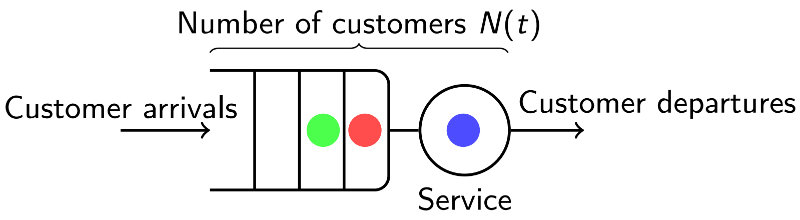
M/M/1 queue

Introduction to the M/M/1 queue

M/M/1 queue description



- Poisson arrivals (λ),
- Exponential service times (μ),
- 1 server,
- infinite buffer capacity,
- FIFO (First In First Out) service discipline

Hello, my name is Hind. I propose to study the basic queueing system denoted by M/M/1. First we begin with an introduction.
Here we can see the M/M/1 queue. It is described by the following specificities. Customer arrivals follow a Poisson process with rate λ. Service times are exponential with parameter μ. One server. The capacity of the system is infinite. The service discipline is FIFO (First In First Out).

Evolution of the number of customers

1. $N(t)$ is the number of customers at time t
2. Analyze evolution of $N(t)$ in $[t, t + \Delta t]$
3. Transition $N(t) \rightarrow N(t + \Delta t)$, according to different events:
   - **1 arrival**
   - **1 service**
   - **no arrival, no service**

We denote by $N(t)$ the number of customers in the system at time t. The analysis of the system consists in studying the evolution of the process $N(t)$ in the time interval $[t, t + \Delta t]$. We have a transition from $N(t)$ to $N(t + \Delta t)$ according to different events: 1 arrival, 1 service, no arrival/no service.

Probabilities of events occurring

- Probability of one arrival is proportional to Δt:

$$\textit{prob. of } 1 \textit{ arrival in } [t, t + \Delta t] = \lambda \Delta t + o(\Delta t)$$

where $o(\Delta t)$ is very small in comparison with Δt:

$$lim_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t} = 0$$

- Probability of more than one arrival is negligible:

$$\textit{prob. of } k \textit{ arrivals in } [t, t + \Delta t] = o(\Delta t)$$

We explain now how to compute the probabilities of events occurring during Δt. The probability of one arrival during Δt is proportional to Δt. So the probability of one arrival is the sum $\lambda \Delta t + o(\Delta t)$, where $o(\Delta t)$ is very small in comparison with Δt. So the limit of $o(\Delta t)/\Delta t$ when Δt tends to 0, is equal to 0. The probability of more than one arrival is negligible, so it is equal to $o(\Delta t)$.

Probabilities of other events occurring

- Probability of one service:

$$\textit{prob. of } 1 \textit{ service in } [t, t+\Delta t] \; = \mu \Delta t + o(\Delta t)$$

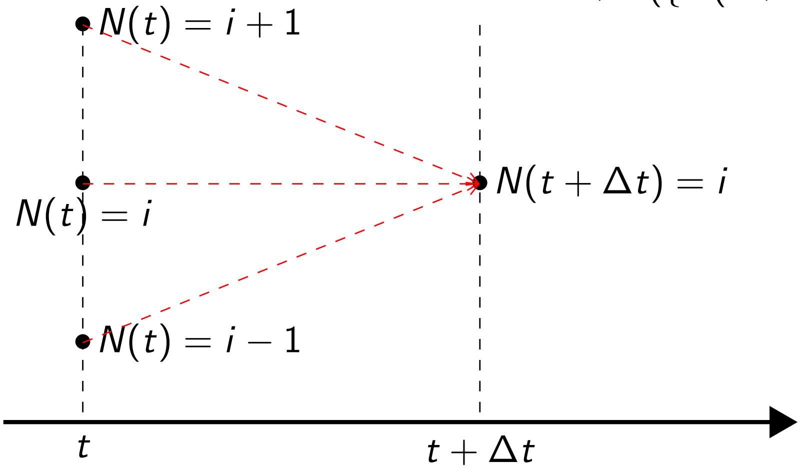
- Probability of no arrival, no service:

$$\textit{prob. of no events} = 1 - (\lambda + \mu)\Delta t + o(\Delta t)$$

For the other events we have the following probabilities: the probability of one service during Δt is the sum $\mu \Delta t + o(\Delta t)$, and the probability of no arrival and no service is $1 - (\lambda + \mu)\Delta t + o(\Delta t)$.

Evolution of $N(t)$

$$\begin{aligned} \mathbf{P}(N(t+\Delta t)=i) &= \mathbf{P}\left(\{N(t+\Delta t)=i\}\cap\{N(t)=i-1\}\right) \\ &+\mathbf{P}\left(\{N(t+\Delta t)=i\}\cap\{N(t)=i\}\right) \\ &+\mathbf{P}\left(\{N(t+\Delta t)=i\}\cap\{N(t)=i+1\}\right) \end{aligned}$$

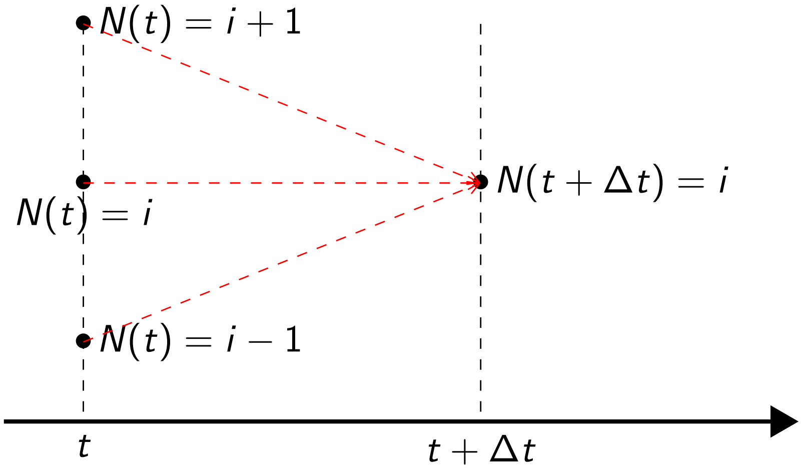


Now, we give the evolution equation of $N(t)$. The probability that $N(t+\Delta t)$ equals i is the probability to have i customers at time $t+\Delta t$, and it is equal to the sum of three probabilities: the probability to have i customers at time $t+\Delta t$ and to have $i-1$ customers at time t, plus the probability to have i customers at time $t+\Delta t$ and to have i customers at time t, plus the probability to have i customers at time $t+\Delta t$ and to have $i+1$ customers at time t.

Evolution of $N(t)$

Then we obtain the following total probability formula:

$$\begin{aligned}\mathbf{P}(N(t+\Delta t)=i) &= \mathbf{P}\left(N(t)=i-1\right)\mathbf{P}\left(N(t+\Delta t)=i|N(t)=i-1\right)\\ &+\mathbf{P}\left(N(t)=i\right)\mathbf{P}\left(N(t+\Delta t)=i|N(t)=i\right)\\ &+\mathbf{P}\left(N(t)=i+1\right)\mathbf{P}\left(N(t+\Delta t)=i|N(t)=i+1\right)\end{aligned}$$

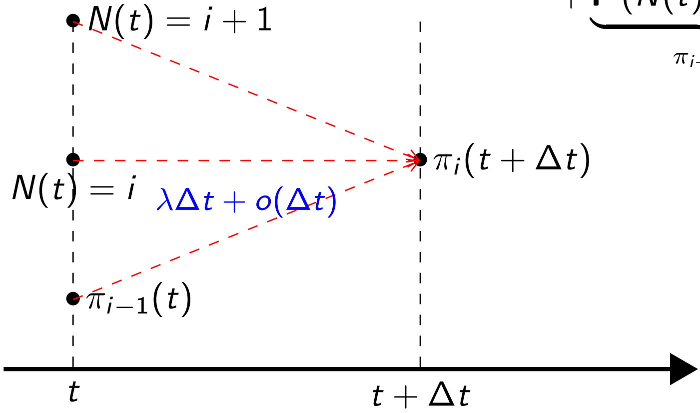


Now, we introduce the conditional probabilities for each term of the sum, and so we obtain the total probability formula for the equation evolution of $N(t)$, which gives the probability to have i customers at time $t+\Delta t$ using the probabilities of the number of customers at time t, and the transition probabilities between t and $t+\Delta t$.

Evolution of the number of customers

We define $\pi_i(t)$ as $\pi_i(t) = \mathbf{P}(N(t) = i)$.

$$\text{Then,}\quad \underbrace{\mathbf{P}(N(t+\Delta t)=i)}_{\pi_i(t+\Delta t)} = \underbrace{\mathbf{P}\left(N(t)=i-1\right)}_{\pi_{i-1}(t)}\underbrace{\mathbf{P}\left(N(t+\Delta t)=i|N(t)=i-1\right)}_{\lambda\Delta t+o(\Delta t)}$$
$$+\underbrace{\mathbf{P}\left(N(t)=i\right)}_{\pi_i(t)}\underbrace{\mathbf{P}\left(N(t+\Delta t)=i|N(t)=i\right)}_{1-(\lambda+\mu)\Delta t+o(\Delta t)}$$
$$+\underbrace{\mathbf{P}\left(N(t)=i+1\right)}_{\pi_{i+1}(t)}\underbrace{\mathbf{P}\left(N(t+\Delta t)=i|N(t)=i+1\right)}_{\mu\Delta t+o(\Delta t)}$$



In order to simplify the equations, now we introduce the probability distribution $\pi_i(t)$, the probability of having i customers at time t. Then we replace in the previous total probability formula, and we obtain: $\pi_i(t+\Delta t)$ equals the sum of three probabilities: $\pi_{i-1}(t)$ multiplied by the probability $\lambda\Delta t + o(\Delta t)$ of having one arrival, plus $\pi_i(t)$ multiplied by the probability $1-(\lambda+\mu)\Delta t + o(\Delta t)$ of having no arrival and no service, plus $\pi_{i+1}(t)$ multiplied by the probability $\mu\Delta t + o(\Delta t)$ of having one service.

Evolution of the number of customers

For $i \geq 1$, we obtain:

$$\begin{aligned}
\pi_i(t+\Delta t) &= \pi_{i-1}(t)\,\lambda\Delta t \\
&+\pi_i(t)(1-(\lambda+\mu)\Delta t) \\
&+\pi_{i+1}(t)\mu\Delta t + o(\Delta t)
\end{aligned}$$

So for i greater than 1, we obtain the following equations for the probabilities of the number of customers.

Evolution of the number of customers

For $i = 0$, we have:

$$\begin{aligned}\underbrace{\mathbf{P}(N(t+\Delta t)=0)}_{\pi_0(t+\Delta t)} &= \underbrace{\mathbf{P}(N(t)=0)}_{\pi_0(t)}\underbrace{\mathbf{P}(N(t+\Delta t)=0|N(t)=0)}_{1-\lambda\Delta t+o(\Delta t)} \\ &+ \underbrace{\mathbf{P}(N(t)=1)}_{\pi_1(t)}\underbrace{\mathbf{P}(N(t+\Delta t)=0|N(t)=1)}_{\mu\Delta t+o(\Delta t)}\end{aligned}$$

So we obtain:

$$\begin{aligned}\pi_0(t+\Delta t) &= \pi_0(t)\,(1-\lambda\Delta t) \\ &+\pi_1(t)\,\mu\,\Delta t + o(\Delta t)\end{aligned}$$

For $i = 0$, the probability $\pi_0(t+\Delta t)$ of having zero customers at time $t+\Delta t$, is the sum of only two probabilities: the probability $\pi_0(t)$ of having 0 customers at time t multiplied by the probability of having no arrival during Δt, $(1-\lambda\Delta t + o(\Delta t))$, plus the probability $\pi_1(t)$ of having 1 customer at time t multiplied by the probability of having one service during Δt, $(\mu\Delta t + o(\Delta t))$.

Chapman Kolmogorov equations

So the probabilities $\pi_i(t)$, for $i \geq 0$ satisfy the following equations:

$$\begin{cases} \pi_i(t+\Delta t) & = & \pi_{i-1}(t)\,\lambda\,\Delta t + \pi_i(t)\,(1-(\lambda+\mu)\Delta t) \\ & & +\pi_{i+1}(t)\,\mu\,\Delta t + o(\Delta t), \;\; i \geq 1 \\ \\ \pi_0(t+\Delta t) & = & \pi_0(t)(1-\lambda\Delta t) + \pi_1(t)\,\mu\,\Delta t + o(\Delta t) \end{cases}$$

So we obtain the following system of equations for the probabilities $\pi(t)$ called Chapman Kolmogorov equations.