M/M/1 queue

Steady state probabilities

We continue the analysis of the M/M/1 queue with the computation of the steady state probabilities.

Stability condition for the steady state behavior

- under what conditions is a steady state established ?
- the stability condition is:

$$\underbrace{\lambda}_{\textit{Arrival rate}} < \underbrace{\mu}_{\textit{Service rate}}$$

- $\rho = \lambda/\mu$ is called *traffic intensity* or *load*.

The queue is stable if $\rho < 1$, otherwise it is overloaded.

We focus on the stability conditions to establish a steady-state behavior. So the question is: under what conditions does the steady-state behavior exist?
The response is that the system must verify the stability condition. In the case of the M/M/1 queue, the stability condition is: "the arrival rate λ is lower than the service rate μ".
We denote by ρ the fraction $\frac{\lambda}{\mu}$. It is called the traffic intensity or the load. So the M/M/1 queue is stable if ρ is strictly lower than 1, otherwise it is unstable, or overloaded.

Steady state probabilities of the M/M/1

- $\pi_i = (\frac{\lambda}{\mu})^i \pi_0 = \rho^i \pi_0$, for $i \geq 1$
- from the normalizing condition $\sum_{i=0}^{\infty} \pi_i = 1$ we have: $\sum_{i=0}^{\infty} \pi_0 \rho^i = 1$
- we obtain:

$$\pi_0 = \frac{1}{\sum_{i=0}^{+\infty} \rho^i}$$

- if $\rho < 1$:
  - the sum $\sum_{i=0}^{+\infty} \rho^i$ converges to $1/(1-\rho)$.
  - so

$$\pi_0 = \frac{1}{\sum_{i=0}^{+\infty} \rho^i} = 1 - \rho$$

  - the steady state probabilities of the number of customers is:

$$\pi_i = (1-\rho)\rho^i, i = 0, 1, 2, \ldots$$

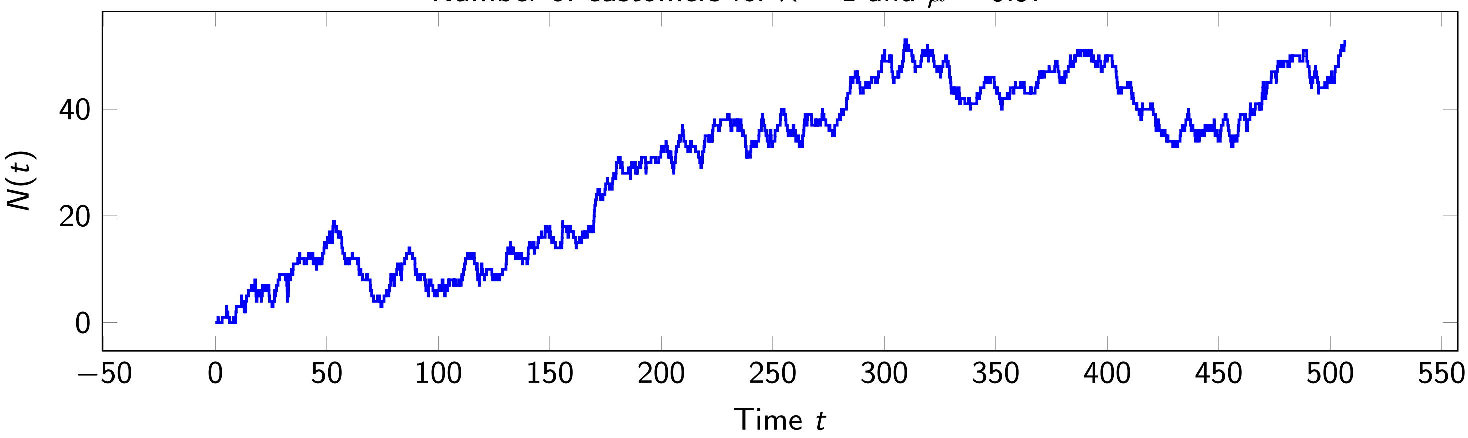
The steady-state probabilities π_i for all i, are equal to $(\lambda/\mu)^i$ multiplied by π_0, or ρ^i multiplied by π_0, for i greater than or equal to 1.
From the normalizing condition, the sum of the probabilities π_i, for i varying from 0 to infinity is equal to 1. So we have π_0 times the sum of ρ^i over all nonnegative i equals 1, and we deduce that π_0 equals 1 divided by that sum.
We know that if ρ is lower than 1, then the sum to $\frac{1}{1-\rho}$.
So we can deduce that π_0 equals $1-\rho$. It results that the steady-state probability π_i is $(1-\rho)\rho^i$, for all i, corresponding to the geometric probability distribution.

What is an unstable queue ?

Unstable (Overloaded) queue

- an M/M/1 queue is unstable if $\rho \geq 1$
- customers arrive faster than the server's processing speed ($\lambda \geq \mu$)
- the queue is unstable, the number of customers in the queue increases indefinitely, there is no stationary behavior

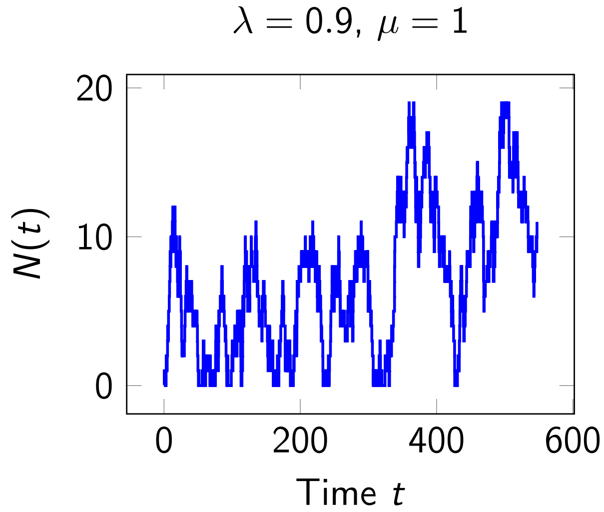
Number of customers for $\lambda = 1$ and $\mu = 0.9$.

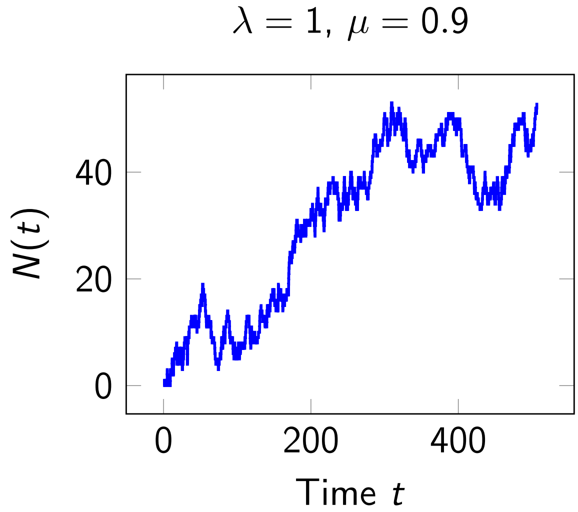


The question now is what is an unstable M/M/1 queue or an overloaded M/M/1 queue?
The M/M/1 queue is unstable if ρ is greater than or equal to 1. In this case, customers arrive faster than the server's processing speed, so λ is greater than or equal to μ. The number of customers in the queue increases indefinitely, so there is no stationary behavior when time goes to infinity.
We can see in the figure, that when time t tends to infinity, the number of customers $N(t)$ increases indefinitely.

Stability of the M/M/1 queue

$N(t)$ the number of customers, for stable ($\lambda < \mu$), and unstable ($\lambda \geq \mu$) queues





To finish with the stability of the M/M/1 queue, in the present figures, we compare the behavior of a stable queue on the left where λ is lower than μ, and the unstable queue on the right when λ is greater than or equal to μ. We can see that the number of customers on the left is finite, while on the right it goes to infinity.