CONTROL ON HILBERT SPACES AND APPLICATION TO MEAN FIELD TYPE CONTROL THEORY

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Abstract

We propose a new approach to studying classical solutions of the Bellman equation and Master equation for mean field type control problems, using a novel form of the "lifting" idea introduced by P.-L. Lions. Rather than studying the usual system of Hamilton-Jacobi/Fokker-Planck PDEs using analytic techniques, we instead study a stochastic control problem on a specially constructed Hilbert space, which is reminiscent of a tangent space on the Wasserstein space in optimal transport. On this Hilbert space we can use classical control theory techniques, despite the fact that it is infinite dimensional. A consequence of our construction is that the mean field type control problem appears as a special case. Thus we preserve the advantages of the lifiting procedure, while removing some of the difficulties. Our approach extends previous work by two of the coauthors, which dealt with a deterministic control problem for which the Hilbert space could be generic [2].

1 INTRODUCTION

Mean Field Game/Control theory has made remarkable progress in the recent years, thanks to important contributions, particularly the recent books of R. Carmona and F. Delarue [7] and P. Cardaliaguet, F. Delarue, J.-M Lasry, and P.-L Lions [5]. Many additional concepts, techniques and results can be found in the papers of A. Cosso and H. Pham [8], H. Pham and X. Wei [15], M.F. Djete, D. Possamai, and X. Tan [9], R. Buckdahn, J. Li, S. Peng, and C. Rainer [4], R. Carmona and F. Delarue [6], C. Mou and J. Zhang [14], and Gangbo and Mészáros [11]. All these results contribute to the rigorous treatment of the Bellman and Master equations of Mean Field Games and Mean Field Field Type Control Theory. In this article we contribute to this objective with a different vision, extending the theory developed in the paper of two of the authors, A. Bensoussan, S. C. P. Yam [2], inspired by the paper of W. Gangbo and A. Święch [10]. In [2] we have considered an abstract control problem for a dynamical system whose state space is a Hilbert space. It was a purely deterministic control problem. The fact that the state space is infinite dimensional does not prevent the methodology of control theory to be applicable. We used a simple dynamics (since a major objective was to compare with the approach of W. Gangbo and A. Święch [10]) when the Hilbert space

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is the space of L^2 random variables. Our approach is to implement the interesting idea of "lifting," first introduced by P.-L. Lions [13] for the purpose of studying derivatives on the space of measures. This lifting turns out to be quite powerful in obtaining the Bellman equation and the master equation of mean field games. Conversely, the classical approach is to use the Wasserstein metric space of probability measures, since in mean field games and mean field type control problems, the key aspect is that the payoff functional involves probability of states. However, a dynamical system whose state space is not a vector space leads to challenging difficulties.

Using the Hilbert space of square integrable random variables simplifies considerably the mathematical technicalities and makes the problem more transparent. The difficulty is to keep track of the original problem. Is the new control problem in a Hilbert space of square-integrable random variables equivalent to the original one? For instance, one has to check whether the dependence of the value function with respect to the random variable is only through its probability measure. This, of course, is not needed, when one solves the problem directly on the Wasserstein space.

Another difficulty arises when there are several sources of randomness, one from the state, and another from a Wiener process of disturbances. So we cannot simply generalize the "deterministic case" dealt with in our previous paper [2]. That is, we cannot just consider a stochastic control problem for a system whose state space is an arbitrary Hilbert space. There is an interaction between the randomness of the elements of the Hilbert space and the additional randomness from the stochastic dynamics.

We have nevertheless tried to implement the lifting approach, in a previous version of this work, posted on ArXiv [3], in order to keep most of the advantages of the deterministic theory, though the Hilbert space cannot be arbitrary. We have greatly benefited of some quite useful technical results obtained by R. Carmona and F. Delarue [7]. We keep the Wasserstein space of probability measures, but we do not use the concept of Wasserstein gradient, which seems difficult to extend to the second derivative. We use the concept of functional derivative, which extends to the second order. Using the lifting concept of P.-L. Lions, it is possible to work with the Hilbert space of square integrable random variables. The concept of gradient in the Hilbert space is the Gâteaux derivative. At this stage an "almost" complete equivalence is possible. The situation does not carry over so nicely to second order derivatives. So we have a second order Gâteaux derivative in the Hilbert space and a second order functional derivative, but not a full equivalence. On the other hand, when both exist, we have formulas to transform one concept into the other. In the ArXiv paper [3], we called these formulas *rules of correspondence*. The advantage of the Hilbert space approach is that equations can be written in a more synthetic way. From these equations and the rules of correspondence, the equations with functional derivatives can be written, but their rigorous study requires a direct approach and specific assumptions. This is not fully satisfactory, because our use of the lifting approach was meant precisely to circumvent the direct study of these equations.

In the present work, we proceed differently. We develop a framework that allows us to work in a Hilbert space and use the concept of Gâteaux derivatives in this Hilbert space. At the same time, we get the equations with functional derivatives as a particular case. We do not claim originality in the result, but in the method. For instance, in the book by P. Cardialaguet et al. [5], a fully analytic approach is taken, consisting of starting with the system of Hamilton-Jacobi-Bellman and Fokker-Planck equations, and the lifting concept is not used. Our method is new, even with respect to the lifting concept, since we consider a different Hilbert space. The advantage of the new formulation is that, in this way, we can derive all the results using only optimal control theory on a Hilbert space; the original mean field control problem appears as a particular case. We intend to develop the methodology for more general problems, including those with common noise, in future work.

The rest of this article is organized as follows. In Section 2, we introduce the Wasserstein space of measures and derivatives of functionals defined on this space. It is here that we introduce a new idea for "lifting" functionals to define them on a Hilbert space, which is the foundation for our arguments. In Section 3 we introduce an optimal control problem on the Hilbert space introduced in Section 2, with a detailed discussion of the stochastic dynamics. In Section 4, we prove several important properties of the value function, which sets up our study of the Bellman equation in Section 5. Finally, we give an existence result for the Master Equation in Section 6. Proofs of technical results from Sections 3, 4, and 5 are contained in Appendices A, B, and C, respectively.

2 FORMALISM

2.1 WASSERSTEIN SPACE

We denote by $\mathcal{P}_2(\mathbb{R}^n)$ the Wasserstein space of Borel probability measures m on $\mathbb{R}^n$ such that $\int_{\mathbb{R}^n} |x|^2 \,\mathrm{d}m(x) < \infty$, which is endowed with the metric

$$W_2(\mu,\nu) = \sqrt{\inf\left\{\int |x-y|^2 \,\mathrm{d}\pi(x,y) : \pi \in \Pi(\mu,\nu)\right\}}, \tag{2.1}$$

where $\Pi(\mu,\nu)$ is the space of all Borel probability measures on $\mathbb{R}^n \times \mathbb{R}^n$ whose first and second marginals are μ and ν, respectively. We will often use an alternative definition of W_2, given as follows. Consider an atomless probability space $(\Omega, \mathcal{A}, \mathbb{P})$, and on it the space $L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^n)$ of square integrable random variables with values in $\mathbb{R}^n$. For $X \in L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^n)$ we denote by $\mathcal{L}_X$ the law of X, given by $\mathcal{L}_X(A) = \mathbb{P}(X \in A)$. To any m in $\mathcal{P}_2(\mathbb{R}^n)$, one can find a random variable X_m in $L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^n)$ such that $\mathcal{L}_{X_m} = m$. We then have

$$W_2^2(m,m') = \inf_{\mathcal{L}_{X_m}=m,\ \mathcal{L}_{X_{m'}}=m'} \mathbb{E}[|X_m - X_{m'}|^2] \tag{2.2}$$

The infimum is attained, so we can find $\hat{X}_m$ and $\hat{X}_{m'}$ in $L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^n)$ such that

$$W_2^2(m,m') = \mathbb{E}[|\hat{X}_m - \hat{X}_{m'}|^2] \tag{2.3}$$

A sequence $\{m_k\}$ converges to m in $\mathcal{P}_2(\mathbb{R}^n)$ if and only if it converges in the sense of weak convergence and

$$\int_{\mathbb{R}^n} |x|^2 \,\mathrm{d}m_k(x) \to \int_{\mathbb{R}^n} |x|^2 \,\mathrm{d}m(x) \tag{2.4}$$

It is also important to indicate the following compactness result of the space $\mathcal{P}_2(\mathbb{R}^n)$:

$$\text{A family } \{m_k\} \text{ is relatively compact in } \mathcal{P}_2(\mathbb{R}^n) \text{ if } \sup_k \mathbb{E}[|X_{m_k}|^\beta] < +\infty, \text{for some } \beta > 2. \tag{2.5}$$

We refer to Carmona-Delarue [7] for details.

2.2 FUNCTIONALS AND THEIR DERIVATIVES

Consider a functional $F(m)$ on $\mathcal{P}_2(\mathbb{R}^n)$. Continuity is clearly defined by the metric. For the concept of derivative in $\mathcal{P}_2(\mathbb{R}^n)$, we use the concept of functional derivative.

Definition 2.1. *We say F is* continuously differentiable *provided there exists a continuous function* $\dfrac{\mathrm{d}F}{\mathrm{d}m}$: $\mathcal{P}_2 \times \mathbb{R}^n \to \mathbb{R}$ *such that, for some* $c : \mathcal{P}_2(\mathbb{R}^n) \to [0,\infty)$ *that is bounded on bounded subsets, we have*

$$\left|\frac{\mathrm{d}F}{\mathrm{d}m}(m,x)\right| \le c(m)\left(1 + |x|^2\right) \tag{2.6}$$

and

$$\lim_{\epsilon \to 0} \frac{F(m + \epsilon(m'-m)) - F(m)}{\epsilon} = \int \frac{\mathrm{d}F}{\mathrm{d}m}(m,x)\,\mathrm{d}(m'-m)(x) \tag{2.7}$$

for any $m' \in \mathcal{P}_2$. Since $\dfrac{\mathrm{d}F}{\mathrm{d}m}$ is unique only up to a constant, we require the normalization condition

$$\int \frac{\mathrm{d}F}{\mathrm{d}m}(m,x)\,\mathrm{d}m(x) = 0, \tag{2.8}$$

which in particular ensures the functional derivative of a constant is 0.

Thanks to (2.6), $x \mapsto \dfrac{\mathrm{d}F}{\mathrm{d}m}(m,x)$ is in $L^2_m(\mathbb{R}^n) := \{\varphi(\cdot) | \int_{\mathbb{R}^n} |\varphi(x)|^2 \,\mathrm{d}m(x)\}$, and by a slight abuse of notation we will denote $\dfrac{\mathrm{d}F}{\mathrm{d}m}(m)(x) := \dfrac{\mathrm{d}F}{\mathrm{d}m}(m,x)$. Note that the definition (2.7) implies

$$\frac{\mathrm{d}}{\mathrm{d}\theta} F(m+\theta(m'-m)) = \int_{\mathbb{R}^n} \frac{\mathrm{d}F}{\mathrm{d}m}(m+\theta(m'-m))(x)\,\mathrm{d}(m'-m)(x) \tag{2.9}$$

and

$$F(m') - F(m) = \int_0^1 \int_{\mathbb{R}^n} \frac{\mathrm{d}F}{\mathrm{d}m}(m+\theta(m'-m))(x)\,\mathrm{d}(m'-m)(x)\,\mathrm{d}\theta \tag{2.10}$$

We prefer the notation $\dfrac{\mathrm{d}F}{\mathrm{d}m}(m)(x)$ to $\dfrac{\delta F}{\delta m}(m)(x)$ used in R.Carmona-F. Delarue [7], because there is no risk of confusion and it works pretty much like an ordinary Gâteaux derivative. We can proceed with the second order functional derivative.

Definition 2.2. *We say F is* twice continuously differentiable *provided there exists a continuous function $\dfrac{\mathrm{d}^2F}{\mathrm{d}m^2} : \mathcal{P}_2 \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that, for some $c : \mathcal{P}_2(\mathbb{R}^n) \to [0,\infty)$ that is bounded on bounded subsets,*

$$\left| \frac{\mathrm{d}^2F}{\mathrm{d}m^2}(m,x,\tilde{x}) \right| \le c(m)\left(1+|x|^2+|\tilde{x}|^2\right) \tag{2.11}$$

and

$$\begin{aligned}\lim_{\epsilon\to 0} \frac{1}{\epsilon} \int \left(\frac{\mathrm{d}F}{\mathrm{d}m}(m+\epsilon(\tilde{m}'-m),x) - \frac{\mathrm{d}F}{\mathrm{d}m}(m,x) \right) \mathrm{d}(m'-m)(x) \\ = \iint \frac{\mathrm{d}^2F}{\mathrm{d}m^2}(m,x,\tilde{x})\,\mathrm{d}(m'-m)(x)\,\mathrm{d}(\tilde{m}'-m)(\tilde{x})\end{aligned} \tag{2.12}$$

for any $m', \tilde{m}' \in \mathcal{P}_2$. To ensure $\dfrac{\mathrm{d}^2F}{\mathrm{d}m^2}(m,x,\tilde{x})$ is uniquely defined, we will use the normalization convention

$$\int \frac{\mathrm{d}^2F}{\mathrm{d}m^2}(m,x,\tilde{x})\,\mathrm{d}m(\tilde{x}) = 0 \ \forall x, \qquad \int \frac{\mathrm{d}^2F}{\mathrm{d}m^2}(m,x,\tilde{x})\,\mathrm{d}m(x) = 0 \ \forall \tilde{x}. \tag{2.13}$$

Again, we will write $\dfrac{\mathrm{d}^2F}{\mathrm{d}m^2}(m,x,\tilde{x}) = \dfrac{\mathrm{d}^2F}{\mathrm{d}m^2}(m)(x,\tilde{x})$, where we note that $\dfrac{\mathrm{d}^2F}{\mathrm{d}m^2}(m) \in L^2_{m\times m}$. We have also

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2} F(m+\theta(m'-m)) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\mathrm{d}^2F}{\mathrm{d}m^2}(m+\theta(m'-m))(x,\tilde{x})\,\mathrm{d}(m'-m)(x)\,\mathrm{d}(m'-m)(\tilde{x}) \tag{2.14}$$

and

$$\begin{aligned}F(m') - F(m) = \int_{\mathbb{R}^n} \frac{\mathrm{d}F}{\mathrm{d}m}(m)(x)\,\mathrm{d}(m'-m)(x) \\ + \int_0^1\int_0^1 \theta \frac{\mathrm{d}^2F}{\mathrm{d}m^2}(m+\lambda\theta(m'-m))(x,\tilde{x})\,\mathrm{d}(m'-m)(x)\,\mathrm{d}(m'-m)(\tilde{x})\,\mathrm{d}\lambda\,\mathrm{d}\theta.\end{aligned} \tag{2.15}$$

If F is twice continuously differentiable, then standard arguments show that $\dfrac{\mathrm{d}^2F}{\mathrm{d}m^2}(m)$ is symmetric, i.e.

$$\frac{\mathrm{d}^2F}{\mathrm{d}m^2}(m)(x,\tilde{x}) = \frac{\mathrm{d}^2F}{\mathrm{d}m^2}(m)(\tilde{x},x).$$

We conclude this subsection with a remark on notation of derivatives with respect to variables in $\mathbb{R}^n$. If $\mathbb{R}^n \ni x \mapsto \dfrac{\mathrm{d}F}{\mathrm{d}m}(m)(x)$ is differentiable, we denote its derivative by $D\dfrac{\mathrm{d}F}{\mathrm{d}m}(m)(x)$. If $\mathbb{R}^n \times \mathbb{R}^n \ni (x_1,x_2) \mapsto \dfrac{\mathrm{d}F}{\mathrm{d}m}(m)(x_1,x_2)$ is differentiable, we denote by $D_1\dfrac{\mathrm{d}F}{\mathrm{d}m}(m)(x_1,x_2)$ and $D_2\dfrac{\mathrm{d}F}{\mathrm{d}m}(m)(x_1,x_2)$ its derivatives with respect to x_1 and x_2, respectively. Higher-order derivatives for $\dfrac{\mathrm{d}F}{\mathrm{d}m}(m)(\cdot)$ will simply denoted D^k for $k = 1,2,\ldots$, while for $\dfrac{\mathrm{d}^2F}{\mathrm{d}m^2}(m)(\cdot)$ we will use the partial derivatives D_1^k and D_2^ℓ.

2.3 HILBERT SPACE

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be an atomless probability space. For $m \in \mathcal{P}_2$, let $\mathcal{H}_m := L^2(\Omega, \mathcal{A}, \mathbb{P}; L^2_m(\mathbb{R}^n; \mathbb{R}^n))$ where $L^2_m(\mathbb{R}^n; \mathbb{R}^n)$ is simply the set of all measurable vector fields Φ such that $\int |\Phi(x)|^2 \,\mathrm{d}m(x) < \infty$. On $\mathcal{H}_m$ we define the inner product

$$\langle X, Y \rangle_{\mathcal{H}_m} = \mathbb{E} \int X(x) \cdot Y(x) \,\mathrm{d}m(x) = \int_\Omega \int_{\mathbb{R}^n} X(\omega, x) \cdot Y(\omega, x) \,\mathrm{d}m(x) \,\mathrm{d}\mathbb{P}(\omega). \tag{2.16}$$

The corresponding norm is given by $\|X\|_{\mathcal{H}_m} = \sqrt{\langle X, X \rangle_{\mathcal{H}_m}}$. When it is sufficiently clear which inner product we mean, we will often drop the subscript $\mathcal{H}_m$.

There is a natural isometric isomorphism between $\mathcal{H}_m$ and $L^2(\Omega \times \mathbb{R}^n, \mathcal{A} \otimes \mathcal{B}, \mathbb{P} \times m; \mathbb{R}^n)$, where $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}^n$. Note that the product measure $\mathbb{P} \times m$ is a probability measure on $\Omega \times \mathbb{R}^n$. If we consider the law of $X \in \mathcal{H}_m$ viewed as an L^2 random variable on this product space, we arrive at the following definition:

Definition 2.3. *Let $m \in \mathcal{P}_2, X \in \mathcal{H}_m$. We define $X \otimes m \in \mathcal{P}_2$ by duality: for all continuous functions $\phi : \mathbb{R}^n \to \mathbb{R}$ such that $x \mapsto \frac{|\phi(x)|}{1+|x|^2}$ is bounded, we have*

$$\int \phi(x) \,\mathrm{d}(X \otimes m)(x) = \mathbb{E}\left[\int \phi\left(X(x)\right) \mathrm{d}m(x)\right] = \int_\Omega \int_{\mathbb{R}^n} \phi\left(X(\omega, x)\right) \mathrm{d}m(x) \,\mathrm{d}\mathbb{P}(\omega). \tag{2.17}$$

Let us notice that if X is deterministic, i.e. not ω-dependent, then $X \otimes m = X \sharp m$, where $X \sharp m(E) := m(X^{-1}(E))$ is the push-forward of m through X. In fact, the following lemma helps show how much $\otimes$ behaves like the push-forward operator:

Lemma 2.4. *Let $X, Y \in \mathcal{H}_m$, and suppose $X \circ Y \in \mathcal{H}_m$. Then $(X \circ Y) \otimes m = X \otimes (Y \otimes m)$.*

Proof. Let ϕ be a bounded, Borel measurable function. Then by Fubini's Theorem, $x \mapsto \mathbb{E}[\phi(X(x))]$ is also bounded and Borel measurable, and moreover

$$\int \phi \,\mathrm{d}((X \circ Y) \otimes m) = \int \mathbb{E}\left[\phi\left(X\left(Y(x)\right)\right)\right] \mathrm{d}m(x) = \int \mathbb{E}\left[\phi\left(X\left(x\right)\right)\right] \mathrm{d}(Y \otimes m)(x) = \int \phi \,\mathrm{d}(X \otimes (Y \otimes m)). \tag{2.18}$$

$\square$

Using properties of the push-forward, we exhibit two useful examples:

Example 2.5. *(i) If $X(x) = x$ is the identity map, then $X \otimes m = m$.*

(ii) If $X(x) = a$ is a constant map, then $X \otimes m = \delta_a$, the Dirac delta mass concentrated at a.

As before, let $F : \mathcal{P}_2 \to \mathbb{R}$. For every $m \in \mathcal{P}_2$, the map $X \mapsto F(X \otimes m)$ is a functional on $\mathcal{H}_m$. By an abuse of notation, we will now think of $F(X \otimes m)$ as a function of two variables, taking care to remember that X is always attached to m in the sense that $X \in \mathcal{H}_m$.

Lemma 2.6. *The map $X \mapsto X \otimes m$ is 1-Lipschitz from $\mathcal{H}_m$ to $\mathcal{P}_2$. Thus if $F : \mathcal{P}_2 \to \mathbb{R}$ is continuous, then for a fixed $m \in \mathcal{P}_2$, the map $\mathcal{H}_m \to \mathbb{R}$ given by $X \mapsto F(X \otimes m)$ is also continuous.*

Proof. Let $X, Y \in \mathcal{H}_m$. Let $\pi \in \Pi(X \otimes m, Y \otimes m)$ be given by

$$\int \phi(x, y) \,\mathrm{d}\pi(x, y) = \mathbb{E} \int \phi\left(X(x), Y(x)\right) \mathrm{d}m(x). \tag{2.19}$$

Since the right-hand side is a bounded, non-negative linear functional on the space of continuous functions, by the Riesz representation theorem this defines a unique measure on $\mathbb{R}^n \times \mathbb{R}^n$. Moreover, the first marginal of π is indeed $X \otimes m$ because

$$\int \phi(x) \,\mathrm{d}\pi(x, y) = \mathbb{E} \int \phi\left(X(x)\right) \mathrm{d}m(x) = \int \phi(x) \,\mathrm{d}(X \otimes m)(x) \tag{2.20}$$

and likewise the second marginal of π is $Y \otimes m$ because

$$\int \phi(y) \, \mathrm{d}\pi(x,y) = \mathbb{E} \int \phi\left(Y(x)\right) \mathrm{d}m(x) = \int \phi(x) \, \mathrm{d}(Y \otimes m)(x). \tag{2.21}$$

It follows that

$$\mathrm{d}_2^2(X \otimes m, Y \otimes m) \leq \int |x-y|^2 \, \mathrm{d}\pi(x,y) = \mathbb{E} \int |X(x) - Y(x)|^2 \, \mathrm{d}m(x) = \|X - Y\|_{\mathcal{H}_m}^2 . \tag{2.22}$$

□

Definition 2.7. *Let $F : \mathcal{P}_2 \to \mathbb{R}$, and consider its extension $F(X \otimes m)$. For any $m \in \mathcal{P}_2$, we define the "partial derivative" of F with respect to $X \in \mathcal{P}_2$ as the unique element $D_X F(X \otimes m)$ of $\mathcal{H}_m$, if it exists, such that*

$$\lim_{\epsilon \to 0} \frac{F\left((X + \epsilon Y) \otimes m\right) - F(X \otimes m)}{\epsilon} = \langle D_X F(X \otimes m), Y \rangle \;\; \forall Y \in \mathcal{H}_m. \tag{2.23}$$

Note that the "partial derivative" in Definition 2.7 is the usual Gâteaux derivative in a Hilbert space. It is not a true partial derivative in the sense that it is not independent of the variable m; in this sense it is more akin to a tangent vector, where $\mathcal{H}_m$ is likened to a tangent space. The following proposition characterizes $D_X F$ in a more elementary way.

Proposition 2.8. *Let $F : \mathcal{P}_2 \to \mathbb{R}$ be continuously differentiable and assume $x \mapsto \frac{\mathrm{d}F}{\mathrm{d}m}(m,x)$ is continuously differentiable in $\mathbb{R}^n$. Assume that its derivative $D\frac{\mathrm{d}F}{\mathrm{d}m}(m)(x)$ is continuous in both m and x with*

$$\left| D \frac{\mathrm{d}F}{\mathrm{d}m}(m)(x) \right| \leq c(m) \left(1 + |x|\right) \tag{2.24}$$

for some constant $c(m)$ depending only on m. Then

$$D_X F(X \otimes m) = D \frac{\mathrm{d}F}{\mathrm{d}m}(X \otimes m)(X(\cdot)). \tag{2.25}$$

Proof. Note that, by (2.24), $D\frac{\mathrm{d}F}{\mathrm{d}m}(X \otimes m, X(\cdot)) \in \mathcal{H}_m$ for any $X \in \mathcal{H}_m$. Let $Y \in \mathcal{H}_m$ be arbitrary. For $\epsilon \neq 0$, let $\mu = (X + \epsilon Y) \otimes m, \nu = X \otimes m$, and for $t \in [0,1]$ set $\nu_t = \nu + t(\mu - \nu)$. Then we have

$$\begin{aligned}
\frac{1}{\epsilon}\Big(F\left((X+\epsilon Y)\otimes m\right) - F(X \otimes m)\Big) &= \frac{1}{\epsilon} \int_0^1 \int_{\mathbb{R}^n} \frac{\mathrm{d}F}{\mathrm{d}m}(\nu_t, x) \, \mathrm{d}(\mu - \nu)(x) \, \mathrm{d}t \\
&= \frac{1}{\epsilon} \mathbb{E} \int_0^1 \int_{\mathbb{R}^n} \left(\frac{\mathrm{d}F}{\mathrm{d}m}(\nu_t, X(x) + \epsilon Y(x)) - \frac{\mathrm{d}F}{\mathrm{d}m}(\nu_t, X(x)) \right) \mathrm{d}m(x) \, \mathrm{d}t \\
&\to \mathbb{E} \int_{\mathbb{R}^n} D \frac{\mathrm{d}F}{\mathrm{d}m}(X \otimes m, X(x)) \cdot Y(x) \, \mathrm{d}m(x) \\
&= \left\langle D \frac{\mathrm{d}F}{\mathrm{d}m}(X \otimes m, X(\cdot)), Y \right\rangle_{\mathcal{H}_m}
\end{aligned} \tag{2.26}$$

using the continuity of $D\frac{\mathrm{d}F}{\mathrm{d}m}$. □

A special case of (2.25) is when X is the identity, i.e. $X(x) = x$. In this case $X \otimes m = m$ (Example 2.5), and thus (2.25) implies

$$D_X F(m) = D \frac{\mathrm{d}F}{\mathrm{d}m}(m)(X(\cdot)). \tag{2.27}$$

This is precisely the L-derivative, cf. [7, 5].

We now want to consider $F(X \otimes m)$ as m varies but X is fixed. To do this we should restrict X so that $X \in \cap_{m \in \mathcal{P}_2} \mathcal{H}_m$. The following observation is useful.

Lemma 2.9. *Let $X : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ be a $(\mathcal{A} \otimes \mathcal{B}, \mathcal{B})$ measurable vector field (where $\mathcal{B}$ is the Borel σ-algebra on $\mathbb{R}^n$) such that*

$$\mathbb{E}|X(x)|^2 \leq c(X)\left(1 + |x|^2\right) \;\; \forall x \in \mathbb{R}^n, \tag{2.28}$$

where $c(X)$ is a constant depending only on X. Then $X \in \cap_{m \in \mathcal{P}_2} \mathcal{H}_m$.

Proof. Observe that if $m \in \mathcal{P}_2$ then

$$\|X\|^2_{\mathcal{H}_m} = \mathbb{E}\int |X(x)|^2 \, \mathrm{d}m(x) = \int \mathbb{E}|X(x)|^2 \, \mathrm{d}m(x) \leq \int c(X)\left(1 + |x|^2\right) \mathrm{d}m(x) < \infty, \tag{2.29}$$

and thus $X \in \mathcal{H}_m$ for arbitrary $m \in \mathcal{P}_2$. $\square$

Definition 2.10. *Let $F : \mathcal{P}_2 \to \mathbb{R}$ and let $X \in \mathcal{H}$. We define the partial derivative of $F(X \otimes m)$ with respect to m, denoted $\frac{\partial F}{\partial m}(X \otimes m)(x)$, to be the derivative of $m \mapsto F(X \otimes m)$ in the sense of Definition 2.1.*

Proposition 2.11. *Let $F : \mathcal{P}_2 \to \mathbb{R}$ be continuously differentiable and let $X \in \mathcal{H}$. Then*

$$\frac{\partial F}{\partial m}(X \otimes m)(x) = \mathbb{E}\frac{\mathrm{d}F}{\mathrm{d}m}(X \otimes m)(X(x)). \tag{2.30}$$

Proof. For $\epsilon \neq 0$ let $\mu = X \otimes (m + \epsilon(m' - m))\,, \nu = X \otimes m$, and for $t \in [0,1]$ set $\nu_t = \nu + t(\mu - \nu)$. We have, as $\epsilon \to 0$,

$$\begin{aligned}
\frac{1}{\epsilon}\left(F\left(X \otimes \left(m + \epsilon(m' - m)\right)\right) - F\left(X \otimes m\right)\right) &= \frac{1}{\epsilon}\int_0^1 \int_{\mathbb{R}^n} \tfrac{\mathrm{d}F}{\mathrm{d}m}(\nu_t, x) \, \mathrm{d}(\mu - \nu)(x) \\
&= \mathbb{E}\int_0^1 \int_{\mathbb{R}^n} \frac{\mathrm{d}F}{\mathrm{d}m}\left(\nu_t, X(x)\right) \mathrm{d}(m' - m)(x) \\
&\to \mathbb{E}\int_{\mathbb{R}^n} \frac{\mathrm{d}F}{\mathrm{d}m}\left(X \otimes m, X(x)\right) \mathrm{d}(m' - m)(x),
\end{aligned} \tag{2.31}$$

using the continuity of $\frac{\mathrm{d}F}{\mathrm{d}m}$. The claim follows. $\square$

We conclude this subsection with a remark on notation. For a functional $F : \mathcal{P}_2 \to \mathbb{R}$, whenever the symbols $D_X F$ and $\frac{\partial F}{\partial m}$ are used, they should be thought of as partial derivatives of $F(X \otimes m)$, which can be evaluated at any elements (X, m) such that $m \in \mathcal{P}_2$ and $X \in \cap_{\mu \in \mathcal{P}_2} \mathcal{H}_\mu$. In particular, $\frac{\partial F}{\partial m}$ should not be confused with $\frac{\mathrm{d}F}{\mathrm{d}m}$; the relation between the two is clarified by Proposition 2.11. Moreover, standard usage of partial derivative notation applies to compositions of functions. Thus if $X \mapsto Y_X$ is a map $\mathcal{H}_m \to \mathcal{H}_m$, then the symbol $D_X F(Y_X \otimes m)$ should be interpreted as $D_X F$ evaluated at the point $Y_X \otimes m$, rather than as the derivative of the composite function $X \mapsto F(Y_X \otimes m)$. When it is necessary to differentiate a composite function, we will clearly state that we are doing so.

2.4 MORE ELABORATE DERIVATIVES, CHAIN RULE

Consider a random vector field $X = X(m, x)$, with the equivalent of (2.28)

$$\mathbb{E}\left[|X(m,x)|^2\right] \leq c(X)(1 + |x|^2) \tag{2.32}$$

where $c(X)$ is a constant, not depending on m, x, but only on X. We define the functional derivative of X with respect to m in a way analogous to Definition 2.1. Namely, we say X is continuously differentiable with respect to m if there exists a random field $\frac{\partial X}{\partial m} : \mathcal{P}_2(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n \to L^2(\Omega, \mathcal{A}, \mathbb{P}; \mathbb{R}^n)$, which is continuous in all variables, such that

$$\mathbb{E}\left[\left|\frac{\partial X}{\partial m}(m, x, \tilde{x})\right|^2\right] \leq c(X, m)(1 + |x|^2 + |\tilde{x}|^2) \tag{2.33}$$

and

$$\left\| \frac{X(m+\theta(m'-m),\cdot)-X(m,\cdot)}{\theta} - \int_{\mathbb{R}^n} \frac{\partial X}{\partial m}(m,\cdot,\tilde{x})\,\mathrm{d}(m'-m)(\tilde{x}) \right\|_{L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^n)} \to 0, \text{ as } \theta \to 0 \tag{2.34}$$

Following our usual convention, we can write $\frac{\partial X}{\partial m}(m,x,\tilde{x}) = \frac{\partial X}{\partial m}(m,x)(\tilde{x})$, where $\frac{\partial X}{\partial m}(m,x)$ is viewed as an element of $L^2_m\left(L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^n)\right)$.

We can envisage the functional derivative of $m \mapsto F(X(m,.)\otimes m)$. The calculation works as for ordinary derivatives, the tensor product acting as an ordinary product. We can then write

$$\frac{\mathrm{d}}{\mathrm{d}m}\left(F(X(m,.)\otimes m)\right)(x) = \frac{\partial F}{\partial m}(X(m,\cdot)\otimes m))(x) + \left\langle D_X F(X(m,\cdot)\otimes m), \frac{\mathrm{d}}{\mathrm{d}m}X(m,.)(x)\right\rangle_{\mathcal{H}_m} \tag{2.35}$$

which we can make explicit as follows

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}m}\left(F(X(m,.)\otimes m)\right)(x) = \mathbb{E}\frac{\mathrm{d}F}{\mathrm{d}m}(X(m,.)\otimes m)(X(m,x)) \\ + \mathbb{E}\int_{\mathbb{R}^n} D\frac{\mathrm{d}F}{\mathrm{d}m}(X(m,.)\otimes m)(X(m,\xi))\cdot\frac{\mathrm{d}X}{\mathrm{d}m}(m,\xi)(x)\,\mathrm{d}m(\xi)\end{aligned} \tag{2.36}$$

We have

$$\left|\frac{\mathrm{d}}{\mathrm{d}m}F(X(m,.)\otimes m)(x)\right| \leq c(X,m)(1+|x|^2). \tag{2.37}$$

2.5 SECOND ORDER GÂTEAUX DERIVATIVE IN THE HILBERT SPACE

The functional $F(X\otimes m)$ has a second order Gâteaux derivative in $\mathcal{H}_m$, denoted $D^2_X F(X\otimes m) \in \mathcal{L}(\mathcal{H}_m;\mathcal{H}_m)$, if

$$\frac{\langle D_X F((X+\epsilon Y)\otimes m) - D_X F(X\otimes m), Y\rangle_{\mathcal{H}_m}}{\epsilon} \to \left\langle D^2_X F(X\otimes m)(Y), Y\right\rangle_{\mathcal{H}_m} \text{ as } \epsilon \to 0, \forall Y \in \mathcal{H}_m, \tag{2.38}$$

and we can define $D^2_X F(X\otimes m)(Z)$ using the parallelogram law:

$$\begin{aligned}&\left\langle D^2_X F(X\otimes m)(Z), W\right\rangle_{\mathcal{H}_m} \\ &\quad = \frac{1}{4}\left(\left\langle D^2_X F(X\otimes m)(Z+W), Z+W\right\rangle_{\mathcal{H}_m} - \left\langle D^2_X F(X\otimes m)(Z-W), Z-W\right\rangle_{\mathcal{H}_m}\right).\end{aligned} \tag{2.39}$$

Note that $D^2_X F(X\otimes m)$ is self-adjoint. It is convenient to define the symmetric bilinear form on $\mathcal{H}_m$, which, by a slight abuse of notation, we may denote

$$D^2_X F(X\otimes m)(Z,W) = \left\langle D^2_X F(X\otimes m)(Z), W\right\rangle. \tag{2.40}$$

Observe that

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\langle D_X F((X+\theta Z)\otimes m), W\rangle = \left\langle D^2_X F((X+\theta Z)\otimes m)(Z), W\right\rangle. \tag{2.41}$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}\theta}F((X+\theta Y)\otimes m) = \langle D_X F((X+\theta Y)\otimes m), Y\rangle$$

we have also

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}F((X+\theta Y)\otimes m) = \left\langle D^2_X F((X+\theta Y)\otimes m)(Y), Y\right\rangle \tag{2.42}$$

Hence

$$F((X+Y)\otimes m)) = F(X\otimes m) + \langle D_X F(X), Y\rangle + \int_0^1\int_0^1 \theta\left\langle D^2_X F((X+\theta\lambda Y)\otimes m)(Y), Y\right\rangle \mathrm{d}\theta\,\mathrm{d}\lambda \tag{2.43}$$

From Proposition 2.11 we have $\langle D_X F(X\otimes m), W\rangle = \mathbb{E}\int_{\mathbb{R}^n} D\frac{\mathrm{d}F}{\mathrm{d}m}(X\otimes m)(X(x))\cdot W(x)\,\mathrm{d}m(x)$. It follows that

$$\frac{\langle D_X F((X+\epsilon Z)\otimes m), W\rangle - \langle D_X F(X\otimes m), W\rangle}{\epsilon}$$
$$= \frac{\mathbb{E}\int_{\mathbb{R}^n}\Big(D\frac{\mathrm{d}F}{\mathrm{d}m}((X+\epsilon Z)\otimes m)(X(x)+\epsilon Z(x)) - D\frac{\mathrm{d}F}{\mathrm{d}m}(X\otimes m)(X(x))\Big)\cdot W(x)\,\mathrm{d}m(x)}{\epsilon},$$

so, assuming the existence and continuity of $\dfrac{\mathrm{d}^2 F}{\mathrm{d}m^2}(m)(x,\tilde{x})$ and its derivatives $D_1 D_2 \dfrac{\mathrm{d}^2 F}{\mathrm{d}m^2}(m)(x,\tilde{x})$, as well as the existence and continuity of $D^2\dfrac{\mathrm{d}F}{\mathrm{d}m}(m)(x)$, we deduce that the above limit is

$$\begin{aligned}\Big\langle D_X^2 F(X\otimes m)(Z), W\Big\rangle &= \mathbb{E}\int_{\mathbb{R}^n} D^2\frac{\mathrm{d}F}{\mathrm{d}m}(X\otimes m)(X(x))Z(x)\cdot W(x)\,\mathrm{d}m(x)\\ &+\mathbb{E}\tilde{\mathbb{E}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} D_1 D_2\frac{\mathrm{d}^2F}{\mathrm{d}m^2}(X\otimes m)(\tilde{X}(\tilde{x}),X(x))\tilde{Z}(\tilde{x})\cdot W(x)\,\mathrm{d}m(\tilde{x})\,\mathrm{d}m(x),\end{aligned} \tag{2.44}$$

in which $\tilde{X}(\tilde{x}), \tilde{Z}(\tilde{x})$ are independent copies of $X(x), Z(x)$. Consequently, we can write

$$D_X^2 F(X\otimes m)(Z)(x) = D^2\frac{\mathrm{d}F}{\mathrm{d}m}(X\otimes m)(X(x))Z(x) + \tilde{\mathbb{E}}\int_{\mathbb{R}^n} D_1 D_2\frac{\mathrm{d}^2F}{\mathrm{d}m^2}(X\otimes m)(\tilde{X}(\tilde{x}),X(x))\tilde{Z}(\tilde{x})\,\mathrm{d}m(\tilde{x}) \tag{2.45}$$

in which the expectation $\tilde{\mathbb{E}}$ is independent of $X(x)$. If we take $X(x)=x$, we obtain (recall Example 2.5)

$$D_X^2 F(m)(Z)(x) = D^2\frac{\mathrm{d}F}{\mathrm{d}m}(m)(x)Z(x) + \int_{\mathbb{R}^n} D_2 D_1\frac{\mathrm{d}^2F}{\mathrm{d}m^2}(m)(\tilde{x},x)\tilde{\mathbb{E}}\tilde{Z}(\tilde{x})\,\mathrm{d}m(\tilde{x}) \tag{2.46}$$

It follows immediately that if Z is independent of X and $\mathbb{E}[Z]=0$, then

$$D_X^2 F(X\otimes m)(Z)(x) = D^2\frac{\mathrm{d}F}{\mathrm{d}m}(X\otimes m)(X(x))Z(x) \tag{2.47}$$

In order to get $D_X^2 F(X\otimes m)\in\mathcal{H}_m$, it will suffice to assume

$$\Big|D\tfrac{\mathrm{d}F}{\mathrm{d}m}(m)(x)\Big|\le c(m),\ \Big|D_2 D_1\tfrac{\mathrm{d}^2F}{\mathrm{d}m^2}(m)(\tilde{x},x)\Big|\le c(m) \tag{2.48}$$

where $|\cdot|$ in (2.48) is the matrix norm.

3 CONTROL PROBLEM WITH STATE VARIABLE IN $\mathcal{H}_m$

For this entire section we will fix a measure $m\in\mathcal{P}_2(\mathbb{R}^n)$. We will define an optimal control problem on the Hilbert space $\mathcal{H}_m$ attached to m. Then we will discuss its solution. In the following section, we will study properties of the value function.

3.1 PRELIMINARIES

Let $(\Omega,\mathcal{A},\mathbb{P})$ be a probability space sufficiently large to contain a standard Wiener process in $\mathbb{R}^n$, denoted $w(t)$, with filtration $\mathcal{W}_t=\{\mathcal{W}_t^s\}_{s\ge t}$ where $\mathcal{W}_t^s=\sigma\left((w(\tau)-w(t)) : t\le\tau\le s\right)$. We also assume $(\Omega,\mathcal{A},\mathbb{P})$ is rich enough to support random variables that are independent of the entire Wiener process. For example, we could take $(\Omega,\mathcal{A},\mathbb{P})=(\Omega_0\times\Omega_1,\mathcal{A}_0\otimes\mathcal{A}_1,\mathbb{P}_0\times\mathbb{P}_1)$ where $\mathcal{W}_t^s\subset\mathcal{A}_0$ and $(\Omega_1,\mathcal{A}_1,\mathbb{P}_1)$ is itself a sufficiently rich probability space. For a given $t\ge 0$, we denote by $\mathcal{H}_{m,t}$ the space of all $X=X_t\in\mathcal{H}_m$ such that X is independent of $\mathcal{W}_t$ (say, $\sigma(X)\subset\mathcal{A}_1$). For $X\in\mathcal{H}_{m,t}$ we define σ-algebras $\mathcal{W}_{Xt}^s=\sigma(X)\vee\mathcal{W}_t^s$, and the filtration generated by these will be denoted $\mathcal{W}_{Xt}$.

For the remainder of this section we will fix $t\ge 0$ and $X\in\mathcal{H}_{m,t}$. For the control problem stated below in Section 3.2, the space of controls will be $L^2_{\mathcal{W}_{Xt}}(t,T;\mathcal{H}_m)$, the set of all processes in $L^2(t,T;\mathcal{H}_m)$ that are adapted to $\mathcal{W}_{Xt}$. Because X is independent of $\mathcal{W}_t$, we have an important observation:

Lemma 3.1. *Let $X \in \mathcal{H}_{m,t}$. Then there exists a natural linear isometry between $L^2_{\mathcal{W}_{Xt}}(t,T;\mathcal{H}_m)$ and $L^2_{\mathcal{W}_t}(t,T;\mathcal{H}_{X\otimes m})$, obtained by inserting the random variable X in place of the argument x in the vector field v.*

Proof. Without loss of generality we will assume $(\Omega,\mathcal{A},\mathbb{P}) = (\Omega_0\times\Omega_1, \mathcal{A}_0\otimes\mathcal{A}_1, \mathbb{P}_0\times\mathbb{P}_1)$ as above. Let $v \in L^2_{\mathcal{W}_{Xt}}(t,T;\mathcal{H}_m)$. We will show there exists a unique $\tilde{v} \in L^2_{\mathcal{W}_t}(t,T;\mathcal{H}_{X\otimes m})$ such that for any $\omega = (\omega_0,\omega_1)\in\Omega$ and any $s\in[t,T]$, we have $v(\omega,s,x) = \tilde{v}(\omega_0,s,X(\omega_1,x))$. First, note that because $(\omega_0,\omega_1,x)\mapsto v(\omega_0,\omega_1,s,x)$ is $(\mathcal{W}_t^s\otimes\sigma(X)\otimes\mathcal{B},\mathcal{B})$ measurable, it follows that there exists a $(\mathcal{W}_t^s\otimes\mathcal{B},\mathcal{B})$-measurable function $(\omega_0,x)\mapsto\tilde{v}(\omega_0,s,x)$ such that $v(\omega,s,x) = \tilde{v}(\omega_0,s,X(\omega_1,x))$. Thus $\tilde{v}$ is adapted to $\mathcal{W}_t$. Now observe that

$$\begin{aligned}
\|v\|^2_{L^2_{\mathcal{W}_{Xt}}(t,T;\mathcal{H}_m)} &= \int_t^T\int_\Omega\int_{\mathbb{R}^n}|v(\omega,s,x)|^2\,\mathrm{d}m(x)\,\mathrm{d}\mathbb{P}(\omega)\,\mathrm{d}s\\
&= \int_t^T\int_{\Omega_0}\int_{\Omega_1}\int_{\mathbb{R}^n}|\tilde{v}(\omega_0,s,X(\omega_1))|^2\,\mathrm{d}m(x)\,\mathrm{d}\mathbb{P}_1(\omega_1)\,\mathrm{d}\mathbb{P}_0(\omega_0)\,\mathrm{d}s\\
&= \int_t^T\int_{\Omega_0}\int_{\mathbb{R}^n}|\tilde{v}(\omega_0,s,x)|^2\,\mathrm{d}(X\otimes m)(x)\,\mathrm{d}\mathbb{P}_0(\omega_0)\,\mathrm{d}s\\
&= \|\tilde{v}\|^2_{L^2_{\mathcal{W}_t}(t,T;\mathcal{H}_{X\otimes m})},
\end{aligned}\tag{3.1}$$

which proves $\tilde{v}\in L^2_{\mathcal{W}_t}(t,T;\mathcal{H}_{X\otimes m})$ and also that $v\mapsto\tilde{v}$ is an isometry. To see that $\tilde{v}$ is unique, observe that if $\tilde{v}'$ is another element of $L^2_{\mathcal{W}_t}(t,T;\mathcal{H}_{X\otimes m})$ such that $v(\omega,s,x) = \tilde{v}'(\omega_0,s,X(\omega_1,x))$, then for any other $u\in L^2_{\mathcal{W}_t}(t,T;\mathcal{H}_{X\otimes m})$, we have

$$\begin{aligned}
&\int_t^T\int_{\Omega_0}\int_{\mathbb{R}^n}u(\omega_0,s,x)\cdot\tilde{v}'(\omega_0,s,x)\,\mathrm{d}(X\otimes m)(x)\,\mathrm{d}\mathbb{P}_0(\omega_0)\,\mathrm{d}s\\
&\quad= \int_t^T\int_{\Omega_0}\int_{\Omega_1}\int_{\mathbb{R}^n}u(\omega_0,s,X(\omega_1))\cdot\tilde{v}'(\omega_0,s,X(\omega_1))\,\mathrm{d}m(x)\,\mathrm{d}\mathbb{P}_1(\omega_1)\,\mathrm{d}\mathbb{P}_0(\omega_0)\,\mathrm{d}s\\
&\quad= \int_t^T\int_{\Omega_0}\int_{\Omega_1}\int_{\mathbb{R}^n}u(\omega_0,s,X(\omega_1))\cdot\tilde{v}(\omega_0,s,X(\omega_1))\,\mathrm{d}m(x)\,\mathrm{d}\mathbb{P}_1(\omega_1)\,\mathrm{d}\mathbb{P}_0(\omega_0)\,\mathrm{d}s\\
&\quad= \int_t^T\int_{\Omega_0}\int_{\mathbb{R}^n}u(\omega_0,s,x)\cdot\tilde{v}(\omega_0,s,x)\,\mathrm{d}(X\otimes m)(x)\,\mathrm{d}\mathbb{P}_0(\omega_0)\,\mathrm{d}s
\end{aligned}\tag{3.2}$$

where we have used the fact that $f\mapsto f\circ X$ is an isometry from $L^2_{X\otimes m}$ to $L^2_{\mathbb{P}\times m}$ to see that $u(\omega,s,X(\omega_1))$ defines an element in $L^2_{W_{Xt}}(t,T;\mathcal{H}_m)$. It follows that $\tilde{v}'=\tilde{v}$, so $v\mapsto\tilde{v}$ is a well-defined isometry. Linearity is easily checked. ☐

Now for a given $v\in L^2_{\mathcal{W}_{Xt}}(t,T;\mathcal{H}_m)$, which we also denote v_{Xt} to emphasis the measurability constraint, we consider the SDE

$$X(s) = X + \int_t^s v(\tau)\,\mathrm{d}\tau + \sigma(w(s)-w(t)),\tag{3.3}$$

where σ is a fixed, deterministic $n\times n$ matrix, which is symmetric and positive definite. Equation (3.3) defines a process $X(\cdot)\in L^2_{\mathcal{W}_{Xt}}(t,T;\mathcal{H}_m)$, which we will denote $X_{Xt}(s) = X_{Xt}(s;v_{Xt}(\cdot))$. Indeed,

$$\begin{aligned}
\mathbb{E}\|X(s)\|^2_{\mathcal{H}_m} &\le 3\|X\|^2_{\mathcal{H}_m} + 3(s-t)\mathbb{E}\int_t^s\|v(\tau)\|^2_{\mathcal{H}_m}\,\mathrm{d}\tau + 3|\sigma|^2\,\mathbb{E}|w(s)-w(t)|^2\\
&\le 3\|X\|^2_{\mathcal{H}_m} + 3(s-t)\|v\|^2_{L^2_{\mathcal{W}_{Xt}}(t,T;\mathcal{H}_m)} + 3|\sigma|^2\,(s-t),
\end{aligned}\tag{3.4}$$

where $|\sigma|$ denotes the matrix norm of σ.

Remark 3.2. Let $X(\cdot)$ be the solution of (3.3). It is critical to observe that $X(s)\in\mathcal{H}_{m,\tau}$ whenever $\tau\ge s\ge t$, i.e. $X(s)$ is independent of $\mathcal{W}_\tau$. To see this, notice that $X(s)$ is $\mathcal{W}^s_{Xt}$-measurable, where $\mathcal{W}^s_{Xt} = \sigma(X)\vee\mathcal{W}^s_t$ by definition. As $\mathcal{W}^s_t$ is independent of $\mathcal{W}_\tau$ by independent increments and $\sigma(X)$ is independent of $\mathcal{W}_t$ by assumption, we conclude that $X(s)$ is indeed independent of $\mathcal{W}_\tau$.

We can also interpret (3.3) as a finite-dimensional SDE. Let $\tilde{v} \in L^2_{\mathcal{W}_t}(t, ; \mathcal{H}_{X\otimes m})$ be the representative of v given by Lemma 3.1. For m-a.e. x, consider

$$x(s) = x + \int_t^s \tilde{v}(\tau, x)\,\mathrm{d}\tau + \sigma(w(s) - w(t)). \tag{3.5}$$

This defines a unique solution $x(\cdot) \in L^2_{\mathcal{W}_t}(t, T; \mathbb{R}^n)$, which we denote $x(s; x, \tilde{v}(\cdot, x))$. By viewing each term in (3.3) as an element in $L^2_m(\mathbb{R}^n; \mathbb{R}^n)$ and evaluating at x, we have the relation

$$X_{Xt}(s; v_{Xt}(\cdot))(x) = x(s; X(x), \tilde{v}(\cdot, X(x))), \tag{3.6}$$

for m-a.e. x. More precisely, using the decomposition $\Omega = \Omega_0 \times \Omega_1$ as above, we can write for $\omega = (\omega_0, \omega_1)$

$$X_{Xt}(\omega, s; v_{Xt}(\cdot))(x) = x(\omega_0, s; X(\omega_1, x), \tilde{v}(\cdot, X(\omega_1, x))) \tag{3.7}$$

for m-a.e. x.

Lemma 3.3. *The law of $X_{Xt}(s; v_{Xt}(\cdot))$, considered as a random variable on the product space $\Omega \times m$, is $x(s; \cdot, \tilde{v}(\cdot, \cdot)) \otimes (X \otimes m)$.*

Proof. First, notice from (3.5) that

$$\mathbb{E}\int_{\mathbb{R}^n} |x(s; x, \tilde{v}(\cdot, x))|^2\,\mathrm{d}m(x) \le 3\int_{\mathbb{R}^n} |x|^2\,\mathrm{d}m(x) + 3(s-t)\|\tilde{v}\|^2_{L^2_{\mathcal{W}_t}(t,T;\mathcal{H}_{X\otimes m})} + 3|\sigma|^2 (s-t), \tag{3.8}$$

and thus, $\mathbb{P}_0$-a.s., $x(s; \cdot, \tilde{v}(\cdot, \cdot)) \in L^2_m(\mathbb{R}^n; \mathbb{R}^n)$. Now let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function such that $x \mapsto \frac{\phi(x)}{1+|x|^2}$ is bounded. Then, using Fubini's Theorem, we have

$$\begin{aligned}\mathbb{E}\int_{\mathbb{R}^n} \phi\left(X_{Xt}(s; v)(x)\right)\mathrm{d}m(x) &= \int_\Omega \int_{\mathbb{R}^n} \phi\left(x(\omega_0, s; X(\omega_1, x), \tilde{v}(\cdot, X(\omega_1, x)))\right)\mathrm{d}m(x)\,\mathrm{d}\mathbb{P}_0(\omega_0, \omega_1)\\ &= \int_{\Omega_0}\int_{\mathbb{R}^n} \phi\left(x(\omega_0, s; x, \tilde{v}(\cdot, x))\right)\mathrm{d}(X \otimes m)(x)\,\mathrm{d}\mathbb{P}_0(\omega_0).\end{aligned} \tag{3.9}$$

In particular, using the fact that the left-hand side is finite by (3.4), we deduce that $x(s; \cdot, \tilde{v}(\cdot, \cdot)) \in L^2_{X\otimes m}(\mathbb{R}^n; \mathbb{R}^n)$. Therefore, we deduce

$$\mathbb{E}\int_{\mathbb{R}^n} \phi\left(X_{Xt}(s; v)(x)\right)\mathrm{d}m(x) = \int_{\mathbb{R}^n} \phi(x)\,\mathrm{d}\left(x(s; \cdot, \tilde{v}(\cdot, \cdot)) \otimes (X \otimes m)\right)(x), \tag{3.10}$$

which completes the proof. $\square$

As a result of Lemma 3.3, we write

$$X_{Xt}(s; v_{Xt}(\cdot)) \otimes m := x(s; \cdot, \tilde{v}(\cdot, \cdot)) \otimes (X \otimes m), \tag{3.11}$$

even though this is an abuse of notation, as $X_{Xt}(s; v_{Xt}(\cdot))$ is not an element of $\mathcal{H}_m$. (Note, however, that it is *almost surely* an element of $\mathcal{H}_m$.)

To conclude this subsection, we introduce the following conventions.

- The symbol $v_{\xi t}(\cdot)$ (or possibly $v_{xt}(\cdot)$), with a lower-case letter as its first subscript, will actually refer to the vector field $\tilde{v} \in L^2_{\mathcal{W}_t}(t, T; \mathcal{H}_{X\otimes m})$. Meanwhile v itself will always be denoted by inserting the argument X, i.e. $v_{Xt}(\cdot)$. This accords with Lemma 3.1. Note that $v_{Xt}(\cdot)$ here refers to an element of $L^2_{\mathcal{W}_{Xt}}(t, T; \mathcal{H}_m)$, and we may write $v_{X.t}(\cdot)$ to emphasize this point.

- In a similar spirit, the symbol $X_{\xi t}(\cdot)$ (or any other lower-case letter in place of ξ) will actually refer to $x(\cdot; \xi, \tilde{v}(\cdot, \xi))$, i.e. an element of $L^2_{\mathcal{W}_t}(t, T; \mathbb{R}^n)$. If we plug in the random variable X, we recover $X_{Xt}(\cdot)$, the trajectory in $L^2_{\mathcal{W}_{Xt}}(t, T; \mathcal{H}_m)$ driven by the control $v_{Xt}(\cdot)$. Thus (3.11) becomes

$$X_{Xt}(s) \otimes m \;(= X_{Xt}(s; v_{Xt}(\cdot)) \otimes m) = X_{\cdot t}(s; v_\cdot(\cdot)) \otimes (X_{\cdot t} \otimes m). \tag{3.12}$$

Although we risk some confusion in using these conventions, which are technically an abuse of notation, we nevertheless believe that their use in the following arguments are sufficiently clear. They are also evocative, in that v_{Xt} and $v_{\xi t}$ are, by Lemma 3.1, not essentially distinct objects, but merely the same object expressed in different spaces (the same remark applies to X_{Xt} and $X_{\xi t}$).

3.2 CONTROL PROBLEM

Recall that $t \geq 0$ and $X \in \mathcal{H}_{m,t}$ are fixed. Consider a state process $X_{Xt}(s) = X_{Xt}(s; v_{Xt}(\cdot))$ associated to a control $v_{Xt}(\cdot)$. Define the cost functional $J_{X\otimes m,t} : L^2_{\mathcal{W}_t}(t, T; \mathcal{H}_{X\otimes m}) \to \mathbb{R}$ by

$$\begin{aligned} J_{X\otimes m,t}(v_{\cdot t}(\cdot)) = \frac{\lambda}{2} \int_t^T \int_{\mathbb{R}^n} \mathbb{E}|v_{\xi t}(s)|^2 \,\mathrm{d}(X \otimes m)(\xi)\,\mathrm{d}s \\ + \int_t^T F(X_{\cdot t}(s; v_{\cdot t}(\cdot)) \otimes (X_{\cdot t} \otimes m))\,\mathrm{d}s + F_T(X_{\cdot t}(T; v_{\cdot t}(\cdot)) \otimes (X_{\cdot t} \otimes m)). \end{aligned} \tag{3.13}$$

An equivalent and more condensed version of $J_{X\otimes m,t}$ is the functional $J_{Xt} : L^2_{\mathcal{W}_{X_t}}(t, T; \mathcal{H}_m) \to \mathbb{R}$ given by

$$J_{Xt}(v_{Xt}(\cdot)) = \frac{\lambda}{2} \int_t^T ||v_{Xt}(s)||^2\,\mathrm{d}s + \int_t^T F(X_{Xt}(s; v_{Xt}(\cdot)) \otimes m)\,\mathrm{d}s + F_T(X_{Xt}(T; v_{Xt}(\cdot)) \otimes m) \tag{3.14}$$

We shall make precise assumptions to guarantee the strict convexity of the functional $J_{Xt}(v_{Xt}(\cdot))$, its coeciveness, hence existence and uniqueness of an optimal minimum, for which we shall write the necessary and sufficient optimality conditions.

3.3 ASSUMPTIONS ON COST FUNCTIONAL

We describe assumptions on the functionals $X \mapsto F(X \otimes m)$ and $X \mapsto F_T(X \otimes m)$ on $\mathcal{H}_m$. Throughout these assumptions, c, c_T, c', and c'_T are fixed positive constants. We first assume a growth bound:

$$|F(X \otimes m)| \leq c(1 + \|X\|^2),\ |F_T(X \otimes m)| \leq c_T(1 + \|X\|^2). \tag{3.15}$$

We also assume the functionals have Lipschitz continuous Gâteaux derivatives satisfying

$$||D_X F(X \otimes m)|| \leq c(1 + \|X\|),\ ||D_X F_T(X \otimes m)|| \leq c_T(1 + \|X\|) \tag{3.16}$$

$$||D_X F(X_1 \otimes m) - D_X F(X_2 \otimes m)|| \leq c||X_1 - X_2||,\ ||D_X F_T(X_1 \otimes m) - D_X F_T(X_2 \otimes m)|| \leq c_T||X_1 - X_2|| \tag{3.17}$$

as well as the following monotonicity conditions:

$$\langle D_X F(X_1 \otimes m) - D_X F(X_2 \otimes m), X_1 - X_2 \rangle \geq -c'||X_1 - X_2||^2 \tag{3.18}$$

$$\langle D_X F_T(X_1 \otimes m) - D_X F_T(X_2 \otimes m), X_1 - X_2 \rangle \geq -c'_T||X_1 - X_2||^2. \tag{3.19}$$

If the second order derivatives $D^2_X F(X \otimes m)(Z)$ and $D^2_X F_T(X \otimes m)(Z)$ exist, we can verify (3.17),(3.18), (3.19) by assuming

$$||D^2_X F(X \otimes m)(Z)|| \leq c||Z||,\ ||D^2_X F_T(X \otimes m)(Z)|| \leq c_T||Z||,\ \forall X, Z \in \mathcal{H}_m \tag{3.20}$$

$$D^2_X F(X \otimes m)(Y, Y) \geq -c'||Y||^2,\ D^2_X F_T(X \otimes m)(Y, Y) \geq -c'_T||Y||^2 \tag{3.21}$$

and we recall (2.48), so (3.20) means

$$\left| D^2 \frac{\mathrm{d}F}{\mathrm{d}m}(m)(x) \right| \leq c,\ \left| D_2 D_1 \frac{\mathrm{d}^2 F}{\mathrm{d}m^2}(m)(\tilde{x}, x) \right| \leq c \tag{3.22}$$

If we recall (2.45), (3.21) is equivalent to

$$\begin{aligned} \mathbb{E} \int_{\mathbb{R}^n} D^2 \frac{\mathrm{d}F}{\mathrm{d}m}(X \otimes m)(X(x))Y(x) \cdot Y(x)\,\mathrm{d}m(x) \\ + \mathbb{E}\tilde{\mathbb{E}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} D_2 D_1 \frac{\mathrm{d}^2 F}{\mathrm{d}m^2}(X \otimes m)(\tilde{X}(\tilde{x}), X(x))\tilde{Y}(\tilde{x}) \cdot Y(x)\,\mathrm{d}m(\tilde{x})\,\mathrm{d}m(x) \geq -c'\mathbb{E} \int_{\mathbb{R}^n} |Y(x)|^2\,\mathrm{d}m(x), \end{aligned} \tag{3.23}$$

$$
\begin{aligned}
&\mathbb{E}\int_{\mathbb{R}^n} D^2\,\frac{\mathrm{d}F_T}{\mathrm{d}m}\,(X\otimes m)(X(x))Y(x)\cdot Y(x)\,\mathrm{d}m(x)\\
&\quad+\mathbb{E}\tilde{\mathbb{E}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} D_2D_1\,\frac{\mathrm{d}^2F_T}{\mathrm{d}m^2}\,(X\otimes m)(\tilde{X}(\tilde{x}),X(x))\tilde{Y}(\tilde{x})\cdot Y(x)\,\mathrm{d}m(\tilde{x})\,\mathrm{d}m(x)\geq -c_T'\mathbb{E}\int_{\mathbb{R}^n}|Y(x)|^2\,\mathrm{d}m(x).
\end{aligned}
\tag{3.24}
$$

Considering the map $v_{Xt}(\cdot)\to J_{Xt}(v_{Xt}(\cdot))$ as a functional on the Hilbert space $L^2_{\mathcal{W}_{Xt}}(t,T;\mathcal{H}_m)$, we get the

Lemma 3.4. *Under the assumptions (3.15),(3.16) the functional $J_{Xt}(v_{Xt}(\cdot))$ has a Gâteaux derivative, given by*

$$
\begin{aligned}
&D_vJ_{Xt}(v_{Xt}(\cdot))(s)=\lambda v_{Xt}(s)\\
&\qquad\qquad+\mathbb{E}\left[\int_s^T D_XF(X_{Xt}(\tau;v_{Xt}(\cdot))\otimes m)\,\mathrm{d}\tau+D_XF(X_{Xt}(T;v_{Xt}(\cdot))\otimes m)\middle|\,\mathcal{W}^s_{Xt}\right].
\end{aligned}
\tag{3.25}
$$

The proof can be found in Appendix A.

Remark 3.5. We can replace in (3.25) the conditional expectation with respect to $\mathcal{W}^s_{Xt}=\sigma(X)\vee\mathcal{W}^s_t$ by the conditional expectation with respect to $\mathcal{B}\vee\mathcal{W}^s_t$, for some $\mathcal{B}$ independent of $\mathcal{W}_s$ such that $\sigma(X)\subset\mathcal{B}$. Indeed, the random variable $\int_s^T D_XF(X_{Xt}(\tau;v_{Xt}(.))\otimes m)\,\mathrm{d}\tau+D_XF(X_{Xt}(T;v_{Xt}(.))\otimes m)$ is $\sigma(X)\vee\mathcal{W}^T_t$ measurable, and $\sigma(X)\vee\mathcal{W}^T_t=\sigma(X)\vee\mathcal{W}^s_t\vee\mathcal{W}^T_s$. But $\sigma(X)\vee\mathcal{W}^s_t\subset\mathcal{B}\vee\mathcal{W}^s_t$, which is independent of $\mathcal{W}^T_s$. It follows that the conditional expetation with respect to $\mathcal{B}\vee\mathcal{W}^s_t$ is the same as the conditional expectation with respect to $\sigma(X)\vee\mathcal{W}^s_t$. This remark will be very useful for comparison purposes.

3.4 CONVEXITY OF THE OBJECTIVE FUNCTIONAL

The following result gives conditions that imply the existence of a unique solution to the optimal control problem. From now on we will assume that these conditions hold.

Proposition 3.6. *Assume 3.15,3.16,3.18 and*

$$
\lambda-T(c_T'+\frac{c'T}{2})>0. \tag{3.26}
$$

Then the functional $J_{Xt}(v_{Xt}(.))$ is strictly convex. It is also coercive, i.e. $J_{Xt}(v_{Xt}(.))\to+\infty$, as $\int_t^T||v_{Xt}(s)||^2\,\mathrm{d}s\to+\infty$. Consequently, there exists one and only one minimizer of $J_{Xt}(v_{Xt}(.))$.

The proof can be found in Appendix A.

3.5 NECESSARY AND SUFFICIENT CONDITION FOR OPTIMALITY

Here and from now on, assumptions of Section 3.3 and Proposition 3.6 are in force. According to Proposition 3.6, there exists one and only one optimal control $\hat{v}_{Xt}(s)$. It must satisfy the necessary and sufficient condition $D_vJ_{Xt}(\hat{v}_{Xt}(.))(s)=0$. Calling $Y_{Xt}(s)$ the corresponding optimal state and $Z_{Xt}(s)=-\lambda\hat{v}_{Xt}(s)$, the pair $(Y_{Xt}(s),Z_{Xt}(s))$ is the unique solution of the system

$$
Y_{Xt}(s)=X-\frac{1}{\lambda}\int_t^s Z_{Xt}(\tau)\,\mathrm{d}\tau+\sigma(w(s)-w(t)), \tag{3.27}
$$

$$
Z_{Xt}(s)=\mathbb{E}\left[\int_s^T D_XF(Y_{Xt}(\tau)\otimes m)\,\mathrm{d}\tau+D_XF_T(Y_{Xt}(T)\otimes m)\middle|\,\mathcal{W}^s_{Xt}\right]. \tag{3.28}
$$

Moreover, since $L^2_{\mathcal{W}_{Xt}}(t,T;\mathcal{H}_m)$ is isometric to $L^2_{\mathcal{W}_t}(t,T;\mathcal{H}_{X\otimes m})$ by Lemma 3.1, using the convention outline in Section 3.1 there exists $Y_{\xi t}(s)$,$Z_{\xi t}(s)$ belonging to $L^2_{\mathcal{W}_t}(t,T;\mathcal{H}_{X\otimes m})$ such that $Y_{Xt}(s)=Y_{\xi t}(s)|_{\xi=X}$, $Z_{Xt}(s)=$

$Z_{\xi t}(s)|_{\xi=X}$. The pair of random fields $(Y_{\cdot t}(s), Z_{\cdot t}(s))$ is the solution of

$$Y_{\xi t}(s) = \xi - \frac{1}{\lambda}\int_t^s Z_{\xi t}(\tau)\,\mathrm{d}\tau + \sigma(w(s)-w(t)), \tag{3.29}$$

$$Z_{\xi t}(s) = \mathbb{E}\left[\int_s^T D\frac{\mathrm{d}F}{\mathrm{d}m}(Y_{\cdot t}(\tau)\otimes(X\otimes m))(Y_{\xi t}(\tau))\,\mathrm{d}\tau + D\frac{\mathrm{d}F_T}{\mathrm{d}m}(Y_{\cdot t}(T)\otimes(X\otimes m))(Y_{\xi t}(T))\middle|\,\mathcal{W}_t^s\right]\cdot \tag{3.30}$$

Indeed, by Proposition 2.8 we have $D\dfrac{\mathrm{d}F}{\mathrm{d}m}(Y_{\cdot t}(\tau)\otimes(X\otimes m))(Y_{\xi t}(\tau))|_{\xi=X} = D_XF(Y_{Xt}(\tau)\otimes m)$, and similarly for F_T. We notice that $Y_{\xi t}(s)$ and $Z_{\xi t}(s)$ depend on m only through $X\otimes m$, so we can write them $Y_{\xi,X\otimes m,t}(s)$ and $Z_{\xi,X\otimes m,t}(s)$, respectively.

Remark 3.7. The well-posedness of forward-backward systems such as (3.27)-(3.28) or (3.29)-(3.30) follows from the estimates assumed on F and F_T using standard arguments, because they are necessary conditions for a strictly convex minimization problem.

We can express the value function as

$$V(X,t) := J_{Xt}(\hat{v}_{Xt}(\cdot)) = \frac{1}{2\lambda}\int_t^T ||Z_{Xt}(s)||^2\,\mathrm{d}s + \int_t^T F(Y_{Xt}(s)\otimes m)\,\mathrm{d}s + F_T(Y_{Xt}(T)\otimes m), \tag{3.31}$$

which depends only on the probability measure $X\otimes m$ and t. Equivalently, by a slight abuse of notation, the value function can be written as follows:

$$\begin{aligned} V(X\otimes m,t) := J_{X\otimes m,t}(\hat{v}_{\cdot t}(\cdot)) = \frac{1}{2\lambda}\int_t^T \mathbb{E}\int_{\mathbb{R}^n} |Z_{\xi,X\otimes m,t}(s)|^2\,\mathrm{d}(X\otimes m)(\xi)\,\mathrm{d}s \\ + \int_t^T F(Y_{\cdot,X\otimes m,t}(s)\otimes(X\otimes m))\,\mathrm{d}s + F_T(Y_{\cdot,X\otimes m,t}(T)\otimes(X\otimes m)). \end{aligned} \tag{3.32}$$

Cf. Section 3.2.

3.6 DYNAMIC OPTIMALITY PRINCIPLE

For a fixed $h>0$, consider $Y_{Xt}(t+h)$, which is an element of $\mathcal{H}_{m,t+h}$ (i.e. independent of $\mathcal{W}_{t+h}$–see Section 3.1) that is also $\mathcal{W}_{Xt}^{t+h}$ measurable. We can then consider a control problem starting at $t+h$ instead of t with initial value $Y_{Xt}(t+h)$, or simply $Y(t+h)$ if there is no danger of ambiguity. To compute the optimal trajectory, we find the unique solution $\left(Y_{Y(t+h),t+h}(s), Z_{Y(t+h),t+h}(s)\right)$, $s\in[t+h,T]$ to a forward-backward system, similar to (3.27)- (3.28) (see Remark 3.7):

$$Y_{Y(t+h),t+h}(s) = Y(t+h) - \frac{1}{\lambda}\int_{t+h}^s Z_{Y(t+h),t+h}(\tau)\,\mathrm{d}\tau + \sigma(w(s)-w(t+h)) \tag{3.33}$$

$$Z_{Y(t+h),t+h}(s) = \mathbb{E}\left[\int_s^T D_XF(Y_{Y(t+h),t+h}(\tau)\otimes m)\,\mathrm{d}\tau + D_XF(Y_{Y(t+h),t+h}(T)\otimes m)\middle|\,\mathcal{W}^s_{Y(t+h),t+h}\right] \tag{3.34}$$

where $\mathcal{W}^s_{Y(t+h),t+h} = \sigma(Y(t+h))\vee\mathcal{W}^s_{t+h}$. Using Remark 3.5, we can replace $\mathcal{W}^s_{Y(t+h),t+h}$ with $\mathcal{W}^s_{Xt}$ in the conditional expectation, and we see that $(Y_{Xt}(s), Z_{Xt}(s))$ is still a solution. By uniqueness of the solution we deduce

$$Y_{Y(t+h),t+h}(s) = Y_{Xt}(s) \text{ and } Z_{Y(t+h),t+h}(s) = Z_{Xt}(s)\ \forall s>t+h. \tag{3.35}$$

Consequently, we also have

$$Y_{\cdot\,Y(t+h)\otimes m,t+h}(s)\otimes(Y(t+h)\otimes m) = Y_{\cdot\,X\otimes m,t}(s)\otimes(X\otimes m) \tag{3.36}$$

Therefore, from (3.32), we get

$$\begin{aligned} V(X\otimes m,t) &= \frac{1}{2\lambda}\int_t^{t+h} ||Z_{Xt}(s)||^2\,\mathrm{d}s + \int_t^{t+h} F(Y_{Xt}(s)\otimes m)\,\mathrm{d}s \\ &+ \frac{1}{2\lambda}\int_{t+h}^T \mathbb{E}\int_{\mathbb{R}^n} |Z_{\xi\,Y(t+h)\otimes m\,t+h}(s)|^2\,\mathrm{d}\big(Y(t+h)\otimes m\big)(\xi) + \int_{t+h}^T F(Y_{\cdot\,Y(t+h)\otimes m,t+h}(s)\otimes(Y(t+h)\otimes m))\,\mathrm{d}s \\ &+ F_T(Y_{\cdot\,Y(t+h)\otimes m,t+h}(s)\otimes(Y(t+h)\otimes m)), \end{aligned}$$

and by substituting $(Y(t+h), t+h)$ for (X,t) in (3.32) and applying (3.36), we finally deduce

$$V(X\otimes m,t) = \frac{1}{2\lambda}\int_t^{t+h} ||Z_{Xt}(s)||^2 \,\mathrm{d}s + \int_t^{t+h} F(Y_{Xt}(s)\otimes m)\,\mathrm{d}s + V(Y_{Xt}(t+h)\otimes m, t+h), \tag{3.37}$$

which is the dynamic optimality principle.

4 PROPERTIES OF THE VALUE FUNCTION

In this section we systematically study the regularity of $V(X\otimes m,t)$ defined in (3.32), beginning with pointwise estimates, then proceeding to derivatives with respect to X and m, and finishing with continuity in time. Recall that $V(X\otimes m,t)$ is well-defined for any $m\in\mathcal{P}_2, t\geq 0$, and $X\in\mathcal{H}_{m,t}$, where $\mathcal{H}_{m,t}$ is the closed subspace of $\mathcal{H}_m$ of random fields $X_{\cdot t}$ independent of $\mathcal{W}_t$ (see Section 3.1).

4.1 BOUNDS

We begin with the following estimates, which express the growth rate of the value function and optimal trajectory with respect to $\|X\|$.

Proposition 4.1. *Assume (3.15), (3.16), (3.17), (3.18), and (3.26). Let $m\in\mathcal{P}_2(\mathbb{R}^n), t\geq 0$, and $X\in\mathcal{H}_{m,t}$, and let (Y_{Xt}, Z_{Xt}) be the solution of* (3.27)-(3.28)*. Then we have*

$$||Y_{Xt}(s)||,\ ||Z_{Xt}(s)|| \leq C_T(1+\|X\|),\ \forall s\in(t,T), \tag{4.1}$$

$$|V(X\otimes m,t)| \leq C_T(1+\|X\|^2)\ \forall t\geq 0, \tag{4.2}$$

where C_T is a constant depending only on the data and T, independent of X, m, s, and t.

The proof can be found in Appendix B

4.2 REGULARITY OF $V(X\otimes m,t)$ WITH RESPECT TO X

Proposition 4.2. *Assume (3.15), (3.16), (3.17), (3.18), and (3.26). Let $m\in\mathcal{P}_2(\mathbb{R}^n)$ and $t\geq 0$. Then the functional $\mathcal{H}_{m,t}\ni X\mapsto V(X\otimes m,t)$ is Gâteaux differentiable and*

$$D_X V(X\otimes m,t) = Z_{Xt}(t). \tag{4.3}$$

We also have the Lipschitz property: for all $X^1, X^2\in\mathcal{H}_{m,t}$, we have

$$||D_X V(X^1\otimes m,t) - D_X V(X^2\otimes m,t)|| \leq C_T||X^1-X^2|| \tag{4.4}$$

where C_T is a constant depending only on the data and T, independent of X^1, X^2, m, and t.

The proof can be found in Appendix B.

4.3 FUNCTIONAL DERIVATIVE OF $V(m,t)$

If we take $X_{xt}=x$, recalling that $X\otimes m = m$, we get $D_X V(m,t) = Z_{xt}(t) = Z_{xmt}(t)$, where the pair $(Y_{xmt}(s), Z_{xmt}(s))$ is the unique solution of

$$Y_{xmt}(s) = x - \frac{1}{\lambda}\int_t^s Z_{xmt}(\tau)\,\mathrm{d}\tau + \sigma(w(s)-w(t)), \tag{4.5}$$

$$Z_{xmt}(s) = \mathbb{E}\left[\int_s^T D\frac{\mathrm{d}F}{\mathrm{d}m}(Y_{\cdot mt}(\tau)\otimes m)(Y_{xmt}(\tau))\,\mathrm{d}\tau + D\frac{\mathrm{d}F_T}{\mathrm{d}m}(Y_{\cdot mt}(T)\otimes m)(Y_{xmt}(T))\middle|\mathcal{W}_t^s\right], \tag{4.6}$$

cf. (3.29)-(3.30) and Remark 3.7. Equations (4.5) and (4.6) form the system of optimality conditions for the following control problem, obtained by appropriately specifying the dynamics (3.3) and objective functional (3.13):

$$X_{xt}(s) = x + \int_t^s v_{xt}(\tau)\,\mathrm{d}\tau + \sigma(w(s) - w(t)), \tag{4.7}$$

$$J_{mt}(v_{\cdot t}(.)) = \frac{\lambda}{2}\int_t^T ||v_{\cdot t}(s)||^2_{\mathcal{H}_m}\,\mathrm{d}s + \int_t^T F(X_{\cdot t}(s; v_{\cdot t}(.)) \otimes m)\,\mathrm{d}s + F_T(X_{\cdot t}(T; v_{\cdot t}(.)) \otimes m) \tag{4.8}$$

with $v_{\cdot t}(.) \in L^2_{\mathcal{W}_t}(t, T; \mathcal{H}_m)$. From (3.32) it follows that

$$\begin{aligned} V(m,t) &= \inf_{v_{\cdot t}(\cdot)} J_{m,t}(v_{\cdot t}(\cdot)) \\ &= \frac{1}{2\lambda}\int_t^T \mathbb{E}\int_{\mathbb{R}^n} |Z_{xmt}(s)|^2\,\mathrm{d}m(x)\,\mathrm{d}s + \int_t^T F(Y_{\cdot mt}(s) \otimes m)\,\mathrm{d}s + F_T(Y_{\cdot mt}(T) \otimes m). \end{aligned} \tag{4.9}$$

Applying Proposition 4.1 (under the assumptions stated there), we have

$$\mathbb{E}|Y_{xmt}(s)|^2 \le C_T(1 + |x|^2),\ \mathbb{E}|Z_{xmt}(s)|^2 \le C_T(1 + |x|^2). \tag{4.10}$$

Equation (4.10) implies that the optimal control is an element of the set

$$\mathscr{V} := \left\{ v_{\cdot t}(\cdot) : x \mapsto \frac{v_{xt}(\cdot)}{1 + |x|^2} \in L^\infty(\mathbb{R}^n; L^2_{\mathcal{W}_t}(t, T; \mathbb{R}^n)) \right\}, \tag{4.11}$$

and therefore the value function $V(m,t)$ remains unchanged if we restrict the domain of $J_{m,t}$ to $\mathscr{V}$. A crucial fact is that

$$\mathscr{V} \subset \cap_{\mu \in \mathcal{P}_2} L^2_{\mathcal{W}_t}(t, T; \mathcal{H}_\mu), \tag{4.12}$$

which is proved in the same way as Lemma 2.9. Thus, for a given $v_{\cdot t}(\cdot) \in \mathscr{V}$, $J_{m,t}(v_{\cdot t}(\cdot))$ is defined for *all* $m \in \mathcal{P}_2$. This will be used to prove the following:

Proposition 4.3. *Assume (3.15), (3.16), (3.17), (3.18), and (3.26). Then the value function $V(m,t)$ has a functional derivative $\frac{\mathrm{d}}{\mathrm{d}m}V(m,t)(x)$, given by*

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}m}V(m,t)(x) \\ &= \frac{1}{2\lambda}\int_t^T \mathbb{E}|Z_{xmt}(s)|^2\,\mathrm{d}s + \int_t^T \mathbb{E}\frac{\mathrm{d}F}{\mathrm{d}m}(Y_{\cdot mt}(s) \otimes m)(Y_{xmt}(s))\,\mathrm{d}s + \mathbb{E}\frac{\mathrm{d}F_T}{\mathrm{d}m}(Y_{\cdot mt}(T) \otimes m)(Y_{xmt}(T)). \end{aligned} \tag{4.13}$$

Moreover,

$$D\frac{\mathrm{d}}{\mathrm{d}m}V(m,t)(x) = Z_{xmt}(t) \text{ and } D_X V(X \otimes m, t) = D\frac{\mathrm{d}}{\mathrm{d}m}V(X \otimes m, t)(X). \tag{4.14}$$

Proof. Consider $J_{m,t}(v_{\cdot t}(\cdot))$, defined in (4.8), for an arbitrary $v_{\cdot t}(\cdot) \in \mathscr{V}$. It is straightforward to see that $m \mapsto J_{m,t}(v_{\cdot t}(\cdot))$ is continuously differentiable and has a functional derivative given by the expression

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}m}J_{m,t}(v_{\cdot t}(\cdot))(x) = \frac{\lambda}{2}\int_t^T \mathbb{E}|v_{xt}(s)|^2\,\mathrm{d}s + \int_t^T \mathbb{E}\frac{\mathrm{d}F}{\mathrm{d}m}(X_{\cdot t}(s; v_{\cdot t}(\cdot)) \otimes m)(X_{xt}(s; v_{xt}(\cdot)))\,\mathrm{d}s \\ + \mathbb{E}\frac{\mathrm{d}F_T}{\mathrm{d}m}(X_{\cdot t}(T; v_{\cdot t}(\cdot)) \otimes m)(X_{xt}(s; v_{xt}(\cdot))). \end{aligned} \tag{4.15}$$

Thus (4.13) follows by a simple application of the Envelope theorem. Next we consider the following ordinary stochastic control problem depending parametrically on the function $s \mapsto Y_{\cdot mt}(s) \otimes m$ with values in $\mathcal{P}_2(\mathbb{R}^n)$. We take controls in $L^2_{\mathcal{W}_t}(t, T; \mathbb{R}^n)$ and the state is defined by

$$x_{xt}(s) = x + \int_t^s v(\tau)\,\mathrm{d}\tau + \sigma(w(s) - w(t)) \tag{4.16}$$

The cost to minimize is given by

$$K_{xt}(v(\cdot))=\frac{\lambda}{2}\mathbb{E}\int_t^T|v(s)|^2\,\mathrm{d}s+\int_t^T\mathbb{E}\,\frac{\mathrm{d}F}{\mathrm{d}m}(Y_{\cdot mt}(s)\otimes m)(x_{xt}(s))\,\mathrm{d}s+\mathbb{E}\,\frac{\mathrm{d}F_T}{\mathrm{d}m}(Y_{\cdot mt}(T)\otimes m)(x_{xt}(T)).\quad(4.17)$$

It is straightforward to check that the optimal control coincides with $-\frac{1}{\lambda}Z_{xmt}(s)$ and the optimal state is $Y_{xmt}(s)$; indeed, one writes the necessary conditions of optimality and observes that they coincide with system (4.5)-(4.6), whose solution is unique. Therefore, (4.13) implies

$$\frac{\mathrm{d}}{\mathrm{d}m}V(m,t)(x)=\inf_{v(\cdot)}K_{xt}(v(\cdot)).\tag{4.18}$$

But $x\mapsto K_{xt}(v(\cdot))$ is differentiable, with

$$D_1K_{xt}(v(\cdot))=\int_t^T\mathbb{E}\,D\frac{\mathrm{d}F}{\mathrm{d}m}(Y_{\cdot mt}(s)\otimes m)(x_{xt}(s))\,\mathrm{d}s+\mathbb{E}\,D\frac{\mathrm{d}F_T}{\mathrm{d}m}(Y_{\cdot mt}(T)\otimes m)(x_{xt}(T))\tag{4.19}$$

Applying the envelope theorem again, we deduce

$$D\frac{\mathrm{d}}{\mathrm{d}m}V(m,t)(x)=\int_t^T\mathbb{E}\,D\frac{\mathrm{d}F}{\mathrm{d}m}(Y_{\cdot mt}(s)\otimes m)(Y_{xmt}(s))\,\mathrm{d}s+\mathbb{E}\,D\frac{\mathrm{d}F_T}{\mathrm{d}m}(Y_{\cdot mt}(T)\otimes m)(Y_{xmt}(T)),$$

which proves the first part of (4.14). The second part follows immediately from Proposition 2.8. This completes the proof. □

We can also extend the result (4.14) as follows

Proposition 4.4. *Assume (3.15), (3.16), (3.17), (3.18), and (3.26). Then for any $x\in\mathbb{R}^n$, we have*

$$Z_{xmt}(s)=D\frac{\mathrm{d}V}{\mathrm{d}m}(Y_{\cdot mt}(s)\otimes m,s)(Y_{xmt}(s))\quad\forall s\in[t,T].\tag{4.20}$$

Proof. We consider first the processes $Y_{\xi,Y_{\cdot mt}(s)\otimes m,s}(\tau),Z_{\xi,Y_{\cdot mt}(s)\otimes m,s}(\tau)$, defined as the solution of the system

$$Y_{\xi,Y_{\cdot mt}(s)\otimes m,s}(\tau)=\xi-\frac{1}{\lambda}\int_s^\tau Z_{\xi,Y_{\cdot mt}(s)\otimes m,s}(\theta)\,\mathrm{d}\theta+\sigma(w(\tau)-w(s)),\tag{4.21}$$

$$\begin{aligned}Z_{\xi,Y_{\cdot mt}(s)\otimes m,s}(\tau)=\mathbb{E}\Bigg[&\int_\tau^T D\frac{\mathrm{d}F}{\mathrm{d}m}(Y_{\cdot,Y_{\cdot mt}(s)\otimes m,s}(\theta)\otimes(Y_{\cdot mt}(s)\otimes m))(Y_{\xi,Y_{\cdot mt}(s)\otimes m,s}(\theta))\,\mathrm{d}\theta\\&+D\frac{\mathrm{d}F_T}{\mathrm{d}m}(Y_{\cdot,Y_{\cdot mt}(s)\otimes m,s}(T)\otimes(Y_{\cdot mt}(s)\otimes m))(Y_{\xi,Y_{\cdot mt}(s)\otimes m,s}(T))\Bigg|\mathcal{W}_s^\tau\Bigg],\end{aligned}\tag{4.22}$$

cf. (4.5)-(4.6) Then, applying the first part of (4.14) we have

$$Z_{\xi,Y_{\cdot mt}(s)\otimes m,s}(s)=D\frac{\mathrm{d}V}{\mathrm{d}m}(Y_{\cdot mt}(s)\otimes m,s)(\xi)\tag{4.23}$$

Next, define

$$\tilde{Y}_{xmt}(s,\tau)=Y_{Y_{xmt}(s),Y_{\cdot mt}(s)\otimes m,s}(\tau),\ \tilde{Z}_{xmt}(s,\tau)=Z_{Y_{xmt}(s),Y_{\cdot mt}(s)\otimes m,s}(\tau)\tag{4.24}$$

then, from (4.23) we can write

$$\tilde{Z}_{xmt}(s,s)=D\frac{\mathrm{d}V}{\mathrm{d}m}(Y_{\cdot mt}(s)\otimes m,s)(\tilde{Y}_{xmt}(s,s))\tag{4.25}$$

We are going to show that

$$\tilde{Y}_{xmt}(s,\tau)=Y_{xmt}(\tau),\ \tilde{Z}_{xmt}(s,\tau)=Z_{xmt}(\tau)\quad\forall t\le s\le\tau\le T.\tag{4.26}$$

Equation (4.26) implies $\tilde{Y}_{xmt}(s,s) = Y_{xmt}(s)$, $\tilde{Z}_{xmt}(s,s) = Z_{xmt}(s)$, and so the result (4.20) will follow immediately from (4.25).

It remains to check (4.26). We notice that $Y_{\cdot,Y_{\cdot mt}(s)\otimes m,s}(\theta)\otimes(Y_{\cdot mt}(s)\otimes m) = \tilde{Y}_{\cdot mt}(s,\theta)\otimes m$ (see Equation (3.12)) and

$$\mathbb{E}\left[\int_\tau^T D\frac{\mathrm{d}F}{\mathrm{d}m}(\tilde{Y}_{\cdot mt}(s,\theta)\otimes m)(\tilde{Y}_{xmt}(s,\theta))\,\mathrm{d}\theta + D\frac{\mathrm{d}F_T}{\mathrm{d}m}(\tilde{Y}_{\cdot mt}(s,T)\otimes m)(\tilde{Y}_{xmt}(s,T))\middle|\mathcal{W}_t^\tau\right]$$
$$= \mathbb{E}\left[\int_\tau^T D\frac{\mathrm{d}F}{\mathrm{d}m}(\tilde{Y}_{\cdot mt}(s,\theta)\otimes m)(Y_{\xi,Y_{\cdot mt}(s)\otimes m,s}(\theta))\,\mathrm{d}\theta + \right.$$
$$\left. + \; D\frac{\mathrm{d}F_T}{\mathrm{d}m}(\tilde{Y}_{\cdot mt}(s,T)\otimes m)(Y_{\xi,Y_{\cdot mt}(s)\otimes m,s}(T))\middle|\mathcal{W}_s^\tau\right]_{|\xi=Y_{xmt}(s)}.$$

Therefore, the pair $\left(\tilde{Y}_{xmt}(s,\tau),\tilde{Z}_{xmt}(s,\tau)\right)$ is the solution to the system

$$\tilde{Y}_{xmt}(s,\tau) = x - \frac{1}{\lambda}\int_t^s Z_{xmt}(\theta)\,\mathrm{d}\theta - \frac{1}{\lambda}\int_s^\tau \tilde{Z}_{xmt}(s,\theta)\,\mathrm{d}\theta + \sigma(w(\tau)-w(t))$$
$$\tilde{Z}_{xmt}(s,\tau) = \mathbb{E}\left[\int_\tau^T D\frac{\mathrm{d}F}{\mathrm{d}m}(\tilde{Y}_{\cdot mt}(s,\theta)\otimes m)(\tilde{Y}_{xmt}(s,\theta))\,\mathrm{d}\theta + D\frac{\mathrm{d}F_T}{\mathrm{d}m}(\tilde{Y}_{\cdot mt}(s,T)\otimes m)(\tilde{Y}_{xmt}(s,T))\middle|\mathcal{W}_t^\tau\right].$$

By taking $s=\tau$ in (4.5)- (4.6), we see that $\left(Y_{xmt}(\tau), Z_{zmt}(\tau)\right)$ is another solution, and so by uniqueness we deduce (4.26). The proof is complete. □

By (2.25) we can also write (4.20) as

$$Z_{\cdot mt}(s) = D_X V(Y_{\cdot mt}(s)\otimes m, s). \tag{4.27}$$

In fact, by a similar argument, we can derive the identity

$$Z_{Xt}(s) = D_X V(Y_{Xt}(s)\otimes m, s) \quad \forall s\in[t,T],\ \forall X\in\mathcal{H}_m. \tag{4.28}$$

The proof is quite similar to that of Proposition 4.4. Indeed, one considers the solution $\left(Y_{Xt}(s), Z_{Xt}(s)\right)$ of the system (3.27)-(3.28) and then repeats the same reasoning as above; we omit the details. It follows that $Y_{Xt}(s)$ can be expressed as the solution of a stochastic differential equation in $\mathcal{H}_m$, namely

$$Y_{Xt}(s) = X - \frac{1}{\lambda}\int_t^s D_X V(Y_{Xt}(\tau)\otimes m,\tau)\,\mathrm{d}\tau + \sigma(w(s)-w(t)) \tag{4.29}$$

4.4 REGULARITY IN TIME

Proposition 4.5. *Assume (3.15), (3.16), (3.17), (3.18), and (3.26). Let $m\in\mathcal{P}_2(\mathbb{R}^n)$ and $t\geq 0$. Then for any $X = X_{xt}\in\mathcal{H}_{m,t}$ and $h>0$ we have*

$$|V(X\otimes m,t+h) - V(X\otimes m,t)| \leq C_T h(1+\|X\|^2), \tag{4.30}$$
$$\|D_X V(X\otimes m,t+h) - D_X V(X\otimes m,t)\| \leq C_T(h^{\frac{1}{2}}+h)(1+\|X\|), \tag{4.31}$$

where C_T is a constant depending only on the data and T, independent of X, m, t, and h.

The proof can be found in Appendix B.

5 BELLMAN EQUATION

5.1 FURTHER REGULARITY ASSUMPTIONS

In order to show that the value function defined in (3.32) is a classical solution to the Bellman equation (see Equation (5.15) below), we will need additional regularity assumptions on the data. We will assume that the functionals $X \mapsto F(X \otimes m)$ and $X \mapsto F_T(X \otimes m)$ have second order Gâteaux derivatives in $\mathcal{H}_m$ (which take values in $\mathcal{L}(\mathcal{H}_m; \mathcal{H}_m)$), denoted $D_X^2 F(X \otimes m)$ and $D_X^2 F_T(X \otimes m)$, respectively. From formula (2.45) we have

$$D_X^2 F(X \otimes m)(Z) = \Phi(X)Z + \Psi_{XZ}(X) \tag{5.1}$$

where $\Phi : \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^n; \mathbb{R}^n)$ and $\Psi_{XZ} : \mathbb{R}^n \to \mathbb{R}^n$ are measurable functions. The map $\mathcal{H}_m \ni Z \to \Psi_{XZ}(x)$ is linear for each $X \in \mathcal{H}_m$ and $x \in \mathbb{R}^n$.

We also assume the following estimates:

$$||D_X^2 F(X \otimes m)|| \leq c, \ ||D_X^2 F_T(X \otimes m)|| \leq c_T, \tag{5.2}$$

$$\left\langle D_X^2 F(X \otimes m)(Z), Z \right\rangle + c'||Z||^2 \geq 0, \ \left\langle D_X^2 F_T(X \otimes m)(Z), Z \right\rangle + c'_T||Z||^2 \geq 0, \ \forall Z \in \mathcal{H}_m, \tag{5.3}$$

$$\begin{aligned} &\mathbb{E} \int_{\mathbb{R}^n} |D_X^2 F(X \otimes m)(Z)| \, \mathrm{d}m(x) \leq cE \int_{\mathbb{R}^n} |Z(x)| \, \mathrm{d}m(x), \ \forall Z \in L^1(\Omega, \mathcal{A}, P; L_m^1(\mathbb{R}^n; \mathbb{R}^n)) \\ &\mathbb{E} \int_{\mathbb{R}^n} |D_X^2 F_T(X \otimes m)(Z)| \, \mathrm{d}m(x) \leq c_T \mathbb{E} \int_{\mathbb{R}^n} |Z(x)| \, \mathrm{d}m(x), \ \forall Z \in L^1(\Omega, \mathcal{A}, P; L_m^1(\mathbb{R}^n; \mathbb{R}^n)) \end{aligned} \tag{5.4}$$

We also assume the following continuity:

$$\begin{aligned} &X \mapsto D_X^2 F(X \otimes m)(Z) \text{ is continuous from } \mathcal{H}_m \text{ to } \mathcal{H}_m, \text{for each } Z \in \mathcal{H}_m, \\ &X \mapsto D_X^2 F_T(X \otimes m)(Z) \text{ is continuous from } \mathcal{H}_m \text{ to } \mathcal{H}_m, \text{for each } Z \in \mathcal{H}_m, \end{aligned} \tag{5.5}$$

and

$$\begin{aligned} &\forall \epsilon, M > 0, \ \forall X, Z \in \mathcal{H}_m \text{ s.t. } \|Z\| \leq M, \exists \delta_X(\epsilon, M) > 0 \text{ s.t. } \|X_k - X\| \leq \delta_X(\epsilon, M) \\ &\qquad \Longrightarrow \begin{cases} \mathbb{E} \int_{\mathbb{R}^n} |D_X^2 F(X_k \otimes m)(Z) - D_X^2 F(X \otimes m)(Z)| \, \mathrm{d}m(x) \leq \epsilon, \\ \mathbb{E} \int_{\mathbb{R}^n} |D_X^2 F_T(X_k \otimes m)(Z) - D_X^2 F_T(X \otimes m)(Z)| \, \mathrm{d}m(x) \leq \epsilon. \end{cases} \end{aligned} \tag{5.6}$$

We have seen in (3.22),(3.23),(3.24) how to achieve (5.2),(5.3). We have seen in (2.48) that to get (5.2) it suffices to assume

$$\begin{aligned} &\left| D^2 \frac{\mathrm{d}F}{\mathrm{d}m}(m)(x) \right| \leq c, \left| DD_2 \frac{\mathrm{d}^2 F}{\mathrm{d}m^2}(m)(x, \tilde{x}) \right| \leq c, \\ &\left| D \frac{dF_T}{dm}(m)(x) \right| \leq c_T, \left| D_2 D_1 \frac{d^2 F_T}{dm^2}(m)(x, \tilde{x}) \right| \leq c_T, \forall x, \tilde{x} \in \mathbb{R}^n, \end{aligned} \tag{5.7}$$

where $|\cdot|$ denotes the matrix norm. We may fulfill (5.3) by assuming that

$$\begin{aligned} &D^2 \frac{\mathrm{d}F}{\mathrm{d}m}(m)(x)\xi \cdot \xi + D_2 D_1 \frac{\mathrm{d}^2 F}{\mathrm{d}m^2}(m)(x, \tilde{x})\xi \cdot \tilde{\xi} \geq -c'|\xi|(|\xi| + |\tilde{\xi}|), \\ &D^2 \frac{\mathrm{d}F}{\mathrm{d}m}(m)(x)\xi \cdot \xi + D_2 D_1 \frac{\mathrm{d}^2 F}{\mathrm{d}m^2}(m)(x, \tilde{x})\xi \cdot \tilde{\xi} \geq -c'|\xi|(|\xi| + |\tilde{\xi}|), \forall x, \tilde{x}, \xi, \tilde{\xi}. \end{aligned} \tag{5.8}$$

The bounds (5.7) also suffice to imply (5.4). Finally, to get the continuity properties in (5.5) and (5.6), it is sufficient to assume

$$\begin{aligned} &(m, x) \mapsto D^2 \frac{\mathrm{d}F}{\mathrm{d}m}(m)(x), \ (m, x, \tilde{x}) \mapsto D_2 D_1 \frac{\mathrm{d}^2 F}{\mathrm{d}m^2}(m)(x, \tilde{x}) \text{ are continuous from } \mathcal{P}_2(\mathbb{R}^n) \times \mathbb{R}^n \\ &\qquad \text{and } \mathcal{P}_2(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^n; \mathbb{R}^n), \text{ respectively,} \end{aligned} \tag{5.9}$$

and likewise for F_T. To see this, we use the identity (2.45) and apply standard arguments. Let us provide the details for the most difficult step, leaving the rest of the argument to the reader. We will show that if $X_k \to X$ in $\mathcal{H}_m$ and $||Z|| \le M$, $\forall \epsilon$, $\exists \delta_X(\epsilon, M)$ such that $||X_k - X|| \le \delta_X(\epsilon, M)$ implies

$$I_k := \\ \mathbb{E} \int_{\mathbb{R}^n} \left| \tilde{\mathbb{E}} \int_{\mathbb{R}^n} D_2 D_1 \frac{\mathrm{d}^2 F}{\mathrm{d}m^2} (X_k \otimes m)(X_k(x), \tilde{X}_k(\tilde{x})) - D_2 D_1 \frac{\mathrm{d}^2 F}{\mathrm{d}m^2} (X \otimes m)(X(x), \tilde{X}(\tilde{x})) \tilde{Z}(\tilde{x}) \, \mathrm{d}m(\tilde{x}) \right| \mathrm{d}m(x) \\ \le \epsilon. \quad (5.10)$$

By the Cauchy-Scwhartz inequality, we have $I_k \le C_M J_k$ where

$$J_k := \sqrt{\mathbb{E}\tilde{\mathbb{E}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left| D_2 D_1 \frac{\mathrm{d}^2 F}{\mathrm{d}m^2} (X_k \otimes m)(X_k(x), \tilde{X}_k(\tilde{x})) - D_2 D_1 \frac{\mathrm{d}^2 F}{\mathrm{d}m^2} (X \otimes m)(X(x), \tilde{X}(\tilde{x})) \right|^2 \mathrm{d}m(x) \, \mathrm{d}m(\tilde{x})}.$$

We claim $J_k \to 0$ as $k \to \infty$. Indeed, we can assume, by extracting a subsequence, that $X_k(x) \to X(x)$ a.s. a.e. Then by the continuity assumption (5.9) and the uniform bound (5.7), J_k tends to 0. This is true for any converging subsequence. Hence the full sequence $J_k \to 0$. Thus for any $\eta > 0$, there exists $\beta_X(\eta) > 0$ such that $\|X_k - X\| \le \beta_X(\eta) \Rightarrow J_k \le \eta$. Setting $\delta_X(\epsilon, M) = \beta_X\left(\frac{\epsilon}{C_M}\right)$ the result (5.10) is obtained.

5.2 EXISTENCE OF SECOND DERIVATIVE OF THE VALUE FUNCTION

The following crucial result provides the existence and continuity of $D_X^2 V$.

Proposition 5.1. *Assume (3.15), (3.16), (3.17), (3.18), (3.26), (5.2), (5.3), (5.5), and (5.6).*

(i) The value function $V(X \otimes m, t)$, defined in (3.32)*, has a second order Gâteaux derivative with respect to $X \in \mathcal{H}_{m,t}$, which we denote $D_X^2 V(X \otimes m, t) \in \mathcal{L}(\mathcal{H}_m; \mathcal{H}_m)$. It satisfies*

$$||D_X^2 V(X \otimes m, t)(\mathcal{X})|| \le C_T ||\mathcal{X}|| \quad (5.11)$$

where C_T is a constant not depending on X or t.

(ii) We have the following continuity property. Let $t_k \downarrow t$, and let $X_k, \mathcal{X}_k \in \mathcal{H}_{m,t_k}$ converge in $\mathcal{H}_m$ to $X, \mathcal{X}$ respectively. Then

$$D_X^2 V(X_k \otimes m, t_k)(\mathcal{X}_k) \to D_X^2 V(X \otimes m, t)(\mathcal{X}) \text{ in } \mathcal{H}_m. \quad (5.12)$$

The limits $X, \mathcal{X}$ are independent of $\mathcal{W}_t$, i.e. $X, \mathcal{X} \in \mathcal{H}_{m,t}$.

(iii) We also have a formula for the second order Gâteaux derivative. Let $\left(Y_{Xt}(s), Z_{Xt}(s)\right)$ be the solution of the system (3.27), (3.28). We define $\left(\mathcal{Y}_{X\mathcal{X}t}(s), \mathcal{Z}_{X\mathcal{X}t}(s)\right)$ to be the unique solution of the system

$$\begin{aligned} \mathcal{Y}_{X\mathcal{X}t}(s) &= \mathcal{X} - \frac{1}{\lambda} \int_t^s \mathcal{Z}_{X\mathcal{X}t}(\tau) \, \mathrm{d}\tau \\ \mathcal{Z}_{X\mathcal{X}t}(s) &= \mathbb{E}\left[\int_s^T D_X^2 F(Y_{Xt}(\tau) \otimes m)(\mathcal{Y}_{X\mathcal{X}t}(\tau)) \, \mathrm{d}\tau + D_X^2 F_T(Y_{Xt}(T) \otimes m)(\mathcal{Y}_{X\mathcal{X}t}(T)) \,\middle|\, \mathcal{W}_{X\mathcal{X}t}^s \right] \end{aligned} \quad (5.13)$$

where $\mathcal{W}_{X\mathcal{X}t}^s = \sigma(X, \mathcal{X}) \vee \mathcal{W}_t^s$. Then

$$D_X^2 V(X \otimes m, t)(\mathcal{X}) = \mathcal{Z}_{X\mathcal{X}t}(t) \quad (5.14)$$

The proof can be found in Appendix C.

5.3 BELLMAN EQUATION

The Bellman equation for the optimal control problem stated in Section 3.2 is given by

$$\begin{aligned}&\frac{\partial V}{\partial t}(X\otimes m,t)+\frac{1}{2}\left\langle D_X^2V(X\otimes m,t)(\sigma N),\sigma N\right\rangle-\frac{1}{2\lambda}||D_XV(X\otimes m,t)||^2+F(X\otimes m)=0,\\&V(X\otimes m,T)=F_T(X\otimes m).\end{aligned}\tag{5.15}$$

Let us now define what we mean by solutions to (5.15).

Definition 5.2. *Let $V:\mathcal{P}_2(\mathbb{R}^n)\times[0,T]\to\mathbb{R}$ be any function satisfying the following regularity properties:*

- *V is continuous and satisfies the estimates (4.2) and (4.30);*
- *$D_XV(X\otimes m,t)$ exists for each $X\in\mathcal{H}_{m,t}$ and is continuous, satisfying the estimates (4.4) and (4.31);*
- *$D_X^2V(X\otimes m,t)$ exists for each $X\in\mathcal{H}_{m,t}$ and is continuous, satisfying both properties (5.11) and (5.12);*
- *the continuity property (C.24) is satisfied; and*
- *$t\mapsto V(X\otimes m,t)$ is right-differentiable.*

We say that V is a classical solution *to the Bellman equation (5.15) provided that, for any $t\geq 0,m\in\mathcal{P}_2(\mathbb{R}^n)$, and $X\in\mathcal{H}_{m,t}$, and for any standard Gaussian variable N in $\mathbb{R}^n$ that is independent of the filtration $\mathcal{W}_0$ and of X, Equation (5.15) holds, where $\frac{\partial V}{\partial t}(X\otimes m,t)$ stands for the right-hand derivative.*

Theorem 5.3. *Assume (3.15), (3.16), (3.17), (3.18), (3.26), (5.2), (5.3), (5.5), (5.6), and (5.4). We also assume*

$$\lambda-T(c_T+c\frac{T}{2})>0\tag{5.16}$$

where c,c_T are the constants appearing in (5.4). Let V be the value function defined in (3.32). Then V is the unique classical solution to the Bellman equation (5.15).

The proof is given in Appendix C.

5.4 CASE $X_{xt}=x$

By inserting $X_{xt}=x$ into the Bellman equation (5.15), we can reduce it to a PDE on the space of measures. Recall that $X\otimes m=m$ (Example 2.5). By (4.14) we have $D_XV(m,t)=D\frac{\mathrm{d}}{\mathrm{d}m}V(m,t)(x)$. Therefore

$$||D_XV(X\otimes m,t)||^2=\int_{\mathbb{R}^n}\left|D\frac{\mathrm{d}}{\mathrm{d}m}V(m,t)(x)\right|^2\mathrm{d}m(x).\tag{5.17}$$

We need next to interpret $\left\langle D_X^2V(m,t)(\sigma N),\sigma N\right\rangle$. Consider the system

$$\begin{aligned}&\mathcal{Y}(s)=\sigma N-\frac{1}{\lambda}\int_t^s\mathcal{Z}(\tau)\,\mathrm{d}\tau,\\&\mathcal{Z}(s)=\mathbb{E}\left[\int_s^T D_X^2F(Y_{\cdot t}(\tau)\otimes m)(\mathcal{Y}(\tau))\,\mathrm{d}\tau+D_X^2F_T(Y_{\cdot t}(T)\otimes m)(\mathcal{Y}(T))\middle|\mathcal{W}_{Nt}^s\right],\end{aligned}\tag{5.18}$$

which we derive by taking $\mathcal{X}=\sigma N$ in (5.13). Applying Proposition 5.1, we deduce

$$\mathcal{Z}(t)=D_X^2V(m,t)(\sigma N)\tag{5.19}$$

Recall the solution $\big(Y_{xmt}(s),\, Z_{xmt}(s)\big)$ of system (4.5)-(4.6). From the regularity of F and F_T we can differentiate in x. Define

$$\mathcal{Y}_{xmt}(s) = DY_{xmt}(s),\ \mathcal{Z}_{xmt}(s) = DZ_{xmt}(s). \tag{5.20}$$

Then from (4.5)-(4.6) it follows

$$\mathcal{Y}_{xmt}(s) = I - \frac{1}{\lambda}\int_t^s \mathcal{Z}_{xmt}(\tau)\,\mathrm{d}\tau \tag{5.21}$$

$$\mathcal{Z}_{xmt}(s) = \mathbb{E}\left[\int_s^T D^2\frac{\mathrm{d}F}{\mathrm{d}m}(Y_{\cdot mt}(\tau)\otimes m)(Y_{xmt}(\tau))\mathcal{Y}_{xmt}(\tau)\,\mathrm{d}\tau + D^2\frac{\mathrm{d}F_T}{\mathrm{d}m}(Y_{\cdot mt}(T)\otimes m)(Y_{xmt}(T))\mathcal{Y}_{xmt}(T)\middle|\,\mathcal{W}_t^s\right]. \tag{5.22}$$

We claim that

$$\mathcal{Y}(s) = \mathcal{Y}_{xmt}(s)\sigma N,\ \mathcal{Z}(s) = \mathcal{Z}_{xmt}(s)\sigma N \tag{5.23}$$

is the solution of (5.18). It suffices to show that

$$D_X^2F(Y_{\cdot t}(\tau)\otimes m)(\mathcal{Y}(\tau)) = D^2\frac{\mathrm{d}}{\mathrm{d}m}F(Y_{\cdot mt}(\tau)\otimes m)(Y_{xmt}(\tau))\mathcal{Y}_{xmt}(\tau) \tag{5.24}$$

Now since $\widetilde{N}$ is independent of $\widetilde{Y}_{xmt}(\tau)$ and $\widetilde{\mathcal{Y}}_{xmt}(s)$ and has mean 0, we have

$$\widetilde{\mathbb{E}}\int_{\mathbb{R}^n} D_2D_1\frac{\mathrm{d}F}{\mathrm{d}m}(Y_{\cdot mt}(\tau)\otimes m)(Y_{xmt}(\tau),\widetilde{Y}_{xmt}(\tau))\widetilde{\mathcal{Y}}_{xmt}(s)\sigma\widetilde{N} = 0, \tag{5.25}$$

and (5.24) follows from (5.25) plugged into formula (2.45). An analogous statement holds for F_T, and so the claim follows.

Combining (5.19) and (5.23) we deduce

$$D_X^2V(m,t)(\sigma N) = \mathcal{Z}_{xmt}(t)\sigma N = D^2\frac{\mathrm{d}}{\mathrm{d}m}V(m,t)(x)\sigma N \tag{5.26}$$

and thus

$$\begin{aligned}\Big\langle D_X^2V(m,t)(\sigma N),\sigma N\Big\rangle &= \mathbb{E}\int_{\mathbb{R}^n} D^2\frac{\mathrm{d}}{\mathrm{d}m}V(m,t)(x)\sigma N\cdot(\sigma N)\,\mathrm{d}m(x)\\ &= \int_{\mathbb{R}^n}\mathrm{tr}\,\left(\sigma\sigma * D^2\frac{\mathrm{d}}{\mathrm{d}m}V(m,t)(x)\right)\mathrm{d}m(x).\end{aligned} \tag{5.27}$$

We now plug (5.17) and (5.27) into the Bellman equation (5.15) to obtain a PDE for $V(m,t)$. Introducing the second order differential operator

$$A_x\varphi(x) = -\frac{1}{2}\mathrm{tr}\,\Big(\sigma\sigma * D^2\varphi(x)\Big) \tag{5.28}$$

we obtain

$$\begin{aligned}&-\frac{\partial V}{\partial t}(m,t) + \int_{\mathbb{R}^n} A_x\frac{\mathrm{d}}{\mathrm{d}m}V(m,t)(x)\,\mathrm{d}m(x) + \frac{1}{2\lambda}\int_{\mathbb{R}^n}\left|D\frac{\mathrm{d}}{\mathrm{d}m}V(m,t)(x)\right|^2\mathrm{d}m(x) = F(m),\\ &V(m,T) = F_T(m).\end{aligned} \tag{5.29}$$

6 MASTER EQUATION

6.1 THE EQUATION

The Master Equation is given by

$$\begin{aligned}&-\frac{\partial U}{\partial t}(x,m,t) + A_xU(x,m,t) + \int_{\mathbb{R}^n} A_\xi\frac{\mathrm{d}}{\mathrm{d}m}U(\xi,m,t)(x)\,\mathrm{d}m(\xi) + \frac{1}{2\lambda}|DU(x,m,t)|^2\\ &+\frac{1}{\lambda}\int_{\mathbb{R}^n} D_\xi U(\xi,m,t)\cdot D_\xi\frac{\mathrm{d}}{\mathrm{d}m}U(\xi,m,t)(x)\,\mathrm{d}m(\xi) = \frac{\mathrm{d}}{\mathrm{d}m}F(m)(x),\\ &U(x,m,T) = \frac{\mathrm{d}}{\mathrm{d}m}F_T(m)(x).\end{aligned} \tag{6.1}$$

Equation (6.1) can be derived from (5.29) by taking a functional derivative. Indeed, let us define U to be the functional derivative of V, i.e.

$$U(x,m,t) = \frac{\mathrm{d}}{\mathrm{d}m} V(m,t)(x). \tag{6.2}$$

Formally, one differentiates Equation (5.29) with respect to m to see that (6.1) is satisfied. We will justify this calculation below in Section 6.2.

The goal of this section is to establish that the Master Equation (6.1) has a solution, which is given by (6.2). We use some additional regularity on the data, following the definition below.

Definition 6.1. *Let $c>0$ be a fixed constant. We say that a functional $F : \mathcal{P}_2(\mathbb{R}^n) \to \mathbb{R}$ is of class $\mathcal{S}_c$ provided that $D\frac{\mathrm{d}}{\mathrm{d}m}F$, $D^2\frac{\mathrm{d}}{\mathrm{d}m}F$, $D_1\frac{\mathrm{d}^2}{\mathrm{d}m^2}F$, $D_2D_1\frac{\mathrm{d}^2}{\mathrm{d}m^2}F$, $D^3\frac{\mathrm{d}}{\mathrm{d}m}F$, $D_1^2\frac{\mathrm{d}^2}{\mathrm{d}m^2}F$, and $D_1^2D_2\frac{\mathrm{d}^2}{\mathrm{d}m^2}F$ all exist, are continuous, and satisfy the following estimates for all $m \in \mathcal{P}_2(\mathbb{R}^n)$ and all $x,\tilde{x} \in \mathbb{R}^n$:*

$$\begin{gathered}
\left|D\frac{\mathrm{d}}{\mathrm{d}m}F(m)(x)\right| \le c(1+|x|), \quad \left|D^2\frac{\mathrm{d}}{\mathrm{d}m}F(m)(x)\right| \le c, \quad \left|D^3\frac{\mathrm{d}}{\mathrm{d}m}F(m)(x)\right| \le c,\\
\left|D_1\frac{\mathrm{d}^2}{\mathrm{d}m^2}F(m)(x,\tilde{x})\right| \le c(1+|\tilde{x}|), \quad \left|D_2D_1\frac{\mathrm{d}^2}{\mathrm{d}m^2}F(m)(x,\tilde{x})\right| \le c,\\
\left|D_1^2\frac{\mathrm{d}^2}{\mathrm{d}m^2}F(m)(x,\tilde{x})\right| \le c(1+|\tilde{x}|), \quad \left|D_1^2D_2\frac{\mathrm{d}^2}{\mathrm{d}m^2}F(m)(x,\tilde{x})\right| \le c.
\end{gathered} \tag{6.3}$$

We will now state our main result.

Theorem 6.2. *Assume (3.15), (3.16), (3.17), (3.18), (3.26), (5.2), (5.3), (5.5), (5.6), and (5.4). Assume also that F is of class $\mathcal{S}_c$ and F_T is of class $\mathcal{S}_{c_T}$ for some constants $c, c_T > 0$. Then there exists a λ_T large enough, depending on c, c_T, and T, such that if $\lambda \ge \lambda_T$, the following assertion holds.*

Let V be the value function defined in (3.32)*, and let U be given by* (6.2)*. Then U is a solution of Equation* (6.1) *in a pointwise sense. The derivative $\frac{\partial U}{\partial t}$ is a right-hand derivative, while all the other derivatives appearing in the equation exist and are continuous.*

The master equation (6.1) is interpreted in mean field game theory as the limiting equation for a Nash system [5]. Here we cannot interpret it in terms of Nash equilibrium, unless the corresponding Nash game is *potential* [7]. Instead we focus on the importance of U defined in (6.2) as a decoupling field for the system of necessary and sufficient conditions (4.5)-(4.6). To see this, let $\big(Y_{xmt}(s), Z_{xmt}(s)\big)$ be the solution of System (4.5)-(4.6). Combine Propositions 4.3 and 4.4 to see that $Y_{xmt}(s)$ is the solution of a stochastic differential equation

$$Y_{xmt}(s) = x - \frac{1}{\lambda}\int_t^s DU(Y_{xmt}(\tau), Y_{\cdot mt}(\tau)\otimes m, \tau)\,\mathrm{d}\tau + \sigma(w(s)-w(t)). \tag{6.4}$$

In other words, with the function $U = \frac{\mathrm{d}V}{\mathrm{d}m}$ in hand, we can find the optimal trajectory by solving (6.4) for $Y_{xmt}(s)$, after which the adjoint state $Z_{xmt}(s)$ is given by Equation (4.20) from Proposition 4.4.

Before proving Theorem 6.2, we make a remark about uniqueness. Under certain assumptions, one can show that U is unique using the Lasry-Lions monotonicity argument [12], which goes as follows. Let U_1, U_2 be two solutions, and for $i = 1,2$ let $m_i(t)$ be the weak solution of the Fokker-Planck equation

$$\frac{\partial m_i}{\partial t} + A_x m_i - \frac{1}{\lambda}\mathrm{div}\left(DU_i(x, m_i(t), t)m_i\right) = 0, \ \ m_i(\tau) = m \in \mathcal{P}_2(\mathbb{R}^n) \tag{6.5}$$

for some $\tau \in [0,T]$. One checks that

$$\begin{aligned}
&\int_{\mathbb{R}^n}\left(\frac{\mathrm{d}F_T}{\mathrm{d}m}(m_1(T))(x) - \frac{\mathrm{d}F_T}{\mathrm{d}m}(m_2(T))(x)\right)\mathrm{d}(m_1(T)-m_2(T))(x)\\
&\qquad + \int_\tau^T\int_{\mathbb{R}^n}\left(\frac{\mathrm{d}F}{\mathrm{d}m}(m_1(t))(x) - \frac{\mathrm{d}F_T}{\mathrm{d}m}(m_2(t))(x)\right)\mathrm{d}(m_1(t)-m_2(t))(x)\,\mathrm{d}t\\
&\qquad\qquad + \frac{1}{2\lambda}\int_\tau^T\int_{\mathbb{R}^n}\left|DU_1(x,m_1(t),t) - DU_2(x,m_2(t),t)\right|^2\mathrm{d}(m_1(t)+m_2(t))\,\mathrm{d}t = 0.
\end{aligned} \tag{6.6}$$

Formally, (6.6) is derived by using $DU_j(x, m_j(t), t)$, $j = 1, 2$ as a test function in (6.5), $i = 1, 2$ and then subtracting. A typical assumption would be that F and F_T are monotone in the sense that

$$\int_{\mathbb{R}^n} \left(\frac{\mathrm{d}F}{\mathrm{d}m}(m_1)(x) - \frac{\mathrm{d}F_T}{\mathrm{d}m}(m_2)(x) \right) \mathrm{d}(m_1 - m_2)(x) \geq 0 \quad \forall m_1, m_2 \in \mathcal{P}_2(\mathbb{R}^n). \tag{6.7}$$

In this case, (6.6) immediately implies $DU_1(x, m, \tau) = DU_2(x, m, \tau)$ (at least on the support of m). One can exploit this to deduce that $U_1 = U_2$, provided both solutions are sufficiently regular.

In our framework, sufficient regularity to prove uniqueness in this way is a delicate issue. Although uniqueness for the Bellman equation (5.15) is obtained using convexity of the underlying control problem, we do not find such a straightforward path to uniqueness for the Master equation (6.1). We leave this issue for future study.

6.2 EXISTENCE OF A SOLUTION TO THE MASTER EQUATION

In this subsection we prove Theorem 6.2. The whole argument consists in differentiating the Bellman equation (5.29) with respect to m. We just need to show that this step is justified. Recall that $\big(Y_{xmt}(s), Z_{xmt}(s)\big)$ is the solution of System (4.5)-(4.6). Consider also the solution $\big(\mathcal{Y}_{xmt}(s), \mathcal{Z}_{xmt}(s)\big)$ of System (5.21)-(5.22), which is in fact the derivative of $\big(Y_{xmt}(s), Z_{xmt}(s)\big)$ with respect to x (see Equation (5.20)). By Proposition 4.4 we have

$$Z_{xmt}(t) = DU(x, m, t), \;\; \mathcal{Z}_{xmt}(t) = D^2U(x, m, t). \tag{6.8}$$

By (6.8), we can write Bellman equation (5.29) as

$$\begin{aligned} -\frac{\partial V}{\partial t}(m, t) - \frac{1}{2} \int_{\mathbb{R}^n} \mathrm{tr} \sigma \ast \mathcal{Z}_{\xi mt}(t) \, \mathrm{d}m(\xi) + \frac{1}{2\lambda} \int_{\mathbb{R}^n} |Z_{\xi mt}(t)|^2 \, \mathrm{d}m(\xi) = F(m), \\ V(m, T) = F_T(m). \end{aligned} \tag{6.9}$$

The key step now is to check that $Z_{\xi mt}(t)$ and $\mathcal{Z}_{\xi mt}(t)$ have functional derivatives with respect to m. In fact we will show the differentiability with respect to m of $Y_{\xi mt}(s), Z_{\xi mt}(s), \mathcal{Y}_{\xi mt}(s)$, and $\mathcal{Z}_{\xi mt}(s)$, for any $s > t$. We shall label their derivatives $\bar{Y}_{mt}(s, \xi, x)$, $\bar{Z}_{mt}(s, \xi, x)$, $\bar{\mathcal{Y}}_{mt}(s, \xi, x)$, and $\bar{\mathcal{Z}}_{mt}(s, \xi, x)$, respectively. We obtain them by taking the derivative in m of the systems (4.5)-(4.6) and (5.21)-(5.22), being careful to replace x with ξ in these expressions. See Section 6.2.1 below.

Given the existence of these derivatives, it is straightforward to differentiate (6.9) with respect to m, from which we obtain

$$\begin{aligned} -\frac{\partial U}{\partial t}(x, m, t) - \frac{1}{2} \mathrm{tr} \sigma \ast \mathcal{Z}_{xmt}(t) - \frac{1}{2} \int_{\mathbb{R}^n} \mathrm{tr} \sigma \ast \bar{\mathcal{Z}}_{mt}(t, \xi, x) \, \mathrm{d}m(\xi) \\ + \frac{1}{2\lambda} |Z_{xmt}(t)|^2 + \frac{1}{\lambda} \int_{\mathbb{R}^n} Z_{\xi mt}(t) \cdot \bar{Z}_{mt}(t, \xi, x) \, \mathrm{d}m(\xi) = \frac{\mathrm{d}}{\mathrm{d}m} F(m)(x). \end{aligned} \tag{6.10}$$

Substituting (6.8) into (6.10) and using the definition of $\bar{Z}_{mt}$ and $\bar{\mathcal{Z}}_{mt}$, we see that the Master Equation (6.1) holds.

6.2.1 LINEAR SYSTEMS FOR THE DERIVATIVES

We define the pair $\Big(\bar{Y}_{mt}(s,\xi,x),\ \bar{Z}_{mt}(s,\xi,x)\Big)$ to be the solution of the linear system

$$\bar{Y}_{mt}(s,\xi,x) = -\frac{1}{\lambda}\int_t^s \bar{Z}_{mt}(\tau,\xi,x)\,\mathrm{d}\tau, \tag{6.11}$$

$$\begin{aligned}
\bar{Z}_{mt}(s,\xi,x) = \mathbb{E}\Bigg[&\int_s^T D^2\frac{\mathrm{d}F}{\mathrm{d}m}(Y_{\cdot mt}(\tau)\otimes m)(Y_{\xi mt}(\tau))\bar{Y}_{mt}(\tau,\xi,x)\,\mathrm{d}\tau \\
&+ D^2\frac{\mathrm{d}}{\mathrm{d}m}F_T(Y_{\cdot mt}(T)\otimes m)(Y_{\xi mt}(T))\bar{Y}_{mt}(T,\xi,x) \\
&+ \widetilde{\mathbb{E}}\int_s^T\int_{\mathbb{R}^n} D_2D_1\frac{\mathrm{d}^2F}{\mathrm{d}m^2}(Y_{\cdot mt}(\tau)\otimes m)(Y_{\xi mt}(\tau),\widetilde{Y}_{\eta mt}(\tau))\widetilde{\bar{Y}}_{mt}(\tau,\eta,x)\,\mathrm{d}m(\eta)\,\mathrm{d}\tau \\
&+ \widetilde{\mathbb{E}}\int_{\mathbb{R}^n} D_2D_1\frac{\mathrm{d}^2F_T}{\mathrm{d}m^2}(Y_{\cdot mt}(T)\otimes m)(Y_{\xi mt}(T),\widetilde{Y}_{\eta mt}(T))\widetilde{\bar{Y}}_{mt}(T,\eta,x)\,\mathrm{d}m(\eta) \\
&+ \int_s^T \widetilde{\mathbb{E}}D_1\frac{\mathrm{d}^2F}{\mathrm{d}m^2}(Y_{\cdot mt}(\tau)\otimes m)(Y_{\xi mt}(\tau),\widetilde{Y}_{xmt}(\tau))\,\mathrm{d}\tau + \widetilde{\mathbb{E}}D_1\frac{\mathrm{d}^2F_T}{\mathrm{d}m^2}(Y_{\cdot mt}(T)\otimes m)(Y_{\xi mt}(T),\widetilde{Y}_{xmt}(T))\Bigg|\,\mathcal{W}_t^s\Bigg],
\end{aligned} \tag{6.12}$$

and the pair $\Big(\bar{\mathcal{Y}}_{mt}(s,\xi,x),\bar{\mathcal{Z}}_{mt}(s,\xi,x)\Big)$ is the solution of

$$\bar{\mathcal{Y}}_{mt}(s,\xi,x) = -\frac{1}{\lambda}\int_t^s \bar{\mathcal{Z}}_{mt}(\tau,\xi,x)\,\mathrm{d}\tau, \tag{6.13}$$

$$\begin{aligned}
\bar{\mathcal{Z}}_{mt}(s,\xi,x) = \mathbb{E}\Bigg[&\int_s^T D^2\frac{\mathrm{d}}{\mathrm{d}m}F(Y_{\cdot mt}(\tau)\otimes m)(Y_{\xi mt}(\tau))\bar{\mathcal{Y}}_{mt}(\tau,\xi,x)\,\mathrm{d}\tau \\
&\qquad + D^2\frac{\mathrm{d}}{\mathrm{d}m}F_T(Y_{\cdot mt}(T)\otimes m)(Y_{\xi mt}(T))\bar{\mathcal{Y}}_{mt}(T,\xi,x)\Bigg|\,\mathcal{W}_t^s\Bigg] \\
+\mathbb{E}\Bigg[&\int_s^T\Bigg(D^3\frac{\mathrm{d}}{\mathrm{d}m}F(Y_{\cdot mt}(\tau)\otimes m)(Y_{\xi mt}(\tau))\bar{Y}_{mt}(\tau,\xi,x) + \widetilde{\mathbb{E}}D_1^2\frac{\mathrm{d}^2}{\mathrm{d}m^2}F(Y_{\cdot mt}(\tau)\otimes m)(Y_{\xi mt}(\tau),\widetilde{Y}_{xmt}(\tau)) \\
&+ \widetilde{\mathbb{E}}\int_{\mathbb{R}^n} D_1^2D_2\frac{\mathrm{d}^2}{\mathrm{d}m^2}F(Y_{\cdot mt}(\tau)\otimes m)(Y_{\xi mt}(\tau),\widetilde{Y}_{\eta mt}(\tau))\widetilde{\bar{Y}}_{mt}(\tau,\eta,x)\,\mathrm{d}m(\eta)\Bigg)\mathcal{Y}_{\xi mt}(\tau)\,\mathrm{d}\tau \\
&+\Bigg(D^3\frac{\mathrm{d}}{\mathrm{d}m}F_T(Y_{\cdot mt}(T)\otimes m)(Y_{\xi mt}(T))\bar{Y}_{mt}(\tau,\xi,x) + \widetilde{\mathbb{E}}D_1^2\frac{\mathrm{d}^2}{\mathrm{d}m^2}F_T(Y_{\cdot mt}(T)\otimes m)(Y_{\xi mt}(T),\widetilde{Y}_{xmt}(T)) \\
&+ \widetilde{\mathbb{E}}\int_{\mathbb{R}^n} D_1^2D_2\frac{\mathrm{d}^2}{\mathrm{d}m^2}F_T(Y_{\cdot mt}(T)\otimes m)(Y_{\xi mt}(T),\widetilde{Y}_{\eta mt}(T))\widetilde{\bar{Y}}_{mt}(T,\eta,x)\,\mathrm{d}m(\eta)\Bigg)\mathcal{Y}_{\xi mt}(T)\Bigg|\,\mathcal{W}_t^s\Bigg].
\end{aligned} \tag{6.14}$$

We now provide a result on the well-posedness of these systems.

Proposition 6.3. *Assume that F is of class $\mathcal{S}_c$ and F_T is of class $\mathcal{S}_{c_T}$ for some constants $c, c_T > 0$. Then there exists a λ_T large enough, depending on c, c_T, and T, such that if $\lambda \geq \lambda_T$, each of the systems (6.11)-(6.12) and (6.13)-(6.14) has a unique solution, satisfying the estimates*

$$\mathbb{E}\int_{\mathbb{R}^n}|\bar{Y}_{mt}(s,\xi,x)|^2\,\mathrm{d}m(\xi) \leq C_T(1+|x|^2),\ \mathbb{E}\int_{\mathbb{R}^n}|\bar{Z}_{mt}(s,\xi,x)|^2\,\mathrm{d}m(\xi) \leq C_T(1+|x|^2), \tag{6.15}$$

$$\mathbb{E}\int_{\mathbb{R}^n}|\bar{\mathcal{Y}}_{mt}(s,\xi,x)|^2\,\mathrm{d}m(\xi) \leq C_T(1+|x|^2),\ \mathbb{E}\int_{\mathbb{R}^n}|\bar{\mathcal{Z}}_{mt}(s,\xi,x)|^2\,\mathrm{d}m(\xi) \leq C_T(1+|x|^2). \tag{6.16}$$

Proof. First we establish (6.15) and (6.16) as a priori estimates. We begin with (6.11)-(6.12). Note that (4.10) holds for sufficiently large λ. Using (4.10) and the estimates (6.3) applied to both F and F_T in (6.12), we deduce

$$\mathbb{E}\int_{\mathbb{R}^n} |\bar{Z}_{mt}(s,\xi,x)|^2 \,\mathrm{d}m(\xi)$$
$$\leq C_T\left(\int_s^T \mathbb{E}\int_{\mathbb{R}^n} |\bar{Y}_{mt}(\tau,\xi,x)|^2 \,\mathrm{d}m(\xi)\,\mathrm{d}\tau + \mathbb{E}\int_{\mathbb{R}^n} |\bar{Y}_{mt}(T,\xi,x)|^2 \,\mathrm{d}m(\xi)\right) + C_T(1+|x|^2).$$

On the other hand, by (6.11) it follows from the Cauchy-Schwartz inequality that

$$\mathbb{E}\int_{\mathbb{R}^n} \left|\bar{Y}_{mt}(s,\xi,x)\right|^2 \mathrm{d}m(\xi) \leq \frac{s-t}{\lambda^2}\int_t^s \mathbb{E}\int_{\mathbb{R}^n} \left|\bar{Z}_{mt}(\tau,\xi,x)\right|^2 \mathrm{d}m(\xi).$$

Combining these two inequalities and taking λ sufficiently large, we obtain the estimates (6.15). For (6.16), we use the estimates (6.3) for both F and F_T in (5.21)-(5.22) to see that the solution $\big(\mathcal{Y}_{\xi mt}(s), \mathcal{Z}_{\xi mt}(s)\big)$ satisfies

$$\mathbb{E}|\mathcal{Y}_{xmt}(s)|^2 \leq C_T,\ \mathbb{E}|\mathcal{Z}_{xmt}(s)|^2 \leq C_T. \tag{6.17}$$

Using these estimates, as well as (4.10) and (6.15), in the system (6.13)- (6.14), we argue similarly as for (6.11)-(6.12) to see that (6.16) holds for λ sufficiently large.

We have thus obtained a priori estimates for the solutions of (6.11)-(6.12) and (6.13)-(6.14). Since these systems are linear, the existence and uniqueness of the solutions are obtained by a standard fixed point argument. The proof is complete. □

Remark 6.4. In fact, the a priori estimates (6.15) and (6.16) can be improved. We get, for λ large enough,

$$\mathbb{E}|\bar{Y}_{mt}(s,\xi,x)|^2 \leq C_T(1+|x|^2),\ \mathbb{E}|\bar{Z}_{mt}(s,\xi,x)|^2 \leq C_T(1+|x|^2), \tag{6.18}$$
$$\mathbb{E}|\bar{\mathcal{Y}}_{mt}(s,\xi,x)|^2 \leq C_T(1+|x|^2),\ \mathbb{E}|\bar{\mathcal{Z}}_{mt}(s,\xi,x)|^2 \leq C_T(1+|x|^2). \tag{6.19}$$

To see this, we again use (4.10) and the estimates (6.3) applied to both F and F_T in (6.12), but this time also appealing to (6.15), already proved. By using the same argument as in the proof of Proposition 6.3, we derive (6.18). The proof of (6.19) is similar.

6.2.2 Differentiating $Y_{\xi mt}(s)$, $Z_{\xi mt}(s)$, $\mathcal{Y}_{\xi mt}(s)$, and $\mathcal{Z}_{\xi mt}(s)$ with respect to m

Using Proposition 6.3, it is now possible to verify that $\bar{Y}_{mt}(s,\xi,x)$, $\bar{Z}_{mt}(s,\xi,x)$, $\bar{\mathcal{Y}}_{mt}(s,\xi,x)$, and $\bar{\mathcal{Z}}_{mt}(s,\xi,x)$ are indeed the functional derivatives of $Y_{\xi mt}(s)$, $Z_{\xi mt}(s)$, $\mathcal{Y}_{\xi mt}(s)$, and $\mathcal{Z}_{\xi mt}(s)$, respectively. First, we observe that these functionals are continuous with respect to (m,x). Indeed, if we take $(m_n,x_n)\to(m,x)$ and consider differences, e.g. $\bar{Y}_{m_nt}(s,\xi,x_n)-\bar{Y}_{mt}(s,\xi,x)$, and consider the resulting system of equations satisfied by these differences, it is straightforward (but tedious) to show that these differences converge to zero. (We also use Remark 6.4, which mean that our estimates will not depend on m.)

Now we take $\tilde{m}, m \in \mathcal{P}_2(\mathbb{R}^n)$ and $\epsilon > 0$ arbitrary, then define

$$\begin{aligned}
&m_\epsilon := m+\epsilon(\tilde{m}-m),\\
&Y_\xi^\epsilon(s) = \frac{1}{\epsilon}\Big(Y_{\xi m_\epsilon t}(s) - Y_{\xi mt}(s)\Big) - \int_{\mathbb{R}^n} \bar{Y}_{mt}(s,\xi,x)\,\mathrm{d}(\tilde{m}-m)(x),\\
&Z_\xi^\epsilon(s) = \frac{1}{\epsilon}\Big(Z_{\xi m_\epsilon t}(s) - Z_{\xi mt}(s)\Big) - \int_{\mathbb{R}^n} \bar{Z}_{mt}(s,\xi,x)\,\mathrm{d}(\tilde{m}-m)(x),\\
&\mathcal{Y}_\xi^\epsilon(s) = \frac{1}{\epsilon}\Big(\mathcal{Y}_{\xi m_\epsilon t}(s) - \mathcal{Y}_{\xi mt}(s)\Big) - \int_{\mathbb{R}^n} \bar{\mathcal{Y}}_{mt}(s,\xi,x)\,\mathrm{d}(\tilde{m}-m)(x),\\
&\mathcal{Z}_\xi^\epsilon(s) = \frac{1}{\epsilon}\Big(\mathcal{Z}_{\xi m_\epsilon t}(s) - \mathcal{Z}_{\xi mt}(s)\Big) - \int_{\mathbb{R}^n} \bar{\mathcal{Z}}_{mt}(s,\xi,x)\,\mathrm{d}(\tilde{m}-m)(x).
\end{aligned}$$

Our goal is to show that $Y_\cdot^\epsilon$,$Z_\cdot^\epsilon$, $\mathcal{Y}_\cdot^\epsilon$, and $\mathcal{Z}_\cdot^\epsilon$ all converge to zero in $\mathcal{H}_m$ as $\epsilon\to 0$. We will focus on Y^ϵ and Z^ϵ, the proof for $\mathcal{Y}^\epsilon$, and $\mathcal{Z}^\epsilon$ being very similar. First, observe that $Y_\xi^\epsilon(s) = -\frac{1}{\lambda}\int_t^s Z_\xi^\epsilon(\tau)\,\mathrm{d}\tau$. Next, we will further divide Z^ϵ into two parts, using the following definitions:

$$Z^1_{\xi mt}(s) := \mathbb{E}\left[\int_s^T D\,\frac{\mathrm{d}F}{\mathrm{d}m}\,(Y_{\cdot mt}(\tau)\otimes m)(Y_{\xi mt}(\tau))\,\mathrm{d}\tau\,\middle|\,\mathcal{W}^s_t\right],$$
$$Z^2_{\xi mt}(s) := \mathbb{E}\left[D\,\frac{\mathrm{d}F_T}{\mathrm{d}m}\,(Y_{\cdot mt}(T)\otimes m)(Y_{\xi mt}(T))\,\middle|\,\mathcal{W}^s_t\right],$$

$$\begin{aligned}\bar{Z}^1_{mt}(s,\xi,x) := \mathbb{E}\Bigg[&\int_s^T D^2\,\frac{\mathrm{d}F}{\mathrm{d}m}\,(Y_{\cdot mt}(\tau)\otimes m)(Y_{\xi mt}(\tau))\bar{Y}_{mt}(\tau,\xi,x)\,\mathrm{d}\tau\\ &+\widetilde{\mathbb{E}}\int_s^T\int_{\mathbb{R}^n} D_2D_1\,\frac{\mathrm{d}^2F}{\mathrm{d}m^2}\,(Y_{\cdot mt}(\tau)\otimes m)(Y_{\xi mt}(\tau),\widetilde{Y}_{\eta mt}(\tau))\widetilde{\bar{Y}}_{mt}(\tau,\eta,x)\,\mathrm{d}m(\eta)\,\mathrm{d}\tau\\ &+\int_s^T \widetilde{\mathbb{E}}D_1\,\frac{\mathrm{d}^2F}{\mathrm{d}m^2}\,(Y_{\cdot mt}(\tau)\otimes m)(Y_{\xi mt}(\tau),\widetilde{Y}_{xmt}(\tau))\,\mathrm{d}\tau\,\Bigg|\,\mathcal{W}^s_t\Bigg],\end{aligned}$$

$$\begin{aligned}\bar{Z}^2_{mt}(s,\xi,x) := \mathbb{E}\Bigg[&D^2\,\frac{\mathrm{d}}{\mathrm{d}m}\,F_T(Y_{\cdot mt}(T)\otimes m)(Y_{\xi mt}(T))\bar{Y}_{mt}(T,\xi,x)\\ &+\widetilde{\mathbb{E}}\int_{\mathbb{R}^n} D_2D_1\,\frac{\mathrm{d}^2F_T}{\mathrm{d}m^2}\,(Y_{\cdot mt}(T)\otimes m)(Y_{\xi mt}(T),\widetilde{Y}_{\eta mt}(T))\widetilde{\bar{Y}}_{mt}(T,\eta,x)\,\mathrm{d}m(\eta)\\ &+\widetilde{\mathbb{E}}D_1\,\frac{\mathrm{d}^2F_T}{\mathrm{d}m^2}\,(Y_{\cdot mt}(T)\otimes m)(Y_{\xi mt}(T),\widetilde{Y}_{xmt}(T))\,\Bigg|\,\mathcal{W}^s_t\Bigg].\end{aligned}$$

Then we can write $Z^\epsilon_\xi = Z^{\epsilon,1}_\xi + Z^{\epsilon,2}_\xi$ where for $i=1,2$ we define

$$Z^{\epsilon,i}(s) := \frac{1}{\epsilon}\left(Z^i_{\xi m_\epsilon t}(s) - Z^i_{\xi mt}(s)\right) - \int_{\mathbb{R}^n}\bar{Z}^i_{mt}(s,\xi,x)\,\mathrm{d}(\tilde{m}-m)(x). \tag{6.20}$$

Using the Fundamental Theorem of Calculus and rules for differentiation from Section 2, we can rewrite

$$\begin{aligned}Z^1_{\xi m_\epsilon t}(s) - Z^1_{\xi mt}(s) = \mathbb{E}\Bigg[&\int_s^T\int_0^1 D^2\,\frac{\mathrm{d}F}{\mathrm{d}m}\,(\mu^\theta_\epsilon)(\chi^{\epsilon,\theta}_\xi(\tau))\left(Y_{\xi m_\epsilon t}(\tau) - Y_{\xi m_\epsilon t}(\tau)\right)\mathrm{d}\theta\,\mathrm{d}\tau\\ &+\widetilde{\mathbb{E}}\int_s^T\int_{\mathbb{R}^n}\int_0^1\int_0^1 D_2D_1\,\frac{\mathrm{d}^2F}{\mathrm{d}m^2}\,(\mu^\theta_\epsilon)(\chi^{\epsilon,\theta}_\xi(\tau),\widetilde{\chi}^{\epsilon,\theta'}_\eta(\tau))\left(\tilde{Y}_{\eta m_\epsilon t}(\tau) - \tilde{Y}_{\eta mt}(\tau)\right)\mathrm{d}\theta'\,\mathrm{d}\theta\,\mathrm{d}m(\eta)\,\mathrm{d}\tau\\ &+\epsilon\int_s^T\widetilde{\mathbb{E}}\int_{\mathbb{R}^n}\int_0^1 D_1\,\frac{\mathrm{d}^2F}{\mathrm{d}m^2}\,(\mu^\theta_\epsilon)(\chi^{\epsilon,\theta}_\xi(\tau),\widetilde{Y}_{xm_\epsilon t}(\tau))\,\mathrm{d}\theta\,\mathrm{d}(\tilde{m}-m)(x)\,\mathrm{d}\tau\,\Bigg|\,\mathcal{W}^s_t\Bigg],\end{aligned}$$

where

$$\chi^{\epsilon,\theta}_\xi(\tau) := \theta Y_{\xi m_\epsilon t}(\tau) + (1-\theta)Y_{\xi mt}(\tau),$$
$$\mu^\theta_\epsilon := \theta Y_{\cdot m_\epsilon t}(\tau)\otimes m_\epsilon + (1-\theta)Y_{\cdot mt}(\tau)\otimes m.$$

We now define

$$\Phi^\epsilon_1(\xi,\tau) := \int_0^1\left(D^2\,\frac{\mathrm{d}F}{\mathrm{d}m}\,(\mu^\theta_\epsilon)(\chi^{\epsilon,\theta}_\xi(\tau)) - D^2\,\frac{\mathrm{d}F}{\mathrm{d}m}\,(Y_{\cdot mt}(\tau)\otimes m)(Y_{\xi mt}(\tau))\right)\mathrm{d}\theta,$$

$$\Phi^\epsilon_2(\xi,\eta,\tau) := \int_0^1\int_0^1\left(D_2D_1\,\frac{\mathrm{d}^2F}{\mathrm{d}m^2}\,(\mu^\theta_\epsilon)(\chi^{\epsilon,\theta}_\xi(\tau),\widetilde{\chi}^{\epsilon,\theta'}_\eta(\tau)) - D_2D_1\,\frac{\mathrm{d}^2F}{\mathrm{d}m^2}\,(Y_{\cdot mt}(\tau)\otimes m)(Y_{\xi mt}(\tau),\widetilde{Y}_{\eta mt}(\tau))\right)\mathrm{d}\theta'\,\mathrm{d}\theta,$$

$$\Phi^\epsilon_3(\xi,\tau) := \widetilde{\mathbb{E}}\int_{\mathbb{R}^n}\int_0^1\left(D_1\,\frac{\mathrm{d}^2F}{\mathrm{d}m^2}\,(\mu^\theta_\epsilon)(\chi^{\epsilon,\theta}_\xi(\tau),\widetilde{Y}_{xm_\epsilon t}(\tau)) - D_1\,\frac{\mathrm{d}^2F}{\mathrm{d}m^2}\,(Y_{\cdot mt}(\tau)\otimes m)(Y_{\xi mt}(\tau),\widetilde{Y}_{xmt}(\tau))\right)\mathrm{d}\theta\,\mathrm{d}(\tilde{m}-m)(x).$$

By the a priori estimates from Propostion 6.3, arguing as in Section 5.1, we see that as $\epsilon \to 0$, $\Phi_1^\epsilon(\cdot,\tau), \Phi_3^\epsilon(\cdot,\tau) \to 0$ in $\mathcal{H}_m$ uniformly in τ, and similarly $\Phi_2^\epsilon(\cdot,\cdot,\tau) \to 0$ in $\mathcal{H}_{m\times m}$, uniformly in τ.

Now $Z_\xi^{\epsilon,1}$ can be written in the form

$$
\begin{aligned}
Z_\xi^{\epsilon,1}(s) = \mathbb{E}\Bigg[&\int_s^T\int_0^1 D^2\frac{\mathrm{d}F}{\mathrm{d}m}(\mu_\epsilon^\theta)(\chi_\xi^{\epsilon,\theta}(\tau))Y^\epsilon(\tau)\,\mathrm{d}\theta\,\mathrm{d}\tau \\
&+\widetilde{\mathbb{E}}\int_s^T\int_{\mathbb{R}^n}\int_0^1\int_0^1 D_2D_1\frac{\mathrm{d}^2F}{\mathrm{d}m^2}(\mu_\epsilon^\theta)(\chi_\xi^{\epsilon,\theta}(\tau),\widetilde{\chi}_\eta^{\epsilon,\theta'}(\tau))Y^\epsilon(\tau)\,\mathrm{d}\theta'\,\mathrm{d}\theta\,\mathrm{d}m(\eta)\,\mathrm{d}\tau \\
&+\int_s^T\int_{\mathbb{R}^n}\Phi_1^\epsilon(\xi,\tau)\bar{Y}_{mt}(\tau,\xi,x)\,\mathrm{d}(\tilde{m}-m)(x)\,\mathrm{d}\tau \\
&+\widetilde{\mathbb{E}}\int_s^T\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\Phi_2^\epsilon(\xi,\eta,\tau)\widetilde{\bar{Y}}_{mt}(\tau,\eta,x)\,\mathrm{d}m(\eta)\,\mathrm{d}(\tilde{m}-m)(x)\,\mathrm{d}\tau+\int_s^T\Phi_3^\epsilon(\xi,\tau)\,\mathrm{d}\tau\Bigg|\mathcal{W}_t^s\Bigg].
\end{aligned}
$$

By the same argument, $Z^{\epsilon,2}$ can be written in an analogous way, with F replaced by F_T and integrals in time replaced by evaluation at T. Indeed, we can write

$$
\begin{aligned}
Z_\xi^{\epsilon}(s) = \mathbb{E}\Bigg[&\int_s^T\int_0^1 D^2\frac{\mathrm{d}F}{\mathrm{d}m}(\mu_\epsilon^\theta)(\chi_\xi^{\epsilon,\theta}(\tau))Y^\epsilon(\tau)\,\mathrm{d}\theta\,\mathrm{d}\tau+\int_0^1 D^2\frac{\mathrm{d}F_T}{\mathrm{d}m}(\mu_\epsilon^\theta)(\chi_\xi^{\epsilon,\theta}(T))Y^\epsilon(T)\,\mathrm{d}\theta \\
&+\widetilde{\mathbb{E}}\int_s^T\int_{\mathbb{R}^n}\int_0^1\int_0^1 D_2D_1\frac{\mathrm{d}^2F}{\mathrm{d}m^2}(\mu_\epsilon^\theta)(\chi_\xi^{\epsilon,\theta}(\tau),\widetilde{\chi}_\eta^{\epsilon,\theta'}(\tau))Y^\epsilon(\tau)\,\mathrm{d}\theta'\,\mathrm{d}\theta\,\mathrm{d}m(\eta)\,\mathrm{d}\tau \\
&\widetilde{\mathbb{E}}\int_{\mathbb{R}^n}\int_0^1\int_0^1 D_2D_1\frac{\mathrm{d}^2F_T}{\mathrm{d}m^2}(\mu_\epsilon^\theta)(\chi_\xi^{\epsilon,\theta}(T),\widetilde{\chi}_\eta^{\epsilon,\theta'}(T))Y^\epsilon(T)\,\mathrm{d}\theta'\,\mathrm{d}\theta\,\mathrm{d}m(\eta)+\Phi^\epsilon(\xi)\Bigg|\mathcal{W}_t^s\Bigg],
\end{aligned}
$$

where $\Phi^\epsilon \to 0$ in $\mathcal{H}_m$. We use the fact that $Y_\xi^\epsilon(s) = -\frac{1}{\lambda}\int_t^s Z_\xi^\epsilon(\tau)\,\mathrm{d}\tau$ and λ is sufficiently large, as in the proof of Proposition 6.3, to deduce that $Z_\cdot^\epsilon(s) \to 0$ in $\mathcal{H}_m$, uniformly in s. Therefore the same holds for Y^ϵ. This concludes the proof.

A PROOFS FROM SECTION 3

A.1 PROOF OF LEMMA 3.4

Consider a control $v_{Xt}(s)+\epsilon\tilde{v}_{Xt}(s)$. The corresponding state is $X_{Xt}(s;v_{Xt}(.))+\epsilon\int_t^s\tilde{v}_{Xt}(\tau)\,\mathrm{d}\tau$. Therefore

$$
\begin{aligned}
J_{Xt}(v_{Xt}(.)+\epsilon\tilde{v}_{Xt}(.))-J_{Xt}(v_{Xt}(.)) &= \lambda\epsilon\int_t^T\langle v_{Xt}(s),\tilde{v}_{Xt}(s)\rangle\,\mathrm{d}s+\epsilon^2\frac{\lambda}{2}\int_t^T||\tilde{v}_{Xt}(s)||^2\,\mathrm{d}s \\
&+\int_t^T\left(F\left(\left(X_{Xt}(s;v_{Xt}(.))+\epsilon\int_t^s\tilde{v}_{Xt}(\tau)\,\mathrm{d}\tau\right)\otimes m\right)-F(X_{Xt}(s;v_{Xt}(.))\otimes m)\right)\mathrm{d}s \\
&+F_T\left(\left(X_{Xt}(T;v_{Xt}(.))+\epsilon\int_t^T\tilde{v}_{Xt}(\tau)\,\mathrm{d}\tau\right)\otimes m\right)-F_T(X_{Xt}(T;v_{Xt}(.))\otimes m) \\
= \lambda\epsilon\int_t^T\langle v_{Xt}(s),\tilde{v}_{Xt}(s)\rangle\,\mathrm{d}s &+\epsilon\int_t^T\int_0^1\left\langle D_XF\left(\left(X_{Xt}(s;v_{Xt}(.))+\theta\epsilon\int_t^s\tilde{v}_{Xt}(\tau)\,\mathrm{d}\tau\right)\otimes m\right),\int_t^s\tilde{v}_{Xt}(\tau)\,\mathrm{d}\tau\right\rangle\mathrm{d}s\,\mathrm{d}\theta \\
&+\epsilon\int_0^1\left\langle D_XF_T\left(\left(X_{Xt}(T;v_{Xt}(.))+\theta\epsilon\int_t^T\tilde{v}_{Xt}(\tau)\,\mathrm{d}\tau\right)\otimes m\right),\int_t^T\tilde{v}_{Xt}(\tau)\,\mathrm{d}\tau\right\rangle\mathrm{d}\theta+o(\epsilon).
\end{aligned}
$$

From the continuity assumptions (3.17) and the membership $v_{Xt}, \tilde{v}_{Xt} \in L^2_{\mathcal{W}_{Xt}}(t,T;\mathcal{H}_m)$, we obtain

$$\frac{J_{Xt}(v_{Xt}(.)+\epsilon\tilde{v}_{Xt}(.)) - J_{Xt}(v_{Xt}(.))}{\epsilon} \to \lambda \int_t^T \langle v_{Xt}(s), \tilde{v}_{Xt}(s)\rangle \,\mathrm{d}s$$

$$+\int_t^T \left\langle D_X F(X_{Xt}(s;v_{Xt}(.))\otimes m), \int_t^s \tilde{v}_{Xt}(\tau)\,\mathrm{d}\tau\right\rangle \mathrm{d}s + \left\langle D_X F_T(X_{Xt}(T;v_{Xt}(.))\otimes m), \int_t^T \tilde{v}_{Xt}(\tau)\,\mathrm{d}\tau\right\rangle$$

$$= \lambda \int_t^T \langle v_{Xt}(s), \tilde{v}_{Xt}(s)\rangle \,\mathrm{d}s$$

$$+\int_t^T \left\langle \int_s^T D_X F(X_{Xt}(\tau;v_{Xt}(.))\otimes m)\,\mathrm{d}\tau + D_X F_T(X_{Xt}(T;v_{Xt}(.))\otimes m), \tilde{v}_{Xt}(s)\right\rangle \mathrm{d}s.$$

Using the fact that $\tilde{v}_{Xt}(s)$ is arbitrary and $\mathcal{W}^s_{Xt}$ measurable, we immediately obtain formula 3.25. ■

A.2 PROOF OF PROPOSITION 3.6

We take two controls v^1_{Xt} and v^2_{Xt}. We are going to check that

$$\int_t^T \left\langle D_v J_{Xt}(v^1_{Xt}(.))(s) - D_v J_{Xt}(v^2_{Xt}(.))(s), v^1_{Xt}(s) - v^2_{Xt}(s)\right\rangle \mathrm{d}s$$

$$\geq \left(\lambda - T\left(c'_T + \frac{c'T}{2}\right)\right)\int_t^T ||v^1_{Xt}(s) - v^2_{Xt}(s)||^2\,\mathrm{d}s. \quad \text{(A.1)}$$

Then from the assumption (3.26) the result will follow immediately. To simplify notation, we set $v^1(s) = v^1_{Xt}(s)$, $v^2(s) = v^2_{Xt}(s)$ and

$$X^1(s) = X_{Xt}(s;v^1_{Xt}(.)),\; X^2(s) = X_{Xt}(s;v^2_{Xt}(.))$$

From formula (3.25) we have

$$\int_t^T \left\langle D_v J_{Xt}(v^1_{Xt}(.))(s) - D_v J_{Xt}(v^2_{Xt}(.))(s), v^1_{Xt}(s) - v^2_{Xt}(s)\right\rangle \mathrm{d}s = \lambda\int_t^T ||v^1(s)-v^2(s)||^2\,\mathrm{d}s$$

$$+\int_t^T \left\langle \int_s^T (D_X F(X^1(\tau)\otimes m) - D_X F(X^2(\tau)\otimes m))\,\mathrm{d}\tau, v^1(s)-v^2(s)\right\rangle \mathrm{d}s$$

$$+\int_t^T \left\langle D_X F(X^1(T)\otimes m) - D_X F(X^2(T)\otimes m), v^1(s)-v^2(s)\right\rangle \mathrm{d}s$$

using the fact that $v^1(s)-v^2(s)$ is $\mathcal{W}^s_{Xt}$ measurable. Next since $v^1(s)-v^2(s) = \frac{\mathrm{d}}{\mathrm{d}s}(X^1(s)-X^2(s)$ and $X^1(t)-X^2(t) = 0$, we have

$$\int_t^T \left\langle \int_s^T (D_X F(X^1(\tau)\otimes m) - D_X F(X^2(\tau)\otimes m))\,\mathrm{d}\tau, v^1(s)-v^2(s)\right\rangle \mathrm{d}s$$

$$+\int_t^T \left\langle D_X F(X^1(T)\otimes m) - D_X F(X^2(T)\otimes m), v^1(s)-v^2(s)\right\rangle \mathrm{d}s$$

$$= \int_t^T \left\langle D_X F(X^1(s)\otimes m) - D_X F(X^2(s)\otimes m), X^1(s)-X^2(s)\right\rangle \mathrm{d}s$$

$$+\left\langle D_X F(X^1(T)\otimes m) - D_X F(X^2(T)\otimes m), X^1(T)-X^2(T)\right\rangle$$

$$\geq -c'\int_t^T ||X^1(s)-X^2(s)||^2\,\mathrm{d}s - c'_T||X^1(T)-X^2(T)||^2$$

by the monotonicity conditions (3.18). We next use

$$X^1(s)-X^2(s) = \int_t^s (v^1(\tau)-v^2(\tau))\,\mathrm{d}\tau$$

to deduce

$$||X^1(T) - X^2(T)||^2 \leq T \int_t^T ||v^1(s) - v^2(s)||^2 \, \mathrm{d}s$$

and

$$\int_t^T ||X^1(s) - X^2(s)||^2 \, \mathrm{d}s \leq \frac{T^2}{2} \int_t^T ||v^1(s) - v^2(s)||^2 \, \mathrm{d}s.$$

Collecting results, we obtain (A.1), as desired. Equation A.1 also implies

$$\int_t^T \langle D_v J_{Xt}(v_{Xt}(.))(s) - D_v J_{Xt}(0)(s), v_{Xt}(s)\rangle \, \mathrm{d}s \geq c_0 \int_t^T ||v_{Xt}(s)||^2 \, \mathrm{d}s \tag{A.2}$$

But

$$J_{Xt}(v_{Xt}(.)) - J_{Xt}(0) = \int_0^1 \langle D_v J_{Xt}(\theta v_{Xt}(.))(s), v_{Xt}(s)\rangle \, \mathrm{d}s \, \mathrm{d}\theta,$$

which, when combined with (A.2), implies

$$J_{Xt}(v_{Xt}(.)) - J_{Xt}(0) \geq \int_t^T D_v J_{Xt}(0)(s) v_{Xt}(s) \, \mathrm{d}s + \frac{c_0}{2} \int_t^T ||v_{Xt}(s)||^2 \, \mathrm{d}s.$$

This implies that J_{Xt} is both strictly convex and coercive, from which we deduce the existence and uniqueness of a minimizer of $J_{Xt}(v_{Xt}(.))$. This completes the proof. ■

B PROOFS FROM SECTION 4

B.1 PROOF OF PROPOSITION 4.1

To simplify notation, we omit the indices Xt in $Y_{Xt}(s), Z_{Xt}(s)$. Recall that the inner product $\langle Y(s), Z(s)\rangle$ is defined as an expected value. Using the tower property of iterated expectation, (3.27)-(3.28) together imply

$$\begin{aligned}
&\langle Y(s+\epsilon), Z(s+\epsilon)\rangle - \langle Y(s), Z(s)\rangle = \langle Y(s+\epsilon), Z(s+\epsilon) - Z(s)\rangle \\
&\quad + \langle Y(s+\epsilon) - Y(s), Z(s)\rangle = -\left\langle Y(s+\epsilon), \int_s^{s+\epsilon} D_X F(Y_{Xt}(\tau) \otimes m) \, \mathrm{d}\tau \right\rangle - \frac{1}{\lambda} \left\langle \int_s^{s+\epsilon} Z(\tau) \, \mathrm{d}\tau, Z(s) \right\rangle.
\end{aligned}$$

Divide by ϵ and let ϵ tend to 0. As the necessary continuity to pass to the limit is easily checked, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}s} \langle Y(s), Z(s)\rangle = -\frac{1}{\lambda} \|Z(s)\|^2 - \langle Y(s), D_X F(Y(s) \otimes m)\rangle)$$

Integrating between t and T, we obtain

$$\langle X, Z(t)\rangle = \frac{1}{\lambda} \int_t^T \|Z(s)\|^2 \, \mathrm{d}s + \int_t^T \langle D_X F(Y(s) \otimes m), Y(s)\rangle \, \mathrm{d}s + \langle D_X F_T(Y(T) \otimes m), Y(T)\rangle . \tag{B.1}$$

We also have

$$\langle X, Z(t)\rangle = \left\langle X, \int_t^T D_X F(Y(s) \otimes m) \, \mathrm{d}s + D_X F_T(Y(T) \otimes m) \right\rangle \tag{B.2}$$

simply by taking the inner product of X with (3.28). Combining (B.1) and (B.2), we get

$$\begin{aligned}
&\frac{1}{\lambda} \int_t^T \|Z(s)\|^2 \, \mathrm{d}s + \int_t^T \langle D_X F(Y(s) \otimes m) - D_X F(\delta), Y(s)\rangle \, \mathrm{d}s + \langle D_X F_T(Y(T) \otimes m) - D_X F_T(\delta), Y(T)\rangle \\
&\qquad = \left\langle X, \int_t^T (D_X F(Y(s) \otimes m) - D_X F(\delta)) \, \mathrm{d}s + D_X F_T(Y(T) \otimes m) - D_X F_T(\delta) \right\rangle \\
&\qquad\qquad + \langle X - Y(T), D_X F_T(\delta)\rangle + \int_t^T \langle X - Y(s), D_X F(\delta)\rangle \, \mathrm{d}s, \qquad \text{(B.3)}
\end{aligned}$$

where δ is the Dirac measure concentrated at the origin. We proceed to estimate the right-hand side of (B.3). Using the fact that $D_X F(\delta)$ and $D_X F_T(\delta)$ are deterministic, we use (3.27) to obtain

$$\begin{aligned}\langle X - Y(T), D_X F_T(\delta)\rangle + \int_t^T \langle X - Y(s), D_X F(\delta)\rangle \\ = \frac{1}{\lambda}\left\langle D_X F_T(\delta), \int_t^T Z(\tau)\,\mathrm{d}\tau\right\rangle + \frac{1}{\lambda}\int_t^T \left\langle D_X F(\delta), \int_t^s Z(\tau)\,\mathrm{d}\tau\right\rangle \mathrm{d}s,\end{aligned}$$

and then by applying Cauchy-Schwartz we get

$$\begin{aligned}\left|\langle X - Y(T), D_X F_T(\delta)\rangle + \int_t^T \langle X - Y(s), D_X F(\delta)\rangle\right| \\ \leq \frac{1}{\lambda}\sqrt{T}\left(||D_X F_T(\delta)|| + \frac{2}{3}T||D_X F(\delta)||\right)\sqrt{\int_t^T \|Z(s)\|^2\,\mathrm{d}s}.\end{aligned} \tag{B.4}$$

On the other hand, using the Lipschitz property (3.17) and writing $\delta = 0 \otimes m$ (Example 2.5), we have

$$\begin{aligned}\left|\left\langle X, \int_t^T (D_X F(Y(s)\otimes m) - D_X F(\delta))\,\mathrm{d}s + D_X F_T(Y(T)\otimes m) - D_X F_T(\delta)\right\rangle\right| \\ \leq \|X\|\left(c_T||Y(T)|| + c\int_t^T \|Y(s)\|\,\mathrm{d}s\right).\end{aligned} \tag{B.5}$$

Applying Cauchy-Schwartz directly to Equation (3.27) we have

$$\|Y(s)\| \leq \|X\| + \frac{1}{\lambda}\sqrt{s-t}\sqrt{\int_t^T \|Z(\tau)\|^2\,\mathrm{d}\tau} + \|\sigma\|\sqrt{s-t}, \tag{B.6}$$

which is plugged into (B.5) to get, after some simple estimates,

$$\begin{aligned}\left|\left\langle X, \int_t^T (D_X F(Y(s)\otimes m) - D_X F(\delta))\,\mathrm{d}s + D_X F_T(Y(T)\otimes m) - D_X F_T(\delta)\right\rangle\right| \\ \leq (c_T + cT)\|X\|^2 + \sqrt{T}\left(c_T + \frac{2}{3}cT\right)\left(\|\sigma\| + \frac{1}{\lambda}\sqrt{\int_t^T \|Z(s)\|^2\,\mathrm{d}s}\right)\|X\|.\end{aligned} \tag{B.7}$$

Combining (B.2) with inequalities (B.4) and (B.7), then using assumption (3.18), we obtain

$$\begin{aligned}\frac{1}{\lambda}\int_t^T \|Z(s)\|^2\,\mathrm{d}s - c'_T||Y(T)||^2 - c'\int_t^T \|Y(s)\|^2\,\mathrm{d}s \leq (c_T + cT)\|X\|^2 + \|\sigma\|\sqrt{T}\left(c_T + \frac{2}{3}cT\right) \\ + \frac{1}{\lambda}\sqrt{T}\left(\left(c_T + \frac{2}{3}cT\right)\|X\| + ||D_X F_T(\delta)|| + \frac{2}{3}T||D_X F(\delta)||\right)\sqrt{\int_t^T \|Z(s)\|^2\,\mathrm{d}s}\end{aligned} \tag{B.8}$$

Now using a weighted Young's inequality in (B.6) we derive, for arbitrary $\epsilon > 0$,

$$\|Y(s)\|^2 \leq \frac{1}{\lambda^2}(s-t)\left(1 + \frac{1}{\epsilon}\right)\int_t^T \|Z(s)\|^2\,\mathrm{d}s + (\|X\| + \|\sigma\|\sqrt{s-t})^2\left(1 + \frac{1}{\epsilon}\right)$$

and therefore, (B.8) implies

$$\begin{aligned}\frac{1}{\lambda}\int_t^T \|Z(s)\|^2\,\mathrm{d}s \leq (c_T + cT)\|X\|^2 + \|\sigma\|\sqrt{T}\left(c_T + \frac{2}{3}cT\right) + (\|X\| + \|\sigma\|\sqrt{s-t})^2\left(1 + \frac{1}{\epsilon}\right)(c'_T + c'T) \\ + \frac{1}{\lambda}\sqrt{T}\left(\left(c_T + \frac{2}{3}cT\right)\|X\| + ||D_X F_T(\delta)|| + \frac{2}{3}T||D_X F(\delta)||\right)\sqrt{\int_t^T \|Z(s)\|^2\,\mathrm{d}s} \\ + \frac{1}{\lambda^2}T(1+\epsilon)\left(c'_T + c'\frac{T}{2}\right)\int_t^T \|Z(s)\|^2\,\mathrm{d}s.\end{aligned}$$

Then from the assumption (3.26) taking ϵ small enough, it follows that

$$\frac{1}{\lambda}\int_t^T \|Z(s)\|^2\,\mathrm{d}s \leq C_T(1+\|X\|^2) \tag{B.9}$$

Plugging (B.9) back into (B.5) we get $\|Y(s)\| \leq C_T(1+\|X\|)$. From formula (3.28) and assumption (3.16) we get also $\|Z(s)\| \leq C_T(1+\|X\|)$. From (3.31) and assumption (3.15) we get immediately (4.2). The proof is complete. ■

B.2 PROOF OF PROPOSITION 4.2

Let $X^1, X^2 \in \mathcal{H}_m$, independent of $\mathcal{W}_t$. We consider the corresponding systems

$$Y_{X^1t}(s) = X^1 - \frac{1}{\lambda}\int_t^s Z_{X^1t}(\tau)\,\mathrm{d}\tau + \sigma(w(s)-w(t)), \tag{B.10}$$

$$Z_{X^1t}(s) = \mathbb{E}\left[\int_s^T D_XF(Y_{X^1t}(\tau)\otimes m)\,\mathrm{d}\tau + D_XF(Y_{X^1t}(T)\otimes m)\middle|\mathcal{W}^s_{X^1t}\right], \tag{B.11}$$

$$Y_{X^2t}(s) = X^2 - \frac{1}{\lambda}\int_t^s Z_{X^2t}(\tau)\,\mathrm{d}\tau + \sigma(w(s)-w(t)), \tag{B.12}$$

$$Z_{X^2t}(s) = \mathbb{E}\left[\int_s^T D_XF(Y_{X^2t}(\tau)\otimes m)\,\mathrm{d}\tau + D_XF(Y_{X^2t}(T)\otimes m)\middle|\mathcal{W}^s_{X^2t}\right]. \tag{B.13}$$

To simplify notation we write $Y^1(s) = Y_{X^1t}(s)$, $Z^1(s) = Z_{X^1t}(s)$ and $Y^2(s) = Y_{X^2t}(s)$, $Z^2(s) = Z_{X^2t}(s)$. According to Remark 3.5, we can replace both $\mathcal{W}^s_{X^1t}$ in (B.11) and $\mathcal{W}^s_{X^2t}$ in (B.13) by $\mathcal{W}^s_{X^1X^2t} := \sigma(X^1,X^2)\vee\mathcal{W}^s_t$. Similarly, we denote by $\mathcal{W}_{X^1X^2t}$ the filtration generated by the familiy of $\sigma-$algebras $\mathcal{W}^s_{X^1X^2t}$. Then for $i=1,2$, $Y^i(s)$ can be interpreted as the optimal trajectory and $Z^i(s)$ the corresponding adjoint state for the following optimal control problem:

$$\begin{aligned}
X^i_{X^1X^2t}(s) &= X^i + \int_t^s v_{X^1X^2t}(\tau)\,\mathrm{d}\tau + \sigma(w(s)-w(t)),\ s>t,\\
J^i_{X^1X^2t}(v_{X^1X^2t}(.)) &= \frac{\lambda}{2}\int_t^T ||v_{X^1X^2t}(s)||^2\,\mathrm{d}s + \int_t^T F(X^i_{X^1X^2t}(s;v_{X^1X^2t}(.))\otimes m)\,\mathrm{d}s\\
&\quad + F_T(X^i_{X^1X^2t}(T;v_{X^1X^2t}(.))\otimes m),\quad \forall v_{X^1X^2t}(.)\in L^2_{\mathcal{W}_{X^1X^2t}}(t,T;\mathcal{H}_m),
\end{aligned} \tag{B.14}$$

where as usual $J_{X^1X^2t}$ denotes the cost to be minimized. Note that $X^1_{X^1X^2t}(s)$ and $X^2_{X^1X^2t}(s)$ belong to $\mathcal{H}_m$, so the probability measures $X^1_{X^1X^2t}(s;v_{X^1X^2t}(.))\otimes m$ and $X^2_{X^1X^2t}(s;v_{X^1X^2t}(.))\otimes m$ are well defined. The optimal controls are respectively $-\frac{1}{\lambda}Z^1(s)$ and $-\frac{1}{\lambda}Z^2(s)$. Thanks to the optimality in the control space $L^2_{\mathcal{W}_{X^1X^2t}}(t,T;\mathcal{H}_m)$, we have that $-\frac{1}{\lambda}Z^2(s)$ is an admissible control for the problem starting with initial condition X^1 and thus sub-optimal. The trajectory corresponding to this control is

$$X^1 - \frac{1}{\lambda}\int_t^s Z^2(\tau)\,\mathrm{d}\tau + \sigma(w(s)-w(t)) = Y^2(s) + X^1 - X^2$$

and the sub-optimality allows to write the inequality

$$\begin{aligned}
V(X^1\otimes m,t) - V(X^2\otimes m,t) &\leq \int_t^T \Big(F((Y^2(s)+X^1-X^2)\otimes m) - F(Y^2(s)\otimes m)\Big)\,\mathrm{d}s\\
&\quad + F_T((Y^2(T)+X^1-X^2)\otimes m) - F(Y^2(T)\otimes m)\\
&= \int_0^1\int_t^T \Big\langle D_XF((Y^2(s)+\theta(X^1-X^2))\otimes m), X^1-X^2\Big\rangle\,\mathrm{d}s\,\mathrm{d}\theta\\
&\quad + \int_0^1 \Big\langle D_XF_T((Y^2(T)+\theta(X^1-X^2))\otimes m), X^1-X^2\Big\rangle\,\mathrm{d}\theta\\
&\leq \left\langle \int_t^T D_XF(Y^2(s)\otimes m)\,\mathrm{d}s + D_XF(Y^2(T)\otimes m), X^1-X^2\right\rangle + \frac{1}{2}(c_T+cT)\Big\|X^1-X^2\Big\|^2,
\end{aligned} \tag{B.15}$$

where the last inequality comes from the assumption (3.17). Combining (B.13) with (B.15), we obtain

$$V(X^1 \otimes m, t) - V(X^2 \otimes m, t) \leq \left\langle Z^2(t), X^1 - X^2 \right\rangle + \frac{1}{2}(c_T + cT)\left\|X^1 - X^2\right\|^2 \tag{B.16}$$

To obtain a lower bound, we exchange the roles of X^1 and X^2 to get

$$V(X^1 \otimes m, t) - V(X^2 \otimes m, t) \geq \left\langle Z^2(t), X^1 - X^2 \right\rangle - \frac{1}{2}(c_T + cT)\left\|X^1 - X^2\right\|^2 + \left\langle Z^1(t) - Z^2(t), X^1 - X^2 \right\rangle. \tag{B.17}$$

We now need to bound $\left\langle Z^1(t) - Z^2(t), X^1 - X^2 \right\rangle$ from below. Arguing as in the proof of Proposition 4.1, we have

$$\begin{aligned}\left\langle Z^1(s) - Z^2(s), Y^1(s) - Y^2(s) \right\rangle &= \left\langle \int_s^T (D_X F(Y^1(\tau) \otimes m) - D_X F(Y^2(\tau) \otimes m))\, \mathrm{d}\tau, Y^1(s) - Y^2(s) \right\rangle \\ &\quad + \left\langle D_X F_T(Y^1(T) \otimes m) - D_X F_T(Y^2(T) \otimes m), Y^1(s) - Y^2(s) \right\rangle \end{aligned} \tag{B.18}$$

Differentiate this formula in s, substitute the identity $\frac{\mathrm{d}}{\mathrm{d}s}\left(Y^1(s) - Y^2(s)\right) = -\frac{1}{\lambda}\left(Z^1(s) - Z^2(s)\right)$, and then reintegrate the resulting equation from t to T. Using (B.11) and (B.13), we obtain

$$\begin{aligned}\left\langle Z^1(t) - Z^2(t), X^1 - X^2 \right\rangle &= \frac{1}{\lambda}\int_t^T \left\|Z^1(s) - Z^2(s)\right\|^2 \mathrm{d}s \\ &\quad + \int_t^T \left\langle D_X F(Y^1(s) \otimes m) - D_X F(Y^2(s) \otimes m), Y^1(s) - Y^2(s) \right\rangle \mathrm{d}s \\ &\quad + \left\langle D_X F_T(Y^1(T) \otimes m) - D_X F(Y^2(T) \otimes m), Y^1(T) - Y^2(T) \right\rangle \\ &\geq \frac{1}{\lambda}\int_t^T \left\|Z^1(s) - Z^2(s)\right\|^2 \mathrm{d}s - c'\int_t^T \left\|Y^1(s) - Y^2(s)\right\|^2 \mathrm{d}s - c'_T\left\|Y^1(T) - Y^2(T)\right\|^2. \end{aligned} \tag{B.19}$$

Using estimates similar to those in the proof of Proposition 4.1, we get

$$\begin{aligned}\int_t^T \left\|Y^1(s) - Y^2(s)\right\|^2 \mathrm{d}s &\leq \frac{1}{\lambda^2}(1+\epsilon)\frac{T^2}{2}\int_t^T \left\|Z^1(s) - Z^2(s)\right\|^2 \mathrm{d}s + T\left(1 + \frac{1}{\epsilon}\right)\left\|X^1 - X^2\right\|^2, \\ \left\|Y^1(T) - Y^2(T)\right\|^2 &\leq \frac{1}{\lambda^2}(1+\epsilon)T\int_t^T \left\|Z^1(s) - Z^2(s)\right\|^2 \mathrm{d}s + \left(1 + \frac{1}{\epsilon}\right)\left\|X^1 - X^2\right\|^2, \end{aligned}$$

which we plug into (B.19) to obtain

$$\begin{aligned}\left\langle Z^1(t) - Z^2(t), X^1 - X^2 \right\rangle &\geq \frac{1}{\lambda}\left(1 - \frac{1}{\lambda}T(1+\epsilon)(c'\frac{T}{2} + c'_T)\right)\int_t^T \left\|Z^1(s) - Z^2(s)\right\|^2 \mathrm{d}s \\ &\quad - \left(1 + \frac{1}{\epsilon}\right)(c'T + c'_T)\left\|X^1 - X^2\right\|^2 \end{aligned} \tag{B.20}$$

From the assumption (3.26) and choosing ϵ sufficiently small, the first term in the right hand side is positive.

Combining (B.20) with (B.17) and taking (B.16) into account, it follows that

$$|V(X^1 \otimes m, t) - V(X^2 \otimes m, t) - \left\langle Z^2(t), X^1 - X^2 \right\rangle| \leq C_T\left\|X^1 - X^2\right\|^2, \tag{B.21}$$

which implies that $X \mapsto V(X \otimes m, t)$ is Gâteaux differentiable and that Equation (4.3) holds. Next, we use (B.18) with $s = t$ and the Lipschitz estimates (3.17) to deduce

$$\left\langle Z^1(t) - Z^2(t), X^1 - X^2 \right\rangle \leq \left\|X^1 - X^2\right\|\left(c\int_t^T \left\|Y^1(s) - Y^2(s)\right\| \mathrm{d}s + c_T\left\|Y^1(T) - Y^2(T)\right\|\right).$$

Using estimates as before, cf. (B.6) and (B.7), we derive

$$\left\langle Z^1(t)-Z^2(t),X^1-X^2\right\rangle \le (c_T+cT)\left\|X^1-X^2\right\|^2+\frac{\sqrt{T}}{\lambda}(c_T+\frac{2}{3}cT)\left\|X^1-X^2\right\|\sqrt{\int_t^T \|Z^1(s)-Z^2(s)\|^2\,\mathrm{d}s},$$

which we plug into (B.20) to obtain

$$\frac{1}{\lambda}\left(1-\frac{1}{\lambda}T(1+\epsilon)(c'\frac{T}{2}+c'_T)\right)\int_t^T\left\|Z^1(s)-Z^2(s)\right\|^2\mathrm{d}s \le (c_T+cT+\left(1+\frac{1}{\epsilon}\right)(c'T+c'_T))\left\|X^1-X^2\right\|^2 \\ +\frac{\sqrt{T}}{\lambda}(c_T+\frac{2}{3}cT)\left\|X^1-X^2\right\|\sqrt{\int_t^T \|Z^1(s)-Z^2(s)\|^2\,\mathrm{d}s}. \quad \text{(B.22)}$$

From (B.22) we deduce

$$\int_t^T\left\|Z^1(s)-Z^2(s)\right\|^2\mathrm{d}s \le C_T\left\|X^1-X^2\right\|^2. \quad \text{(B.23)}$$

Returning to Equations (B.10)-(B.13), applying (B.23) and using the Lipschitz estimates (3.17), we obtain

$$\left\|Y^1(s)-Y^2(s)\right\|\le C_T\left\|X^1-X^2\right\|,\ \left\|Z^1(s)-Z^2(s)\right\|\le C_T\left\|X^1-X^2\right\|$$

and thus (4.4) has been proven, which completes the proof. ■

B.3 PROOF OF PROPOSITION 4.5

We begin with (4.30). From the optimality principle (3.37) we have

$$V(X\otimes m,t)-V(X\otimes m,t+h)=\frac{1}{2\lambda}\int_t^{t+h}||Z_{Xt}(s)||^2\,\mathrm{d}s+\int_t^{t+h}F(Y_{Xt}(s)\otimes m)\,\mathrm{d}s \\ +V(Y_{Xt}(t+h)\otimes m,t+h)-V(X\otimes m,t+h), \quad \text{(B.24)}$$

and from (4.4) we obtain

$$|V(Y_{Xt}(t+h)\otimes m,t+h)-V(X\otimes m,t+h)-\langle D_XV(X\otimes m,t+h),Y_{Xt}(t+h)-X\rangle\,|\le C_T||Y_{Xt}(t+h)-X||^2. \quad \text{(B.25)}$$

Since $D_XV(X\otimes m,t+h)$ is $\sigma(X)$ measurable, while X is independent of $\mathcal{W}_t$, we can multiply (3.27) by $D_XV(X\otimes m,t+h)$ and integrate to get

$$\langle D_XV(X\otimes m,t+h),Y_{Xt}(t+h)-X\rangle=-\frac{1}{\lambda}\left\langle D_XV(X\otimes m,t+h),\int_t^{t+h}Z_{Xt}(s)\,\mathrm{d}s)\right\rangle,$$

which implies, using Propositions 4.1 and 4.2,

$$\left|\langle D_XV(X\otimes m,t+h),Y_{Xt}(t+h)-X\rangle\right|\le C_Th\left(1+\|X\|^2\right). \quad \text{(B.26)}$$

Finally, using Equation (3.27) as when we derived (B.6), we get

$$\|Y_{Xt}(t+h)-X\|\le\frac{1}{\lambda}\sqrt{h}\sqrt{\int_t^T\|Z(\tau)\|^2\,\mathrm{d}\tau}+\|\sigma\|\sqrt{h}\le C_T\sqrt{h}\left(1+\|X\|\right), \quad \text{(B.27)}$$

where the second inequality follows from Proposition 4.1. Combine inequalities (B.27), (B.26), and (B.25) with (B.24) to conclude (4.30).

We turn to (4.31), which by (4.3) is equivalent to

$$\left\|Z_{X,t+h}(t+h)-Z_{Xt}(t)\right\|\le C_T(h^{\frac{1}{2}}+h)(1+\|X\|). \quad \text{(B.28)}$$

From (3.28) we have

$$Z_{Xt}(t) = \mathbb{E}\left[\int_t^{t+h} D_X F(Y_{Xt}(s)\otimes m)\,\mathrm{d}s \middle| \sigma(X)\right] + \mathbb{E}\left[Z_{Xt}(t+h)|\sigma(X)\right]. \tag{B.29}$$

We use the assumption (3.16), the estimate (4.1) from Proposition 4.1, and the tower property to get

$$\left\|\mathbb{E}\left[\int_t^{t+h} D_X F(Y_{Xt}(s)\otimes m)\,\mathrm{d}s \middle| \sigma(X)\right]\right\| \le C_T h(1+\|X\|). \tag{B.30}$$

Subtract (B.29) from $Z_{X,t+h}(t+h)$ and use (B.30) to get

$$\left\|Z_{X,t+h}(t+h) - Z_{Xt}(t)\right\| \le C_T h(1+\|X\|) + \left\|Z_{X,t+h}(t+h) - \mathbb{E}[Z_{Xt}(t+h)|\sigma(X)]\right\|,$$

and since $Z_{X,t+h}(t+h)$ is $\sigma(X)$ measurable, the tower property implies

$$\left\|Z_{X,t+h}(t+h) - Z_{Xt}(t)\right\| \le C_T h(1+\|X\|) + \left\|Z_{X,t+h}(t+h) - Z_{Xt}(t+h)\right\| \tag{B.31}$$

Note that $Z_{Xt}(t+h) = Z_{Y_{Xt}(t+h),t+h}(t+h)$, so by the proof of Proposition 4.2 combined with (B.27),

$$||Z_{Xt}(t+h)(s) - Z_{X,t+h}(s)|| \le C_T||Y_{Xt}(t+h) - X|| \le C_T\sqrt{h}(1+\|X\|),\ \forall s\in[t+h,T]. \tag{B.32}$$

Combining (B.32) and (B.31) we obtain (B.28), which implies (4.31). ■

C PROOFS FROM SECTION 5

C.1 PROOF OF PROPOSITION 5.1.

We connect the system (5.13) to a control problem. The space of controls is $L^2_{\mathcal{W}_{X\mathcal{X}t}}(t,T;\mathcal{H}_m)$ where $\mathcal{W}_{X\mathcal{X}t}$ is the filtration generated by the σ-algebras $\mathcal{W}^s_{X\mathcal{X}t}$. If $\mathcal{V}_{X\mathcal{X}t}(s)$ is a control, the state is defined by

$$\mathcal{X}_{X\mathcal{X}t}(s) = \mathcal{X} + \int_t^s \mathcal{V}_{X\mathcal{X}t}(\tau)\,\mathrm{d}\tau \tag{C.1}$$

and the payoff is

$$\mathcal{J}_{X\mathcal{X}t}(\mathcal{V}_{X\mathcal{X}t}(.)) = \frac{\lambda}{2}\int_t^T ||\mathcal{V}_{X\mathcal{X}t}(s)||^2\,\mathrm{d}s + \frac{1}{2}\int_t^T \left\langle D^2F(Y_{Xt}(s)\otimes m)(\mathcal{X}_{X\mathcal{X}t}(s)), \mathcal{X}_{X\mathcal{X}t}(s)\right\rangle \mathrm{d}s$$
$$+ \frac{1}{2}\left\langle D^2F(Y_{Xt}(T)\otimes m)(\mathcal{X}_{X\mathcal{X}t}(T)), \mathcal{X}_{X\mathcal{X}t}(T)\right\rangle. \tag{C.2}$$

Thanks to the assumption (3.26) this is a strictly convex linear quadratic problem, which has a unique optimal control. The system (5.13) has a unique solution and the optimal control is $\hat{\mathcal{V}}_{X\mathcal{X}t}(s) = -\dfrac{1}{\lambda}\mathcal{Z}_{X\mathcal{X}t}(s)$. The optimal state is $\mathcal{Y}_{X\mathcal{X}t}(s)$. Moreover, we have

$$\inf_{\mathcal{V}_{X\mathcal{X}t}(.)}\frac{1}{2}\left\langle \mathcal{Z}_{X\mathcal{X}t}(t), \mathcal{X}\right\rangle = \frac{1}{2\lambda}\int_t^T ||\mathcal{Z}_{X\mathcal{X}t}(s)||^2\,\mathrm{d}s + \frac{1}{2}\int_t^T \left\langle D^2F(Y_{Xt}(s)\otimes m)(\mathcal{Y}_{X\mathcal{X}t}(s)), \mathcal{Y}_{X\mathcal{X}t}(s)\right\rangle \mathrm{d}s$$
$$+ \frac{1}{2}\left\langle D^2F(Y_{Xt}(T)\otimes m)(\mathcal{V}_{X\mathcal{X}t}(T)), \mathcal{V}_{X\mathcal{X}t}(T)\right\rangle \tag{C.3}$$

where in (C.3) $(\mathcal{Y}_{X\mathcal{X}t}(s), \mathcal{Z}_{X\mathcal{X}t}(s))$ is the solution of (5.13). Thanks to (3.26) we check easily that

$$||\mathcal{Y}_{X\mathcal{X}t}(s)||,\ ||\mathcal{Z}_{X\mathcal{X}t}(s)|| \le C_T||\mathcal{X}||. \tag{C.4}$$

Now $\mathcal{X} \mapsto \mathcal{Y}_{X\mathcal{X}t}(s)$ and $\mathcal{X} \mapsto \mathcal{Z}_{X\mathcal{X}t}(s)$ are linear. Indeed, the conditional expectation in the definition of $\mathcal{Z}_{X\mathcal{X}t}(s)$ does not introduce nonlinearities, since taking an initial condition $\alpha\mathcal{X}_1 + \beta\mathcal{X}_2$, one can extend the conditioning σ-algebra to contain both $\mathcal{X}_1, \mathcal{X}_2$ and the linearity follows easily. Therefore the maps $\mathcal{X} \mapsto \mathcal{Y}_{X\mathcal{X}t}(s)$ and $\mathcal{X} \mapsto \mathcal{Z}_{X\mathcal{X}t}(s)$ belong to $\mathcal{L}(\mathcal{H}_m, \mathcal{H}_m)$.

The next important step is to check the convergence

$$\frac{Y_{X+\epsilon\mathcal{X},t}(s) - Y_{Xt}(s)}{\epsilon} \to \mathcal{Y}_{X\mathcal{X}t}(s), \text{ and } \frac{Z_{X+\epsilon\mathcal{X},t}(s) - Z_{Xt}(s)}{\epsilon} \to \mathcal{Z}_{X\mathcal{X}t}(s) \text{ as } \epsilon \to 0, \ \forall s \in [t, T]. \tag{C.5}$$

Define

$$Y^{\epsilon}_{X\mathcal{X}t}(s) = \frac{Y_{X+\epsilon\mathcal{X},t}(s) - Y_{Xt}(s)}{\epsilon}, \ Z^{\epsilon}_{X\mathcal{X}t}(s) = \frac{Z_{X+\epsilon\mathcal{X},t}(s) - Z_{Xt}(s)}{\epsilon}.$$

Then the following relations follow from (3.27) and (3.28):

$$\begin{aligned} Y^{\epsilon}_{X\mathcal{X}t}(s) &= \mathcal{X} - \frac{1}{\lambda}\int_t^s Z^{\epsilon}_{X\mathcal{X}t}(\tau)\,\mathrm{d}\tau, \\ Z^{\epsilon}_{X\mathcal{X}t}(s) &= \mathbb{E}\left[\int_s^T \frac{D_X F(Y_{X+\epsilon\mathcal{X},t}(\tau)\otimes m) - D_X F(Y_{Xt}(\tau)\otimes m)}{\epsilon}\,\mathrm{d}\tau \right. \\ &\qquad \left. + \frac{D_X F_T(Y_{X+\epsilon\mathcal{X},t}(T)\otimes m) - D_X F_T(Y_{Xt}(T)\otimes m)}{\epsilon} \middle| \mathcal{W}^s_{X,\mathcal{X},t}\right]. \end{aligned} \tag{C.6}$$

The second relation in (C.6) can be written

$$\begin{aligned} Z^{\epsilon}_{X\mathcal{X}t}(s) = \mathbb{E}\Bigg[&\int_s^T \int_0^1 D_X^2 F((Y_{Xt}(\tau) + \theta\epsilon Y^{\epsilon}_{X\mathcal{X}t}(\tau))\otimes m)(Y^{\epsilon}_{X\mathcal{X}t}(\tau))\,\mathrm{d}\theta\,\mathrm{d}\tau \\ &+ \int_0^1 D_X^2 F_T((Y_{Xt}(T) + \theta\epsilon Y^{\epsilon}_{X\mathcal{X}t}(T))\otimes m)(Y^{\epsilon}_{X\mathcal{X}t}(T))\,\mathrm{d}\theta \Bigg| \mathcal{W}^s_{X,\mathcal{X},t}\Bigg]. \end{aligned} \tag{C.7}$$

Using an argument analogous to the proof of Proposition 4.1, we conclude that $Y^{\epsilon}_{X\mathcal{X}t}(s), Z^{\epsilon}_{X\mathcal{X}t}(s)$ are bounded in $L^{\infty}_{\mathcal{W}_{X\mathcal{X}t}}(t,T;\mathcal{H}_m)$. Passing to a subsequence, we can take $Z^{\epsilon}_{X\mathcal{X}t}(s)$ to converge weakly in $L^2_{\mathcal{W}_{X\mathcal{X}t}}(t,T;\mathcal{H}_m)$ to $\mathcal{Z}_{X\mathcal{X}t}(s)$. It follows that

$$Y^{\epsilon}_{X\mathcal{X}t}(s) \rightharpoonup \mathcal{Y}_{X\mathcal{X}t}(s) = \mathcal{X} - \frac{1}{\lambda}\int_t^s \mathcal{Z}_{X\mathcal{X}t}(\tau)\,\mathrm{d}\tau, \text{weakly in } L^2(\Omega, \mathcal{W}^s_{X,\mathcal{X},t}, \mathbb{P}; L^2_m(\mathbb{R}^n;\mathbb{R}^n)), \forall s. \tag{C.8}$$

Define

$$\begin{aligned} J^{\epsilon}_{xt}(s) = \mathbb{E}\Bigg[&\int_s^T \left(\int_0^1 D_X^2 F((Y_{Xt}(\tau) + \theta\epsilon Y^{\epsilon}_{X\mathcal{X}t}(\tau))\otimes m)(Y^{\epsilon}_{X\mathcal{X}t}(\tau))\,\mathrm{d}\theta - D_X^2 F(Y_{Xt}(\tau)\otimes m)(\mathcal{Y}_{X\mathcal{X}t}(\tau))\right)\mathrm{d}\tau \\ &+ \int_0^1 D_X^2 F_T((Y_{Xt}(T) + \theta\epsilon Y^{\epsilon}_{X\mathcal{X}t}(T))\otimes m)(Y^{\epsilon}_{X\mathcal{X}t}(T))\,\mathrm{d}\theta - D_X^2 F_T(Y_{Xt}(T)\otimes m)(\mathcal{Y}_{X\mathcal{X}t}(T)) \Bigg| \mathcal{W}^s_{X,\mathcal{X},t}\Bigg], \end{aligned}$$

which is an element of $L^2(\Omega, \mathcal{W}^s_{X,\mathcal{X},t}, \mathbb{P}; L^2_m(\mathbb{R}^n;\mathbb{R}^n))$. We are going to show that it converges weakly to 0. We write $J^{\epsilon}_{xt}(s) = I^{\epsilon}_{xt}(s) + II^{\epsilon}_{xt}(s)$ with

$$\begin{aligned} I^{\epsilon}_{xt}(s) := \mathbb{E}\Bigg[&\int_s^T \left(\int_0^1 D_X^2 F((Y_{Xt}(\tau) + \theta\epsilon Y^{\epsilon}_{X\mathcal{X}t}(\tau))\otimes m)(Y^{\epsilon}_{X\mathcal{X}t}(\tau))\,\mathrm{d}\theta - D_X^2 F(Y_{Xt}(\tau)\otimes m)(Y^{\epsilon}_{X\mathcal{X}t}(\tau))\right)\mathrm{d}\tau \\ &+ \int_0^1 D_X^2 F_T((Y_{Xt}(T) + \theta\epsilon Y^{\epsilon}_{X\mathcal{X}t}(T))\otimes m)(Y^{\epsilon}_{X\mathcal{X}t}(T))\,\mathrm{d}\theta - D_X^2 F_T(Y_{Xt}(T)\otimes m)(Y^{\epsilon}_{X\mathcal{X}t}(T)) \Bigg| \mathcal{W}^s_{X,\mathcal{X},t}\Bigg] \end{aligned}$$

and

$$\begin{aligned} II^{\epsilon}_{xt}(s) := \mathbb{E}\Bigg[&\int_s^T \left(D_X^2 F(Y(\tau)\otimes m)(Y^{\epsilon}_{X\mathcal{X}t}(\tau)) - D_X^2 F(Y_{Xt}(\tau)\otimes m)(\mathcal{Y}_{X\mathcal{X}t}(\tau))\right)\mathrm{d}\tau \\ &+ D_X^2 F_T(Y_{Xt}(T)\otimes m)(Y^{\epsilon}_{X\mathcal{X}t}(T)) - D_X^2 F_T(Y_{Xt}(T)\otimes m)(\mathcal{Y}_{X\mathcal{X}t}(T)) \Bigg| \mathcal{W}^s_{X,\mathcal{X},t}\Bigg]. \end{aligned}$$

Then by (C.8) it follows that $II^{\epsilon}_{xt}(s)$ converges weakly to 0 in $L^2(\Omega, \mathcal{W}^s_{X,\mathcal{X},t}, \mathbb{P}; L^2_m(\mathbb{R}^n;\mathbb{R}^n))$, since $D_X^2F(Y(\tau)\otimes m)$ and $D_X^2F_T(Y(T)\otimes m)$ are in $\mathcal{L}(\mathcal{H}_m, \mathcal{H}_m)$. We turn our attention to $I^{\epsilon}_{xt}(s)$. We claim that

$$\mathbb{E}\int_{\mathbb{R}^n} |I^{\epsilon}_{xt}(s)|\,\mathrm{d}m(x) \to 0,\ \forall s > t \tag{C.9}$$

Indeed,

$$\begin{aligned}
&\mathbb{E}\int_{\mathbb{R}^n} |I^{\epsilon}_{xt}(s)|\,\mathrm{d}m(x) \\
&\leq \mathbb{E}\int_{\mathbb{R}^n} \Bigg| \int_s^T \left(\int_0^1 D_X^2F((Y_{Xt}(\tau)+\theta\epsilon Y^{\epsilon}_{X\mathcal{X}t}(\tau))\otimes m)(Y^{\epsilon}_{X\mathcal{X}t}(\tau))\,\mathrm{d}\theta - D_X^2F(Y_{Xt}(\tau)\otimes m)(Y^{\epsilon}_{X\mathcal{X}t}(\tau)) \right)\mathrm{d}\tau \\
&\qquad + \int_0^1 D_X^2F_T((Y_{Xt}(T)+\theta\epsilon Y^{\epsilon}_{X\mathcal{X}t}(T))\otimes m)(Y^{\epsilon}_{X\mathcal{X}t}(T))\,\mathrm{d}\theta - D_X^2F_T(Y_{Xt}(T)\otimes m)(Y^{\epsilon}_{X\mathcal{X}t}(T))\Bigg|\,\mathrm{d}m(x) \\
\leq &\int_s^T\int_0^1 \mathbb{E}\int_{\mathbb{R}^n}\left| D_X^2F((Y_{Xt}(\tau)+\theta\epsilon Y^{\epsilon}_{X\mathcal{X}t}(\tau))\otimes m)(Y^{\epsilon}_{X\mathcal{X}t}(\tau)) - D_X^2F(Y_{Xt}(\tau)\otimes m)(Y^{\epsilon}_{X\mathcal{X}t}(\tau))\right|\mathrm{d}m(x)\,\mathrm{d}\theta\,\mathrm{d}\tau + \\
&\int_0^1 \mathbb{E}\int_{\mathbb{R}^n}\left| D_X^2F_T((Y_{Xt}(T)+\theta\epsilon Y^{\epsilon}_{X\mathcal{X}t}(T))\otimes m)(Y^{\epsilon}_{X\mathcal{X}t}(T)) - D_X^2F_T(Y_{Xt}(T)\otimes m)(Y^{\epsilon}_{X\mathcal{X}t}(T))\right|\mathrm{d}m(x)\,\mathrm{d}\theta.
\end{aligned}$$

From the assumption (5.6) we can assert that

$$\mathbb{E}\int_{\mathbb{R}^n}\left| D_X^2F((Y_{Xt}(\tau)+\theta\epsilon Y^{\epsilon}_{X\mathcal{X}t}(\tau))\otimes m)(Y^{\epsilon}_{X\mathcal{X}t}(\tau)) - D_X^2F(Y_{Xt}(\tau)\otimes m)(Y^{\epsilon}_{X\mathcal{X}t}(\tau))\right|\mathrm{d}m(x) \to 0, \forall\theta,\tau$$

and this function of θ,τ,ϵ is bounded by a constant, thanks to (5.2) and the uniform bound on $Y^{\epsilon}_{X\mathcal{X}t}(\tau)$ in $\mathcal{H}_m$. Similar assertions apply to the term involving F_T. Thus (C.9) follows from the bounded convergence theorem. On the other hand, I^{ϵ} is bounded in $L^2(\Omega, \mathcal{W}^s_{X,\mathcal{X},t}, \mathbb{P}; L^2_m(\mathbb{R}^n;\mathbb{R}^n))$. From (C.9), 0 is the unique weak limit point. Hence $I^{\epsilon}_{xt}(s)$ converges weaky to 0 in $L^2(\Omega, \mathcal{W}^s_{X,\mathcal{X},t}, \mathbb{P}; L^2_m(\mathbb{R}^n;\mathbb{R}^n))$. Therefore $J^{\epsilon}_{xt}(s)$ converges weakly to 0 in $L^2(\Omega, \mathcal{W}^s_{X,\mathcal{X},t}, \mathbb{P}; L^2_m(\mathbb{R}^n;\mathbb{R}^n))$. We deduce that

$$\begin{aligned}
Z^{\epsilon}_{X\mathcal{X}t}(s) = \mathbb{E}\Bigg[\int_0^1 \bigg(\int_s^T & D_X^2F((Y_{Xt}(\tau)+\theta\epsilon Y^{\epsilon}_{X\mathcal{X}t}(\tau))\otimes m)(Y^{\epsilon}_{X\mathcal{X}t}(\tau))\,\mathrm{d}\tau \\
&+ D_X^2F_T((Y_{Xt}(T)+\theta\epsilon Y^{\epsilon}_{X\mathcal{X}t}(T))\otimes m)(Y^{\epsilon}_{X\mathcal{X}t}(T))\bigg)\,\mathrm{d}\theta\,\Bigg|\,\mathcal{W}^s_{X,\mathcal{X},t}\Bigg] \\
\to \mathbb{E}\left[\int_s^T D_X^2F(Y_{Xt}(\tau)\otimes m)(\mathcal{Y}_{X\mathcal{X}t}(\tau))\,\mathrm{d}\tau + D_X^2F_T(Y_{Xt}(T)\otimes m)(\mathcal{Y}_{X\mathcal{X}t}(T))\,\middle|\,\mathcal{W}^s_{X,\mathcal{X},t}\right]& \\
\text{weakly in } L^2(\Omega, \mathcal{W}^s_{X,\mathcal{X},t}, \mathbb{P}; L^2_m(\mathbb{R}^n;\mathbb{R}^n)),\ \forall s.&
\end{aligned} \tag{C.10}$$

Necessarily the weak limit $\mathcal{Z}_{X\mathcal{X}t}(s)$ of $Z^{\epsilon}_{X\mathcal{X}t}(s)$ in the Hilbert space $L^2_{\mathcal{W}_{X\mathcal{X}t}}(t,T;\mathcal{H}_m)$ coincides with the right hand side of (C.10). But then the weak limits $\mathcal{Y}_{X\mathcal{X}t}(s)$, $\mathcal{Z}_{X\mathcal{X}t}(s)$ coincide with the solution of the system (5.13), which is unique. Therefore the whole sequence converges weakly. We also obtain weak convergence for any s, by the same argument as in (C.8). Let us check that the convergence is strong. We first argue as in the proof of Proposition 4.1 to derive the identities

$$\begin{aligned}
\frac{1}{2}\langle Z^{\epsilon}_{X\mathcal{X}t}(t), \mathcal{X}\rangle = &\frac{1}{2\lambda}\int_t^T ||Z^{\epsilon}_{X\mathcal{X}t}(s)||^2\,\mathrm{d}s \\
&+ \frac{1}{2}\int_t^T \left\langle \int_0^1 D_X^2F((Y_{Xt}(s)+\theta\epsilon Y^{\epsilon}_{X\mathcal{X}t}(s))\otimes m)(Y^{\epsilon}_{X\mathcal{X}t}(s))\,\mathrm{d}\theta, Y^{\epsilon}_{X\mathcal{X}t}(s)\right\rangle \mathrm{d}s \\
&\qquad + \frac{1}{2}\left\langle \int_0^1 D_X^2F_T((Y_{Xt}(T)+\theta\epsilon Y^{\epsilon}_{X\mathcal{X}t}(T))\otimes m)(Y^{\epsilon}_{X\mathcal{X}t}(T))\,\mathrm{d}\theta, Y^{\epsilon}_{X\mathcal{X}t}(T)\right\rangle.
\end{aligned} \tag{C.11}$$

and

$$\frac{1}{2}\left\langle \mathcal{Z}_{X\mathcal{X}t}(t),\mathcal{X}\right\rangle = \frac{1}{2\lambda}\int_t^T ||\mathcal{Z}_{X\mathcal{X}t}(s)||^2\,\mathrm{d}s$$
$$+\frac{1}{2}\int_t^T \left\langle D^2F(Y_{Xt}(s)\otimes m)(\mathcal{Y}_{X\mathcal{X}t}(s)),\mathcal{Y}_{X\mathcal{X}t}(s)\right\rangle \mathrm{d}s + \frac{1}{2}\left\langle D^2F_T(Y_{Xt}(T)\otimes m)(\mathcal{V}_{X\mathcal{X}t}(T)),\mathcal{V}_{X\mathcal{X}t}(T)\right\rangle. \tag{C.12}$$

By continuity of D_X^2F and $D_X^2F_T$, we have

$$\int_t^T\left\{\left\langle\int_0^1 D_X^2F((Y_{Xt}(s)+\theta\epsilon Y^\epsilon_{X\mathcal{X}t}(s))\otimes m)(\mathcal{Y}_{X\mathcal{X}t}(s))\,\mathrm{d}\theta,\mathcal{Y}_{X\mathcal{X}t}(s)\right\rangle\right.$$
$$\left.-\left\langle D_X^2F(Y_{Xt}(s)\otimes m)(\mathcal{Y}_{X\mathcal{X}t}(s)),\mathcal{Y}_{X\mathcal{X}t}(s)\right\rangle\right\}\mathrm{d}s$$
$$+\left\langle\int_0^1 D_X^2F_T((Y_{Xt}(T)+\theta\epsilon Y^\epsilon_{X\mathcal{X}t}(T))\otimes m)(\mathcal{Y}_{X\mathcal{X}t}(T))\,\mathrm{d}\theta,\mathcal{Y}_{X\mathcal{X}t}(T)\right\rangle$$
$$-\left\langle D_X^2F_T(Y_{Xt}(T)\otimes m)(\mathcal{Y}_{X\mathcal{X}t}(T)),\mathcal{Y}_{X\mathcal{X}t}(T)\right\rangle\to 0 \tag{C.13}$$

and

$$\int_t^T\left\{\left\langle\int_0^1 D_X^2F((Y_{Xt}(s)+\theta\epsilon Y^\epsilon_{X\mathcal{X}t}(s))\otimes m)(Y^\epsilon_{X\mathcal{X}t}(s))\,\mathrm{d}\theta,\mathcal{Y}_{X\mathcal{X}t}(s)\right\rangle\right.$$
$$\left.-\left\langle D_X^2F(Y_{Xt}(s)\otimes m)(\mathcal{Y}_{X\mathcal{X}t}(s)),\mathcal{Y}_{X\mathcal{X}t}(s)\right\rangle\right\}\mathrm{d}s$$
$$+\left\langle\int_0^1 D_X^2F_T((Y_{Xt}(T)+\theta\epsilon Y^\epsilon_{X\mathcal{X}t}(T))\otimes m)(Y^\epsilon_{X\mathcal{X}t}(T))\,\mathrm{d}\theta,\mathcal{Y}_{X\mathcal{X}t}(T)\right\rangle$$
$$-\left\langle D_X^2F_T(Y_{Xt}(T)\otimes m)(\mathcal{Y}_{X\mathcal{X}t}(T)),\mathcal{Y}_{X\mathcal{X}t}(T)\right\rangle\to 0 \tag{C.14}$$

Putting together (C.11), (C.12), (C.13), and (C.14), and using the fact that $Z_{X\mathcal{X}t}(s)\rightharpoonup\mathcal{Z}_{X\mathcal{X}t}(s)$ weakly, we deduce

$$\frac{1}{2\lambda}\int_t^T\left\|Z^\epsilon_{X\mathcal{X}t}(s)-\mathcal{Z}_{X\mathcal{X}t}(s)\right\|^2\mathrm{d}s$$
$$+\frac{1}{2}\int_t^T\left\langle\int_0^1 D_X^2F((Y_{Xt}(s)+\theta\epsilon Y^\epsilon_{X\mathcal{X}t}(s))\otimes m)(Y^\epsilon_{X\mathcal{X}t}(s)-\mathcal{Y}_{X\mathcal{X}t}(s))\,\mathrm{d}\theta,Y^\epsilon_{X\mathcal{X}t}(s)-\mathcal{Y}_{X\mathcal{X}t}(s)\right\rangle\mathrm{d}s$$
$$+\frac{1}{2}\left\langle\int_0^1 D_X^2F_T((Y_{Xt}(T)+\theta\epsilon Y^\epsilon_{X\mathcal{X}t}(T))\otimes m)(Y^\epsilon_{X\mathcal{X}t}(T)-\mathcal{Y}_{X\mathcal{X}t}(T))\,\mathrm{d}\theta,Y^\epsilon_{X\mathcal{X}t}(T)-\mathcal{Y}_{X\mathcal{X}t}(T)\right\rangle\mathrm{d}s\to 0 \tag{C.15}$$

From the assumption (5.3), it follows that

$$\limsup_{\epsilon\to 0}\left(\frac{1}{\lambda}\int_t^T\left\|Z^\epsilon_{X\mathcal{X}t}(s)-\mathcal{Z}_{X\mathcal{X}t}(s)\right\|^2\mathrm{d}s-c'\int_t^T\left\|Y^\epsilon_{X\mathcal{X}t}(s)-\mathcal{Y}_{X\mathcal{X}t}(s)\right\|^2\mathrm{d}s-c'_T\left\|Y^\epsilon_{X\mathcal{X}t}(T)-\mathcal{Y}_{X\mathcal{X}t}(T)\right\|^2\right)\le 0.$$

Using $Y^\epsilon_{X\mathcal{X}t}(s)-\mathcal{Y}_{X\mathcal{X}t}(s)=-\dfrac{1}{\lambda}\int_t^s(Z^\epsilon_{X\mathcal{X}t}(\tau)-\mathcal{Z}_{X\mathcal{X}t}(\tau))\,\mathrm{d}\tau$ and the condition (3.26) on λ, we get

$$\int_t^T\left\|Z^\epsilon_{X\mathcal{X}t}(s)-\mathcal{Z}_{X\mathcal{X}t}(s)\right\|^2\mathrm{d}s\to 0.$$

This implies immediately $Y^\epsilon_{X\mathcal{X}t}(s)\to\mathcal{Y}_{X\mathcal{X}t}(s)$ in $L^2(\Omega,\mathcal{W}^s_{X,\mathcal{X},t},\mathbb{P};L^2_m(\mathbb{R}^n;\mathbb{R}^n)),\forall s$, and by arguments already used $Z^\epsilon_{X\mathcal{X}t}(s)\to\mathcal{Z}_{X\mathcal{X}t}(s)$ in $L^2(\Omega,\mathcal{W}^s_{X,\mathcal{X},t},\mathbb{P};L^2_m(\mathbb{R}^n;\mathbb{R}^n))$, $\forall s$.

Since we have $Z_{Xt}(t) = D_X V(X \otimes m, t)$ by Proposition 4.2, $Z^{\epsilon}_{X\mathcal{X}t}(t) \to \mathcal{Z}_{X\mathcal{X}t}(t)$ implies that $X \mapsto V(X \otimes m, t)$ is twice Gâteaux differentiable and that the relation (5.14) holds. From (C.4), we derive property (5.11). It remains to prove the continuity property (5.12). By (5.14), this is equivalent to showing that if $t_k \downarrow t$ and $X_k, \mathcal{X}_k \in \mathcal{H}_{m,t_k}$ converge in $\mathcal{H}_m$ to $X, \mathcal{X}$ respectively, then $\mathcal{Z}_{X_k\mathcal{X}_kt_k}(t_k) \to \mathcal{Z}_{X\mathcal{X}t}(t)$ in $\mathcal{H}_m$. From (C.4) and the linearity of $\mathcal{Y}_{X\S t}$ and $\mathcal{Z}_{X\S t}$ with respect to $\mathcal{X}$, we have $||\mathcal{Y}_{X_k\mathcal{X}_kt_k}(s) - \mathcal{Y}_{X_k\mathcal{X},t_k}(s)|| \leq C_T||\mathcal{X}_k - \mathcal{X}||$, $||\mathcal{Z}_{X_k\mathcal{X}_kt_k}(s) - \mathcal{Z}_{X_k\mathcal{X},t_k}(s)|| \leq C_T||\mathcal{X}_k - \mathcal{X}||$. So we can assume, without loss of generality, that $\mathcal{X}_k = \mathcal{X}$, i.e. it is enough to show that $\mathcal{Z}_{X_k\mathcal{X}t_k}(t_k) \to \mathcal{Z}_{X\mathcal{X}t}(t)$ in $\mathcal{H}_m$.

By definition,

$$\begin{aligned}
&\mathcal{Y}_{X_k\mathcal{X},t_k}(s) = \mathcal{X} - \frac{1}{\lambda}\int_{t_k}^{s} \mathcal{Z}_{X_k\mathcal{X},t_k}(\tau)\,\mathrm{d}\tau,\ s > t_k,\\
&\mathcal{Z}_{X_k\mathcal{X},t_k}(s) = \mathbb{E}\left[\int_s^T D_X^2F(Y_{X_kt_k}(\tau)\otimes m)(\mathcal{Y}_{X_k\mathcal{X},t_k}(\tau))\,\mathrm{d}\tau + D_X^2F_T(Y_{X_kt_k}(T)\otimes m)(\mathcal{Y}_{X_k\mathcal{X},t_k}(T))\middle|\, \mathcal{W}^s_{X_k\mathcal{X},t_k}\right].
\end{aligned}\tag{C.16}$$

with

$$\begin{aligned}
Y_{X_kt_k}(s) &= X_k - \frac{1}{\lambda}\int_{t_k}^{s} Z_{X_kt_k}(\tau)\,\mathrm{d}\tau + \sigma(w(s) - w(t_k)),\\
Z_{X_kt_k}(s) &= \mathbb{E}\left[\int_s^T D_XF(Y_{X_kt_k}(\tau)\otimes m)\,\mathrm{d}\tau + D_XF(Y_{X_kt_k}(T)\otimes m)\middle|\, \mathcal{W}^s_{X_kt_k}\right].
\end{aligned}\tag{C.17}$$

We fix $s > t$. We can assume that $s > t_k$. From the proof of Proposition 4.5, we obtain

$$\begin{aligned}
\|Y_{X_kt_k}(s) - Y_{Xt}(s)\| &\leq \|Y_{X_kt_k}(s) - Y_{Xt_k}(s)\| + \|Y_{Xt_k}(s) - Y_{Xt}(s)\|\\
&\leq C_T\|X_k - X\| + C_T\left((t_k - t)^{\frac{1}{2}} + (t_k - t)\right)(1 + \|X\|)
\end{aligned}\tag{C.18}$$

and a similar estimate for $||Z_{X_kt_k}(s) - Z_{Xt}(s)||$. We fix $s > t$, and define $k(s) = \min\{k|\ t_k < s\}$. The function $k(s)$ is monotone decreasing. We define the σ-algebra

$$\widetilde{\mathcal{W}}^s_{X\mathcal{X}t} = \vee_{j\geq k(s)}(\mathcal{W}^s_{X_j\mathcal{X},t_j} \vee \mathcal{W}^{t_j}_t)\tag{C.19}$$

which is increasing in s. Note that X is $\widetilde{\mathcal{W}}^s_{X\mathcal{X}t}$ measurable. We call $\widetilde{\mathcal{W}}_{X\mathcal{X}t}$ the filtration generated by the sequence $\widetilde{\mathcal{W}}^s_{X\mathcal{X}t}$. For $k \geq k(s)$, $\widetilde{\mathcal{W}}^s_{X\mathcal{X}t}$ is an extension of $\mathcal{W}^s_{X_k\mathcal{X},t_k}$, independent of $\mathcal{W}_s$. Therefore, according to Remark 3.5 , we can change the conditioning σ algebra $\mathcal{W}^s_{X_k\mathcal{X},t_k}$ to $\widetilde{\mathcal{W}}^s_{X\mathcal{X}t}$. Then (C.16) becomes

$$\begin{aligned}
&\mathcal{Y}_{X_k\mathcal{X},t_k}(s) = \mathcal{X} - \frac{1}{\lambda}\int_{t_k}^{s} \mathcal{Z}_{X_k\mathcal{X},t_k}(\tau)\,\mathrm{d}\tau,\ s > t_k,\\
&\mathcal{Z}_{X_k\mathcal{X},t_k}(s) = \mathbb{E}\left[\int_s^T D_X^2F(Y_{X_kt_k}(\tau)\otimes m)(\mathcal{Y}_{X_k\mathcal{X},t_k}(\tau))\,\mathrm{d}\tau + D_X^2F_T(Y_{X_kt_k}(T)\otimes m)(\mathcal{Y}_{X_k\mathcal{X},t_k}(T))\middle|\, \widetilde{\mathcal{W}}^s_{X\mathcal{X}t}\right].
\end{aligned}\tag{C.20}$$

Note that $s > t_k$ implies $k \geq k(s)$. To define the processes for $t < s < t_k$ we set

$$\begin{cases}\mathcal{Y}_{X_k\mathcal{X},t_k}(s) = \mathcal{X},\\ \mathcal{Z}_{X_k\mathcal{X},t_k}(s) = \mathbb{E}[\mathcal{Z}_{X_k\mathcal{X},t_k}(t_k)|\widetilde{\mathcal{W}}^s_{X\mathcal{X}t}],\end{cases}\qquad \text{whenever } t < s < t_k.\tag{C.21}$$

To simplify notation, we shall denote $\mathcal{Y}^k(s) = \mathcal{Y}_{X_k\mathcal{X},t_k}(s)$ and $\mathcal{Z}^k(s) = \mathcal{Z}_{X_k\mathcal{X},t_k}(s)$. We use a similar argument as in the first part of the proof to show that these sequences converge strongly in $\mathcal{H}_m$.

We first see that $\mathcal{Y}^k(s)$ and $\mathcal{Z}^k(s)$ remain bounded in $L^\infty_{\widetilde{\mathcal{W}}_{X\mathcal{X}t}}(t,T;\mathcal{H}_m)$. We pick a subsequence of $\mathcal{Z}^k(.)$, which converges weakly to $\widetilde{\mathcal{Z}}(.)$ in $L^2_{\widetilde{\mathcal{W}}_{X\mathcal{X}t}}(t,T;\mathcal{H}_m)$. Then

$$\mathcal{Y}^k(s) \rightharpoonup \mathcal{X} - \frac{1}{\lambda}\int_t^s \widetilde{\mathcal{Z}}(\tau)\,\mathrm{d}\tau = \widetilde{\mathcal{Y}}(s)\ \text{ weakly in } L^2(\Omega, \widetilde{\mathcal{W}}^s_{X\mathcal{X}t}, \mathbb{P}; L^2_m(\mathbb{R}^n,\mathbb{R}^n)),\ \forall s \in (t,T].$$

As above, we write

$$J_{xt}^k(s) = \mathbb{E}\left[\int_s^T (D_X^2 F(Y_{X_k t_k}(\tau) \otimes m)(\mathcal{Y}^k(\tau)) - D_X^2 F(Y_{Xt}(\tau) \otimes m)(\widetilde{\mathcal{Y}}(\tau)))\, \mathrm{d}\tau \right.$$
$$\left. + D_X^2 F_T(Y_{X_k t_k}(T)) \otimes m)(\mathcal{Y}^k(T)) - D_X^2 F_T(Y_{Xt}(T) \otimes m)(\widetilde{\mathcal{Y}}(T)) | \widetilde{\mathcal{W}}_{X\mathcal{X}t}^s \right],$$

which is an element of $L^2(\Omega, \widetilde{\mathcal{W}}_{X\mathcal{X}t}^s, \mathbb{P}; L_m^2(\mathbb{R}^n;\mathbb{R}^n))$. We show that it converges weakly to 0. We write $J_{xt}^k(s) = I_{xt}^k(s) + II_{xt}^k(s)$ with

$$I_{xt}^k(s) = \mathbb{E}\left[\int_s^T (D_X^2 F(Y_{X_k t_k}(\tau) \otimes m)(\mathcal{Y}^k(\tau)) - D_X^2 F(Y_{Xt}(\tau) \otimes m)(\mathcal{Y}^k(\tau)))\, \mathrm{d}\tau \right.$$
$$\left. + D_X^2 F_T(Y_{X_k t_k}(T)) \otimes m)(\mathcal{Y}^k(T)) - D_X^2 F_T(Y_{Xt}(T) \otimes m)(\mathcal{Y}^k(T)) | \widetilde{\mathcal{W}}_{X\mathcal{X}t}^s \right]$$

and

$$II_{xt}^k(s) = \mathbb{E}\left[\int_s^T (D_X^2 F(Y_{Xt}(\tau) \otimes m)(\mathcal{Y}^k(\tau)) - D_X^2 F(Y_{Xt}(\tau) \otimes m)(\widetilde{\mathcal{Y}}(\tau)))\, \mathrm{d}\tau \right.$$
$$\left. + D_X^2 F_T(Y_{Xt}(T) \otimes m)(\mathcal{Y}^k(T)) - D_X^2 F_T(Y_{Xt}(T) \otimes m)(\widetilde{\mathcal{Y}}(T)) \middle| \mathcal{W}_{X,\mathcal{X},t}^s \right].$$

Once again, it is easy to see that $II_{xt}^k(s) \rightharpoonup 0$ weakly in $L^2(\Omega, \widetilde{\mathcal{W}}_{X\mathcal{X}t}^s, \mathbb{P}; L_m^2(\mathbb{R}^n;\mathbb{R}^n))$, for any s. Using (C.18) and similar reasoning as for $I_{xt}^\epsilon(s)$, we have

$$\mathbb{E}\int_{\mathbb{R}^n} |I_{xt}^k(s)|\, \mathrm{d}m(x) \to 0,\ \forall s$$

and as above it follows that $I_{xt}^k(s) \rightharpoonup 0$ weakly in $L^2(\Omega, \widetilde{\mathcal{W}}_{X\mathcal{X}t}^s, \mathbb{P}; L_m^2(\mathbb{R}^n;\mathbb{R}^n))$, for any $s > t$. It follows that

$$\mathcal{Z}^k(s) \rightharpoonup \mathbb{E}\left[\int_s^T D_X^2 F(Y_{Xt}(\tau) \otimes m(\widetilde{\mathcal{Y}}(\tau))\, \mathrm{d}\tau + D_X^2 F_T(Y_{Xt}(T) \otimes m)(\widetilde{\mathcal{Y}}(T)) \middle| \mathcal{W}_{X,\mathcal{X},t}^s \right]$$

weakly in $L^2(\Omega, \widetilde{\mathcal{W}}_{X\mathcal{X}t}^s, \mathbb{P}; L_m^2(\mathbb{R}^n;\mathbb{R}^n))$, for any $s > t$. Then the weak limits $\widetilde{\mathcal{Y}}(s)$ and $\widetilde{\mathcal{Z}}(s)$ satisfy

$$\begin{aligned} \widetilde{\mathcal{Y}}(s) &= \mathcal{X} - \frac{1}{\lambda}\int_{t_k}^s \widetilde{\mathcal{Z}}(\tau)\, \mathrm{d}\tau,\ s > t, \\ \widetilde{\mathcal{Z}}(s) &= \mathbb{E}\left[\int_s^T D_X^2 F(Y_{Xt}(\tau) \otimes m)(\widetilde{\mathcal{Y}}(\tau))\, \mathrm{d}\tau + D_X^2 F_T(Y_{Xt}(T) \otimes m)(\widetilde{\mathcal{Y}}(T)) | \widetilde{\mathcal{W}}_{X\mathcal{X}t}^s \right]. \end{aligned} \tag{C.22}$$

By uniqueness of solutions to the forward-backward system, $\widetilde{\mathcal{Y}}(s) = \mathcal{Y}_{X\mathcal{X}t}(s)$, $\widetilde{\mathcal{Z}}(s) = \mathcal{Z}_{X\mathcal{X}t}(s)$. From the uniqueness of the limit, the whole sequence converges weakly, for any s. The convergence is strong, by a reasoning identical to that for Y^ϵ, Z^ϵ above. The continuity (5.14) is obtained and the proof of Proposition 5.1 is completed. ■

C.2 PROOF OF THEOREM 5.3

We begin a lemma reducing the Brownian increments appearing in certain inner products to generic Gaussian random variables.

Lemma C.1. *Let $V : \mathcal{P}_2(\mathbb{R}^n) \times [0,T] \to \mathbb{R}$ be such that $D_X^2 V(X \otimes m, t)$ is continuous. Let $X \in \mathcal{H}_{m,t}$, i.e. let $X \in \mathcal{H}_m$ be independent of $\mathcal{W}_t$, and let N be any standard Gaussian N in $\mathbb{R}^n$ that is independent of both X and the filtration $\mathcal{W}_t$. Then we have*

$$\left\langle D_X^2 V(X \otimes m, t+h)(\sigma\frac{w(t+h) - w(t)}{\sqrt{h}}), \sigma\frac{w(t+h) - w(t)}{\sqrt{h}} \right\rangle = \left\langle D_X^2 V(X \otimes m, t+h)(\sigma N), \sigma N \right\rangle. \tag{C.23}$$

Proof. Using the definition of the second derivative as a limit and the representation of the first derivative, see (4.14) we have

$$\left\langle D_X^2 V(X\otimes m, t+h)\left(\sigma\frac{w(t+h)-w(t)}{\sqrt{h}}\right), \sigma\frac{w(t+h)-w(t)}{\sqrt{h}}\right\rangle$$
$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\left\langle D\frac{\mathrm{d}V}{\mathrm{d}m}\left(\left(X+\epsilon\sigma\frac{w(t+h)-w(t)}{\sqrt{h}}\right)\otimes m, t+h\right)(X+\epsilon\sigma\frac{w(t+h)-w(t)}{\sqrt{h}}), \sigma\frac{w(t+h)-w(t)}{\sqrt{h}}\right\rangle\right.$$
$$\left. - \left\langle D\frac{\mathrm{d}V}{\mathrm{d}m}(X\otimes m, t+h)(X), \sigma\frac{w(t+h)-w(t)}{\sqrt{h}}\right\rangle\right).$$

Now X and $\frac{w(t+h)-w(t)}{\sqrt{h}}$ are independent, as are X and N, so $X+\epsilon\sigma\frac{w(t+h)-w(t)}{\sqrt{h}}$ and $X+\epsilon\sigma N$ have the same law. In like manner, the probability measure $\left(X+\epsilon\sigma\frac{w(t+h)-w(t)}{\sqrt{h}}\right)\otimes m$ depends only on the marginals, and thus it is equal to $(X+\epsilon\sigma N)\otimes m$. The right-hand side is therefore equal to

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\left\langle D\frac{\mathrm{d}V}{\mathrm{d}m}((X+\epsilon\sigma N)\otimes m, t+h)(X+\epsilon\sigma N), \sigma N\right\rangle - \left\langle D\frac{\mathrm{d}V}{\mathrm{d}m}(X\otimes m, t+h)(X), \sigma N\right\rangle\right),$$

which is equal to the right-hand side of (C.23). ■ □

We next obtain a variant of the continuity property (5.12) from Proposition 5.1.

Lemma C.2. *Let $t_k \downarrow t$, and let $X_k, \mathcal{X}_k \in \mathcal{H}_{t_k}$, where $X_k \to X$ in $\mathcal{H}_m$ and $\mathcal{X}_k$ is bounded in $\mathcal{H}_m$. Then*

$$\mathbb{E}\int_{\mathbb{R}^n}\left|D_X^2 V(X_k\otimes m, t_k)(\mathcal{X}_k) - D_X^2 V(X\otimes m, t_k)(\mathcal{X}_k)\right|\mathrm{d}m(x) \to 0 \tag{C.24}$$

Proof. We consider the system

$$\begin{aligned}
\mathcal{Y}_{X_k\mathcal{X}_k t_k}(s) &= \mathcal{X}_k - \frac{1}{\lambda}\int_{t_k}^s \mathcal{Z}_{X_k\mathcal{X}_k t_k}(\tau)\,\mathrm{d}\tau,\ s>t_k,\\
\mathcal{Z}_{X_k\mathcal{X}_k t_k}(s) &= \mathbb{E}\left[\int_s^T D_X^2 F(Y_{X_k t_k}(\tau)\otimes m)(\mathcal{Y}_{X_k\mathcal{X}_k t_k}(\tau))\,\mathrm{d}\tau + D_X^2 F_T(Y_{X_k t_k}(T)\otimes m)(\mathcal{Y}_{X_k\mathcal{X}_k t_k}(T))\middle|\mathcal{W}^s_{X_k\mathcal{X}_k t_k}\right]
\end{aligned} \tag{C.25}$$

and the equivalent for $\mathcal{Y}_{X\mathcal{X}_k t_k}(s), \mathcal{Z}_{X\mathcal{X}_k t_k}(s)$. By Remark 3.5, we can condition on the common σ-algebra $\mathcal{W}^s_{X_k X\,\mathcal{X}_k t_k} = \mathcal{W}^s_{X_k\,\mathcal{X}_k t_k}\vee\mathcal{W}^s_{X\,\mathcal{X}_k t_k}$. So we can write

$$\begin{aligned}
\mathcal{Y}_{X_k\mathcal{X}_k t_k}(s) &= \mathcal{X}_k - \frac{1}{\lambda}\int_{t_k}^s \mathcal{Z}_{X_k\mathcal{X}_k t_k}(\tau)\,\mathrm{d}\tau,\ s>t_k\\
\mathcal{Z}_{X_k\mathcal{X}_k t_k}(s) &= \mathbb{E}\left[\int_s^T D_X^2 F(Y_{X_k t_k}(\tau)\otimes m)(\mathcal{Y}_{X_k\mathcal{X}_k t_k}(\tau))\,\mathrm{d}\tau + D_X^2 F_T(Y_{X_k t_k}(T)\otimes m)(\mathcal{Y}_{X_k\mathcal{X}_k t_k}(T))\middle|\mathcal{W}^s_{X_k X\mathcal{X}_k t_k}\right]
\end{aligned} \tag{C.26}$$

and

$$\begin{aligned}
\mathcal{Y}_{X\mathcal{X}_k t_k}(s) &= \mathcal{X}_k - \frac{1}{\lambda}\int_{t_k}^s \mathcal{Z}_{X\mathcal{X}_k t_k}(\tau)\,\mathrm{d}\tau,\ s>t_k\\
\mathcal{Z}_{X\mathcal{X}_k t_k}(s) &= \mathbb{E}\left[\int_s^T D_X^2 F(Y_{X t_k}(\tau)\otimes m)(\mathcal{Y}_{X\mathcal{X}_k t_k}(\tau))\,\mathrm{d}\tau + D_X^2 F_T(Y_{X t_k}(T)\otimes m)(\mathcal{Y}_{X\mathcal{X}_k t_k}(T))\middle|\mathcal{W}^s_{X_k X\mathcal{X}_k t_k}\right].
\end{aligned} \tag{C.27}$$

By Proposition 5.1, we have

$$\mathcal{Z}_{X_k\mathcal{X}_kt_k}(t_k) = D_X^2V(X_k\otimes m, t_k)(\mathcal{X}_k),\ \mathcal{Z}_{X\mathcal{X}_kt_k}(t_k) = D_X^2V(X\otimes m, t_k)(\mathcal{X}_k). \tag{C.28}$$

Therefore the result $(C.24)$ will be a consequence of

$$\sup_{s\in[t_k,T]}\mathbb{E}\int_{\mathbb{R}^n}|\mathcal{Y}_{X_k\mathcal{X}_kt_k}(s)-\mathcal{Y}_{X\mathcal{X}_kt_k}(s)|\,\mathrm{d}m(x),\ \sup_{s\in[t_k,T]}\mathbb{E}\int_{\mathbb{R}^n}|\mathcal{Z}_{X_k\mathcal{X}_kt_k}(s)-\mathcal{Z}_{X\mathcal{X}_kt_k}(s)|\,\mathrm{d}m(x)\to 0,\text{ as}\,k\to+\infty \tag{C.29}$$

We define

$$\widetilde{\mathcal{Y}}_{X_kX\mathcal{X}_kt_k}(s) = \mathcal{Y}_{X_k\mathcal{X}_kt_k}(s)-\mathcal{Y}_{X\mathcal{X}_kt_k}(s),\ \widetilde{\mathcal{Z}}_{X_kX\mathcal{X}_kt_k}(s) = \mathcal{Z}_{X_k\mathcal{X}_kt_k}(s)-\mathcal{Z}_{X\mathcal{X}_kt_k}(s)$$

Then the pair $\left(\widetilde{\mathcal{Y}}_{X_kX\mathcal{X}_kt_k}(s), \widetilde{\mathcal{Z}}_{X_kX\mathcal{X}_kt_k}(s)\right)$ is the solution of the system

$$\begin{aligned}
&\widetilde{\mathcal{Y}}_{X_kX\mathcal{X}_kt_k}(s) = -\frac{1}{\lambda}\int_{t_k}^s\widetilde{\mathcal{Z}}_{X_kX\mathcal{X}_kt_k}(\tau)\,\mathrm{d}\tau,\\
&\widetilde{\mathcal{Z}}_{X_kX\mathcal{X}_kt_k}(s) = \mathbb{E}\left[\int_s^T D_X^2F(Y_{Xt_k}(\tau)\otimes m)(\widetilde{\mathcal{Y}}_{X_kX\mathcal{X}_kt_k}(\tau))\,\mathrm{d}\tau\right.\\
&\qquad\left.+D_X^2F_T(Y_{Xt_k}(T)\otimes m)(\widetilde{\mathcal{Y}}_{X_kX\mathcal{X}_kt_k}(T))\right|\mathcal{W}^s_{X_kX\mathcal{X}_kt_k}\Big] + I_{X_kX\mathcal{X}_kt_k}(s),
\end{aligned} \tag{C.30}$$

where

$$\begin{aligned}
I_{X_kX\mathcal{X}_kt_k}(s) := \mathbb{E}&\left[\int_s^T(D_X^2F(Y_{X_kt_k}(\tau)\otimes m)-D_X^2F(Y_{Xt_k}(\tau)\otimes m))(\mathcal{Y}_{X_k\mathcal{X}_kt_k}(\tau))\,\mathrm{d}\tau\right.\\
&\left.+(D_X^2F_T(Y_{X_kt_k}(T)\otimes m)-D_X^2F_T(Y_{Xt_k}(T)\otimes m))(\mathcal{Y}_{X_k\mathcal{X}_kt_k}(T))\right|\mathcal{W}^s_{X_kX\mathcal{X}_kt_k}\Big].
\end{aligned} \tag{C.31}$$

We have

$$\begin{aligned}
&\mathbb{E}\int_{\mathbb{R}^n}|I_{X_kX\mathcal{X}_kt_k}(s)|\,\mathrm{d}m(x)\leq\\
&\quad\int_s^T\mathbb{E}\int_{\mathbb{R}^n}\Big|(D_X^2F(Y_{X_kt_k}(\tau)\otimes m)-D_X^2F(Y_{Xt_k}(\tau)\otimes m))(\mathcal{Y}_{X_k\mathcal{X}_kt_k}(\tau))\Big|\,\mathrm{d}m(x)\,\mathrm{d}\tau\\
&\qquad+\mathbb{E}\int_{\mathbb{R}^n}\Big|(D_X^2F_T(Y_{X_kt_k}(T)\otimes m)-D_X^2F_T(Y_{Xt_k}(T)\otimes m))(\mathcal{Y}_{X_k\mathcal{X}_kt_k}(T))\Big|\,\mathrm{d}m(x).
\end{aligned} \tag{C.32}$$

By the inequality (C.4) and the fact that $\mathcal{X}_k$ is bounded, we have

$$\sup_{s\in[t_k,T]}||\mathcal{Y}_{X_k\mathcal{X}_kt_k}(s)||\leq C_T$$

Since both $Y_{X_kt_k}(s)$ and $Y_{Xt_k}(s)$ converge to $Y_{Xt}(s)$ in $\mathcal{H}_m$ for any s, we can then use the continuity property (5.6) and the bounds to deduce that

$$l_k(s) := \mathbb{E}\int_{\mathbb{R}^n}|I_{X_kX\mathcal{X}_kt_k}(s)|\,\mathrm{d}m(x)\to 0\ \forall s\in(t,T]$$

and that $l_k(s)$ is bounded.

Now, we make use of the assumption (5.4) to write

$$\mathbb{E}\int_{\mathbb{R}^n}|\widetilde{\mathcal{Z}}_{X_kX\mathcal{X}_kt_k}(s)|\,\mathrm{d}m(x)\leq c\int_s^T\mathbb{E}\int_{\mathbb{R}^n}|\widetilde{\mathcal{Y}}_{X_kX\mathcal{X}_kt_k}(\tau)|\,\mathrm{d}m(x)\,\mathrm{d}\tau + c_T\mathbb{E}\int_{\mathbb{R}^n}|\widetilde{\mathcal{Y}}_{X_kX\mathcal{X}_kt_k}(T)|\,\mathrm{d}m(x)$$

and, from the definition (C.30),

$$|\widetilde{\mathcal{Y}}_{X_kX\mathcal{X}_kt_k}(\tau)|\leq\frac{1}{\lambda}\int_{t_k}^\tau|\widetilde{\mathcal{Z}}_{X_kX\mathcal{X}_kt_k}(\theta)|\,\mathrm{d}\theta,\ |\widetilde{\mathcal{Y}}_{X_kX\mathcal{X}_kt_k}(T)|\leq\frac{1}{\lambda}\int_{t_k}^T|\widetilde{\mathcal{Z}}_{X_kX\mathcal{X}_kt_k}(\theta)|\,\mathrm{d}\theta$$

Combining the two previous inequalities yields

$$\left(1-\frac{1}{\lambda}(c\frac{T^2}{2}+c_TT)\right)\int_{t_k}^{T}\mathbb{E}\int_{\mathbb{R}^n}|\widetilde{\mathcal{Z}}_{X_kX\mathcal{X}_kt_k}(s)|\,\mathrm{d}m(x)\,\mathrm{d}s\leq\int_{t_k}^{T}l_k(s)\,\mathrm{d}s\to 0.$$

Thanks to (5.16) we obtain

$$\sup_{s\in[t_k,T]}\mathbb{E}\int_{\mathbb{R}^n}|\widetilde{\mathcal{Y}}_{X_kX\mathcal{X}_kt_k}(s)|\,\mathrm{d}m(x),\ \mathbb{E}\int_{\mathbb{R}^n}|\widetilde{\mathcal{Z}}_{X_kX\mathcal{X}_kt_k}(s)|\,\mathrm{d}m(x)\to 0$$

and thus (C.29) is proven. The proof of the Lemma is complete. □

Our final lemma proves that, under the regularity properties that $V(X,t)$ satisfies, we have a formula that "lifts" the usual Itô formula to our Hilbert space setting.

Lemma C.3. *Suppose $V:\mathcal{P}_2(\mathbb{R}^n)\times[0,T]\to\mathbb{R}$ is any function such that*

- *V is continuous and satisfies the estimates* (4.2) *and* (4.30)*;*
- *$D_XV(X\otimes m,t)$ exists for each $X\in\mathcal{H}_{m,t}$ and is continuous, satisfying the estimates* (4.4) *and* (4.31)*;*
- *$D_X^2V(X\otimes m,t)$ exists for each $X\in\mathcal{H}_{m,t}$ and is continuous, satisfying both properties* (5.11) *and* (5.12)*; and*
- *the continuity property* (C.24) *is satisfied.*

Let $X\in\mathcal{H}_{m,t}$, let $v\in L^2_{\mathcal{W}_{Xt}}(t,T;\mathcal{H}_m)\cap\mathcal{C}([t,T];\mathcal{H}_m)$, and let $X(s)=X_{Xt}(s)$ be given by the SDE (3.3)*. Then for $h>0$ small enough, we have*

$$V(X_{Xt}(t+h)\otimes m,t+h)=V(X\otimes m,t+h)+\left\langle D_XV(X\otimes m,t),\int_t^{t+h}v(s)\,\mathrm{d}s\right\rangle\\+\frac{h}{2}\left\langle D_X^2V(X\otimes m,t)\,(\sigma N)\,,\sigma N\right\rangle+o(h).\quad\text{(C.33)}$$

Proof. Since $V(X\otimes m,t)$ has a second derivative with respect to X, we can begin with the following expansion:

$$\begin{aligned}V(X_{Xt}(t+h)\otimes m,t+h)&=V(X\otimes m,t+h)\\&+\left\langle D_XV(X\otimes m,t+h),\int_t^{t+h}v(s)\,\mathrm{d}s+\sigma(w(t+h)-w(t))\right\rangle\\&+\left\langle\int_0^1\int_0^1\theta D_X^2V\left(X+\theta\mu\left(\int_t^{t+h}v(s)\,\mathrm{d}s+\sigma(w(t+h)-w(t))\right)\right)\otimes m,t+h)\,\mathrm{d}\theta d\mu\right.\\&\left.\left(\int_t^{t+h}v(s)\,\mathrm{d}s+\sigma(w(t+h)-w(t))\right),\int_t^{t+h}v(s)\,\mathrm{d}s+\sigma(w(t+h)-w(t))\right\rangle.\end{aligned}\quad\text{(C.34)}$$

Since $D_XV(X\otimes m,t+h)$ is $\sigma(X)$-measurable and X is independent of $\mathcal{W}_t$, we have

$$\langle D_XV(X\otimes m,t+h),\sigma(w(t+h)-w(t))\rangle=0.\quad\text{(C.35)}$$

From the Hölder-in-time property (4.31) and the fact that $v(s)$ is bounded in $\mathcal{H}_m$, we deduce

$$\frac{1}{h}\left\langle D_XV(X\otimes m,t+h)-D_XV(X\otimes m,t),\int_t^{t+h}v(s)\,\mathrm{d}s\right\rangle\to 0,\text{ as }h\to 0\quad\text{(C.36)}$$

Also, by estimate (5.11) and again using the bound on $v(s)$, we deduce

$$\frac{1}{h}\left\langle \int_0^1\int_0^1 \theta D_X^2 V\left((X+\theta\mu\left(\int_t^{t+h} v(s)\,\mathrm{d}s+\sigma(w(t+h)-w(t))\right)\right)\otimes m,t+h)\,\mathrm{d}\theta d\mu\right.$$
$$\left.\left(\int_t^{t+h} v(s)\,\mathrm{d}s+\sigma(w(t+h)-w(t))\right),\int_t^{t+h} v(s)\,\mathrm{d}s\right\rangle \to 0,\ h\to 0. \quad \text{(C.37)}$$

We next prove that

$$\left\langle \int_0^1\int_0^1 \theta D_X^2 V\left((X+\theta\mu\left(\int_t^{t+h} v(s)\,\mathrm{d}s+\sigma(w(t+h)-w(t))\right)\right)\otimes m,t+h)\,\mathrm{d}\theta d\mu\right.$$
$$\left.\left(\sigma\frac{w(t+h)-w(t)}{\sqrt{h}}\right),\sigma\frac{w(t+h)-w(t)}{\sqrt{h}}\right\rangle$$
$$-\frac{1}{2}\left\langle D_X^2V(X\otimes m,t+h)\left(\sigma\frac{w(t+h)-w(t)}{\sqrt{h}}\right),\sigma\frac{w(t+h)-w(t)}{\sqrt{h}}\right\rangle \to 0,\ h\to 0. \quad \text{(C.38)}$$

Set

$$X_h(\theta,\mu)=X+\theta\mu\left(\int_t^{t+h} v(s)\,\mathrm{d}s+\sigma(w(t+h)-w(t))\right),\ \mathcal{X}_h=\sigma\frac{w(t+h)-w(t)}{\sqrt{h}}.$$

Then the expression (C.38) is equivalent to

$$\int_0^1\int_0^1 \theta L_h(\theta,\mu)\,\mathrm{d}\theta\,\mathrm{d}\mu \to 0. \quad \text{(C.39)}$$

where

$$L_h(\theta,\mu):=\mathbb{E}\int_{\mathbb{R}^n}(D_X^2V(X_h(\theta,\mu)\otimes m,t+h)(\mathcal{X}_h)-D_X^2V(X\otimes m,t+h)(\mathcal{X}_h))\cdot\mathcal{X}_h\,\mathrm{d}m(x).$$

By estimate (5.11) we have

$$|L_h(\theta,\mu)|\le C_T\|\mathcal{X}_h\|^2=C_T\mathrm{tr}\ \sigma\sigma^*,$$

i.e. $L_h(\theta,\mu)$ is bounded. So to prove (C.39) it is enough to show that $L_h(\theta,\mu)\to 0$ pointwise as $h\to 0$. Now from the continuity estimate (C.24) we have

$$\mathbb{E}\int_{\mathbb{R}^n}|D_X^2V(X_h(\theta,\mu)\otimes m,t+h)(\mathcal{X}_h)-D_X^2V(X\otimes m,t+h)(\mathcal{X}_h)|\,\mathrm{d}m(x)\to 0. \quad \text{(C.40)}$$

Define

$$L_{h\epsilon}(\theta,\mu):=\mathbb{E}\int_{\mathbb{R}^n}(D_X^2V(X_h(\theta,\mu)\otimes m,t+h)(\mathcal{X}_h)-D_X^2V(X\otimes m,t+h)(\mathcal{X}_h))\cdot\frac{\mathcal{X}_h}{1+\epsilon|\mathcal{X}_h|}\,\mathrm{d}m(x).$$

Then by (C.40) we have $L_{h\epsilon}(\theta,\mu)\to 0$ as $h\to 0$, for fixed $\epsilon>0,\theta,\mu\in[0,1]$. Notice that

$$L_h(\theta,\mu)-L_{h\epsilon}(\theta,\mu):=\epsilon\mathbb{E}\int_{\mathbb{R}^n}(D_X^2V(X_h(\theta,\mu)\otimes m,t+h)(\mathcal{X}_h)-D_X^2V(X\otimes m,t+h)(\mathcal{X}_h))\cdot\frac{\mathcal{X}_h|\mathcal{X}_h|}{1+\epsilon|\mathcal{X}_h|}\,\mathrm{d}m(x).$$

By estimate (5.11) in Proposition 5.1 we have

$$|L_h(\theta,\mu)-L_{h\epsilon}(\theta,\mu)|\le\epsilon C_T\|\mathcal{X}_h\|\left\|\frac{\mathcal{X}_h|\mathcal{X}_h|}{1+\epsilon|\mathcal{X}_h|}\right\|.$$

Taking the fourth moment of a Gaussian random variable, we have $\left\|\frac{\mathcal{X}_h|\mathcal{X}_h|}{1+\epsilon|\mathcal{X}_h|}\right\|\le\mathbb{E}\left[|\mathcal{X}_h|^4\right]^{1/2}\le\sqrt{3}\|\sigma\|^2$, and we conclude $|L_h(\theta,\mu)-L_{h\epsilon}(\theta,\mu)|\le C_T'\epsilon$. Since $\epsilon>0$ is arbitrary, we deduce that $L_h(\theta,\mu)\to 0$ pointwise, and property (C.39) follows.

Combining (C.35), (C.36), (C.37), and (C.38) with (C.34), we obtain

$$V(X_{Xt}(t+h)\otimes m, t+h) = V(X\otimes m, t+h) + \left\langle D_X V(X\otimes m, t), \int_t^{t+h} v(s)\,\mathrm{d}s \right\rangle$$
$$+ \frac{1}{2}\left\langle D_X^2 V(X\otimes m, t+h)\left(\sigma(w(t+h)-w(t))\right), \sigma(w(t+h)-w(t))\right\rangle + o(h).$$

Using Lemma C.1 to rewrite the last term and applying the continuity of $D_X^2 V(X\otimes m, t)$ with respect to t, we deduce (C.33). □

We can now proceed with the proof of Theorem 5.3. To prove that $V(X\otimes m, t)$ solves the Bellman equation (5.15), first note that V satisfies the hypotheses of Lemma C.3 by Propositions 4.1, 4.2, 4.5, and 5.1 as well as Lemma C.2. So we take (C.33) with $X_{Xt}(s) = Y_{Xt}(s)$ (the optimal trajectory) and $v(s) = -\frac{1}{\lambda}Z_{Xt}(s)$ (the optimal control, which is continuous by (B.32)), and we combine it with the optimality principle (3.37) to get

$$V(X\otimes m, t) - V(X\otimes m, t+h) = \frac{1}{2\lambda}\int_t^{t+h} ||Z_{Xt}(s)||^2\,\mathrm{d}s + \int_t^{t+h} F(Y_{Xt}(s)\otimes m)\,\mathrm{d}s$$
$$-\left\langle D_X V(X\otimes m, t), \frac{1}{\lambda}\int_t^{t+h} Z_{Xt}(s)\,\mathrm{d}s\right\rangle + \frac{h}{2}\left\langle D_X^2 V(X\otimes m, t+h)(\sigma N), \sigma N\right\rangle + o(h).$$

Letting $h\to 0$, we see that V is right-differentiable and that the Bellman equation (5.15) is satisfied.

Conversely, suppose V is any other classical solution to the Bellman equation (5.15). Note that it satisfies the regularity properties assumed in Lemma C.3. We will show that V must be equal to the the value function defined by (3.31). Take any $v \in L^2_{\mathcal{W}_{Xt}}(t,T;\mathcal{H}_m)\cap\mathcal{C}([t,T];\mathcal{H}_m)$, and let $X(s) = X_{Xt}(s)$ be given by the SDE (3.3). Then by taking $t = s$ and $X = X_{Xt}(s)$ in (C.33), we get

$$V(X_{Xt}(s+h)\otimes m, s+h) = V(X_{Xt}(s)\otimes m, s+h) + \left\langle D_X V(X_{Xt}(s)\otimes m, t), \int_s^{s+h} v(\tau)\,\mathrm{d}\tau\right\rangle$$
$$+ \frac{h}{2}\left\langle D_X^2 V(X_{Xt}(s)\otimes m, s)\left(\sigma N\right), \sigma N\right\rangle + o(h).$$

Subtract $V(X_{Xt}(s)\otimes m, s)$ from both sides and divide by h, then send $h\to 0$. Using the fact that $V(X\otimes m, t)$ is right-differentiable with respect to t, we see that $V(X_{Xt}(s)\otimes m, s)$ is differentiable and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\left(V(X_{Xt}(s)\otimes m, s)\right) &= \frac{\partial V}{\partial t}\left(X_{Xt}(s)\otimes m, s\right) + \langle D_X V(X_{Xt}(s)\otimes m, s), v(s)\rangle \\ &+ \frac{1}{2}\left\langle D_X^2 V(X_{Xt}(s)\otimes m, s)\left(\sigma N\right), \sigma N\right\rangle = \langle D_X V(X_{Xt}(s)\otimes m, s), v(s)\rangle \\ &+ \frac{1}{2\lambda}\left\|D_X V(X_{Xt}(s)\otimes m, s)\right\|^2 - F(X_{Xt}(s)\otimes m) \geq -\frac{\lambda}{2}\left\|v(s)\right\|^2 - F(X_{Xt}(s)\otimes m).\end{aligned} \tag{C.41}$$

Integrating from t to T reveals $V(X,t) \leq J_{Xt}(v(\cdot))$, where J_{Xt} is the objective functional defined in (3.14). In particular, we can take v to be the optimal control $\hat{v}$, and thus $V(X,t) \leq J_{Xt}(\hat{v}(\cdot))$. On the other hand, we can first solve the SDE

$$X(s) = X - \frac{1}{\lambda}\int_t^s D_X(X(s)\otimes m, s)\,\mathrm{d}s + \sigma(w(s)-w(t))$$

and then take as a candidate control $v(s) = -\frac{1}{\lambda}D_X(X(s)\otimes m, s)$. Then all the inequalities in (C.41) become equalities, and we see that $V(X,t) = J_{Xt}(v(\cdot))$. It follows that $v(\cdot)$ must in fact be optimal and then V is the value function.

This completes the proof of Theorem 5.3. ■

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