# Inverse Using Partial Fraction

(1) $$\frac{4s+5}{(s-1)^2(s+2)}$$

Soln $$F(s) = \frac{4s+5}{(s-1)^2(s+2)}$$

$$= \frac{A}{(s-1)} + \frac{B}{(s-1)^2} + \frac{C}{(s+2)}$$

$$\Rightarrow \quad A(s-1)(s+2) + B(s+2) + C(s-1)^2 = 4s+5$$

For $s=1$, $3B = 9 \Rightarrow B = 3$

$s=-2$, $9C = -3 \Rightarrow C = -\frac{1}{3}$

coeff $s^2$, $A + C = 0 \Rightarrow A = \frac{1}{3}$

$$\therefore \quad F(s) = \frac{1}{3}\frac{1}{(s-1)} + \frac{3}{(s-1)^2} - \frac{1}{3}\frac{1}{(s+2)}$$

$$L^{-1}\{F(s)\} = \frac{1}{3}e^{t} + 3e^{t}L^{-1}\left\{\frac{1}{s^2}\right\} - \frac{1}{3}e^{-2t}$$

$$= \frac{1}{3}e^{t} + 3e^{t}t - \frac{1}{3}e^{-2t}$$

(2) $$\frac{5s+3}{(s+3)(s^2+2s+5)}$$

Sol$^n$

$$F(s) = \frac{5s+3}{(s+3)(s^2+2s+5)}$$

$$= \frac{A}{(s+3)} + \frac{Bs+C}{(s^2+2s+5)}$$

$$\Rightarrow \quad A(s^2+2s+5) + (Bs+C)(s+3) = 5s+3$$

$\Rightarrow$ coeff $s^2$, $\quad A + B = 0$

$s$, $\quad 2A + 3B + C = 5$

constant, $\quad 5A + 3C = 3$

$$\Rightarrow \quad A = -\frac{3}{2}, \quad B = \frac{3}{2}, \quad C = \frac{7}{2}$$

$$\therefore F(s) = -\frac{3}{2}\frac{1}{(s+3)} + \frac{\frac{3}{2}s + \frac{7}{2}}{s^2+2s+5}$$

$$= -\frac{3}{2}\underbrace{\frac{1}{(s+3)}}_{F_1} + \frac{1}{2}\underbrace{\frac{(3s+7)}{s^2+2s+5}}_{F_2}$$

$$F_1 = \frac{1}{s+3}$$

$$L^{-1}\{F_1\} = e^{-3t}$$

$$F_2 = \frac{3s+7}{s^2+2s+5}$$

$$= \frac{3s+7}{(s+1)^2+4}$$

$$= \frac{3(s+1-1)+7}{(s+1)^2+4}$$

$$= \frac{3(s+1)+4}{(s+1)^2+4}$$

$$L^{-1}\{F_2\} = e^{-t} L^{-1}\left[\frac{3s+4}{s^2+4}\right]$$

$$= e^{-t} L^{-1}\left\{3\frac{s}{s^2+2^2} + \frac{4}{s^2+2^2}\right]$$

$$= e^{-t}\left[3\cos 2t + 2\sin 2t\right]$$

$$\therefore L^{-1}\{F(s)\} = \frac{-3}{2}e^{-3t} + \frac{e^{-t}}{2}\left[3\cos 2t + 2\sin 2t\right]$$

③ $$\frac{2}{(s+1)^2(s^2+4)}$$

Solⁿ $$F(s) = \frac{2}{(s+1)^2(s^2+4)}$$

$$= \frac{A}{(s+1)} + \frac{B}{(s+1)^2} + \frac{Cs+D}{(s^2+4)}$$

$\Rightarrow$ $A(s+1)(s^2+4) + B(s^2+4) + (Cs+D)(s+1)^2$
$= 2$

$\Rightarrow$ $A(s+1)(s^2+4) + B(s^2+4) + (Cs+D)(s^2+2s+1)$
$= 2$

$\Rightarrow$ $s=-1$, $5B = 2 \Rightarrow B = \frac{2}{5}$

coeff $s^3$, $A + C = 0$

$s^2$, $A + B + 2C + D = 0 \Rightarrow A + 2C + D = -\frac{2}{5}$

$s$, $4A + C + 2D = 0$

$\Rightarrow$ $A = \frac{4}{25}$, $C = -\frac{4}{25}$, $D = -\frac{6}{25}$

$\therefore$ $$F(s) = \frac{4}{25}\frac{1}{(s+1)} + \frac{2}{5}\frac{1}{(s+1)^2} - \frac{4}{25}\frac{s}{(s^2+4)} - \frac{6}{25}\frac{1}{(s^2+4)}$$

$$L^{-1}\{F(s)\} = \frac{4}{25}e^{-t} + \frac{2}{5}e^{-t}t - \frac{4}{25}\cos 2t - \frac{3}{25}\sin 2t$$

(4) $\dfrac{s}{s^4+4a^4}$

Sol$^n$ $F(s) = \dfrac{s}{s^4+4a^4}$

$= \dfrac{s}{(s^2)^2+(2a^2)^2}$

$= \dfrac{s}{(s^2)^2+(2a^2)^2+2\cdot 2a^2s^2-2\cdot 2a^2s^2}$

$= \dfrac{s}{(s^2+2a^2)^2-(2as)^2}$

$= \dfrac{s}{\underbrace{(s^2+2a^2+2as)}_{X}\underbrace{(s^2+2a^2-2as)}_{Y}}$

$= \dfrac{1}{4a}\dfrac{(X-Y)}{XY}$

$\boxed{\begin{array}{l} X-Y=4as \\ \frac{1}{4a}(X-Y)=s \end{array}}$

$= \dfrac{1}{4a}\left[\dfrac{1}{Y}-\dfrac{1}{X}\right]$

$= \dfrac{1}{4a}\left[\underbrace{\dfrac{1}{s^2-2as+2a^2}}_{F_1}-\underbrace{\dfrac{1}{s^2+2as+2a^2}}_{F_2}\right]$ —— (1)

$$F_1 = \frac{1}{s^2 - 2as + 2a^2}$$

$$= \frac{1}{s^2 - 2as + a^2 + a^2}$$

$$= \frac{1}{(s-a)^2 + a^2}$$

$$L^{-1}\{F_1\} = e^{at} L^{-1}\left\{\frac{1}{s^2+a^2}\right\}$$

$$= e^{at} \frac{1}{a} \sin at$$

$$F_2 = \frac{1}{s^2 + 2as + 2a^2}$$

$$= \frac{1}{(s+a)^2 + a^2}$$

$$L^{-1}\{F_2\} = e^{-at} \frac{1}{a} \sin at$$

$$\therefore L^{-1}\{F(s)\} = \frac{1}{4a}\left[\frac{1}{a} e^{at} \sin at - \frac{1}{a} e^{-at} \sin at\right]$$

$$= \frac{1}{4a^2} \sin at \left[e^{at} - e^{-at}\right]$$

$$= \frac{1}{4a^2} \sin at \cdot 2 \sinh at$$

$$= \frac{1}{2a^2} \sin at \sinh at$$

5 $\dfrac{s^3}{s^4 - a^4}$

$X+Y = 2s^2$

$\frac{1}{2}(X+Y) = s^2$

Sol$^n$ $F(s) = \dfrac{s^3}{(s^2)^2 - (a^2)^2}$

$= \dfrac{s^3}{\underset{X}{(s^2+a^2)}\,\underset{Y}{(s^2-a^2)}}$

$= s\,\dfrac{1}{2}\,\dfrac{(X+Y)}{XY}$

$= \dfrac{1}{2}\,s\left(\dfrac{1}{Y} + \dfrac{1}{X}\right)$

$= \dfrac{1}{2}\,s\left(\dfrac{1}{s^2-a^2} + \dfrac{1}{s^2+a^2}\right)$

$= \dfrac{1}{2}\left(\dfrac{s}{s^2-a^2} + \dfrac{s}{s^2+a^2}\right)$

$L^{-1}\{F(s)\} = \dfrac{1}{2}[\cosh at + \cos at]$

6 $\dfrac{s^2+6}{(s^2+1)(s^2+4)}$

Sol$^n$ $F(s) = \dfrac{s^2+6}{(s^2+1)(s^2+4)}$

Sol$^n$

Method I:

$$F(s) = \frac{s^2+1+5}{(s^2+1)(s^2+4)}$$

$y-x=3$

$\frac{1}{3}(y-x)=1$

$$= \frac{1}{s^2+4} + \frac{5}{(s^2+1)(s^2+4)}$$

$x \quad y$

$$= \frac{1}{(s^2+4)} + \frac{5}{3}\frac{(y-x)}{xy}$$

$$= \frac{1}{(s^2+4)} + \frac{5}{3}\left(\frac{1}{x} - \frac{1}{y}\right)$$

$$= \frac{1}{(s^2+4)} + \frac{5}{3}\frac{1}{(s^2+1)} - \frac{5}{3}\frac{1}{(s^2+4)}$$

$$= -\frac{2}{3}\frac{1}{(s^2+4)} + \frac{5}{3}\frac{1}{(s^2+1)}$$

$$L^{-1}\{F(s)\} = -\frac{2}{3}\frac{1}{2}\sin 2t + \frac{5}{3}\sin t$$

$$= -\frac{1}{3}\sin 2t + \frac{5}{3}\sin t$$

or Method II:

$$F(s) = \frac{s^2+6}{(s^2+1)(s^2+4)}$$

put $s^2 = x$

$$F(S) = \frac{X+6}{(X+1)(X+4)}$$

$$= \frac{A}{X+1} + \frac{B}{X+4}$$

$$\Rightarrow \quad A(X+4) + B(X+1) = X+6$$

$X=-1, \quad 3A = 5 \Rightarrow A = 5/3$

$X=-4, \quad -3B = 2 \Rightarrow B = -2/3$

$$\therefore \quad F(S) = \frac{5}{3}\frac{1}{s^2+1} - \frac{2}{3}\frac{1}{s^2+4}$$

$$L^{-1}\{F(s)\} = \frac{5}{3}\sin t - \frac{1}{3}\sin 2t$$

(6) $\quad F(s) = \dfrac{s^2+2s-4}{(s^2+2s+2)(s^2+2s+5)}$

Sol^n put $s^2+2s = X$

$$\therefore \quad F(S) = \frac{X-4}{(X+2)(X+5)}$$

$$= \frac{A}{X+2} + \frac{B}{X+5}$$

$$\Rightarrow \quad A(X+5) + B(X+2) = X-4$$

$x = -2, \quad 3A = -6 \quad \Rightarrow A = -2$

$x = -5, \quad -3B = -9 \quad \Rightarrow B = 3$

$$\therefore \quad F(s) = \frac{-2}{s^2+2s+2} + \frac{3}{s^2+2s+5}$$

$$= \frac{-2}{(s+1)^2+1} + \frac{3}{(s+1)^2+4}$$

$$L^{-1}\{F(s)\} = e^{-t} L^{-1}\left[\frac{-2}{s^2+1} + \frac{3}{s^2+4}\right]$$

$$= e^{-t}\left[-2\sin t + \frac{3}{2}\sin 2t\right]$$

(7) $\dfrac{s}{s^4+s^2+1}$

Solⁿ $F(s) = \dfrac{s}{s^4+s^2+1}$

$$= \frac{s}{(s^2)^2+s^2+1+s^2-s^2}$$

$$= \frac{s}{(s^2+1)^2 - s^2}$$

$$= \frac{s}{\underset{x}{(s^2+1+s)}\ \underset{y}{(s^2+1-s)}}$$

$$F(s) = \frac{1}{2}\,\frac{(X-Y)}{XY}$$

$X - Y = 2s$

$$= \frac{1}{2}\left(\frac{1}{Y} - \frac{1}{X}\right)$$

$$= \frac{1}{2}\left(\underbrace{\frac{1}{s^2-s+1}}_{F_1} - \underbrace{\frac{1}{s^2+s+1}}_{F_2}\right)$$

$$F_1 = \frac{1}{s^2-s+1} = \frac{1}{s^2-s+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+1}$$

$$= \frac{1}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$L^{-1}\{F_1\} = e^{t/2}\,L^{-1}\left\{\frac{1}{s^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\}$$

$$= e^{t/2}\,\frac{2}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t$$

$$F_2 = \frac{1}{s^2+s+1} = \frac{1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$L^{-1}\{F_2\} = e^{-t/2}\,L^{-1}\left\{\frac{1}{s^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\}$$

$$= e^{-t/2}\,\frac{2}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t$$

$$\therefore L^{-1}\{F(s)\} = \frac{1}{2}\left[\frac{2}{\sqrt{3}} e^{t/2} \sin \frac{\sqrt{3}}{2}t - \frac{2}{\sqrt{3}} e^{-t/2} \sin \frac{\sqrt{3}}{2}t\right]$$

$$= \frac{1}{\sqrt{3}} \sin \frac{\sqrt{3}}{2}t \left[e^{t/2} - e^{-t/2}\right]$$

$$= \frac{1}{\sqrt{3}} \sin \sqrt{3}/2\, t \cdot 2 \sinh t/2$$

$$= \frac{2}{\sqrt{3}} \sin\left(\frac{\sqrt{3}}{2}t\right) \sinh\left(t/2\right)$$

(8) $$F(s) = \frac{as(s^2 - 2a^2)}{s^4 + 4a^4}$$

$$= \frac{s(s^2 - 2a^2)}{(s^2)^2 + (2a^2)^2 + 2\cdot 2a^2 s^2 - 2\cdot 2a^2 s^2}$$

$$= \frac{as(s^2 - 2a^2)}{(s^2 + 2a^2)^2 - (2as)^2}$$

$$= \frac{s(s^2 - 2a^2)}{\underbrace{(s^2 + 2a^2 + 2as)}_{X}\underbrace{(s^2 + 2a^2 - 2as)}_{Y}}$$

$$= \frac{s(s^2 + 2a^2 - 4a^2)}{XY}$$

$$\boxed{\begin{aligned} &X + Y \\ &= 2s^2 + 4a^2 \\ &= 2(s^2 + 2a^2) \end{aligned}}$$

$$F(s) = \frac{s\left(\frac{1}{2}(x+y) - 4a^2\right)}{xy}$$

$$x - y = 4as$$
$$\frac{1}{4a}(x-y) = s$$

$$= \frac{\frac{s}{2}(x+y) - 4a^2 s}{xy}$$

$$= \frac{s}{2}\frac{(x+y)}{xy} - \frac{4a^2}{4a}\frac{(x-y)}{xy}$$

$$= \frac{s}{2}\left(\frac{1}{y} + \frac{1}{x}\right) - a\left(\frac{1}{y} - \frac{1}{x}\right)$$

$$= \frac{1}{2}\left(\frac{s-2a}{y} + \frac{s+2a}{x}\right)$$

$$= \frac{1}{2}\left[\underbrace{\frac{s-2a}{s^2-2as+2a^2}}_{F_1} + \underbrace{\frac{s+2a}{s^2+2as+2a^2}}_{F_2}\right]$$

$$F_1 = \frac{s-2a}{s^2-2as+2a^2} = \frac{s-2a}{(s-a)^2+a^2}$$

$$= \frac{(s-a)-a}{(s-a)^2+a^2}$$

$$L^{-1}\{F_1\} = e^{at} L^{-1}\left[\frac{s}{s^2+a^2} - \frac{a}{s^2+a^2}\right]$$

$$= e^{at}\left[\cos at - \sin at\right]$$

$$F_2 = \frac{s+2a}{s^2+2as+2a^2}$$

$$= \frac{s+2a}{(s+a)^2+a^2} = \frac{(s+a)+a}{(s+a)^2+a^2}$$

$$L^{-1}\{F_2\} = e^{-at} L^{-1}\left[\frac{s}{s^2+a^2} + \frac{a}{s^2+a^2}\right]$$

$$= e^{-at}\left[\cos at + \sin at\right]$$

$$L^{-1}\{F(s)\} = \frac{1}{2}\Big[e^{at}\left[\cos at - \sin at\right]$$
$$+ e^{-at}\left[\cos at + \sin at\right]\Big]$$

$$= \frac{1}{2}(e^{at}+e^{-at}) \cos at$$
$$- \frac{1}{2}(e^{at}-e^{-at}) \sin at$$

$$= \cosh at \cos at - \sinh at \sin at$$