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Coursework 2

1) (a) Let q be an arbitrary initial state in M, where π is a path from q.

$\pi \models \varphi R \psi$ iff $\pi[j..\infty] \models \psi$ for all $0 \leq j \leq i$ if $\pi[i..\infty] \models \varphi$ for some $i \geq 0$, otherwise $\pi[k..\infty] \models \psi$ for all $k \geq 0$

(b) $\varphi R \psi \overset{?}{\equiv} [\psi \wedge (X\psi) U \varphi] \vee [\neg F\varphi \wedge G\psi]$

(c) $\pi \models [\psi \wedge (X\psi) U \varphi] \vee [\neg F\varphi \wedge G\psi]$ iff $\pi \models \psi \wedge (X\psi) U \varphi$ or $\pi \models \neg F\varphi \wedge G\psi$

iff $\pi \models \psi$, and $\pi \models (X\psi) U \varphi$;
or $\pi \models \neg F\varphi$, and $\pi \models G\psi$

iff $\pi \models \psi$, and $\pi[i..\infty] \models \varphi$ for some $i \geq 0$, and
$\pi[j..\infty] \models X\psi$ for all $0 \leq j < i$;
or $\pi \not\models F\varphi$, and $\pi[k..\infty] \models \psi$ for all $k \geq 0$

iff $\pi \models \psi$, and $\pi[i..\infty] \models \varphi$ for some $i \geq 0$, and
$\pi[j..\infty][1..\infty] \models \psi$ for all $0 \leq j < i$;
or $\pi[l..\infty] \not\models \varphi$ for any $l \geq 0$, and
$\pi[k..\infty] \models \psi$ for all $k \geq 0$

iff $\pi[j..\infty] \models \psi$ for all $0 \leq j \leq i$, and
$\pi[i..\infty] \models \varphi$ for some $i \geq 0$;
or $\pi[j..\infty] \not\models \varphi$ for any $l \geq 0$, and
$\pi[k..\infty] \models \psi$ for all $k \geq 0$

iff $\pi[j..\infty] \models \psi$ for all $0 \leq j \leq i$
if $\pi[i..\infty] \models \varphi$ for some $i \geq 0$,
otherwise $\pi[k..\infty] \models \psi$ for all $k \geq 0$

iff $\pi \models \varphi R \psi$

(d) $\pi \models \bot R \psi$ iff $\pi[j..\infty] \models \psi$ for all $0 \leq j \leq i$ if $\pi[i..\infty] \models \bot$ for some $i \geq 0$,
otherwise $\pi[k..\infty] \models \psi$ for all $k \geq 0$

iff $\pi[j..\infty] \models \psi$ for all $0 \leq j \leq i$ if $\pi[i..\infty] \not\models \text{true}$ for some $i \geq 0$,
otherwise $\pi[k..\infty] \models \psi$ for all $k \geq 0$

iff $\pi[k..\infty] \models \psi$ for all $k \geq 0$ [as $\lambda \models \text{true}$ for any path λ]

iff $\pi \models G\psi$

2) $(M,q) \models EF\Phi$ iff $(M,q) \models E(\text{true}\, U\, \Phi)$

iff for some path λ from q, $(M,\lambda) \models \text{true}\, U\, \Phi$

iff for some path λ from q, $(M,\lambda[i]) \models \Phi$ for some $i \geq 0$, and $(M,\lambda[j]) \models \text{true}$ for all $0 \leq j < i$

iff for some path λ from q, for some $j \geq 0$, $(M,\lambda[j]) \models \Phi$ [as true is valid]

$(M,q) \models AF\Phi$ iff $(M,q) \models A(\text{true}\, U\, \Phi)$

iff for every path λ from q, $(M,\lambda) \models \text{true}\, U\, \Phi$

iff for every path λ from q, $(M,\lambda[i]) \models \Phi$ for some $i \geq 0$, and $(M,\lambda[j]) \models \text{true}$ for all $0 \leq j < i$

iff for every path λ from q, for some $j \geq 0$, $(M,\lambda[j]) \models \Phi$ [as true is valid]

$(M,q) \models EG\Phi$ iff $(M,q) \models \neg AF\neg\Phi$

iff $(M,q) \not\models AF\neg\Phi$

iff for <u>not</u> every path λ from q, for some $j \geq 0$, $(M,\lambda[j]) \models \neg\Phi$

iff for <u>not</u> every path λ from q, for some $j \geq 0$, $(M,\lambda[j]) \not\models \Phi$ [using above proof]

iff for <u>not</u> every path λ from q, <u>not</u> for all $j \geq 0$, $(M,\lambda[j]) \models \Phi$ $[\exists x.\neg A \equiv \neg\forall x.A]$

iff for some path λ from q, for all $j \geq 0$, $(M,\lambda[j]) \models \Phi$

$[\neg\forall x.\neg\forall y.A \equiv \forall x.\neg\neg\forall y.A \equiv \forall x,y.A]$

$(M,q) \models AG\Phi$ iff $(M,q) \models \neg EF\neg\Phi$

iff $(M,q) \not\models EF\neg\Phi$

iff for <u>no</u> path λ from q, for some $j \geq 0$, $(M,\lambda[j]) \models \neg\Phi$ [using above proof]

iff for <u>no</u> path λ from q, for some $j \geq 0$, $(M,\lambda[j]) \not\models \Phi$

iff for <u>no</u> path λ from q, <u>not</u> for all $j \geq 0$, $(M,\lambda[j]) \models \Phi$ $[\exists x.\neg A \equiv \neg\forall x.A]$

iff for every path λ from q, for all $j \geq 0$, $(M,\lambda[j]) \models \Phi$

$[\neg\exists x.\neg\forall y.A \equiv \forall x.\neg\neg\forall y.A \equiv \forall x,y.A]$

3) a) We will prove this by induction over the definition of Φ, using the CTL definition from lecture 5.

<u>Base case:</u> $p \in AP$ is a state formula in CTL, and also in CTL* from definition 1.

<u>Inductive hypothesis:</u> Assume Φ_i is a CTL* formula if it is a CTL formula, for arbitrary Φ_i.

In the case of $\neg\Phi_i$: assuming Φ_i is a CTL formula, we know from our induction hypothesis that it is also CTL*, and so from definition 1, so is $\neg\Phi_i$.

In the case of $\Phi_1 \wedge \Phi_2$: from our induction hypothesis both Φ_1 and Φ_2 satisfy the property, so using definition 1 we also know $\Phi_1 \wedge \Phi_2$ is a CTL* formula, as required.

In the case of $E\Phi_i$: from our induction hypothesis when Φ_i is a CTL formula it is a CTL* formula also, thus from definition 1 $E\Phi_i$ is a CTL* formula

In the case of $A\Phi_i$: the reasoning is identical to that of the previous case.

Therefore CTL is a syntactic fragment of CTL*.

(b) We can find a counter-example: From definition 1 we have that $EXXp$ is a state formula in CTL*, but it does not belong to CTL because every temporal operator (such as X) must be immediately preceded by exactly one path quantifier (A or E).

7) In CTL: $\Phi ::= a \mid \neg\Phi \mid \Phi\wedge\Phi \mid EX\Phi \mid AX\Phi \mid E(\Phi U \Phi) \mid A(\Phi U \Phi)$

We can reason inductively over the syntax of CTL formulas, using definition 2, to show that restricting definition 2 to CTL formulas gives us the same truth conditions as in definitions 1.7/1.8.

Base case: $(M,s) \models p$ iff $s \in V(p)$ is the same in CTL

Inductive step: Assume the property we want to show holds for some arbitrary Φ, Φ'.

In the case of $\neg\Phi$: $(M,s) \models \neg\Phi$ iff $(M,s) \not\models \Phi$ is the same in CTL

In the case of $\Phi\wedge\Phi'$: $(M,s) \models \Phi\wedge\Phi'$ iff $(M,s)\models\Phi$ and $(M,s)\models\Phi'$ is the same in CTL

In the case of $EX\Phi$: $(M,s) \models EX\Phi$ iff for some path π from s, $(M,\pi)\models X\Phi$ [From rule 4]
iff for some path π from s, $(M,\pi[1..\infty][0])\models\Phi$
iff for some path π from s, $(M,\pi[1..\infty])\models\Phi$ [From rules 6 and 9]
which is the same in CTL, as a state formula

In the case of $AX\Phi$: $(M,s)\models AX\Phi$ iff for all paths π from s, $(M,\pi)\models X\Phi$ [From rule 5]
iff for all paths π from s, $(M,\pi[1..\infty][0])\models\Phi$
iff for all paths π from s, $(M,\pi[1..\infty])\models\Phi$ [From rules 6 and 9]
which is the same in CTL, as a state formula

In the case of $E(\Phi U \Phi')$: $(M,s)\models E(\Phi U\Phi')$ iff for some path π from s, $(M,\pi)\models \Phi U \Phi'$ [From rule 4]
iff for some path π from s, $(M,\pi[i..\infty][0])\models\Phi'$ for some $i\geq 0$, and $(M,\pi[j..\infty][0])\models\Phi$ for all $0\leq j<i$ [From rules 6 and 10]
iff for some path π from s, $(M,\pi[i..\infty])\models\Phi'$ for some $i\geq 0$, and $(M,\pi[j..\infty])\models\Phi$ for all $0\leq j<i$
which is the same in CTL, as a state formula

In the case of $A(\Phi U\Phi')$: $(M,s)\models A(\Phi U\Phi')$ iff for all paths π from s, $(M,\pi)\models\Phi U\Phi'$ [From rule 5]
iff for all paths π from s, $(M,\pi[i..\infty][0])\models\Phi'$ for some $i\geq 0$, and $(M,\pi[j..\infty][0])\models\Phi$ for all $0\leq j<i$ [From rules 6 and 10]
iff for all paths π from s, $(M,\pi[i..\infty])\models\Phi'$ for some $i\geq 0$, and $(M,\pi[j..\infty])\models\Phi$ for all $0\leq j<i$
which is the same in CTL, as a state formula

Hence restricting definition 2 to formulas in CTL gives us the same truth conditions as definitions 1.7 and 1.8 of lecture 5.

(5) (a) Take an arbitrary formula Φ of CTL, and arbitrary model M and initial state s.
From part (3) we know CTL is a syntactic fragment of CTL*, so Φ is also a CTL* formula.
From part (4) we know the truth conditions for formulas in CTL are exactly the same as those for CTL* (restricted to formulas of CTL), so $(M,s) \models \Phi$ in CTL iff $(M,s) \models \Phi$ in CTL*.
Hence we can find a CTL* formula $\Phi' \equiv \Phi$, equivalent for any CTL formula Φ, as required.

(b) From lecture 5 we know $F(a \wedge Xa)$ is an LTL formula but cannot be expressed in CTL, i.e. there is no CTL formula equivalent to $F(a \wedge Xa) \equiv \text{true}\, U\, (a \wedge Xa)$.
Take an arbitrary model M and initial state q.

[in LTL] $(M,q) \models F(a \wedge Xa)$ iff $\lambda \models F(a \wedge Xa)$ for every path λ in M from q
iff for every path λ in M from q, for some $i \geq 0$, $\lambda[i..\infty] \models a \wedge Xa$
iff for every path λ in M from q, for some $i \geq 0$, $\lambda[i..\infty] \models a$ and $\lambda[i..\infty] \models Xa$
iff for every path λ in M from q, for some $i \geq 0$, $\lambda[i..\infty] \models a$ and $\lambda[i+1..\infty] \models a$

[in CTL*] $(M,q) \models AF(\text{true}\, U\, (a \wedge Xa))$ iff for every path λ from q, $(M,\lambda) \models F(\text{true}\, U (a \wedge Xa))$
iff for every path λ from q, for some $i \geq 0$, $(M, \lambda[i..\infty]) \models \text{true}\, U\, (a \wedge Xa)$
iff for every path λ from q, for some $i \geq 0$, $(M, \lambda[i..\infty][j..\infty]) \models a \wedge Xa$ for some $j \geq 0$, and $(M, \lambda[i..\infty][k..\infty]) \models \text{true}$ for all $0 \leq k < j$
iff for every path λ from q, for some $i \geq 0$, for some $j \geq 0$ $(M, \lambda[i+j..\infty]) \models a$ and $(M, \lambda[i+j+1..\infty]) \models a$ [as true is valid]
iff for every path λ from q, for some $i \geq 0$, $(M, \lambda[i..\infty]) \models a$ and $(M, \lambda[i+1..\infty]) \models a$

From these derivations we can see that the LTL formula $F(a \wedge Xa)$ is equivalent to the CTL* formula $AF(\text{true}\, U\, (a \wedge Xa))$.
Since there is no CTL formula equivalent to the LTL formula $F(a \wedge Xa)$, we have shown that there is a CTL* formula $\Phi = AF(\text{true}\, U\, (a \wedge Xa))$ for which there is no equivalent formula Φ' in CTL.

(6) We will prove by mutual induction on the structure of Φ and ψ that $(M,t) \models \Phi$ iff $(M',t') \models \Phi$ and $(M,\pi) \models \psi$ iff $(M',\pi') \models \psi$.
We assume that (M,t) and (M',t') are bisimilar, and so are (M,π) and (M',π').

Base case: $\Phi = p$

$(M,t) \models p$ iff $t \in V(p)$ [by definition 2]
iff $t' \in V'(p)$ [as (M,t) and (M',t') are bisimilar]
iff $(M',t') \models p$ [by definition 2]

Also we do not have $(M,\pi) \models p$, nor $(M',\pi') \models p$ (by definition 2).

Inductive hypotheses: If (M_0, t_0) and (M_0', t_0') are bisimilar, then $(M_0, t_0) \models \Phi$ iff $(M_0', t_0') \models \Phi$ (Φ state formula)
If (M_0, π_0) and (M_0', π_0') are bisimilar, then $(M_0, \pi_0) \models \psi$ iff $(M_0', \pi_0') \models \psi$ (ψ path formula)

Inductive case: $\Phi = \neg \Phi_1 / \psi = \neg \psi_1$

$(M, t) \models \neg \Phi_1$ iff $(M, t) \not\models \Phi_1$ [by definition 2]
iff $(M, t) \models \Phi_1$ is not true
iff $(M', t') \models \Phi_1$ is not true [by IH (induction hypothesis), (M,t) and (M',t') bisimilar]
iff $(M', t') \models \neg \Phi_1$ [by definition 2]

$(M, \pi) \models \neg \psi_1$ iff $(M, \pi) \models \psi_1$ is not true [by definition 2].
iff $(M', \pi') \models \psi_1$ is not true [by IH]
iff $(M', \pi') \models \neg \psi_1$ [by definition 2]

Inductive case: $\Phi = \Phi_1 \wedge \Phi_2 / \psi = \psi_1 \wedge \psi_2$

$(M, t) \models \Phi_1 \wedge \Phi_2$ iff $(M, t) \models \Phi_1$ and $(M, t) \models \Phi_2$ [by definition 2]
iff $(M', t') \models \Phi_1$ and $(M', t') \models \Phi_2$ [by IH, (M,t) and (M',t') bisimilar]
iff $(M', t') \models \Phi_1 \wedge \Phi_2$ [by definition 2]

The proof for $(M, \pi) \models \psi_1 \wedge \psi_2$ iff $(M', \pi') \models \psi_1 \wedge \psi_2$ is analogous.

Inductive case: $\psi = \Phi_1$

$(M, t) \models \Phi_1$ iff $(M', t') \models \Phi_1$ [by IH]

$(M, \pi) \models \Phi_1$ iff $(M, \pi[0]) \models \Phi_1$ [by definition 2]
iff $(M', \pi'[0]) \models \Phi_1$ [IH, definition 3, (M, π) and (M', π') are bisimilar]
iff $(M', \pi') \models \Phi_1$ [by definition 2]

Inductive case: $\Phi = E\psi_1$

$(M, t) \models E\psi_1$ iff for some π_0 from t, $(M, \pi_0) \models \psi_1$ [by definition 2]
iff for some π_0' from t', $(M', \pi_0') \models \psi_1$ [IH, definition 3, (M,t) and (M',t') bisimilar]
iff $(M', t') \models E\psi_1$ [by definition 2]

We do not have either $(M, \pi) \models E\psi_1$ or $(M', \pi') \models E\psi_1$ from definition 2.

Inductive case: $\Phi = A\psi_1$

$(M, t) \models A\psi_1$ iff for all π_0 from t, $(M, \pi_0) \models \psi_1$ [by definition 2]
iff for all π_0' from t, $(M', \pi_0') \models \psi_1$ [IH, definition 3, (M,t) and (M',t') bisimilar]
iff $(M', t') \models E\psi_1$ [by definition 2]

We do not have either $(M, \pi) \models A\psi_1$ or $(M', \pi') \models A\psi_1$ from definition 2.

Inductive case: $\psi = X\psi_1$

We do not have either $(M,t) \vDash X\psi_1$ or $(M',t') \vDash X\psi_1$ from definition 3.

$(M,\pi) \vDash X\psi_1$ iff $(M,\pi[1..\infty]) \vDash \psi_1$ [by definition 2]
iff $(M',\pi'[1..\infty]) \vDash \psi_1$ [by IH, using forth/back properties with (M,π) and (M',π') bisimilar]
iff $(M',\pi') \vDash X\psi_1$ [by definition 2]

Inductive case: $\psi = \psi_1 U \psi_2$

We do not have either $(M,t) \vDash \psi_1 U \psi_2$ or $(M',t') \vDash \psi_1 U \psi_2$ from definition 3.

$(M,t) \vDash \psi_1 U \psi_2$ iff $(M,\pi[i..\infty]) \vDash \psi_2$ for some $i \geq 0$, and $(M,\pi[j..\infty]) \vDash \psi_1$ for all $0 \leq j < i$ [definition 2]
iff $(M',\pi'[i..\infty]) \vDash \psi_2$ for some $i \geq 0$, and $(M',\pi'[j..\infty]) \vDash \psi_1$ for all $0 \leq j < i$
[since $(M,\pi[i])$ and $(M',\pi'[i])$ bisimilar, by IH]
iff $(M',\pi') \vDash \psi_1 U \psi_2$ [by definition 2]

We have shown that for arbitrary M, M', t, t', π, π', if (M,t) and (M',t') are bisimilar and (M,π) and (M',π') are bisimilar, then $(M,t) \vDash \Phi$ iff $(M',t') \vDash \Phi$ and $(M,\pi) \vDash \psi$ iff $(M',\pi') \vDash \psi$. Since formulas in CTL* are all and only the state formulas, we can conclude that bisimulations preserve formula truth.

7 Consider that the sets St, St' in models M and M' are finite, and take arbitrary $t \in M$, $t' \in M'$. We need to show that if t and t' are CTL-equivalent, then (M,t) and (M',t') are bisimilar. We shall thus prove that there is a bisimulation B between M and M' such that $B(t,t')$.

Condition 1 of definition 3 is trivially true: equivalent worlds satisfy the same atoms.

For condition 2, assume that $B(t,t')$ and $t \to v$ for some $v \in St$.

Now assume for a contradiction that for no $v' \in St'$, $t' \to v'$ and $B(v,v')$.

Let $S' = \{u_i \in St' \mid t' \to u_i\}$; S' is non-empty (by CTL-equivalence of M, M') and finite.

Now by assumption, for every $u_i \in S'$, there is a formula Φ_i such that $(M,v) \vDash \Phi_i$ but $(M,u_i) \nvDash \Phi_i$.

$\bigwedge_i (M,v) \vDash \Phi_i$ implies $(M,v) \vDash \Phi_1 \wedge \dots \wedge \Phi_n$ [by definition 1.7]
implies $(M,\pi[1]) \vDash \Phi_1 \wedge \dots \wedge \Phi_n$ for some path π from t [since $t \to v$, v arbitrary]
implies $(M,\pi) \vDash X(\Phi_1 \wedge \dots \wedge \Phi_n)$ for some path π from t [by definition 1.8]
implies $(M,t) \vDash EX(\Phi_1 \wedge \dots \wedge \Phi_n)$ [by definition 1.7]

Similarly, we can show $\bigwedge_i (M',u_i) \nvDash \Phi_i$ implies $(M',t') \nvDash EX(\Phi_1 \wedge \dots \wedge \Phi_n)$.

But t and t' are assumed to be equivalent, so we arrive at a contradiction.

Thus there is a $v' \in St'$ with $t' \to v'$ and $B(v,v')$, so condition 2 is satisfied.

Condition 3 can be shown similarly.

Therefore $B(t,t')$, and hence (M,t) and (M',t') are bisimilar.

⑧ To show the $\Rightarrow$ direction, assume (M,t) and (M',t') satisfy the same CTL formulas.
From part (7) this means that (M,t) and (M',t') are bisimilar.
But then from part (6) bisimulation preserves the truth of CTL* formulas, so (M,t) and (M',t') must also satisfy the same CTL* formulas.
For the $\Leftarrow$ direction, assume (M,t) and (M',t') satisfy the same CTL* formulas.
For each CTL* formula Φ satisfied by (M,t) and (M',t'), we know from part (3) that either Φ is also a CTL formula or not, because CTL is a syntactic fragment of CTL*.
Also, from part (5) we know each Φ that is a CTL formula has an equivalent CTL* formula Φ'.
But since restricting the truth conditions of CTL* to formulas in CTL gives the same truth conditions as CTL itself, by part (4), we can conclude that $\Phi = \Phi'$.
Therefore (M,t) and (M',t') satisfy the same CTL formulas.

To elaborate on this, we can say that CTL* builds on CTL by allowing any number of path quantifiers and temporal operators within a state formula, making it more expressive than CTL. However we have seen that the subset of CTL* that follows CTL syntax also has identical semantics (equivalence of formulas), so it makes sense that the two logics would have the same distinguishing power.
We can use this to our advantage, given that satisfiability checking in CTL is PSPACE-complete but 2EXPTIME-complete in CTL*. For a given model, if our formula to check (SAT) is CTL, we can get better performance using a solver, while obtaining the same result as with CTL*. For those that are only CTL*, it will still be slower though.