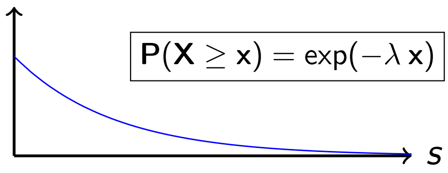
What is a queue?

Exponential distribution

Exponential distribution

Let us consider $X \sim Exp(\lambda)$.

- **Tail distribution**



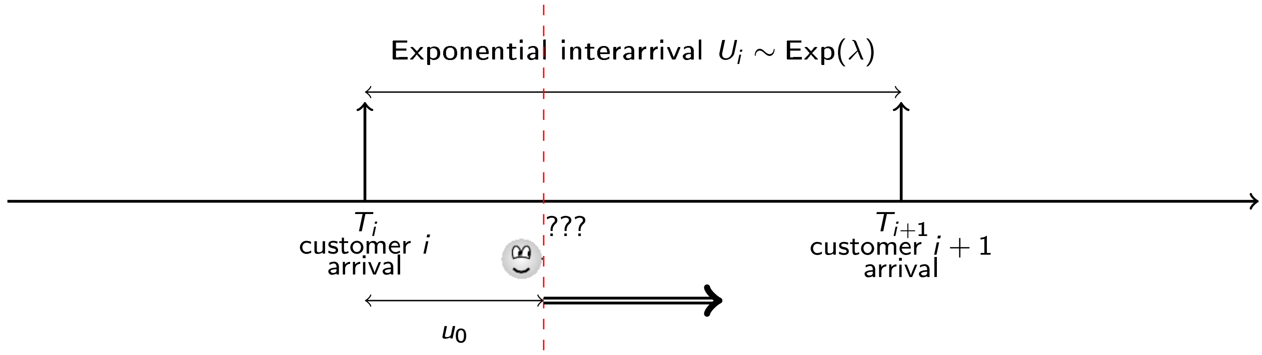
Exponential tail distribution

- **Mean,variance and squared coefficient of variation :**

Mean	$\mathbf{E}(X) = 1/\lambda$
Variance	$\text{var}(X) = \mathbf{E}(X^2) - (\mathbf{E}(X))^2$ $= \mathbf{E}((X - \mathbf{E}(X))^2) = 1/\lambda^2$
Squared Coefficient of Variation	$\text{Cv}^2 = \frac{\text{var}(X)}{(\text{E}(X))^2} = 1, \text{for all } \lambda$

Let's remember a few results about the exponential distribution. We'll assume that the random variable X is distributed according to the exponential law with parameter λ. This means that the probability that X is greater than a threshold « small x » equals exponential of minus λx. It can be shown that the expected value, or mean value, of X equals one divided by λ.
It can also be proven that the variance of X, defined as the difference between the expected value of X squared and the square of the mean value of X, equals one divided by λ^2. The squared coefficient of variation of X is defined as the ratio of the variance of X to the square of its mean value. Note that, whatever the value of the parameter λ, the squared coefficient of variation of an exponential random variable is equal to one.

Memoryless property of the Exponential distribution



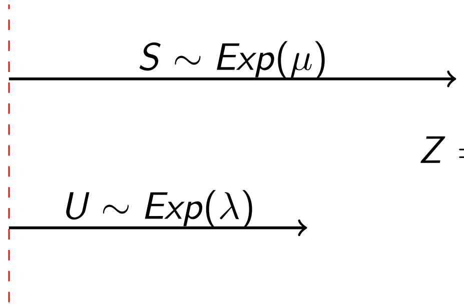
$$\boxed{\mathbf{P}(U_i \geq u + u_0 \mid U_i \geq u_0) = \mathbf{P}(U_i \geq u) = \exp(-\lambda\, u)}$$

where $\mathbf{P}(A \mid B) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)}$ when A and B are two events such that $\mathbf{P}(B) \neq 0$.

One very important property of the exponential distribution is the so called "memoryless property". Consider an electric bulb with an exponential lifetime. Whether it has been used for one day or for one year, the time that remains until it fails is distributed according to the same exponential distribution. This means that the past lifetime has no influence on the future lifetime of the bulb; it will not wear out. Now let's go back to our customers. If they arrive according to a Poisson process, then the interarrival time between two consecutive customers is an exponential with parameter λ. Assume that an external observer knows that the last customer has arrived u_0 time units ago. From the point of view of this observer, the time until the next customer arrives is distributed according to an exponential distribution with parameter λ. Knowing that the last customer arrived u_0 time units ago does not bring any useful information to predict when the next customer will arrive.
The memoryless property of the exponential distribution can be stated mathematically. Before we do so, let's remember that if A and B are two events, then the probability of A given B is defined as the ratio of the probability of (A and B), or A intersect B, to the probability of B. If U_i is an exponential random variable with rate λ, then the probability that U_i is greater than u plus u_0 given that U_i is greater than u_0 is equal to the probability that U_i is greater than u. This quantity equals the exponential of minus λu. This is the memoryless property of the exponential distribution.

Minimum of two exponential random variables

Let $U \sim Exp(\lambda)$ and $S \sim Exp(\mu)$. Assume that U and S are independent.



$Z = \min(U, S)$, so that
$Z = U$ if $U \leq S$
$Z = S$ if $U > S$

$$\boxed{Z = \min(U, S) \sim Exp(\lambda + \mu)}$$

$$\boxed{\begin{aligned} \mathbf{P}(Z = U) &= \mathbf{P}(U \leq S) = \tfrac{\lambda}{\lambda+\mu} \quad \text{and} \\ \mathbf{P}(Z = S) &= \mathbf{P}(U > S) = \tfrac{\mu}{\lambda+\mu} \end{aligned}}$$

Another important property of the exponential distribution is the law of the minimum of two exponential random variables.
Let's assume that U and S are two independent random variables, such that U is exponential with parameter λ and S is exponential with parameter μ. Let's consider Z, defined as the minimum of U and S.
In particular, in a queuing system context, if U is the time until the next arrival and S is the time until the next departure (in other words, an end of service time) Z is the time until the next event in this system, whether it's a departure or an arrival.
One remarkable property is that Z is distributed according to an exponential law and that the parameter of this exponential law is the sum of λ and μ.
Let's look at the probability that the next event is an arrival. Another way to say that is that Z is equal to U. It can be demonstrated that this probability is λ divided by (λ plus μ).
And what about the probability that the next event is a departure? Well, this probability corresponds to Z equals S, and it turns out to be μ divided by (λ plus μ).
The first case corresponds to U less than S, or arrival before departure, while the second case corresponds to U greater than S, or departure before arrival.