Continuous-Time Markov Chains

Steady state distribution

Transient behavior

Discrete Time Markov Chain

- $\pi(n) = (P(X_n = 0), P(X_n = 1), \cdots)$
- P, transition probability matrix
- $\pi(n+1) = \pi(n)P \Longleftrightarrow \pi(n+1) - \pi(n) = \pi(n)(P - \mathrm{I})$

Continuous Time Markov Chain

- $\pi(t) = (P(X_t = 0), P(X_t = 1), \cdots)$
- Q, transition rate matrix
- $\frac{d}{dt}\pi(t) = \pi(t)Q$

As for discrete-time Markov chains, we are mainly interested in the asymptotic behavior of any continuous-time Markov chain. However, we will first have a look at the transient evolution. Recall that in a discrete-time Markov chain, we define a vector $\pi(n)$ whose components are the probability of being in each state at time n. The evolution of $\pi(n)$ is guided by the equation $\pi(n+1) = \pi(n)P$ where P is the transition matrix.
In order to stress the similarity between discrete-time Markov chain and continuous-time Markov chain, let's rewrite this equation in the following form $\pi(n+1) - \pi(n) = \pi(n)(P?I)$. Notice that $(P - I)$ is a matrix for which the sum of the coefficients on every line is zero. For a continuous-time Markov chain, we introduce $\pi(t)$ which is the exact analog of $\pi(n)$ except that it is defined for any time t instead of being defined only for integer time indexes. So, it is not surprising that the equation governing $\pi(t)$ is $\frac{d}{dt}\pi(t) = \pi(t)Q$ where Q is the transition rate matrix.

Steady state distribution

- Consider a Continuous Time Markov Chain of transition rate matrix Q,
- And assume that $\pi(t) \xrightarrow[t \to \infty]{} \pi$
- As $\frac{d}{dt}\pi(t) = \pi(t)Q$ it holds that

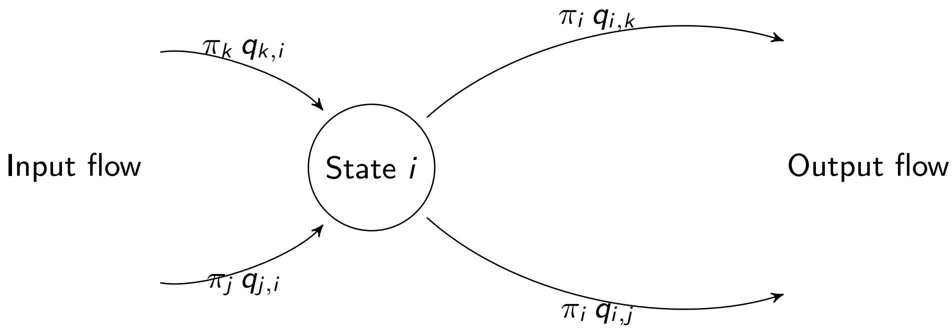
  $$\boxed{\pi Q = 0}$$

  and moreover, $\sum_{i \in \text{States}} \pi_i = 1$
- **Definition:** flow $\phi_{i,j}$= mean number of transitions per unit of time from i to j at steady state

  $$\phi_{i,j} = \pi_i \, q_{i,j}$$

By definition an invariant probability measure should be invariant when time goes on. The equation $\frac{d}{dt}\pi(t) = \pi(t)Q$ means that this can only happen if $\pi(t) = \pi$ with $\pi Q = 0$ plus the normalizing condition $\sum_i \pi_i = 1$.
There is a very interesting and very useful interpretation of this system of equations. Let's define the flow from state i to state j as the product $\pi_i q_{i,j}$ which represents the mean number of transitions from i to j per time unit. We interpret this quantity as the flow that enters state j coming from state i in the transition diagram.

Balance equations

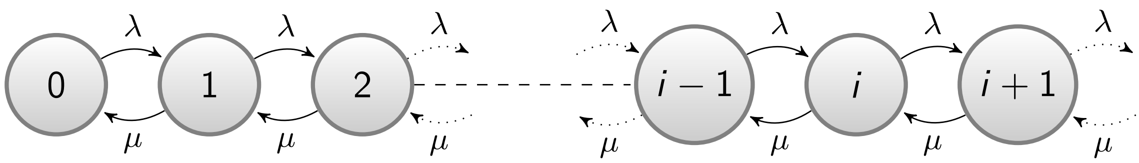


$$\underbrace{\sum_{j \neq i} \pi_j \; q_{j,i}}_{\text{Input flow}} \qquad \underbrace{\sum_{j \neq i} \pi_i \; q_{i,j}}_{\text{Output flow}} = -\pi_i \, q_{i,i} \text{ since } \sum_j q_{i,j} = 0$$

$$\boxed{\underbrace{\sum_{j \neq i} \pi_j \; q_{j,i}}_{\text{Input flow}} = \underbrace{-\pi_i \, q_{i,i}}_{\text{Output flow}} \iff \pi Q = 0}$$

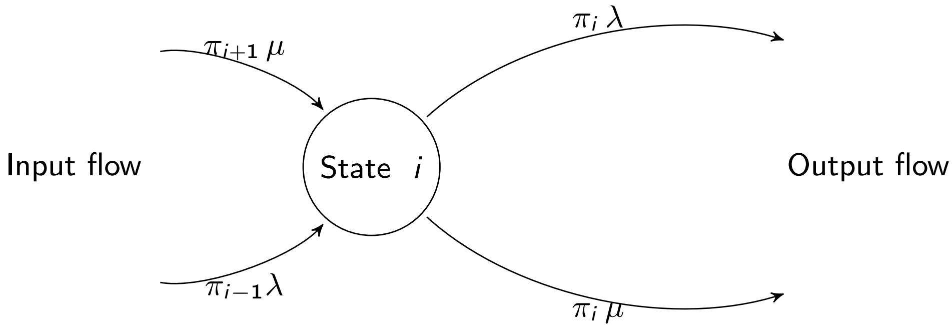
The total input flow in state i is the sum of all the flows coming from the other states into i. That means that the input flow is the sum over j of $\pi_j q_{j,i}$. The output flow from i is the sum of all the flows coming from i to the other states, it is represented by the sum over j of $\pi_i q_{i,j}$. With a few manipulations, it can also be written as $-\pi_i q_{i,i}$. The basic idea is to say that "what goes in is equal to what goes out" as usual if we are in an equilibrium state. This means that the input flow has to be equal to the output flow. The equations we obtain are exactly the expanded version of the system $\pi Q = 0$.

M/M/1



We will now retrieve the equations we obtained for the M/M/1 queue. The state space is the set of integers, λ is the arrival rate of the Poisson process and $1/\mu$ is the mean service time. The most important thing to notice is that when the queue is in state i, the next event may be either an arrival or a departure. This means that the input flows into state i come only from state $i+1$ or state $i-1$. Likewise, the flows coming out of state i can only go to state $i-1$ or to state $i+1$. Taking into account the value of the $q_{i,j}$ we obtain this diagram. For state 1, the input flow from 2 to 1 is $\pi_2\mu$.

M/M/1 continued



$$\mu\,\pi_1 = \lambda\,\pi_0$$
$$\mu\,\pi_2 + \lambda\,\pi_0 = \lambda\,\pi_1 + \mu\,\pi_1$$
$$\mu\,\pi_3 + \lambda\,\pi_1 = \lambda\,\pi_2 + \mu\,\pi_2$$
$$\vdots$$
$$\mu\,\pi_{i+1} + \lambda\,\pi_{i-1} = \lambda\,\pi_i + \mu\,\pi_i$$

The input flow from 0 to 1 corresponds to an arrival, so it is $\pi_0\lambda$. Thus the total input flow in state 1 is $\mu\pi_2 + \lambda\pi_0$. The output flow is the sum of the flow from 1 to 2, which corresponds to an arrival and is equal to $\lambda\pi_1$ and the flow from 1 to 0, which corresponds to a departure and is equal to $\mu\pi_1$. In virtue of the equilibrium physicists' definition "what goes in equals what comes out", we have the equation $\mu\pi_2 + \lambda\pi_0 = \lambda\pi_1 + \mu\pi_1$.
The very same reasoning holds for state 2, the input flow is $\mu\pi_3$ for the part coming from state 3 plus $\lambda\pi_1$ for the part corresponding to an arrival when in state 1 and the output flow is $\lambda\pi_2 + \mu\pi_2$ and we can repeat the same analysis for any state i. This gives the equation $\lambda\pi_{i-1} + \mu\pi_{i+1} = \lambda\pi_i + \mu\pi_i$. There is one particular state, namely state 0 since we can have neither a departure from this state nor an arrival from another state. So the input flow comes only from state 1 and is equal to $\mu\pi_1$ and the output flow goes only to state 1 and is equal to $\lambda\pi_0$.