Inverse Using convolution Theorem

① $\dfrac{s}{(s^2+a^2)^2}$

Solⁿ $F(s) = \dfrac{s}{(s^2+a^2)^2}$

$= \dfrac{s}{(s^2+a^2)} \times \dfrac{1}{(s^2+a^2)}$

$L^{-1}\left\{\dfrac{s}{s^2+a^2}\right\} = \cos at$

$L^{-1}\left\{\dfrac{1}{s^2+a^2}\right\} = \dfrac{1}{a}\sin at$

$\therefore L^{-1}\{F(s)\} = \text{conv}\left(\cos at, \dfrac{1}{a}\sin at\right)$

$= \int_0^t \cos au \, \dfrac{1}{a} \sin a(t-u) \, du$

$= \dfrac{1}{a}\int_0^t \cos au \sin(at-au) \, du$

$= \dfrac{1}{a}\int_0^t \dfrac{1}{2}\left(\sin at - \sin(2au-at)\right) du$

$= \dfrac{1}{2a}\left[u \sin at + \dfrac{\cos(2au-at)}{2a}\right]_0^t$

$= \dfrac{1}{2a}\left[t \sin at + \dfrac{\cos at}{2a} - \dfrac{\cos at}{2a}\right]$

$= \dfrac{1}{2a} t \sin at$

(2) $$\frac{1}{(s^2+4s+13)^2}$$

Sol<sup></sup>n

Solⁿ

$$F(s) = \frac{1}{(s^2+4s+13)^2}$$

$$= \frac{1}{((s+2)^2+9)^2}$$

$$L^{-1}\{F(s)\} = e^{-2t} L^{-1}\left\{\frac{1}{(s^2+9)^2}\right\} \quad \text{--- (1)}$$

$$\frac{1}{(s^2+9)^2} = \frac{1}{(s^2+9)} \times \frac{1}{(s^2+9)}$$

$$L^{-1}\left\{\frac{1}{s^2+9}\right\} = \frac{1}{3}\sin 3t$$

$$L^{-1}\left\{\frac{1}{(s^2+9)^2}\right\} = \text{conv}\left(\frac{1}{3}\sin 3t, \frac{1}{3}\sin 3t\right)$$

$$= \int_0^t \frac{1}{3}\sin 3u \cdot \frac{1}{3}\sin 3(t-u)\, du$$

$$= \frac{1}{9}\int_0^t \sin 3u \sin(3t-3u)\, du$$

$$= \frac{1}{9}\int_0^t \frac{1}{2}(\cos(6u-3t) - \cos 3t)\, du$$

$$= \frac{1}{18}\left[\frac{\sin(6u-3t)}{6} - u\cos 3t\right]_0^t$$

$$= \frac{1}{18}\left[\frac{\sin 3t}{6} - t\cos 3t - \frac{\sin(-3t)}{6}\right]$$

$$= \frac{1}{18}\left[\frac{1}{3}\sin 3t - t\cos 3t\right]$$

$$\therefore L^{-1}\{F(s)\} = \frac{e^{-2t}}{18}\left[\frac{1}{3}\sin 3t - t\cos 3t\right]$$

(3) $\left(\frac{s+2}{s^2+4s+8}\right)^2$

<u>Sol^n</u> $F(s) = \left(\frac{s+2}{s^2+4s+8}\right)^2$

$$= \frac{(s+2)^2}{(s^2+4s+8)^2}$$

$$= \frac{(s+2)^2}{((s+2)^2+4)^2}$$

$$L^{-1}\{F(s)\} = e^{-2t} L^{-1}\left\{\frac{s^2}{(s^2+4)^2}\right\}$$

$$\frac{s^2}{(s^2+4)^2} = \frac{s}{(s^2+4)} \times \frac{s}{(s^2+4)}$$

$$L^{-1}\left\{\frac{s}{s^2+4}\right\} = \cos 2t$$

$$\therefore L^{-1}\left\{\frac{s^2}{(s^2+2^2)^2}\right\}$$

$$= \text{conv}(\cos 2t, \cos 2t)$$

$$= \int_0^t \cos 2u \cos 2(t-u)\, du$$

$$= \int_0^t \cos 2u \cos(2t-2u)\, du$$

$$= \int_0^t \frac{1}{2}\left[\cos 2t + \cos(4u-2t)\right] du$$

$$= \frac{1}{2}\left[u \cos 2t + \frac{\sin(4u-2t)}{4}\right]_0^t$$

$$= \frac{1}{2}\left[t \cos 2t + \frac{1}{4}\sin 2t - \frac{1}{4}\sin(-2t)\right]$$

$$= \frac{1}{2}\left[t \cos 2t + \frac{1}{2}\sin 2t\right]$$

$$\therefore L^{-1}\{F(s)\} = \frac{e^{-2t}}{2}\left[t \cos 2t + \frac{1}{2}\sin 2t\right]$$

4) $\dfrac{1}{(s+2)^2\,(s+3)^2}$

Soln: $F(s) = \dfrac{1}{(s+2)^2\,(s+3)^2}$

$= \dfrac{1}{(s+2)^2} \times \dfrac{1}{(s+3)^2}$

$L^{-1}\left\{\dfrac{1}{(s+2)^2}\right\} = e^{-2t}\,t$

$L^{-1}\left\{\dfrac{1}{(s-3)^2}\right\} = e^{-3t}\,t$

| | |
|---|---|
| $ut-u^2$ | $e^u$ |
| $t-2u$ | $\searrow e^u$ |
| $-2$ | $\searrow e^u$ |
| $0$ | $\searrow e^u$ |

$\therefore\ L^{-1}\{F(s)\} = \text{conv}(e^{-2t}t,\ e^{-3t}t)$

$= \int_0^t e^{-2u}.u \cdot e^{-3(t-u)}(t-u)\,du$

$= e^{-3t}\int_0^t e^{u}\,(ut-u^2)\,du$

$= e^{-3t}\left[(ut-u^2)e^u - (t-2u)e^u - 2e^u\right]_0^t$

$= e^{-3t}\left[t e^t - 2e^t + t + 2\right]$

$= e^{-2t}(t-2) + e^{-3t}(t+2)$

(5) $F(s) = \dfrac{1}{(s-5)\left((s-3)^2+8\right)}$

Sol$^n$ $F(s) = \dfrac{1}{(s-3-2)\left((s-3)^2+8\right)}$

$L^{-1}\{F(s)\} = e^{3t} L^{-1}\left\{\dfrac{1}{(s-2)(s^2+8)}\right\}$

$\dfrac{1}{(s-2)(s^2+8)} = \dfrac{1}{(s-2)} \times \dfrac{1}{(s^2+8)}$

$L^{-1}\left\{\dfrac{1}{(s-2)}\right\} = e^{2t}$

$L^{-1}\left\{\dfrac{1}{s^2+8}\right\} = \dfrac{1}{\sqrt{8}} \sin\sqrt{8}\,t$

$\therefore L^{-1}\left\{\dfrac{1}{(s-2)(s^2+8)}\right\} = \text{conv}\left(e^{2t}, \dfrac{1}{\sqrt{8}}\sin\sqrt{8}\,t\right)$

$= \int_0^t e^{2(t-u)} \cdot \dfrac{1}{\sqrt{8}} \sin\sqrt{8}\,u \; du$

$= \dfrac{e^{2t}}{\sqrt{8}} \int_0^t e^{-2u} \sin\sqrt{8}\,u \; du$

$= \dfrac{e^{2t}}{\sqrt{8}} \left. \dfrac{e^{-2u}}{(4+8)} \left[-2\sin\sqrt{8}u - \sqrt{8}\cos\sqrt{8}u\right] \right|_0^t$

$= \dfrac{e^{2t}}{\sqrt{8}\;12}\left[e^{-2t}\left(-2\sin\sqrt{8}t - \sqrt{8}\cos\sqrt{8}t\right) - \left[-\sqrt{8}\right]\right]$

$$= \frac{1}{12\sqrt{8}}\left[-2\sin\sqrt{8}t - \sqrt{8}\cos\sqrt{8}t\right]$$

$$+ \frac{e^{2t}}{12}$$

$$\therefore L^{-1}\{F(s)\} = \frac{e^{3t}}{12\sqrt{8}}\left[-2\sin\sqrt{8}t - \sqrt{8}\cos\sqrt{8}t\right]$$

$$+ \frac{e^{5t}}{12}$$

(6) $\frac{1}{(s^2-q^2)^2}$

Solⁿ $F(s) = \frac{1}{(s^2-q^2)^2}$

$$= \frac{1}{(s^2-q^2)} \times \frac{1}{(s^2-q^2)}$$

$$L^{-1}\left\{\frac{1}{s^2-q^2}\right\} = \frac{1}{q}\sinh qt$$

$$\therefore L^{-1}\{F(s)\} = \text{conv}\left(\frac{1}{q}\sinh qt, \frac{1}{q}\sinh qt\right)$$

$$= \int_0^t \frac{1}{q} \sinh qu \cdot \frac{1}{q} \sinh q(t-u)\, du$$

$$= \frac{1}{q^2} \int_0^t \sinh qu \, \sinh(qt - qu)\, du$$

$$= \frac{1}{q^2} \int_0^t \frac{1}{2}\left[\cosh qt - \cosh(2qu - qt)\right] du$$

$$= \frac{1}{2q^2}\left[u \cosh qt - \frac{\sinh(2qu - qt)}{2q}\right]_0^t$$

$$= \frac{1}{2q^2}\left[t \cosh qt - \frac{\sinh qt}{2q} + \frac{\sinh(-qt)}{2q}\right]$$

$$= \frac{1}{2q^2}\left[t \cosh qt - \frac{1}{q} \sinh qt\right]$$