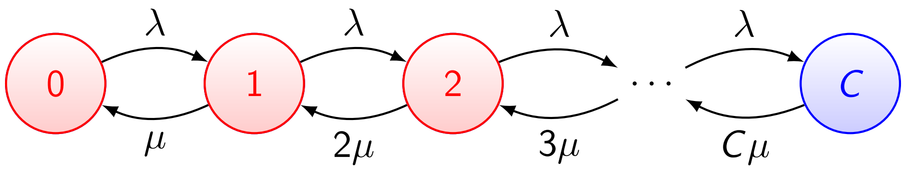
Multi-Server Systems

M/M/C/C – Erlang-B formula

M/M/C/C – Steady-State Distribution



1. Local Balance Equations
$$\lambda\pi_0 = \mu\pi_1$$
$$\lambda\pi_1 = 2\mu\pi_2$$
$$\cdots$$
$$\lambda\pi_{j-1} = j\mu\pi_j,$$
$j = 1, \ldots, C\pi_j = \frac{\rho}{j}\pi_{j-1}, j = 1, \ldots, C$

2. Recursion $\Rightarrow$
$\pi_j = \frac{\rho^j}{j!}\pi_0, \; j = 0, 1, \ldots, C$

3. Normalization $\Rightarrow$
$$\pi_0 = \frac{1}{\sum_{j=0}^{C} \frac{\rho^j}{j!}}$$

4. Steady-State Distribution

$$\pi_j = \frac{\frac{\rho^j}{j!}}{\sum_{i=0}^{C} \frac{\rho^i}{i!}}, \; j = 0, 1, \ldots, C$$

Now let's compute the so-called Erlang-B formula. To do that, we first have to find the steady-state distribution of our system, which can be interpreted as the fraction of time the system is in each state in the long-run. Let's consider the state transition diagram that we saw in the previous video. We'll go through 4 steps in order to solve the steady state distribution

- First, we have to solve the local balance equations. We consider every pair of consecutive states and establish that the mean outgoing flow from one of these states towards the second state equals the mean flow going from the second state to the first one. For instance, for states 0 and 1 we have $\lambda\pi_0 = \mu\pi_1$. For states 1 and 2 we have $\lambda\pi_1 = 2\mu\pi_2$. In general, for any pair $(j-1, j)$ of consecutive states we have $\lambda\pi_{j-1} = j\mu\pi_j$, or what is the same, $\pi_j = \frac{\rho}{j\pi_{j-1}}$, for j from 1 to C.
- Second, we apply a recursion in order to express every steady state probability as a function of π_0. We obtain $\pi_j = \frac{\rho^j}{j!\pi_0}$, and this is valid for j from 0 to C.
- Third, we apply the Normalization condition, meaning that since we are working with probabilities, they must sum up to one. So we obtain an expression for π_0.
- Forth and last, we substitute the obtained results to get the steady state distribution

M/M/C/C – Blocking Probability

- The probability of the system being busy is given by π_C
- PASTA property $\Rightarrow$ customers arriving according to a Poisson process see the system in the steady state $\Rightarrow$
- The blocking probability is given by the probability of having all servers busy: i.e. π_C

$$\boxed{\mathbf{P}_{blocking} = \pi_C = \frac{\frac{\rho^C}{C!}}{\sum_{i=0}^{C} \frac{\rho^i}{i!}}}$$

We are now able to define the blocking probability of our system.

- On the one hand; we have just computed the steady-state probabilities, and in particular the probability of the system being busy; that is, of having C servers busy, which is given by π_C
- On the other hand, we have the result of the PASTA property. Let's remember what the PASTA property is. This property tells us that if customers arrive according to a Poisson process, they see the system in the steady state.
- All in all, the blocking probability, that is, the probability that an arriving customer sees the system busy, isgiven by π_C

Erlang-B

- The blocking probability of an M/M/C/C system is referred to as the Erlang-B formula
- It is given by π_C
- It is expressed as a function of the offered load ($\rho = \lambda/\mu$) and the system capacity (C)

$$\boxed{E_B(\rho, C) = \frac{\frac{\rho^C}{C!}}{\displaystyle\sum_{i=0}^{C} \frac{\rho^i}{i!}}}$$

And we finally arrive at the definition of Erlang-B. Actually, we already did, since the blocking probability of an M/M/C/C system has received the name "Erlang-B formula". So it is given by the expression of π_C and it is expressed as a function of the offered traffic and the capacity of the system. This formula was proposed by Erlang as early as 1917.