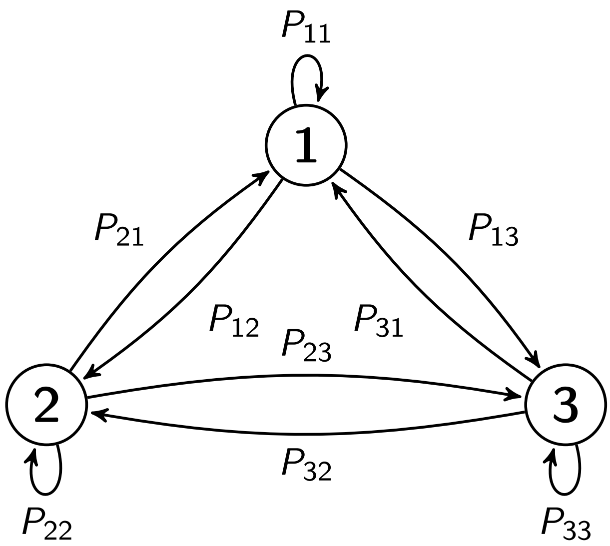
Discrete-Time Markov Chains

Representing a discrete-time Markov chain

Graphical representation: the state transition diagram

Ingredients of a discrete-time Markov chain: initial situation, and

- **states** $\Rightarrow$ *draw one circle per state*
- **transition probabilities** $\Rightarrow$ *label arrow from i to j with P_{ij}*

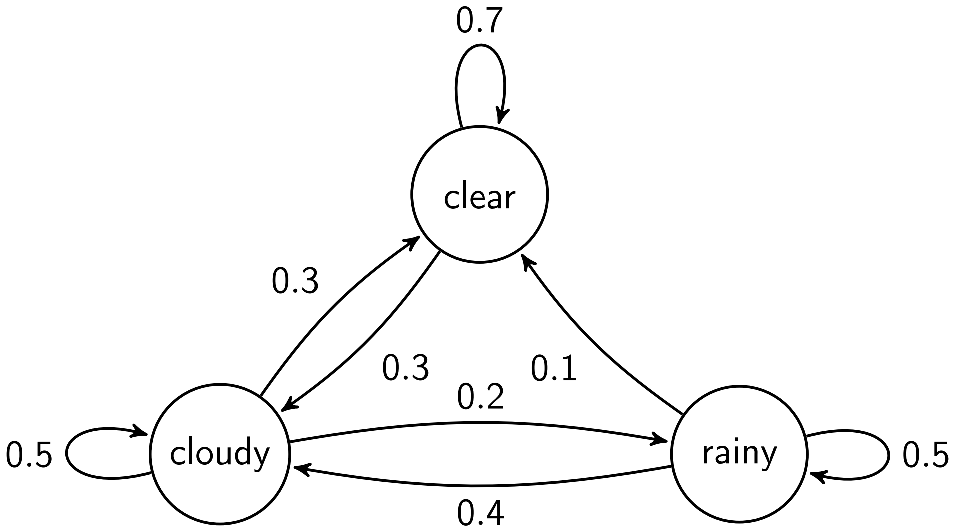


When we have a discrete-time Markov chain, the state at the next period depends only on the current state, and on some random events independent of the past.
So, to characterize the evolution of the process, we just need to specify the states we can have, along with the transition probabilities from each state to each state.
This can be represented graphically: let's just draw a circle for each different state, and an arrow from each state to each state, including itself.
We then have the set of states, and to represent the transition probabilities, we label each arrow from state i to state j with the transition probability P_{ij}. This forms what is called the state transition diagram of our Markov chain, and it's very convenient to draw and read.

State transition diagram: an example

Assume a simple weather model (a period = one hour):

- if currently clear: next hour is clear (prob. 0.7) or cloudy (prob. 0.3);
- if currently cloudy: next hour is cloudy (prob. 0.5), clear (prob. 0.3), or rainy (prob. 0.2);
- if currently rainy: next hour is rainy (prob. 0.5), cloudy (prob. 0.4), or clear (prob. 0.1);



For example, consider a very simple model for the evolution of the weather, with three states: clear, cloudy, rainy. And assume the following probabilities from one hour to the next:

- if the weather is clear then it has a probability 0.7 of remaining that way, and a probability 0.3 of becoming cloudy;
- similarly, cloudy weather has a probability 0.5 of lasting for the next hour, a probability 0.3 of turning to clear, and 0.2 of becoming rainy;
- finally, rainy weather can stay rainy with probability 0.5, can become cloudy with probability 0.4, and clear with probability 0.1.

This process is a Markov chain since it is precisely described as such: the state at the next period (here, the next hour) depends only on the current state.

Let's draw the state transition diagram for that Markov chain: we have our three states, we add the arrows, and then label the arrows with the transition probabilities. When a transition has probability 0, this means it cannot occur, so we can remove the arrow. We're all done, we have our state transition diagram, which completely characterizes the evolution of our Markov chain.

Mathematical representation: the state transition matrix

Ingredients of a discrete-time Markov chain: initial situation, and

- **states** $\Rightarrow$ *order them*
- **transition probabilities** $\Rightarrow$ *write* P_{ij} *on* i^{th} *row,* j^{th} *column*

$$
\begin{array}{cc}
 & \begin{array}{ccc} \text{1:clear} & \text{2:cloudy} & \text{3:rainy} \end{array} \\
\begin{array}{r} \text{clear:1} \\ \text{cloudy:2} \\ \text{rainy:3} \end{array} & \begin{bmatrix} 0.7 & 0.3 & 0 \\ 0.3 & 0.5 & 0.2 \\ 0.1 & 0.4 & 0.5 \end{bmatrix} \begin{array}{l} \rightarrow \textbf{sum} = 1 \\ \rightarrow \textbf{sum} = 1 \\ \rightarrow \textbf{sum} = 1 \end{array}
\end{array}
$$

State transition matrix

Now let's look at another way to represent the same information, which is less convenient to read but much more convenient to exploit mathematically: it involves writing all the transition probabilities in a table, or a matrix.
To do so, we need to order the states. This can be done arbitrarily; let's take the order "clear, cloudy, rainy" for our last example. Then we define a matrix where each line corresponds to one origin state, and each column corresponds to one destination state, using that order. In other words, the term in the ith line and jth column is the transition probability P_{ij} from the ith state to the jth state.
This matrix is called the state transition matrix of the Markov chain. Its size provides the number of states, and the values give the transition probabilities.
The state transition matrix should verify a simple property: on each line, the values must add up to one. Indeed, from any State i, with probability 1 we will be in one of the possible states at the next period.
Similarly, on the state transition diagram, for each state the labels of the outgoing transitions from that state add up to one.