Stability of Gradient Learning Dynamics in Continuous Games: Vector Action Spaces

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***Abstract*— Towards characterizing the optimization landscape of games, this paper analyzes the stability and spectrum of gradient-based dynamics near fixed points of two-player continuous games. We introduce the quadratic numerical range as a method to bound the spectrum of game dynamics linearized about local equilibria. We also analyze the stability of differential Nash equilibria and their robustness to variation in agent's learning rates. Our results show that by decomposing the game Jacobian into symmetric and anti-symmetric components, we can assess the contribution of vector field's potential and rotational components to the stability of the equilibrium. In zero-sum games, all differential Nash equilibria are stable; in potential games, all stable points are Nash. Furthermore, zero-sum Nash equilibria are robust in the sense that they are stable for all learning rates. For continuous games with general costs, we provide a sufficient condition for instability. We conclude with a numerical example that investigates how players with different learning rates can take advantage of rotational components of the game to converge faster.**

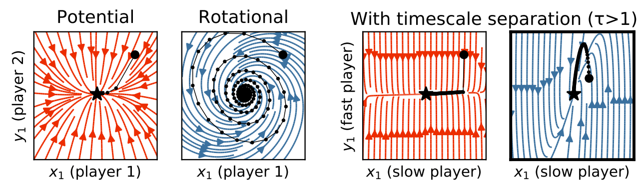

Fig. 1. *Game dynamics with rotational components can converge at a faster rate with timescale separation.* We plot slices of the vector field and learning trajectories of mostly potential and mostly rotational learning dynamics from Example 2. The game Jacobian at the equilibrium decomposes into $J = (1-\varepsilon)S + \varepsilon A$, where $S = S^\top$ is symmetric and $A = -A^\top$ is anti-symmetric. For the mostly potential system (red, $\varepsilon = 0.1$), players converge to the equilibrium without cycling. For the mostly rotational system (blue, $\varepsilon = 0.9$), players without timescale separation cycle around the equilibrium. Players with timescale separation take advantage of the rotational vector field to converge faster to the equilibrium, as shown in the right-most plot. See Fig. 5 for a continuation of this example.

I. Introduction

The study of learning in games is experiencing a resurgence in the control theory [20], [22], [23], optimization [12], [14], and machine learning [4], [6], [7], [9], [15] communities. Partly driving this resurgence is the prospect for game-theoretic analysis to yield machine learning algorithms that generalize better or are more robust. A natural paradigm for learning in games is gradient play since updates in large decision spaces can be performed locally with minimal information, while still guaranteeing local convergence in many problems [6], [14].

Towards understanding the optimization landscape in such formulations, dynamical systems theory is emerging as a principal tool for analysis and ultimately synthesis [1]–[3], [12], [13]. One of the primary means to understand the optimization landscape of games is the eigenstructure and spectrum of the Jacobian of the learning dynamics in a neighborhood of a stationary point. However, as has been demonstrated [12], not all attractors of the learning dynamics are game theoretically meaningful. Furthermore, structural heterogeneity in the learning algorithms employed by players can drastically change the convergence behavior.

The local stability of a hyperbolic fixed point in a nonlinear system can be assessed by examining the eigenstructure of the linearized dynamics [10], [21]. However, in a game context there are extra structure coming from the underlying game—that is, players are constrained to move only along directions over which they have control. They can only control their individual actions, as opposed to the entire state of the dynamical system corresponding to the learning rules being applied by the agents. It has been observed in earlier work that not all stable attractors of gradient play are local Nash equilibria and not all local Nash equilibria are stable attractors of gradient play [12]. Furthermore, changes in players' learning rates—which corresponds to scaling rows of the Jacobian—can change an equilibrium from being stable to unstable and vice versa [6].

To summarize, there is a subtle but extremely important difference between game dynamics and traditional nonlinear dynamical systems: alignment conditions are important for distinguishing between equilibria that have game-theoretic meaning versus those which are simply stable attractors of learning rules, and features of learning dynamics such as learning rates can play an important role in shaping not only equilibria but also alignment properties. Motivated by this observation along with the recent resurgence of applications of learning in games in control, optimization, and machine learning, in this paper we provide an in-depth analysis of the spectral properties of gradient-based learning in two-player continuous games.

Contributions: Our main results are bounds on the spectrum of gradient-based game dynamics near equilibria (Theorem 1, Theorem 2) and robustness guarantees of differential Nash equilibria to variations in learning rates (Theorem 3, Theorem 4) These results consider two-player zero-sum

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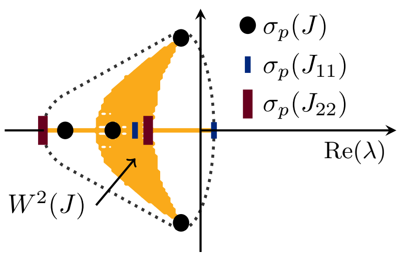


Fig. 2. *A stable equilibrium that is not Nash.* The spectrum of J, J_{11}, and J_{22} in Example 1 are contained in the numerical range (convex dashed region) and quadratic numerical range (non-convex region) of J. The eigenvalues of J are in the left plane, hence the fixed point is stable under gradient play (1). However, the first player's J_{11} is indefinite, hence the fixed point is not a Nash equilibrium.

(adversarial) and potential (cooperative) games with twice continuously-differentiable costs. For continuous games in general, we prove a sufficient condition for instability of equilibria of the learning dynamics (Theorem 5).

This paper extends the authors' conference paper on scalar games [5] to continuous games with vector action spaces. By using the quadratic numerical range to analyze the spectrum and stability of game dynamics, the block Jacobian in the vector case reduces to the analysis of a two-by-two matrix. The added complexity in the vector case means that the authors could not get the same sort of necessary and sufficient conditions as they did in the scalar case. Despite this challenge, tight results are obtained for the specialized cases of zero-sum and potential games, with an instability result for general games to conclude the analysis.

II. Preliminaries

This section contains game-theoretic preliminaries, mathematical formalism, and a description of the gradient-based learning paradigm studied in this paper.

A. Game-Theoretic Preliminaries

A 2-player *continuous game* $\mathcal{G} = (f_1, f_2)$ is a collection of costs defined on $X = X_1 \times X_2$ where player (agent) $i \in \mathcal{I} = \{1, 2\}$ has cost $f_i : X \to \mathbb{R}$. In this paper, the results apply to games with sufficiently smooth costs $f_i \in C^r(X, \mathbb{R})$ for some $r \geq 0$. Agent i's set of feasible strategies is the d_i-dimensional precompact set $X_i \subseteq \mathbb{R}^{d_i}$. The notation x_{-i} denotes the strategy of player i's competitor; that is, $x_{-i} = x_j$ where $j \in \mathcal{I}\backslash\{i\}$.[1]

The most common and arguably natural notion of an equilibrium in continuous games is due to Nash [17].

Definition 1 (Local Nash equilibrium): A joint action profile $x = (x_1, x_2) \in W_1 \times W_2 \subset X_1 \times X_2$ is a local Nash equilibrium on $W_1 \times W_2$ if, for each player $i \in \mathcal{I}$, $f_i(x_i, x_{-i}) \leq f_i(x_i', x_{-i})$, $\forall x_i' \in W_i$.

A local Nash equilibrium can equivalently be defined as in terms of best response maps: $x_i \in \arg\min_y f_i(y, x_{-i})$. From this perspective, local optimality conditions for players' optimization problems give rise to the notion of a differential Nash equilibrium [19], [20]; non-degenerate differential Nash are known to be generic and structurally stable amongst local Nash equilibria in sufficiently smooth games [18]. Let $D_i f_i$ denote the derivative of f_i with respect to x_i and, analogously, let $D_i(D_i f_i) \equiv D_i^2 f_i$ be player i's individiaul Hessian.

Definition 2: For continuous game $\mathcal{G} = (f_1, f_2)$ where $f_i \in C^2(X_1 \times X_2, \mathbb{R})$, a joint action profile $(x_1, x_2) \in X_1 \times X_2$ is a *differential Nash equilibrium* if $D_i f_i(x_1, x_2) = 0$ and $D_i^2 f_i(x_1, x_2) > 0$ for each $i \in \mathcal{I}$.

A differential Nash equilibrium is a strict local Nash equilibrium [19, Thm. 1]. Furthermore, the conditions $D_i f_i(x) = 0$ and $D_i^2 f_i(x) \geq 0$ are necessary for a local Nash equilibrium [19, Prop. 2].

Learning processes in games, and their study, arose as one of the explanations for how players grapple with one another in seeking an equilibrium [8]. In the case of sufficiently smooth games, gradient-based learning is a natural learning rule for myopic players.

B. Gradient-based Learning as a Dynamical System

At time t, a myopic agent i updates its current action $x_i(t)$ by following the gradient of its individual cost f_i given the decisions of its competitors x_{-i}. The synchronous adaptive process that arises is the discrete-time dynamical system

$$x_i(t+1) = x_i(t) - \gamma_i D_i f_i(x_i(t), x_{-i}(t)) \tag{1}$$

for each $i \in \mathcal{I}$ where $D_i f_i$ is the gradient of player i's cost with respect to x_i and γ_i is player i's learning rate.

a) Stability: Recall that a matrix A is called Hurwitz if its spectrum lies in the open left-half complex plane $\mathbb{C}_-^\circ$. Furthermore, we often say such a matrix is *stable* in particular when A corresponds to the dynamics of a linear system $\dot{x} = Ax$ or the linearization of a nonlinear system around a fixed point of the dynamics.[2]

It is known that (1) will converge locally asymptotically to a differential Nash equilibrium if the local linearization is a contraction [6]. Let

$$g(x) = (D_1 f_1(x), D_2 f_2(x)) \tag{2}$$

be the vector of individual gradients and let $Dg(x)$ be its Jacobian—i.e., the *game Jacobian*. Further, let $\sigma_p(A) \subset \mathbb{C}$ denote the *spectrum* of the matrix A, and $\rho(A)$ its *spectral radius*. Then, x is *locally exponentially stable* if and only if $\rho(I - \Gamma Dg(x)) < 1$, where $\Gamma = \mathrm{blockdiag}(\gamma_1 I_{d_1}, \gamma_2 I_{d_2})$ is a diagonal matrix and I_{d_i} is the identity matrix of dimension d_i. The map $I - \Gamma Dg(x)$ is the local linearization of (1). Hence, to study stability (and, in turn, convergence) properties it is useful to analyze the spectrum of not only the map $I - \Gamma Dg(x)$ but also $Dg(x)$ itself.

[1] For 2-player games, $x_{-1} = x_2$ and $x_{-2} = x_1$.

[2] The Hartman-Grobman theorem [21] states that around any hyperbolic fixed point of a nonlinear system, there is a neighborhood on which the nonlinear system is stable if the spectrum of Jacobian lies in $\mathbb{C}_-^\circ$.

b) Partitioning the Game Jacobian: Let $x = (x_1, x_2)$ be a joint action profile such that $g(x) = 0$. Towards better understanding the spectral properties of $Dg(x)$ (respectively, $\Lambda Dg(x)$), we partition $Dg(x)$ into blocks:

$$J(x) = \begin{bmatrix} -D_1^2 f_1(x) & -D_{12} f_1(x) \\ -D_{21} f_2(x) & -D_2^2 f_2(x) \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}. \quad (3)$$

A differential Nash equilibrium (the second order conditions of which are sufficient for a local Nash equilibrium) is such that $J_{11} < 0$ and $J_{22} < 0$. On the other hand, as noted above, J is Hurwitz or stable if $\sigma_p(J) \subset \mathbb{C}_-^{\circ}$. Moreover, since the diagonal blocks are symmetric, J is similar to the matrix in Fig 3. For the remainder of the paper, we will study the Dg at a given fixed point x as defined in (3).

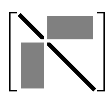
$J(x, y) \sim$ 

Fig. 3. *Similarity*: the game Jacobian in (3) is similar to a matrix with diagonal block-diagonals.

c) Classes of Games: Different classes of games can be characterized via J. For instance, a *zero-sum game*, where $f_1 \equiv -f_2$, is such that $J_{12} = -J_{21}^\top$. On the other hand, a game $\mathcal{G} = (f_1, f_2)$ is a *potential game* if and only if $D_{12} f_1 \equiv D_{21} f_2^\top$ [16, Thm. 4.5], which implies that $J_{12} = J_{21}^\top$.

C. Spectrum of Block Matrices

One useful tool for characterizing the spectrum of a block operator matrix is the numerical range and quadratic numerical range, both of which contain the operator's spectrum [24] and therefore all of its eigenvalues. The *numerical range* of J is defined by

$$W(J) = \{\langle Jz, z\rangle : \ z \in \mathbb{C}^{d_1+d_2}, \ \|z\| = 1\},$$

and is convex. Given a block operator J, let

$$J_{v,w} = \begin{bmatrix} \langle J_{11} v, v\rangle & \langle J_{12} w, v\rangle \\ \langle J_{21} v, w\rangle & \langle J_{22} w, w\rangle \end{bmatrix} \quad (4)$$

where $v \in \mathbb{C}^{d_1}$ and $w \in \mathbb{C}^{d_2}$. The *quadratic numerical range* of J, defined by

$$W^2(J) = \bigcup_{v \in \mathcal{S}_1, w \in \mathcal{S}_2} \sigma_p(J_{v,w}), \quad (5)$$

is the union of the spectra of (4) where $\sigma_p(\cdot)$ denotes the (point) spectrum of its argument and $\mathcal{S}_i = \{z \in \mathbb{C}^{d_i} : \ \|z\| = 1\}$, It is, in general, a non-convex subset of $\mathbb{C}$. The quadratic numerical range (5) is equivalent to the set of solutions of the characteristic polynomial

$$\begin{aligned} \lambda^2 - \lambda(\langle J_{11} v, v\rangle + \langle J_{22} w, w\rangle) + \langle J_{11} v, v\rangle \langle J_{22} w, w\rangle \\ - \langle J_{12} v, w\rangle \langle J_{21} w, v\rangle = 0 \end{aligned} \quad (6)$$

for $v \in \mathcal{S}_1$ and $w \in \mathcal{S}_2$. We use the notation $\langle Jx, y\rangle = x^* J y$ to denote the inner product. Note that $W^2(J)$ is a subset of $W(J)$ and, as previously noted, contains $\sigma_p(J)$. Albeit non-convex, $W^2(J)$ provides a tighter characterization of the spectrum[3].

[3]There are numerous computational approaches for estimating the $W(\cdot)$ and $W^2(\cdot)$ (see, e.g., [11, Sec. 6]).

Example 1: Consider the game Jacobian of the zero-sum game $(f, -f)$ defined by cost $f : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$,

$$f(x, y) = -\tfrac{1}{2} x_1^2 + \tfrac{5}{2} x_2^2 + 7 y_1 x_1 - 3 y_2 x_2 - 2 y_1^2 - 6 y_2^2.$$

The numerical range, quadratic numerical range, spectrum and diagonal entries of J, defined using the origin as the fixed point, are plotted in Fig. 2. In this example, the origin is not a differential Nash equilibrium since $D_1^2 f_1(0, 0)$ is indefinite, yet it is an exponentially stable equilibrium of $\dot{x} = -g(x)$ since all the eigenvalues of J are all negative.

Observing that the quadratic numerical range for a block 2×2 matrix J derived from a game on a finite dimensional Euclidean space reduces to characterizing the spectrum of 2×2 matrices, we first characterize stability properties of scalar 2-player continuous games.

III. Stability of 2-Player Continuous Games

In this section, we give stability results for 2-player continuous games on vector action spaces. Consider a game (f_1, f_2). Recall from the preliminaries that $f_1, f_2 \in C^2(X_1 \times X_2, \mathbb{R})$ and $X_1 \subseteq \mathbb{R}^{d_1}, X_2 \subseteq \mathbb{R}^{d_2}$, are d_i-dimensional actions spaces.

Let x be a fixed point of (2) such that $g(x) = 0$. We study the resulting gradient learning dynamics of Equation (1) near fixed points x. In particular, we analyze the Jacobian of g and its relation with differential Nash equilibria and stable equilibria.

A. Jacobian Decomposition: Block 2×2 case

In the block 2×2 case, we decompose the game Jacobian (3) by analogy with the decomposition in the scalar case.

We decompose the game Jacobian,

$$J(x) = \begin{bmatrix} J_{11} & P \\ P^\top & J_{22} \end{bmatrix} + \begin{bmatrix} 0 & -Z \\ Z^\top & 0 \end{bmatrix}, \quad (7)$$

where $P = \frac{1}{2}(J_{12} + J_{21}^\top)$ and $Z = \frac{1}{2}(J_{12} - J_{21}^\top)$.

B. Discussion of Decomposition

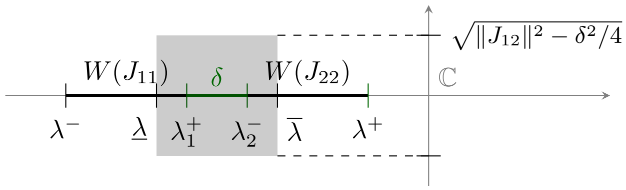
Consider the Jacobian in (7) and its quadratic numerical range $\mathcal{W}^2(J(x))$, we note that the spectrum of $J(x)$ is contained in the spectrum of

$$J_{v,w} = \begin{bmatrix} a & p - z \\ p^* + z^* & d \end{bmatrix} \quad (8)$$

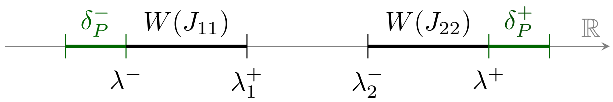
where $a = \langle J_{11} v, v\rangle$, $d = \langle J_{22} w, w\rangle$ $p = \langle P v, w\rangle$, and $z = \langle Z w, v\rangle$ for unit-length complex numbers $v \in \mathcal{S}_1, w \in \mathcal{S}_2$. Hence, to show the stability of a particular fixed point x, we must show that for all v, w, the spectrum of (8) is contained in the left-half complex plane. We directly assess whether a fixed point is Nash by the signs of a, d.

C. Types of Games

The decomposition in (7) naturally leads to games of various types based on either Z or P being 0.



(a) Zero-sum game where $\delta = \lambda_2^- - \lambda_1^+ > 0$ and $\|J_{12}\| > \delta/2$



(b) Potential game where $\lambda_2^- - \lambda_1^+ > 0$.

Fig. 4. *Spectrum of learning dynamics near a fixed point in zero-sum and potential games.* We illustrate Theorem 1 (a, zero-sum game) and Theorem 2 (b, potential game). The highlighted regions contain the spectrum of the linearized dynamics.

1) Potential games ($Z = 0$): All stable fixed points are differential Nash equilibria. Further, we provide a necessary and sufficient condition for a stable Nash equilibrium.

Proposition 1 (Stability in Potential Games): Consider a potential game $\mathcal{G} = (f_1, f_2)$ on finite dimensional action spaces X_1, X_2. If x is a stable equilibrium of $\dot{x} = -g(x)$, then x is a differential Nash equilibrium of $\mathcal{G}$. Moreover, differential Nash equilibria are stable if and only if $J_{11} - PJ_{22}^{-1}P^T < 0$.

Since $Z = 0$ in potential games, $J(x)$ is symmetric and its eigenvalues are directly related to the definitness properties of $J(x)$. That is, $J_{11} < 0$ and $J_{22} < 0$ are necessary conditions for $J < 0$. The result is consistent with our intuition from the scalar case that the interaction terms in potential games only discourage stability.

2) Zero-sum games ($P = 0$): All differential Nash equilibria are stable, and these equilibria are robust to variations of learning rates.

Proposition 2 (Stability in Zero-Sum Games): Consider a zero-sum game $\mathcal{G} = (f, -f)$ on finite dimensional action spaces X_1, X_2. If x is a differential Nash equilibrium of $\mathcal{G}$, then x is a stable equilibrium of $\dot{x} = -g(x)$.

Proof: For zero-sum games, $P = 0$, hence the symmetric component of $J(x)$ at fixed point x is $\frac{1}{2}\big(J(x) + J(x)^\top\big) = \mathrm{blockdiag}(-D_1^2 f_1(x), -D_2^2 f_2(x))$. If x is a differential Nash equilibrium, then $J(x) + J(x)^\top < 0$ and hence x is stable. ■

The result can also be proven directly from Lyapunov theory using Lyapunov function $\|x\|^2$.

In the following section, we bound the spectrum of zero-sum and potential game dynamics using the quadratic numerical range. Future work will look for methods to combine two cases to generalize the results to games with any cost structures.

IV. Spectrum of Game Dynamics Near Equilibria

A. Block Stability: Uniform Learning Rates

We bound the spectrum of zero-sum and potential game dynamics using the norms of the interaction terms Z and P defined in (7).

For game (f_1, f_2) with game Jacobian (3), define the following for $i = 1, 2$: $\lambda_i^- = \min \sigma_p(J_{ii})$, $\lambda_i^+ = \max \sigma_p(J_{ii})$. Additionally, define

$$\begin{aligned} \lambda^- &= \min\{\lambda_1^-, \lambda_2^-\}, & \underline{\lambda} &= \tfrac{1}{2}(\lambda_1^- + \lambda_2^-), \\ \lambda^+ &= \max\{\lambda_1^+, \lambda_2^+\}, & \overline{\lambda} &= \tfrac{1}{2}(\lambda_1^+ + \lambda_2^+). \end{aligned}$$

These terms will be useful in deriving bounds on the spectrum of J. The next two theorems are our main results, giving tight bounds on the spectrum of J (i.e. bounds on the real and imaginary eigenvalues) for game dynamics of zero-sum and potential games. Recall that for zero-sum game $(f, -f)$, the interaction term $Z = D_{12}f(x)$.

Theorem 1 (Spectrum of Zero-Sum Game Dynamics): Consider a zero-sum game $\mathcal{G} = (f, -f)$. The Jacobian $J(x) = Dg(x)$ of the dynamics $\dot{x} = -g(x)$ at fixed points x is such that

$$\sigma_p(J(x)) \cap \mathbb{R} \subset \big[\lambda^-, \lambda^+\big] \tag{9}$$

and $\sigma_p(J(x)) \backslash \mathbb{R}$ is contained in

$$\big\{z \in \mathbb{C} : \ \mathrm{Re}(z) \in \big[\underline{\lambda}, \overline{\lambda}\big]\,, \ |\mathrm{Im}(z)| \le \|Z\|\big\}\,. \tag{10}$$

Furthermore, if $\lambda_2^+ < \lambda_1^-$ or $\lambda_1^+ < \lambda_2^-$ then the following two implications hold for $\delta = \lambda_1^- - \lambda_2^+$ or $\delta = \lambda_2^- - \lambda_1^+$, respectively: (i) $\|Z\| \le \delta/2 \implies \sigma_p(J(x)) \subset \mathbb{R}$; (ii) $\|Z\| > \delta/2 \implies \sigma_p(J(x)) \backslash \mathbb{R} \subset \{z \in \mathbb{C} : \ |\mathrm{Im}(z)| \le \sqrt{\|Z\|^2 - \delta^2/4}\}$.

Proof: Observe that $\overline{\det(J_{v,w}(x) - \lambda I)} = \det(J_{v,w}(x) - \bar{\lambda} I)$ for $v \in \mathcal{S}_1$ and $w \in \mathcal{S}_2$ since $D_1^2 f(x)$ and $-D_2^2 f(x)$ are symmetric, which implies that $W^2(J(x)) = W^2(J(x))^*$. Since $-w^* D_{12} f(x)^T v v^* D_{12} f(x) w \le 0$, (9) and (10) follow from [24, Prop. 1.2.6], and (i) and (ii) follow from [25, Lem. 5.1-(ii)]. ■

Recall that for potential games with potential function ϕ, the interaction term $P = D_{12}\phi(x)$. Define

$$\delta_P^\pm = \|P\| \tan\left(\frac{1}{2} \arctan \frac{2\|P\|}{|\lambda_1^\pm - \lambda_2^\pm|}\right).$$

Theorem 2 (Spectrum of Potential Game Dynamics): Consider a potential game $\mathcal{G} = (f_1, f_2)$. The Jacobian $J(x) = Dg(x)$ of the dynamics $\dot{x} = -g(x)$ at fixed points x is such that $\sigma_p(J(x)) \subset \mathbb{R}$ and

$$\begin{aligned} \lambda^- - \delta_P^- \le \min \sigma_p(J(x)) \le \lambda^- \\ \lambda^+ \le \max \sigma_p(J(x)) \le \lambda^+ + \delta_P^+. \end{aligned} \tag{11}$$

Furthermore, if $\lambda_2^+ < \lambda_1^-$, then $\sigma_p(J(x)) \cap (\lambda_2^+, \lambda_1^-)$ is empty. If $\lambda_1^+ < \lambda_2^-$, then $\sigma_p(J(x)) \cap (\lambda_1^-, \lambda_2^+)$ is empty.

Proof: Inequalities in (11) follow from [24, Prop. 1.2.4] and last statements follow from [24, Cor. 1.2.3]. ■

B. Block Stability: Non-Uniform Learning Rates

Theorem 3 (Zero-sum Nash are Robust): Consider a zero-sum game $(f_1, f_2) = (f, -f)$ with game Jacobian J. Suppose that x is a differential Nash equilibrium. Then, x is a locally stable equilibrium of $\dot{x} = -\Lambda g(x)$ for any learning rate ratio τ.

Proof: First, observe that $a = \langle J_{11}v, v\rangle$ and $d = \langle J_{22}w, w\rangle$ are negative real numbers for any $v \in \mathcal{S}_1$ and $w \in \mathcal{S}_2$ by assumption that x is a differential Nash equilibrium, i.e. $-D_i^2 f_i(x) < 0$ for each $i \in \{1, 2\}$. Second, observe that for zero-sum games, $z = -\langle J_{12}w, v\rangle = \langle J_{21}v, w\rangle^*$. Therefore, for x to be stable, the eigenvalues of

$$J_{v,w} = \begin{bmatrix} a & -z \\ \tau z^* & \tau d \end{bmatrix}$$

must all be negative. Hence, we compute the trace and determinant conditions to be $\operatorname{tr}(J_{v,w}) = \lambda_1+\lambda_2 = a+\tau d$ and $\det(J_{v,w}) = \lambda_1\lambda_2 = \tau(ad+|z|^2)$. Notice that, $\tau(ad+|z|^2) > 0 \iff ad + |z|^2 > 0$, and $a + \tau d < 0 \iff a + d < 0$. Since $a, d < 0$ and $\tau > 0$, both of the trace and determinant conditions for stability are satisfied, i.e. $\operatorname{tr}(J_{v,w}) < 0$ and $\det(J_{v,w}) > 0$. Hence, x is a stable equilibrium of $\dot{x} = -\Lambda g(x)$. ■

Further, the stability of $\dot{x} = -\Lambda g(x)$ implies that there exists a range of learning rates γ such that $x(t + 1) = x(t) - \gamma\Lambda g(x(t))$ is locally asymptotically stable.

Theorem 4 (Robustness of Potential Games): Suppose that x is a differential Nash equilibrium of a potential game with Jacobian Then, x is a locally stable equilibrium of $\dot{x} = -\Lambda g(x)$ for all learning rate ratio τ if $\sigma_{\max}(J_{11})\sigma_{\max}(J_{22}) > \sigma_{\max}(J_{12})^2$.

Proof: First, observe that $a = \langle J_{11}v, v\rangle$ and $d = \langle J_{22}w, w\rangle$ are both negative real numbers for any $v \in \mathcal{S}_1$ and $w \in \mathcal{S}_2$ by assumption that x is a differential Nash equilibrium, i.e. $-D_i^2 f_i(x) < 0$ for each $i \in \{1, 2\}$. Second, observe that for potential games, $p = \langle J_{12}w, v\rangle = \langle J_{21}v, w\rangle$. Therefore, for x to be stable, the eigenvalues of

$$J_{v,w} = \begin{bmatrix} a & p \\ \tau p^* & \tau d \end{bmatrix}$$

must all be negative. Hence, we compute the the trace and determinant conditions to be $\operatorname{tr}(J_{v,w}) = \lambda_1 + \lambda_2 = a + \tau d$ and $\det(J_{v,w}) = \lambda_1\lambda_2 = \tau(ad - |p|^2)$. Notice that $a + \tau d < 0 \iff a + d < 0$ and $\tau(ad - |p|^2) > 0 \iff ad - |p|^2 > 0 \iff ad > |p|^2 > 0$. In terms of the original matrix we have $\sigma_{\max}(J_{11})\sigma_{\max}(J_{22}) > \sigma_{\max}(P)^2$. ■

We have shown that Nash equilibria in zero-sum games are robust in variation in learning rates, whereas Nash equilibria of potential games are not robust to variation in learning rates in general. For the latter case, we provide a sufficient condition that guarantees its robustness.

V. Instability in General-Sum Games

We provide a sufficient condition for the instability of fixed points of continuous games on finite dimensional spaces in general. Our results quantifies the contribution of the off-diagonal interaction terms of (3) in destabilizing equilibria in games.

We begin by expressing the game Jacobian as the sum of symmetric and skew-symmetric matrices, $J = \frac{1}{2}(J + J^\top) + \frac{1}{2}(J - J^\top)$. Let R be a rotation that diagonalizes $\frac{1}{2}(J + J^\top)$ and sorts the eigenvalues so that J decomposes into

$$RJR^\top = \begin{bmatrix} M_+ & 0 \\ 0 & M_- \end{bmatrix} + \begin{bmatrix} Z_1 & Z_2 \\ -Z_2^\top & Z_3 \end{bmatrix} \tag{12}$$

where $M_+ > 0$, $M_- \leq 0$ are diagonal and Z_1 and Z_3 are skew-symmetric. Let $\lambda^-(M_+) > 0$ be the minimum eigenvalue of M_+ and $\lambda^+(M_-) \leq 0$ be the maximum eigenvalue of M_-.

Theorem 5 (Sufficient Conditions for Instability in Games): Consider general-sum game (f_1, f_2) with $f_i \in C^2(X_1 \times X_2, \mathbb{R})$ where X_i is d_i-dimensional for each $i = 1, 2$. At a fixed point x, $\sigma_p(J(x)) \not\subset \mathbb{C}_-^\circ$ if

$$\|Z_2\| < \tfrac{1}{2}\big(|\lambda^+(M_-)| + |\lambda^-(M_+)|\big) < |\lambda^-(M_+)| \tag{13}$$

with M_+, M_- and Z_2 defined in (12).

Proof: Since Z_1 and Z_3 are skew-symmetric we have that $\operatorname{Re}\big(M_- + Z_3)\big) \leq \lambda^+(M_-) \leq 0$ and $0 \leq \lambda^-(M_+) \leq \operatorname{Re}\big(W(M_+ + Z_1)\big)$ [25, Prop. 1.1.12]. ■

The result above works by bounding a non-empty subset of the eigenvalues of J in $\mathbb{C}_+^\circ$ to guarantee instability. The inequalities in (13) are the block matrix equivalent of being inside the circle of radius $\sqrt{h^2 + p^2}$ in the scalar case [5]. The left inequality is analogous to being inside the circle in the vertical (z) direction; the right inequality is analogous to being right enough in the horizontal (m) direction. See

VI. Numerical Example

Example 2: In this example, we explore how timescale separation can improve the convergence rate of game dynamics. In particular, we show that when a game has larger rotational components, timescale separation can lead to well-conditioned game dynamics and thus faster convergence.

Consider a zero-sum game $\mathcal{G} = (f, -f)$ on $\mathbb{R}^2 \times \mathbb{R}^2$ with cost given by

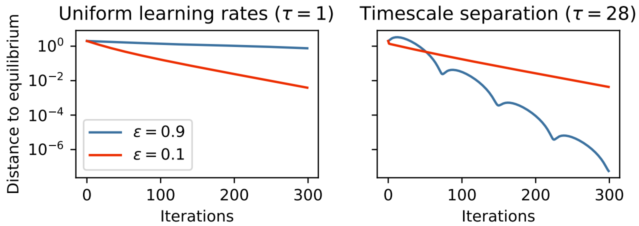
$$f(x, y) = (1 - \varepsilon)\left(x_1^2 + \tfrac{3}{2}x_2^2 - 2y_1^2 - \tfrac{5}{2}y_2^2\right) + \varepsilon x^\top By$$

and the matrix B is such that each entry is $B_{ij} = 1$ for each i, j except for $B_{22} = -1$. The parameter $0 \leq \varepsilon \leq 1$ controls the amount of rotational component in the game. Note that when $\varepsilon = 0$, the game Jacobian is symmetric; when $\varepsilon = 1$, the game Jacobian is skew-symmetric. The decomposition of the Jacobian is $J = (1 - \varepsilon)S + \varepsilon A$ where $S = S^\top$ and $A = -A^\top$. Suppose agents descend their individual gradient with individual learning rates $\gamma_1 = \gamma, \gamma_2 = \tau\gamma$, expressed relative to base learning rate γ, yielding learning dynamics

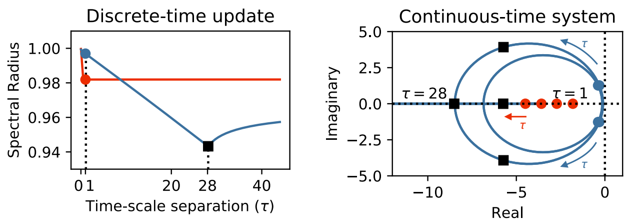
$$\begin{aligned} x(t + 1) &= x(t) - \gamma D_1 f(x(t), y(t)) \\ y(t + 1) &= y(t) + \gamma\tau D_2 f(x(t), y(t)). \end{aligned} \tag{14}$$

Recall that the spectrum of $\Lambda_\tau J(x, y)$ at an equilibrium (x, y) determines its stability and that the spectral radius of $I + \gamma\Lambda_\tau J(x, y)$ determines the convergence convergence rate of the discrete-time system above, where $\Lambda_\tau = \operatorname{blockdiag}(I_1, \tau I_2)$ with identity matrices I_1, I_2.

By using a learning update with timescale separation between the players, players can take advantage of the rotational component of a vector field to converge at a faster rate. We simulate (14) from $(1, 1, 1, 1)$ and show that for $\varepsilon = 0.9$, the system converges fastest with $\tau = 28$ as shown in Fig. 5(a). Timescale separation also warps the vector field of potential-like and rotation-like systems in qualitatively differently, shown in Fig. 1. For $\tau > 0$, the spectral radius



(a) The rotational system (blue) with timescale separation (right) achieves the fastest convergence by taking advantage of the rotational vector field.



(b) The spectral radius of $I + \Gamma_\tau J(z)$ for the discrete-time update and the eigenvalues of $\Lambda_\tau J(z)$ for the continuous-time system $\dot{z} = \Lambda_\tau g(z)$ at equilibrium $z = (x, y) = 0$ for increasing learning rate ratio $\tau > 0$.

Fig. 5. *Faster convergence of rotational learning dynamics with time-scale separation.* Time-scale separation can be used to speed up convergence of systems with mostly rotational components, shown in (a). The spectral radius of the discrete-time update and the eigenvalues of the corresponding continuous-time system are plotted in (b), showing that at $\tau = 28$, the mostly-rotational system achieves fastest convergence because it is able to take advantage of the imaginary components of the eigenvalues to achieve a smaller spectral radius.

of the discrete-time update for base learning rate $\gamma = 10^{-3}$ is shown in Fig. 5(b). It achieves a minimum at $\tau = 28$ for the mostly-rotational system. The eigenvalues of the corresponding continuous-time system is plotted in Fig. 5(b). This example shows that timescale separation can be used to speed up convergence of learning dynamics by taking advantage of the imaginary components of the eigenvalues of the linearized Jacobian near the equilibrium.

VII. Conclusion

We provide a comprehensive characterization of the local stability and Nash optimality for fixed points of two-player gradient learning dynamics. We assess the contribution of the interaction terms of the game Jacobian in stabilizing a non-Nash equilibrium or destabilizing a Nash. Such results give valuable insights into the interaction of algorithms in settings most accurately modeled as games.

In the numerical example, we demonstrate that there is an important trade-off between the rotational component of the learning dynamics and timescale that each player learns at: timescale separation can be introduced to learning rules to improve convergence when the vector field has enough rotational component. As a future direction, we will look at how to optimize convergence speed given the strength of the anti-symmetric component of the game.

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