AM III

TUTORIAL 01

LAPLACE TRANSFORM

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1) . $f(t) = \int_0^t u e^{-3u} . \sin^2 u \, du$

$$L\{f(t)\} = L . \left\{ \int_0^t u . e^{-3u} . \sin^2 u \, du \right\}$$

$$= \frac{1}{s} L . \left\{ u . e^{-3u} \sin^2 u \right\}$$ .... transform of integral of f(t)

now, $L\{\sin^2 u\} = L \left\{ \frac{1}{2} (1 - \cos 2u) \right\}$

$$= \frac{1}{2} \left[\frac{1}{s} - \frac{s}{s^2+4} \right]$$ ... $L\{1\} = 1/s$

now, $L\{u . \sin^2 u\} = -\frac{1}{2} \frac{d}{ds} \left[\frac{1}{s} - \frac{s}{s^2+4} \right]$

$$= -\frac{1}{2} \left[\frac{-1}{s^2} - \left[\frac{(s^2+4) - s(2s)}{(s^2+4)^2} \right] \right]$$

$$= \frac{1}{2} \left[\frac{+1}{s^2} + \frac{4 - s^2}{(s^2+4)^2} \right]$$

now, $L\{e^{-3u} u \sin^2 u\} = \frac{1}{2} \left[\frac{+1}{(s+3)^2} + \frac{4 - (s+3)^2}{[(s+3)^2+4]^2} \right]$

now $\boxed{L\{f(t)\} = \frac{1}{2s} \left[\frac{+1}{(s+3)^2} + \frac{4 - (s+3)^2}{[(s+3)^2+4]^2} \right]}$

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2) • $f(t) = \dfrac{\cos a\sqrt{t}}{\sqrt{t}} = \dfrac{\cos\sqrt{a^2 t}}{\sqrt{t}} = a \cdot \dfrac{\cos\sqrt{a^2 t}}{\sqrt{a^2 t}}$

Applying chang of scale property

$$L\left\{\frac{\cos a\sqrt{t}}{\sqrt{t}}\right\} = a \cdot \frac{1}{a^2}\, L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\}\Bigg|_{s = s/a^2}$$

now, we know, $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dots$ (cosine series)

$$\therefore \frac{\cos t^{1/2}}{t^{1/2}} = \frac{1}{t^{1/2}}\left[1 - \frac{t}{2!} + \frac{t^2}{4!} - \dots\right]$$

$$= t^{-1/2} - \frac{t^{1/2}}{2!} + \frac{t^{3/2}}{4!} - \dots$$

$$\therefore L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \frac{\Gamma(1/2)}{s^{1/2}} - \frac{1}{2!}\frac{\Gamma(3/2)}{s^{3/2}} + \frac{1}{4!}\cdot\frac{\Gamma(5/2)}{s^{5/2}} - \frac{1}{6!}\frac{\Gamma(7/2)}{s^{7/2}} + \dots \quad (\Gamma(n+1) = n\Gamma(n))$$

$$= \frac{\Gamma(1/2)}{s^{1/2}}\left[1 - \frac{1}{2!}\cdot\frac{1}{2}\cdot\frac{1}{s} + \frac{1}{4!}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\frac{1}{s^2} - \frac{1}{6!}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\frac{1}{s^3}\dots\right]$$

$$= \sqrt{\frac{\pi}{s}}\left[1 - \frac{1}{4s} + \frac{1}{2}\,\frac{1}{(4s)^2} - \frac{1}{6}\,\frac{1}{(4s)^3} + \dots\right]$$

$$= \sqrt{\frac{\pi}{s}} \cdot e^{-1/4s}$$

$$\therefore L\{f(t)\} = \frac{1}{a}\sqrt{\frac{\pi}{(s/a^2)}} \cdot e^{-1/4(s/a^2)}$$

$$\therefore \boxed{L\{f(t)\} = \sqrt{\frac{\pi}{s}} \cdot e^{-\frac{a}{4s}}}$$

3). ~~f(t)~~ $I = \int_0^\infty \frac{e^{-at} - e^{-bt}}{t}\, dt$

$= \int_0^\infty \frac{e^{-at}}{t}\, dt - \int_0^\infty \frac{e^{-bt}}{t}\, dt$

$= \int_{s=a}^\infty L\left\{\int_0^\infty e^{-at}\, dt\right\} - \int_{s=b}^\infty L\left\{\int_0^\infty \frac{e^{-bt}}{t}\, dt\right\}$ .... $\int_0^\infty e^{-at} f(t)\, dt = L\{f(t)\}|_{s=a}$

$= \int_{s=a}^\infty L\{1\}|_{s=a} - \int_{s=b}^\infty L\{1\}|_{s=b}$

$= \int_{s=a}^\infty \frac{1}{a} - \int_{s=b}^\infty \frac{1}{b}$

$= [\log a]_a^\infty - [\log b]_b^\infty$

$= -\log a + \log b$

$= \log\left(\frac{b}{a}\right)$

$\therefore \boxed{\int_0^\infty \frac{e^{-at} - e^{-bt}}{t}\, dt = \log\left(\frac{b}{a}\right)}$

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4) • $I = \int_0^\infty e^{-2t} \cos^5 t \, dt$

$L\{\cos^5 t\}$:-

let $x = e^{it}$, $x^{-1} = e^{-it}$

$x + x^{-1} = 2\cos t$

$2^5 \cos^5 t = (x + x^{-1})^5$

$= {}^5C_0\, x^5 + {}^5C_1\, x^3 + {}^5C_2\, x + {}^5C_3\, x^{-1} + {}^5C_4\, x^{-3} + {}^5C_5\, x^{-5}$

$= (x^5 + x^{-5}) + 5(x^3 + x^{-3}) + 10(x + x^{-1})$

$= 2\cos 5t + 5 \cdot 2\cos 3t + 10 \cdot 2\cos t$

$\therefore \cos^5 t = \frac{1}{2^4}\left[\cos 5t + 5\cos 3t + 10\cos t\right]$ ....

$L\{\cos^5 t\} = \frac{1}{2^4}\left[\frac{s}{s^2+25} + \frac{5s}{s^2+9} + \frac{10s}{s^2+1}\right]$ .... $L\{\cos at\} = \frac{s}{s^2+a^2}$

$\int_0^\infty e^{-2t} \cos^5 t \, dt = L\{\cos^5 t\}\Big|_{s=2}$

$I = \frac{1}{2^4}\left[\frac{2}{29} + \frac{10}{13} + \frac{20}{5}\right]$

$\therefore \boxed{I = \frac{114}{377}}$

Teacher's Sign.: ____________

5). $f(t) = \begin{cases} 2t & 0 < t < 1 \\ 3t^2 - 1 & t > 1 \end{cases}$

$f(t) = 2t\,[1 - H(t-1)]$
$+ 3t^2 - 1\,[H(t-1) - 0]$

$= 2t + [H(t-1)[-2t + 3t^2 - 1]]$ ... using second shifting property

$L\{2t\} = \frac{2}{s^2}$ .... $L\{t\} = 1/s^2$

$L\{H(t-1)[-2t + 3t^2 - 1]\}$

$= e^{-s} L\{-2(t+1) + 3(t+1)^2 - 1\}$

$= e^{-s} . L\{-2t - 2 + 3t^2 + 6t + 3 - 1\}$

$= e^{-s} L\{3t^2 + 4t\}$

$= e^{-s} . \left[\frac{6}{s^3} + \frac{4}{s^2}\right]$ .... $L\{t^2\} = 2/s^3$

$\therefore \boxed{L\{f(t)\} = \frac{2}{s^2} + e^{-s}\left[\frac{6}{s^3} + \frac{4}{s^2}\right]}$