World Scientific
www.worldscientific.com

ON THE MATHEMATICAL THEORY OF THE DYNAMICS
OF SWARMS VIEWED AS COMPLEX SYSTEMS

N. BELLOMO

Department of Mathematics, Politecnico Torino,
Corso Duca degli Abruzzi 24, 10129 Torino, Italy
nicola.bellomo@polito.it

J. SOLER*

Departamento de Matemática Aplicada,
University of Granada, 18071 Granada, Spain
jsoler@ugr.es

This paper deals with the modeling and simulation of swarms viewed as a living, hence complex, system. The approach is based on methods of kinetic theory and statistical mechanics, where interactions at the microscopic scale are nonlinearly additive and modeled by stochastic games.

Keywords: Living systems; swarming; collective behavior; self-organization; nonlinear interactions; learning; large deviations; flocking.

1. Introduction and Plan of the Project

Modeling swarm dynamics is a challenging research objective, which requires mathematical tools suitable to capture the complexity features of these systems. The approach needs to describe them as living systems due to the individual ability of the entities that form a swarm to develop specific strategies by means of their interaction with other entities.

There is a common attraction generated by the observation of the beauty of the shapes formed by birds which appear in the sky during spring and autumn periods. Analogous phenomena are observed in other animal systems such as fish schools[77] or cells which aggregate forming particular patterns. However, the dynamics of interactions differs substantially for each type of system due to the peculiar strategy that the individuals generate from the formation of biofilms in the case of bacteria to the escape from the attack of a predator, among others.

*Corresponding author

Various modeling approaches have been recently proposed increasing hugely the literature on this topic. A first classification could be done by attending to the observation and representation of the selected scales for the modeling approach: stochastic differential equations,[1] macroscopic equations derived from stochastic perturbation of individual dynamics,[26,33,34] modeling swarming patterns,[14,86,90] kinetic theory methods,[21,38,41,67] dynamical systems,[29,85] and other approaches such as structured population models.[2,39] One of the challenging objectives is the derivation of a mathematical theory of living fluids.[54,60] However, much is unknown about why animals exhibit collective behavior. Usually, on the basis of any collective strategy there is the intention of preserving the survival, and therefore the individual's genetic. Behind this problem there are deep fundamental questions of biology, physics, and mathematics: the multiple interaction (not just binary) between individuals to define criteria for identifying partners (flock-mates), to precise the different domains of interactions, processes of synesthetic congruency, and synchronization of external or internal group or individual signals, among others. Progress on these issues requires a confluence between the more theoretical aspects, modeling and experimentation.

Let us briefly review the literature on some of the recent advances in these topics. Flocking phenomena, that constitute a typical characteristic of swarms, have been treated in Refs. 30, 31, 66 and 84, while various types of animal behaviors have been examined in Refs. 5, 71 and 72. Indeed, a deep insight on emerging strategies needs to be specifically referred to the type of individuals composing the swarm,[35] as well as on the interaction between individual behavior and dynamics.[40] A specific feature is that the swarm has the ability to express a collective intelligence[27] or memory to reproduce some patterns related to the environmental conditions,[15] which can evolve by learning processes. This feature can be used to drive learning processes in modern technology of robots.[58]

The experimental investigation on swarms differs from that developed in the case of pedestrian crowds. In fact, it is mainly focused on understanding the dynamics (and topology) of the interactions corresponding to different animal species. Furthermore, experiments are addressed to understand emerging behaviors, such as flocking phenomena, break up and aggregation of swarms, or reaction to a predator. All of them should be depicted by the models. Therefore, the collection of empirical data is generally focused on qualitative, rather than quantitative, aspects. The interested reader can find additional bibliography in the last section of the review paper.[12]

This paper basically focuses on methodological issues to provide mathematical structures suitable to capture the complexity features of swarms, thus offering a unified background toward the derivation of specific models. Hopefully, the approach can unify various research contributions that take into account only a limited number of the variety of the several features exhibited by the class of systems under consideration, while the analogies and differences with respect to the modeling of crowds are critically analyzed in Ref. 6.

The structure of this paper is as follows: Sec. 2 presents some speculations on flocking phenomena based on the interpretation of their specific phenomenological behaviors. These reasonings are preliminary to Sec. 3, which focuses on the complexity features of swarms to be viewed as specific requirements for the modeling approach. Section 4 proposes some mathematical structures that are appropriate, according to the authors' bias, to derive specific models. Section 5 aims at showing how the aforesaid structures can be properly specialized to derive specific aspects of swarming phenomena. Section 6 looks ahead to research perspectives.

2. Flocks are Dynamics by Nature

The idea of this section is to try to give some insight on the underlying problems concerning flocking phenomena, such as those represented schematically in Fig. 1. The aim consists in pointing out some features of the complexity of the aforesaid dynamics simply by phenomenological interpretation. Moreover, it will be shown that specific features of the dynamics depend on the type of species under consideration, namely from cells to micro-organisms, up to vertebrates. These considerations may contribute to extract, as we shall see in the next section, some common complexity features.

It is plain that flocks are dynamic by nature. This means that the graph representing their topology and interaction links are changing as it goes on. Flocks are formed by a large group of autonomous mobile individuals with local interactions between subgroups and a possible group objective or additional property of the swarm. In fact, the aggregation of individuals has a global property, which is not observed in the behavior of an isolated individual, through the interaction of the individual units. Social behavior of groups of animals can be understood with the help of complexity science. Although there are many similarities, the study of each species requires knowledge of the specifics of each group. For example, in studies of self-organization of large flocks of birds, the density is similar between back and front and flock density is independent of flock size,[4] while in fish schools high frontal density is observed and also that group density depends on the number of individuals.[49]

The hypothesis of binary interactions does not allow to recover the dynamics of effective multiple interactions between individuals and subgroups.[4] The dynamical domains of influence of individuals and groups inside a flock are based on multisensory integration of auditory, smell, visual, taste, . . . ; information and/or the capacity of perception and integration of swimming, flight moving waves or (short or long range) signals of different nature coming from other individuals inside or outside the group and from the environment, such as temperature gradients and electromagnetic, gravity or pressure gradients, see for example Ref. 80. These signals can promote a stronger coupling motivated by synesthetic congruency[76] or by simple synchronization mechanisms[19] and, as a consequence, the individuals adapt continuously their domain of influence. The result is a nonlinear complex system.

In Reynolds' pioneering paper on flocking[82] the individuals of the simulated flock, which are called *boids*, adjust their dynamical behavior by following three simple rules: Separation (collision avoidance from neighbors too close), Cohesion (attraction to distant neighbors) and Alignment (in particular, velocity alignments
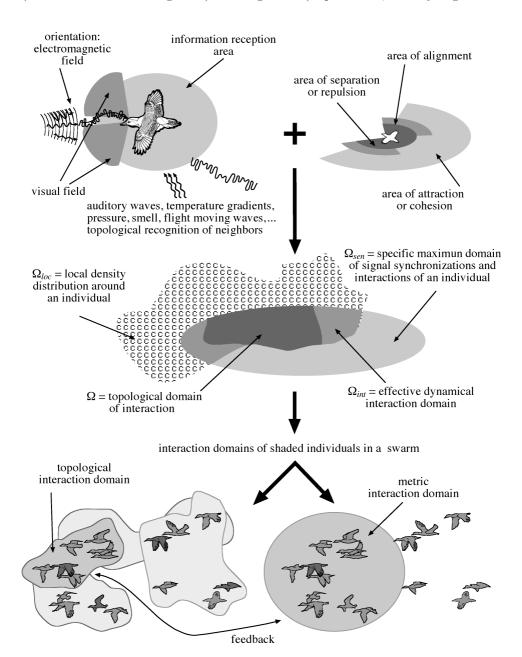

Fig. 1. Scheme of the different domains of interaction in the collective behavior of birds.

with neighbors). Reynolds calculates for each boid the average of the summed up direction vectors of the boids that are in the angle of view and in range of sight. The final direction of a boid's movement is calculated as the weighted sum of the results of these three rules. The variation of the sizes of these zones could reproduce various aggregation patterns, see for example Refs. 28, 49 and 52. The area of blindness also plays an important role in the collective behavior (see Fig. 1). The above results also show the existence of a collective memory in the patterns formation of the flock.

The overlap of inputs caused by the intersection of the domain of sensory signals and the cohesion, attraction or alignment regions determines the domain of local interaction with other individuals. This domain is induced by a metric distance criterion. As we have pointed out previously, assuming that the local interaction domain of an individual only depends on its static visual field, it could be too simple to describe the effective multisensory integration of the received information even in the classical examples of birds, fishes or insects.

2.1. *Metric versus topological interaction domains*

In a metric-distance domain, an individual interacts with all other individuals in its neighborhood. The number of members in the metric-domain fluctuates dynamically as nearby individuals enter and leave its neighborhood. In this model, a fixed limit of perception corresponding to their interaction neighborhoods is assumed and should be given by the researcher in advance, for example by defining an interaction range or radius. Outside this neighborhood, agents do not interact with each other. The metric-distance model could provide a good explanation for collective, synchronized motions when the density of individual agents is high.

However, when the density is low, topological domains play a relevant role.[4,24,27,42,50] In topological-distance domains the neighborhood of each individual is elastic and expands and shrinks continuously. The perception limit is defined by the number of neighbors independently of the absolute distance to the reference individual. This requires a criterion for identifying the flock-mates. For example, the number of members in a topological domain of birds ranges between 6–7 neighbors. Ballerini *et al.*[4] suggested that the topological range of interactions rather than the metric distance should model collective motion of birds. The concept of topological distance becomes important to model collective behavior.

In our opinion, topological distance and metric distance are two aspects of the same problem. Note the interdependence between both types of cognition concerning the capacity of decision of an individual: topological, which requires a criterion for identifying the reference flock-mates of an individual, while metric distance refers to the capacity of the individuals of incorporating signals coming from the interior or exterior of the swarm. At every time step, each individual updates its information, switching between metric-domain and topological-domain. Two recent approaches to this problem have been done in Ref. 50, where each agent can adjust

its interaction range when its density of neighbors changes, and by Ref. 69, where the idea is to focus on the influence of both criteria in the degree of alignment of the swarm. The objective, that should be included in the modeling processes, must be to coordinate both interaction domains in a continuous feedback (see Fig. 1) in order to improve the capacities of the individuals in accordance with those of the group.

2.2. *The example of self-interaction bacteria and biofilms*

Other individuals such as cells (bacteria), strategy-games players or computing algorithms, where swarming intelligence can be applied to govern migration in the distributed genetic programming algorithm governed by Darwinian natural selection, are also involved in movement forming groups of organisms in motion and could open the window to modeling interaction, strategies, possible leaders of subgroups, and various others. These new inputs should be incorporated in a realistic modeling of such swarming processes. As mentioned before, bacteria are among these examples. Since the collective motion of other entities (birds, insects, fishes, . . .) has been treated more deeply in the literature, let us give an introduction to the main general characteristics of motion in bacteria. Swarming is the fastest known bacterial mode of surface translocation and enables the rapid colonization of a nutrient-rich environment and host tissues. Although bacteria are unicellular organisms, they often show group behavior: e.g. in living biofilms (sessile communities with micro-organisms embedded within a matrix and attached to a surface), where individual cells at different locations may have different activities.[55] This led to the proposal that biofilm communities may represent an evolutionary step between unicellular non-specialized organisms and multicellular organisms that possess specialized cells.[32,88] This complex multicellular behavior requires the integration of chemical and physical signals, which leads to the physiological and morphological differentiation of the bacteria into swarming cells.[65,87]

Decision-making between rapidly colonizing a surface and biofilm formation is central to bacterial survival among competitors. Bacteria might select between motility, such as swarming, and sessile biofilm formation at certain stages. In the process of swarming and biofilm formation, flagella enable bacteria to move towards favorable environments during swimming and contribute to the virulence of pathogens through adhesion and biofilm formation on host surfaces. Since nutrient availability is crucial to sustain the energy demanding process of swarming, one would expect the chemotaxis sensory system to be important for swarming in bacteria. However, chemotaxis is not necessarily required. The composition of the extracellular matrix is tightly controlled and determines whether motility of the colony can occur or not. Cell density is centrally important in swarming and a critical cell mass is necessary to initiate and sustain the swarming process. It is, therefore, not surprising that in many bacteria, swarming is coupled to quorum sensing.[80]

Quorum sensing regulation of swarming presumably allows optimal dissemination of bacterial cells when a population is getting too large to inhabit a single given niche. Quorum sensing is the regulation of gene expression in response to fluctuations in cell-population density. Quorum sensing bacteria produce and release chemical signal molecules called auto-inducers that increase in concentration as a function of cell density. This effectively expands the range of environmental signals that influence target gene expression, such as those for biosurfactant production beyond population density. In fact, the internal and external signals received by vertebrates as described before play a similar role that chemical signals or quorum sensing in the bacteria case.

Complex patterns of swarming have also been reported for bacteria in the literature such as Bacteria, *E. Coli*, or swarms of elementary organisms,[25,36,37,53,68] among others. In every of these examples linear diffusion does not play any role and swarming and self-organization into domain structures is a result of nonequilibrium, nonadiabatic, biologically-driven processes that require social objectives and multiagent interaction motility systems to occur. From an applied point of view, influencing the swarming behavior of bacteria may help to control root colonization and containment, and this may have important applications in agriculture. Quorum-sensing-mediated swarming control may also have implications on biofilms in industrial and ecological settings (e.g. potable water distribution systems) and in environments more relevant for public health (such as indwelling medical devices, cystic fibrosis, periodontitis) and may provide an alternative therapeutic strategy to combat microbial contamination, see, for example, Ref. 87.

3. Complexity Features of Swarms

As already anticipated in Sec. 2, the modeling approach should retain, as far as possible, the complexity features that characterize the dynamics of swarms, viewed as a class of living (hence complex) systems. More precisely, the mathematical structure selected to derive specific models should be consistent with the aforesaid features.

Accordingly, five specific characteristics are selected, among various ones, that can be considered particularly important according to the authors' bias. The analysis focuses on swarming phenomena in the animal world, while some particularizations and generalizations will be discussed in the last section.

(1) *Ability to express a strategy.* Individuals in a swarm are living entities capable to develop a specific *organization ability* that depends on the state of the other individuals in their interaction domain (metric or topological). This strategy can be expressed without the application of any external organizing principle, while it can be modified by external actions. It is generated both by the ability to communicate with the other entities and to organize the dynamics according also to the interpretation of the strategy expressed by the other entities. Such

strategy includes a clustering ability (flocking) that prevents the fragmentation of the domain occupied by the swarm. Moreover, when a fragmentation of such domain occurs, the clustering ability induces re-aggregation dynamics. The strategy is related to the ability to produce patterns, such as biofilms in the case of bacteria. It is important here to emphasize the group's ability to interact in feedback with others and retrain each other's patterns. This may be the case of the spatial distribution of certain flowers and the organization of flying pollinators.

(2) *Interactions.* The aforesaid strategy is expressed by nonlinearly additive interactions, which can involve immediate neighbors, but also distant ones (topological domain). In some cases, a fixed number of neighbors can play a prominent role in the development of the strategy and interactions as proposed in Ref. 4, also on the basis of empirical data.[23] Interactions modify the state of individuals according to the strategy they develop. The output is not due to deterministic causality principles. Indeed, it depends on the strategy mentioned above, which evolves in time according to the fact that living systems receive a feedback from their environments, which enables them to learn from their experiences.

(3) *Heterogeneity and hierarchy.* The ability to express a strategy may be heterogeneously distributed among individuals, although all of them generally express it, in normal conditions, with almost equal intensity. However, panic conditions can induce relevant heterogeneity in such expression. Moreover, the strategy differs for different categories of individuals in the same swarm. This feature is called *hierarchy.* For instance, the strategy expressed by the first category has a strong influence on that of the second one. In some animal species, for instance ants or bombo-bees,[46,51] the hierarchy corresponds to a sharp division into dominant and dominated. More in general one can talk about hierarchical functions, for instance a leader who controls a flock.[17]

(4) *Learning.* In general a swarm has the ability to express a collective intelligence that is generated by a cooperative strategy.[15,66] Such intelligence is also related to the fact that living systems receive feedback from their environment, which enables them to learn from their experiences. Similarly to the previously discussed characteristics of the interactions, the learning process is not linearly additive.

(5) *Large deviations.* The expression of the strategic ability and the characteristics of interactions among individuals in the swarm can be largely modified when panic conditions occur. Namely, a swarm in normal conditions has a well-defined objective, for instance reaching a certain zone starting from a localization. However, panic conditions can modify the overall strategy to pursue this objective, which consequently modify the overall dynamics. Thus, emerging behaviors very different from those observed in normal flow conditions can appear. Among these, fragmentation phenomena that are not sufficiently contrasted by the natural trend to flocking can show up.

Some additional comments can be made in view of a modeling approach that takes into account the aforesaid complexity features.

— The dynamics of interactions differs in the various zones of the swarm. For instance, from the border to the center of the domain occupied by the swarm. Small stochastic behaviors could be an important characteristics for the dynamics to take into account small fluctuations induced by heterogeneous behaviors.

— Validation of models should be based on their ability to depict observed emerging behaviors, which depends on the type of individuals composing the swarm.[44] Various emerging behaviors can be studied, such as milling formation,[62] flocking phenomena,[30,31] and many others. Generally, the qualitative behavior is preserved in the case of similar input data; however, small deviations of them may induce large deviations of the quantitative values of the dynamics.

— The concept of swarm can be extended to various types of micro-organisms and ultimately to cells in a multicellular system. In this case the strategy expressed by the interacting entities depends on the biological functions that characterize the population. Moreover, the modeling approach should include proliferative and/or destructive events, while heterogeneity of individual behaviors plays an important role in the dynamics of swarms of cells in biology.[11]

— The number of individuals involved in the swarm is not large enough to justify the approach of continuum mechanics. Moreover, even the continuity assumption of the probability distribution over the microscopic state of the individuals of the swarm needs to be put in discussion and to be treated as an approximation of physical reality.[6]

The mathematical approach should consider all the above reasonings to be developed more extensively. The main difficulty consists, as we shall see, in introducing the modeling of nonlinearly additive interactions consistent with the conjecture,[4,23] while the existing literature is mainly based on linearly additive interactions as the classical kinetic theory.

4. Mathematical Structures

Let us consider, with reference to Fig. 2, a large system of entities interacting in a domain Σ_t, which evolves in time starting from the initial shape Σ_{t_0}, and let us look for a general mathematical structure suitable to offer the conceptual background for the derivation of specific models. The strategy presented in the next subsections can be developed, toward the modeling of the class of systems under consideration, as follows:

(i) Representation by a probability density over the state, to be properly modeled at the microscopic scale, of the individual beings in the swarm;

(ii) Derivation, on the basis of suitable models of interactions at the micro-scale, of an evolution equation of the aforesaid probability density;
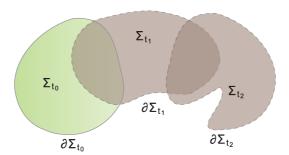
Fig. 2.   Geometry of the domain occupied by the swarm.

(iii) Critical analysis focusing on the consistency of such equation with the complexity features presented in Sec. 3.

Plane motions will be considered for simplicity of notations, leaving to additional technical calculations the generalization to dynamics in three-space dimensions.

The contents of the following subsections are based on the mathematical kinetic theory for active particles.[10] Therefore, the state of each individual, hereinafter called *active particle*, or simply *particle*, includes, in addition to classical and geometrical variables, also an additional variable, called *activity*, suitable to represent the strategy expressed by active particles. The interpretation of this class of equations referred to Markov processes is given in Ref. 59.

More in detail, we consider a system constituted of a fixed number N of particles and use dimensionless quantities by referring positions to the largest dimension ℓ of the domain Σ_{t_0} at the initial time $t = t_0$; and the velocity to the highest velocity v_M, which can be reached by individuals in a free isolated motion. This strategy contributes to generalize the approach to different types of animal behaviors.

4.1. *Representation*

Let us consider a system of active particles whose mechanical characteristics are modeled as rigid bars with center C and orientation θ, which is also the direction of the motion. Their microscopic state is identified by the following variables:

— $\mathbf{x} = \{x, y\}$ is the localization (referred to ℓ) of the center C of each particle;
— θ is the angular orientation of the particles, which also identifies the direction $\boldsymbol{\nu} = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$ of the motion, where $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors of a system of Cartesian axes;
— $\mathbf{v} = v\boldsymbol{\nu}$ is the velocity along $\boldsymbol{\nu}$ with v referred to v_M;
— $u \in D_u$ is the activity variable, which models the strategy. It can also be used to identify a hierarchy. In extreme cases, $u = 1$ corresponds to dominant, while $u = 0$ to dominated.

The overall state of the system is described by the probability density distribution over the micro-state:

$$f = f(t, \mathbf{x}, v, \theta, u) : [0, T] \times \Sigma_t \times [0, 1] \times [0, 2\pi) \times D_u \to \mathbb{R}_+, \qquad (4.1)$$

which is positively defined, referred to N, and provides via $f\, d\mathbf{x}\, vdv\, d\theta\, du$ the probability of finding an active particle in the elementary volume of the space of the microscopic states $[\mathbf{x}, \mathbf{x} + d\mathbf{x}] \times [v, v + dv] \times [\theta, \theta + d\theta] \times [u, u + du]$.

If the particles are subdivided into *functional subsystems* by a hierarchy, the following representation can be used:

$$f = f(t, \mathbf{x}, v, \theta, u) = \sum_{i=1}^{p} f_i(t, \mathbf{x}, v, \theta)\delta(u - u_i), \qquad (4.2)$$

wherein the particular case that has been mentioned above has $p = 2$.

Remark 4.1. The concept of functional subsystem needs to be made precise. Namely it corresponds to aggregation of particles that expresses the same activity. Therefore, if the system is made of particles of the same species, the hierarchy identifies the subdivision into subsystems. If the number of species is larger than one, each species identifies an aggregated subsystem to be further subdivided into functional subsystems when a hierarchy has to be taken into account.

Remark 4.2. If f, or f_i, are known, macroscopic quantities can be computed by weighted moments by standard calculations[10] that are not repeated here. However, it is worth mentioning that the representations (4.1) and (4.2) approximate a discrete system by a continuous one. Therefore, conceivable methods to overcome this approximation needs to be discussed as we shall see in the last section.

Remark 4.3. The above representation is such that particles move along the direction θ that coincides with the orientation of the bar. If the direction of the motion differs from that of the bar, an additional angular variable has to be inserted $f = f(t, \mathbf{x}, \theta, \mathbf{v}, u)$. If the length is not considered, the representation is simply obtained by using the vector velocity in its microscopic state: $f = f(t, \mathbf{x}, \mathbf{v}, u)$.

4.2. *From interaction models to mathematical structures*

Let us now look for a mathematical structure, consistent with the complexity features presented in Sec. 3, and suitable to define the evolution in time and space for the probability density introduced in Sec. 4.1. More precisely, we consider the simpler case of a unique functional subsystem with equally expressed activity for all particles. Some conceivable generalizations are introduced in the next subsection concerning particles with heterogeneous behavior.

The strategy to derive these equations follows the guidelines of the classical kinetic theory. Namely, by using a balance equation for net flow of particles in the elementary volume of the space of the microscopic state taking into account

transport and interactions. Specifically both long range interactions with the whole swarm and local interactions are taken into account referring to the following formal equation:

$$\partial_t f + \mathbf{v} \cdot \partial_\mathbf{x} f + \partial_\mathbf{v}(\mathcal{F}[f]f) = J[f], \tag{4.3}$$

where $f = f(t, \mathbf{x}, \mathbf{v})$, long range interactions within the swarm are modeled by $\mathcal{F}[f](\mathbf{x})$, the left-hand side models the transport of particles, and where the right-hand side $J[f]$ represents the short range interaction among particles. The following active particles are involved in the interactions:

- *Test* particles with microscopic state, at the time t, defined by the variable $(\mathbf{x}, \mathbf{v})$, whose distribution function is $f = f(t, \mathbf{x}, \mathbf{v})$;
- *Field* particles with microscopic state, at the time t, defined by the variable $(\mathbf{x}^*, \mathbf{v}^*)$, whose distribution function is $f^* = f(t, \mathbf{x}^*, \mathbf{v}^*)$;
- *Candidate* particles with microscopic state, at the time t, defined by the variable $(\mathbf{x}_*, \mathbf{v}_*)$, whose distribution function is $f_* = f(t, \mathbf{x}_*, \mathbf{v}_*)$.

The probability distribution is the dependent variable in Eq. (4.3), to be obtained as solution of an initial value problem in unbounded domains.

The main problem consists in modeling the aforesaid interactions. Some perspective ideas will be here proposed towards this objective that is alternative to the traditional binary interaction schemes. These ideas have to be specifically particularized for each type of swarm being aware that specific characteristics are related to different types of them. Bearing all above in mind, three types of domains can be defined at the micro-scale:

— Ω_{sen} is for each particle a specific theoretical maximum domain of sensorial signal synchronization (vision, auditory waves, smell, temperature, pressure, and other stimuli) and of local interaction (alignment, separation, attraction), see Fig. 1;
— Ω_{loc} is the support of the local density distribution;
— Ω_{int} is the effective interaction domain. In general $\Omega_{\text{int}} \subseteq \Omega_{\text{sen}}$,

$$\Omega_{\text{int}} = \Omega_{\text{sen}}|_{\text{supp}(\rho)} = \Omega_{\text{sen}} \cap \Omega_{\text{loc}};$$

— Ω is the topological domain $\Omega \subseteq \Omega_{\text{int}}$.

Modeling interactions at the microscopic scale implies a study of the role of the aforesaid domains and of the velocity fields of such dynamics. In fact, each active particle decides the movement strategy according to that of the other particles within an interaction domain to be properly defined, as well as according to the velocity field of the whole swarm and of that of the entities within such domain. This problem can be treated at a practical level only if referred to well-defined types of swarms. Therefore, some general guidelines are given in the sequel.

As usual, we define

$$\rho(t,x) = \int_{D_v} f(t,\mathbf{x},\mathbf{v})d\mathbf{v}, \quad \rho(t) = \int_{\Sigma_t} \rho(t,x)d\mathbf{x} \tag{4.4}$$

and

$$\rho_\Omega(t) = \int_\Omega \rho(t,x)d\mathbf{x}. \tag{4.5}$$

Equation (4.5) constitutes a strongly nonlinear coupling relation between Ω and ρ that defines the multiple interaction process of the individuals (not only binary) in Ω. Of course Eq. (4.5), and the equivalents for Ω_{sen} and Ω_{int}, become simple integrals when the shapes of the domains Ω, Ω_{sen} and Ω_{int} are static and known in advance. However, in general this assumption seems not to be too realistic. In fact, the different domains defined above are also time-dependent upon the density ρ.

A way towards understanding the role of the aforesaid domains consists in imposing a restriction induced by a critical density. For instance:

$$\rho_\Omega^c(t) = \int_{D_v} \int_\Omega f(t,\mathbf{x},\mathbf{v})d\mathbf{x}\,d\mathbf{v}, \tag{4.6}$$

where ρ^c is a critical density assigned *a priori*. In a similar way we can define $\rho_{\Omega_{\mathrm{sen}}}^c(t)$ or $\rho_{\Omega_{\mathrm{int}}}^c(t)$. Focusing on Ω_{sen}, these assumptions state that each active particle is subject to short range interactions at most with a fixed number of particles in its sensorial domain Ω_{sen}. Usually the only constraint (4.6) is far to define the domain Ω. Other additional possibility to specify the domain consists in fixing different moments in velocity of the distribution function such as the direction of the relative velocity of the aggregate with respect to Ω,

$$\Theta_\Omega^c(t) = \langle \mathbf{v} \rangle = \int_{D_v} \int_\Omega \mathbf{v}f(t,\mathbf{x},\mathbf{v})d\mathbf{x}\,d\mathbf{v}, \tag{4.7}$$

or similarly in fixing the mean velocity in Ω,

$$\mathbf{V}_\Omega(t) = \frac{\langle \mathbf{v} \rangle}{\rho_\Omega},$$

or the direction velocity in Ω,

$$\boldsymbol{\nu}_\Omega(t) = \frac{\mathbf{V}_\Omega(t)}{|\mathbf{V}_\Omega(t)|}.$$

Figure 3 shows a conceivable domain of interaction of an individual, where $\boldsymbol{\nu}_S$ is the direction of the motion of the swarm (in this case the integral in (4.7) must be taken on Σ_t). Other moments or combination of them (such as the variance) or other specific properties of each swarm are needed to define the domains.

Of course, as we pointed out in Sec. 2, the domain Ω is dynamic, as well as Ω_{sen} and Ω_{int} are. Some individuals in the domain Ω_{sen} may go out of the interaction domain Ω_{int} or the local variables of motion of the selected topological flock-mates are not related with those of the swarm. This is the case when particles do not receive sufficient information, namely the received information contrasts with that
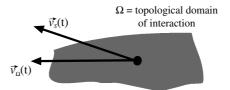
Fig. 3.   Interaction zone relating the direction of motion $\boldsymbol{\nu}_\Omega(t)$ in the topological domain to those of the swarm $\boldsymbol{\nu}_S(t)$.

of the metric domain and of the collective motion domain Σ_t. Therefore, individuals need to re-evaluate their topological domain Ω.

This process of re-evaluation and feedback between metric, topological and collective information must be continuous in time. This is a difficult and very stimulating open problem where some recent contributions have been done.[4,50,69] We propose, at least formally, that a function $\Omega_\beta = \beta(\Omega, \Omega_{\text{int}})$ can be introduced to model the continuous feedback between metric and topological domains by testing how the individuals in the topological domain are in the metric domain and how the local variables, such as direction, density, mean velocity or variance, for instance, associated with the topological domain are in agreement with the global ones. In the case where the density population is high, one can imagine a stable situation for which β is the characteristic function of the domain Ω_{sen}. Moment-in-velocity calculations must be related to this function β.

One of the possible tests that may carry out the function β is to compare the velocity direction of the topological domain with that of the swarm. More precisely, the individual should check the direction of the motion $\boldsymbol{\nu}_\Omega(t)$, which is the variable at the microscopic scale, with those of the whole population $\boldsymbol{\nu}_S(t)$ in order to reorganize his topological domain by matching both directions introducing individuals from his interaction domain Ω_{int}. From now on, we will call Ω_β the domain of reference of an individual.

As an example, we can mention the simple case in which the domain is previously assigned and for instance consists in an arc of circle identified by two visual angles, say θ^+ and θ^-. Then, detailed calculations are possible. In fact, Ω_{loc} can be computed and if $\Omega_{\text{loc}} \subseteq \Omega_{\text{sen}}$ one has $\Omega_{\text{int}} = \Omega_{\text{loc}}$, while if $\Omega_{\text{sen}} \subset \Omega_{\text{loc}}$, then $\Omega_{\text{int}} = \Omega_{\text{sen}}$. In this case individuals do not receive sufficient information. In particular, the angle of interaction domain may not be symmetric with respect to the velocity and may be narrower or wider according to the specific cases object of modeling.

In order to simplify the exposition we have not included the mean variations in velocities, but an analogous reasoning can be developed concerning the velocity domain.

The density within the interaction domain Ω_β is computed as follows:

$$\rho_{\Omega_\beta}(t) = \rho_{\Omega_\beta}[f](t) = \int_{D_v} \int_{\Omega_\beta} f(t, \mathbf{x}, \mathbf{v}) d\mathbf{x} \, d\mathbf{v}. \tag{4.8}$$

Moreover, the mean velocity of the particles included in Ω is computed by

$$\mathbf{V}_{\Omega_\beta}(t) = \mathbf{V}_{\Omega_\beta}[f](t) = \frac{1}{\rho_{\Omega_\beta}(t)} \int_{D_v} \int_{\Omega_\beta} \mathbf{v} f(t, \mathbf{x}, \mathbf{v}) d\mathbf{x}\, d\mathbf{v}. \tag{4.9}$$

This is the mean velocity within the cluster and we define the direction

$$\boldsymbol{\nu}_{\Omega_\beta}(t) = \frac{\mathbf{V}_{\Omega_\beta}(t)}{|\mathbf{V}_{\Omega_\beta}(t)|}.$$

An additional quantity of interest is the mean velocity of the whole storm:

$$\mathbf{U}(t) = \mathbf{U}[f](t) = \frac{1}{\rho(t)} \int_{D_v} \int_{\Sigma_t} \mathbf{v} f(t, \mathbf{x}, \mathbf{v}) d\mathbf{x}\, d\mathbf{v}, \tag{4.10}$$

where

$$\rho(t) = \rho[f](t) = \int_{D_v} \int_{\Sigma_t} f(t, \mathbf{x}, \mathbf{v}) d\mathbf{x}\, d\mathbf{v}.$$

Bearing all above in mind, let us now look for some general rules concerning the interaction dynamics. Nonlinearly additive interactions will be considered and the modeling approach is based on the assumption that each particle is subject to individual actions due to the action at the microscopic scale of the other particles, and to an overall one applied by the system as a whole at the macroscopic scale.

H.1. The *test* particle $\mathbf{x}, \mathbf{v}$ is subject to an action of the whole storm of the type:

$$\mathcal{F}[f](t, \mathbf{x}) = \frac{1}{\rho_M} \int_{\Sigma_t} \varphi(\mathbf{x}, \mathbf{x}^*, \mathbf{v}, \mathbf{v}^*, \mathbf{U}[f](t)) f(t, \mathbf{x}^*, \mathbf{v}^*) d\mathbf{x}^*\, d\mathbf{v}^*, \tag{4.11}$$

where φ is the individual action of the field particle $(\mathbf{x}^*, \mathbf{v}^*)$ over the test particle. This acceleration term depends on the distance between particles inducing a flocking action, but also induces an attraction towards the mean direction of the swarm:

$$\varphi = \frac{1}{\varepsilon + |\mathbf{x} - \mathbf{x}^*|^a} \psi(\mathbf{v}, \boldsymbol{\nu}_S), \quad a > 0, \quad \boldsymbol{\nu}_S = \frac{\mathbf{U}}{|\mathbf{U}|}, \tag{4.12}$$

where ε is a small positive dimensionless quantity. The term ψ models the attraction of individuals toward the main stream of the swarm. It increases with increasing angle between the individual direction and the mean direction of the swarm.

H.2. There is an interaction rate η related to ρ_{Ω_β} as follows: $\eta[f](t, \mathbf{x}) = \eta_0(t, \mathbf{x})\mu[f]$ where, depending on time and position,

$$\mu[f] = \left(\frac{\rho_{\Omega_\beta}}{\rho_{\Omega_{\text{int}}}}\right)^b, \quad b > 0, \tag{4.13}$$

and where $\rho_{\Omega_{\text{int}}}(t)$ represents the mean density on Ω_{int} and η_0 represents the interaction rate when the two domains coincide.

H.3. The *candidate* particle with microscopic state $(\mathbf{x}_*, \mathbf{v}_*)$ at the time t interacts with *field* particles $(\mathbf{x}^*, \mathbf{v}^*)$ with rate η and acquires the state of the *test* particles. The candidate particle modifies its state according to the probability density $\mathcal{A}$ which depends on the state of the interacting particles and on the mean velocity $\mathbf{V}$:

$$\mathcal{A}(\mathbf{v}_* \to \mathbf{v}|\mathbf{v}_*, \mathbf{v}^*, \mathbf{V}[f]), \quad \int_{D_\mathbf{v}} \mathcal{A}(\mathbf{v}_* \to \mathbf{v}|\mathbf{v}_*, \mathbf{v}^*, \mathbf{V}[f])d\mathbf{v} = 1, \qquad (4.14)$$

for all conditioning inputs, while *test* particles interact with field particles and lose their state. More precisely, candidate particles have a trend, in short range interactions, to adjust their velocity both to the mean velocity of the cluster and to that of the field particles.

These assumptions state that each individual feels an action of the whole swarm and a local action of the effective interaction domain. Conservation of particles in the elementary volume of the phase space yields:

$$(\partial_t + \mathbf{v} \cdot \partial_\mathbf{x} + \partial_\mathbf{v}\mathcal{F}[f])f(t, \mathbf{x}, \mathbf{v})$$

$$= J[f](t, \mathbf{x}, \mathbf{v})$$

$$= \int_{\Omega_\beta} \int_{(D_\mathbf{v})^2} \mu[f]\mathcal{A}(\mathbf{v}_* \to \mathbf{v}|\mathbf{v}_*, \mathbf{v}^*, \mathbf{V}[f])f(t, \mathbf{x}, \mathbf{v}_*)f(t, \mathbf{x}^*, \mathbf{v}^*)d\mathbf{x}^* d\mathbf{v}_* \, d\mathbf{v}^*$$

$$- f(t, \mathbf{x}, \mathbf{v}) \int_{\Omega_\beta} \int_{D_\mathbf{v}} \mu[f]f(t, \mathbf{x}^*, \mathbf{v}^*)d\mathbf{x}^* \, d\mathbf{v}^*. \qquad (4.15)$$

This structure can be rapidly generalized to the case of (4.2), corresponding to the presence of a hierarchy, as follows:

$$\left(\partial_t + \mathbf{v} \cdot \partial_\mathbf{x} + \sum_{j=1}^p \partial_\mathbf{v}\mathcal{F}_{ij}[f]\right)f_i(t, \mathbf{x}, \mathbf{v}) = J_i[f](t, \mathbf{x}, \mathbf{v}) = \sum_{j=1}^p J_{ij}[f](t, \mathbf{x}, \mathbf{v}), \qquad (4.16)$$

where

$$J_{ij} = \int_{\Omega_\beta} \int_{(D_\mathbf{v})^2} \mu[\mathbf{f}]\mathcal{A}_{ij}(\mathbf{v}_* \to \mathbf{v}|\mathbf{v}_*, \mathbf{v}^*, \mathbf{f})f_i(t, \mathbf{x}, \mathbf{v}_*)f_j(t, \mathbf{x}^*, \mathbf{v}^*)d\mathbf{x}^* \, d\mathbf{v}_* \, d\mathbf{v}^*$$

$$- f_i(t, \mathbf{x}, \mathbf{v}) \int_{\Omega_\beta} \int_{D_\mathbf{v}} \mu[\mathbf{f}]f_j(t, \mathbf{x}^*, \mathbf{v}^*)d\mathbf{x}^* \, d\mathbf{v}^*. \qquad (4.17)$$

If the velocity is given in polar coordinates, rewriting Eqs. (4.11)–(4.13) is immediate. On the other hand, dealing with random distributions also over the activity variable, with interactions that modify such distribution, needs additional nontrivial work.

The mathematical structures (4.15)–(4.17) offer the basis for the derivation of specific models to be designed by a particularization of Assumptions H.1–H.3 according to the phenomenology of the system under consideration, which depends on the specificity of the system itself. Their generalization to the case of particles with heterogeneous distribution of the activity variable is presented in the next subsection.

4.3. *Particles with heterogeneous behavior*

The approach presented in the preceding subsection can be generalized to the case of systems where particles express heterogeneously their strategy, namely when the representation is delivered by the probability density (4.1). The main problem consists in modeling the transition probability density (4.14), which includes also the activity variable

$$\mathcal{A}(\mathbf{v}_* \to \mathbf{v}, u_* \to u | u_*, u^*, \mathbf{v}_*, \mathbf{v}^*, \mathbf{V}[f], E(u)), \tag{4.18}$$

to be normalized with respect to integration over both outputs $\mathbf{v}$ and u.

A heuristic hypothesis proposed in Ref. 10 suggests to factorize $\mathcal{A}$ as follows:

$$\mathcal{A} = \mathcal{B}(\mathbf{v}_* \to \mathbf{v} | u_*, u^*, \mathbf{v}_*, \mathbf{v}^*, \mathbf{V}[f])\mathcal{C}(u_* \to u | u_*, u^*, E(u)), \tag{4.19}$$

which corresponds to assume that variation of velocity depends on the activity variable of the interacting individuals, while modification of the activity is not influenced by the velocity variable. The mathematical structure can be modified, with respect to (4.15), as follows:

$$
\begin{aligned}
(\partial_t + \mathbf{v} \cdot \partial_{\mathbf{x}} &+ \partial_{\mathbf{v}} \mathcal{F}[f]) f(t, \mathbf{x}, \mathbf{v}, u) = Q[f](t, \mathbf{x}, \mathbf{v}, u) \\
&= \int_{\Omega_\beta} \int_{(D_{\mathbf{v}})^2} \mu[f]\mathcal{B}(\mathbf{v}_* \to \mathbf{v} | u_*, u^*, \mathbf{v}_*, \mathbf{v}^*, \mathbf{V}[f])\mathcal{C}(u_* \to u | u_*, u^*, E(u)) \\
&\quad \cdot f(t, \mathbf{x}, \mathbf{v}_*, u_*) f(t, \mathbf{x}^*, \mathbf{v}^*, u^*) d\mathbf{x}^* \, d\mathbf{v}_* \, d\mathbf{v}^* \, du_* \, du^* \\
&\quad - f(t, \mathbf{x}, \mathbf{v}, u) \int_{\Omega_\beta} \int_{D_{\mathbf{v}}} \mu[f] f(t, \mathbf{x}^*, \mathbf{v}^*, u^*) d\mathbf{x}^* d\mathbf{v}^* du^* \tag{4.20}
\end{aligned}
$$

and similarly in the case of several functional subsystems.

It is worth mentioning that the modeling of the term $\mathcal{C}$ can take advantage of the evolutive game theory,[43,47,70] where players modify their state according to games involving two or more players. However, the classical game theory cannot be straightforwardly applied in this specific case. In fact, the state of the players is linked to a probability distribution; and the output of the game is also modeled by a probability density. This feature justifies the use of the term *stochastic games*.

4.4. *Critical analysis*

The mathematical structure presented in Secs. 4.2 and 4.3 offers the conceptual background for the derivation of specific models. In other words, these structures are not yet a model that is obtained only after having properly identified the kernels of the various parameters that model interactions, namely ρ_{Ω_β} and $\mathcal{A}$. These structures have the ability to include, at least in part, some of the complexity features presented in Sec. 2. As we shall see, not all of them can be included, while a more detailed analysis is proposed later focusing on specific models.

(1) *Ability to express a strategy.* The strategy is expressed by interaction rules which need the definition of the interaction domains as well as specific mechanical rules. This strategy includes the clustering ability (flocking).

(2) *Interactions.* The output is not due to deterministic causality principles. Indeed, it depends on the strategy mentioned above, which evolves in time according to the fact that living systems receive a feedback from their environments, which enable them to learn from their experiences. This feature has to be specifically included in the interactions terms.

(3) *Heterogeneity and hierarchy.* The ability to express a strategy can be heterogeneously distributed among individuals, although all of them generally express it, in normal conditions, with almost equal intensity. However, panic conditions can induce an important heterogeneity in such expression. Moreover, the strategy differs for different categories of individuals in the same swarm. This is the case of the presence of a *hierarchy*. In a broad sense, the hierarchy is sometimes only a matter of strength in dealing, for example, with direct wind action when an individual is placed first in a flock of migrating birds.

(4) *Learning.* The mathematical structures (4.15)–(4.16) and (4.20) do not explicitly include learning processes, which need a timescale greater than that related to interactions valid for short time intervals. This topic is important and should be properly developed focusing on specific models.

(5) *Large deviations.* The dynamics depicted by models related to (4.15)–(4.16) and (4.20) generate emerging behaviors that depend on the specific parameters of the model and problems. We argue that these behaviors are very sensitive to the aforesaid parameters. Therefore, small deviations of them can generate large deviations of the emerging behaviors. In addition, we should also take into account large deviations related to modifications of environmental conditions. In particular, transitions from normal to panic conditions.

5. Perspectives on the Derivation of Models

The mathematical structures derived in the preceding section constitute the basis for the derivation of models corresponding to specific systems belonging to the more general class under consideration. This section presents a preliminary analysis of the modeling of some case studies. More precisely, the following topics, selected among several ones according to the authors' bias, will be treated:

(i) Hierarchy and homogeneous and heterogeneous behaviors;
(ii) Modeling an attack from a predator;
(iii) Small fluctuations perturbing deterministic behaviors.

Some introductory reasonings on the modeling approach related to these case studies will be briefly presented in the following subsections, while some perspective ideas towards future research programs are brought to the reader's attention as conceivable perspectives. It is worth stressing that looking at specific applications

can induce further developments of the mathematical structures derived in the previous section. For instance, this is the case of (ii), which requires the modeling of open systems. The assumption of continuity of the distribution function needs to be put in discussion considering that the number of interacting entities is not large enough, to justify this assumption that is typical of the kinetic theory and which refers to a huge number of particles. This matter is analyzed in the last section.

5.1. *Hierarchy, ordered and heterogeneous behaviors*

A hierarchy is often present in swarms. In some cases it is roughly a division between dominant and dominated individuals, for example in colonies of insects (see the empirical data presented by Hogeweg and coworkers[46,51]). Jager and Segel proposed in Ref. 57 a modeling approach. These important papers can be considered pioneering in the field and have motivated the research activity on these topics. The interested reader is referred to the book by Schweitzer[83] for methods and bibliography. In general, the presence of a hierarchy can affect in various ways the dynamics of a swarm by selecting groups of individuals characterized by different behaviors, or by a different expression of a strategy.

Bearing all the above in mind, let us select the specific case of the dynamics observed in some swarms of birds, such as ducks, where the leader is followed by a number of individuals as depicted in the last low representation in the previously shown Fig. 1. The dynamics can be depicted as a consequence of the following interactions:

(i) The leader has a visibility large enough to cover the whole landscape on the front. For instance $D_\theta = [-\frac{\pi}{2}, \frac{\pi}{2}]$;

(ii) The dynamics of the leader is independent of that of the others. Indeed, the strategy consists in developing a personal selection of the trajectory to follow;

(iii) Gregarious entities select a one-sided small visibility angle $D_\theta = -[\alpha, 0]$ or $D_\theta = [0, \alpha]$, while the critical density which defines the interaction domain Ω_c is simply identified by one particle;

(iv) The field acting on each gregarious entity simply depends on the distance and induces an acceleration for a distance greater than a critical one and a deceleration for a smaller one.

In this case, the presence of heterogeneous behavior is not relevant, while three subsystems can be identified, namely leader and left/right-sided gregarious particles. Or equivalently, we have two subsystems where heterogeneity simply refers to the left and right sides. The selection of a certain trajectory can be induced by a specific ability of the leader, for instance, to perceive electromagnetic or temperature fields. The gregarious entities follow interaction rules such as those indicated in the preceding sections. Technically, the mathematical structure (4.15) can be used to model the aforesaid dynamics where the term $\mathcal{F}$ includes attraction/repulsion

actions by means of an appropriate potential, while the right-hand side term $J[f]$ induces a trend to unify the velocities.

Analogous reasonings can be used to model the transition from disorder to order, which can be generated when one or more leaders decide to follow a certain trajectory. Here, the dynamics of the leader is independent upon that of the other entities, which learn to follow that of the trajectory of the leader.

5.2. *Modeling an attack from a predator*

Predation and competition are between the key factors determining the natural selection of species; in particular predation is one of the most fundamental inter-specific interactions in ecology. Density distribution of prey may have consequences for the capacity of predators to act on their population. The net outcome of predation efficiency, prey population abundance, aggregation, dispersion, panic situations and prey distribution pattern, in such situations, has not been well studied.

Theories based only on time-dependent dynamical systems, without any input on spatial density distribution, suggests that increasing the productivity of prey leads the equilibrium density of the prey and the predator to move to limit cycles which characterize time-periodic oscillations in population densities. However, experimental data are not consistent with the prediction of such dynamical system models, see Refs. 64 and 89 and references therein. Furthermore, the conventional non-spatial models do not allow the implementation of different survival strategies that can lead to the spatial heterogeneity of species distribution. The abundance and spatial distribution of prey and predators are linked with feedback mechanisms. The spatial organization of the prey determines the location and foraging efficiency of the predator. Between the different domains, which are called patches (or aggregates) in this context, there are migration of individuals and the whole system is considered as a predator–prey metapopulation.[45]

Some spatial predation models were constructed based on PDEs models coupled (or not) to time-dynamical systems or coupled to integro-differential equations or lattice models, see Refs. 68, 72 and 78 and references therein. Space plays an important role in preventing population oscillations and stabilizing the predation system. An important concept in spatially coupled populations is synchrony, which indicates that the population sizes in adjacent different patches tend to fluctuate to the same direction.[63] Also, time synchronization plays an important role in the stabilization process.[16] Early theories regarding the predator–prey systems often implicitly assume well-mixed homogeneous populations in space, namely the mean-field assumption and thus violate the reality that the spatial distribution of species is rarely homogeneous. Self-organization exhibits a tendency towards forming dynamic large-scale and coherent spatial structures. Complex spatial patterns associated with heterogeneous spatial distribution show asynchronous dynamics and improve aggregate structures persistence, see Ref. 81 and references therein.
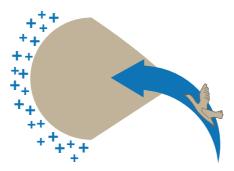
Fig. 4.   Presence of a predator attacking the swarm.

The presence of an external attack of a predator can induce large deviations with respect to the normal behavior of the swarm. This problem has been studied by various authors. Among others, we refer specifically to Ref. 61 and the bibliography therein. Two specific issues seem especially interesting in this field in order to be thoroughly examined:

(i) Modifications of the structure (4.15);
(ii) Modifications of the interaction rules.

The first topic basically means that the system is not closed in the presence of an external action. Therefore (4.15) needs to be implemented by a term suitable to take into account the interactions between the inner system, namely the swarm, and predator, see Fig. 4, where the presence of the predator should be assumed known for each position and velocity values. The following mathematical structure is proposed:

$$(\partial_t + \mathbf{v} \cdot \partial_{\mathbf{x}} + \partial_{\mathbf{v}} \mathcal{F}[f]) f(t, \mathbf{x}, \mathbf{v}) = J[f](t, \mathbf{x}, \mathbf{v}) + Q[f, g](t, \mathbf{x}, \mathbf{v}), \tag{5.1}$$

where the operators $J[f]$ are given by (4.15), while the operator $Q[f, g]$ models the interactions with the outer system corresponding to the distribution function g. The detailed expression of this term presents a structure analogous to that of the closed system as follows:

$$\begin{aligned}
Q[f, g](t, \mathbf{x}, \mathbf{v}) = &\int_{\Omega_\beta} \int_{(D_{\mathbf{v}})^2} \nu[f, g] \mathcal{B}(\mathbf{v}_* \to \mathbf{v} | \mathbf{v}_*, \mathbf{v}^*, \mathbf{V}[f]) \\
&\cdot f(t, \mathbf{x}, \mathbf{v}_*) g(t, \mathbf{x}^*, \mathbf{v}^*) d\mathbf{x}^* \, d\mathbf{v}_* \, d\mathbf{v}^* \\
&- f(t, \mathbf{x}, \mathbf{v}) \int_{\Omega_\beta} \int_{D_{\mathbf{v}}} \nu[f, g] g(t, \mathbf{x}^*, \mathbf{v}^*) d\mathbf{x}^* \, d\mathbf{v}^*,
\end{aligned} \tag{5.2}$$

with obvious meaning of notations.

The modeling approach should consider the dynamics of the predator, its representation in terms of probability distribution, and the interactions between the

inner and outer system. Some ideas proposed in Ref. 61 will be related to the mathematical structure reported above.

If position and velocity of the predator are denoted by X_P and V_P, one can look for a dynamical model of the type:

$$\begin{cases} \dfrac{dX_P}{dt} = V_P, \\[2mm] \dfrac{dV_P}{dt} = \varphi(X_P, V_P, \omega(f))|f, g), \end{cases}$$

where φ models the interaction between the predator (X_P, V_P) and some essential features of the swarm represented by the term ω, which collects the main features of the swarm: for instance position and velocity of the center of mass, or more precisely of the border in contact with the predator. Arguably, the predator attempts to adapt one's individual position and velocity to those expressed by the swarm. Accordingly, this equation should be coupled with Eq. (5.1).

Assuming that the trajectory of the predator is known, the modeling approach should first describe its presence, namely the distribution g defined over its influence area visualized in Fig. 4, and subsequently the dynamical response of the swarm. It is worth stressing, towards research perspectives, that the presence of danger can induce relevant modification in the behavior of individuals.[48]

5.3. *Fluctuations perturbing behaviors of closed systems*

Once the dynamics of the swarm has been delivered by models consistent with the structures of Sec. 3 and, if the case, with the developments proposed in this section, fluctuations should be taken into account. More precisely, we refer to fluctuations induced by small variations of the environment such as wind streams, temperature variations, and similar actions. Let us restrict our attention to this type of actions, namely excluding large deviations induced by a predator.

A conceivable way to take into account this feature was suggested, although in a different context, by Othmer, Dunbar, and Alt.[74] Subsequently, Othmer and coworkers[75] have used this modeling approach to derive macroscopic equations from the underlying description at the microscopic scale as documented in the wide literature reported in Refs. 7–9. More precisely, this approach consists in perturbing the equation for an isolated system by a velocity jump process as follows:

$$(\partial_t + \mathbf{v} \cdot \nabla_{\mathbf{x}})f = \nu \mathcal{L}[f] + J[f], \tag{5.3}$$

where $J[f]$ is the operator defined in (4.15). Moreover,

- ν is the turning rate or turning frequency, hence $\tau = \frac{1}{\nu}$ is the mean run time.

- The linear transport term describes the dynamics of biological organisms modeled by a velocity-jump process,

$$\mathcal{L}[f] = \int_{D_\mathbf{v}} \int_{D_u} (T(\mathbf{v}^* \to \mathbf{v}, u^* \to u) f(t, \mathbf{x}, \mathbf{v}^*, u^*)$$
$$- T(\mathbf{v} \to \mathbf{v}^*, u \to u^*) f(t, \mathbf{x}, \mathbf{v}, u)) d\mathbf{v}^* \, du^*, \tag{5.4}$$

where $T(\mathbf{v}^* \to \mathbf{v}, u^* \to u)$ is the probability kernel for the new velocity and state $\mathbf{v} \in D_\mathbf{v}$ and $u \in D_u$ assuming that the previous velocity was u^*. This structure can be rapidly generalized to the case of models with a hierarchy. The term $\mathcal{L}$ is linear, more precisely the perturbation T does not depend on f for instance through the distance of the average state and that which is perturbed.

This modeling approach can be used to derive macroscopic equations, as documented in the already cited papers,[7–9] for a dynamics where the perturbation modifies the spatially homogeneous case. However, additional work is needed to extend the analysis to the case of swarms. More precisely, dealing with models characterized by nonlinear interactions is an open problem which should start from a deeper analysis of the multiscale essence of living systems.[37]

The literature shows that the derivation of macroscopic equations has been developed using linear perturbation operators. The derivation in the case of nonlinear operators appears to be a challenging research perspective. In this framework, let us mention the derivation of flux-limited approaches to this problem in the context of chemotaxis.[7] In fact, sometimes the effects of linear diffusion cause large variations when what was intended was to modeling small fluctuations. In this sense, the role of nonlinear flux-limiter diffusion terms to model fluctuations becomes increasingly important because they allow controlled perturbations of the solutions. In the case of small fluctuations in velocity and activation variables the operator $\mathcal{L}$ could take the form of a flux-limited equation of type

$$\mathcal{L}[f] = \sigma \operatorname{div}_{(u,v)} \left(\frac{f^m \nabla_{(u,v)} f}{\sqrt{f^2 + \frac{\sigma^2}{c^2} |\nabla_{(u,v)} f|^2}} \right), \quad m \geq 1, \tag{5.5}$$

where σ is a kinematic viscosity and c the velocity of propagation of the fluctuation of the solution in the variables (v, u). We refer to Refs. 3, 7, 20 and 22 for a summary about this kind of models.

In general, the derivation should be specifically related to the type of swarm, which is object of the modeling approach. Substantial differences occur from case-to-case as documented in Sec. 2.

6. Looking Forward

A line of search towards a unified mathematical structure suitable to underlay the derivation of swarms has been presented in this paper. The derivation of such

structure has been obtained in Sec. 4 consistently with the paradigms of complexity proposed in Sec. 3. Three case studies have been briefly analyzed in the preceding section and proposed as research perspectives. Some hints were given being aware that each of these topics could be exhaustively treated only within a self-consistent research program, which should include the development of appropriate computational schemes.[56]

As we have seen, the specialization of the mathematical structure has to be referred to the specific type of swarm which is object of the modeling approach. Additional work is needed when the swarm is constituted by entities of a smaller size such as micro-organisms. In fact, interactions are also related to signaling processes that can induce rules on the dynamics more complex than those introduced in Sec. 3. Therefore, the contents of this paper need to be considered essentially as methodological. In fact, it provides mathematical structures consistent with the complexity features of living systems in general and, at least in part, with swarms in particular. However, these structures should be adapted to the particular nature of the swarm under consideration.

The activity variable models the strategy expressed by particles. In most cases it is heterogeneously distributed and has a remarkable influence over the dynamics of interactions that is nonlinear and involves various types of distances, namely not only the geometrical one. For instance, particles follow topological rules that take into account a critical density or an upper bound for the number of particles involved in the interaction. This problem is treated in Sec. 4.2 by considering various alternatives that have a practical answer when the domain has a well-defined shape modified by the local density in size, but not in the shape.

An additional problem, that has been mentioned several times, is the validity of the continuity assumption of the probability distribution over the microscopic state for systems where the number of particles is not large enough to justify this assumption. Such a problem has not yet a fully satisfactory answer in the literature. Arguably, the first criticism on this matter was posed by Daganzo focusing on vehicular traffic,[18] while various approaches to overcome it have been proposed to model the dynamics of vehicles or pedestrians as documented, respectively, in Refs. 6 and 18, and references cited therein. The approach of these papers consists in the discretization of the velocity space, while it can possibly be extended to the full phase space. An alternative approach, worth to be mentioned, consists in a hybrid method,[79] where the mass conservation equation is closed by a velocity field represented by suitable probability measure related to the density distribution.

However, all aforesaid approaches need further substantial developments to be extended to the class of systems under consideration. Conversely, some of the reasonings presented in this paper can hopefully contribute to the modeling of crowd dynamics. Hence, this specific problem can be considered, as various ones mentioned in this paper, as a research perspective for future projects.

References

1. S. Albeverio and W. Alt, Stochastic dynamics of viscoelastic skeins: Condensation waves and continuum limit, *Math. Models Methods Appl. Sci.* **18**(Suppl.) (2008) 1149–1191.
2. B. P. Ayati, A structured-population model of proteus mirabilis swarm-colony development, *J. Math. Biol.* **22** (2006) 93–114.
3. F. Andreu, V. Caselles, J. M. Mazón, J. Soler and M. Verbeni, Radially symmetric solutions of a tempered diffusion equation. A porous media–flux–limited case, to appear in *SIAM J. Math. Anal.*
4. M. Ballerini, N. Cabibbo, R. Candelier, A. Cavagna, E. Cisbani, I. Giardina, V. Lecomte, A. Orlandi, G. Parisi, A. Procaccini, M. Viale and V. Zdravkovic, Interaction ruling animal collective behavior depends on topological rather than metric distance: Evidence from a field study, *Proc. Natl. Acad. Sci.* **105** (2008) 1232–1237.
5. R. N. Bearon and K. L. Grünbaum, From individual behavior to population models: A case study using swimming algae, *J. Theor. Biol.* **251** (2008) 33–42.
6. N. Bellomo and A. Bellouquid, On the modeling of crowd dynamics: Looking at the beautiful shapes of swarms, *Netw. Heter. Media* **6** (2011) 383–399.
7. N. Bellomo, A. Bellouquid, J. Nieto and J. Soler, Multiscale biological tissue models and flux-limited chemotaxis for multicellular growing systems, *Math. Models Methods Appl. Sci.* **20** (2010) 1179–1207.
8. N. Bellomo, A. Bellouquid, J. Nieto and J. Soler, Complexity and mathematical tools towards the modelling multicellular growing systems in biology, *Math. Comput. Model.* **51** (2010) 441–451.
9. N. Bellomo, A. Bellouquid, J. Nieto and J. Soler, On the asymptotic theory from microscopic to macroscopic growing tissue models: An overview with perspectives, *Math. Models Methods Appl. Sci.* **22** (2012) 1130001.
10. N. Bellomo, C. Bianca and M. Delitala, Complexity analysis and mathematical tools towards the modelling of living systems, *Phys. Life Rev.* **6** (2009) 144–175.
11. N. Bellomo and M. Delitala, From the mathematical kinetic, and stochastic game theory to modelling mutations, onset, progression and immune competition of cancer cells, *Phys. Life Rev.* **5** (2008) 183–206.
12. N. Bellomo and C. Dogbè, On the modelling of traffic and crowds — A survey of models, speculations, and perspectives, *SIAM Rev.* **53** (2011) 409–463.
13. A. Bellouquid, E. De Angelis and L. Fermo, Towards the modeling of vehicular traffic as a complex system: A kinetic theory approach, *Math. Models Methods Appl. Sci.* **22** (2012) 1140003.
14. A. Bertozzi, D. Grunbaum, P. S. Krishnaprasad and I. Schwartz, Swarming by nature and by design, www.ipam.ucla.edu/programs/swa2006.
15. E. Bonabeau, M. Dorigo and G. Theraulaz, *Swarm Intelligence*: *From Natural to Artificial Systems* (Oxford Univ. Press, 1999).
16. L. L. Bonilla, C. Pérez-Vicente, F. Ritort and J. Soler, Exactly solvable phase oscillator models with synchronization dynamics, *Phys. Rev. Lett.* **81** (1998) 3643–3646.

17. R. W. Brockett, On the control of a flock by a leader, *Proc. Steklov Inst. Math.* **268** (2010) 49–57.

18. C. F. Daganzo, Requiem for second order fluid approximations of traffic flow, *Transp. Res. B* **29** (1995) 277–286.

19. A. Q. Cai, K. A. Landman and B. D. Hughes, Modelling directional guidance and motility regulation in cells, *Bull. Math. Biol.* **68** (2006) 25–52.

20. J. Calvo, J. Mazón, J. Soler and M. Verbeni, Qualitative properties of the solutions of a nonlinear flux-limited equation arising in the transport of morphogens, *Math. Models Methods Appl. Sci.* **21**(Suppl.) (2011) 893–937.

21. J. A. Carrillo, M. Fornasier, J. Rosado and G. Toscani, Asymptotic flocking dynamics for the kinetic Cucker–Smale model, *SIAM J. Math. Anal.* **42** (2010) 218–236.

22. V. Caselles, On the entropy conditions for some flux limited diffusion equations, *J. Differential Equations* **250** (2011) 3311–3348.

23. A. Cavagna, A. Cimarelli, I. Giardina, G. Parisi, R. Santagati, F. Stefanini and R. Tavarone, From empirical data to inter-individual interactions: Unveiling the rules of collective animal behavior, *Math. Models Methods Appl. Sci.* **20** (2010) 1491–1510.

24. A. Cavagna, A. Cimarelli, I. Giardina, G. Parisi, R. Santagati, F. Stefanini and M. Viale, Scale-free correlation in the bird flocks, *Proc. Natl. Acad. Sci. USA* **107** (2010) 11865–11870.

25. F. Cerreti, B. Perthame, C. Schmeiser, M. Tang and V. Vauchelet, Waves for the hyperbolic Keller–Segel model, *Math. Models Methods Appl. Sci.* **21** (2011) 825–842.

26. Y. Chjang, M. D'Orsogna, D. Marthaler, A. Bertozzi and L. Chaves, State transition and the continuum limit for 2D interacting, self-propelled particles system, *Physica D* **232** (2007) 33–47.

27. I. D. Couzin, Collective minds, *Nature* **445** (2007) 715.

28. I. D. Couzin, J. Krause, R. James, G. D. Ruxtion and N. R. Franks, Collective memory and spatial sorting in animal groups, *J. Theor. Biol.* **218** (2002) 1–11.

29. E. Cristiani, P. Frasca and B. Piccoli, Effects of anisotropic interactions in the structure of animal groups, *J. Math. Biol.* **62** (2011) 569–588.

30. F. Cucker and J. G. Dong, On the critical exponent for flocks under hierarchical leadership, *Math. Models Methods Appl. Sci.* **19** (2009) 1391–1404.

31. F. Cucker and S. Smale, Emergent behavior in flocks, *IEEE Trans. Automatic Control* **52** (2007) 853–862.

32. B. D'Acunto and L. Frunzo, Qualitative analysis and simulations of a free boundary problem for multispecies biofilm models, *Math. Comput. Model.* **53** (2011) 1596–1606.

33. P. Degond and S. Motsch, Continuum limit of self-driven particles with orientation interaction, *Math. Models Methods Appl. Sci.* **18** (2008) 1193–1217.

34. P. Degond and J.-G. Liu, Hydrodynamics of self-alignment interactions with precession and derivation of the Landau–Lifschitz–Gilbert equation, *Math. Models Methods Appl. Sci.* **22** (2012) 1140001.

35. C. Detrain and J.-L. Doneubourg, Self-organized structures in a superorganism: Do ants "behave" like molecules? *Phys. Life Rev.* **3** (2006) 162–187.

36. C. Dombrowsky, L. Cisneros, S. Chatkaew, R. E. Goldstein and J. O. Kessler, Self-concentration and large scale coherence in bacterial dynamics, *Phys. Rev. Lett.* **93** (2004) 098103.

37. H. Du, Z. Xu, J. D. Shrout and M. Alber, Multiscale modeling of *pseudomona aeruginosa* swarming, *Math. Models Methods Appl. Sci.* **21** (2011) 939–954.

38. R. Duan, M. Fornasier and G. Toscani, A kinetic flocking model with diffusion, *Commun. Math. Phys.* **300** (2010) 95–145.

39. A. Ducrot, F. L. Foll, P. Magal, H. Mukarawa, J. Pasquier and G. F. Webb, *In vitro* cell population dynamics model incorporating cell size, quiescence, and contact inhibition, *Math. Models Methods Appl. Sci.* **21** (2011) 871–892.

40. G. Flierl, D. Gruñbaum, S. Levin and D. Olson, From individuals to aggregations: The interplay between behavior and physics, *J. Theor. Biol.* **196** (1999) 397–454.

41. M. Fornasier, J. Haskovec and G. Toscani, Fluid dynamic description of flocking via the Povzner–Boltzmann equation, *Physica D* **240** (2011) 21–31.

42. F. Ginelli and H. Chaté, Relevance of metric-free interaction in flocking phenomena, *Phys. Rev. Lett.* **105** (2010) 168103.

43. H. Gintis, *Game Theory Evolving*, 2nd edn. (Princeton Univ. Press, 2009).

44. D. Gruñbaum, K. Chan, E. Tobin and M. T. Nishizaki, Non-linear advection diffusion equations approximate swarms but not schooling population, *Math. Biosci.* **214** (2008) 38–48.

45. I. Hanski, Metapopulation dynamics, *Nature* **396** (1998) 42–49.

46. P. Hogeweg and B. Hesper, Socioinformatic processes: MIRROR modelling methodology, *J. Theor. Sociobiol.* **9** (1981) 111–119.

47. D. Helbing, *Quantitative Sociodynamics* (Springer, 2010).

48. D. Helbing, I. Farkas and T. Vicsek, Simulating dynamical feature of escape panic, *Nature* **407** (2000) 487–490.

49. C. K. Hemelrijk and H. Hildenbrandt, Self-organized shape and frontal density of fish schools, *Ethology* **114** (2008) 245–254.

50. H. Hildenbrandt, C. Carere and C. K. Hemelrijk, Self-organized aerial displays of thousands of starlings: A model, *Behav. Ecol.* **21** (2010) 1349–1359.

51. C. van Honk and P. Hogeweg, The ontogeny of the social structure in a captive Bombus terrestris colony, *Behav. Ecol. Biol.* **113** (1985) 311–330.

52. A. Huth and C. Wissel, The simulation of the movement of fish school, *J. Theor. Biol.* **156** (1992) 365–385.

53. E. F. Keller and L. A. Segel, Traveling bands of chemotactic bacteria: A theoretical analysis, *J. Theor. Biol.* **30** (1971) 235–248.

54. D. L. Koch and G. Subramanian, Collective hydrodynamics of swimming microorganisms: Living fluids, *Annu. Rev. Fluid Mech.* **43** (2011) 637–659.

55. I. Klapper and J. Dockeryt, Mathematical description of microbial biofilms, *SIAM Rev.* **52** (2010) 221–265.

56. P. Koumoutsakos, B. Bayati, F. Milke, G. Tauriello, Particle simulations of morphogenesis, *Math. Models Methods Appl. Sci.* **21** (2011) 955–1006.

57. E. Jager and L. Segel, On the distribution of dominance in populations of social organisms, *SIAM J. Appl. Math.* **52** (1992) 1442–1468.

58. K. Jin, P. Liang and G. Beni, Stability of synchronized distributed control of discrete swarm structures, in *IEEE Int. Conf. Robotics and Automation* (IEEE, 1994), pp. 1033–1038.

59. M. Lachowicz, Individually-based Markov processes modeling nonlinear systems in mathematical biology, *Nonlinear Anals*: *Real World Appl.* **12** (2010) 2396–2407.

60. E. Lauga and T. R. Powers, The hydrodynamics of swimming microorganisms, *Rep. Prog. Phys.* **72** (2009) 096601.

61. S.-H. Lee, H.-K. Pak and T.-S. Chon, Dynamics of prey-flock escaping behavior in response to predator's attack, *J. Theor. Biol.* **240** (2006) 250–259.

62. R. Lukemann, Y.-X. Li and L. Edelstein-Keshet, A conceptual model for milling formation in biological aggregates, *Bull. Math. Biol.* **71** (2009) 352–382.

63. S. F. Matter, Synchrony, extinction, and dynamics of spatially segregated, heterogeneous populations, *Ecol. Model.* **141** (2001) 217–226.

64. E. McCauley, W. G. Wilson and A. M. De Roos, Dynamics of age-structured and spatial structured predator-prey population interactions: Individual-based models and population level formulations, *Am. Nat.* **142** (1993) 412–442.

65. M. B. Miller and B. L. Bassler, Quorum sensing in bacteria, *Annu. Rev. Microbiol.* **55** (2001) 165–199.

66. A. Mogilner, L. Edelstein-Keshet, L. Bent and A. Spiros, Mutual interactions, potentials, and individual distance in a social aggregation, *J. Math. Biol.* **47** (2003) 353–389.

67. S. Motsch and E. Tadmor, A new model for self-organized dynamics and its flocking behavior, *J. Stat. Phys.* **144** (2011) 923–947.

68. J. D. Murray, *Mathematical Biology I. An Introduction, Mathematical Biology II. Spatial Models and Biomedical Applications* (Springer, 2002).

69. T. Niizatoa and Y.-P. Gunj, Metric-topological interaction model of collective behavior, *Ecol. Model.* **222** (2011) 3041–3049.

70. M. A. Nowak, *Evolutionary Dynamics* (Belknap Harvard Univ. Press, 2006).

71. A. Okubo, Dynamical aspects of animal grouping: Swarms, schools, flocks, and herds, *Adv. Biophys.* **22** (1986) 1–94.

72. A. Okubo and S. A. Levin, *Diffusion and Ecological Problems*: *Modern Perspectives* (Springer, 2001).

73. A. Okubo, D. Grunbaum and L. Edelstein-Keshet, The dynamics of animal grouping, in *Diffusion and Ecological Problems*, 2nd edn., eds. A. Okubo and S. Levin, Interdiscip. Appl. Math., Vol. 14 (Springer, 1999), pp. 197–237.

74. H. Othmer, S. R. Dunbar and W. Alt, Models of dispersal in biological systems, *J. Math. Biol.* **26** (1988) 263–298.

75. H. Othmer and T. Hillen, The diffusion limit of transport equations II: Chemotaxis equations, *SIAM J. Appl. Math.* **62** (2002) 1222–1250.

76. C. V. Parise and C. Spence, "When birds of a feather flock together": Synesthetic correspondences modulate audiovisual integration in non-synesthetes, PLoS ONE, **4** (2009) 1–7.

77. K. J. Parrish, Complexity, pattern, and evolutionary trade-off in animal aggregation, *Science* **284** (1999) 99–101.

78. S. V. Petrovskii, *Exactly Solvable Models of Biological Invasion* (Chapman & Hall/CRC, 2006).

79. B. Piccoli and A. Tosin, Time evolving measures and macroscopic modeling of pedestrian flows, *Arch. Rational Mech. Anal.* **199** (2010) 707–738.

80. T. Piersma, R. van Aelst, K. Kurk, H. Berkhoudt and L. R. M. Maas, A new pressure sensory mechanism for prey detection in birds: The use of principles of seabed dynamics? *Proc. R. Soc. Lond. B* **265** (1998) 1377–1383.

81. A. Ramanantoanina, C. Huic and A. Ouhinoua, Effects of density-dependent dispersal behaviours on the speed and spatial patterns of range expansion in predator–prey metapopulations, *Ecol. Model.* **222** (2011) 3524–3530.

82. C. W. Reynolds, Flocks, herds, and schools: A distributed behavioral model, *Comput. Graph.* **21** (1987) 25–34.

83. F. Schweitzer, *Brownian Agents and Active Particles* (Springer, 2003).

84. J. Toner and Y. Tu, Flocks, herds, and schools: A quantitative theory of flocking, *Phys. Rev. E.* **58** (1998) 4828–4858.

85. J. R. Touma, A. Shreim and L. I. Klushin, Self-organization in two dimensional swarms, *Phys. Rew. E.* **81** (2010) 066106-1-7.

86. C. M. Topaz and A. Bertozzi, Swarming patterns in a two dimensional kinematic model for biological groups, *SIAM J. Appl. Math.* **65** (2004) 152–174.

87. N. Verstraeten, K. Braeken, B. Debkumari, M. Fauvart, J. Fransaer, J. Vermant and J. Michiels, Living on a surface: Swarming and biofilm formation, *Trends in Microbiol.* **16** (2008) 496–506.

88. Y. Yang, V. Marceau and G. Gompper, Swarm behavior of self-propelled rods and swimming flagella, *Phys. Rev. E* **82** (2004) 031904.

89. C. Xu and Z. Li, Effect of diffusion and spatially varying predation risk on the dynamics and equilibrium density of a predatorprey system, *J. Theor. Biol.* **219** (2002) 73–82.

90. M. Zheng, Y. Kashimori, O. Hoshino, K. Fujita and T. Kambara, Behavior pattern (innate action) of individuals in fish schools generating collective evasion from predation, *J. Theor. Biol.* **235** (2005) 153–157.