E.x. Find the Laplace of following.

(1) $\sin\sqrt{t}$

Sol^n $f(t) = \sin\sqrt{t}$

We have,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$\therefore$ $f(t) = \sin t^{1/2}$

$$= t^{1/2} - \frac{1}{3!} t^{3/2} + \frac{1}{5!} t^{5/2} - \cdots$$

$$\boxed{\begin{aligned}&\Gamma_{n+1}\\ &= n\,\Gamma_n\end{aligned}}$$

$\therefore$ $L\{f(t)\}$

$$= \frac{\Gamma_{3/2}}{s^{3/2}} - \frac{1}{3!}\frac{\Gamma_{5/2}}{s^{5/2}} + \frac{1}{5!}\frac{\Gamma_{7/2}}{s^{7/2}} - \cdots$$

$$= \frac{\Gamma_{3/2}}{s^{3/2}}\left[1 - \frac{1}{3!}\cdot\frac{3}{2}\frac{1}{s} + \frac{1}{5!}\cdot\frac{5}{2}\frac{3}{2}\frac{1}{s^2} - \frac{1}{7!}\frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{s^3} + \cdots\right]$$

$$= \frac{\frac{1}{2}\Gamma_{1/2}}{s^{3/2}}\left[1 - \frac{1}{2\cdot 2}\frac{1}{s} + \frac{1}{4\cdot 2\cdot 2\cdot 2}\frac{1}{s^2} - \frac{1}{6\cdot 4\cdot 2\cdot 2\cdot 2\cdot 2}\frac{1}{s^3} + \cdots\right]$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\left[1 - \cancel{8}\,\frac{1}{4s} + \frac{1}{2}\frac{1}{(4s)^2} - \frac{1}{6}\frac{1}{(4s)^3} + \cdots\right]$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\left[1 - \frac{1}{4s} + \frac{1}{2!}\left(\frac{1}{4s}\right)^2 - \frac{1}{3!}\left(\frac{1}{4s}\right)^3 + \cdots\right]$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\, e^{-\frac{1}{4s}}$$

(2) $\dfrac{\cos\sqrt{t}}{\sqrt{t}}$

Sol^n We have

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$\frac{\cos\sqrt{t}}{\sqrt{t}} = \frac{1}{t^{1/2}} \cos t^{1/2}$$

$$= \frac{1}{t^{1/2}}\left[1 - \frac{t}{2!} + \frac{t^2}{4!} - \cdots\right]$$

$$= t^{-1/2} - \frac{t^{1/2}}{2!} + \frac{t^{3/2}}{4!} - \cdots$$

$$L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \frac{\Gamma 1/2}{s^{1/2}} - \frac{1}{2!}\frac{\Gamma 3/2}{s^{3/2}} + \frac{1}{4!}\frac{\Gamma 5/2}{s^{5/2}} - \cdots$$

$$= \frac{\Gamma 1/2}{s^{1/2}}\left[1 - \frac{1}{2!}\frac{1}{2}\frac{1}{s} + \frac{1}{4!}\frac{3}{2}\frac{1}{2}\frac{1}{s^2} - \frac{1}{6!}\frac{5}{2}\frac{3}{2}\frac{1}{2}\frac{1}{s^3} + \cdots\right]$$

$$= \frac{\sqrt{\pi}}{s^{1/2}}\left[1 - \frac{1}{2\cdot 2}\frac{1}{s} + \frac{1}{4\cdot 2\cdot 2\cdot 2}\frac{1}{s^2} - \frac{1}{6\cdot 4\cdot 2\cdot 2\cdot 2\cdot 2}\frac{1}{s^3} + \cdots\right]$$

$$= \frac{\sqrt{\pi}}{s^{1/2}}\left[1 - \frac{1}{4s} + \frac{1}{2}\frac{1}{(4s)^2} - \frac{1}{6}\frac{1}{(4s)^3} + \cdots\right]$$

$$= \frac{\sqrt{\pi}}{s^{1/2}}\left[1 - \frac{1}{4s} + \frac{1}{2!}\left(\frac{1}{4s}\right)^2 - \frac{1}{3!}\left(\frac{1}{4s}\right)^3 + \cdots\right]$$

$$= \frac{\sqrt{\pi}}{s^{1/2}} e^{-1/4s}$$

Or

Let $g(t) = \sin\sqrt{t}$; then

$$g'(t) = \cos\sqrt{t} \cdot \frac{1}{2} t^{-1/2} = \frac{1}{2} \frac{\cos\sqrt{t}}{\sqrt{t}}$$

Applying Laplace on derivative,

$$L\{g'(t)\} = sL\{g(t)\} - g(0)$$

$$\Rightarrow \frac{1}{2} L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = sL\{\sin\sqrt{t}\} - \sin(0)$$

Now, $L\{\sin\sqrt{t}\} = \ldots$

$$= \frac{\sqrt{\pi}}{2 s^{3/2}} e^{-1/4s}$$

$$\Rightarrow \frac{1}{2} L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = s \frac{\sqrt{\pi}}{2 s^{3/2}} e^{-1/4s}$$

$$\Rightarrow L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \frac{\sqrt{\pi}}{s^{1/2}} e^{-\frac{1}{4s}}$$

(3) $\sin a\sqrt{t}$

Sol^n $f(t) = \sin a\sqrt{t} = \sin\sqrt{a^2 t}$

Applying change of scale property,

$$L\{\sin a\sqrt{t}\} = \frac{1}{a^2} L\{\sin\sqrt{t}\}_{s \to \frac{s}{a^2}}$$

$$= \frac{1}{a^2}\left[\frac{\sqrt{\pi}}{2 s^{3/2}} e^{-\frac{1}{4s}}\right]_{s \to \frac{s}{a^2}}$$

$$= \frac{1}{a^2} \frac{\sqrt{\pi}}{2 s^{3/2}} \times a^3 \; e^{-\frac{a^2}{4s}}$$

$$= \frac{\sqrt{\pi}\, a}{2 s^{3/2}} e^{-\frac{a^2}{4s}}$$

(4) $\dfrac{\cos 3\sqrt{t}}{\sqrt{t}}$

Sol^n $\dfrac{\cos 3\sqrt{t}}{\sqrt{t}} = 3\dfrac{\cos 3\sqrt{t}}{3\sqrt{t}} = 3\dfrac{\cos\sqrt{9t}}{\sqrt{9t}}$

∴ Applying change of scale property

$$L\left\{\frac{\cos 3\sqrt{t}}{\sqrt{t}}\right\} = 3\,\frac{1}{9}\, L\left\{\frac{\cos\sqrt{8t}}{\sqrt{t}}\right\}_{s \to \frac{s}{9}}$$

$$= \frac{1}{3}\left[\frac{\sqrt{\pi}}{s^{1/2}} e^{-\frac{1}{4s}}\right]_{s \to \frac{s}{9}}$$

$$= \frac{1}{3} \frac{\sqrt{\pi}}{s^{1/2}} \times 3 \; e^{-\frac{9}{4s}}$$

$$= \frac{\sqrt{\pi}}{s^{1/2}} e^{-\frac{9}{4s}}$$

(5) $\cos^8 t$

Sol^n Let $x = e^{it}$, then $x^{-1} = e^{-it}$

$\therefore \quad x + x^{-1} = 2\cos t$

$\therefore \quad 2^8 \cos^8 t = (x + x^{-1})^8$

$$= x^8 + {}^8C_1 x^6 + {}^8C_2 x^4 + {}^8C_3 x^2 + {}^8C_4 + {}^8C_5 x^{-2}$$
$$+ {}^8C_6 x^{-4} + {}^8C_7 x^{-6} + {}^8C_8 x^{-8}$$

$$= (x^8 + x^{-8}) + {}^8C_1 (x^6 + x^{-6}) + {}^8C_2 (x^4 + x^{-4})$$
$$+ {}^8C_3 (x^2 + x^{-2}) + {}^8C_4$$

$$= 2\cos 8t + {}^8C_1 \cdot 2\cos 6t + {}^8C_2 \cdot 2\cos 4t$$
$$+ {}^8C_3 \cdot 2\cos 2t + {}^8C_4$$

$$\Rightarrow \cos^8 t = \frac{2}{2^8}\left[\cos 8t + {}^8C_1 \cos 6t + {}^8C_2 \cos 4t + {}^8C_3 \cos 2t + \frac{1}{2}\, {}^8C_4\right]$$

$$= \frac{1}{2^7}\left[\cos 8t + 8\cos 6t + 28\cos 4t + 56\cos 2t + 35\right]$$

$$\therefore L\{\cos^8 t\} = \frac{1}{2^7}\left[\frac{s}{s^2+8^2} + 8\frac{s}{s^2+6^2} + 28\frac{s}{s^2+4^2} + 56\frac{s}{s^2+2^2} + \frac{35}{s}\right]$$

(6) $\cos^5 t \sin^3 t$

Solⁿ Let $x = e^{it}$ then $x^{-1} = e^{-it}$,

$$x + x^{-1} = 2\cos t, \quad x - x^{-1} = 2i \sin t$$

$$\therefore\ 2^5 \cos^5 t \cdot (2i)^3 \sin^3 t$$

$$= (x + x^{-1})^5 (x - x^{-1})^3$$

$$= (x+x^{-1})^2 (x + x^{-1})^3 (x - x^{-1})^3$$

$$= (x + x^{-1})^2 (x^2 - x^{-2})^3$$

$$= (x^2 + 2 + x^{-2})(x^6 - 3x^2 + 3x^{-2} - x^{-6})$$

$$= x^8 - 3x^4 + 3 - x^{-4}$$
$$+ 2x^6 - 6x^2 + 6x^{-2} - 2x^{-6}$$
$$+ x^{\cancel{-}4} - 3 + 3x^{-4} - x^{-8}$$

$$= (x^8 - x^{-8}) - 3(x^4 - x^{-4}) + (x^4 - x^{-4}) + 2(x^6 - x^{-6})$$
$$- 6(x^2 - x^{-2})$$

$$= \cancel{2i}\,(x^8 - x^{-8}) + 2(x^6 - x^{-6}) - 2(x^4 - x^{-4})$$
$$- 6(x^2 - x^{-2})$$

$$= 2i \sin 8t + 2\cdot 2i \sin 6t - 2\cdot 2i \sin 4t$$
$$- 6\cdot 2i \sin 2t$$

$\therefore \cos^5 t \sin^3 t$

$$= \frac{2i}{2^5 (2i)^3}\left[\sin 8t + 2\sin 6t - 2\sin 4t - 6\sin 2t\right]$$

$$= -\frac{1}{2^7}\left[\sin 8t + 2\sin 6t - 2\sin 4t - 6\sin 2t\right]$$

$\therefore L\{\cos^5 t \sin^3 t\}$

$$= -\frac{1}{2^7}\left[\frac{8}{s^2+8^2} + 2\cdot\frac{6}{s^2+6^2} - 2\cdot\frac{4}{s^2+4^2} - 6\cdot\frac{2}{s^2+2^2}\right]$$

$$= -\frac{1}{2^7}\left[\frac{8}{s^2+64} + \frac{12}{s^2+36} - \frac{8}{s^2+16} - \frac{12}{s^2+4}\right]$$