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CATE

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Exercise Information

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Student Declaration - Version 1

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① a) $\lambda \models \varphi R \psi$ iff $(\exists i \geq 0.\ \lambda[i \dots \infty] \models \varphi$ and $\forall j,\ 0 \leq j \leq i,\ \lambda[j \dots \infty] \models \psi)$
or
$(\forall j \geq 0.\ \lambda[j \dots \infty] \models \psi)$

b) $\varphi R \psi = G\psi \vee (\psi U (\varphi \wedge \psi))$

c) $\lambda \models \varphi R \psi \Leftrightarrow \lambda \models G\psi \vee (\psi U (\varphi \wedge \psi)) \Leftrightarrow \lambda \models G\psi$ or $\lambda \models \psi U (\varphi \wedge \psi)$
$\Leftrightarrow (\forall j \geq 0.\ \lambda[j \dots \infty] \models \psi)$ or $(\exists i \geq 0.\ \lambda[i \dots \infty] \models \varphi \wedge \psi$ and $\forall j.\ 0 \leq j < i$ s.t. $\lambda[j \dots \infty] \models \psi) \Leftrightarrow (\forall j \geq 0.\ \lambda[j \dots \infty] \models \psi)$ or $(\exists i \geq 0.\ \lambda[i \dots \infty] \models \varphi$ and $\lambda[i \dots \infty] \models \psi$ and $\forall j.\ 0 \leq j < i.\ \lambda[j \dots \infty] \models \psi) \Longleftrightarrow$
$\Leftrightarrow (\forall j \geq 0.\ \lambda[j \dots \infty] \models \psi)$ or $(\exists i \geq 0.\ \lambda[i \dots \infty] \models \varphi$ and $\forall j.\ 0 \leq j \leq i.\ \lambda[j \dots \infty] \models \psi)$

d) $\bot R \psi \equiv G\psi \vee (\psi U (\bot \wedge \psi)) \equiv G\psi \vee (\psi U \bot) \equiv G\psi \vee \bot \equiv G\psi$

⑧ "$\Leftarrow$" ~~[illegible]~~ (M, t) and (M', t') satisfy the same CTL* formulas. Assume (M, t) satisfies some CTL formula ϕ that (M', t') does not. By 5(a), $\exists \phi'$ a CTL* formula such that is equivalent to ϕ. So (M, t) satisfies ϕ', but by our assumption, (M', t') satisfies ϕ'. So (M', t') satisfies ϕ ~~do~~. (same for why (M', t') does not satisfy any different CTL formula)

"$\Rightarrow$" (M, t) and (M', t') satisfy the same CTL formulas, then by ⑦, (M, t) is bisimilar to (M', t'). Since the truth of CTL* formulas is preserved by bisimulation (by ⑥), (M, t) & (M', t') satisfy the same CTL* formulas

This interesting result shows that any two models that express the same CTL formulas ~~have~~ cannot express different CTL* formulas, ~~even though~~ mostly because the additional syntax and semantics don't account for any choice ~~of~~ w.r.t. the truth of formulas that is determined through atom valuations.

(M', t') does not satisfy any different CTL formula)

$\Rightarrow$ (M, t) and (M', t') satisfy the same CTL formulas

Then by ⓐ, (M, t) is bisimilar to (M', t'). Since the truth of CTL* formulas is preserved by bisimulation (by ⓑ), (M, t) & (M', t') satisfy the same CTL* formulas

This interesting result shows that any two models that agree on the same CTL formulas ~~they~~ cannot agree on different CTL* formulas, ~~even though~~ mostly because the additional hybridity and formulation does it account for any choice of path in the truth of formulas that is determined through other [illegible].

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a) Consider the syntax for CTL state formulas:

$$\phi = a \mid \neg\phi \mid \phi \wedge \phi \mid EX\phi \mid AX\phi \mid E(\phi U \phi) \mid A(\phi U \phi)$$

Will show that CTL is a syntactic fragment of CTL* by structural induction on state formulas:

Case $\phi = a \mid \neg\phi \mid \phi \wedge \phi$. Then ϕ is also a CTL* formula, as this 3 cases are exactly as in the def. of state formulas in CTL*.

Case $\phi = EX\phi_1$. Then $\phi = EX\phi_1 = E\gamma = \phi_{CTL*}$, since ϕ_1 is a CTL* formula (by ind. hypothesis) and $X\phi_1$ is in the syntax of CTL* path formulas (γ is a path formula).

Case $\phi = AX\phi_1$. Then $\phi = AX\phi_1 = A\gamma = \phi_{CTL*}$ by the same argument as above (analogue).

Case $\phi = E(\phi_1 U \phi_2)$. Then $\phi = E\gamma = \phi_{CTL*}$ by the fact that ϕ_1 & ϕ_2 are CTL* formulas (ind. hypothesis) and $\phi_1 U \phi_2$ is in the syntax of CTL* path formulas.

Case $\phi = A(\phi_1 U \phi_2)$ similarly as above.

Those are all the possible cases. Induction is complete. □

b) $EXXp$ is a formula in CTL* but not in CTL (where p is an atom): In CTL, we cannot have consecutive 'X'.

②

i) $(M, g) \models EF\phi \iff (M, g) \models E(\text{true}\, U\, \phi) \overset{def}{\iff}$ for some path λ starting from g, $(M, \lambda) \models \text{true}\, U\, \phi \iff$ for some path λ starting from g, for some $j \geq 0$, $(M, \lambda[j]) \models \phi$ and for all i s.t. $0 \leq i < j$, $(M, \lambda[i]) \models \text{true} \overset{*}{\iff}$ for some path λ starting from g, for some $j \geq 0$, $(M, \lambda[j]) \models \phi$ $\square$

ii) $(M, g) \models AF\phi \iff (M, g) \models A(\text{true}\, U\, \phi) \overset{def}{\iff}$ for all paths λ starting from g, $(M, \lambda) \models \text{true}\, U\, \phi \iff$ for all paths λ starting from g, for some $j \geq 0$, $(M, \lambda[j]) \models \phi$ and for all i s.t. $0 \leq i < j$, $(M, \lambda[i]) \models \text{true} \overset{*}{\iff}$ for all paths λ starting from g, for some $j \geq 0$, $(M, \lambda[j]) \models \phi$

iii) $(M, g) \models EG\phi \iff (M, g) \models \neg AF\neg\phi \overset{def}{\iff} (M, g) \not\models AF\neg\phi$ $\overset{ii)}{\iff}$ it is not the case that for all paths λ starting from g, for some $j \geq 0$, $(M, \lambda[j]) \models \neg\phi \iff$ for some λ starting from g, for all $j \geq 0$, $(M, \lambda[j]) \not\models \neg\phi \iff$ for some λ starting from g, for all $j \geq 0$, $M, \lambda[j] \models \phi$. $\square$

iv) $(M, g) \models AG\phi \iff (M, g) \models \neg EF\neg\phi \overset{def}{\iff} (M, g) \not\models EF\neg\phi$ $\overset{i)}{\iff}$ it is not the case that for some path λ from g, for some $j \geq 0$, $(M, \lambda[j]) \models \neg\phi \iff$ for all λ from g, for all $j \geq 0$, $M, \lambda[j]) \not\models \neg\phi \iff$ for all λ from g, for all $j \geq 0$, $(M, \lambda[j]) \models \phi$!

Note: * is using the facts that $A \land T \equiv A$ and $(M, \lambda[i]) \models \text{true}$ for any i and any M, λ.

(4) Restricting Def 2 to CTL means we no longer have the rules for path formulas that are not syntactically path formulas in CTL. Thus, we ~~drop~~ the following rules:

$(M, \pi) \models \phi$ iff ...
$(M, \pi) \models \neg\psi$ iff ...
$(M, \pi) \models \psi \wedge \psi$

The state formulas are the same as in Def 1.7, so they're unchanged

$(M, s) \models \top$
$(M, s) \models \neg\phi$
$(M, s) \models \phi \wedge \phi'$
$(M, s) \models E\psi$
$(M, s) \models A\psi$.

We note that in Def 2, $(M, \pi) \models X\psi$ holds only if we can prove ψ. ψ is a state formula, so it can only be proved by the state rules (i.e. ones above & in Def 1.7) Thus for $X\psi$, we only need the first state, $\pi[1]$, to prove ψ. So $(M, \pi) \models X\psi$ is the same as in Def 1.8.
For a similar reason, $(M, \pi) \models \psi \, U \psi'$ has to be the same as in Def 1.8. The restriction is completed.

⑤ b) Consider $A(\text{true}\ U(a \wedge Xa))$ a CTL* formula.
This formula is equivalent to $F(a \wedge Xa)$ in LTL:

$M \models F(a \wedge Xa) \overset{LTL}{\underset{\text{Def } 1.5}{\Longleftrightarrow}} (M, g_0) \models F(a \wedge Xa)$ for every (initial) state g_0 in $M \overset{LTL}{\underset{\text{Def } 1.5}{\Longleftrightarrow}}$ for arb. init state g_0, for every path λ in M. where $\lambda[0] = g_0$, $\lambda \models F(a \wedge Xa)$

$\Longleftrightarrow$ for every λ in M where $\lambda[0] = g_0$, $\lambda \models F(a \wedge Xa)$

$\Longleftrightarrow$ —||—, $(\exists)\ i \geq 0$, s.t. $\lambda[i \ldots \infty] \models a \wedge Xa$

$\Longleftrightarrow$ —||—, #||#, s.t. $\lambda[i \ldots \infty] \models a$ and $\lambda[i \ldots \infty] \models Xa$

$\Longleftrightarrow$ —||—, #||#, s.t. $\lambda[i \ldots \infty] \models a$ and $\lambda[i \ldots \infty][1 \ldots \infty] \models a$

$\Longleftrightarrow$ —||—, #||#, s.t. $\lambda[i \ldots \infty] \models a$ and $\lambda[i+1 \ldots \infty] \models a$

—||—, #||#, s.t. M

$\Longleftrightarrow$ —||—, $(M, \lambda) \models \text{true}\ U\ (a \wedge Xa)$

$\Longleftrightarrow (M, g_0) \models A(\text{true}\ U(a \wedge Xa))$, for all paths starting from g_0.

But since $F(a \wedge Xa)$ has no equivalent formula in CTL, then "$A(\text{true}\ U(a \wedge Xa))$" has no equivalent. ▽

a) Consider an arbitrary formula in CTL. Take an arbitrary model and a state, M & s, s.t. $(M, s) \models \phi$. Since CTL is a syntactic fragment of CTL*, we can consider $\phi' = \phi$, where ϕ' is a CTL* formula. Then $(M, s) \models \phi'$, because in order to show $(M, s) \models \phi'$, we only need the semantic rules of CTL* restricte to CTL & replicate the proof for $(M, s) \models \phi$ in CTL. Thus for every CTL formula we have an equivalent formula (itself).

⑥ We will show by structural induction on the following:
$(M,f) \models \phi$ iff $(M',f') \models \phi$, for arbitrary M, M' models,
f, f' states or paths (both of the same type) and $(M,f) \approx (M',f')$

→ Case $\phi = p$. $(M,t) \models p \Leftrightarrow t \in V(p) \underset{\text{bisim. \& (a)}}{\overset{B \text{ is a}}{\Longleftrightarrow}} t' \in V'(p) \Leftrightarrow (M',t') \models p$. □

→ Case $\phi = \neg\psi$. $(M,t) \models \neg\psi \overset{def}{\Longleftrightarrow} (M,t) \not\models \psi \underset{\text{Hypothesis}}{\overset{\text{Ind.}}{\Longleftrightarrow}} (M',t') \not\models \psi \Leftrightarrow (M',t') \models \neg\psi$.

→ Case $\phi = \phi_1 \wedge \phi_2$. $(M,t) \models \phi_1 \wedge \phi_2 \Leftrightarrow (M,t) \models \phi_1$ and $(M,t) \models \phi_2 \Leftrightarrow$
$\underset{\text{Applied twice}}{\overset{\text{Ind. Hyp}}{\Longleftrightarrow}} (M',t') \models \phi_1$ and $(M',t') \models \phi_2 \Leftrightarrow (M',t') \models \phi_1 \wedge \phi_2$. □

→ Case $\phi = E\psi$. We'll show "$(M,t) \Rightarrow (M',t')$" and "$\Leftarrow$" is going to be similar (with the difference that we'll use the back property instead of the forth):

$(M,t) \models E\psi \Rightarrow$ ~~for~~ some path π_E starting from t, we have $(M,\pi_E) \models \psi$. Let $\pi_E[i] \overset{not}{=} t_i$, $\forall i \geq 1$. By applying the forth property, we know that $\exists t_1'$ s.t. $t' \to t_1'$ & $B(t_1, t_1')$ (since $B(t,t')$ & $t \to t_1$). By doing this countably many times, we obtain the states $t_1', t_2', \dots$ s.t. $B(t_i, t_i')$, $\forall i \geq 1$.
Then we have a path $\pi_E' = [t', t_1', \dots]$ in M' s.t. $(M, \pi_E) \approx (M', \pi_E')$
By Ind. hypothesis, we have obtained a path π_E' starting from t' such that $(M', \pi_E') \models \psi$. Thus by definition, $(M',t') \models E\psi$.

→ Case $\phi = A\psi$ is argued similarly as the previous case.

→ Cases $\phi = \psi, \psi \wedge \psi, \neg\psi$ for (M,π) are argued similarly as the cases for state formulas above.

→ Case $\phi = X\psi$. $(M,\pi) \models X\psi \Leftrightarrow (M, \pi[1\dots\infty]) \models \psi \underset{\text{and } (*)}{\overset{\text{Ind Hyp}}{\Longleftrightarrow}} (M', \pi'[1\dots\infty]) \models \psi \Leftrightarrow (M',\pi') \models X\psi$

Where $(*)$: $(M,\pi) \approx (M',\pi') \Leftrightarrow \forall i \geq 0.\ (M,\pi[i]) \approx (M',\pi'[i]) \Rightarrow \forall i \geq 1.$
$(M,\pi[i]) \approx (M',\pi'[i]) \Leftrightarrow (M, \pi[1\dots\infty]) \approx (M', \pi'[1,\dots\infty])$

→ Case $\phi = \psi U \psi'$ follows from applying $(*)$ and Ind Hypothesis twice.
Those are all cases, thus induction is complete. □

7. (a) holds, since t & t' are CTL-equivalent, so they satisfy the same atoms. □

(b ≡ forth) Take $v \in St$ such that $t \to v$.

If there are no paths starting from t, then $(M, t) \models \neg E T$ so $(M', t') \models \neg E T$, so there are no paths starting from t' either. Then, in both models, we can only construct formulas with $\neg$, $\wedge$, and atoms & bisimilarity is proven similarly to ~~thm~~ thm. 35.

If there is a path from t, consider the path π s.t. $\pi[0] = t$ & $\pi[1] = v$. Then $(M, \pi) \models X T$ holds, for any π' s.t. $\pi'[0] = t'$ (modally equivalence). This implies that $\exists\, v'$ s.t. $t' \to v'$ & $\pi'[1] = v'$ & exists a path π' starting from t'.

Now, for contradiction, assume that for no $v_i' \in St'$ with $t' \to v'$, we have (M, v) & (M', v') CTL-equivalent. Then, for each v_i' there $\exists$ a formula ϕ_i s.t. $(M, v) \models \phi_i$ & $(M', v') \not\models \phi_i$. Then, $(M, \pi) \models X(\phi_1 \wedge \phi_2 \wedge \ldots)$ but $(M', \pi') \not\models X(\phi_1 \wedge \ldots)$. Thus we have a contradiction with t & t' being CTL-equiv. □

(c ≡ back) Similarly as above. □