal{T}} \left(\rho_i \left\|u^i_t - \hat{u}^i_t\right\|^2 + p(\bar{\sigma}_t) u^i_t\right) \\ \text{s.t.} \quad & s^i_{t+1} = a^i s^i_t + b^i u^i_t \quad \forall t \in \mathcal{T} \\ & [s^i(u^i); u^i] \in (\mathcal{S}^i \times \mathcal{U}^i) \cap \mathcal{C}^i, \end{aligned} \tag{13}$$

where $s^i_t = s^i_t(u^i)$ is the state of the load $i$ at time $t$ (e.g., its temperature in case of thermostatically controlled loads or its state of charge in case of plug-in electric vehicles), $s^i_1 \in \mathbb{R}$ is its given initial state, $a^i, b^i \in \mathbb{R}\backslash\{0\}$ are parameters modeling its dynamics and efficiency, while $\bar{\sigma}_t = \frac{1}{N}\sum_{i=1}^N u^i_t \in \mathbb{R}$ is the average energy demand at time $t$, $\bar{\sigma} := [\bar{\sigma}_1; \ldots; \bar{\sigma}_T] \in \mathbb{R}^T$. The energy consumption $u^i$ and state vector $s^i(u^i)$ are constrained by the sets $\mathcal{U}^i \subset \mathbb{R}^T$ and $\mathcal{S}^i \subset \mathbb{R}^T$ and by the coupling constraint set $\mathcal{C}^i \subset \mathbb{R}^{2T}$. The first term in the cost function of (13) models the curtailment cost that each agent encounters for deviating from its nominal energy schedule $\hat{u}^i_t$, according to the Taguchi loss function (Ma et al., 2014), where $\rho_i > 0$ is a constant weighting parameter. The second term models the demand-response mechanism: the price that each agent has to pay for the required energy varies according to a price function $p(\bar{\sigma}_t)$ that is assumed to be an affine increasing function of the energy demand (Ma et al., 2014, Eq. (15))

$$p(\bar{\sigma}_t) := \lambda \bar{\sigma}_t + p_0, \quad \lambda > 0. \tag{14}$$

Approaches to find a Nash equilibrium via a central coordinator have been proposed in Grammatico et al. (2016) and Ma et al. (2013), Ma et al. (2014). Algorithm 2 provides a distributed alternative.

**Corollary 4.** *Suppose that Assumptions 1, 3 and 4 hold, and let $p(\bar{\sigma}_t)$ be as in (14). The following iterations and conditions guarantee convergence of the strategies computed according to Algorithm 2, from any initial point, to an $\varepsilon_{N,\nu}$-Nash equilibrium for (13).*

| | | | $(\nu_1, \nu_2)$ | *Cost* $(\forall i)$ | *Network* |
|---|---|---|---|---|---|
| *1.* | $\Phi^{P-B}$ | *In* (8) | $(0, \nu)$ | $\rho_i > \lambda/2$ | $\|P\| \leq 1$ |
| *2.* | $\Phi^{K}$ | *In* (9) | $(0, \nu)$ | $\rho_i \geq \lambda/2$ | $\|P\| \leq 1$ |
| *3.* | $\Phi^{M}_k$ | *In* (10) | $(\nu/2, \nu/2)$ | | $P = P^\top$ □ |

The model given in (13) can be used for example to describe demand-response methods for heating ventilation air conditioning (HVAC) systems in smart buildings, as suggested in Ma et al. (2014), by selecting $\rho_i = \theta\gamma_i^2$, where $\theta > 0$ is the cost coefficient of the Taguchi loss function and $\gamma_i > 0$ specifies the thermal

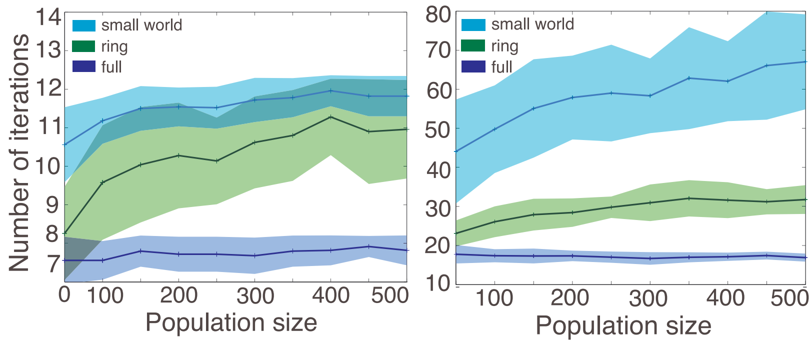


**Fig. 1.** Average number of iterations (solid line) and 90% confidence intervals as functions of the population size $N$ for three different network topologies. The left plot refers to a population where only stubborn agents are present and update their opinion using the BR scheme ($\Phi^{\text{P-B}}$), while the right plot refers to a population composed by half stubborn and half follower agents, using the scheme with memory ($\Phi^{\text{K}}$). In each case, 50 different networks and populations were simulated with $x^i_{(0)} \in \mathcal{U}([0, 1]^2)$. The stopping criterion is $\|\boldsymbol{z}_{(k)} - \boldsymbol{z}_{(k-1)}\|_\infty \le 10^{-5}$.

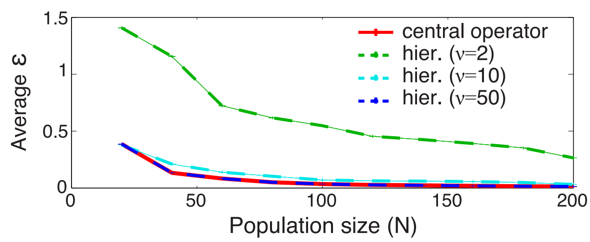


**Fig. 2.** Comparison between the approach with central operator and the distributed hierarchical approach with $M = 5$ and $\nu \in \{2, 10, 50\}$. $P_M$ corresponds to a symmetric ring. The plot shows the average cost improvement that an agent can achieve by unilateral deviations at convergence. We set $\lambda = 2$, $\theta\gamma_i^2 = 0.1$ for all $i$, $T = 24$ hrs and we assume a baseline energy consumption $\sigma_0$ as in Ma et al. (2013, Figure 1). We set the baseline energy price to $p_0 := \lambda\sigma_0$. The average is computed over 10 different populations with $\hat{x}^i$ uniformly sampled in $[0, 1]^T$ and $\mathcal{X}^i := \{x \mid \sum_{t=1}^T x^i_t = \sum_{t=1}^T \hat{x}^i_t, x^i_t = 0 \text{ if } t \notin [T^i_{\text{start}}, T^i_{\text{end}}]\}$, with $T^i_{\text{start}}$ uniformly sampled in $\{1, T\}$, $T^i_{\text{end}}$ uniformly sampled in $\{T^i_{\text{start}} + 1, T\}$. We use $\Phi^{\text{M}}_k$ and $\nu_1 = \nu_2 = \frac{\nu}{2}$. The stopping criterion is $\|\boldsymbol{\mathcal{A}}_\nu(\boldsymbol{\mathcal{A}}_{\nu_1,0}(\boldsymbol{z}_{(k)})) - \boldsymbol{\mathcal{A}}_{\nu_1,0}(\boldsymbol{z}_{(k)})\|_\infty \le 10^{-3}$.

characteristic of the HVAC system. In (Ma et al., 2014, Section V.B) it is shown that, for $N > 3$, if $\gamma_i = \gamma > 0$ for all $i$, $\mathcal{U}^i = [u^i_{\min}, u^i_{\max}] \subset \mathbb{R}^n$ with $u^i_{\min}, u^i_{\max} \in \mathbb{R}^n$, and $\lambda \le \frac{2\theta\gamma^2}{N-3}$, then the Nash equilibrium is unique and can be computed using a control algorithm involving a central coordinator. Corollary 4 proves that, Algorithm 2, can instead be used to find an $\varepsilon$-Nash equilibrium in a distributed fashion, under arbitrary convex constraints $u^i \in \mathcal{U}^i$ and for all values of $\lambda$.

As a particular case, we consider a hierarchical communication structure that groups buildings managed by the same company. For simplicity, we assume that there are $M$ companies and each one manages $B$ buildings, for a total of $N = MB$ buildings. At every communication step the managers compute the aggregate power demand of their buildings, then communicate among each other using a network $P_M \in \mathbb{R}^{M\times M}$ and finally broadcast the price signal to their buildings. Note that the corresponding network $P = P_M \otimes \frac{1}{B}\mathbb{1}_B\mathbb{1}_B^\top$ satisfies Assumption 4, $\|P\| \le 1$ and $P = P^\top$ if and only if $P_M$ does. Fig. 2 illustrates the value of $\varepsilon$ obtained at convergence using Algorithm 2, for several values of $\nu$, in comparison with the value of $\varepsilon$ obtained at convergence using the approach with central operator described in Grammatico et al. (2016). Consistently with our theory $\varepsilon$ becomes smaller for larger populations ($N \to \infty$) and for higher number of communications ($\nu \to \infty$). In fact, this simulation suggests that $\nu = 50$ communications suffice to recover the performance of the scheme with central operator.

We conclude this section by noting that the model given in (13) can also be used to compute the optimal charging strategy for large populations of plug-in electrical vehicles (Grammatico et al., 2016; Ma et al., 2013). The term $\sum_{t\in\mathcal{T}} p(\bar{\sigma}_t)u^i_t$ in that case represents the total price encountered by agent $i$ for selecting the charging profile $u^i$ over a given horizon of time $\mathcal{T}$, while $\sum_{t\in\mathcal{T}} \rho_i \left\|u^i_t - \hat{u}^i_t\right\|^2$ is a regularization term first suggested in Ma et al. (2013). The sets $\mathcal{U}^i$ and $\mathcal{S}^i$ can be used to model constraints on the charging input and on the battery state (such as availability for charging, total charge needed, etc.). Algorithms to reach a $\varepsilon$-Nash equilibrium of such game by using a central coordinator are suggested e.g. in Grammatico et al. (2016) and Ma et al. (2013). Under the same conditions of (Grammatico et al., 2016, Corollary 3), Corollary 4 (point 3) allows one to recover an $\varepsilon$-Nash equilibrium by using a symmetric network $P$ instead of a central coordinator.

## 6. Conclusion and outlook

In this work, we have proposed a new class of distributed algorithms that, under different parameter choices, guarantee convergence for NAGs and AAGs. Our technical results are derived for agents that update their strategies synchronously and over a fixed network. Moreover, in the case of Mann updates, they require the agents to use the same sequence $\alpha_k$. As future work, it would be interesting to study whether similar convergence results can be achieved when the sequence $\alpha_k$ is agent dependent or when asynchronous updates and time-varying or random communications are employed. Additionally, an important assumption of our work is that the agents communicate truthfully with their neighbors. A neighborhood could be defined as the set of players with whom an agent is willing to communicate truthfully. Moreover, in technical applications (as e.g. demand response schemes) one may envision these communications to be performed by an automatic device. Nonetheless, a future research direction is the analysis of the robustness of our scheme when part of the agents are allowed to cheat.

## Appendix A. Proof of Theorem 1

Let $\boldsymbol{a}_{(k)} := \left[\mathcal{A}^1_{\nu_1,0}(\boldsymbol{z}_{(k)}); \ldots; \mathcal{A}^N_{\nu_1,0}(\boldsymbol{z}_{(k)})\right]$ and note that the set $\boldsymbol{\mathcal{P}}_{\nu_2}\mathcal{X}_{1\times N}$ is compact and convex. The main idea behind the proof of Theorem 1 is to derive sufficient conditions guaranteeing that $\boldsymbol{\mathcal{A}}_{\nu_1,\nu_2}$ in (7) possesses one of the following regularity properties (Bauschke & Combettes, 2010).

**Definition 2** (*Regularity Properties*). Fix $S \in \mathbb{R}^{n\times n}$, $S \succ 0$. A mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ is

(1) Strongly monotone (SMON) in $\mathcal{H}_S$ if $\exists\epsilon > 0$ s.t. $(f(x) - f(y))^\top S\,(x - y) \ge \epsilon\,\|x - y\|^2_S$, $\forall x, y \in \mathbb{R}^n$.

(2) Monotone (MON) if $(f(x)-f(y))^\top S\,(x-y)\geq 0$.
(3) Anti-monotone (AMON) if $-f(\cdot)$ is MON.
(4) a Contraction (CON) in $\mathcal{H}_S$ if $\exists\delta\in(0,1]$ s.t. $\|f(x)-f(y)\|_S\leq(1-\delta)\|x-y\|_S$, $\forall x,y\in\mathbb{R}^n$.
(5) Non-Expansive (NE) in $\mathcal{H}_S$ if $\|f(x)-f(y)\|_S\leq\|x-y\|_S$, $\forall x,y\in\mathbb{R}^n$.
(6) Firmly Non-Expansive (FNE) Grammatico et al. (2016, Lemma 5) in $\mathcal{H}_S$ if $\|f(x)-f(y)\|_S^2\leq(x-y)^\top S\,(f(x)-f(y))$, $\forall x,y$.
(7) Strongly Pseudo-Contractive (SPC) in $\mathcal{H}_S$ if $\mathrm{Id}-f$ is SMON in $\mathcal{H}_S$.

**Proof of statements 1 and 2 of Theorem 1.** From Proposition 3 provided below, under the assumption of statement 1 (or 2), $\mathcal{A}_{0,\nu}(\cdot)\equiv\mathcal{A}_\nu(\cdot)$ is a CON (or NE) in $\mathcal{H}_{I_N\otimes S}$. Hence, by using $\Phi^{\text{P–B}}$ (or $\Phi^{\text{K}}$) $z_{(k)}$ converges to a fixed point of the mapping $\mathcal{A}_\nu$ (Berinde, 2007, Thm. 2.1 (or Thm. 3.2)). The proof follows since for $\nu_1=0$ $\boldsymbol{a}_{(k)}:=\mathcal{P}_{\nu_1}\boldsymbol{z}_{(k)}\equiv\boldsymbol{z}_{(k)}$. A CON mapping has a unique fixed point by (Berinde, 2007, Thm. 2.1). ∎

**Proof of statements 3 and 4 of Theorem 1.** From Proposition 3 provided below, under the assumption of statement 3, the mapping $\boldsymbol{z}_{(k)}\mapsto\mathcal{A}_{\frac{\nu}{2},\frac{\nu}{2}}(\boldsymbol{z}_{(k)})$ is FNE. Therefore, by using $\Phi^{\text{P–B}}$, $\boldsymbol{z}_{(k)}$ converges to a fixed point $\bar{\boldsymbol{z}}$ of the mapping $\mathcal{A}_{\frac{\nu}{2},\frac{\nu}{2}}$, (Combettes & Pennanen, 2002, Section 1, p. 522). On the other hand, under the assumption of statement 4, $\boldsymbol{z}_{(k)}\mapsto\mathcal{A}_{\frac{\nu}{2},\frac{\nu}{2}}(\boldsymbol{z}_{(k)})$ is SPC and Lipschitz (by Assumption 2). Consequently, it has a unique fixed point (Berinde, 2007, Thm. 4.11). Convergence can be guaranteed by using $\Phi^{\text{M}}_k$ ((Berinde, 2007, Thm. 4.11)) or, if the Lipschitz constant $\mu$ of $\mathcal{A}_{\frac{\nu}{2},\frac{\nu}{2}}$ is known, by using $\Phi^{\text{K}}$ with $\lambda\in(0,\frac{2}{1+\mu^2})$ (Berinde, 2007, Thm. 3.6, eqs. (11) and (13) pg. 72, with $r:=0$ and $s:=\mu$). In both cases, $\boldsymbol{a}_{(k)}:=\mathcal{P}_{\nu_1}\boldsymbol{z}_{(k)}$ hence, for $\nu_1=\frac{\nu}{2}$, $\boldsymbol{a}_{(k)}$ converges to $\bar{\boldsymbol{a}}:=\mathcal{P}_{\frac{\nu}{2}}\bar{\boldsymbol{z}}$, which is a fixed point of the mapping $\mathcal{A}_\nu$ since $\bar{\boldsymbol{z}}=\mathcal{A}_{\frac{\nu}{2},\frac{\nu}{2}}(\bar{\boldsymbol{z}})\Rightarrow\bar{\boldsymbol{z}}=\mathcal{P}_{\frac{\nu}{2}}\boldsymbol{x}^\star(\mathcal{P}_{\frac{\nu}{2}}(\bar{\boldsymbol{z}}))\Rightarrow\mathcal{P}_{\frac{\nu}{2}}\bar{\boldsymbol{z}}=\mathcal{P}_\nu\boldsymbol{x}^\star(\mathcal{P}_{\frac{\nu}{2}}(\bar{\boldsymbol{z}}))\Rightarrow\bar{\boldsymbol{a}}=\mathcal{P}_\nu\boldsymbol{x}^\star(\bar{\boldsymbol{a}})\Rightarrow\bar{\boldsymbol{a}}=\mathcal{A}_\nu(\bar{\boldsymbol{a}})$. ∎

**Lemma 1.** *If (A2.a)/(A2.b)/(A2.c)/(A2.d) holds, then the mapping $\boldsymbol{x}^\star$ in (4) is a CON/NE/FNE/AMON in $\mathcal{H}_{I_N\otimes S}$, respectively.*

**Proof.** (a) For all $i\in\mathbb{Z}[1,N]$, by (A2.a) the mapping $x^{i\star}$ is a CON in $\mathcal{H}_S$, with some rate $\delta_i\in(0,1]$. Therefore, for any $\boldsymbol{r},\boldsymbol{s}\in\mathbb{R}^{Nn}$ we have $\|\boldsymbol{x}^\star(\boldsymbol{r})-\boldsymbol{x}^\star(\boldsymbol{s})\|^2_{I_N\otimes S}=\|[x^{1\star}(r^1)-x^{1\star}(s^1);\ldots;x^{N\star}(r^N)-x^{N\star}(s^N)]\|^2_{I_N\otimes S}=\|x^{1\star}(r^1)-x^{1\star}(s^1)\|_S^2+\ldots+\|x^{N\star}(r^N)-x^{N\star}(s^N)\|^2_S\leq(1-\delta_1)^2\|r^1-s^1\|^2_S+\ldots+(1-\delta_N)^2\|r^N-s^N\|^2_S\leq(1-\min_{i\in\mathbb{Z}[1,N]}\delta_i)^2\|\boldsymbol{r}-\boldsymbol{s}\|^2_{I_N\otimes S}$. Note that $\delta:=\min_{i\in\mathbb{Z}[1,N]}\delta_i$ is strictly positive since $N$ is finite. (b) As in the previous point, with $\delta_i=0$ for all $i$. (c) If (A2.c) holds then, the mappings $x^{i\star}$ are FNE in $\mathcal{H}_S$. Therefore, for all $\boldsymbol{r},\boldsymbol{s}\in\mathbb{R}^{Nn}$ we have $\|\boldsymbol{x}^\star(\boldsymbol{r})-\boldsymbol{x}^\star(\boldsymbol{s})\|^2_{I_N\otimes S}=\sum_{i=1}^N\|x^{i\star}(r^i)-x^{i\star}(s^i)\|^2_S\leq\sum_{i=1}^N(r^i-s^i)^\top S(x^{i\star}(r^i)-x^{i\star}(s^i))=(\boldsymbol{r}-\boldsymbol{s})^\top(I_N\otimes S)(\boldsymbol{x}^\star(\boldsymbol{r})-\boldsymbol{x}^\star(\boldsymbol{s}))$. (d) If (A2.d) holds, the mappings $-x^{i\star}$ are MON in $\mathcal{H}_S$. Hence, for any $\boldsymbol{r},\boldsymbol{s}\in\mathbb{R}^{Nn}$ we have $(-\boldsymbol{x}^\star(\boldsymbol{r})+\boldsymbol{x}^\star(\boldsymbol{s}))^\top(I_N\otimes S)(\boldsymbol{r}-\boldsymbol{s})=\sum_{i=1}^N\left(-x^{i\star}(r^i)+x^{i\star}(s^i)\right)^\top S\left(r^i-s^i\right)\geq 0$. □

**Lemma 2.** *Consider $P\in\mathbb{R}^{N\times N}$, $\nu\in\mathbb{N}$ and $\mathcal{P}_\nu=P^\nu\otimes I_n$. For any $S\in\mathbb{R}^{n\times n}$, $S\succ 0$, let $\mathcal{S}:=I_N\otimes S$. If $\|P\|\leq 1$, then $\|\mathcal{P}_\nu\|_{\mathcal{S}}\leq 1$.*

**Proof.** The condition $\|P\|\leq 1$ implies $\|P^\nu\|\leq\|P\|^\nu\leq 1$. By (Parise et al., 2015, Lemma 1) (with $R=I_N$) this implies $\left(P^{\nu\top}P^\nu-I_N\right)\preccurlyeq 0$. By (Parise et al., 2015, Lemma 1) with $R=\mathcal{S}$, $\|\mathcal{P}_\nu\|_{\mathcal{S}}\leq 1\Leftrightarrow\mathcal{P}_\nu{}^\top\mathcal{S}\mathcal{P}_\nu-\mathcal{S}\preccurlyeq 0\Leftrightarrow(P^\nu\otimes I_n)^\top(I_N\otimes S)(P^\nu\otimes I_n)-I_N\otimes S\preccurlyeq 0\Leftrightarrow(P^{\nu\top}P^\nu-I_N)\otimes S\preccurlyeq 0$. Since $S\succ 0$ and $(P^{\nu\top}P^\nu-I_N)\preccurlyeq 0$, it holds $(P^{\nu\top}P^\nu-I_N)\otimes S\preccurlyeq 0$ and thus $\|\mathcal{P}_\nu\|_{\mathcal{S}}\leq 1$. □

**Proposition 3** (*Regularity of $\mathcal{A}_{\nu_1,\nu_2}$*). *The following statements hold.*

*(1) If (A2.a) holds and $\|P\|\leq 1$, then the mapping $\mathcal{A}_{0,\nu}\equiv\mathcal{A}_\nu$ in (6) is a CON in $\mathcal{H}_{I_N\otimes S}$;*
*(2) If (A2.b) holds and $\|P\|\leq 1$, then the mapping $\mathcal{A}_{0,\nu}\equiv\mathcal{A}_\nu$ in (6) is NE in $\mathcal{H}_{I_N\otimes S}$;*
*(3) If (A2.c) holds, $\nu\in 2\mathbb{N}$ and $P=P^\top$, then the mapping $\mathcal{A}_{\frac{\nu}{2},\frac{\nu}{2}}$ in (7) is FNE in $\mathcal{H}_{I_N\otimes S}$;*
*(4) If (A2.d) holds, $\nu\in 2\mathbb{N}$ and $P=P^\top$, then the mapping $\mathcal{A}_{\frac{\nu}{2},\frac{\nu}{2}}$ in (7) is SPC in $\mathcal{H}_{I_N\otimes S}$.*

**Proof.** Let $\mathcal{S}:=I_N\otimes S$. (1) By Lemma 1, $\boldsymbol{x}^\star$ is a CON and, by (Parise et al., 2015, Lemma 1) and Lemma 2, $\mathcal{P}_\nu$ is NE. Hence the mapping $\mathcal{A}_\nu=\mathcal{P}_\nu\boldsymbol{x}^\star$, composition of a CON mapping and a NE one, is a CON. (2) Analogous to the proof of point 1, with $\boldsymbol{x}^\star$ NE. (3) By Lemma 1, $\boldsymbol{x}^\star$ is FNE in $\mathcal{H}_{\mathcal{S}}$. Since $P$ is row-stochastic and $P=P^\top$, $P$ is doubly stochastic. From Hölder's inequality, $\|P\|\leq\sqrt{\|P\|_1\cdot\|P\|_\infty}=1$. Therefore, from Lemma 2, $\|\mathcal{P}_{\frac{\nu}{2}}\|_{\mathcal{S}}\leq 1$. Moreover, $P=P^\top$ implies $\mathcal{P}_{\frac{\nu}{2}}=\mathcal{P}_{\frac{\nu}{2}}{}^\top$. Hence $\forall\boldsymbol{r},\boldsymbol{s}\in\mathbb{R}^{Nn}$, $\|\mathcal{A}_{\frac{\nu}{2},\frac{\nu}{2}}(\boldsymbol{r})-\mathcal{A}_{\frac{\nu}{2},\frac{\nu}{2}}(\boldsymbol{s})\|^2_{\mathcal{S}}=\|\mathcal{P}_{\frac{\nu}{2}}(\boldsymbol{x}^\star(\mathcal{P}_{\frac{\nu}{2}}(\boldsymbol{r})))-\mathcal{P}_{\frac{\nu}{2}}(\boldsymbol{x}^\star(\mathcal{P}_{\frac{\nu}{2}}(\boldsymbol{s})))\|^2_{\mathcal{S}}\leq\|\boldsymbol{x}^\star(\mathcal{P}_{\frac{\nu}{2}}(\boldsymbol{r}))-\boldsymbol{x}^\star(\mathcal{P}_{\frac{\nu}{2}}(\boldsymbol{s}))\|^2_{\mathcal{S}}\leq\left(\mathcal{P}_{\frac{\nu}{2}}\boldsymbol{r}-\mathcal{P}_{\frac{\nu}{2}}\boldsymbol{s}\right)^\top\mathcal{S}\left(\boldsymbol{x}^\star(\mathcal{P}_{\frac{\nu}{2}}(\boldsymbol{r}))-\boldsymbol{x}^\star(\mathcal{P}_{\frac{\nu}{2}}(\boldsymbol{s}))\right)=(\boldsymbol{r}-\boldsymbol{s})^\top\mathcal{S}\left(\mathcal{A}_{\frac{\nu}{2},\frac{\nu}{2}}(\boldsymbol{r})-\mathcal{A}_{\frac{\nu}{2},\frac{\nu}{2}}(\boldsymbol{s})\right)$, where the last inequality derives from $\boldsymbol{x}^\star$ being FNE and we used $\mathcal{P}_{\frac{\nu}{2}}{}^\top\mathcal{S}=\mathcal{P}_{\frac{\nu}{2}}\mathcal{S}=(P^{\frac{\nu}{2}}\otimes I_n)(I_N\otimes S)=\mathcal{S}\mathcal{P}_{\frac{\nu}{2}}$. This implies that $\mathcal{A}_{\frac{\nu}{2},\frac{\nu}{2}}(\cdot)$ is FNE in $\mathcal{H}_{\mathcal{S}}$. (4) From Lemma 1, $-\boldsymbol{x}^\star(\cdot)$ is MON in $\mathcal{H}_{\mathcal{S}}$. Since $\mathcal{P}_{\frac{\nu}{2}}{}^\top\mathcal{S}=\mathcal{S}\mathcal{P}_{\frac{\nu}{2}}$, for any $\boldsymbol{r},\boldsymbol{s}\in\mathbb{R}^{Nn}$, $\left(-\mathcal{A}_{\frac{\nu}{2},\frac{\nu}{2}}(\boldsymbol{r})+\mathcal{A}_{\frac{\nu}{2},\frac{\nu}{2}}(\boldsymbol{s})\right)^\top\mathcal{S}\,(\boldsymbol{r}-\boldsymbol{s})=\left(-\mathcal{P}_{\frac{\nu}{2}}(\boldsymbol{x}^\star(\mathcal{P}_{\frac{\nu}{2}}(\boldsymbol{r})))+\mathcal{P}_{\frac{\nu}{2}}(\boldsymbol{x}^\star(\mathcal{P}_{\frac{\nu}{2}}(\boldsymbol{s})))\right)^\top\mathcal{S}\,(\boldsymbol{r}-\boldsymbol{s})=\left(-\boldsymbol{x}^\star(\mathcal{P}_{\frac{\nu}{2}}(\boldsymbol{r}))+\boldsymbol{x}^\star(\mathcal{P}_{\frac{\nu}{2}}(\boldsymbol{s}))\right)^\top\mathcal{S}\left(\mathcal{P}_{\frac{\nu}{2}}\boldsymbol{r}-\mathcal{P}_{\frac{\nu}{2}}\boldsymbol{s}\right)=\left(-\boldsymbol{x}^\star(\tilde{\boldsymbol{r}})+\boldsymbol{x}^\star(\tilde{\boldsymbol{s}})\right)^\top\mathcal{S}\left(\tilde{\boldsymbol{r}}-\tilde{\boldsymbol{s}}\right)\geq 0$. Hence $-\mathcal{A}_{\frac{\nu}{2},\frac{\nu}{2}}$ is MON, $(\mathcal{I}-\mathcal{A}_{\frac{\nu}{2},\frac{\nu}{2}})$ is SMON ($\epsilon=1$) and $\mathcal{A}_{\frac{\nu}{2},\frac{\nu}{2}}$ is SPC. □

## Appendix B. Proofs of remaining statements

**Lemma 3.** *Under Assumptions 1 and 2 the mapping $\mathcal{A}_\nu$ in (6) admits at least one fixed point.*

**Proof.** By Assumption 2, $\boldsymbol{x}^\star(\boldsymbol{z})$ in (4) is continuous in $\boldsymbol{z}$ and thus $\mathcal{A}_\nu(\boldsymbol{z})=\mathcal{P}_\nu\boldsymbol{x}^\star(\boldsymbol{z})$ is continuous for any $\nu$; $\mathcal{A}_\nu$ maps $\mathcal{P}_\nu\mathcal{X}_{1\times N}$ into itself (since $\boldsymbol{x}^\star(\boldsymbol{z})\in\mathcal{X}_{1\times N}$ for any $\boldsymbol{z}$). By Assumption 1, $\mathcal{P}_\nu\mathcal{X}_{1\times N}$ is compact and convex. By Brouwer fixed point theorem (Smart, 1974, Thm. 4.1.5) $\mathcal{A}_\nu$ admits at least one fixed point. □

**Proof of Proposition 1.** Existence of a fixed point follows from Lemma 3. Consider now an arbitrary fixed point $\bar{\boldsymbol{z}}=\left[\bar{z}^1;\ldots;\bar{z}^N\right]\in\mathbb{R}^{Nn}$ of $\mathcal{A}_1$ in (6), that is $\bar{z}^i=\sum_{j=1}^N P_{ij}x^{j\star}(\bar{z}^j)$. By definition of fixed point, $\bar{x}^i:=x^{i\star}(\bar{z}^i)=\arg\min_{x^i\in\mathcal{X}^i}J^i(x^i,\bar{z}^i)=\arg\min_{x^i\in\mathcal{X}^i}J^i(x^i,\sum_{j=1}^N P_{ij}\bar{x}^j)=\arg\min_{x^i\in\mathcal{X}^i}J^i(x^i,P_{ii}x^i+\sum_{j\neq i}^N P_{ij}\bar{x}^j)$, where we used $P_{ii}=0$. Hence $\left\{\bar{x}^i\right\}_{i=1}^N$ is a Nash equilibrium. ∎

**Proof of Proposition 2.** Existence of a fixed point follows from Lemma 3. For any $(N,\nu)\in\mathbb{N}^2$, consider an arbitrary fixed point $\bar{\boldsymbol{z}}=\left[\bar{z}^1;\ldots;\bar{z}^N\right]\in\mathbb{R}^{Nn}$ of the aggregation mapping $\mathcal{A}_\nu$ in (6), that is $\bar{z}^i=\sum_{j=1}^N P^\nu_{ij}x^{j\star}(\bar{z}^j)$ and define $\bar{x}^i:=x^{i\star}(\bar{z}^i)$. By definition $J^i(\bar{x}^i,\sum_{j=1}^N P^\nu_{ij}\bar{x}^j)\leq J^i(x^i,\sum_{j=1}^N P^\nu_{ij}\bar{x}^j)$ for all $x^i\in\mathcal{X}^i$. Hence $J^i(\bar{x}^i,\bar{\sigma})=J^i(\bar{x}^i,\sum_j\frac{1}{N}\bar{x}^j)\leq J^i(\bar{x}^i,\sum_j P^\nu_{ij}\bar{x}^j)+\bar{L}_J\|\sum_j(\frac{1}{N}-P^\nu_{ij})\bar{x}^j\|_2\leq J^i(\bar{x}^i,\sum_j P^\nu_{ij}\bar{x}^j)+\bar{L}_J\sum_j|\frac{1}{N}-P^\nu_{ij}|\|\bar{x}^j\|_2\leq J^i(\bar{x}^i,\sum_j P^\nu_{ij}\bar{x}^j)+\bar{L}_JD\sum_j|\frac{1}{N}-P^\nu_{ij}|\leq J^i(x^i,P^\nu_{ii}\bar{x}^i+\sum_{j\neq i}P^\nu_{ij}\bar{x}^j)+\bar{L}_JDd_\infty(\nu)\leq J^i(x^i,\frac{1}{N}\bar{x}^i+\sum_{j\neq i}\frac{1}{N}\bar{x}^j)+2\bar{L}_JDd_\infty(\nu)\leq J^i(x^i,\frac{1}{N}x^i+\sum_{j\neq i}\frac{1}{N}\bar{x}^j)+2\bar{L}_JD\frac{1}{N}+2\bar{L}_JDd_\infty(\nu)\leq J^i(x^i,\frac{1}{N}x^i+\sum_{j\neq i}\frac{1}{N}\bar{x}^j)+K(\frac{1}{N}+d_\infty(\nu))$, where

$D := \max_{x\in\mathcal{X}} \|x\|_2$, $d_\infty(\nu) := \|P^\nu - \frac{1}{N}\mathbb{1}_N\mathbb{1}_N^\top\|_\infty$ and $K := 2\bar{L}_J D$. Agent $i$ can thus improve its cost at most by $\varepsilon_{N,\nu} := K(\|P^\nu - \frac{1}{N}\mathbb{1}_N\mathbb{1}_N^\top\|_\infty + \frac{1}{N})$ and $\{\bar{x}^i\}_{i=1}^N$ is an $\varepsilon_{N,\nu}$-Nash equilibrium for the AAG in (2). $K$ is a constant that does not depend on $N$, $P$ or $\nu$ and for any fixed $N$ we have $\|P^\nu - \frac{1}{N}\mathbb{1}_N\mathbb{1}_N^\top\|_\infty \to 0$ as $\nu \to \infty$. Hence, for all $\varepsilon > 0$ and for any fixed $N > \bar{N} := \frac{K}{\varepsilon}$, there exists $\bar{\nu}$ such that for all $\nu \geq \bar{\nu}$, we have $\varepsilon_{N,\nu} < \varepsilon$. ■

**Proof of Corollary 3.** The cost function in (12) can be rewritten, up to constant terms that do not depend on $x^i$, as $J^i(x^i, \sigma^i) := (1+\theta_i)\|x^i\|^2 - 2(\sigma^i + \theta_i x^i_{(0)})^\top x^i$. The game in (12) is thus a NAG game with $\nu = 1$ and quadratic cost. It follows from (Grammatico et al., 2016, Thm. 2), that if $\theta_i > 0$ then (A2.a) holds while if $\theta_i \geq 0$ then (A2.b) holds. Consequently, by Theorem 1 the given conditions guarantee convergence of Algorithm 2 to a fixed point $\bar{\boldsymbol{z}}$ of the aggregation mapping $\boldsymbol{\mathcal{A}}_1$. Finally, Proposition 1 guarantees that the set $\{x^{i\star}(\bar{z}^i)\}_{i=1}^N$ is a Nash equilibrium. ■

**Proof of Corollary 4.** The cost function in (13) with $p(\bar{\sigma}_t) := \lambda\bar{\sigma}_t + p_0$ is quadratic. It follows from (Grammatico et al., 2016, Thm. 2) that if $\rho_i - \frac{\lambda}{2} > 0$, then (A2.a) holds while if $\rho_i - \frac{\lambda}{2} \geq 0$ then (A2.b) holds; (A2.c) always holds. Consequently, by Theorem 1 the given conditions guarantee convergence of Algorithm 2 to a fixed point $\bar{\boldsymbol{z}}$ of the aggregation mapping $\boldsymbol{\mathcal{A}}_\nu$. The conclusion follows from Proposition 2. ■

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