Continuous-Time Markov Chains

What is a continuous-time Markov chain?

Continuous-time Markov chain

Definition (CTMC)

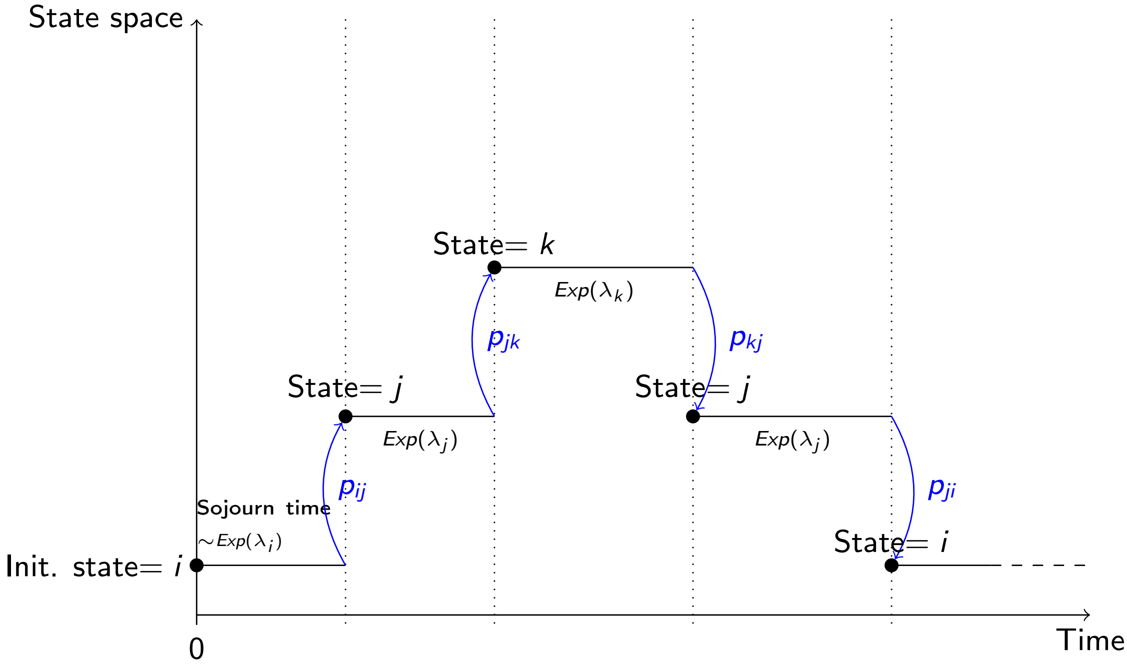
Continuous-time MC = Discrete-time MC + random sojourn time in each state

Ingredients of a continuous-time Markov chain: initial situation, and

- **states**
- **transition probabilities** : $p_{i,j}$
- **sojourn time in each state** : $\mathrm{Exp}(\lambda_i)$

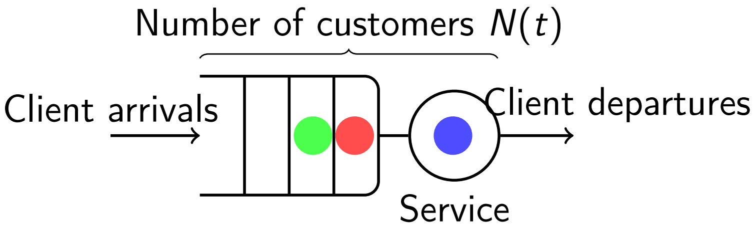
This week is devoted to continuous time Markov chains. These stochastic processes are the combination of a discrete time Markov chain and a random sojourn time. Namely, we have as before a state space, the transition probabilities $p_{i,j}$, to which we add random sojourn times, which are supposed to be exponentially distributed.

Construction of a sample-path



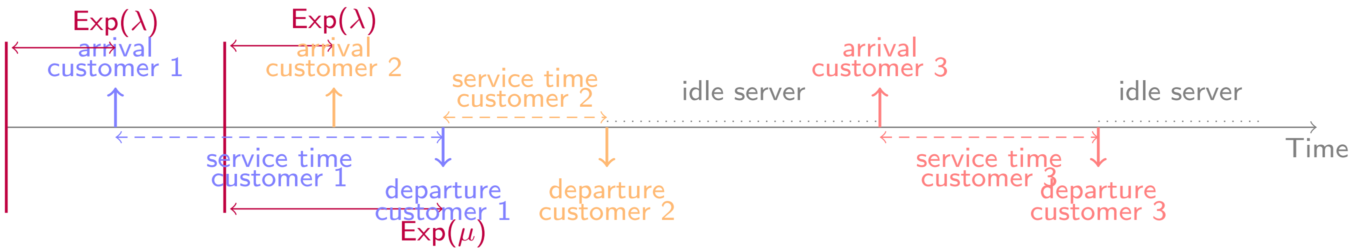
Let's look at how the trajectory of a Continuous Time Markov Chain is built.
We start with an initial state i. The process stays there for a random amount of time, which is an exponential random variable of parameter λ_i. Then, with probability $p_{i,j}$, it jumps to state j. And it stays there for an exponential random time of parameter λ_j. Then jumps with probability $p_{j,k}$ to state k where it stays for an exponential random time of parameter λ_k. And the process continues forever, following these rules.

M/M/1



- 1 server,
- infinite buffer capacity,
- Poisson arrivals (λ),
- Exponential services (μ),
- FIFO (First In First Out) service discipline

Let's see now how it works for the M/M/1 queue. Recall that we have one server, an infinite buffer, Poisson arrivals of intensity λ, exponentially distributed service times, and the discipline is First In First Out. The process we are interested in is the number of customers in the system.



- $p_{i,j} = 0$ if $j > i + 1$ or $j < i - 1$
- Until next event,
$$\min\Big(\mathrm{Exp}(\mu),\ \mathrm{Exp}(\lambda)\Big) = \mathrm{Exp}(\lambda + \mu) \Longrightarrow \lambda_i = \lambda + \mu$$
- For $j = i + 1$,
$$p_{i,i+1} = \mathbf{P}\Big(\mathrm{Exp}(\lambda) \le \mathrm{Exp}(\mu)\Big) = \frac{\lambda}{\lambda + \mu}$$
- For $j = i - 1$,
$$p_{i,i-1} = \mathbf{P}\Big(\mathrm{Exp}(\lambda) \ge \mathrm{Exp}(\mu)\Big) = \frac{\mu}{\lambda + \mu}$$
- $\lambda_0 = \lambda,\ p_{0,1} = 1$

This number changes only when a new customer arrives or when a service ends. Since there are no two simultaneous arrivals or simultaneous departures, the process can only increase or decrease by one unit at each change. This means $p_{i,j}$ is zero if the gap between i and j is more than 2. If we have i clients present in the system, the next event happens either when there is an arrival or when there is a departure. The next arrival happens in an exponential time of parameter λ since the arrivals form a Poisson process of intensity λ. The next departure happens in an exponential time of intensity μ. So, the next event happens at the minimum of these two time lapses, which, as we know, is distributed as an exponential random variable of parameter $\lambda + \mu$. This means that λ_i is equal to $\lambda + \mu$. The process jumps from i to $i + 1$ if the arrival happens before the departure, in other words... if an exponential random variable of parameter λ is smaller than an exponential random variable of parameter μ. Again, we know that the probability of this event is $\frac{\lambda}{\lambda+\mu}$. The process jumps from i to $i - 1$ with the complementary probability which is $\frac{\mu}{\lambda+\mu}$. If i is zero, no departure can occur, so the next event happens in an exponential time of parameter λ. This means $\lambda_0 = \lambda$ and $p_{0,1} = 1$.