Discrete-Time Markov Chains

Computing the steady-state distribution

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Given a Markov chain with transition matrix P,

how to compute its steady-state distribution π?

A simple numerical solution: simply apply

$$\pi(n) \quad = \quad \pi(0) \times P^n \xrightarrow{n \text{ large}} \boxed{\pi}$$

But... when exactly to stop?

Assume we are given a Markov chain, and we are looking for its steady-state distribution. We have already seen a numerical way of computing it: just start from any initial distribution, and then repeatedly multiply it on the right with the transition matrix P.
The problem is that this can be long, and we do not know precisely when to stop iterating so that the values are close enough to the steady-state values. We want to do better than that.

An exact solution

Recall the recursive relation for probability vectors

$$
\begin{array}{ccc}
\pi(n+1) & = & \pi(n) \times P \\
& \Big\downarrow \text{let } n \to \infty & \\
\pi & = & \pi \times P
\end{array}
$$

What we can do is use the recursive expression for state distribution vectors, which we saw earlier. Now let n tend to infinity: both $\pi(n)$ and $\pi(n+1)$ converge to π, so the steady-state distribution vector should satisfy the simple matrix equation $\pi = \pi P$.

Balance equations

Decompose the relation $\pi = \pi P$: for each state i,

$$\begin{aligned} \pi_i &= \sum_j \pi_j P_{ji} \\ &= \pi_i P_{ii} + \sum_{j \neq i} \pi_j P_{ji}, \end{aligned}$$

which gives

$$\pi_i (1 - P_{ii}) = \sum_{j \neq i} \pi_j P_{ji}$$

$$\boxed{\pi_i \sum_{j \neq i} P_{ij} = \sum_{j \neq i} \pi_j P_{ji}}$$

Balance equation

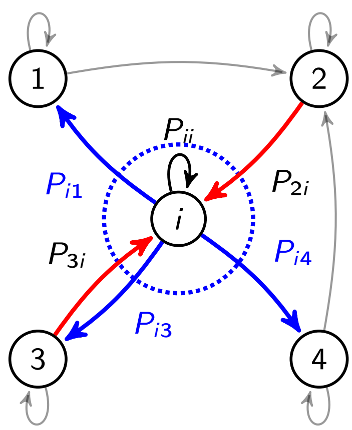
Let's see what this implies on a given state i: the steady-state probability π_i should equal the sum of $\pi_j P_{ji}$ over all states j. Or equivalently, π_i multiplied by one minus P_{ii} equals the sum, over all other states j different from i, of the product $\pi_j P_{ji}$. Finally, recalling that the terms in each row of the transition matrix add up to one, the "one minus P_{ii}" can be replaced with the sum of P_{ij} over all states j different from i.

So for each state i we have this equality, which is called a balance equation.

Interpretation of balance equations

For each state i,
$$\pi_i \sum_{j \neq i} P_{ij} = \sum_{j \neq i} \pi_j P_{ji}$$

Example (highlighting transitions involving State i):



Probability of leaving State i *Probability of entering State i*

$$\boxed{\pi_i \times (P_{i1} + P_{i3} + P_{i4}) = \pi_2 \times P_{2i} + \pi_3 \times P_{3i}}$$

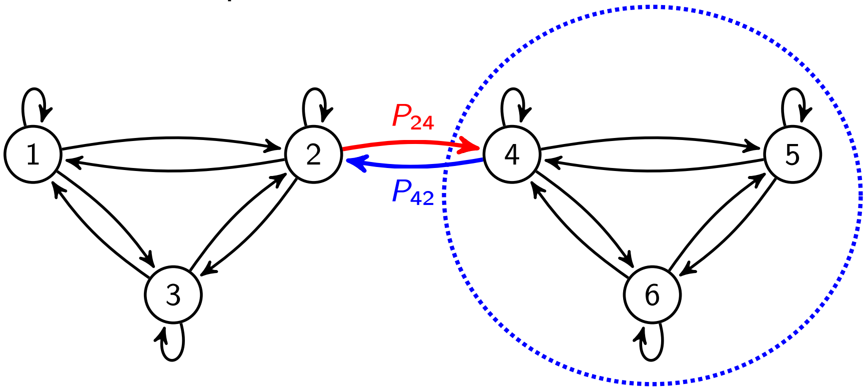
That equation actually has a nice interpretation: when we are in the steady-state regime, so, when the probabilities aren't moving anymore, then the process must visit each state i from time to time, but on average it should leave state i as frequently as it enters it. Otherwise, that would mean that the proportion of time in State i is changing.

So how often do we leave State i? More precisely, from one time period to another, what is the probability of leaving State i? Well, first we need to be in State i, which occurs with probability π_i. And we also need to have a transition from i to another state, resulting in the sum of transition probabilities here.
Similarly, how often do we enter State i? Well, before, we needed to be in a state j different from i, and to have a transition from j to i, which occurs with probability P_{ji}.

So this reasoning "we enter as often as we exit" gives us the balance equation for each state.

Balance equations for groups of states

Another example:



$\Rightarrow$ here $\quad \pi_4 P_{42} = \pi_2 P_{24}$

Note that this reasoning also works for any closed region in the transition diagram, which may contain several states: the sum of the frequencies of outgoing transitions must equal the sum of the frequencies of incoming transitions. And the frequency of a transition is just the steady-state probability of the origin state multiplied by the transition probability. Sometimes, selecting the closed regions to use smartly, gives us simpler equations than reasoning on a per-state basis.

To summarize: computing the steady-state distribution π

1. Balance equations, or equivalently relation $\pi = \pi P$
2. $\sum_{\text{States } i} \pi_i = 1$: **normalization condition**

$\Rightarrow$ A system of equations to solve to find π.

So to summarize, either the matrix form $\pi = \pi P$ or certain balance equations give us relations verified by the steady-state probabilities.

But those equations alone do not completely characterize π, because if we have a solution π and multiply it by a constant, then we obtain another solution to those equations.
We therefore need to use the additional condition that the vector π contains the steady-state probabilities of all states, and thus its values must add up to one. This new equation is called the *normalization condition*.

This normalization condition, together with what we obtain from the balance equations, finally completely characterizes π. We obtain a system with a unique solution, which is the steady-state distribution. Depending on the complexity of our Markov chain, we can compute this solution either analytically, or numerically.