Discrete-Time Markov Chains

Steady-state distribution

Convergence

Our weather forecast example:

$$\begin{cases} \text{State 1: clear} \\ \text{State 2: cloudy} \\ \text{State 3: rainy} \end{cases} \qquad P = \begin{bmatrix} 0.7 & 0.3 & 0 \\ 0.3 & 0.5 & 0.2 \\ 0.1 & 0.4 & 0.5 \end{bmatrix}$$

Start with clear weather:

$$\begin{aligned}
\pi(0) &= [1.000 \quad 0.000 \quad 0.000] \\
\pi(1) &= [0.700 \quad 0.300 \quad 0.000] \\
\pi(2) &= [0.580 \quad 0.360 \quad 0.060] \\
\pi(3) &= [0.520 \quad 0.378 \quad 0.102] \\
\pi(4) &= [0.488 \quad 0.386 \quad 0.127] \\
\pi(10) &= [0.449 \quad 0.394 \quad 0.157] \\
\pi(15) &= [0.447 \quad 0.395 \quad 0.158] \\
\pi(20) &= [0.447 \quad 0.395 \quad 0.158]
\end{aligned}$$

Start with rainy weather:

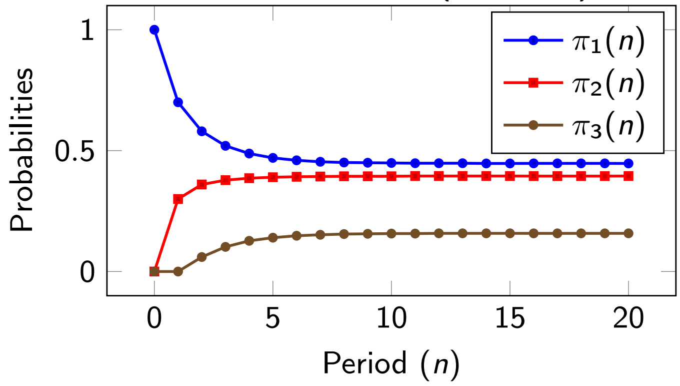
$$\begin{aligned}
\pi(0) &= [0.000 \quad 0.000 \quad 1.000] \\
\pi(1) &= [0.100 \quad 0.400 \quad 0.500] \\
\pi(2) &= [0.240 \quad 0.430 \quad 0.330] \\
\pi(3) &= [0.330 \quad 0.419 \quad 0.251] \\
\pi(4) &= [0.382 \quad 0.409 \quad 0.209] \\
\pi(10) &= [0.445 \quad 0.395 \quad 0.159] \\
\pi(15) &= [0.447 \quad 0.395 \quad 0.158] \\
\pi(20) &= [0.447 \quad 0.395 \quad 0.158]
\end{aligned}$$

In the previous video we saw how to compute the state probability vector at any time period, knowing it at period 0. Let's see what this gives us over time on our weather-forecast model. So assume we start with clear weather: we then compute $\pi(1)$, $\pi(2)$, $\pi(3)$, $\pi(4)$, $\pi(5)$... What do we observe here? Well, the state probability values seem to converge! And they actually do.

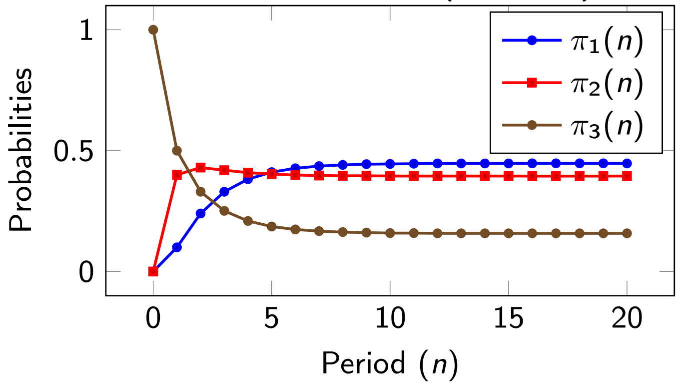
Let's try again, starting with another initial distribution, say, rainy weather. Again we can compute the successive values of the state probabilities over time. And again they converge, to the same values.

Convergence

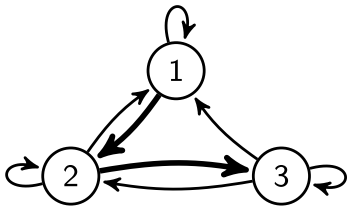
Start with clear weather (State 1):



Start with rainy weather (State 3):



Necessary condition for convergence:
From each state, we can reach any other state



We can also see that convergence graphically.
This is actually a general property with Markov chains. Under certain mild conditions, called ergodicity conditions, the state probabilities converge to values that do not depend on the initial state probabilities! We do not detail those conditions here, but we give one necessary condition: from each state we must be able to reach any other state with positive probability, possibly in several steps, as shown here from State 1. When that condition is satisfied, we are looking good to have convergence of the state probabilities.

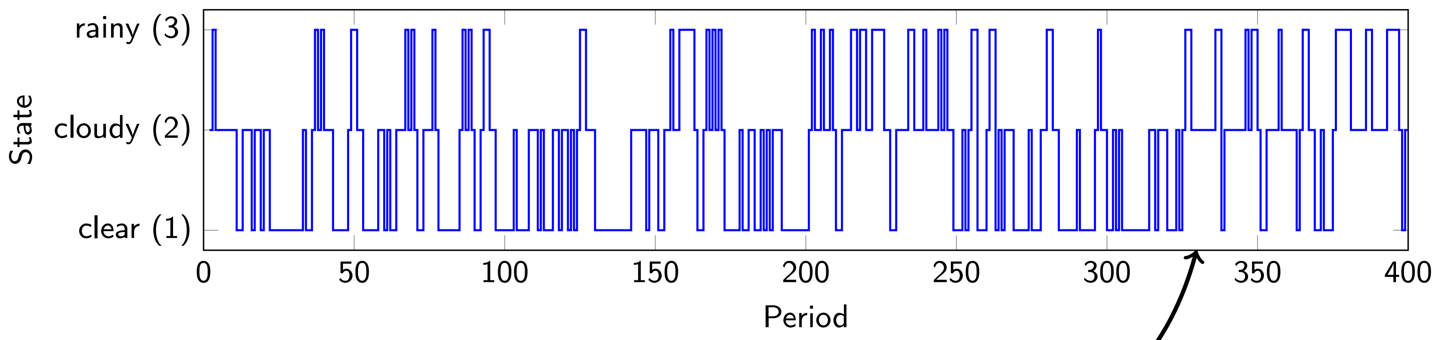
Steady-state probabilities

$$
\begin{array}{lll}
\text{Probabilities} & \pi(n) = [\pi_1(n) \quad \pi_2(n) \quad \pi_3(n) \quad \pi_4(n) \quad \ldots] \\
\text{converge to} & \big\downarrow n \to \infty \\
\textbf{Steady state probabilities} & \pi \quad = [\pi_1 \quad \pi_2 \quad \pi_3 \quad \pi_4 \quad \ldots]
\end{array}
$$

Those limit probabilities are called the *steady-state* probabilities of the Markov chain. Since they do not depend on time, we can remove the reference to the period and denote them by π_i for each state i. Similarly we denote the vector of the steady-state probabilities, by π. We can also call it the steady-state distribution of the Markov chain.

Interpretation of steady-state probabilities

A trajectory (obtained by simulation):



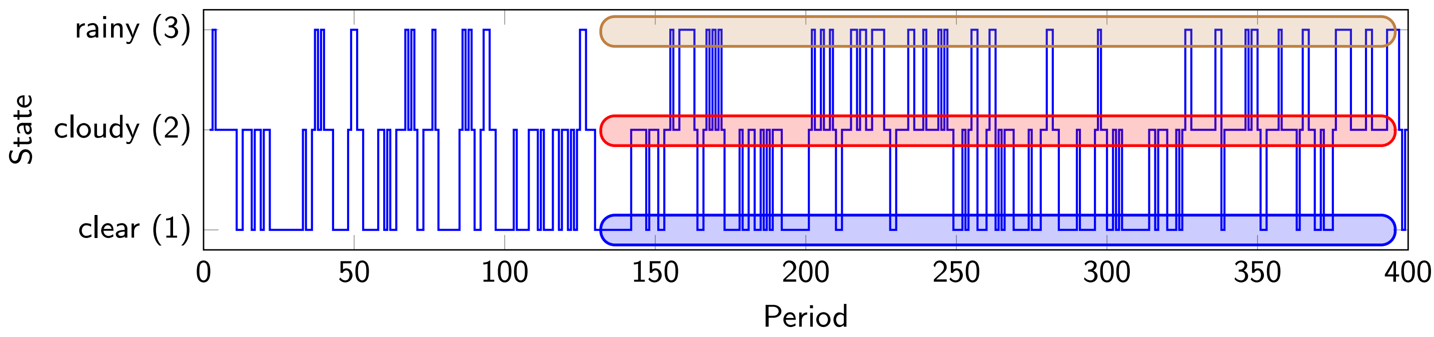
Probability of state i?

$\Rightarrow \pi_i$

What do those probabilities mean? Well, they mean that if I start from any state distribution and let the process evolve long enough, then I have a probability π_i of finding the process in State i, independently of the specific date at which I look at the system.

Interpretation of steady-state probabilities

A trajectory (obtained by simulation):



Proportion of time in each state (frequency of visit)?

$$\Rightarrow \begin{cases} \pi_3 \text{ of rainy weather} \\ \pi_2 \text{ of cloudy weather} \\ \pi_1 \text{ of clear weather} \end{cases}$$

This also means that after some time, the process is in State i a proportion π_i of the periods.
The steady-state distribution of the Markov chain will turn out to be very useful: in our weather forecast model, it will give us the proportion of time the weather is clear, cloudy, or rainy, which may for example be used to predict how much solar energy we can expect to produce on average per year.
In other queueing systems, the steady-state distribution will be the key to computing performance metrics, such as the probability of being blocked, or the average waiting time.
The next video will explain how we can compute that steady-state distribution.