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- I declare that this final submitted version is my unaided work.

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C499 Modal Logic for Strategic Reasoning in AI Coursework 2

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1.

(a)

$\pi \vDash \varphi R \psi$ iff

$\forall j \geq 0. \pi[j..\infty] \vDash \psi$ or $\exists i \geq 0 \left((\pi[i..\infty] \vDash \varphi) \wedge \forall\, 0 \leq k \leq i\, (\pi[k..\infty] \vDash \psi)\right)$

(b)

$\varphi R \psi \equiv \neg\left((\neg\varphi) U (\neg\psi)\right)$

(c)

Truth conditions of the LTL formula in (b), which is $\neg\left((\neg\varphi) U (\neg\psi)\right)$

iff (definition of R)

$\neg\exists j \geq 0. \left((\pi[j..\infty] \vDash \neg\psi) \wedge \forall 0 \leq i < j. (\pi[i..\infty] \vDash \neg\varphi)\right)$

iff (semantics of negation)

$\neg\exists j \geq 0. \left((\pi[j..\infty] \nvDash \psi) \wedge \forall 0 \leq i < j. (\pi[i..\infty] \nvDash \varphi)\right)$

iff (duality of $\exists$ and $\forall$)

$\forall j \geq 0. \neg\left((\pi[j..\infty] \nvDash \psi) \wedge \forall 0 \leq i < j. (\pi[i..\infty] \nvDash \varphi)\right)$

iff (de Morgan's law)

$\forall j \geq 0. \left(\neg(\pi[j..\infty] \nvDash \psi) \vee \neg\forall 0 \leq i < j. (\pi[i..\infty] \nvDash \varphi)\right)$

iff (semantics of negation)

$\forall j \geq 0. (\pi[j..\infty] \vDash \psi \vee \exists 0 \leq i < j. \pi[i..\infty] \vDash \varphi)$

iff $\forall j \geq 0. \pi[j..\infty] \vDash \psi$ or $\exists i \geq 0. \left((\pi[i..\infty] \vDash \varphi) \wedge \forall\, 0 \leq k \leq i. (\pi[k..\infty] \vDash \psi)\right)$

iff Truth conditions provided in (a)

(d)

$\perp R\psi$

iff $(\forall i \geq 0, \pi[i..\infty] \vDash \psi)\ \vee\ \exists i \geq 0, \left(\pi[i..\infty] \vDash \perp \wedge (\forall 0 \leq j \leq i, \pi[j..\infty] \vDash \psi)\right)$

iff $(\forall i \geq 0, \pi[i..\infty] \vDash \psi)\ \vee\ \exists i \geq 0, \left(\perp \wedge (\forall 0 \leq j \leq i, \pi[j..\infty] \vDash \psi)\right)$

iff $(\forall i \geq 0, \pi[i..\infty] \vDash \psi)\ \vee\ \perp$

iff $(\forall i \geq 0, \pi[i..\infty] \vDash \psi)$

iff $G\psi$

2.

$(M, q) \vDash EF\Phi$

iff $(M, q) \vDash E(trueU\Phi)$

iff for some path λ from q, $(M, \lambda) \vDash (trueU\Phi)$

iff for some path λ from q, for some $j \geq 0$, $(M, \lambda[j]) \vDash \Phi$ and $(M, \lambda[k]) \vDash \top$ for all $0 \leq k < j$

iff for some path λ from q, for some $j \geq 0$, $(M, \lambda[j]) \vDash \Phi$

$(M, q) \vDash AF\Phi$

iff $(M, q) \vDash A(trueU\Phi)$

iff for every path λ from q, $(M, \lambda) \vDash (trueU\Phi)$

iff for every path λ from q, for some $j \geq 0$, $(M, \lambda[j]) \vDash \Phi$ and $(M, \lambda[k]) \vDash \top$ for all $0 \leq k < j$

iff for every path λ from q, for some $j \geq 0$, $(M, \lambda[j]) \vDash \Phi$

$(M, q) \vDash EG\Phi$

iff $(M, q) \vDash \neg AF\neg\Phi$

iff It's not the case that, for all path λ from q, $(M, \lambda) \vDash F\neg\Phi$

iff for some path λ from q, it's not the case that, $(trueU\neg\Phi)$

iff for some path λ from q, it's not the case that, for some $j \geq 0$, $(M, \lambda[j]) \vDash \neg\Phi$
and $(M, \lambda[k]) \vDash \top$, for all $0 \leq k < j$

iff for some path λ from q, for all $j \geq 0$, $(M, \lambda[j]) \vDash \Phi$

$(M, q) \vDash AG\Phi$

iff $(M, q) \vDash \neg EF\neg\Phi$

iff It's not the case that, for some path λ from q, $(M, \lambda) \vDash F\neg\Phi$

iff for every path λ from q, it's not the case that, $(trueU\neg\Phi)$

iff for every path λ from q, it's not the case that, for some $j \geq 0$, $(M, \lambda[j]) \vDash \neg\Phi$
and $(M, \lambda[k]) \vDash \top$, for all $0 \leq k < j$

iff for every path λ from q, for all $j \geq 0$, $(M, \lambda[j]) \vDash \Phi$

3.

(a)

Temporal logic on infinite trees [Clarke & Emerson 1981]

- State formulas Φ, Ψ:

(1) - $a \in AP$ atoms

(2) - $\neg\Phi$ negation

(3) - $\Phi \wedge \Psi$ conjunction

(4) - $E\phi$ for some path ϕ is true

(5) - $A\phi$ for every path ϕ is true

- Path formulas ϕ:

(6) - $X\Phi$ *neXt* Φ

(7) - $\Phi U \Psi$ Φ *Until* Ψ

Definition 1 (Syntax of CTL*) *State (Φ) and path (ψ) formulas in CTL* are defined in Backus-Naur form as follows, where p is an atom:*

$$\begin{array}{lcl} \Phi & ::= & \overset{(1)}{p} \mid \overset{(2)}{\neg\Phi} \mid \overset{(3)}{\Phi \wedge \Phi} \mid \overset{(4)}{E\psi} \mid \overset{(5)}{A\psi} \\ \psi & ::= & \underset{(6)}{\Phi} \mid \underset{(7)}{\neg\psi} \mid \underset{(8)}{\psi \wedge \psi} \mid \underset{(9)}{X\psi} \mid \underset{(10)}{\psi U \psi} \end{array}$$

The formulas of CTL are all and only the* state *formulas.*

We need to show that

For every formula Φ, if Φ is a formula of CTL, then Φ is also a formula in CTL*

Proof by induction

Base case:

For a formula $\Phi = p, p \in AP$,

if Φ is a formula of CTL as condition (1) of CTL syntax,

then Φ is also a formula of CTL* since condition (1) of CTL* syntax.

Inductive cases:

Let state formula Ψ is a CTL and CLT* formula.

For a formula $\Phi = \neg\Psi$,

if Φ is a formula of CTL as condition (2) of CTL syntax,

then Φ is also a formula of CTL* since condition (2) of CTL* syntax.

Let both state formulas $\Psi 1$ and $\Psi 2$ are CTL and CLT* formulas.

For a formula $\Phi = \Psi 1 \wedge \Psi 2$,

if Φ is a formula of CTL as condition (3) of CTL syntax,

then Φ is also a formula of CTL* since condition (3) of CTL* syntax.

Let γ be a path formula, for a formula $\Phi = E\gamma$,

if Φ is a formula of CTL as condition (4) of CTL syntax,

it means that γ can only be $X\,\Psi 1$ or $\Psi 1\, U\, \Psi 2$ where both $\Psi 1$ and $\Psi 2$ are state formulas as conditions (6) and (7) of CTL syntax,

then Φ is also a formula of CTL* since conditions (4), (9) and (10) of CTL* syntax.

Let γ be a path formula, for a formula $\Phi = A\gamma$,

if Φ is a formula of CTL as condition (5) of CTL syntax,

it means that γ can only be $X\,\Psi 1$ or $\Psi 1\, U\, \Psi 2$ where both $\Psi 1$ and $\Psi 2$ are state formulas as conditions (6) and (7) of CTL syntax,

then Φ is also a formula of CTL* since conditions (5), (9) and (10) of CTL* syntax.

In conclusion, for every formula Φ, if Φ is a formula of CTL, then Φ is also a formula in CTL*

(b)

Let $\Phi = A(Xp \wedge Xq)$

Φ is a CTL* formula, since $\Phi = A\psi$, $\psi = Xp \wedge Xq$ and condition (5) of CTL* syntax,
since $\psi = Xp \wedge Xq$, $\varphi_1 = Xp$, $\varphi_2 = Xp$ and condition (9) of CTL* syntax,
since $\varphi_1 = \Psi_1$, $\Psi_1 = p$ and conditions (6) and (1) of CTL* syntax, and
$\varphi_2 = \Psi_2$, $\Psi_2 = q$ and conditions (6) and (1) of CTL* syntax.

Φ is NOT a CTL formula, since $\Phi = A\psi$, $\psi = Xp \wedge Xq$ and condition (5) of CTL syntax,
since $(Xp \wedge Xq)$ is NOT a path formula of CTL syntax.

NOTE:

There exists some formula Φ' in CTL* such that $\Phi = A(Xp \wedge Xq)$ and $\Phi' = AX(p \wedge q)$ are equivalent, BUT $\Phi = A(Xp \wedge Xq)$ itself is NOT a CTL formula.

4.

Definition 2 (Semantics of CTL*) *Let M be a model, s a state, π a path, Φ and Φ' state formulas, and ψ' path formulas. Then,*

(1)	$(M,s) \models p$	*iff*	$s \in V(p)$
(2)	$(M,s) \models \neg\Phi$	*iff*	$(M,s) \not\models \Phi$
(3)	$(M,s) \models \Phi \wedge \Phi'$	*iff*	$(M,s) \models \Phi$ *and* $(M,s) \models \Phi'$
(4)	$(M,s) \models E\psi$	*iff*	*for some path* π *starting from* s, $(M,\pi) \models \psi$
(5)	$(M,s) \models A\psi$	*iff*	*for all paths* π *starting from* s, $(M,\pi) \models \psi$
(6)	$(M,\pi) \models \Phi$	*iff*	$(M,\pi[0]) \models \Phi$, *where* $\pi[0]$ *is the initial state in path* π.
(7)	$(M,\pi) \models \neg\psi$	*iff*	$(M,\pi) \not\models \psi$
(8)	$(M,\pi) \models \psi \wedge \psi'$	*iff*	$(M,\pi) \models \psi$ *and* $(M,\pi) \models \psi'$
(9)	$(M,\pi) \models X\psi$	*iff*	$(M,\pi[1\ldots\infty]) \models \psi$
(10)	$(M,\pi) \models \psi U \psi'$	*iff*	$(M,\pi[i\ldots\infty]) \models \psi'$ *for some* $i \geq 0$, *and* $(M,\pi[j\ldots\infty]) \models \psi$ *for all* $0 \leq j < i$

Slide from Lecture 5:

Let

- $M = \langle St, \longrightarrow, V\rangle$ be a model defined on a transition system
- Φ, Ψ be state formulas
- γ be a path formula.

Definition 1.7 (Semantics of CTL: state formulas)

(1)	$(M,q) \models a$	iff	$q \in V(a)$
(2)	$(M,q) \models \neg\Phi$	iff	$(M,q) \not\models \Phi$
(3)	$(M,q) \models \Phi \wedge \Psi$	iff	$(M,q) \models \Phi$ and $(M,q) \models \Psi$
(4)	$(M,q) \models \mathrm{E}\gamma$	iff	for *some* path λ starting from q, $(M,\lambda) \models \gamma$
(5)	$(M,q) \models \mathrm{A}\gamma$	iff	for *all* paths λ starting from q, $(M,\lambda) \models \gamma$

Definition 1.8 (Semantics of CTL: path formulas)

(6)	$(M,\lambda) \models \mathrm{X}\Phi$	iff	$(M,\lambda[1]) \models \Phi$
(7)	$(M,\lambda) \models \Phi\mathrm{U}\Psi$	iff	$(M,\lambda[i]) \models \Psi$ for some $i \geq 0$, and $(M,\lambda[j]) \models \Phi$ for all $0 \leq j < i$

By restricting Def. 2 to formulas in CTL, we **freeze** the conditions (6), (7) and (8). We need to show that the Def. 2 after freezing will obtain the same truth conditions as the semantics of CTL.

After restricting, a user can only call the conditions (1)-(5) directly to build a CTL formula, which are the entailment from a **state** s, but NOT allowed to call the conditions (6)-(10) directly, which are the entailment from a **path** π, hence CTL only can be applied on a state.

For **conditions (1), (2) and (3)** in both CTL and CLT* semantics, these pairs are exactly same. Hence, after restricting, the conditions (1), (2) and (3) of CTL* semantics will obtain the same truth conditions as conditions (1), (2) and (3) semantics of CTL.

For **condition** (4) in CTL* semantics:

$(M, s) \vDash E\varphi$ in CTL* semantics

iff for some path π starting from s, $(M, \pi) \vDash \varphi$ in CTL* semantics

iff for some path π starting from s, $(M, \pi) \vDash X\psi$ in CTL* semantics or $(M, \pi) \vDash \psi U \psi'$ in CTL* semantics

iff for some path π starting from s, $(M, \pi[1 \cdots \infty]) \vDash \psi$ or $(M, \pi[i \cdots \infty]) \vDash \psi'$ for some $i \geq 0$, and $(M, \pi[j \cdots \infty]) \vDash \psi$ for all $0 \leq j < i$

(at following step, we applied the condition (6) of CTL semantics)*

iff for some path π starting from s,
$(M, \pi[1]) \vDash \Phi$, where $\pi[1]$ is the initial state of path $\pi[1 \cdots \infty]$ and $\Phi = \psi$ or $(M, \pi[i]) \vDash \Phi'$ and $\Phi' = \psi'$ for some $i \geq 0$, and $(M, \pi[j]) \vDash \Phi$ and $\Phi = \psi$ for all $0 \leq j < i$

iff for some path π starting from s,
$(M, \pi[1]) \vDash X\Phi$ in CTL semantics such that state formula $\Phi = \psi$ or
$(M, \pi[i]) \vDash \Phi U \Phi'$ in CTL semantics such that state formula $\Phi = \psi$ and $\Phi' = \psi'$.

iff for some path π starting from s, $(M, \pi) \vDash \varphi$ in CTL semantics.

iff $(M, \pi) \vDash E\varphi$ in CTL semantics.

For **condition** (5) in CTL* semantics:

$(M, s) \vDash A\varphi$ in CTL* semantics

iff for all path π starting from s, $(M, \pi) \vDash \varphi$ in CTL* semantics

iff for all path π starting from s, $(M, \pi) \vDash X\psi$ in CTL* semantics or $(M, \pi) \vDash \psi U \psi'$ in CTL* semantics

iff for all path π starting from s, $(M, \pi[1 \cdots \infty]) \vDash \psi$ or $(M, \pi[i \cdots \infty]) \vDash \psi'$ for some $i \geq 0$, and $(M, \pi[j \cdots \infty]) \vDash \psi$ for all $0 \leq j < i$

(at following step, we applied the condition (6) of CTL semantics)*

iff for all path π starting from s,
$(M, \pi[1]) \vDash \Phi$, where $\pi[1]$ is the initial state of path $\pi[1 \cdots \infty]$ and $\Phi = \psi$ or $(M, \pi[i]) \vDash \Phi'$ and $\Phi' = \psi'$ for some $i \geq 0$, and $(M, \pi[j]) \vDash \Phi$ and $\Phi = \psi$ for all $0 \leq j < i$

iff for all path π starting from s,
$(M, \pi[1]) \vDash \Phi\psi$ in CTL semantics such that state formula $\Phi = \psi$ or
$(M, \pi[i]) \vDash \Phi U \Phi'$ in CTL semantics such that state formula $\Phi = \psi$ and $\Phi' = \psi'$.

iff for all path π starting from s, $(M, \pi) \vDash \varphi$ in CTL semantics.

iff $(M, \pi) \vDash A\varphi$ in CTL semantics.

In conclusion, the Def. 2 after freezing will obtain the same truth conditions as in Def. 1.7 and 1.8 in Lecture 5 (semantics of CTL).

5.

(a)

Proof by induction

Base case:

For condition (1) in CTL semantics:

$(M, s) \vDash p,\ p \in AP$, w.r.t. CTL semantics

iff $s \in N(p)$

$(M, s) \vDash p,\ p \in AP$, w.r.t. CTL* semantics

Inductive cases:

For condition (2) in CTL semantics:

Let Φ be a state formula of CTL, by the conclusion of question (3), Φ is also a CTL* state formula.

$(M, s) \vDash \neg\Phi$, w.r.t. CTL semantics

iff $(M, s) \nvDash \Phi$

$(M, s) \vDash \neg\Phi$, w.r.t. CTL* semantics

For condition (3) in CTL semantics:

Let Φ and Φ' be state formulas of CTL, by the conclusion of question (3), Φ and Φ' are also CTL* state formulas.

$(M, s) \vDash \Phi \wedge \Phi'$, w.r.t. CTL semantics

iff $(M, s) \vDash \Phi$ and $(M, s) \vDash \Phi'$

$(M, s) \vDash \Phi \wedge \Phi'$, w.r.t. CTL* semantics

For condition (4) in CTL semantics:

$(M, \pi) \vDash E\varphi$ w.r.t. CTL semantics.

iff for some path π starting from s, $(M, \pi) \vDash \varphi$ w.r.t. CTL semantics.

iff for some path π starting from s,

$(M, \pi[1]) \vDash X\Phi$ w.r.t. CTL semantics such that state formula $\Phi = \psi$ or

$(M, \pi[i]) \vDash \Phi U \Phi'$ w.r.t. CTL semantics such that state formula $\Phi = \psi$ and $\Phi' = \psi'$.

iff for some path π starting from s,

$(M, \pi[1]) \vDash \Phi$, where $\pi[1]$ is the initial state of path $\pi[1 \cdots \infty]$ and $\Phi = \psi$ or

$(M, \pi[i]) \vDash \Phi'$ and $\Phi' = \psi'$ for some $i \geq 0$, and $(M, \pi[j]) \vDash \Phi$ and

iff for some path π starting from s, $(M, \pi[1 \cdots \infty]) \vDash \psi$ or

$(M, \pi[i \cdots \infty]) \vDash \psi'$ for some $i \geq 0$, and

$(M, \pi[j \cdots \infty]) \vDash \psi$ for all $0 \leq j < i$

iff for some path π starting from s, $(M, \pi) \vDash X\psi$ w.r.t. CTL* semantics or

$(M, \pi) \vDash \psi U \psi'$ w.r.t. CTL* semantics

iff for some path π starting from s, $(M, \pi) \vDash \varphi$ w.r.t. CTL* semantics

$\Phi = \psi$ for all $0 \leq j < i$

iff $(M, s) \vDash E\varphi$, w.r.t CTL semantics

For **condition** (5) in CTL semantics:

$(M, \pi) \vDash A\varphi$ w.r.t. CTL semantics.

iff for all path π starting from s, $(M, \pi) \vDash \varphi$ w.r.t. CTL semantics.

iff for all path π starting from s,
$(M, \pi[1]) \vDash X\Phi$ w.r.t. CTL semantics such that state formula $\Phi = \psi$ or
$(M, \pi[i]) \vDash \Phi U \Phi'$ w.r.t. CTL semantics such that state formula $\Phi = \psi$ and $\Phi' = \psi'$.

iff for all path π starting from s,
$(M, \pi[1]) \vDash \Phi$, where $\pi[1]$ is the initial state of path $\pi[1 \cdots \infty]$ and $\Phi = \psi$ or
$(M, \pi[i]) \vDash \Phi'$ and $\Phi' = \psi'$ for some $i \geq 0$, and $(M, \pi[j]) \vDash \Phi$ and

iff for all path π starting from s, $(M, \pi[1 \cdots \infty]) \vDash \psi$ or
$(M, \pi[i \cdots \infty]) \vDash \psi'$ for some $i \geq 0$, and
$(M, \pi[j \cdots \infty]) \vDash \psi$ for all $0 \leq j < i$

iff for all path π starting from s, $(M, \pi) \vDash X\psi$ w.r.t. CTL* semantics or
$(M, \pi) \vDash \psi U \psi'$ w.r.t. CTL* semantics

iff for all path π starting from s, $(M, \pi) \vDash \varphi$ w.r.t. CTL* semantics
$\Phi = \psi$ for all $0 \leq j < i$

iff $(M, s) \vDash A\varphi$, w.r.t CTL semantics

In conclusion, for every formula Φ of CTL, there exists some formula Φ' in CTL* such that Φ and Φ' are equivalent.

(b)

We can embed LTL in CTL* by making the implicit universal path quantifiers of LTL explicit. Given a state s we have:

$$s \vDash_{LTL} \varphi \Leftrightarrow s \vDash_{CTL*} A\varphi$$

Suppose φ is LTL formula $\boldsymbol{FGp}$, the equivalent CTL* formula is $\boldsymbol{AFGp}$.
As we proved above, CTL is a fragment of CTL*. However, there is no way to express LTL formula $\boldsymbol{FGp}$ in CTL, hence, CTL* formula $\boldsymbol{AFGp}$ cannot be expressed in CTL.

In conclusion, there exists some formula $\Phi = \boldsymbol{AFGp}$ in CTL* for which there exists no equivalent formula Φ' in CTL

32 **Out of** **49**

1			
a/**2**	b/**2**	c/**3**	d/**3**
			Solution correct, but the removal of the false statement (though intuitive) is not provided. I've given the mark due to the strength of the rest of the explanation and because the removal of false is obvious, but it would be better to explain this next time
2	2	3	3

2			
a/**2**	b/**2**	c/**2**	d/**2**
2	2	2	2

3	
a/**3**	b/**2**
3	2

4
/5
5

5	
a/**2**	b/**2**
2	2

6	7	8
/6	**/6**	**/5**
0	0	0