AM III

TUTORIAL 04

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D10A 01

1) $z = x + iy$

$\therefore \bar{z} = x - iy$

let $u = x$ and $v = -y$

$\therefore u_x = 1$ $\therefore v_x = 0$

$u_y = 0$ $v_y = -1$

Cauchy Reimann eqⁿ : $u_x = v_y$ and $u_y = -v_x$

but $u_x \neq v_y$

$\because$ C.R eqⁿ are not satisfied

$\therefore \bar{z}$ is not analytic at any point

i.e Set of points where $\bar{z}$ is analytic $= \phi$

2) $f(x) = u + iv$

$\therefore |f(z)| = c \implies v^2 + u^2 = c^2$ — ①

Diff ① w.r.t x, $u u_x + v v_x = 0$ — ②

Diff ① w.r.t y, $u u_y + v v_y = 0$ — ③

$\because f(z)$ is analytic, CR eqⁿ are satisfied

$\therefore u_x = v_y$, $u_y = -v_x$

$\therefore$ eqⁿ ②, ③ becomes

$u u_x - v u_y = 0$ and $u u_y \pm v u_x = 0$

Eliminating u_y, $(u^2 + v^2) u_x = 0$ $\therefore c^2 u_x = 0$ $\therefore u_x = 0$

Similarly, $u_y = 0$, $v_x = 0$, $v_y = 0$

$\therefore f(z)$ is analytic, $f'(z) = u_x + i v_x = 0$ $[\because u_x = v_x = 0)$

$\therefore f(z) =$ constant

3) let $v = e^{2x} \sin 2y - 2xy$

$\therefore v_x = 2e^{2x} \sin 2y - 2y$

$v_{xx} \; 4e^{2x} \sin 2y$

$\therefore v_y = 2e^{2x} \cos 2y - 2x$

$v_{yy} = -4e^{2x} \sin 2y$

$\because v_{xx} + v_{yy} = 0$

$\therefore v$ is harmonic

$\therefore$ It can be harmonic conjugate of some function

Let $f(z) = u + iv$ be analytic

$\therefore u_x = v_y$ and $u_y = -v_x$

$\therefore u_x = 2e^{2x} \cos 2y - 2x$

$u_y = 2y - 2e^{2x} \sin 2y$

$\therefore u = \int u_x \, dx + \int [\text{terms in } u_y \text{ without } x] \, dy$

$= \int (2e^{x} \cos 2y - 2x) \, dx + \int 2y \, dy$

$\therefore u = e^{2x} \cos 2y - x^2 + y^2 = c$

$\therefore$ Function whose harmonic conjugate is $e^{2x} \sin 2y - 2xy$ is

$e^{2x} \cos 2y - x^2 + y^2 = c$

4) $u = \dfrac{2\sin 2x}{e^{2y} + e^{-2y} - 2\cos 2x}$

$u_x = \dfrac{(e^{2y} + e^{-2y} - 2\cos 2x)(4\cos 2x) - (2\sin 2x)(4\sin 2x)}{(e^{2y} + e^{-2y} - 2\cos 2x)^2}$

Putting $x = z$, $y = 0$

$\therefore u_x = \dfrac{8\cos 2z - 8\cos^2 2z - 8(1-\cos^2 2z)}{(2 - 2\cos 2z)^2}$

$= \dfrac{8\cos 2z - 8}{(2 - 2\cos 2z)^2}$

$= \dfrac{8(\cos 2z - 1)}{4(1 - \cos 2z)^2}$

$\therefore u_x = \dfrac{-2}{(1 - \cos 2z)}$

$\therefore u_y = \dfrac{(2\sin 2x)(2e^{2y} - 2e^{-2y})}{(e^{2y} + e^{-2y} - 2\cos 2x)^2}$

Putting $x = z$, $y = 0$

$\therefore u_y = 0$

By Milne Thompson method, $f'(z) = u_x$

$\therefore f(z) = \int u_x \, dz + 0$

$= \int \dfrac{-2}{1 - \cos 2z} \, dz$

$= -\int \operatorname{cosec}^2 z \, dz$

$\therefore f(z) = \cot z + c$

FOR EDUCATIONAL USE

5) $f(z) = u + iv$

$if(z) = iu - v$

$\therefore (1+i) f(z) = (u-v) + i(u+v) = U + iV$

$U_x = \frac{\partial U}{\partial x} = \frac{\partial (u-v)}{\partial x} = e^x (\cos y - \sin y)$

$U_y = \frac{\partial U}{\partial y} = \frac{\partial (u-v)}{\partial y} = e^x (-\sin y - \cos y)$

$\therefore (1+i) f'(z) = U_x - iU_y$

$= e^x (\cos y - \sin y) - i[e^x (-\sin y - \cos y)]$

putting $x = z, \quad y = 0$

$\therefore (1+i) f'(z) = e^z + ie^z$

$(1+i) f(z) = \int (e^z + e^z) dz$

$= (1+i) \int e^z dz$

$\therefore \underline{\underline{f(z) = e^z}} + c$

6) $x^3 y - xy^3 + x^2 - y^2 + x = c$

For orthogonal trajectory of this, we have to find the harmonic conjugate of it

Let $u = x^3 y - xy^3 + x^2 - y^2 + x$

$\therefore u_x = 3x^2 y - y^3 + 2x + 1$

$\therefore u_y = x^3 - 3xy^2 - 2y$

By CR eq^n, $u_x = v_y$ & $u_y = -v_x$

$\therefore v_x = 3xy^2 + 2y - x^3$

$\therefore v_y = 3x^2 y - y^3 + 2x + 1$

$\therefore v = \int v_x \, dx + \int [\text{terms in } v_y \text{ without } x] \, dy$

$\therefore v = \int (3xy^2 + 2y - x^3) \, dx + \int (-y^3 + 1) \, dy$

$\therefore v = \frac{3}{2} x^2 y^2 + 2xy - \frac{x^4}{4} - \frac{y^4}{4} + y + c$

$\therefore$ Orthogonal trajectory required:

$$\frac{3}{2} x^2 y^2 + 2xy - \frac{x^4}{4} - \frac{y^4}{4} + y = c$$

7) let $P(r, \theta)$ be points of intersection of given curves

$n \log r = \log a + \log \sec n\theta$

$\therefore \frac{n}{r} \frac{dr}{d\theta} = \frac{1}{\sec n\theta} \cdot \sec n\theta \cdot \tan \theta \cdot n$

$\therefore \frac{dr}{d\theta} = r \tan n\theta$

$\tan \phi_1 = \frac{r\, d\theta}{dr} = \frac{r}{r \tan n\theta} = \cot n\theta$

$\therefore \phi_1 = \frac{\pi}{2} - n\theta$

$n \log r = \log b + \log \operatorname{cosec} n\theta$

$\therefore \frac{n}{r} \frac{dr}{d\theta} = \frac{-1}{\operatorname{cosec} n\theta} \cdot \operatorname{cosec} n\theta \cdot \cot n\theta \cdot n$

$\therefore \frac{dr}{d\theta} = -r \cot n\theta$

$\therefore \tan \phi_2 = \frac{r\, d\theta}{dr} = \frac{-r}{r \cot n\theta} = -\tan n\theta$

$\therefore \phi_2 = -n\theta$

$\therefore \phi_1 - \phi_2 = \frac{\pi}{2} - n\theta - (-n\theta) = \frac{\pi}{2}$

$\therefore$ The curves intersect each other orthogonally and thus form orthogonal trajectories to each other

8) u is harmonic

$\therefore u_{xx} + u_{yy} = 0$ — ①

$f(z) = u_x - i u_y = u + iv$

$\therefore v_x = \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2}$ .... from ①

ie $v_x = u_{xx} = -u_{yy}$

$v_y = u_{xy}$

$\therefore v_x = -u_{yx}$

$\therefore v_y = -u_{yy}$

$\therefore u_x = v_y$ and $u_y = v_x$

$\therefore f(z) = u_x - i u_y$ is analytic