Statistical Properties of the Buddy System

PAUL W. PURDOM, JR.,* AND STEPHEN M. STIGLER

University of Wisconsin,† Madison, Wisconsin

ABSTRACT. The utilization of space and the running speed of the buddy system are considered Equations are derived that give various statistical properties of the buddy system. For the bottom level with Poisson requests and exponential service times the expected amount of space wasted by pairing full cells with empty cells is about $0.513\ \rho^{\frac{1}{2}}$ and the mean time between requests from the bottom level to the next level is about $1.880\ \rho^{\frac{1}{2}}\lambda^{-1}$, where ρ is the mean number of blocks in use on the bottom level and λ^{-1} is the mean time between requests for blocks on the bottom level. The results of a number of simulations of the buddy system are also given and compared with the analytical studies.

KEY WORDS AND PHRASES: buddy system, dynamic storage allocation, operations research, statistics, analysis of algorithms, Markov processes, simulation

CR CATEGORIES: 3.89, 4.32, 4.39, 5.5

1. *Introduction*

Modern computing systems often contain dynamic storage allocation algorithms, which do the bookkeeping required for making available various size blocks of memory to other routines. There are many algorithms which can provide dynamic storage allocation. Several of these are described by Knuth [1]. Since there are several dynamic storage algorithms, an analysis of the efficiency of these algorithms is useful for deciding which algorithm to put into a system. In this paper we analyze the efficiency of the buddy system of dynamic storage allocation.

The buddy system was devised by Knowlton [2]. It provides storage in blocks whose size is a power of two times some fixed basic size (to simplify the discussion we assume that basic size is one). The three basic ideas of the buddy system are as follows. (1) A separate list of available blocks is kept for each size 2^k, for $0 \le k \le m$, where 2^m is the total amount of space. (2) When a block of size 2^k is requested, it is taken from the available space list for that size, or if no block of that size is available, the system requests a block of size 2^{k+1} which it splits into two equal parts. The resulting blocks are called buddies of each other. One half is used to fill the original request and the other is put on the available space list for size 2^k. If there is no space of size 2^{k+1} available then the request for size 2^{k+1} will of course result in additional system requests for larger sizes until either space is found or the method fails because there is no block large enough for the original request. (3) When a block of size 2^k is returned it is combined with its buddy if its buddy is not in use. The resulting block of size 2^{k+1} is then combined with its buddy if its

* Present address: Bell Telephone Laboratories, Naperville, Ill.

† Computer Science Department and Statistics Department. This research was supported in part by the National Science Foundation Grant GP7069, the Wisconsin Alumni Research Foundation, and the Office of Naval Research under contract number Nonr 1202(17), Project number NR 042-222.

buddy is free. This is continued until a block is formed whose buddy is in use. The resulting block is added to the available space list for its size. A detailed description of the algorithm for the buddy system is given by Knowlton [2] and also by Knuth [1]. The algorithm is such that its running time depends only on the number of blocks that are requested by the user and on the number of additional requests that the system makes to itself because it does not have available a block of the required size.

In the buddy system there are two sources of inefficiency in memory utilization. First, memory is provided only in amounts which are a power of two, so the system must provide more memory than is asked for when the request is not a power of two. Second, two empty blocks of size 2^k cannot be used to fill requests for blocks of size 2^{k+1} unless they are buddies. If the distribution of size of requests is known, the inefficiency of the first type is easy to calculate, and we will say nothing more about it. Rather, it is the purpose of this paper to analyze the contribution of the inefficiency of the second type. Our calculations show that the inefficiency of the second type is so small that often the inefficiency of the first type is most important.

In this paper we therefore assume the requests are always for blocks which are a power of two in size. First we consider the relation between the demand for blocks on the bottom level (blocks of the smallest or basic size are said to be on the bottom level) and the number of cells that this demand makes unavailable on the next level (that is, all blocks of twice the basic size). The results for the bottom level can be used on any level provided the demand for cells on the lower levels is so low that they do not have a significant effect. Once the various equations are derived we investigate the solutions of the equations, again concentrating on the bottom level. Finally the results of some simulations of the buddy system are given. These throw light on the behavior of the upper levels.

2. *Analysis*

We will first consider a stochastic model where we keep track only of the pairing of blocks on the bottom level. In this model, which we shall call the restricted model, there are always $2n$ cells for filling requests on the bottom level. Thus in this model we ignore the fact that, in the original system, orders on the upper levels can change the amount of space available for use by the bottom level. Orders are ignored when all of the $2n$ cells are in use. When the number of cells available is unlimited, we have what we call the unrestricted model. The two models behave in nearly the same way when $2n$ is sufficiently larger than the mean number of cells needed to fill requests. We shall assume that the requests for blocks follow a Poisson process and that the lengths of time the blocks are used (service times) are given by independent random variables with an exponential distribution.[1] These assumptions cause the future development of the system to depend only on the present state of the system and not on its previous history. An analysis of this model will permit one to determine how much space is tied up in the available space list for the bottom level thus not being available for requests on the next level, and how often the bottom level requests space

[1] A stochastic process X_t is called a *Poisson process* with intensity λ if the probability Prob $\{X_{t+s} - X_s = K\} = e^{-\lambda t}(\lambda t)^K/K!$ for any $s, t \geq 0$, and $K = 0, 1, \cdots$ and also the random variables $X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \cdots, X_{t_n} - X_{t_{n-1}}$ are independent for any real numbers $0 \leq t_0 \leq t_1 \leq \cdots \leq t_n$. A random variable Y is said to have an *exponential distribution* with decay rate μ if Prob $\{Y > t\} = e^{-\mu t}$.

from the next level. Also, the model can be used for levels other than the bottom level so long as the rate of requests to levels below the one being investigated is so low that they have only a small effect on the level being studied.

The state of the model at time t is given by the bivariate stochastic process $Z_t = (E_t, F_t)$, where E_t is the number of pairs in which one block is in use and its buddy is available, and F_t is the number of pairs where both blocks (which are buddies of each other) are in use. We shall call E_t the number of empty pairs and F_t the number of full pairs. Let $p_{kl} = \lim \text{Prob}\,\{Z_t = (k, l)\}$, the stationary distribution of the process. Then the p_{kl} satisfy the equations:

$$[\lambda p_{kl}]_{\text{if } l \neq n} + (k + 2l)\mu p_{kl}$$
$$= (k+1)\mu p_{k+1\,l} + [2(l+1)\mu p_{k-1\,l+1}]_{\text{if } k \geq 1} + [\lambda p_{k+1\,l-1}]_{\text{if } l \geq 1} + [\lambda p_{0l}]_{\text{if } k=1} \tag{1}$$
$$\text{for } k, l \geq 0 \text{ and } k + l \leq n,$$

$$\sum_{k,l \geq 0} p_{kl} = 1, \tag{2}$$

and

$$p_{kl} = 0 \qquad \text{for } k + l > n. \tag{3}$$

The left-hand side of the first equation gives the rate at which the system leaves state (k, l) and the left-hand side of the second equation gives the rate at which it enters state (k, l). A derivation of the equations is given in Appendix 1.

Similar equations hold for the unrestricted model and can be found by setting $n = \infty$ with the side condition $p_{kl} \geq 0$. Numerical solutions to these equations are given later. By using an exponential generating function (see Appendix 2), we can get relations among the moments $a_{kl} = \sum_{i,j \geq 0} i^k j^l p_{ij}$, and the moments along $k = 0$, $b_l = \sum_{i \geq 0} i^l p_{0i}$ such as

$$a_{10} = \rho b_0, \tag{4}$$

$$a_{01} = \tfrac{1}{2}\rho - \tfrac{1}{2}\rho b_0, \tag{5}$$

$$a_{20} = \tfrac{2}{3}\rho b_1 - \tfrac{1}{3}\rho^2 b_0 + \tfrac{1}{3}\rho + \tfrac{2}{3}\rho b_0, \tag{6}$$

$$a_{11} = \tfrac{1}{3}\rho b_1 + \tfrac{1}{3}\rho^2 b_0 - \tfrac{1}{3}\rho + \tfrac{1}{3}\rho b_0, \tag{7}$$

$$a_{02} = \tfrac{1}{4}\rho^2 - \tfrac{1}{2}\rho b_1 - \tfrac{1}{4}\rho^2 b_0 + \tfrac{1}{2}\rho - \tfrac{1}{2}\rho b_0, \tag{8}$$

where $\rho = \lambda/\mu$. Since

$$a_{10} + b_0 = \sum_{l \geq 0} p_{0l} + \sum_{k,l \geq 0} k p_{kl} \geq \sum_{k,l \geq 0} p_{kl} = 1,$$

the equation $a_{10} - \rho b_0$ leads to the limits $b_0 \geq 1/(1+\rho)$ and $a_{10} \geq \rho/(1+\rho)$, which are useful when ρ is small.

When the bottom level is out of empty cells and has a request, it generates a request from the next level to obtain space to fill the original request. The mean time between requests to the next level is

$$\sum_{l \geq 0} a_{1l} q_l \tag{9}$$

and the variance is

$$\sum_{l \geq 0} b_{1l} q_l - \left(\sum_{l \geq 0} a_{1l} q_l\right)^2; \tag{10}$$

where

$$q_l = [p_{0l}\lambda/(\lambda + 2l\mu)]/[\sum_{i\geq 0} p_{0i}\lambda/(\lambda + 2i\mu)], \tag{11}$$

$$(\lambda + (k + 2l)\mu)a_{kl} = [\lambda a_{k-1\ l+1} + k\mu a_{k-1\ l}]_{\text{if } k\geq 1} + 2l\mu a_{k+1\ l-1} + 1, \tag{12}$$

and

$$(\lambda + (k + 2l)\mu)b_{kl} - 2a_{0l} = [\lambda b_{k-1\ l+1} + k\mu b_{k-1\ l}]_{\text{if } k\geq 1} + 2l\mu b_{k+1\ l-1}. \tag{13}$$

(See Appendix 3.) Numerical results are given later.

Other models for the behavior of the bottom level are considered in Appendix 4.

The analytical techniques of this paper can also be applied to study several levels at a time [3]. The resulting equations, however, take too long to solve numerically in interesting cases.

3. *Numerical Results*

The equations for the stationary distribution of the bottom level of the restricted model with Poisson requests and exponential service times can be solved in a time time proportional to the number of variables, $\binom{n+2}{2}$. It was therefore possible to investigate the model under many conditions.

Table I shows the results of varying the ratio of the request rate and the service time for systems which have enough cells to meet nearly all requests. For each value $\rho = \lambda/\mu$, which we shall call the traffic intensity, we have checked to be sure that increasing the number of cells ($2n$) by 20 percent does not change the tabulated numbers by more than a few parts in 10^8. For all values of ρ of 0.4 or above, we have found that reducing n by 20 percent will change the tabulated numbers by at least

TABLE I. THE RESULTS OF CALCULATIONS OF THE STATIONARY BEHAVIOR OF THE RESTRICTED MODEL FOR THE BOTTOM LEVEL OF THE BUDDY SYSTEM

Poisson requests are at rate λ and exponential service times have decay rate μ. The number of pairs of cells in the system is n. The expected number of full blocks paired with empty blocks is M_E. The value of $\sum_l l^i p_{0l}$ is given by b_i. The mean time between requests to the next level is W, and V is the variance of this time.

$\rho = \lambda/\mu$	n	M_E	b_0	b_1	b_2	b_3	$W\mu$	$V\mu^2$
0.001	4	0.000999	0.9990	0.00000050	0.00000050	0.00000050	1001.	1000001.
0.002	4	0.001996	0.9980	0.00000199	0.00000199	0.00000199	501.0	250001.
0.004	4	0.003984	0.9960	0.00000796	0.00000796	0.00000796	251.0	62501.
0.01	4	0.009901	0.9901	0.00004934	0.00004934	0.00004934	101.0	10001.
0.02	4	0.01961	0.9804	0.0001948	0.0001948	0.0001948	51.00	2501.
0.04	4	0.03846	0.9615	0.0007588	0.0007590	0.0007594	26.00	626.0
0.1	4	0.09092	0.9092	0.004388	0.004395	0.004410	11.00	101.1
0.2	4	0.1668	0.8341	0.01549	0.01559	0.01580	5.983	26.13
0.4	5	0.2875	0.7186	0.04898	0.05033	0.05303	3.444	7.386
1.	8	0.5235	0.5235	0.1691	0.1971	0.2558	1.813	1.922
2.	10	0.7690	0.3845	0.3312	0.5176	0.9623	1.193	0.9182
4.	16	1.099	0.2748	0.5437	1.397	4.165	0.8434	0.5459
10.	20	1.735	0.1735	0.9133	5.273	32.84	0.5476	0.3172
20.	32	2.441	0.1220	1.295	14.35	165.6	0.3945	0.2210
40.	64	3.433	0.08583	1.811	39.05	859.4	0.2833	0.1569
100.	100	5.394	0.05394	2.810	147.6	7821.	0.1820	0.1007
200.	150	7.600	0.03800	3.922	406.5	42315.	0.1298	0.0721
400.	300	10.72	0.02680	5.488	1126.	231627.	0.0924	0.05154

several parts in 10^8. Thus these results should reflect the behavior of the unrestricted model.

We let M_E denote the expected number of blocks in use paired with empty blocks, which measures how much unused space is tied up in the available space list for the bottom level and thus not available for use on the next level. The values of $b_i = \sum_l l^i p_{0l}$ can be used in formulas (4)–(8) to compute the various moments of the number of empty pairs and the number of full pairs. The mean waiting time for orders to the next level, W, indicates how fast the buddy system runs. The variance, V, of this time may be useful in calculating the effect of the bottom level on the next level.

For $\rho = 10$ and $\rho = 100$ the probabilities ρ_{kl} of various values of k and l are given in Figures 1 and 2.

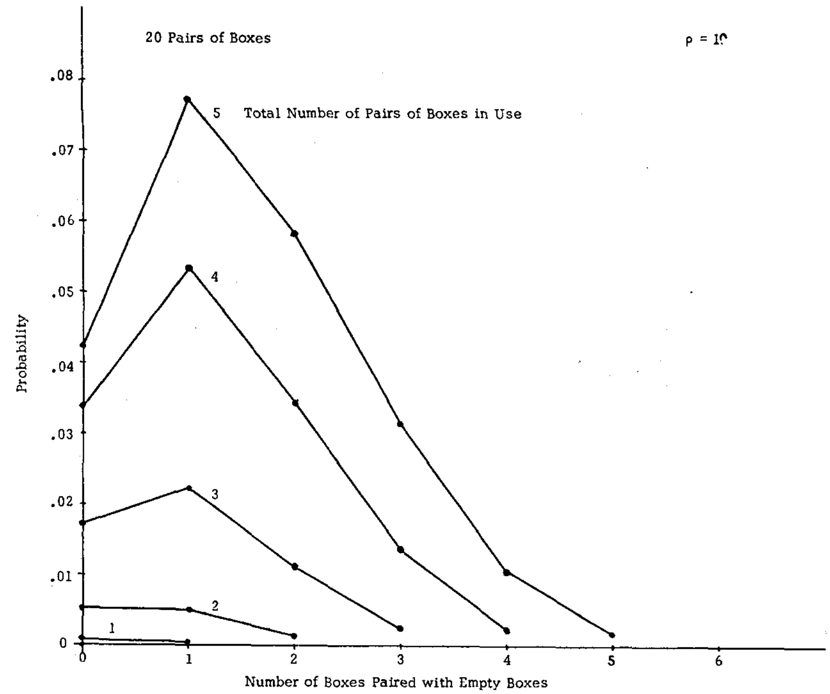
One is usually interested in various characteristics of the buddy system for large values of ρ. Table II shows the results of fitting the original eight-significant-figure data that was used to prepare Table I with polynomials in $\rho^{\frac{1}{2}}$. The leading term was selected by noting which half-integer power of ρ gave the best fit. Then fits were made with 1, 2, 3, 4, and 5 terms of a decreasing power series in $\rho^{\frac{1}{2}}$ to the last 1, 2, 3, 4, and 5 values in Table I for each item in Table II, the number of values always being the same as the number of terms. Fits were also made to the next to last 1, 2, 3, 4, and 5 values. Table II shows the results of the 4-parameter fit. The number of figures reported has been chosen so that the coefficients do not differ by more than 3 in the last figure in the three following situations: (1) 3 parameters are used in the fit, (2) 5 parameters are used in the fit, and (3) 4 parameters are used, but the data starting with the next to last entry ($\rho = 200$) is used. We therefore feel safe in using these series with values of ρ other than those used in the fit. Table III gives various moments of the numbers of full and empty pairs which can be calculated with the fits to b_0 and b_1. It should be noted that the process cannot be asymptotically normal because the mean and standard deviation of the number of empty cells are of the same order for large ρ, while the probability that the number of empty pairs is negative is zero.

If one wishes formulas for small values of ρ, he should find algebraic solutions to the equations for a system with a small number of cells. Table I shows that for $\rho \leq .2$, four pairs of cells is enough to obtain very accurate solutions for any system with four or more pairs of cells.

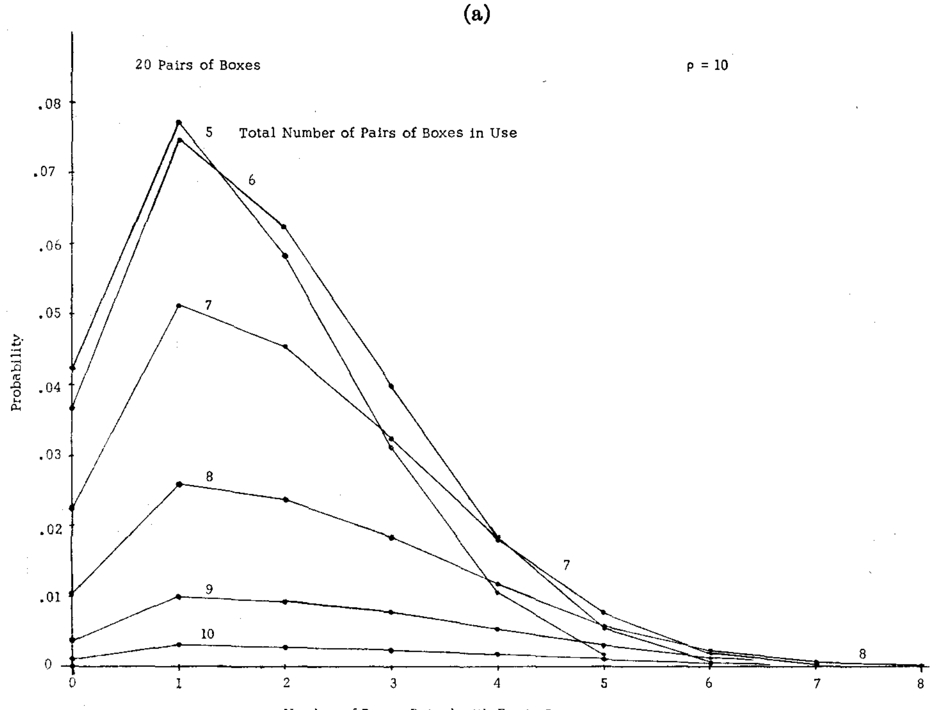
4. *Simulations*

A number of simulations were run to investigate features of the buddy system that were inconvenient or impossible to calculate directly. For the simulations, a random number generator of the form $x_{i+1} \leftarrow (7577 \times x_i + c) \bmod 2^{25}$ was used. Various odd numbers were used for c. In all of the simulations, the arrivals followed a Poisson process and the service times of blocks were exponentially distributed.

Since the simulation gives information about the system as a function of time, it was necessary to investigate how rapidly the system approaches stationarity so that the results of this section could be compared with those of the previous section. In an earlier report [3], it is shown that by 10,000 orders for $\rho = 10$ and by 50,000 orders for $\rho = 100$ we are so close to a stationary distribution that we can neglect any error caused by the fact that we have not yet achieved stationarity; the variation from

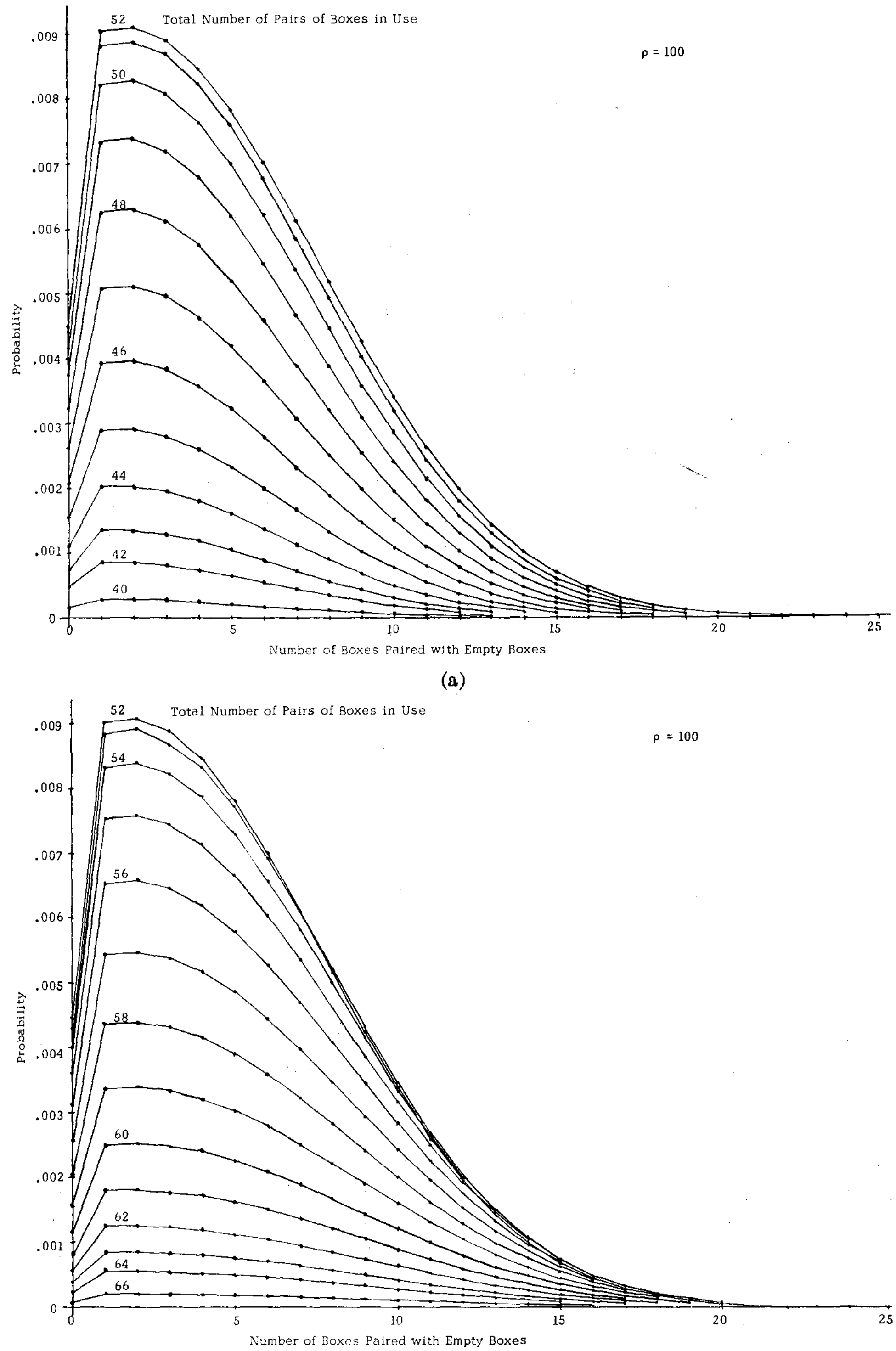


(a)



(b)

FIG. 1. The probabilities of various pairings of cells on the bottom level of the buddy system for $\rho = 10$. Each curve connects together the points where the total number of pairs of cells being used is constant. The probability of each pairing on the bottom level is shown as a function of the number of cells paired with empty cells.



(b)

FIG. 2. The probabilities of various pairings of cells on the bottom level of the buddy system for $\rho = 100$. Each curve connects together the points where the total number of pairs of cells being used is constant. The probability of each pairing on the bottom level is shown as a function of the number of cells paired with empty cells.

TABLE II. THE FORMULAS WHICH GIVE THE BEST FIT TO THE CALCULATIONS OF b_i, W, AND V FOR LARGE VALUES OF ρ

Quantity	Fit for large $\rho = \lambda/\mu$		
b_0	$0.53188\,\rho^{-1/2}$	$+\ 0.087\,\rho^{-1}$	$-\ 0.122\,\rho^{-3/2}$
b_1	$0.26594\,\rho^{1/2}$	$+\ 0.187$	$-\ 0.37\,\rho^{-1/2}$
b_2	$0.13296\,\rho^{3/2}$	$+\ 0.166\,\rho$	$-\ 0.199\,\rho^{1/2}$
b_3	$0.06647\,\rho^{5/2}$	$+\ 0.119\,\rho^2$	$-\ 0.03\,\rho^{3/2}$
$W\mu$	$1.8800\,\rho^{-1/2}$	$-\ 0.68\,\rho^{-1}$	$+\ 0.8\,\rho^{-3/2}$
$V\mu^2$	$1.065\,\rho^{-1/2}$	$-\ 0.80\,\rho^{-1}$	$+\ 2.4\,\rho^{-3/2}$

TABLE III. VARIOUS MOMENTS OF THE NUMBERS OF FULL AND EMPTY PAIRS WHICH CAN BE CALCULATED WITH THE FITS TO b_0 AND b_1

Shown are the mean of E, the number of pairs on the bottom level where one block is in use and its buddy is not; the mean of F, the number of pairs on the bottom level where both blocks of the buddy pair are in use; the mean of E^2; the mean of EF; the mean of F^2; the variance of E; the variance of F; and the product of the standard deviations and the correlation of E and F.

Quantity	Fit from b_0 and b_1
M_E	$0.53188\,\rho^{1/2} + 0.087 - 0.122\,\rho^{-1/2}$
M_F	$\frac{1}{2}\rho - 0.26594\,\rho^{1/2} - 0.043 + 0.061\,\rho^{-1/2}$
M_{E^2}	$0.429\,\rho + 0.15\,\rho^{1/2}$
M_{EF}	$0.26596\,\rho^{3/2} - 0.242\,\rho + 0.01\,\rho^{1/2}$
M_{F^2}	$\frac{1}{4}\rho^2 - .26594\,\rho^{3/2} + 1.539\,\rho + 1.92\,\rho^{1/2}$
$\sigma_E{}^2$	$0.157\,\rho + 0.07\,\rho^{1/2}$
$\sigma_F{}^2$	$1.511\,\rho + 1.87\,\rho^{1/2}$
$\rho_{EF}\sigma_E\sigma_F$	$0.154\,\rho + 0.103\,\rho^{1/2}$

one run to the next is much more significant than the error due to a lack of stationarity.

Table IV gives various statistics for $\rho = 1, 10$, and 100, where all of the orders were coming in on the bottom level. Tables V–VII give statistics for $\lambda_0 + \lambda_1 = 10$, $\mu_0 = 1$, and $\mu_1 = 1$, where λ_0 is the arrival rate on the bottom level, λ_1 is the arrival rate on the next level, μ_0 is the decay rate for the bottom level, and μ_1 is the decay rate for the next level (for blocks that were ordered from outside, thus μ_1 does not count the decay of blocks that were split up to meet orders on the bottom level). Most of those results of the simulation which can be compared with the calculations of Section 3 are in good agreement. The results for W_1, the mean time between orders from the bottom level to the next level, and for V_1, the variance of that time, however, while close to the calculated values, are different by several standard deviations. We suspect the difference is caused by a slight nonrandom behavior of the random number generator.

There are three sets of results from the simulations that we would like to call

TABLE IV. E_i, W_i, AND V_i FOR ORDERS COMING IN ONLY ON THE BOTTOM LEVEL AND WITH $\rho = 1$, 10, AND 100

E_i, mean number of blocks of size 2^i which are paired with empty blocks; W_i, the mean time that passes between each order to level i from level $i - 1$; V_i, the variance of this time. All runs were sufficiently long to achieve stationarity within a very close approximation.

	$\rho = 1$; 10,000 *orders*	$\rho = 10$; 10,000 *orders*	$\rho = 100$; 50,000 *orders*
E_0	$0.524 \pm 0.004(0.014)$	$1.724 \pm 0.015(0.037)$	$5.400 \pm 0.032(0.087)$
E_1	$0.519 \pm 0.004(0.014)$	$1.221 \pm 0.018(0.044)$	$4.930 \pm 0.051(0.134)$
E_2	$0.616 \pm 0.005(0.017)$	$0.806 \pm 0.018(0.045)$	$2.804 \pm 0.092(0.243)$
E_3	$0.626 \pm 0.005(0.015)$	$0.553 \pm 0.066(0.162)$	$1.47 \pm 0.15(0.39)$
E_4	$0.626 \pm 0.005(0.015)$	$0.640 \pm 0.042(0.102)$	$1.032 \pm 0.064(0.170)$
W_1	$1.921 \pm 0.020(0.063)$	$0.576 \pm 0.005(0.012)$	$0.1827 \pm 0.0011(0.0030)$
W_2	$2.623 \pm 0.023(0.072)$	$2.276 \pm 0.05(0.124)$	$1.633 \pm 0.014(0.038)$
W_3	$2.693 \pm 0.020(0.063)$	$8.15 \pm 0.37(0.90)$	$11.28 \pm 0.55(1.45)$
W_4	$2.693 \pm 0.020(0.063)$	$18.3 \pm 1.1(2.6)$	$23.8 \pm 5.1(13.4)$
V_1	$2.048 \pm 0.050(0.161)$	$0.343 \pm 0.005(0.012)$	$0.1044 \pm 0.0019(0.0051)$
V_2	$4.42 \pm 0.20(0.63)$	$4.32 \pm 0.21(0.51)$	$6.72 \pm 0.12(0.32)$
V_3	$5.01 \pm 0.21(0.65)$	$56.1 \pm 6.0(14.6)$	$287 \pm 40(107)$
V_4	$5.01 \pm 0.21(0.65)$	$325 \pm 44(109)$	$3132 \pm 899(2379)$

TABLE V. E_i, THE MEAN NUMBER OF BLOCKS OF SIZE 2^i, PAIRED WITH EMPTY BLOCKS AS A FUNCTION OF λ_0 AND λ_1, THE ARRIVAL RATES ON THE BOTTOM AND NEXT LEVELS

In all cases the decay rates were equal to one and each run consisted of 10,000 orders. Each entry gives the expected value plus or minus the standard error with the standard deviation expected from one run in parentheses.

λ_0	λ_1	E_0	E_1	E_2	E_3	E_4
10	0	$1.724 \pm 0.015(0.037)$	$1.221 \pm 0.018(0.044)$	$0.806 \pm 0.018(0.045)$	$0.553 \pm 0.066(0.162)$	$0.640 \pm 0.042(0.102)$
9	1	$1.656 \pm 0.003(0.008)$	$1.334 \pm 0.012(0.026)$	$0.959 \pm 0.022(0.055)$	$0.762 \pm 0.033(0.081)$	$0.421 \pm 0.018(0.044)$
8	2	$1.552 \pm 0.014(0.034)$	$1.396 \pm 0.018(0.043)$	$0.994 \pm 0.025(0.060)$	$0.928 \pm 0.029(0.072)$	$0.267 \pm 0.022(0.053)$
7	3	$1.459 \pm 0.013(0.033)$	$1.456 \pm 0.009(0.021)$	$1.005 \pm 0.009(0.023)$	$0.923 \pm 0.021(0.053)$	$0.211 \pm 0.016(0.039)$
6	4	$1.356 \pm 0.013(0.030)$	$1.486 \pm 0.011(0.025)$	$1.078 \pm 0.021(0.047)$	$0.892 \pm 0.015(0.033)$	$0.178 \pm 0.010(0.023)$
5	5	$1.224 \pm 0.013(0.028)$	$1.558 \pm 0.008(0.018)$	$1.076 \pm 0.016(0.036)$	$0.884 \pm 0.010(0.021)$	$0.217 \pm 0.038(0.084)$
4	6	$1.092 \pm 0.013(0.029)$	$1.591 \pm 0.011(0.025)$	$1.120 \pm 0.020(0.044)$	$0.837 \pm 0.013(0.029)$	$0.201 \pm 0.019(0.043)$
3	7	$0.958 \pm 0.014(0.030)$	$1.627 \pm 0.007(0.015)$	$1.154 \pm 0.022(0.048)$	$0.802 \pm 0.008(0.017)$	$0.332 \pm 0.061(0.137)$
2	8	$0.776 \pm 0.005(0.013)$	$1.678 \pm 0.015(0.037)$	$1.182 \pm 0.009(0.021)$	$0.758 \pm 0.022(0.054)$	$0.423 \pm 0.052(0.127)$
1	9	$0.518 \pm 0.004(0.011)$	$1.713 \pm 0.007(0.017)$	$1.211 \pm 0.021(0.052)$	$0.818 \pm 0.020(0.050)$	$0.548 \pm 0.035(0.086)$
0	10	0	$1.724 \pm 0.015(0.037)$	$1.221 \pm 0.018(0.044)$	$0.806 \pm 0.018(0.045)$	$0.553 \pm 0.066(0.162)$

attention to in particular. First consider setting

$$\rho_{\text{eff}} = \frac{\lambda_1}{\mu_1} + \left(\frac{E_0 + F_0}{2}\right)\frac{W_1}{\sqrt{V_1}}, \tag{14}$$

where λ_1 is the rate of requests on the next to bottom level, μ_1 is the decay rate on the next to bottom level, $(E_0 + F_0)/2$ is the mean number of pairs of cells in use on the bottom level, W_1 is the mean time between orders from the bottom level to the next level, and V_1 is the variance of this time. If the resulting value of ρ_{eff} is somewhat bigger than one, then the mean number of blocks paired with empties on the next to bottom level can be estimated using ρ_{eff} in place of ρ in the equation for the bottom level, and gives nearly the same answer for E_1 as the simulations. It would be in-

TABLE VI. W_i, THE MEAN TIME BETWEEN ORDERS TO LEVEL i FROM LOWER LEVELS
The form of the table is the same as that of Table V.

λ_0	λ_1	W_1	W_2	W_3	W_4
10	0	0.576 ± 0.005(0.012)	2.276 ± 0.051(0.124)	8.15 ± 0.37(0.90)	18.3 ± 1.1(2.6)
9	1	0.604 ± 0.003(0.008)	1.615 ± 0.023(0.056)	5.57 ± 0.23(0.56)	10.3 ± 0.67(1.64)
8	2	0.641 ± 0.004(0.009)	1.307 ± 0.024(0.059)	4.67 ± 0.20(0.50)	12.3 ± 1.1(2.8)
7	3	0.687 ± 0.004(0.009)	1.068 ± 0.009(0.023)	3.846 ± 0.074(0.182)	10.93 ± 0.51(1.26)
6	4	0.748 ± 0.005(0.010)	0.932 ± 0.009(0.020)	3.326 ± 0.070(0.157)	10.68 ± 0.41(0.92)
5	5	0.800 ± 0.007(0.019)	0.834 ± 0.010(0.021)	3.212 ± 0.059(0.133)	11.72 ± 0.59(1.33)
4	6	0.917 ± 0.008(0.018)	0.756 ± 0.003(0.008)	2.842 ± 0.062(0.140)	9.74 ± 0.32(0.71)
3	7	1.059 ± 0.006(0.012)	0.688 ± 0.004(0.009)	2.696 ± 0.039(0.088)	9.56 ± 0.50(1.11)
2	8	1.301 ± 0.007(0.017)	0.638 ± 0.006(0.015)	2.493 ± 0.036(0.088)	9.45 ± 0.42(1.02)
1	9	1.925 ± 0.020(0.049)	0.597 ± 0.003(0.007)	2.411 ± 0.040(0.097)	9.16 ± 0.41(0.99)
0	10	∞	0.576 ± 0.005(0.012)	2.276 ± 0.051(0.124)	8.15 ± 0.37(0.90)

TABLE VII. V_i, THE VARIANCE OF THE TIME BETWEEN ORDERS TO LEVEL i FROM LOWER LEVELS
The form of the table is the same as that of Table V.

λ_0	λ_1	V_1	V_2	V_3	V_4
10	0	0.343 ± 0.005(0.012)	4.32 ± 0.21(0.51)	56.1 ± 6.0(14.6)	325 ± 44(109)
9	1	0.364 ± 0.011(0.027)	2.481 ± 0.074(0.181)	27.0 ± 2.4(5.8)	157 ± 42(104)
8	2	0.390 ± 0.011(0.027)	1.573 ± 0.044(0.108)	20.9 ± 2.6(6.3)	284 ± 70(173)
7	3	0.428 ± 0.007(0.018)	1.095 ± 0.008(0.019)	13.52 ± 0.63(1.56)	212 ± 22(54)
6	4	0.475 ± 0.006(0.014)	0.819 ± 0.029(0.065)	10.20 ± 0.46(1.02)	179 ± 14(31)
5	5	0.520 ± 0.009(0.019)	0.673 ± 0.025(0.055)	8.73 ± 0.45(1.00)	216 ± 28(63)
4	6	0.625 ± 0.010(0.023)	0.555 ± 0.018(0.040)	7.83 ± 0.23(0.52)	131 ± 12(28)
3	7	0.753 ± 0.024(0.054)	0.483 ± 0.007(0.017)	6.41 ± 0.20(0.45)	110 ± 10(23)
2	8	1.112 ± 0.031(0.075)	0.422 ± 0.006(0.015)	5.61 ± 0.25(0.61)	126 ± 17(41)
1	9	1.998 ± 0.055(0.137)	0.364 ± 0.004(0.011)	5.55 ± 0.34(0.83)	80.0 ± 9.5(23.3)
0	10	∞	0.343 ± 0.005(0.012)	4.32 ± 0.21(0.51)	56.1 ± 6.0(14.6)

teresting to know whether this formula would be useful at values of λ_0, λ_1, μ_0, and μ_1 other than those we have tested. Even from our data, it is clear that the method does not work exactly. Second, while it is not clear what the relation is between ρ on the bottom level and the rate at which the bottom level generates requests which go above the next level, it is evident that the fraction of requests that goes above the next level decreases as ρ increases. Thus for large values of ρ we can get a good idea of how much time the system will spend breaking up blocks by looking only at how much time is spent breaking up blocks from just one level up. Third, the mean time between orders from one level to the one above it depends almost entirely on the rate at which that level receives orders, unless the levels below it receive orders at a much higher rate. From just our one set of runs one cannot tell in general when the lower levels will have a significant effect on this quantity.

5. *Conclusions*

We have given formulas which permit one to calculate the effect of the bottom level of the buddy system on the next level. These show that for Poisson arrivals and exponential service times the mean number of blocks tied up on the available space list for the bottom level is proportional to the square root of the mean number of blocks it is using to fill requests. The rate at which it asks for blocks from the next level is proportional to the square root of the traffic intensity. The simulations indicate that when the average rate of blocks being requested on one level is not small with respect to the rate of blocks being requested from lower levels, the rate at which that level orders blocks from the next higher level is not greatly affected by the rate of orders on the lower levels. Thus it appears a good indication of the running time can be computed by considering each pair of levels independently. There are also indications that the amount of space on the available space lists can be calculated from the results on pairs of levels.

Appendix 1. *Derivation of Eqs.* (1)–(3)

The state of the model at time t is given by the bivariate stochastic process $Z_t = (E_t, F_t)$, where E_t is the number of pairs in which one block is in use and its buddy is available, and F_t is the number of pairs where both blocks (which are buddies of each other) are in use. Let

$$P_{kl}(t) = \text{Prob}\,\{Z_t = (k,l)\} \quad \text{and} \quad Q_{kl}^{ij}(h) = \text{Prob}\,\{Z_{t+h} = (i,j) \mid Z_t = (k,l)\},$$

where $i, j, k, l, t, h \geq 0$. For the restricted model, Z_t is a Markov process with stationary transition probabilities satisfying the following conditions:

(1) $Z_0 = (0, 0)$;

$$(2)\quad Q_{kl}^{ij}(h) = \begin{cases} \lambda h + o(h) & \text{if } i = 1,\ k = 0,\ j = l, \text{ and } j \leq n - 1, \text{ or if } k = i + 1, \\ & j = l + 1, \text{ and } k + l \leq n; \\ k\mu h + o(h) & \text{if } k = i + 1, j = l, \text{ and } k + l \leq n; \\ 2l\mu h + o(h) & \text{if } i = k + 1,\ l = j + 1, \text{ and } k + l \leq n; \\ 1 - [\lambda + (k + 2l)\mu]h + o(h) & \text{if } i = k,\ j = l,\ l \leq n - 1, \text{ and } k + l \leq n; \\ 1 - 2n\mu h + o(h) & \text{if } i = k = 0 \text{ and } j = l = n; \end{cases}$$

where $\lambda > 0$ is the rate of requests, $\mu > 0$ is the decay (or service) rate for a block that is in use, and $o(h)/h \to 0$ as $h \to 0$. Then from the axioms of probability, the process obeys the following set of equations:

$$\begin{aligned} P_{kl}(t + h) = {} & P_{k-1\,l}(t)Q_{k-1\,l}^{kl}(h) + P_{k+1\,l-1}(t)Q_{k+1\,l-1}^{kl}(h) + P_{k+1\,l}(t)Q_{k+1\,l}^{kl}(h) \\ & + P_{k-1\,l+1}(t)Q_{k-1\,l+1}^{kl}(h) + P_{kl}(t)Q_{kl}^{kl}(h) + o(h). \end{aligned}$$

Taking the limit as h goes to zero and replacing the Q's by their values we get the following differential equations for the system:

$$\begin{aligned} P'_{kl}(t) = {} & -(k + 2l)\mu P_{kl}(t) - [\lambda P_{kl}(t)]_{\text{if } l \neq n} + (k + 1)\mu P_{k+1\,l}(t) \\ & + [2(l + 1)\mu P_{k-1\,l+1}(t)]_{\text{if } k \geq 1} + [\lambda P_{k+1\,l-1}(t)]_{\text{if } l \geq 1} + [\lambda P_{0l}(t)]_{\text{if } k=1} \\ & \text{for } k, l \geq 0 \text{ and } k + l \leq n, \end{aligned}$$

subject to the conditions

$$P_{kl}(0) = \begin{cases} 1 & \text{if } k = l = 0, \\ 0 & \text{otherwise;} \end{cases}$$

$$P_{kl}(t) = 0 \quad \text{for } k + l > n.$$

The terms in brackets are included only when their conditions are satisfied. The solutions to the equations also must satisfy the conditions:

$$\sum_{k,l\geq 0} P_{kl}(t) = 1 \qquad \text{for all } t \geq 0,$$

$$P_{kl}(t) \geq 0 \qquad \text{for all } k,\ l,\ t \geq 0.$$

We cannot say much about the distribution of Z_t for finite t. The quantity $E_t + 2F_t$ in its unrestricted case, however, is a simple birth and death process with immigration and has been studied in detail (see [4]).

It is well known that since Z_t is an irreducible continuous parameter Markov chain the limit $\lim_{t\to\infty} P_{kl}(t) = p_{kl}$ exists. These limits, called the stationary probabilities, can be obtained by setting the derivatives in the differential equations equal to zero to obtain the recurrence eqs. (1).

Appendix 2. *Exponential Generating Function*

To study the moments of p_{kl} it is useful to consider the exponential generating function

$$H(s_1, s_2) = \sum_{k,l\geq 0} p_{kl} e^{ks_1 + ls_2}.$$

The function $H(s_1, s_2)$ can also be expanded in a power series, so we can find a_{kl} such that

$$H(s_1, s_2) = \sum_{k,l\geq 0} a_{kl} s_1^k s_2^l.$$

The value of a_{kl} is just the moment $\sum_{i,j\geq 0} i^k j^l p_{ij}$. By multiplying the recurrence equations for the unrestricted case by $e^{ks_1+ls_2}$ and summing, one gets the following partial differential equation:

$$\lambda[1 - e^{s_2 - s_1}]H(s_1, s_2) = [e^{-s_1} - 1]\mu \frac{\partial}{\partial s_1} H(s_1, s_2) + 2\,[e^{(s_1 - s_2)} - 1]\mu \frac{\partial}{\partial s_2} H(s_1, s_2) + \lambda[e^{s_1} - e^{(s_2 - s_1)}]H(-\infty, s_2),$$

where $H(-\infty, s_2) = \sum_l p_{0l} e^{ls_2}$. The presence of this last term makes it difficult to solve this equation. However, by letting $H(-\infty, s_2) = \sum_{l\geq 0} b_l s_2^l$, we can get eqs. (4)-(8).

Appendix 3. *Requests of Space Generated by the Bottom Level*

We now consider how often the bottom level requests cells from the next level. Let the random variable S_{kl} denote the total time elapsed from the time the system enters the state (k, l) until it next requests a block from the next level (i.e. $E_t + F_t$ increases). Since Z_t is a strong Markov process, we can describe S_{kl} as follows:

$$S_{kl} - \inf\{s: \lim_{h\to 0} [E_{t_0+s+h} + F_{t_0+s+h} - E_{t_0+s} - F_{t_0+s}] > 0\}, \qquad \text{where } Z_{t_0} = (k, l).$$

Let $G_{kl}(t)$ be the Laplace transform of the density of S_{kl}, i.e. $G_{kl}(t) = E\{e^{-tS_{kl}}\}$,[2]

[2] We use the notation $E(X)$ or $E\{X\}$ to denote the mathematical expectation (expected value) of the random variable X. We shall denote the variance of X by var (X).

and let T_{kl} be the time the system stays in state (k, l). Then S_{kl} equals T_{kl} plus the time from the next state until the next order. Now, in the unrestricted system, a transition from state (k, l) is to $(k + 1, l - 1)$ with probability $2l\mu/(\lambda + (k + 2l)\mu)$, to $(k - 1, l)$ with probability $k\mu/(\lambda + (k + 2l)\mu)$, if $k > 0$ to $(k - 1, l + 1)$ with probability $\lambda/(\lambda + (k + 2l)\mu)$, and if $k = 0$ to $(k + 1, l)$ (causing a request on the next level) with probability $\lambda/(\lambda + 2l\mu)$. Thus

$$G_{kl}(t) = E\{e^{-tT_{kl}}\}\left[\left(\frac{\lambda}{\lambda + (k + 2l)\mu}\right)G_{k-1\ l+1}(t) + \left(\frac{k\mu}{\lambda + (k + 2l)\mu}\right)G_{k-1\ l}(t) + \left(\frac{2l\mu}{\lambda + (k + 2l)\mu}\right)G_{k+1\ l-1}(t)\right]$$

for $k > 0$ and

$$G_{0l}(t) = E\{e^{-tT_{0l}}\}\left[\left(\frac{\lambda}{\lambda + 2l\mu}\right) + \left(\frac{2l\mu}{\lambda + 2l\mu}\right)G_{1\ l-1}(t)\right].$$

Since Z_t is a Markov process, and thus has exponential waiting times,

$$E\{e^{-tT_{kl}}\} = \frac{\lambda + (k + 2l)\mu}{\lambda + (k + 2l)\mu + t}.$$

Thus we find the equations

$$(\lambda + (k + 2l)\mu + t)G_{kl}(t) = \lambda G_{k-1\ l+1}(t) + k\mu G_{k-1\ l}(t) + 2l\mu G_{k+1\ l-1}(t) \quad \text{for } k > 0,$$

and

$$(\lambda + 2l\mu + t)G_{0l}(t) = \lambda + 2l\mu G_{1\ l-1}(t),$$

which can be solved recursively. By differentiating these equations and letting

$$a_{kl} = E(S_{kl}) = -\frac{d}{dt}G_{kl}(0) \quad \text{and} \quad b_{kl} = E(S_{kl}^2) = \frac{d^2}{dt^2}G_{kl}(0),$$

we obtain eqs. (12) and (13). These equations can be solved to find $E(S_{kl}) = a_{kl}$ and var $(S_{kl}) = b_{kl} - a_{kl}^2$.

To find the mean and variance of the time between orders to the next level once the process has become stationary, let q_l be the stationary probability that we are in the state $(1, l)$ given that an order to the next level has just taken place. In other words,

$$q_l = \lim_{t\to\infty} \text{Prob}\,\{Z_t = (1, l) \mid E_{t-} = 0, E_{t+} = 1\}.$$

Then q_l is proportional to the probability that we arrive at $(1, l)$ from $(0, l)$ given that a transition takes place. This probability is equal to $p_{0l}\lambda/(\lambda + 2l\mu)$. Thus we get eqs. (9)–(11). Higher moments can be calculated in a similar manner.

Appendix 4. *Other Models for the Requests*

Thus far we have assumed that the requests (arrivals) follow a Poisson process; that is, the interarrival times on the bottom level are independent and exponentially distributed. We now drop the assumption that they are exponentially distributed but will assume that they are independently and identically distributed according to some arbitrary distribution function $G(t)$, with $G(0) = 0$. We still assume that the service times are independent and exponentially distributed.

Now if $G(t)$ is not exponential, Z_t is no longer a Markov process. We do, however, have an embedded Markov chain. Let $t_0 = 0$ and t_n be the time of the nth arrival on the bottom level. The difference $t_n - t_{n-1}$ has the distribution G. Let $Z_n = Z_{t_n^-} = (E_{t_n^-}, F_{t_n^-})$ define a discrete time stochastic process. In fact $\{Z_n\}$ is a Markov chain. To see this, let W_n be the number of pairs which are full at time t_n^+, both cells of which complete service before time t_{n+1}; let X_n be the number of pairs which are full at time t_n^+, one cell of which completes service before time t_{n+1}; and let Y_n be the number of pairs which are (half) empty at time t_n^+ and which complete service by time t_{n+1}. Since the service times are exponentially distributed, the conditional probability distribution of (W_n, X_n, Y_n) given Z_n is independent of $Z_1, Z_2, \cdots, Z_{n-1}$. Now

$$Z_{n+1} = \begin{cases} Z_n - (Y_n - X_n + 1, W_n + X_n - 1) & \text{if } E_{t_n^-} > 0, \\ Z_n - (Y_n - X_n - 1, W_n - X_n) & \text{if } E_{t_n^-} = 0, \end{cases}$$

so the conditional probability distribution of Z_{n+1} given Z_n is independent of $Z_1, \cdots, Z_{n-1}$, and the chain $\{Z_n\}$ is Markov.

Let $P_{kl}^{ij} = \text{Prob}\,\{Z_{n+1} = (i, j) \mid Z_n = (k, l)\}$, the one-step transition probabilities for $\{Z_n\}$. Then

$$P_{kl}^{ij} = \int_0^\infty P_{kl}^{ij}(t)\, dG(t),$$

where

$$P_{kl}^{ij}(t) = \text{Prob}\,\{Z_{n+1}(i, j) \mid Z_n = (k, l) \text{ and } t_{n+1} - t_n = t\}.$$

Now for the unrestricted system

$$P_{kl}^{ij}(t) = \begin{cases} \sum_m \binom{l+1}{j}\binom{l+1-j}{m}\binom{k-1}{i-m} 2^m (1 - e^{-\mu t})^{2(l-j)+k-i+1} e^{-\mu(2j+i)t} \\ \qquad \text{if } k > 0,\ l + 1 \geq j, \text{ and } 2l + k + 1 \geq 2j + 1; \\ \sum_m \binom{l}{j}\binom{l-j}{m}\binom{1}{i-m} 2^m (1 - e^{-\mu t})^{2(l-j)+1-i} e^{-\mu(2j+i)t} \\ \qquad \text{if } k = 0,\ l \geq j, \text{ and } 2l + 1 \geq 2j + 1; \\ 0 \qquad \text{otherwise;} \end{cases}$$

where the sums are all finite because the binomal coefficients vanish for extreme values of m. From these equations P_{kl}^{ij} can be found. Letting

$$q_{ij} = \lim_{n\to\infty} \text{Prob}\,\{Z_n = (i, j)\},$$

the stationary distribution for the chain $\{Z_n\}$ can be found by solving

$$q_{ij} = \sum_{k,l\geq 0} q_{kl} P_{kl}^{ij}$$

subject to $\sum_{i,j\geq 0} q_{ij} = 1$.

A similar analysis can be done if the service times are general and the interarrival times are exponential. An embedded Markov chain is then obtained by looking at the process only at times when service is completed. Looking at the system only at these times, however, may not always give an accurate picture of the behavior of the system, since the distribution of times when service is completed will depend on the state of the system.

ACKNOWLEDGMENTS. We wish to thank Mr. Cheam for preparing some of the programs we used. We also wish to thank Professor Cryer, who gave us much advice on the numerical aspects of solving our equations.

REFERENCES

1. KNUTH, DONALD E. *The Art of Computer Programming, Vol. 1.* (2nd printing). Addison-Wesley, Reading, Mass., 1968, pp. 435–455.
2. KNOWLTON, KENNETH C. A fast storage allocator. *Comm. ACM 8* (1965), 623–625.
3. PURDOM, PAUL W., JR., AND STIGLER, STEPHEN M. Statistical properties of the buddy system. Computer Sciences Tech. Rep. #60, U. of Wisconsin, Madison, Wis., 1969.
4. KARLIN, SAMUEL. *A First Course in Stochastic Processes.* Academic Press, New York, 1966, pp. 189–194.

RECEIVED APRIL, 1969; REVISED APRIL, 1970