Discrete-Time Markov Chains

What is a discrete-time Markov chain?

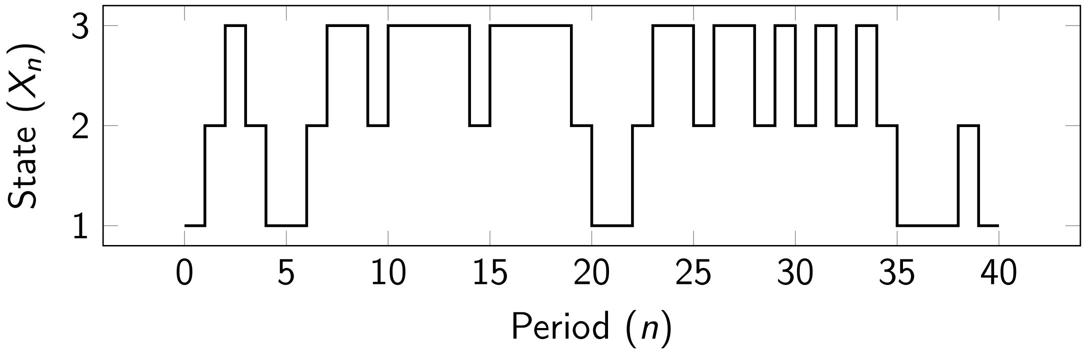
Discrete-time random process

Vocabulary:

- **process**: evolves over time
- **random**: not perfectly predictable
- **discrete-time**: possible changes at *periods*

Process $X = (X_0, X_1, X_2, ...)$

- **state**: a possible value for X_n (ex: 1, 2, 3)



In this course, we are studying random processes (also referred to as stochastic processes). A random process is something that evolves over time, in a way that we cannot predict with certainty. So we use probabilistic models to describe them.
To study queues, it will be very useful to focus on random processes which have a specific structure, where what happens in the future depends only on where we are now, and not where we were before. Such processes are called Markov chains, and we will explain that in more detail in the upcoming videos.

This week, we focus on the case where time is indexed only with integer values, called periods, at which the process can change: we say we have a discrete-time process. So our random process will be a sequence $(X_0, X_1, X_2, ...)$, where the indices represent the time periods, and the value of X_n is called the *state* of the process at period n. We are interested in the evolution of X_n.
We also limit the possible values of X_n to what is called a discrete set: let's just assume that each possible value can be associated with a different integer. So we can talk about State 1, State 2, and so on.

Markov property

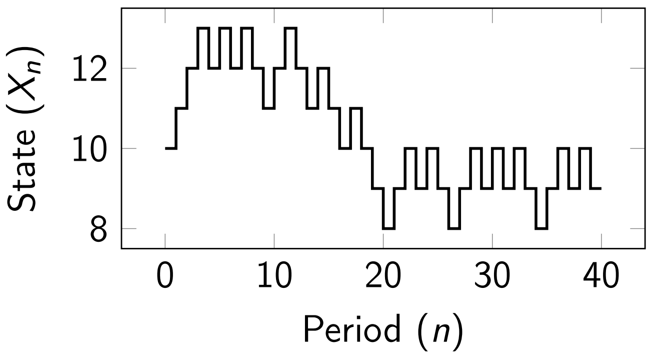
Definition (Markov chain)

The future is independent of the past, given the present. Mathematically,

$$\begin{aligned} & \mathbf{P}(X_{n+1} = j \mid X_n = i) && \text{depends only on } i \text{ and } j \\ = \; & \mathbf{P}(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, X_{n-2} = i_{n-2}, \ldots) \\ = \; & P_{ij} && \textbf{transition probability from } i \textbf{ to } j \end{aligned}$$

Example: coin tosses

- "Heads" = +1
- "Tails" = -1
- $X_0 = 10$



So, when is such a process a Markov chain? Well, when it satisfies the so-called Markov property, whose interpretation is: "the future is independent of the past, given the present"
This formulation may be a bit obscure; let's look at the math characterization: we have a Markov chain when, for any period n and any state i, if I know I am in state i at period n, then the probability that I am in state j (for any j) at the next period depends only on i and j. In particular, it does not depend on the previous states. We can write that like we see it on the screen: knowing the previous states does not change the probability of moving from i to j over one period. And since this probability depends only on i and j, we can call it a transition probability from i to j and simply denote it by P_{ij}.
This is quite a strong assumption: as we will see in the quizzes, in several cases it is not realistic to make this assumption. But for processes like queues, it will turn out to work very well.
Let's look at an example. Assume that I repeatedly toss a coin, when I get "heads" I earn one euro and when I get "tails" I lose one euro. I want to study the evolution of my wealth after each toss. Why is that a Markov chain? Well, if after n tosses I am in state X_n, then after $n+1$ tosses I can only have either $X_n + 1$ euros, with probability 1/2, or X_n?1 euros, with probability 1/2. And this is independent of the previous values of X, before period n: knowing how I reached X_n would not change in any way the future evolution of my wealth. That's precisely the Markov property: knowing the present state, the future is independent of any past states.