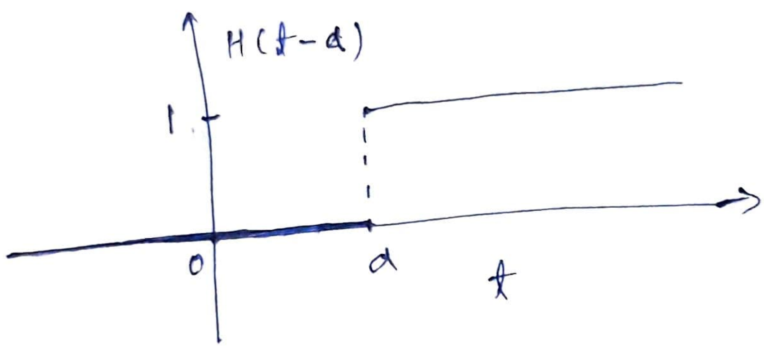
Unit Step Function [Heaviside Function]:-

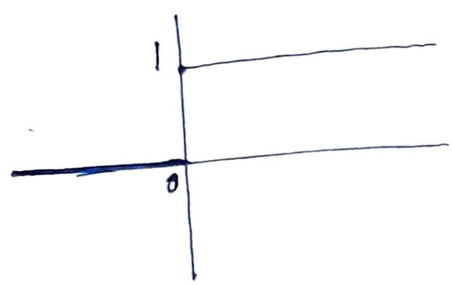
Notations: $u_a(t)$ or $u(t-a)$ or $H_a(t)$ or $H(t-a)$

$$H(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$$



Particular case:

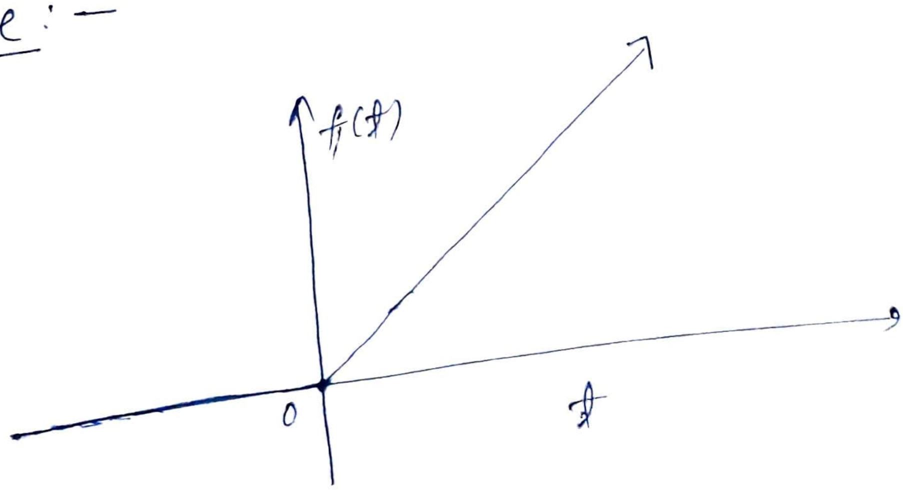
(i) $a = 0$, $H(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$



(ii) $a = \infty$, $H(t-\infty) = 0$, $-\infty < t < \infty$

Note:-

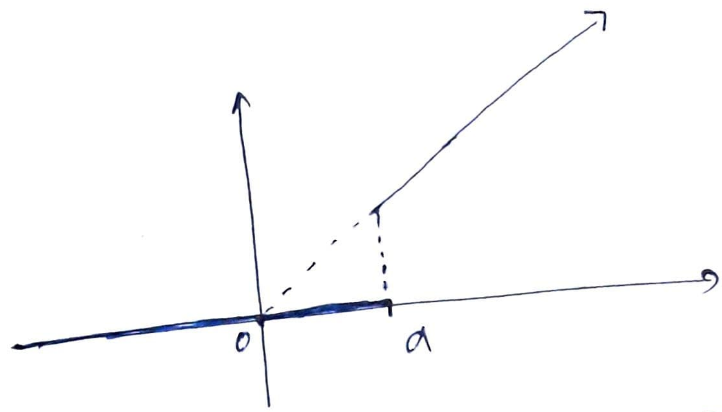
(1)



Then $f(t) = H(t)\, f(t)$

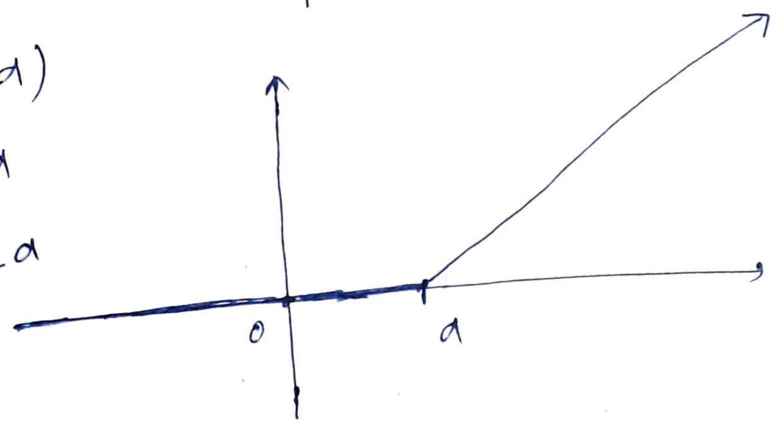
(2) $H(t-a)\, f(t)$

$$= \begin{cases} 0 & t < a \\ f(t) & t \geq a \end{cases}$$

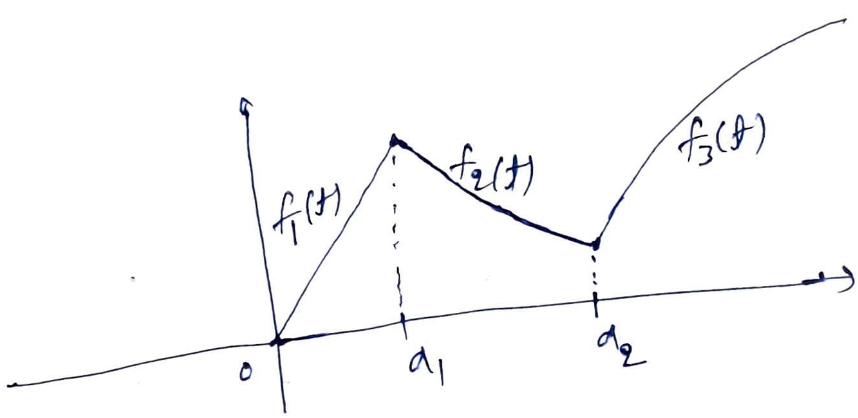


(3) $H(t-a)\, f(t-a)$

$$= \begin{cases} 0 & t < a \\ f(t-a) & t \geq a \end{cases}$$



(4)



$$f(t) = \begin{cases} f_1(t) & 0 \leq t < a_1 \\ f_2(t) & a_1 \leq t < a_2 \\ f_3(t) & t \geq a_2 \end{cases}$$

$$= f_1(t)\left[\overbrace{H(t-0)}^{=1} - H(t-a_1)\right]$$

$$+ f_2(t)\left[H(t-a_1) - H(t-a_2)\right]$$

$$+ f_3(t)\left[H(t-a_2) - \underbrace{H(t-\infty)}_{=0}\right]$$

Second shifting (t-shifting) property:-

Let $g(t) = H(t-a)\, f(t)$

$$= \begin{cases} 0 & t < a \\ f(t) & t \geq a \end{cases}$$

Then

$$\boxed{L\{g(t)\} = L\{H(t-a)\, f(t)\} = e^{-as}\, L\{f(t+a)\}}$$

Ex. 1 Let

$$f(t) = \begin{cases} t & t < 5 \\ 2-3t^2 & 5 \leq t < 7 \\ e^{-t} & t \geq 7 \end{cases}$$

Find $L\{g(t)\}$.

Sol^n

$$f(t) = t\,[\overset{=1}{H(t-0)} - H(t-5)]$$

$$+ (2-3t^2)\,[H(t-5) - H(t-7)]$$

$$+ e^{-t}\,[H(t-7) - \underset{0}{\underset{\|}{H(t-\infty)}}]$$

$$= t + H(t-5)\,(-t+2-3t^2)$$

$$+ H(t-7)\,(-2+3t^2+e^{-t})$$

$$L\{t\} = \frac{1}{s^2}$$

$$L\{H(t-5)(-t+2-3t^2)\}$$

$$= e^{-5s} L\{-(t+5)+2-3(t+5)^2\}$$

$$= e^{-5s} L\{-t-5+2-3(t^2+10t+25)\}$$

$$= e^{-5s} L\{-t-3-3t^2-30t-75\}$$

$$= e^{-5s} L\{-78-31t-3t^2\}$$

$$= e^{-5s}\left\{-\frac{78}{s}-\frac{31}{s^2}-\frac{6}{s^3}\right\}$$

$$L\{H(t-7)(-2+3t^2+e^{-t})\}$$

$$= e^{-7s} L\{-2+3(t+7)^2+e^{-(t+7)}\}$$

$$= e^{-7s} L\{-2+3(t^2+14t+49)+e^{-7}e^{-t}\}$$

$$= e^{-7s} L\{145+42t+3t^2+e^{-7}e^{-t}\}$$

$$= e^{-7s}\left\{\frac{145}{s}+\frac{42}{s^2}+\frac{6}{s^3}+e^{-7}\frac{1}{s+1}\right\}$$

$\therefore$ $L\{f(t)\}$

$$= \frac{1}{s^2} - e^{-5s}\left[\frac{78}{s}+\frac{31}{s^2}+\frac{6}{s^3}\right]$$

$$+ e^{-7s}\left[\frac{145}{s}+\frac{42}{s^2}+\frac{6}{s^3}+\frac{e^{-7}}{s+1}\right]$$

(2) $f(t) = \begin{cases} 2\sin t & 0 \le t < \pi \\ t + \cos 2t & \pi \le t < 2\pi \\ 2\cos 2t - \sin 3t & t \ge 2\pi \end{cases}$

Find $L\{f(t)\}$

Soln

$$f(t) = 2\sin t\,[1 - H(t-\pi)]$$
$$+ (t + \cos 2t)\,[H(t-\pi) - H(t-2\pi)]$$
$$+ (2\cos 2t - \sin 3t)\,[H(t-2\pi) - 0]$$

$$= 2\sin t + H(t-\pi)\,[-2\sin t + t + \cos 2t]$$
$$+ H(t-2\pi)\,[-t - \cos 2t + 2\cos 2t - \sin 3t]$$

$$= 2\sin t + H(t-\pi)\,[t - 2\sin t + \cos 2t]$$
$$+ H(t-2\pi)\,[-t + \cos 2t - \sin 3t]$$

$$L\{2\sin t\} = \frac{2}{s^2+1}$$

$$L\{H(t-\pi)\,[t - 2\sin t + \cos 2t]\}$$
$$= e^{-\pi s} L\{t + \pi - 2\sin(t+\pi) + \cos 2(t+\pi)\}$$
$$= e^{-\pi s} L\{\pi + t + 2\sin t + \cos 2t\}$$
$$= e^{-\pi s}\left\{\frac{\pi}{s} + \frac{1}{s^2} + \frac{2}{s^2+1} + \frac{s}{s^2+4}\right\}$$

$$L\{H(t-2\pi)(-t+\cos 2t-\sin 3t)\}$$

$$= e^{-2\pi s} L\{-(t+2\pi)+\cos 2(t+2\pi)-\sin 3(t+2\pi)\}$$

$$= e^{-2\pi s} L\{-2\pi - t + \cos 2t - \sin 3t\}$$

$$= e^{-2\pi s}\left\{-\frac{2\pi}{s} - \frac{1}{s^2} + \frac{s}{s^2+4} - \frac{3}{s^2+9}\right\}$$

$\therefore$ $L\{f(t)\}$

$$= \frac{2}{s^2+1} + e^{-\pi s}\left[\frac{\pi}{s} + \frac{1}{s^2} + \frac{2}{s^2+1} + \frac{s}{s^2+4}\right]$$

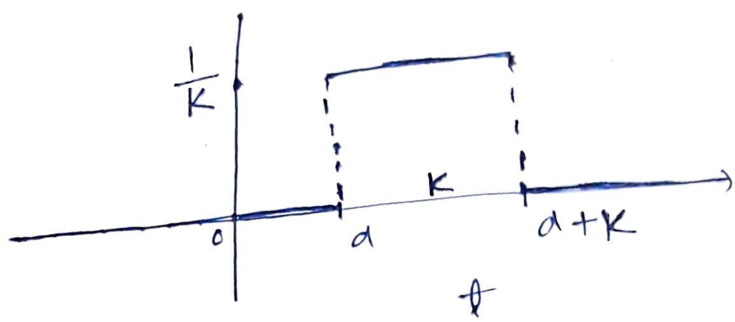
$$+ e^{-2\pi s}\left[-\frac{2\pi}{s} - \frac{1}{s^2} + \frac{s}{s^2+4} - \frac{3}{s^2+9}\right]$$

Unit Impulse (Dirac Delta) Function :-

Notation :- $\delta_a(t)$ or $\delta(t-a)$

$$\delta(t-a) = \lim_{K \to 0} h(t)$$

where
$$h(t) = \begin{cases} \frac{1}{K} & a \le t \le a+K \\ 0 & \text{otherwise} \end{cases}$$



or

$$\delta(t-a) = \begin{cases} \infty & t = a \\ 0 & t \neq a \end{cases}$$

and it satisfy the identity

$$\int_{-\infty}^{\infty} \delta(t-a)\, dt = 1$$

i.e. It gives a impulse one unit at the time instant $t = a$.

Translation property :

$$\int_{-\infty}^{\infty} f(t)\ \delta(t-a)\ dt = f(a)$$

Result :-

$$\boxed{L\{\delta(t-a)\} = e^{-as}}$$

For $a=0$;

$$\boxed{L\{\delta(t)\} = 1}$$

Ex. $f(t) = t^5\ \delta(t-2)$; Find $L\{f(t)\}$

Sol^n

$$L\{f(t)\} = (-1)^5 \frac{d^5}{ds^5} L\{\delta(t-2)\}$$

$$= -\frac{d^5}{ds^5}\left[e^{-2s}\right]$$

$$= -(-2)^5 e^{-2s}$$

$$= 2^5 e^{-2s}$$