Laplace Transform

E.x Find the Laplace Transform of following.

(1) $e^{-t}\sin^2 t$

Soln Let $f(t) = e^{-t}\sin^2 t$

$$= e^{-t}\frac{1}{2}(1-\cos 2t)$$

$$L(1-\cos 2t) = \frac{1}{s} - \frac{s}{s^2+4}$$

$$\therefore L\{f(t)\} = \frac{1}{2}\left[\frac{1}{s+1} - \frac{s+1}{(s+1)^2+4}\right]$$

(2) $t^2 e^{t}\sin 4t$

Soln $f(t) = t^2 e^{t}\sin 4t$

$$L\{\sin 4t\} = \frac{4}{s^2+16}$$

$$L\{t\sin 4t\} = -\frac{d}{ds}\left\{\frac{4}{s^2+16}\right\}$$

$$= -\left(\frac{-4 \times 2s}{(s^2+16)^2}\right) = \frac{8s}{(s^2+16)^2}$$

$$L\{t^2 \sin 4t\} = -\frac{d}{ds}\left[\frac{8s}{(s^2+16)^2}\right]$$

$$L\{t^2 \sin 4t\} = -\frac{8(s^2+16)^2 - 8s\cdot 2(s^2+16)\cdot 2s}{(s^2+16)^4}$$

$$= -\frac{8(s^2+16) - 32s^2}{(s^2+16)^3}$$

$$= -8\,\frac{(-3s^2+16)}{(s^2+16)^3}$$

$$\therefore L\{e^t\, t^2 \sin 4t\} = -8\,\frac{(-3(s-1)^2+16)}{((s-1)^2+16)^3}$$

Ex. 3 $\quad t^3 e^{-3t}$

Sol^n $\quad f(t) = t^3 e^{-3t}$

$$L\{t^3\} = \frac{3!}{s^4} = \frac{6}{s^4}$$

$$\therefore L\{f(t)\} = \frac{6}{(s+3)^4}$$

4 $f(t) = \dfrac{1-\cos 2t}{t}$

Sol^n $L\{1-\cos 2t\} = \dfrac{1}{s} - \dfrac{s}{s^2+4}$

$\therefore L\{f(t)\} = \int_s^\infty \dfrac{1}{s} - \dfrac{s}{s^2+4}\, ds$

$= \log(s) - \frac{1}{2}\log(s^2+4)\Big]_s^\infty$

$= \log\left(\dfrac{s}{\sqrt{s^2+4}}\right)\Big]_s^\infty$

$= 0 - \log\left(\dfrac{s}{\sqrt{s^2+4}}\right)$

$= \log\left(\dfrac{\sqrt{s^2+4}}{s}\right)$

(5) $f(t) = \dfrac{e^{-2t} \sin 2t \cosh t}{t}$

<u>Soln</u> $f(t) = \dfrac{e^{-2t}}{t} \sin 2t \dfrac{(e^{t}+e^{-t})}{2}$

$= \dfrac{\sin 2t}{t} \dfrac{1}{2} (e^{-t}+e^{-3t})$

$= \dfrac{1}{2}\left[\dfrac{\sin 2t}{t} e^{-t} + \dfrac{\sin 2t}{t} e^{-3t}\right]$

$L\{\sin 2t\} = \dfrac{2}{s^2+2^2}$

$\therefore L\left\{\dfrac{\sin 2t}{t}\right\} = 2\int_s^\infty \dfrac{1}{s^2+2^2}\, ds$

$= 2 \times \dfrac{1}{2} \tan^{-1}\left(\dfrac{s}{2}\right)\Big]_s^\infty$

$= \dfrac{\pi}{2} - \tan^{-1}\left(\dfrac{s}{2}\right)$

$= \cot^{-1}\left(\dfrac{s}{2}\right)$

$\therefore L\{f(t)\} = \dfrac{1}{2}\left[\cot^{-1}\left(\dfrac{s+1}{2}\right) + \cot^{-1}\left(\dfrac{s+3}{2}\right)\right]$

6 $\int_0^t u\, e^{-3u} \sin^2 u \, du$

Sol^n $f(t) = \int_0^t u\, e^{-3u} \sin^2 u \, du$

$\sin^2 u = \frac{1}{2}(1 - \cos 2u)$

$L\{\sin^2 u\} = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$

$\therefore L\{u \sin^2 u\} = -\frac{1}{2}\frac{d}{ds}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$

$= -\frac{1}{2}\left[-\frac{1}{s^2} - \frac{s^2+4-s\cdot 2s}{(s^2+4)^2}\right]$

$= \frac{1}{2}\left[\frac{1}{s^2} + \frac{4-s^2}{(s^2+4)^2}\right]$

$\therefore L\{e^{-3u}\, u \sin^2 u\}$

$= \frac{1}{2}\left[\frac{1}{(s+3)^2} + \frac{4-(s+3)^2}{((s+3)^2+4)^2}\right]$

$\therefore L\left\{\int_0^t e^{-3u}\, u \sin^2 u \, du\right\}$

$= \frac{1}{2}\frac{1}{s}\left[\frac{1}{(s+3)^2} + \frac{4-(s+3)^2}{((s+3)^2+4)^2}\right]$

(7) $\int_0^t \frac{1-e^{-au}}{u}\,du$

Sol^n $f(t) = \int_0^t \frac{1-e^{au}}{u}\,du$

$L\{1-e^{au}\} = \frac{1}{s} - \frac{1}{(s-a)}$

$L\left\{\frac{1-e^{au}}{u}\right\} = \int_s^\infty \frac{1}{s} - \frac{1}{(s-a)}\,ds$

$= \log(s) - \log(s-a)\Big]_s^\infty$

$= \log\left(\frac{s}{s-a}\right)\Big]_s^\infty$

$= 0 - \log\left(\frac{s}{s-a}\right)$

$= \log\left(\frac{s-a}{s}\right)$

$\therefore L\left\{\int_0^t \frac{1-e^{au}}{u}\,du\right\}$

$= \frac{1}{s}\log\left(\frac{s-a}{s}\right)$

(8) $\int_0^t \frac{\sin 4x}{x} e^x \, dx$

Sol^n $f(t) = \int_0^t \frac{\sin 4x}{x} e^x \, dx$

$L\{\sin 4x\} = \frac{4}{s^2+4^2}$

$L\left\{\frac{\sin 4x}{x}\right\} = \int_s^\infty \frac{4}{s^2+4^2} \, ds$

$= \tan^{-1}\left(\frac{s}{4}\right)\Big]_s^\infty$

$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{4}\right)$

$= \cot^{-1}\left(\frac{s}{4}\right)$

$\therefore L\left\{e^x \frac{\sin 4x}{x}\right\} = \cot^{-1}\left(\frac{s-1}{4}\right)$

$\therefore L\left\{\int_0^t \frac{\sin 4x}{x} e^x \, dx\right\}$

$= \frac{1}{s} \cot^{-1}\left(\frac{s-1}{4}\right)$

(10) $\int_0^x e^{x-y}\, y \cosh y \, dy$

Soln $f(x) = \int_0^x e^x e^{-y}\, y \cosh y \, dy$

$= e^x \int_0^x e^{-y}\, y \, \frac{(e^y + e^{-y})}{2} \, dy$

$= \frac{1}{2} e^x \int_0^x y\,(1+e^{-2y})\, dy$

$= \frac{1}{2} e^x \int_0^x y + y e^{-2y} \, dy$

$L\{y\} = \frac{1}{s^2}$

$L\{y + y e^{-2y}\} = \frac{1}{s^2} + \frac{1}{(s+2)^2}$

$L\{\int_0^x y + y e^{-2y} dy\} = \frac{1}{s}\left[\frac{1}{s^2} + \frac{1}{(s+2)^2}\right]$

$\therefore L\{f(x)\} = \frac{1}{2}\, \frac{1}{(s-1)} \left[\frac{1}{(s-1)^2} + \frac{1}{(s+1)^2}\right]$

(11) $\dfrac{t\sqrt{1+\sin t}}{2^{3t}}$

Sol^n $f(t) = 2^{-3t}\, t\sqrt{1+\sin t}$

$= e^{\log(2^{-3t})}\, t\sqrt{1+\sin t}$

$= e^{-3t\log(2)}\, t\sqrt{1+\sin t}$

$= e^{-t\log(8)}\, t\sqrt{1+\sin t}$

$\sqrt{1+\sin t} = \sqrt{1+2\sin t/2 \cos t/2}$

$= \sqrt{\cos^2 t/2 + \sin^2 t/2 + 2\sin t/2 \cos t/2}$

$= \sqrt{(\cos t/2 + \sin t/2)^2}$

$= \cos t/2 + \sin t/2$

$L\{\sqrt{1+\sin t}\} = \dfrac{s}{s^2+\frac{1}{4}} + \dfrac{1/2}{s^2+\frac{1}{4}}$

$= \dfrac{s+1/2}{s^2+\frac{1}{4}} = \dfrac{4s+2}{4s^2+1}$

$L\{t\sqrt{1+\sin t}\} = -\dfrac{d}{ds}\left[\dfrac{4s+2}{4s^2+1}\right]$

$= -\dfrac{4(4s^2+1)-(4s+2)\cdot 8s}{(4s^2+1)^2}$

$$= -4 \frac{(4s^2+1+8s^2+4s)}{(4s^2+1)^2}$$

$$= -4 \frac{(12s^2+4s+1)}{(4s^2+1)^2}$$

$$L\{f(t)\} = -4 \frac{(12(s+\log 8)^2+4(s+\log 8)+1)}{(4(s+\log 8)^2+1)^2}$$