Vectors, Linear Algebra, and It's Applications.

a:- properties of vector space

a.1: starting from Basics.

→ what are the vectors in popular notation?

Vectors are the building blocks of everything multivariable. We use them when we want to represent a coordinate in higher dimensional space or more. vectors can be added together and multiplied by scalars to produce another object of the same kind. In popular notion, we can think vectors as a list of numbers and vector algebra as operations performed on the numbers in the list.

There are lots of ways to write vectors. Here are the three most used notations. The arrow on top of $\vec{v}$ is a convention that indicates that $\vec{v}$ refers to a vector.

$$\vec{v} = (1,2,3) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1\hat{i} + 2\hat{j} + 3\hat{k}$$

Above notation extends to any number of dimensions

$\vec{v} = (1,2,3)$ represents the components of the vector $\vec{v}$ along x, y and z axes.

$\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{v}$ represents ~~as~~ column matrix.

$\vec{v} = [1 \ 2 \ 3]$, it represents a row matrix.

$\vec{v} = 1\hat{i} + 2\hat{j} + 3\hat{k}$, represents sum of $\vec{v}$ components along standard unit vectors $\hat{i}$ (x axis), $\hat{j}$ (Y axis) and $\hat{k}$ (along Z axis)

→ Are they a bunch of numbers?

Generally, a vector is a list of ~~numb~~ things, which ends up typically with meaning 'numbers', but not always. While vectors are used to represent and manipulate data, vectors exibit operations beyond traditional arithmetic such as vector scaling, scalar product and cross product. These operations capture geometric relationships, orthogonality and rotational aspects which are not present in basic number operations

→ Let us say there is a 3x3 matrix. Can the set of all 3x3 matrices can be considered as a vector space?

Yes, it can be considered as a vector space if it follows the following properties.

① Commutativity

$x+y=y+x \quad \forall \; x,y \in V$ "V is vector space over field F"

② Associativity

$(x+y)+z = x+(y+z)$

and $(x y_1)c = x_1(y_1 c) \quad \forall \; x,y,z \in V$ and $x_1, y_1 \in F$

③ Additive identity/unique vector

There exists an element $0 \in V$ such that $0+x=x \quad \forall \; x \in V$.

④ Additive inverse

For every $x \in V$, there exists an element ~~or~~ $W \in V$ such that $x+W=0$. or $\vec{x}+(-\vec{x})=0$

(5) Multiplicative identity

$$1 \cdot \vec{V_1} = \vec{V_1} \quad \forall\, V_1 \in V$$

(8) Distributivity

$$C_1(V_1 + V_2) = C_1V_1 + C_1V_2$$

$$(C_1 + C_2)V_1 = C_1V_1 + C_2V_1$$

"The elements $V_1 \in V$ of a vector space are called vectors".

⇒ How many basis vectors would space need?

Every basis for a vector space has same number of vectors.

Basis vectors for 3x3 vector space will be $3 \times 3 = 9$

$$B = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

∴ Dimension of $n \times n$ real matrix is n^2.

2. Subspaces

Q2 :- Subspaces

→ Can the Column matrices with real entries form a vector space?

Yes, all the Column matrices with real entries form a vector space because they satisfy all the properties of a vector space. These properties include closure under addition and scalar multiplication, existence of an additive identity and additive inverses, and the distributive and associative properties. Additionaly, the set of matrices over the real entries is closed under scalar multiplication and matrix addition, making it a vector space.

→ If we consider the space containing the Column vectors with only positive real entries, can these form a subspace of the previously described space?

NO, space containing the column vectors with only positive real entries cannot form a subspace because it does not satisfy the all axioms of vector space. ~~Subspace~~ It lacks zero vector. A subspace must contain zero vector. However, in the space of column vectors with only real positive entries, the zero vector is absent. It also lacks additive inverse because space have only positive real entries, there are no negative numbers to form additive inverses for the positive real entries.

→ Can functions be considered as vector spaces?

Yes, Certain sets of functions can be considered vector spaces, they should satisfy the axioms of vector space.

Q.3:- Basis

→ prove that given n dimensional vector space and 'n' basis vectors, all vectors in this vector space can be represented as a linear combination of these basis vectors.

Proof:

Suppose, we have vector space 'V' over the field 'F' having 'n' dimensions, with basis $x_1, x_2, x_3 \ldots x_n$.

Consider two different representations of vector x in terms of the basis of vector.

$$x = a_1 x_1 + a_2 x_2 + a_3 x_3 + \ldots + a_n x_n \quad —(1)$$

$$x = b_1 x_1 + b_2 x_2 + b_3 x_3 + \ldots + b_n x_n \quad —(2)$$

Subtracting eqn (2) from eqn (1)

$$0 = (a_1 - b_1) x_1 + (a_2 - b_2) x_2 + (a_3 - b_3) x_3 + \ldots + (a_n - b_n) x_n \quad —(3)$$

Here 0 is the zero vector.

Since B is basis vectors i.e., vectors in B are linearly independent, the only way for this linear combination to equal the zero vector is if each coefficient is zero.

From eqn (3)

$(a_1 - b_1) = 0 \qquad a_1 = b_1$

$(a_2 - b_2) = 0 \qquad a_2 = b_2$

$\vdots \qquad\qquad \vdots$

$(a_n - b_n) = 0 \Rightarrow a_n = b_n$

Therefore, the coefficients in both representations are the same.

This shows that any vector 'x' in the vector space 'V' has unique representation in terms of the basis of vectors $x_1, x_2 \dots, x_n$. The coefficients in this representation are unique for each ~~other~~ vector, confirming that any vector can be expressed as a linear combination of the basis vectors in unique way

6:- Matrices and system of Linear equations.

6.1:- Gaussian Elimination

Carry out Gaussian elimination on the following matrix:

$$\begin{bmatrix} 1 & 3 & 6 & 8 \\ 2 & 3 & 9 & 2 \\ 5 & 1 & 3 & 6 \end{bmatrix}$$

which columns have pivots, and which columns do not. What is the column space of the matrix?

$$A = \begin{bmatrix} 1 & 3 & 6 & 8 \\ 2 & 3 & 9 & 2 \\ 5 & 1 & 3 & 6 \end{bmatrix}$$

Augmented matrix will be

$$A = \left[\begin{array}{ccc|c} 1 & 3 & 6 & 8 \\ 2 & 3 & 9 & 2 \\ 5 & 1 & 3 & 6 \end{array}\right]$$

we, will apply elementary row operations to make lower triangle entries as '0'

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 5R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 6 & 8 \\ 0 & -3 & -3 & -14 \\ 0 & -14 & -27 & -34 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 6 & 8 \\ 0 & -3 & -3 & -14 \\ 0 & -14 & -27 & -34 \end{array}\right]$$

$$R_2 \to -\frac{R_2}{3}, \quad R_3 \to -R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 6 & 8 \\ 0 & 1 & 1 & 14/3 \\ 0 & 14 & 27 & 34 \end{array}\right]$$

$$R_3 \to R_3 - 14R_2$$

$$R_1 \to R_1 - 3R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & -6 \\ 0 & 1 & 1 & 14/3 \\ 0 & 0 & 13 & -94/3 \end{array}\right]$$

$$R_3 \to R_3/13$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & -6 \\ 0 & 1 & 1 & 14/3 \\ 0 & 0 & 1 & -94/39 \end{array}\right]$$

$$R_1 \to R_1 - 3R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 16/13 \\ 0 & 1 & 1 & 14/3 \\ 0 & 0 & 1 & -94/39 \end{array}\right]$$

$$R_2 \to R_2 - R_3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 16/13 \\ 0 & 1 & 0 & 92/13 \\ 0 & 0 & 1 & -94/39 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 16/13 \\ 0 & 1 & 0 & 92/13 \\ 0 & 0 & 1 & -94/39 \end{array}\right]$$

$Ax = b$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 16/13 \\ 92/13 \\ -94/39 \end{bmatrix}$$

$x_1 \times 1 + 0 \times x_2 + 0 \times x_3 = 16/13 \Rightarrow x_1 = 16/13$

$x_1 \times 0 + x_2 \times 1 + x_3 \times 0 = 92/13 \Rightarrow x_2 = 92/13$

$0 \times x_1 + 0 \times x_2 + 1 \times x_3 = -94/39 \Rightarrow x_3 = -94/39$

- In the given matrix, we are having pivots in Column 1, column 2 and column 3.

$$C_1 \qquad C_2 \qquad C_3$$

$$C_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Column space for matrix

$$C_1 = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 6 \\ 9 \\ 3 \end{bmatrix}$$

b2:- Null space

→ Find Null space of the following matrices

$$\begin{bmatrix} 1 & 3 & 4 & 5 & 7 \\ 3 & 9 & 6 & 5 & 4 \\ 2 & 5 & 4 & 5 & 2 \end{bmatrix}$$

Applying row operations to get row reduced Echelon form.

$R_1 \to R_2 - 3R_1$

$R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 3 & 4 & 5 & 7 \\ 0 & 0 & -6 & -10 & -17 \\ 0 & -1 & -4 & -5 & -12 \end{bmatrix}$$

$R_3 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & 3 & 4 & 5 & 7 \\ 0 & -1 & -4 & -5 & -12 \\ 0 & 0 & -6 & -10 & -17 \end{bmatrix}$$

$R_3 \to -R_2$

$R_3 \to -R_3/6$

$$\begin{bmatrix} 1 & 3 & 4 & 5 & 7 \\ 0 & 1 & 4 & 5 & 2 \\ 0 & 0 & 1 & 5/3 & 17/6 \end{bmatrix}$$

$R_1 \to R_1 - 4R_3$

$R_2 \to R_2 - 4R_3$

$$\sim \begin{bmatrix} 1 & 3 & 0 & -5/3 & -13/3 \\ 0 & 1 & 0 & -5/3 & 2/3 \\ 0 & 0 & 1 & 5/3 & 17/6 \end{bmatrix}$$

$R_1 \to R_1 - 3R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 10/3 & -19/3 \\ 0 & 1 & 0 & -5/3 & 2/3 \\ 0 & 0 & 1 & 5/3 & 17/6 \end{bmatrix}$$

we know that for null space $Ax = 0$

$$\begin{bmatrix} 1 & 0 & 0 & 10/3 & -19/3 \\ 0 & 1 & 0 & -5/3 & 2/3 \\ 0 & 0 & 1 & 5/3 & 17/6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Pivot columns are C_1, C_2 and C_3 while free columns are C_4 and C_5

Taking only pivots

$$\left.\begin{aligned} x_1 + 10/3\, x_4 - 19/3\, x_5 &= 0 \\ x_2 - 5/3\, x_4 + 2/3\, x_5 &= 0 \\ x_3 + 5/3\, x_4 + 17/6\, x_5 &= 0 \end{aligned}\right\} \text{——} (1)$$

From eq (1), we get

$x_1 = -10/3\, x_4 + 19/3\, x_5$

$x_2 = 5/3\, x_4 - 2/3\, x_5$

$x_3 = -5/3\, x_4 - 17/6\, x_5$

Equation for free variables

$x_4 = x_4$

$x_5 = x_5$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -10/3\,x_4 \\ 5/3\,x_4 \\ -5/3\,x_4 \\ x_4 \\ 0 \end{bmatrix} + \begin{bmatrix} 19/3\,x_5 \\ -2/3\,x_5 \\ -17/6\,x_5 \\ 0 \\ x_5 \end{bmatrix}$$

Taking free variables common

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_4 \begin{bmatrix} -10/3 \\ 5/3 \\ -5/3 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 19/3 \\ -2/3 \\ -17/6 \\ 0 \\ 1 \end{bmatrix}$$

Therefore basis for null space is

$$\left\{ \begin{bmatrix} -10/3 \\ 5/3 \\ -5/3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 19/3 \\ -2/3 \\ -17/6 \\ 0 \\ 1 \end{bmatrix} \right\} \longrightarrow A$$

Using the knowledge of the null matrix, solve the following equation

$$\begin{bmatrix} 1 & 3 & 4 & 5 & 7 \\ 3 & 9 & 6 & 5 & 4 \\ 2 & 5 & 4 & 5 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \\ 9 \end{bmatrix}$$

Augmented matrix is

$$\sim \left[\begin{array}{ccccc|c} 1 & 3 & 4 & 5 & 7 & 2 \\ 3 & 9 & 6 & 5 & 4 & 7 \\ 2 & 5 & 4 & 5 & 2 & 9 \end{array}\right]$$

$R_2 \to R_2 - 3R_1$

$R_3 \to R_3 - 2R_1$

$$\sim \left[\begin{array}{ccccc|c} 1 & 3 & 4 & 5 & 7 & 2 \\ 0 & 0 & -6 & -10 & -17 & 1 \\ 0 & -1 & -4 & -5 & -12 & 5 \end{array}\right]$$

$R_3 \leftrightarrow R_2$

$$\sim \left[\begin{array}{ccccc|c} 1 & 3 & 4 & 5 & 7 & 2 \\ 0 & -1 & -4 & -5 & -12 & 5 \\ 0 & 0 & -6 & -10 & -17 & 1 \end{array}\right]$$

$R_2 \to -R_2$

$R_3 \to -R_3/6$

$$\sim \left[\begin{array}{ccccc|c} 1 & 3 & 4 & 5 & 7 & 2 \\ 0 & 1 & 4 & 5 & 12 & -5 \\ 0 & 0 & 1 & 5/3 & 17/6 & -1/6 \end{array}\right]$$

$R_1 \to R_1 - 4R_3$

$R_2 \to R_2 - 4R_3$

$$\sim \left[\begin{array}{ccccc|c} 1 & 3 & 0 & -5/3 & -13/3 & 8/3 \\ 0 & 1 & 0 & -5/3 & 2/3 & -13/3 \\ 0 & 0 & 1 & 5/3 & 17/6 & -1/6 \end{array}\right]$$

$R_1 \to R_1 - 3R_2$

$$\sim \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 10/3 & -19/3 & 47/3 \\ 0 & 1 & 0 & -5/3 & 2/3 & -13/3 \\ 0 & 0 & 1 & 5/3 & 17/6 & -1/6 \end{array}\right]$$

Since for particular solution

$Ax = b$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 10/3 & -19/3 \\ 0 & 1 & 0 & -5/3 & 2/3 \\ 0 & 0 & 1 & 5/3 & 17/6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 47/3 \\ -13/3 \\ -1/6 \end{bmatrix}$$

$x_1 + 10/3\, x_4 - 19/3\, x_5 = 47/3$

$x_2 - 5/3\, x_4 + 2/3\, x_5 = -13/3$

$x_3 + 5/3\, x_4 + 17/6\, x_5 = -1/6$

Since column 4 and 5 are free ones

Put $x_4 = 0$ and $x_5 = 0$

$\therefore x_1 = 47/3,\ x_2 = -13/3,\ x_3 = -1/6$

∴ Particular Solution i.e.,

$$x_p = \begin{bmatrix} 47/3 \\ -13/3 \\ -1/6 \\ 0 \\ 0 \end{bmatrix}$$

Complete Solution will be

$$x = x_p + x_n$$

$$x = \begin{bmatrix} 47/3 \\ -13/3 \\ -1/6 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -10/3 \\ 5/3 \\ -5/3 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 19/3 \\ -2/3 \\ -17/6 \\ 0 \\ 1 \end{bmatrix}$$

C. Inner products

C.1: Functions Can also have inner products?

→ what is the definition of an inner product space?

let 'V' be the vector space over the field 'F' set of real numbers.

A mapping $\langle , \rangle : V \times V \to F$ is said to be an inner product on 'V'.

If the following Conditions are satisfied

i) $\langle v, v \rangle \geq 0$ And $\langle v, v \rangle = 0$ If and only If $v = 0, \forall v \in V$

ii) $\langle v_1, v_2 \rangle = \langle v_2, v_1 \rangle, \forall v_1, v_2 \in V$

iii) $\langle a v_1 + b v_2, v_3 \rangle = a \langle v_1, v_3 \rangle + b \langle v_2, v_3 \rangle, \forall v_1, v_2, v_3 \in V$
$a, b \in F$

NOTE: The pair $(V, \langle , \rangle)$ is called inner product let $u, v \in R^n$, where $u = (u_1, u_2, u_3, \ldots u_n)$

$v = (v_1, v_2, v_3, \ldots, v_n)$

then the product $\langle u, v \rangle = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$ is an inner product of R^n.

Given $u, v \in R^n$, $u = (u_1, u_2, \ldots u_n)$

$V = (v_1, v_2, \ldots v_n)$

$\langle u, v \rangle = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$

i) $\langle u, u \rangle \geq 0$ and $\langle u, u \rangle = 0$ iff $u = 0$

$\langle u, u \rangle = u_1 \times u_1 + u_2 \times u_2 + \ldots u_n \times u_n$

$= u_1^2 + u_2^2 + \ldots u_n^2$

$\langle u, u \rangle \geq 0$ (because sum of +ve and squared terms is always +ve)

Now, we will prove $\langle u, u \rangle = 0$ iff $u = 0$

Suppose $\langle u, u \rangle = 0$

$\Rightarrow u_1^2 + u_2^2 + u_3^2 + \ldots + u_n^2 = 0$

$\Rightarrow u_1^2 = 0, u_2^2 = 0, u_3^2 = 0, \ldots u_n^2 = 0$

$\Rightarrow u_1 = 0, u_2 = 0, u_3 = 0, \ldots u_n = 0$

$u = (u_1, u_2, u_3, \ldots, u_n)$

$u = (0, 0, 0, \ldots, 0)$

$u = 0$

Conversely, Suppose $u=0$, we shall show $\langle u,u\rangle=0$

$u=0$

$\Rightarrow u=(u_1, u_2, u_3 \ldots, u_n)$

$=(0, 0, 0, \ldots 0)$

$\Rightarrow u_1=0, u_2=0, \ldots u_n=0$

$\Rightarrow u_1^2=0, u_2^2=0, \ldots u_n^2=0$

$\Rightarrow u_1^2+u_2^2+\ldots u_n^2=0$

$\Rightarrow \boxed{\langle u,u\rangle=0}$

ii) $\langle u,v\rangle = u_1v_1+u_2v_2+u_3v_3+\ldots+u_nv_n$

$= v_1u_1+v_2u_2+v_3u_3+\ldots+v_nu_n$

$\Rightarrow \boxed{\langle u,v\rangle=\langle v,u\rangle}$

iii) Let $a,b \in R$ $u,v,w \in R^n$

$u=(u_1, u_2 \ldots u_n)$

$v=(v_1, v_2 \ldots v_n)$

$w=(w_1, w_2 \ldots w_n)$

Now, we shall prove

$\langle au+bv, w\rangle = a\langle u,w\rangle + b\langle v,w\rangle$

Taking LHS

$=\langle au+bv, w\rangle$

$=\langle a(u_1, u_2 \ldots u_n)+b(v_1, v_2 \ldots v_n), (w_1, w_2, \ldots w_n)\rangle$

$=\langle (au_1, au_2 \ldots au_n)+(bv_1, bv_2 \ldots bv_n), (w_1, w_2 \ldots w_n)\rangle$

$(au_1+bv_1)w_1+(au_2+bv_2)w_2+(au_3+bv_3)w_3+\ldots+(au_n+bv_n)w_n$

$(au_1w_1+au_2w_2+au_3w_3+\ldots au_nw_n)+(bv_1w_1+bv_2w_2+bv_3w_3+\ldots+bv_nw_n)$

$= a(u_1 w_1 + u_2 w_2 + \dots + u_n w_n) + b(v_1 w_1 + v_2 w_2 + \dots + v_n w_n)$

$\Rightarrow a\langle u, w\rangle + b\langle v, w\rangle$

$=$ RHS

let us say that we have vector space which is given by the function $f: [-1,1] \to \mathbb{R}$, the inner product is defined as

$$\langle f(x), g(x)\rangle = \int_{-1}^{1} x^2 f(x) g(x)\,dx.$$

there is basis given as $\{1, x, x^2\}$. Find the inner product between these bases. Can you orthogonalize using Gram Schmidt procedure?

Sol:- Inner products between bases $1, x, x^2$ over interval $[-1, 1]$

For $\langle 1, 1\rangle$

$$\int_{-1}^{1} x^2 f(x) g(x)\,dx = \int_{-1}^{1} x^2 \times 1 \times 1\,dx = \frac{1}{3}[x^3]_{-1}^{1} = \frac{1}{3}[1^3 - (-1)^3] = \frac{2}{3}$$

For $\langle 1, x\rangle$

$$\int_{-1}^{1} x^2 f(x) g(x)\,dx = \int_{-1}^{1} x^2 \times 1 \times x\,dx = \frac{1}{4}[x^4]_{-1}^{1} = \frac{1}{4}[1^4 - (-1)^4] = 0$$

For $\langle 1, x^2\rangle$

$$\int_{-1}^{1} x^2 f(x) g(x)\,dx = \int_{-1}^{1} x^2 \times 1 \times x^2\,dx = \frac{1}{5}(x^5)_{-1}^{1} = \frac{1}{5}[1^5 - (-1)^5] = \frac{2}{5}$$

For $\langle x, x\rangle$

$$\int_{-1}^{1} x^2 f(x) g(x)\,dx = \int_{-1}^{1} x^2 \times x \times x\,dx = \frac{1}{5}[1^5 - (-1)^5] = \frac{2}{5}$$

For $\langle x, x^2 \rangle$

$$\int_{-1}^{1} x^2 f(x) g(x) dx = \int_{-1}^{1} x^2 \times x \times x^2 = \frac{1}{6}\left[x^6\right]_{-1}^{1} = \frac{1}{6}\left[1^6 - (-1)^6\right] = 0$$

For $\langle x^2, x^2 \rangle$

$$\int_{-1}^{1} x^2 f(x) g(x) dx = \int_{-1}^{1} x^2 \times x^2 \times x^2 = \frac{1}{7}\left[x^7\right]_{-1}^{1} = \frac{1}{7}\left[1^7 - (-1)^7\right] = 2/7$$

Inner products between

$\langle 1, 1 \rangle = 2/3$

$\langle 1, x \rangle = 0$

$\langle 1, x^2 \rangle = 2/5$

$\langle x, x \rangle = 2/5$

$\langle x, x^2 \rangle = 0$

$\langle x^2, x^2 \rangle = 2/7$

- Orthogonalize using Grahm Schmidt method

we know that bases are $1, x, x^2$

Put $u_1 = 1, u_2 = x, u_3 = x^2$

Now,
$\|u_1\| = (\langle u_1, u_1 \rangle)^{1/2}$

From eqn (1)
we get,
$\|u_1\| = \sqrt{2}$ —— (A)

$\|u_2\| = (\langle u_2, u_2 \rangle)^{1/2}$

From eqn (1)
we get,

$\|u_2\| = \sqrt{2/5}$ —— (B)

$\|u_3\| = (\langle u_3, u_3 \rangle)^{1/2}$ —— ⊗

From eqn ⊗

we get

$\|u_3\| = \sqrt{2/7}$ —— (C)

Take $u_1 = v_1$

$\therefore V_2 = u_2 - \dfrac{\langle u_2, v_1 \rangle v_1}{\| v_1 \|}$

$= x - \dfrac{\left(\int_{-1}^{1} x^2 \times x \times 1\, dx\right)}{\sqrt{2}} \times 1$ (From eqn (A) $\|v_1\| = \sqrt{2}$)

$= x - \dfrac{1}{\sqrt{2}} \int_{-1}^{1} x^3 dx$

$= x - \dfrac{1}{4\sqrt{2}} [1 - 1]$

$= x$

$\therefore V_2 = x$

Now, $V_3 = u_3 - \dfrac{\langle u_3, v_1 \rangle v_1}{\|v_1\|} - \dfrac{\langle u_3, v_2 \rangle}{\| v_2 \|} v_2$

$V_3 = x^2 - \dfrac{\left(\int_{-1}^{1} x^2 \times x^3 \times 1\, dx\right) \times 1}{\sqrt{2}} - \dfrac{\left(\int_{-1}^{1} x^2 \times x^2 \times x\, dx\right)}{\sqrt{2/5}}$

$V_3 = x^2 - \left(\dfrac{\int_{-1}^{1} x^4 dx}{\sqrt{2}}\right) - \left(\dfrac{\int_{-1}^{1} x^5 dx}{\sqrt{2/5}}\right)$

$V_3 = x^2 - \dfrac{2}{5\sqrt{2}}$ —— (D)

$$V_3 = x^2 - \frac{2}{5\sqrt{2}} \quad \text{——} \; (D)$$

Thus we have $V_1 = U_1$, $V_2 = U_2$, $V_3 = x^2 - \frac{2}{5\sqrt{2}}$

Also $\|V_3\| = \langle V_3, V_3 \rangle$

$$= \int_{-1}^{1} x^2 \left(x^2 - \frac{2}{5\sqrt{2}}\right)^2 dx$$

$$= \int_{-1}^{1} \left(x^6 - \frac{2^{3/2} x^4}{5} + \frac{2x^2}{25}\right) dx$$

$$= \int_{-1}^{1} x^6 dx - \frac{1}{5}\int_{-1}^{1} 2^{3/2} x^4 dx + \frac{2}{25}\int_{-1}^{1} x^2 dx$$

$$= \frac{2}{525}\left(89 - 42\sqrt{2}\right)$$

Therefore the required orthonormal Basis are, $\frac{V_1}{\|V_1\|}, \frac{V_2}{\|V_2\|}, \frac{V_3}{\|V_3\|}$

$$= \left\{ \frac{1}{\sqrt{2}}, \frac{\sqrt{5}}{\sqrt{2}} x, \frac{525}{2 \times (89 - 42\sqrt{2})} \times \left(x^2 - \frac{2}{5\sqrt{2}}\right) \right\}$$

C2: Graham Schmidt procedure

There are three vectors $\begin{bmatrix}2\\3\\4\end{bmatrix}, \begin{bmatrix}9\\7\\6\end{bmatrix}, \begin{bmatrix}2\\1\\5\end{bmatrix}$

orthogonalize these using the Graham Schimdt process?

Sol:- Given basis a, b, c are $\begin{bmatrix}2\\3\\4\end{bmatrix}, \begin{bmatrix}9\\7\\6\end{bmatrix}, \begin{bmatrix}2\\1\\5\end{bmatrix}$

Let orthogonal basis be q_1, q_2 and q_n

Let $q_1 = a$

$$q_1 = \begin{bmatrix}2\\3\\4\end{bmatrix}$$

$$\therefore q_2 = b - \frac{b^T \cdot q_1}{\|q_1\|^2} q_1$$

$$\therefore q_2 = \begin{bmatrix}9\\7\\6\end{bmatrix} - \frac{\begin{bmatrix}9 & 7 & 6\end{bmatrix}\begin{bmatrix}2\\3\\4\end{bmatrix}}{\begin{bmatrix}2 & 3 & 4\end{bmatrix}\begin{bmatrix}2\\3\\4\end{bmatrix}} \times \begin{bmatrix}2\\3\\4\end{bmatrix}$$

$$= \begin{bmatrix}9\\7\\6\end{bmatrix} - \frac{63}{29}\begin{bmatrix}2\\3\\4\end{bmatrix}$$

$$= \begin{bmatrix}9\\7\\6\end{bmatrix} - \begin{bmatrix}126/29\\189/29\\252/29\end{bmatrix}$$

$$\therefore q_2 = \begin{bmatrix}135/29\\14/29\\-78/29\end{bmatrix}$$

Also, $q_3 = C - \frac{C^T q_1}{\|q_1\|^2} q_1 - \frac{C^T q_2}{\|q_2\|^2} q_2$

$$q_3 = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} - \frac{\begin{bmatrix} 2 & 1 & 5 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}}{\begin{bmatrix} 2 & 3 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}} \times \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} - \frac{\begin{bmatrix} 2 & 1 & 5 \end{bmatrix}\begin{bmatrix} 135/29 \\ 14/29 \\ -78/29 \end{bmatrix}}{\begin{bmatrix} 135/29 \\ 14/29 \\ -78/29 \end{bmatrix}\begin{bmatrix} \frac{135}{29} & \frac{14}{29} & \frac{-78}{29} \end{bmatrix}} \times \begin{bmatrix} 135/29 \\ 14/29 \\ -78/29 \end{bmatrix}$$

$$q_3 = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} - \frac{27}{29}\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + \frac{106}{\cancel{29}} \times \frac{\cancel{29}^2}{24505} \begin{bmatrix} 135/29 \\ 14/29 \\ -78/29 \end{bmatrix}$$

$$q_3 = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} - \begin{bmatrix} 54/29 \\ 81/29 \\ 108/29 \end{bmatrix} + \frac{106}{845}\begin{bmatrix} 135/29 \\ 14/29 \\ -78/29 \end{bmatrix}$$

$$q_3 = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} - \begin{bmatrix} 54/29 \\ 81/29 \\ 108/29 \end{bmatrix} + \begin{bmatrix} \frac{14310}{24505} \\ \frac{1484}{24505} \\ -\frac{8268}{24505} \end{bmatrix}$$

$$q_3 = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} - \begin{bmatrix} 1.8620 \\ 2.7931 \\ 3.7241 \end{bmatrix} + \begin{bmatrix} 0.5839 \\ 0.0605 \\ -0.3374 \end{bmatrix}$$

$$q_3 = \begin{bmatrix} 0.7219 \\ -1.7326 \\ 0.9386 \end{bmatrix}$$

$\therefore q_1 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \quad q_2 = \begin{bmatrix} 135/29 \\ 14/29 \\ -78/29 \end{bmatrix}, \quad q_3 = \begin{bmatrix} 0.7219 \\ -1.7326 \\ 0.9386 \end{bmatrix}$

Now, Normalize q_1, q_2 and q_3

$$\therefore Q_1 = \frac{q_1}{\|q_2\|} = \frac{1}{\sqrt{2^2+3^2+4^2}} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \frac{1}{\sqrt{29}} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{29} \\ 3/\sqrt{29} \\ 4/\sqrt{29} \end{bmatrix}$$

$$\therefore Q_2 = \frac{q_2}{\|q_2\|} = \frac{1}{\sqrt{\frac{135^2+14^2+78^2}{29^2}}} \begin{bmatrix} 135/29 \\ 14/29 \\ -78/29 \end{bmatrix}$$

$$= \frac{1}{\sqrt{\frac{24505}{29^2}}} \begin{bmatrix} 135/29 \\ 14/29 \\ -78/29 \end{bmatrix} = \frac{1}{\sqrt{845/29}} \begin{bmatrix} 135/29 \\ 14/29 \\ -78/29 \end{bmatrix}$$

$$= \frac{1}{5.398} \begin{bmatrix} 135/29 \\ 14/29 \\ -78/29 \end{bmatrix}$$

$$\therefore Q_2 = \begin{bmatrix} 0.8623 \\ 0.0894 \\ -0.4982 \end{bmatrix}$$

Also, $Q_3 = \dfrac{q_3}{\|q_3\|}$

$$= \frac{1}{\sqrt{(0.7219)^2+(1.7326)^2+(0.9386)^2}} \begin{bmatrix} 0.7219 \\ -1.7326 \\ 0.9386 \end{bmatrix}$$

$$Q_3 = \frac{1}{2.0985}\begin{bmatrix} 0.7219 \\ -1.7326 \\ 0.9386 \end{bmatrix}$$

$$Q_3 = \begin{bmatrix} 0.3340 \\ 0.8256 \\ 0.4472 \end{bmatrix}$$

Therefore orthonormal Basis are

$$\left\{ \begin{bmatrix} 2/\sqrt{29} \\ 3/\sqrt{29} \\ 4/\sqrt{29} \end{bmatrix}, \begin{bmatrix} 0.8623 \\ 0.0894 \\ -0.4982 \end{bmatrix}, \begin{bmatrix} 0.3340 \\ 0.8256 \\ 0.4472 \end{bmatrix} \right\}$$

d: Determinants

→ The determinant of a matrix is a scalar value that represents a sort of volume, which kind of volume?

Ans:- It represents the signed volume and can be used to calculate area of a parallelogram or the volume of parallelepiped. It signifies volume (space) stretch, which gives way to find inverse.

d.1 Properties of determinants

Prove the following.

① $(A^T)^T = A$

let 'A' be matrix i.e,

$$A = [a_{ij}]_{m\times n}$$

Transpose of A is

$$A^T = [a_{ji}]_{n\times m} \longrightarrow ①$$

Transpose on ①

$$(A^T)^T = [a_{ij}]_{m\times n}$$

$$\Rightarrow [(A^T)^T_{ij}] = [(A^T)_{ji}]$$
$$= [A_{ij}]$$

Hence

$(A^T)^T = A$ Hence proved

② $(AC)^T = C^T A^T$

let $A = [a_{ij}]_{m\times n}$ be a matrix, also $B = [b_{ij}]_{m\times n}$ be a matrix

then, Transpose $A^T = [a_{ji}]_{n\times m}$ $\quad B^T = [b_{ji}]_{n\times m'}$

Now, $AB = [a_{ij}]_{m\times n}[b_{ij}]_{n\times m'} = \left[\sum_{k=1}^{n} a_{ik} b_{kj}\right]_{n\times m'}$

Also, $(AB)' = \left[\sum_{k=1}^{n} a_{ki} b_{jk}\right]_{p'\times n}$

$\therefore (AB)^T = \left[\sum_{k=1}^{n} a_{ki} b_{jk} \right]_{m' \times m}$ ——— (i)

Again

$B^T A^T = [b_{ji}]_{m' n} \; [a_{ji}]_{n \times m}$

$= \left[\sum_{k=1}^{n} b_{jk} a_{ki} \right]_{m' \times m}$

$= \left[\sum_{k=1}^{n} a_{ki} b_{jk} \right]_{m' \times m}$ ——— (ii)

From (i) and (ii), we get

$(AB)^T = A^T B^T$

(3) If $\prod_{i=1}^{n} A_i$ is invertible then each of A_i is invertible.

Since $\prod_{i=1}^{n} A_i$ is invertible then there exists an inverse for $\prod_{i=1}^{n} A_i$ and also determinent of $\prod_{i=1}^{n} A_i \neq 0$

$\prod_{i=1}^{n} A_i = A_1 A_2 A_3 \times \ldots \times A_{n-1} A_n$ ——— (i)

$\left(\prod_{i=1}^{n} A_i\right)^{-1} = (A_1 A_2 A_3 \times \ldots \times A_{n-1} A_n)^{-1}$ ——— (ii)

From (i) and (ii), we get

$(A_1 A_2 A_3 \times \ldots A_{n-1} \times A_n) \times (A_1 A_2 A_3 \times \ldots A_{n-1} A_n)^{-1} = I$ ——— (iii) $(\because A^{-1} A = I)$

If determinent of $A_n = 0$, than determinent of equation (iii) will also be equal to ~~zero~~ '0'

From eqn (iii), we get

$(A_1 A_2 A_3 \times \cdots A_{n-1} A_n)(A_n^{-1} A_{n-1}^{-1} \times \cdots \times A_3^{-1} A_2^{-1} A_1^{-1}) = I$

$A_1 (A_2 A_3 \times \cdots A_{n-1} A_n)(A_n^{-1} A_{n-1}^{-1} \cdots A_3^{-1} A_2^{-1}) A_1^{-1} = I$ $\quad (\because A(BC) = (AB)C)$

$\Rightarrow A_1 A_1^{-1} = I$ which is True

Therefore A_1 is invertible, which justifies that each of A_i is invertible of $\prod_{i=1}^{n} A_i$, if above logic is applied for all.

(4) If the determinant of a matrix is zero, then it's not invertible.

Suppose we have a matrix A with $n \times n$ dimensions

therefore inverse will be A^{-1}

we know that

$$A^{-1} A = I \text{ --- (i)}$$

Applying determinant on eqn (i), we get

$\det(A^{-1}A) = \det(I)$

$\det(A^{-1}) \cdot \det(A) = 1$ --- (ii)

If $\det(A) = 0$ --- (iii)

using (iii) in (ii), we get

$0 = 1$

which is not true

Therefore, if determinant of matrix is 'zero' it's inverse doesnot exist, which means it's not invertible.

(5) What the determinant = 0 intuitively mean? Why does it indicate linear dependence between the columns of a matrix?

- We know that matrix represents a transformation and determinant represents degree to which that transformation deforms the area or volume. If the determinant is '0', it means that space is being squished into lower dimension or even to a point ~~and~~ which results in loss of information and once information is destroyed, we cannot invert the transformation to get the information back again.

- It also indicates system of ~~linear~~ equation are linearly dependent, which increases redundancy. It indicates that the system of equations either has no unique solutions or an infinite number of solutions exists. Also one equation of ~~set~~ can be expressed as a linear combination of atleast one other equation or more.

Q2 :- Algebra and Determinents

Find the determinent of the following matrices.

(1) $\begin{bmatrix} a-b & b-c & c-a \\ a^2-b^2 & b^2-c^2 & c^2-a^2 \\ a^3-b^3 & b^3-c^3 & c^3-a^3 \end{bmatrix}$

Here, we will use elementary operations, to get simplyfied, version of above.

$$\begin{bmatrix} a-b & b-c & c-a \\ (a-b)(a+b) & (b-c)(b+c) & (c-a)(c+a) \\ (a-b)(a^2+ab+b^2) & (b-c)(b^2+bc+c^2) & (c-a)(c^2+ca+a^2) \end{bmatrix}$$

$\{\because a^2-b^2=(a-b)(a+b)$
$a^3-b^3=(a-b)(a^2+ab+b^2)\}$

Taking $(a-b)$, $(b-c)$, $(c-a)$ common from C_1, C_2 and C_3

$$(a-b)(b-c)(c-a)\begin{bmatrix} 1 & 1 & 1 \\ a+b & b+c & c+a \\ a^2+ab+b^2 & b^2+bc+c^2 & c^2+ca+a^2 \end{bmatrix}$$

$C_2 \to C_2 - C_1$
$C_3 \to C_3 - C_1$

$$(a-b)(b-c)(c-a)\begin{bmatrix} 1 & 0 & 0 \\ a+b & c-a & c-b \\ a^2+ab+b^2 & c^2-a^2+bc-ab & c^2-b^2+ca-ab \end{bmatrix}$$

$R_2 \to R_2 - (a+b) R_1$

$R_3 \to R_3 - (a^2+ab+b^2) R_1$

$$(a-b)(b-c)(c-a) \begin{bmatrix} 1 & 0 & 0 \\ 0 & c-a & c-b \\ 0 & c^2-a^2-bc-ab & c^2-b^2+ca-ab \end{bmatrix}$$

Expanding along R_1

$$\det(A) = \left[(c-a) \times (c^2-b^2+ca-ab) - (c-b)(c^2-a^2-bc-ab)\right]$$

$$\det(A) = \cancel{c^3} - \cancel{b^2c} + \cancel{c^2a} - \cancel{abc} - \cancel{c^2a} + \cancel{b^2a} - \cancel{a^2c} + \cancel{a^2b}$$
$$- \cancel{c^3} + \cancel{a^2c} - \cancel{c^2b} + \cancel{abc} + \cancel{c^2b} - \cancel{a^2b} + \cancel{b^2c} - \cancel{ab^2}$$

$\det(A) = 0$

(2)
$$\begin{bmatrix} 1 & x & x^2 \\ x & x^2 & x^3 \\ x^2 & x^3 & x^4 \end{bmatrix}$$

$R_2 \to R_2 - x R_1$

$R_3 \to R_3 - x^2 R_1$

$$\begin{bmatrix} 1 & x & x^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Expanding along R_1

$\det(A) = 0$

→ Linear Regression

we have the following data table for house plot price

Area (SQKm)	Price (in 10lakh INR)
1	4
2	7
3	10
4	13
5	14
6	17
8	20
9	22
10	25

Fit a linear curve to this graph and calculate by hand the slope and the intercept of this line.

we know that $y = C + Dt$ —— (i)

while 'C' is intercept and 'D' is slope

$$\Rightarrow \left.\begin{aligned} C+D &= 4 \\ C+2D &= 7 \\ C+3D &= 10 \\ C+4D &= 13 \\ C+5D &= 14 \\ C+6D &= 17 \\ C+8D &= 20 \\ C+9D &= 22 \\ C+10D &= 25 \end{aligned}\right\} \text{—— (i)}$$

From equation-①, we get

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \\ 1 & 6 \\ 1 & 8 \\ 1 & 9 \\ 1 & 10 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} C \\ D \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 4 \\ 7 \\ 10 \\ 13 \\ 14 \\ 17 \\ 20 \\ 22 \\ 25 \end{bmatrix}}_{B}$$

we know, that

$$Ax = b \quad \text{——— Ⓐ}$$

$$A^T A\hat{x} = A^T b$$

$$\hat{x} = (A^T A)^{-1} A^T b \quad \text{——— ②}$$

Now,

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 8 & 9 & 10 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \\ 1 & 6 \\ 1 & 8 \\ 1 & 9 \\ 1 & 10 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 9 & 48 \\ 48 & 336 \end{bmatrix} \quad \text{——— ③}$$

Also,

$$A^T b = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 8 & 9 & 10 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \\ 10 \\ 13 \\ 14 \\ 17 \\ 20 \\ 22 \\ 25 \end{bmatrix}$$

$$\therefore A^T b = \begin{bmatrix} 132 \\ 880 \end{bmatrix} \quad \text{———} (4)$$

From eqn (3), we get

$$(A^T A)^{-1} = \frac{1}{9 \times 336 - 48 \times 48} \begin{bmatrix} 336 & -48 \\ -48 & 9 \end{bmatrix}$$

$$= \frac{1}{720} \begin{bmatrix} 336 & -48 \\ -48 & 9 \end{bmatrix}$$

$$\therefore (A^T A)^{-1} = \begin{bmatrix} 7/15 & -1/15 \\ -1/15 & 1/80 \end{bmatrix}$$

Using eqn (2), we get

$$\hat{x} = \begin{bmatrix} 7/15 & -1/15 \\ -1/15 & 1/80 \end{bmatrix} \begin{bmatrix} 132 \\ 880 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} 44/15 \\ 11/5 \end{bmatrix}$$

From eqn (A), we get

$$\hat{x} = \begin{bmatrix} C \\ D \end{bmatrix}$$

$$\therefore \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 44/15 \\ 11/5 \end{bmatrix}$$

$$\therefore C = \frac{44}{15} = 2.933$$

$$D = \frac{11}{5} = 2.2$$

Therefore intercept is 2.933.
slope is 2.2

From $y = C + Dt$, Projections will be ('x_i')

$$y = \frac{44}{15} + \frac{11}{5} \times t = \frac{44 + 33}{15} \times t = \frac{77}{15} t$$

Projections ($\hat{p}_i$) will be

$$\hat{P}_1 = \frac{77}{15} \times 1 = 5.133$$

$$\hat{P}_2 = \frac{77}{15} \times 2 = 10.266$$

$$\hat{P}_3 = \frac{77}{15} \times 3 = 15.4$$

$$\hat{P}_4 = \frac{77}{15} \times 4 = 20.533$$

$$\hat{P}_5 = \frac{77}{15} \times 5 = 25.666$$

$$\hat{P}_6 = \frac{77}{15} \times 6 = 30.8$$

$$\hat{P}_8 = \frac{77}{15} \times 8 = 41.066$$

$$\hat{P}_9 = \frac{77}{15} \times 9 = 46.2$$

$$\hat{P}_{10} = \frac{77}{15} \times 10 = 51.333$$