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Question 1. Consider the following **release** operator, where ϕ and ψ are LTL formulas.

- ϕ releases ψ, $\phi \mathsf{R} \psi$, if ψ remains true until and including once ϕ becomes true. If ϕ never become true, then ψ must remain true forever.

Then, answer the following questions.

(a) Make the definition of the informally explained operator R precise by providing truth conditions for formulas $\phi \mathsf{R} \psi$ in terms of a model M, path π, and the truth of ϕ and ψ, similarly to Definition 1.4 in Lecture 5.

(b) Now provide an LTL formula, by using atoms, Boolean connectives, and operators next X and until U only, that formalizes the meaning of the release operator R at (a).

(c) Check that the truth conditions provided in (a) match the LTL formula in (b). That is, the LTL formula is true iff the corresponding condition is satisfied.

(d) By using your answers to points (a) – (c), check that the always operator $\mathsf{G}\psi$ can be expressed as $\bot \mathsf{R} \psi$.

Answer 1.

(a) If π is a path in M, then $\pi \vDash \phi \mathsf{R} \psi$ iff

$$\left(\exists i \geq 0,\ \pi\left[i \ldots \infty\right] \vDash \phi,\ \text{and } \forall 0 \leq j \leq i,\ \pi\left[j \ldots \infty\right] \vDash \psi\right),\ \text{or } \forall j \geq 0,\ \pi\left[j \ldots \infty\right] \vDash \psi.$$

(b) Since $\mathsf{G}\psi \equiv \neg \mathsf{F} \neg \psi \equiv \neg\left(\top \mathsf{U} \neg \psi\right)$,

$$\phi \mathsf{R} \psi \equiv \psi \mathsf{U}\left(\psi \wedge \phi\right) \vee \mathsf{G}\psi \equiv \psi \mathsf{U}\left(\psi \wedge \phi\right) \vee \neg\left(\top \mathsf{U} \neg \psi\right).$$

(c) The LTL formula $\psi \mathsf{U}\left(\psi \wedge \phi\right) \vee \mathsf{G}\psi$ is true iff

$$\left(\exists i \geq 0,\ \pi\left[i \ldots \infty\right] \vDash \psi,\ \pi\left[i \ldots \infty\right] \vDash \phi,\ \text{and } \forall 0 \leq j < i,\ \pi\left[j \ldots \infty\right] \vDash \psi\right),\ \text{or } \forall j \geq 0,\ \pi\left[j \ldots \infty\right] \vDash \psi,$$

by semantics of U, $\wedge$, $\vee$, and G, which is true iff the corresponding condition is satisfied.

(d) Since $\bot$ is never true, so is $\psi \mathsf{U} \bot$, so

$$\bot \mathsf{R} \psi \equiv \psi \mathsf{U}\left(\psi \wedge \bot\right) \vee \mathsf{G}\psi \equiv \psi \mathsf{U} \bot \vee \mathsf{G}\psi \equiv \bot \vee \mathsf{G}\psi \equiv \mathsf{G}\psi.$$

Question 2. Recall the following abbreviations in CTL.

$$\begin{aligned}\mathsf{EF}\Phi &= \mathsf{E}\left(\top \mathsf{U} \Phi\right),\\ \mathsf{AF}\Phi &= \mathsf{A}\left(\top \mathsf{U} \Phi\right),\\ \mathsf{EG}\Phi &= \neg\mathsf{AF}\neg\Phi,\\ \mathsf{AG}\Phi &= \neg\mathsf{EF}\neg\Phi.\end{aligned}$$

Then prove the following equivalences by using the definition of satisfaction $\vDash$ for CTL.

(a) $(M, q) \vDash \mathsf{EF}\Phi$ iff for some path λ from q, for some $j \geq 0$, $(M, \lambda[j]) \vDash \Phi$.

(b) $(M, q) \vDash \mathsf{AF}\Phi$ iff for every path λ from q, for some $j \geq 0$, $(M, \lambda[j]) \vDash \Phi$.

(c) $(M, q) \vDash \mathsf{EG}\Phi$ iff for some path λ from q, for all $j \geq 0$, $(M, \lambda[j]) \vDash \Phi$.

(d) $(M, q) \vDash \mathsf{AG}\Phi$ iff for every path λ from q, for all $j \geq 0$, $(M, \lambda[j]) \vDash \Phi$.

Answer 2.

(a) Since $\mathsf{EF}\Phi = \mathsf{E}\left(\top \mathsf{U} \Phi\right)$,

$$\begin{aligned}(M, q) \vDash \mathsf{EF}\Phi &\iff \exists \text{ path } \lambda \text{ from } q,\ (M, \lambda) \vDash \top\mathsf{U}\Phi\\ &\iff \exists \text{ path } \lambda \text{ from } q,\ \exists j \geq 0,\ (M, \lambda[j]) \vDash \Phi,\ \text{and } \forall 0 \leq i < j,\ (M, \lambda[i]) \vDash \top\\ &\iff \exists \text{ path } \lambda \text{ from } q,\ \exists j \geq 0,\ (M, \lambda[j]) \vDash \Phi,\end{aligned}$$

by semantics of E, $\top$, and U.

(b) Since $\mathsf{AF}\Phi = \mathsf{A}\left(\top \mathsf{U} \Phi\right)$,

$$\begin{aligned}(M, q) \vDash \mathsf{AF}\Phi &\iff \forall \text{ path } \lambda \text{ from } q,\ (M, \lambda) \vDash \top\mathsf{U}\Phi\\ &\iff \forall \text{ path } \lambda \text{ from } q,\ \exists j \geq 0,\ (M, \lambda[j]) \vDash \Phi,\ \text{and } \forall 0 \leq i < j,\ (M, \lambda[i]) \vDash \top\\ &\iff \forall \text{ path } \lambda \text{ from } q,\ \exists j \geq 0,\ (M, \lambda[j]) \vDash \Phi,\end{aligned}$$

by semantics of A, $\top$, and U.

(c) Since $\mathsf{EG}\Phi = \neg\mathsf{AF}\neg\Phi = \neg\mathsf{A}\left(\top \mathsf{U} \neg\Phi\right)$,

$$\begin{aligned}(M, q) \vDash \mathsf{EG}\Phi &\iff (M, q) \nvDash \mathsf{A}\left(\top\mathsf{U}\neg\Phi\right)\\ &\iff \exists \text{ path } \lambda \text{ from } q,\ (M, \lambda) \nvDash \top\mathsf{U}\neg\Phi\\ &\iff \exists \text{ path } \lambda \text{ from } q,\ \forall j \geq 0,\ (M, \lambda[j]) \nvDash \neg\Phi,\ \text{or } \exists 0 \leq i < j,\ (M, \lambda[i]) \nvDash \top\\ &\iff \exists \text{ path } \lambda \text{ from } q,\ \forall j \geq 0,\ (M, \lambda[j]) \vDash \Phi,\ \text{or } \exists 0 \leq i < j,\ (M, \lambda[i]) \vDash \bot\\ &\iff \exists \text{ path } \lambda \text{ from } q,\ \forall j \geq 0,\ (M, \lambda[j]) \vDash \Phi,\end{aligned}$$

by semantics of $\neg$, A, $\top$, and U.

(d) Since $\mathsf{AG}\Phi = \neg\mathsf{EF}\neg\Phi = \neg\mathsf{E}\left(\top \mathsf{U} \neg\Phi\right)$,

$$\begin{aligned}(M, q) \vDash \mathsf{AG}\Phi &\iff (M, q) \nvDash \mathsf{E}\left(\top\mathsf{U}\neg\Phi\right)\\ &\iff \forall \text{ path } \lambda \text{ from } q,\ (M, \lambda) \nvDash \top\mathsf{U}\neg\Phi\\ &\iff \forall \text{ path } \lambda \text{ from } q,\ \forall j \geq 0,\ (M, \lambda[j]) \nvDash \neg\Phi,\ \text{or } \exists 0 \leq i < j,\ (M, \lambda[i]) \nvDash \top\\ &\iff \forall \text{ path } \lambda \text{ from } q,\ \forall j \geq 0,\ (M, \lambda[j]) \vDash \Phi,\ \text{or } \exists 0 \leq i < j,\ (M, \lambda[i]) \vDash \bot\\ &\iff \forall \text{ path } \lambda \text{ from } q,\ \forall j \geq 0,\ (M, \lambda[j]) \vDash \Phi,\end{aligned}$$

by semantics of $\neg$, E, $\top$, and U.

Question 3. Consider the following definition of formulas in the temporal logic CTL*.

Definition 1 (Syntax of CTL*)**.** State Φ and path ψ formulas in CTL* are defined in Backus-Naur form as follows, where p is an atom.

$$\Phi ::= p \mid \neg\Phi \mid \Phi \wedge \Phi \mid \mathsf{E}\psi \mid \mathsf{A}\psi,$$
$$\psi ::= \Phi \mid \neg\psi \mid \psi \wedge \psi \mid \mathsf{X}\psi \mid \psi \mathsf{U} \psi.$$

The formulas of CTL* are all and only the state formulas.

Then show that CTL is a strict syntactic fragment of CTL*. That is,

(a) CTL is a syntactic fragment of CTL*, so for every formula Φ, if Φ is a formula of CTL according to the definition in Lecture 5, then Φ is also a formula in CTL* according to Definition 1, and

(b) there exists some formula Φ in CTL* that does not belong to CTL.

Answer 3.

(a) By Lecture 5, the syntax of CTL is

$$\Phi ::= p \mid \neg\Phi \mid \Phi \wedge \Phi \mid \mathsf{EX}\Phi \mid \mathsf{AX}\Phi \mid \mathsf{E}\left(\Phi \mathsf{U} \Phi\right) \mid \mathsf{A}\left(\Phi \mathsf{U} \Phi\right).$$

Then p, $\neg\Phi$, and $\Phi \wedge \Phi$ are state formulas in CTL*. If Φ is a path formula of CTL*, then so are $\mathsf{X}\Phi$ and $\Phi\mathsf{U}\Phi$, so $\mathsf{EX}\Phi$, $\mathsf{AX}\Phi$, $\mathsf{E}\left(\Phi\mathsf{U}\Phi\right)$, and $\mathsf{A}\left(\Phi\mathsf{U}\Phi\right)$ are state formulas in CTL*. Thus CTL is a syntactic fragment of CTL*.

(b) Let p be an atom, and let

$$\Phi = \mathsf{E}p.$$

Then Φ is a state formula in CTL* but not a state formula in CTL.

Question 4. Consider the following definition of the satisfaction relation $\vDash$ for formulas in CTL*.

Definition 2 (Semantics of CTL*)**.** Let M be a model, s a state, π a path, Φ and Φ' state formulas, ψ and ψ' path formulas. Then,

CTL*(1) $(M, s) \vDash p$ iff $s \in V(p)$,

CTL*(2) $(M, s) \vDash \neg\Phi$ iff $(M, s) \nvDash \Phi$,

CTL*(3) $(M, s) \vDash \Phi \wedge \Phi'$ iff $(M, s) \vDash \Phi$ and $(M, s) \vDash \Phi'$,

CTL*(4) $(M, s) \vDash \mathsf{E}\psi$ iff for some path π starting from s, $(M, \pi) \vDash \psi$,

CTL*(5) $(M, s) \vDash \mathsf{A}\psi$ iff for all paths π starting from s, $(M, \pi) \vDash \psi$,

CTL*(6) $(M, \pi) \vDash \Phi$ iff $(M, \pi[0]) \vDash \Phi$, where $\pi[0]$ is the initial state in path π,

CTL*(7) $(M, \pi) \vDash \neg\psi$ iff $(M, \pi) \nvDash \psi$,

CTL*(8) $(M, \pi) \vDash \psi \wedge \psi'$ iff $(M, \pi) \vDash \psi$ and $(M, \pi) \vDash \psi'$,

CTL*(9) $(M, \pi) \vDash \mathsf{X}\psi$ iff $(M, \pi[1 \ldots \infty]) \vDash \psi$, and

CTL*(10) $(M, \pi) \vDash \psi \mathsf{U} \psi'$ iff $(M, \pi[i \ldots \infty]) \vDash \psi'$ for some $i \geq 0$, and $(M, \pi[j \ldots \infty]) \vDash \psi$ for all $0 \leq j < i$.

Show that if we restrict Definition 2 to formulas in CTL, which we can do, as CTL is a fragment of CTL*, then we obtain the same truth conditions as in Definition 1.7 and Definition 1.8 in Lecture 5.

Answer 4. By Lecture 5, the semantics of state formulas of CTL are the first five semantics of CTL*, while the semantics of path formulas of CTL are

- $(M, \pi) \vDash \mathsf{X}\Phi$ iff $(M, \pi[1]) \vDash \Phi$, and
- $(M, \pi) \vDash \Phi \mathsf{U} \Phi'$ iff $(M, \pi[i]) \vDash \Phi'$ for some $i \geq 0$ and $(M, \pi[j]) \vDash \Phi$ for all $0 \leq j < i$.

Since Φ and Φ' are path formulas,

$$\begin{aligned}
(M, \pi) \vDash \mathsf{X}\Phi &\iff (M, \pi[1 \ldots \infty]) \vDash \Phi && \text{by } (9) \\
&\iff (M, \pi[1 \ldots \infty][0]) \vDash \Phi && \text{by } (6) \\
&\iff (M, \pi[1]) \vDash \Phi,
\end{aligned}$$

and

$$\begin{aligned}
(M, \pi) \vDash \Phi \mathsf{U} \Phi' &\iff \exists i \geq 0,\ (M, \pi[i \ldots \infty]) \vDash \Phi',\ \text{and } \forall 0 \leq j < i,\ (M, \pi[j \ldots \infty]) \vDash \Phi && \text{by } (10) \\
&\iff \exists i \geq 0,\ (M, \pi[i \ldots \infty][0]) \vDash \Phi',\ \text{and } \forall 0 \leq j < i,\ (M, \pi[j \ldots \infty][0]) \vDash \Phi && \text{by } (6) \\
&\iff \exists i \geq 0,\ (M, \pi[i]) \vDash \Phi',\ \text{and } \forall 0 \leq j < i,\ (M, \pi[j]) \vDash \Phi.
\end{aligned}$$

Thus the semantics of CTL* restricted to formulas in CTL has same truth conditions as in Lecture 5.

Question 5. Show that CTL* is strictly more expressive than CTL. That is,

(a) CTL* is more expressive than CTL, so for every formula Φ of CTL, there exists some formula Φ' in CTL* such that Φ and Φ' are equivalent, that is for every model M, and initial state s, $(M, s) \models \Phi$ iff $(M, s) \models \Phi'$, and

(b) there exists some formula Φ in CTL* for which there exists no equivalent formula Φ' in CTL, by considering the LTL formulas in Lecture 5 that are not expressible in CTL.

Answer 5.

(a) Since CTL is a syntactic and semantic fragment of CTL*, any formula of CTL is a state formula in CTL* and has the same restricted truth conditions. Thus $\Phi' = \Phi$ is a formula equivalent to Φ.

(b) If ψ is a state formula of CTL, denote $\neg(\psi \wedge \neg\psi)$ by $\top$, denote $\top \mathsf{U} \psi$ by $\mathsf{F}\psi$, and denote $\neg\mathsf{F}\neg\psi$ by $\mathsf{G}\psi$. Let p be an atom, and let

$$\Phi = \mathsf{AFG}p.$$

Then Φ is a state formula in CTL*. Let $M = (St, \to, V)$ be a model defined on a transition system, and let $q \in St$ be a state. Then

$$\begin{aligned}
(M, q) \models \Phi &\iff \forall \text{ path } \lambda \text{ from } q,\ (M, \lambda) \models \mathsf{FG}p && \text{by CTL* (5)} \\
&\iff \forall \text{ path } \lambda \text{ from } q,\ (M, \lambda[0]) \models \mathsf{FG}p && \text{by CTL* (6)} \\
&\iff (M, q) \models \mathsf{FG}p,
\end{aligned}$$

so Φ is equivalent to $\mathsf{FG}p$. By Lecture 5, there exists no CTL formula equivalent to $\mathsf{FG}p$. Thus there exists no CTL formula Φ' equivalent to Φ.

Question 6. Consider the following notion of bisimulation on temporal models.

Definition 3. Let $M = (St, \to, V)$ and $M' = (St', \to', V')$ be models. A **bisimulation** between M and M' is a relation $B \subseteq St \times St'$ such that for every $u \in St$ and $u' \in St'$, if $B(u, u')$ then

(1) for all atoms p, $u \in V(p)$ iff $u' \in V'(p)$,

(2) if $v \in St$ and $u \to v$, then there is $v' \in St'$ such that $u' \to' v'$ and $B(v, v')$, and

(3) if $v' \in St'$ and $u' \to' v'$, then there is $v \in St$ such that $u \to v$ and $B(v, v')$.

Then, (M, t) and (M', t') are **bisimilar**, or $(M, t) \approx (M', t')$, if there exists a bisimulation B between M and M' such that $B(t, t')$. Further, (M, π) and (M', π') are **bisimilar**, or $(M, \pi) \approx (M', \pi')$, if for every $i \geq 0$, $(M, \pi[i])$ and $(M', \pi'[i])$ are bisimilar.

Now assume that (M, t) and (M', t') are bisimilar, (M, π) and (M', π') are also bisimilar, Φ is a state formula, and ψ is a path formula. Then show that

$$(M, t) \vDash \Phi \iff (M', t') \vDash \Phi, \qquad (M, \pi) \vDash \psi \iff (M', \pi') \vDash \psi.$$

The proof is by mutual induction on the structure of Φ and ψ. You will have to prove that given a path π in M, there exists a bisimilar path π' in M', and vice versa. Conclude that the truth of CTL* formulas is preserved by bisimulations.

Answer 6. Let p be an atom. Then

$$\begin{aligned} (M, t) \vDash p &\iff t \in V(p) && \text{by CTL* (1)} \\ &\iff t' \in V'(p) && \text{by (1)} \\ &\iff (M', t') \vDash p && \text{by CTL* (1)}. \end{aligned}$$

Now let Φ and Φ' be state formulas of CTL*, and let ψ and ψ' be path formulas of CTL*. Assume for induction that

IH (1) $(M, t) \vDash \Phi$ iff $(M', t') \vDash \Phi$,

IH (2) $(M, t) \vDash \Phi'$ iff $(M', t') \vDash \Phi'$,

IH (3) $(M, \pi) \vDash \psi$ iff $(M', \pi') \vDash \psi$, and

IH (4) $(M, \pi) \vDash \psi'$ iff $(M', \pi') \vDash \psi'$.

Note that if $\psi = \Phi$ in IH (3), then

$$(M, \pi[0]) \vDash \Phi \iff (M, \pi) \vDash \psi \iff (M', \pi') \vDash \psi \iff (M', \pi'[0]) \vDash \Phi,$$

by CTL* (6). Hence

IH (5) $(M, \pi[0]) \vDash \Phi$ iff $(M', \pi'[0]) \vDash \Phi$.

Also note that if $i \geq 0$ and $\psi = \underbrace{\mathsf{X} \dots \mathsf{X}}_{i} \psi'$ in IH (3), then

$$(M, \pi[i \dots \infty]) \vDash \psi' \iff (M, \pi) \vDash \psi \iff (M', \pi') \vDash \psi \iff (M', \pi'[i \dots \infty]) \vDash \psi',$$

by CTL* (9) and another induction. Hence

IH (6) $(M, \pi[i \dots \infty]) \vDash \psi$ iff $(M', \pi'[i \dots \infty]) \vDash \psi$ for all $i \geq 0$.

Finally note that

IH (7) If λ is a path from t in M such that $(M, \lambda) \vDash \psi$, then there is a path λ' from t' in M' such that $(M', \lambda') \vDash \psi$,

by (2) and another induction, and similarly

IH (8) If λ' is a path from t' in M' such that $(M', \lambda') \vDash \psi$, then there is a path λ from t in M such that $(M, \lambda) \vDash \psi$,

by (3) and another induction.

Then

$$\begin{aligned}(M,t)\vDash\neg\Phi &\iff (M,t)\nvDash\Phi && \text{by CTL* (2)}\\ &\iff (M',t')\nvDash\Phi && \text{by IH (1)}\\ &\iff (M',t')\vDash\neg\Phi && \text{by CTL* (2)},\end{aligned}$$

and

$$\begin{aligned}(M,t)\vDash\Phi\wedge\Phi' &\iff (M,t)\vDash\Phi \text{ and } (M,t)\vDash\Phi' && \text{by CTL* (3)}\\ &\iff (M',t')\vDash\Phi \text{ and } (M',t')\vDash\Phi' && \text{by IH (1) and IH (2)}\\ &\iff (M',t')\vDash\Phi\wedge\Phi' && \text{by CTL* (3)},\end{aligned}$$

and

$$\begin{aligned}(M,t)\vDash\mathsf{E}\psi &\iff \exists \text{ path } \lambda \text{ from } t,\ (M,\lambda)\vDash\psi && \text{by CTL* (4)}\\ &\iff \exists \text{ path } \lambda' \text{ from } t',\ (M',\lambda')\vDash\psi && \text{by IH (7) and IH (8)}\\ &\iff (M',t')\vDash\mathsf{E}\psi && \text{by CTL* (4)},\end{aligned}$$

and

$$\begin{aligned}(M,t)\vDash\mathsf{A}\psi &\iff \forall \text{ path } \lambda \text{ from } t,\ (M,\lambda)\vDash\psi && \text{by CTL* (5)}\\ &\iff \forall \text{ path } \lambda' \text{ from } t',\ (M',\lambda')\vDash\psi && \text{by IH (7) and IH (8)}\\ &\iff (M',t')\vDash\mathsf{A}\psi && \text{by CTL* (5)},\end{aligned}$$

and

$$\begin{aligned}(M,\pi)\vDash\Phi &\iff (M,\pi[0])\vDash\Phi && \text{by CTL* (6)}\\ &\iff (M',\pi'[0])\vDash\Phi && \text{by IH (5)}\\ &\iff (M',\pi')\vDash\Phi && \text{by CTL* (6)},\end{aligned}$$

and

$$\begin{aligned}(M,\pi)\vDash\neg\psi &\iff (M,\pi)\nvDash\psi && \text{by CTL* (7)}\\ &\iff (M',\pi')\nvDash\psi && \text{by IH (3)}\\ &\iff (M',\pi')\vDash\neg\psi && \text{by CTL* (7)},\end{aligned}$$

and

$$\begin{aligned}(M,\pi)\vDash\psi\wedge\psi' &\iff (M,\pi)\vDash\psi \text{ and } (M,\pi)\vDash\psi' && \text{by CTL* (8)}\\ &\iff (M',\pi')\vDash\psi \text{ and } (M',\pi')\vDash\psi' && \text{by IH (3) and IH (4)}\\ &\iff (M',\pi')\vDash\psi\wedge\psi' && \text{by CTL* (8)},\end{aligned}$$

and

$$\begin{aligned}(M,\pi)\vDash\mathsf{X}\psi &\iff (M,\pi[1\ldots\infty])\vDash\psi && \text{by CTL* (9)}\\ &\iff (M',\pi'[1\ldots\infty])\vDash\psi && \text{by IH (6)}\\ &\iff (M',\pi')\vDash\mathsf{X}\psi && \text{by CTL* (9)},\end{aligned}$$

and

$$\begin{aligned}(M,\pi)\vDash\psi\mathsf{U}\psi' &\iff \exists i\geq 0,\ (M,\pi[i\ldots\infty])\vDash\psi',\text{ and }\forall 0\leq j<i,\ (M,\pi[j\ldots\infty])\vDash\psi && \text{by CTL* (10)}\\ &\iff \exists i\geq 0,\ (M',\pi'[i\ldots\infty])\vDash\psi',\text{ and }\forall 0\leq j<i,\ (M',\pi'[j\ldots\infty])\vDash\psi && \text{by IH (6)}\\ &\iff (M',\pi')\vDash\psi\mathsf{U}\psi' && \text{by CTL* (10)}.\end{aligned}$$

Thus $(M,t)\vDash\Phi$ iff $(M',t')\vDash\Phi$ and $(M,\pi)\vDash\psi$ iff $(M',\pi')\vDash\psi$ by induction.

Question 7. Prove a version of the Hennessy-Milner theorem for CTL, in Theorem 35 in Lecture 2.

- If $t \in M$ and $t' \in M'$ are CTL-equivalent, that is they satisfy the same formulas in CTL, then (M, t) and (M', t') are bisimilar.

Consider that the sets St and St' of states in models M and M' are assumed to be finite. You can leverage on the same proof structure as in the proof of Theorem 35 in Lecture 2.

Answer 7. Let $M = (St, \rightarrow, V)$ and $M' = (St', \rightarrow', V')$ be models, and let $t \in St$ and $t' \in St'$ be CTL-equivalent. Then $(M, t) \vDash \Phi$ iff $(M', t') \vDash \Phi$ for all state formulas Φ in CTL. Let

$$B = \{(u, u') \in St \times St' \mid u \text{ is CTL-equivalent to } u'\}.$$

Then $B(t, t')$.

1. Let p be an atom. Then $(M, t) \vDash p$ iff $(M', t') \vDash p$ by CTL-equivalence, so $t \in V(p)$ iff $t' \in V'(p)$ by semantics of atoms.

2. Let $u \in St$ be a state such that $t \rightarrow u$. Then $(M, t) \vDash \mathsf{EX}\top$ by semantics of E and X, so $(M', t') \vDash \mathsf{EX}\top$ by CTL-equivalence, and hence there is a state $u' \in St'$ such that $t' \rightarrow' u'$ by semantics of E and X. Since St' is finite by assumption, there is a finite set of states $u'_1, \dots, u'_n \in St'$ such that $t' \rightarrow' u'_i$ for all i. Suppose for a contradiction that u is not CTL-equivalent to u'_i for all i. Then for all i, there is a state formula Φ_i such that $(M, u) \vDash \Phi_i$ and $(M', u'_i) \nvDash \Phi_i$ by CTL-equivalence, so

$$(M, u) \vDash \Phi_1 \wedge \cdots \wedge \Phi_n, \qquad (M', u'_i) \nvDash \Phi_1 \wedge \cdots \wedge \Phi_n,$$

by semantics of $\wedge$. Hence

$$(M, t) \vDash \mathsf{EX}(\Phi_1 \wedge \cdots \wedge \Phi_n), \qquad (M', t') \nvDash \mathsf{EX}(\Phi_1 \wedge \cdots \wedge \Phi_n),$$

by semantics of E and X, which is a contradiction to CTL-equivalence, so u is CTL-equivalent to u'_i for some i.

3. Let $u' \in St'$ be a state such that $t' \rightarrow' u'$. Then $(M', t') \vDash \mathsf{EX}\top$ by semantics of E and X, so $(M, t) \vDash \mathsf{EX}\top$ by CTL-equivalence, and hence there is a state $u \in St$ such that $t \rightarrow u$ by semantics of E and X. Since St is finite by assumption, there is a finite set of states $u_1, \dots, u_n \in St$ such that $t \rightarrow u_i$ for all i. Suppose for a contradiction that u_i is not CTL-equivalent to u' for all i. Then for all i, there is a state formula Φ_i such that $(M, u_i) \nvDash \Phi_i$ and $(M', u') \vDash \Phi_i$ by CTL-equivalence, so

$$(M, u_i) \nvDash \Phi_1 \wedge \cdots \wedge \Phi_n, \qquad (M', u') \vDash \Phi_1 \wedge \cdots \wedge \Phi_n,$$

by semantics of $\wedge$. Hence

$$(M, t) \nvDash \mathsf{EX}(\Phi_1 \wedge \cdots \wedge \Phi_n), \qquad (M', t') \vDash \mathsf{EX}(\Phi_1 \wedge \cdots \wedge \Phi_n),$$

by semantics of E and X, which is a contradiction to CTL-equivalence, so u_i is CTL-equivalent to u' for some i.

Thus B is a bisimulation such that $B(t, t')$, so (M, t) and (M, t') are bisimilar.

Question 8. By comparing the results at point 5, point 6, and point 7 show that, even though CTL* is strictly more expressive than CTL, the two logics have the same distinguishing power, so (M, t) and (M', t') satisfy the same formulas of CTL iff they satisfy the same formulas of CTL*. Prove this latter fact. Elaborate briefly on these apparently contradictory features of CTL and CTL*.

Answer 8. Let $t \in St$ be a state and π be a path in a model $M = (St, \to, V)$, and let $t' \in St'$ be a state and π' be a path in a model $M' = (St', \to', V')$.

- $\Longrightarrow$ Assume that (M, t) and (M', t') satisfy the same formulas of CTL. Then (M, t) and (M', t') are bisimilar by Question 7. Hence $(M, t) \vDash \Phi$ iff $(M', t') \vDash \Phi$ for all state formulas Φ of CTL* and $(M, \pi) \vDash \psi$ iff $(M', \pi') \vDash \psi$ for all path formulas ψ of CTL* by Question 6. Thus (M, t) and (M', t') satisfy the same formulas of CTL* by definition of satisfaction.

- $\Longleftarrow$ Assume that (M, t) and (M', t') satisfy the same formulas of CTL*. Then $(M, t) \vDash \Phi$ iff $(M', t') \vDash \Phi$ for all state formulas Φ of CTL* and $(M, \pi) \vDash \psi$ iff $(M', \pi') \vDash \psi$ for all path formulas ψ of CTL* by definition of satisfaction. Hence $(M, t) \vDash \Phi$ iff $(M', t') \vDash \Phi$ for all formulas Φ of CTL by Question 5. Thus (M, t) and (M', t') satisfy the same formulas of CTL by definition of satisfaction.

Thus (M, t) and (M', t') satisfy the same formulas of CTL iff they satisfy the same formulas of CTL*. This is not a contradiction, since CTL is a strict syntactic fragment of CTL*, so it can only distinguish models expressible in its language, and in fact satisfiability does not imply expressibility.