Laplace Transform

Let $f(t)$ be a piecewise continuous function on $[0,\infty)$, then

$$L\{f(t)\} = \int_0^\infty e^{-st}\, f(t)\, dt$$

Note: $L\{f(t)\} = F(s)$

Transform of Basic functions:

1. $L\{1\} = \dfrac{1}{s}$
2. $L\{t\} = \dfrac{1}{s^2}$
3. $L\{t^n\} = \dfrac{n!}{s^{n+1}}$, $n = 0,1,2,....$
4. $L\{t^n\} = \dfrac{\Gamma(n+1)}{s^{n+1}}$, $n \neq -1,-2,....$
5. $L\{e^{at}\} = \dfrac{1}{s-a}$
6. $L\{e^{-at}\} = \dfrac{1}{s+a}$
7. $L\{\sin at\} = \dfrac{a}{s^2+a^2}$
8. $L\{\cos at\} = \dfrac{s}{s^2+a^2}$
9. $L\{\sinh at\} = \dfrac{a}{s^2-a^2}$
10. $L\{\cosh at\} = \dfrac{s}{s^2-a^2}$

Properties of Laplace Transform:

1. **Linearity :**

$$L\{a\,f(t)+b\,g(t)\} = a\,L\{f(t)\} + b\,L\{g(t)\}$$

2. **First Shifting :**

$$L\{e^{at} f(t)\} = L\{f(t)\}_{s\to(s-a)}$$

3. **Change of Scale :**

$$L\{f(at)\} = \frac{1}{a}\,L\{f(t)\}_{s\to\left(\frac{s}{a}\right)}$$

4. **Multiplication by ' t ' :**

$$L\{t\,f(t)\} = -\frac{d}{ds}L\{f(t)\}$$

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} L\{f(t)\}, \; n = 1,2,...$$

5. **Division by ' t ' :**

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty L\{f(t)\}\, ds$$

6. **Transform of derivatives of** $f(t)$ **:**

$$L\{f'(t)\} = s L\{f(t)\} - f(0)$$

$$L\left\{f^{(n)}(t)\right\} = s^n L\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{(n-1)}(0)$$

7. **Transform of Integral of** $f(t)$ **:**

$$L\left\{\int_0^t f(u)\, du\right\} = \frac{1}{s} L\{f(t)\}$$

Exercise

Find the Laplace transform of the following.

1. $\cos^2 2t$ 2. $e^{-t} \sin^2 t$ 3. $e^{4t} \sin 2t \cos t$ 4. $\sinh at \sin at$ 5. $t e^{-t} \cosh t$

6. $t^2 e^t \sin 4t$ 7. $t \sin^3 t$ 8. Show that $L\left\{\sinh(t/2) \sin\left(\sqrt{3}\, t/2\right)\right\} = \frac{\sqrt{3}}{2} \frac{s}{s^4 + s^2 + 1}$

9. $t^3 e^{-3t}$ 10. $(t \sin 2t)^2$ 11. $(t \sinh 2t)^2$ 12. $\frac{1-\cos 2t}{t}$ 13. $\frac{1-e^t}{t}$

14. $e^{-3t} \sin 3t \sinh 4t$ 15. $\sin 3t \cos 2t \cos t$ 16. $\frac{\cos at - \cos bt}{t}$

17. $\frac{e^{-2t} \sin 2t \cosh t}{t}$ 18. $\frac{\sin 3t \sinh 2t}{t}$ 19. $\int_0^t e^{-2u} \cos^2 u\, du$

20. $\int_0^t u\, e^{-3u} \sin^2 u\, du$ 21. $\int_0^t u^{-1} e^{-u} \sin u\, du$ 22. $\int_0^t \frac{1-e^{-au}}{u}\, du$

24. $\int_0^t \frac{\sin 4x}{x} e^x dx$ 25. $\int_0^x e^{x-y}\, y \cosh y\, dy$ 26. $\cosh t \int_0^t e^u \cosh u\, du$

27. If $L\{f(t)\} = \frac{20-4s}{s^2 - 4s + 20}$, Find $L\{f(3t)\}$ 28. $\frac{t\sqrt{1+\sin t}}{e^{2t}}$ 29. $2^{3t} + \sqrt{1+\sin t}$

30. $\sin\sqrt{t}$ 31. $\frac{\cos\sqrt{t}}{\sqrt{t}}$ 32. $\sin a\sqrt{t}$ 33. $\frac{\cos 3\sqrt{t}}{\sqrt{t}}$ 34. $t^{-1} \sin a\sqrt{t}$ 35. $t^{-1/2} \cos a\sqrt{t}$

36. $\sin^5 t$ 37. $\cos^8 t$ 38. $\cos^4 t \sin^3 t$ 39. $\sinh^5 t$ 40. $\int_0^t\int_0^t\int_0^t t \sin t\, dt\, dt\, dt$

41. $\int_0^t u e^{u-3t} \sin u\, du$ 42. $e^{-2t} \int_0^t t \cos 3t\, dt$ 43. $\frac{\sinh 2t}{t}$

44. $\left(1-\sqrt[3]{t}\right)^2$ 45. Find the Laplace of $t \sin \omega t$ and hence find the Laplace of $\omega t \cos \omega t + \sin \omega t$

Evaluate the following integrals.

1. $\int_0^\infty \frac{\sin^2 t \;\; e^{-t}}{t}\, dt$ 2. $\int_0^\infty e^{-\sqrt{2}\,t} \; \frac{\sin t \;\; \sinh t}{t}\, dt$ 3. $\int_0^\infty e^{-2t} \sin^3 t \; dt$ 4. $\int_0^\infty e^{-t} \sin^5 t \; dt$

5. $\int_0^\infty \frac{\cos at \; - \; \cos bt}{t}\, dt$ 6. $\int_0^\infty \frac{\cos 6t \; - \; \cos 4t}{t}\, dt$ 7. $\int_0^\infty e^{-t} \; \frac{1 \; - \; \cos 3t}{t}\, dt$

8. $\int_0^\infty \frac{e^{-at} \; - \; e^{-bt}}{t}\, dt$ 9. $\int_0^\infty \int_0^t e^{-2t} e^{-u} \, \frac{\sin u}{u} \, du \; dt$ 10. $\int_0^\infty \int_0^t e^{-t} \, u^4 \, \sinh u \, \cosh u \; du \; dt$

11. $\int_0^\infty e^{-st} \; \frac{\sin at \; + \; \sin bt}{t}\, dt$ 12. $\int_0^\infty e^{-t} \; \frac{\sin 3t \; + \; \sin 2t}{t}\, dt$

13. Find the value of α if $\int_0^\infty e^{-2t} \sin(t+\alpha) \cos(t-\alpha)\, dt = \frac{3}{8}$

Transform of Special Functions:

1. Unit Step Function (Heaviside Function):

Notations: $u(t-a)$, or $u_a(t)$, $H(t-a)$, or $H_a(t)$

$$H(t-a)=\begin{cases} 0 & t<a \\ 1 & t \ge a \end{cases} \quad , \quad \text{for} \quad a=0 \quad H(t)=\begin{cases} 0 & t<0 \\ 1 & t \ge 0 \end{cases}$$

$$L\{H(t-a)\} = \frac{e^{-as}}{s}$$

Note:

If $\quad f(t)=\begin{cases} f_1(t) & t<a_1 \\ f_2(t) & a_1 \le t < a_2 \\ f_3(t) & t \ge a_2 \end{cases}$

Then

$$\begin{aligned} f(t) = & f_1(t)\, \left[1-H(t-a_1)\right] \\ & + f_2(t)\, \left[H(t-a_1)-H(t-a_2)\right] \\ & + f_3(t)\, \left[H(t-a_2)-0\right] \end{aligned}$$

Second Shifting Theorem (t Shifting):

$$L\{H(t-a)\, f(t)\} = e^{-as}\, L\{f(t+a)\}$$

4. Dirac's Delta Function (Unit Impulse function):

$$L\{\delta(t-a)\} = e^{-as}$$

For $a=0$, $L\{\delta(t)\} = e^{-0s} = 1$

Exercise

Evaluate the following:

1. $L\left\{ t^2\, H(t-2) + t^3\, \delta(t-3) \right\}$ 2. $L\left\{ (t-1)^2\, u(t-1) \right\}$ 3. $L\left\{ e^{-3t}\, u(t-2) \right\}$

4. $\int_0^\infty e^{-t}\left(1+t+t^2\right) H(t-3)\, dt$ 5. $\int_0^\infty e^{-2t}\left(1+2t-3t^2+4t^3\right) H(t-1)\, dt$

6. $L\{f(t)\}$, $f(t) = \begin{cases} t^2 & 0<t<2 \\ t-1 & 2<t<3 \\ 7 & t>3 \end{cases}$ 7. $L\{f(t)\}$, $f(t) = |t-1| + |t+1|$

8. $L\{f(t)\}$, $f(t) = \begin{cases} t^2 & 1<t<2 \\ t-1 & 2<t<3 \end{cases}$ 9. $L\{f(t)\}$, $f(t) = \begin{cases} 2t & 0<t<1 \\ 3t^2-1 & t>1 \end{cases}$

10. $L\{f(t)\}$, $f(t) = \begin{cases} t+1 & 0<t<2 \\ 3 & t>2 \end{cases}$ 11. $L\{f(t)\}$, $f(t) = \begin{cases} \sin t & 0<t<\pi \\ \cos t - \sin 2t & \pi<t<2\pi \end{cases}$

12. $L\{f(t)\}$, $f(t) = \begin{cases} \cos t & 0<t<\pi/2 \\ \sin t & t>\pi/2 \end{cases}$ 13. $L\{f(t)\}$, $f(t) = \begin{cases} \sin \omega t & 0<t<\frac{\pi}{\omega} \\ 0 & \frac{\pi}{\omega}<t<\frac{2\pi}{\omega} \end{cases}$

14. $f(t) = \begin{cases} \cos\left(t-\frac{2\pi}{3}\right) & t>\frac{2\pi}{3} \\ 0 & t<\frac{2\pi}{3} \end{cases}$

Inverse Laplace Transform

We have $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s)$

Then **Inverse Transform** of $F(s)$ is given by

$L^{-1}\{F(s)\} = f(t)$

Inverse Transform of Basic functions:

1. $L^{-1}\left\{\frac{1}{s}\right\} = 1$ 2. $L^{-1}\left\{\frac{1}{s^2}\right\} = t$

3. $L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{n!}$, $n = 0,1,2,...$ 4. $L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{\Gamma(n+1)}$, $n \neq -1,-2,....$

5. $L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$ 6. $L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}$

7. $L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin at$ 8. $L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$

9. $L^{-1}\left\{\frac{1}{s^2-a^2}\right\}=\frac{1}{a}\sinh at$

10. $L^{-1}\left\{\frac{s}{s^2-a^2}\right\}=\cosh at$

11. $L^{-1}\left\{e^{-as}\right\}=\delta(t-a)$

12. $L^{-1}\{1\}=\delta(t)$

Inverse Transform Using Properties:

1. $L^{-1}\{F(s-a)\}=e^{at}L^{-1}\{F(s)\}$

2. $L^{-1}\{F(s+a)\}=e^{-at}L^{-1}\{F(s)\}$

3. $L^{-1}\{F(s)\}=-\frac{1}{t}L^{-1}\{F'(s)\}$

Especially useful in case of logarithmic function, inverse trigonometric functions etc.

4. $L^{-1}\left\{\frac{1}{s}F(s)\right\}=\int_0^t L^{-1}\{F(s)\}dt$

5. $L^{-1}\left\{e^{-as}F(s)\right\}=H(t-a)\,L^{-1}\{F(s)\}_{t\to(t-a)}$

Definition: The convolution between any two function $f(t)$ & $g(t)$ is defined as

$$\mathrm{conv}(f,g)=\int_0^t f(u)\,g(t-u)\,du$$

Note: $\mathrm{conv}(f,g)=\mathrm{conv}(g,f)$

Convolution Theorem for Laplace Transform:

$$L\{\mathrm{conv}(f,g)\}=L\{f(t)\}\,L\{g(t)\}$$

where $\mathrm{conv}(f,g)=\int_0^t f(u)\,g(t-u)\,du$

Result: $L^{-1}\{F(s)\,G(s)\}=\mathrm{conv}\left(L^{-1}\{F(s)\},\,L^{-1}\{G(s)\}\right)$

Exercise

Find the Inverse of the following

1. $\frac{2s+4}{s^2+4}$
2. $\frac{s^2-3s+4}{s^3}$
3. $\left(\frac{1-\sqrt{s}}{s^2}\right)^2$
4. $\frac{s+4}{s^2+4s+8}$
5. $\frac{s+b}{s^2+4s+12}$
6. $\frac{3s+7}{s^2-2s-3}$
7. $\frac{s+23}{s^2-4s+13}$
8. $\frac{s}{(s+a)^2}$

Using partial fraction

1. $\dfrac{4s+5}{(s-1)^2(s+2)}$ 2. $\dfrac{5s^2-15s-11}{(s+1)(s-2)^2}$ 3. $\dfrac{2s^2-6s+5}{s^3-6s^2+11s-6}$ 4. $\dfrac{5s+3}{(s-1)\left(s^2+2s+5\right)}$

5. $\dfrac{s+29}{(s+4)\left(s^2+9\right)}$ 6. $\dfrac{3s+1`}{(s+1)\left(s^2+2\right)}$ 7. $\dfrac{s+2`}{(s+3)(s+1)^3}$ 8. $\dfrac{2}{(s+1)^2\left(s^2+4\right)}$

9. $\dfrac{s}{s^4+4a^4}$ 10. $\dfrac{s}{s^4+4}$ 11. $\dfrac{a\left(s^2-2a^2\right)}{s^4+4a^4}$ 12. $\dfrac{s^2+8s+8}{s^4+64}$

13. $\dfrac{s^2+16s-24}{s^4+20s^2+64}$ 14. $\dfrac{s^3}{s^4-a^4}$ 15. $\dfrac{s^2+6}{\left(s^2+1\right)\left(s^2+4\right)}$ 16. $\dfrac{2s^2-1}{\left(s^2+1\right)\left(s^2+4\right)}$

17. $\dfrac{s}{s^4+s^2+1}$ 18. $\dfrac{s^2+2s-4}{\left(s^2+2s+2\right)\left(s^2+2s+5\right)}$

Using Convolution theorem

1. $\dfrac{s}{\left(s^2+a^2\right)^2}$ 2. $\dfrac{s^2}{\left(s^2+a^2\right)^2}$ 3. $\dfrac{1}{\left(s^2+a^2\right)^2}$ 4. $\dfrac{s}{s^4+8s^2+16}$

5. $\dfrac{1}{\left(s^2+4s+13\right)^2}$ 6. $\dfrac{s+3}{\left(s^2+6s+13\right)^2}$ 7. $\dfrac{2s^2-s}{\left(s^2+4\right)^2}$ 8. $\left(\dfrac{s+3}{s^2+6s+13}\right)^2$

9. $\left(\dfrac{s+2}{s^2+4s+8}\right)^2$ 10. $\dfrac{(s-1)^2}{\left(s^2+4s+8\right)^2}$ 11. $\dfrac{s^2+2s+3}{\left(s^2+2s+2\right)\left(s^2+2s+5\right)}$

12. $\dfrac{s^2}{\left(s^2-a^2\right)^2}$ 13. $\dfrac{1}{\left(s^2-q^2\right)^2}$ 14. $\dfrac{1}{s^2\left(s^2+a^2\right)}$ 15. $\dfrac{1}{(s+1)(s+9)^2}$

16. $\dfrac{16}{(s-2)(s+2)^2}$ 17. $\dfrac{1}{(s-5)\left((s-3)^2+8\right)}$ 18. $\dfrac{s^2}{\left(s^2+a^2\right)\left(s^2+b^2\right)}$

19. $\dfrac{s}{\left(s^2+a^2\right)\left(s^2+b^2\right)}$ 20. $\dfrac{9-4s+s^2}{\left(s^2+9\right)^2}$ 21. $\dfrac{s+2}{\left(s^2+2s+3\right)^2}$

22. $\dfrac{s}{\left(s^2+1\right)\left(s^2+4\right)}$ 23. $\dfrac{s^2}{\left(s^2+1\right)\left(s^2+4\right)}$ 24. $\dfrac{1`}{(s+3)^2(s-2)^4}$ 25. $\dfrac{1}{s^3\left(s^2+1\right)}$

Using other properties

1. $\log\left(\frac{s+1}{s-1}\right)$ 2. $\log\left(\frac{s+a}{s+b}\right)$ 3. $\log\left(\frac{s^2+b^2}{(s-a)^2}\right)$ 4. $\log\left(1+\frac{1}{s^2}\right)$ 5. $\cot^{-1}\left(\frac{s}{a}\right)$

6. $\tan^{-1}\left(\frac{s+a}{b}\right)$ 7. $\tan^{-1}\left(\frac{2}{s^2}\right)$ 8. $\cot^{-1}\left(\frac{2}{s^2}\right)$ 9. $\tanh^{-1}\left(\frac{2}{s^2}\right)$ 10. $\frac{8e^{-3s}}{s^2+4}$

11. $\frac{e^{-2s}}{s^2+8s+25}$ 12. $\frac{(s+1)\,e^{-2s}}{s^2+2s+2}$ 13. $\frac{e^{-3s}}{(s^2-2s+5)(s+1)(s-5)}$ 14. $\frac{e^{-\frac{\pi}{2}s}+e^{-\frac{3\pi}{2}s}}{s^2+1}$

15. $\frac{e^{-\pi s}}{s^2(s^2+1)}$ 16. $\frac{e^{-\pi s}}{s^2-2s-2}$ 17. $\frac{e^{-as}}{(s+b)^{\frac{5}{2}}}$ 18. $\frac{e^{-5s}}{(s-2)^2}$ 19. $\frac{e^{4-3s}}{(s+4)^{\frac{5}{2}}}$

20. $e^{-s}\left(\frac{1-\sqrt{s}}{s^2}\right)^2$

Application of Laplace Transform

Initial Value Problems:

1. $\frac{d^2y}{dt^2}+2\frac{dy}{dt}+5y=e^{-t}\sin t$; $y(0)=0$ & $y'(0)=1$
2. $\frac{d^2y}{dt^2}+\frac{dy}{dt}=t^2+2t$; $y(0)=4$ & $y'(0)=-2$
3. $y''+y=t$; $y(0)=1$ & $y'(0)=0$
4. $y''+4y'+3y=e^{-t}$; $y(0)=1$, $y'(0)=1$
5. $y''+4y'+8y=1$; $y(0)=0$, $y'(0)=1$
6. $y''-4y=3e^{t}$; $y(0)=0$, $y'(0)=3$
7. $x''+4x=0$; $x(0)=1$, $x'(0)=-2$
8. $\frac{d^2y}{dt^2}-\frac{dy}{dt}-2y=20\sin t$; $y(0)=1$ & $y'(0)=2$
9. $y''-3y'+2y=4t+e^{3t}$; $y(0)=1$, $y'(0)=-1$
10. $y''+3y'+2y=t\,\delta(t-1)$; $y(0)=0$, $y'(0)=0$
11. $y''+4y=f(t)$; $y(0)=0,\ y'(0)=1,\ f(t)=H(t-2)$

Boundary Value Problems:

1. $\frac{d^2y}{dt^2}+9y=18t$; $y(0)=0$ & $y\left(\frac{\pi}{2}\right)=0$
2. $\frac{d^2y}{dt^2}+9y=\cos 2t$; $y(0)=1$ & $y\left(\frac{\pi}{2}\right)=-1$

Integral Equations:

1. $\frac{dy}{dt} + 2y + \int_0^t y(u)\,du = \sin t\ ;\ y(0) = 1$

2. $y = kt + \int_0^t y(u)\,\sin(t-u)\,du$