Ex. Evaluate the following Integrals

Note that:

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$\therefore \int_0^\infty e^{-at} f(t)\, dt = L\{f(t)\}\Big|_{s=a}$$

(1) $$\int_0^\infty e^{-2t} \sin^3 t\, dt$$

Sol^n

$$I = \int_0^\infty e^{-2t} \sin^3 t\, dt = L\{\sin^3 t\}\Big|_{s=2}$$

$$\sin^3 t = \frac{1}{4}\left[3 \sin t - \sin 3t\right]$$

$$\therefore L\{\sin^3 t\} = \frac{1}{4}\left[3 \times \frac{1}{s^2+1} - \frac{3}{s^2+9}\right]$$

$$\therefore I = \frac{3}{4}\left[\frac{1}{4+1} - \frac{1}{4+9}\right]$$

$$= \frac{3}{4}\left[\frac{1}{5} - \frac{1}{13}\right]$$

$$= \frac{6}{65}$$

(2) $\int_0^\infty e^{-\sqrt{2}t} \frac{\sin t \sinh t}{t} dt$

Sol^n $I = \int_0^\infty e^{-\sqrt{2}t} \frac{\sin t \sinh t}{t} dt$

$= L\left\{ \frac{\sin t \sinh t}{t} \right\} \Big|_{s=\sqrt{2}}$

$f(t) = \frac{\sin t \sinh t}{t} = \frac{\sin t}{t} \frac{1}{2}(e^{t} - e^{-t})$

$= \frac{1}{2}\left[\frac{\sin t}{t} e^{t} - \frac{\sin t}{t} e^{-t} \right]$

$L\{\sin t\} = \frac{1}{s^2+1}$

$\therefore L\left\{ \frac{\sin t}{t} \right\} = \int_s^\infty \frac{1}{s^2+1} ds = \tan^{-1}(s)\Big]_s^\infty$

$= \frac{\pi}{2} - \tan^{-1}(s)$

$L\{f(t)\} = \frac{1}{2}\left[\frac{\pi}{2} - \tan^{-1}(s-1) - \frac{\pi}{2} + \tan^{-1}(s+1) \right]$

$= \frac{1}{2}\left[\tan^{-1}(s+1) - \tan^{-1}(s-1) \right]$

$\therefore I = \frac{1}{2}\left[\tan^{-1}(\sqrt{2}+1) - \tan^{-1}(\sqrt{2}-1) \right]$

$= \frac{1}{2} \tan^{-1}\left(\frac{\sqrt{2}+1-(\sqrt{2}-1)}{1+(\sqrt{2}+1)(\sqrt{2}-1)} \right)$

$= \frac{1}{2} \tan^{-1}(1) = \frac{1}{2} \times \frac{\pi}{4} = \frac{\pi}{8}$

(3) $\int_0^\infty \frac{\cos at - \cos bt}{t} \, dt$

$\underline{\text{Sol}^n}$ $I = \int_0^\infty \frac{\cos at - \cos bt}{t} \, dt$

$= L\left\{ \frac{\cos at - \cos bt}{t} \right\} \Big|_{s=0}$

$L\{ \cos at - \cos bt \} = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$

$\therefore L\left\{ \frac{\cos at - \cos bt}{t} \right\} = \int_s^\infty \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2} \, ds$

$= \frac{1}{2} \log(s^2+a^2) - \frac{1}{2} \log(s^2+b^2) \Big]_s^\infty$

$= \frac{1}{2} \log\left(\frac{s^2+a^2}{s^2+b^2} \right) \Big]_s^\infty$

$= 0 - \frac{1}{2} \log\left(\frac{s^2+a^2}{s^2+b^2} \right)$

$= \frac{1}{2} \log\left(\frac{s^2+b^2}{s^2+a^2} \right)$

$\therefore I = \frac{1}{2} \log\left(\frac{b^2}{a^2} \right)$

$= \log\left(\frac{b}{a} \right)$

(4) $\int_0^\infty \int_0^t e^{-2t} e^{-u} \frac{\sin u}{u} \, du \, dt$

Sol^n $I = \int_0^\infty \int_0^t e^{-2t} e^{u} \frac{\sin u}{u} \, du \, dt$

$$= \int_0^\infty e^{-2t} \left[\int_0^t e^{u} \frac{\sin u}{u} \, du \right] dt$$

$$= L\left\{ \int_0^t e^{u} \frac{\sin u}{u} \, du \right\} \Big|_{s=2}$$

$$L\{\sin u\} = \frac{1}{s^2+1}$$

$$L\left\{\frac{\sin u}{u}\right\} = \int_s^\infty \frac{1}{s^2+1} \, ds = \tan^{-1}(s) \Big]_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}(s)$$

$$L\left\{ e^{u} \frac{\sin u}{u} \right\} = \frac{\pi}{2} - \tan^{-1}(s-1)$$

$$\therefore L\left\{ \int_0^t e^{u} \frac{\sin u}{u} \, du \right\} = \cancel{\frac{1}{s}\left[\frac{\pi}{2} - \tan^{-1}(s+1)\right]}$$

$$= \frac{1}{s}\left[\frac{\pi}{2} - \tan^{-1}(s-1)\right]$$

$$\cancel{\therefore I = \frac{1}{2}\left[\frac{\pi}{2} - \tan^{-1}(1)\right]}$$

$$\therefore I = \frac{1}{2}\left[\frac{\pi}{2} - \tan^{-1}(1)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - \frac{\pi}{4}\right] = \frac{\pi}{8}$$

(5) $\int_0^\infty \int_0^t e^{-t} u^4 \sinh u \cosh u \, du \, dt$

Sol^n $I = \int_0^\infty \int_0^t e^{-t} u^4 \sinh u \cosh u \, du \, dt$

$= \int_0^\infty e^{-t} \left[\int_0^t u^4 \sinh u \cosh u \, du \right] dt$

$= L\left\{ \int_0^t u^4 \sinh u \cosh u \, du \right\} \Big|_{s=1}$

$u^4 \sinh u \cosh u = u^4 \frac{(e^u - e^{-u})}{2} \frac{(e^u + e^{-u})}{2}$

$= \frac{1}{4} u^4 (e^{2u} - e^{-2u})$

$= \frac{1}{4} (u^4 e^{2u} - u^4 e^{-2u})$

$L\{u^4\} = \frac{4!}{s^5}$

$L\{u^4 \sinh u \cosh u\} = \frac{1}{4} \left[\frac{4!}{(s-2)^5} - \frac{4!}{(s+2)^5} \right]$

$L\left\{ \int_0^t u^4 \sinh u \cosh u \, du \right\} = \frac{1}{4} \frac{1}{s} \left[\frac{4!}{(s-2)^5} - \frac{4!}{(s+2)^5} \right]$

$\therefore I = \frac{1}{4} \left[\frac{4!}{(-1)^5} - \frac{4!}{3^5} \right]$

$= \frac{-488}{81}$

(6) Find the value of α ~~and~~ if

$$\int_0^\infty e^{-2t} \sin(t+\alpha)\cos(t-\alpha)\,dt = \frac{3}{8}$$

Solⁿ $I = \int_0^\infty e^{-2t} \sin(\cancel{t+}\alpha)\cos(t-\alpha)\,dt$

$$= L\{\sin(t+\alpha)\cos(t-\alpha)\}\Big|_{s=2}$$

$\sin(t+\alpha)\cos(t-\alpha)$

$$= \frac{1}{2}\left[\sin(2t) + \sin(2\alpha)\right]$$

$\therefore L\{\sin(t+\alpha)\cos(t-\alpha)\}$

$$= \frac{1}{2}\left[\frac{2}{s^2+4} + \sin(2\alpha)\frac{1}{s}\right]$$

$$\therefore I = \frac{1}{2}\left[\frac{1}{4} + \frac{\sin(2\alpha)}{2}\right]$$

$$\therefore \frac{1}{2}\left[\frac{1}{4} + \frac{\sin(2\alpha)}{2}\right] = \frac{3}{8}$$

$$\Rightarrow \frac{1}{4} + \frac{\sin(2\alpha)}{2} = \frac{3}{4}$$

$$\Rightarrow \sin(2\alpha) = 1$$

$$\Rightarrow 2\alpha = \sin^{-1}(1) = \frac{\pi}{2}$$

$$\Rightarrow \alpha = \frac{\pi}{4}$$