M/M/1 queue

Steady-state analysis

We propose for the M/M/1 queue, to present the steady-state analysis.

Chapman Kolmogorov equations

We remark:

$$\begin{cases} \pi_i(t+\Delta t) & = & \pi_{i-1}(t)\,\lambda\,\Delta t + \pi_i(t)\,(1-(\lambda+\mu)\Delta t) \\ & & +\pi_{i+1}(t)\,\mu\,\Delta t + o(\Delta t), \;\; i \geq 1 \\ & & \\ \pi_0(t+\Delta t) & = & \pi_0(t)(1-\lambda\Delta t) + \pi_1(t)\,\mu\,\Delta t + o(\Delta t) \end{cases}$$

which is equivalent to:

$$\begin{cases} \pi_i(t+\Delta t) - \pi_i(t) & = & \pi_{i-1}(t)\,\lambda\Delta t - \pi_i(t)\,(\lambda+\mu)\Delta t \\ & & \\ & & +\pi_{i+1}(t)\,\mu\Delta t, +o(\Delta t) \;\; i \geq 1 \\ & & \\ \pi_0(t+\Delta t) - \pi_0(t) & = & \pi_0(t)\lambda\Delta t + \pi_1(t)\mu\Delta t + o(\Delta t) \end{cases}$$

From the Chapman Kolmogorov equations, we obtain a system of equations of the probabilities $\pi_i(t)$, for i greater than one in the first line, and for $i = 0$ in the second line.
We can notice in red the probabilities that are the same on the left and the right of the equations. We propose to make the product.
So we can therefore transpose them from the right members of the equations to the left members. Then we can divide the right and left members by Δt, and we take the limit when Δt tends to 0.

Chapman Kolmogorov equations

So we obtain after dividing by Δt, and when $\Delta t \to 0$:

$$\begin{cases} \lim_{\Delta t \to 0} \frac{\pi_i(t+\Delta t)-\pi_i(t)}{\Delta t} & = & \pi_{i-1}(t)\,\lambda - \pi_i(t)\,(\lambda+\mu) \\ & & +\pi_{i+1}(t)\,\mu, \ \ i \geq 1 \\ & & \\ \lim_{\Delta t \to 0} \frac{\pi_0(t+\Delta t)-\pi_0(t)}{\Delta t} & = & -\lambda\,\pi_0(t) + \mu\,\pi_1(t) \end{cases}$$

which is equivalent to:

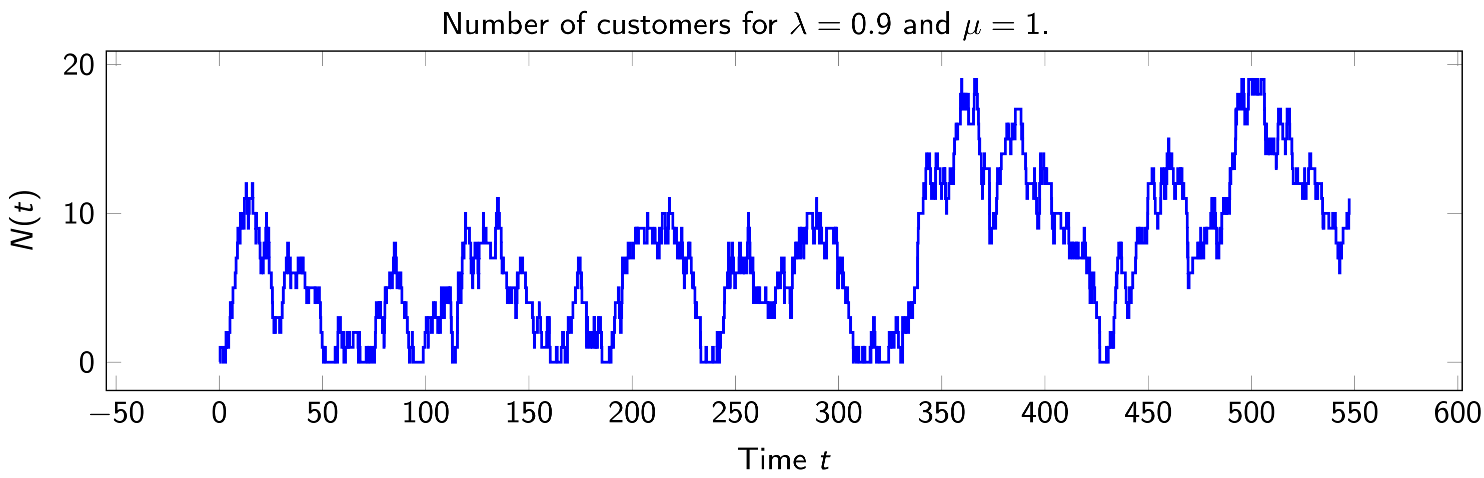
$$\begin{cases} \frac{d}{dt}\pi_i(t) = \pi_{i-1}(t)\,\lambda - \pi_i(t)\,(\lambda+\mu) + \pi_{i+1}(t)\,\mu, \ \ i \geq 1 \\ \\ \frac{d}{dt}\pi_0(t) = -\lambda\,\pi_0(t) + \mu\,\pi_1(t) \end{cases}$$

This system is equivalent to a system with differential equations, corresponding to the derivatives of the probabilities π_i of t.

Steady-state analysis

Under the stability condition, the number of customers in the queue has a limit probability (when $t \to \infty$)

$$\pi_i(t) \overset{t\to\infty}{=} \pi_i, \forall i \geq 0$$



We focus on the steady-state analysis: so we suppose that under the stability condition, the number of customers when the time t is very large (t tends to infinity), has a limiting probability π_i, for all i.
We can see in the present figure the behavior of the number of customers $N(t)$ in a stable system: when t tends to infinity, the number of customers remains finite.

Chapman Kolmogorov equations resolution

We propose to solve the Kolmogorov differential equations system:

$$\frac{d}{dt}\pi_i(t) = 0, i = 0, 1, 2, \ldots$$

In the following system:

$$\begin{cases} \frac{d}{dt}\pi_0(t) = -\lambda\, \pi_0(t) + \mu\, \pi_1(t) \\ \\ \frac{d}{dt}\pi_i(t) = \pi_{i-1}(t)\, \lambda - \pi_i(t)\, (\lambda + \mu) + \pi_{i+1}(t)\, \mu, \;\; i \geq 1 \end{cases}$$

We propose to solve the Chapman Kolmogorov equations in the steady state. So the derivatives in the Chapman Kolmogorov equations are null.
It is important to see that in these equations, the probabilities $\pi_i(t)$ are replaced by the steady-state probabilities π_i.

Steady-state probabilities

The steady-state probabilities $\pi_i,\ i = 0, 1, 2, \ldots$ of the system are derived from:

$$\begin{cases} -\lambda\pi_0 + \mu\pi_1 = 0 \\ \lambda\pi_0 - (\lambda+\mu)\pi_1 + \mu\pi_2 = 0 \\ \ldots \\ \lambda\pi_{i-1} - (\lambda+\mu)\pi_i + \mu\pi_{i+1} = 0,\ i \geq 2 \end{cases}$$

With the following normalizing condition:

$$\sum_{i=0}^{\infty} \pi_i = 1$$

So we obtain a set of equations for the steady-state probabilities π_i, for all i.
We begin with the first equation which considers a relationship between probabilities π_0 and π_1, and after that we consider the other equations when i is greater than 1, between probabilities π_0, π_1, and π_2.
In the general case, it is between π_{i-1}, π_i, and π_{i+1} for i greater than 2.
In order to solve this system, we have to compute the different probabilities $pi_0, pi_1, ...pi_i$, for all i.
We consider the normalizing condition which means that the sum of all the probabilities π_i must be equal to 1.

Resolution of the equations system

$$\begin{cases} -\lambda\pi_0 + \mu\pi_1 = 0 \\ \lambda\pi_0 - \lambda\pi_1 - \mu\pi_1 + \mu\pi_2 = 0 \\ \lambda\pi_1 - \lambda\pi_2 - \mu\pi_2 + \mu\pi_3 = 0 \\ \vdots \\ \lambda\pi_{i-1} - \lambda\pi_i - \mu\pi_i + \mu\pi_{i+1} = 0 \end{cases}$$

$$\begin{cases} \pi_1 = \frac{\lambda}{\mu}\pi_0 \\ \pi_2 = \frac{\lambda}{\mu}\pi_1 = (\frac{\lambda}{\mu})^2\pi_0 \\ \pi_3 = \frac{\lambda}{\mu}\pi_2 = (\frac{\lambda}{\mu})^3\pi_0 \\ \vdots \\ \pi_i = \frac{\lambda}{\mu}\pi_{i-1} = (\frac{\lambda}{\mu})^i\pi_0, i \geq 3 \end{cases}$$

So, the probability to have i customers is:

$$\pi_i = \pi_0\,(\frac{\lambda}{\mu})^i, i = 1, 2, \ldots$$

with the following normalizing condition: $\sum_{i=0}^{\infty} \pi_i = 1$.

Now, we explain how to solve the system of equations. We begin with the first line which is the relationship between probabilities π_0 and π_1.
In the second line, we use the first equation, so we can remove the two terms ($\lambda\pi_0$ and $-\mu\pi_1$). So we can simplify the second equation and we obtain an equation between $-\lambda\pi_1$ and $\mu\pi_2$.
In the third equation, we use the second equation and we can also simplify the equation. And so on for the last equation, we obtain a relationship between λpi_i and μpi_{i+1}.

So the steady-state probability π_1 is expressed in terms of π_0, π_2 is expressed in terms of π_1, and so also π_0, π_i is expressed in terms of π_{i-1} and also π_0. And as a result each π_i is expressed in terms of π_0 multiplied by $\left(\frac{\lambda}{\mu}\right)^i$.
Note that we have also to consider the normalizing condition for the computation of the probabilities π_i.