Probability Space

$(\Omega, \mathcal{F}, \mathbb{P})$

Sample space

$\subseteq 2^{\Omega}$

sigma algebra containing measurable sets

Probability assignment

Moral I : $\Omega$ is abstract (dont have to associate $\Omega$ in a physical sense to any particular experiment)

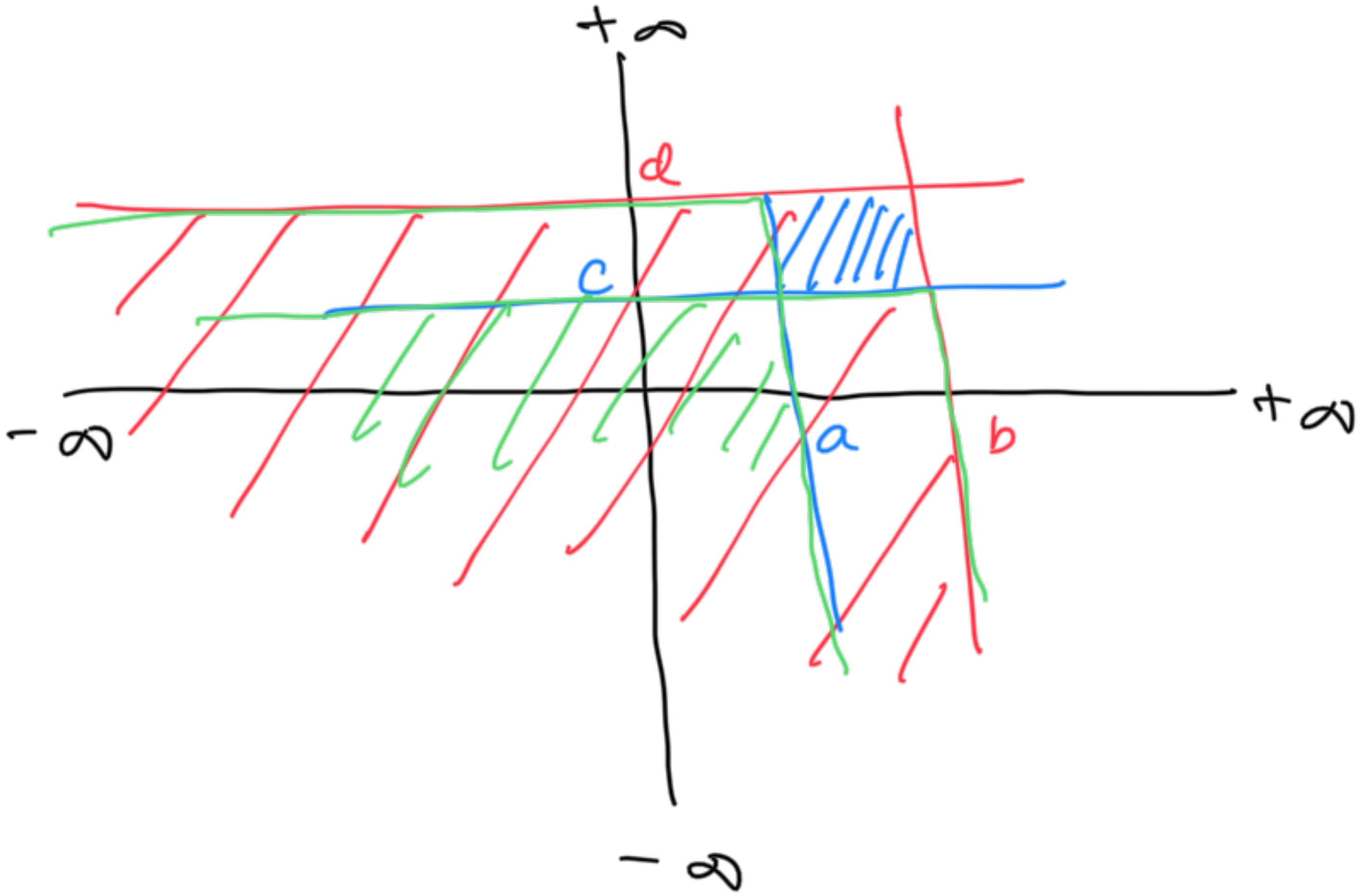
Moral II : $\mathcal{F} \neq 2^{\Omega}$ in general (there exists or it is possible to construct crazy subsets of $\Omega$ such that if we incude it in $\mathcal{F}$ we will not be able to consistently assign probabilities)

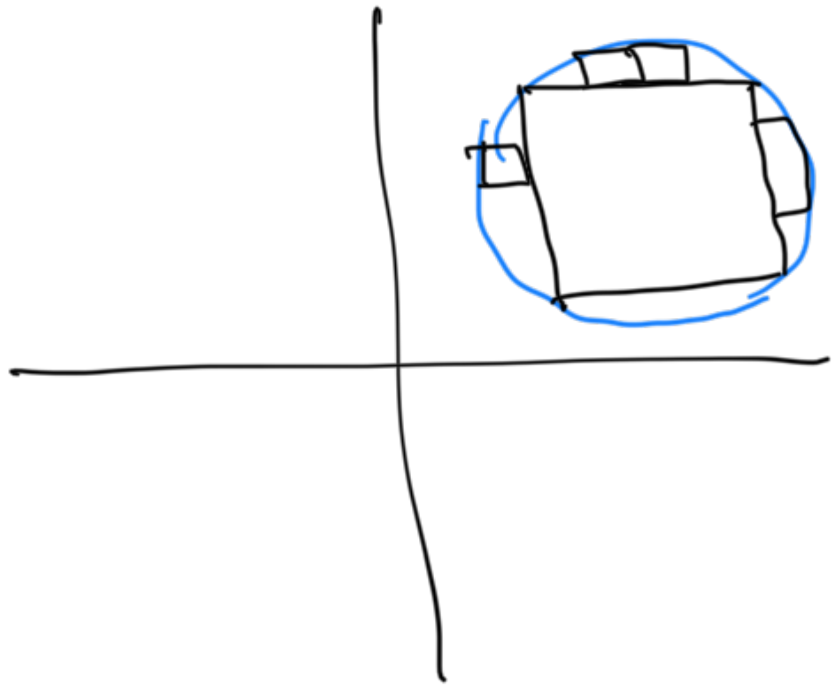
How to construct $(\Omega, \mathcal{F}, \mathbb{P})$ for two real valued random variables random variables

$\Downarrow$

Given: $F_{X_1 X_2}(x_1, x_2) = \text{Prob}(X_1 < x_1, X_2 < x_2)$

Pick $\Omega = \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$



For $\omega = (\omega_1, \omega_2) \in \Omega = \mathbb{R}^2$

$$X_1(\omega) = \omega_1$$
$$X_2(\omega) = \omega_2$$

infinite rectangle

$$\text{Prob}(X_1 < b,\ X_2 < d) = \mathbb{P}\big((-\infty, b) \times (-\infty, d)\big)$$

$$= F_{X_1 X_2}(b, d)$$

Base case I : $\mathbb{P}\big((-\infty, b) \times (-\infty, d)\big)$

Base case II

$$\text{Prob}(a < X_1 < b,\ c < X_2 < d)$$

$$= \mathbb{P}\big((a, b) \times (c, d)\big)$$

$$= F_{X_1 X_2}(b, d) - F_{X_1 X_2}(b, c) - F_{X_1 X_2}(a, d) + F_{X_1 X_2}(a, c)$$

Borel sets : $(-\infty, b) \times (-\infty, d)$

$(a, b) \times (c, d)$ open rectangles

Sigma algebra generated by Borel Sets in $\mathbb{R}^2$

$\mathcal{B}(\mathbb{R}^2)$

* is a sigma algebra

* is the smallest sigma algebra containing Borel sets. Informally, basically sets/shapes that can be obtained as a union of rectangles.

$\Omega = \mathbb{R}^2$, $\mathcal{F} = \mathcal{B}(\mathbb{R}^2)$, $\mathbb{P}((-\infty, b) \times (-\infty, d)) = F_{X_1 X_2}(b, d)$

Fact: Given $X_1, X_2, \ldots, X_n$ $n \geq 0$ and

$F_{X_1 X_2 \cdots X_n}$, we can construct a

$(\Omega, \mathcal{F}, \mathbb{P})$ such that

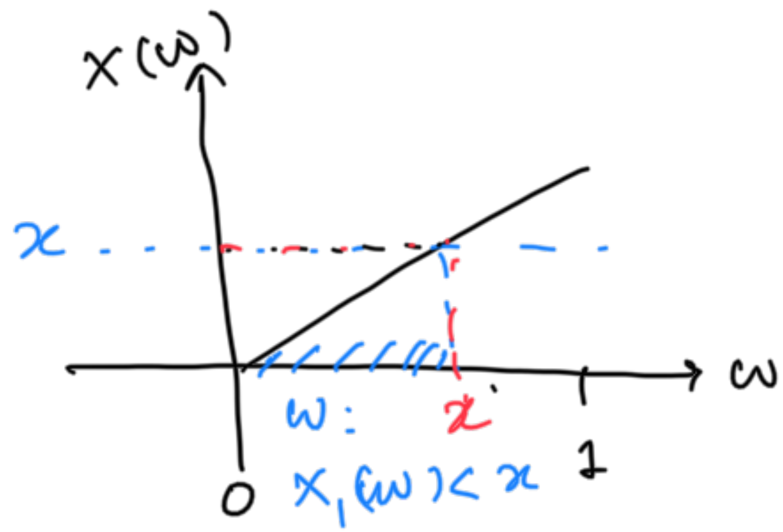
* $X_1, X_2, \ldots, X_n$ reside in $(\Omega, \mathcal{F}, \mathbb{P})$

* $F_{X_1 X_2 \cdots X_n}$ is respected

Example : $\Omega = [0,1]$

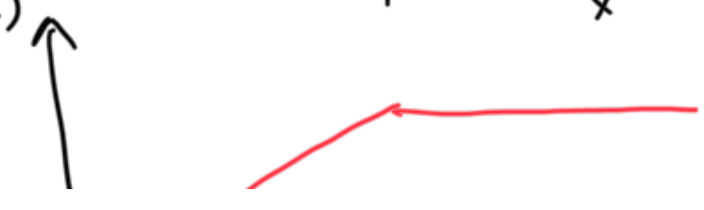
$$\mathcal{F} = \mathcal{B}([0,1])$$

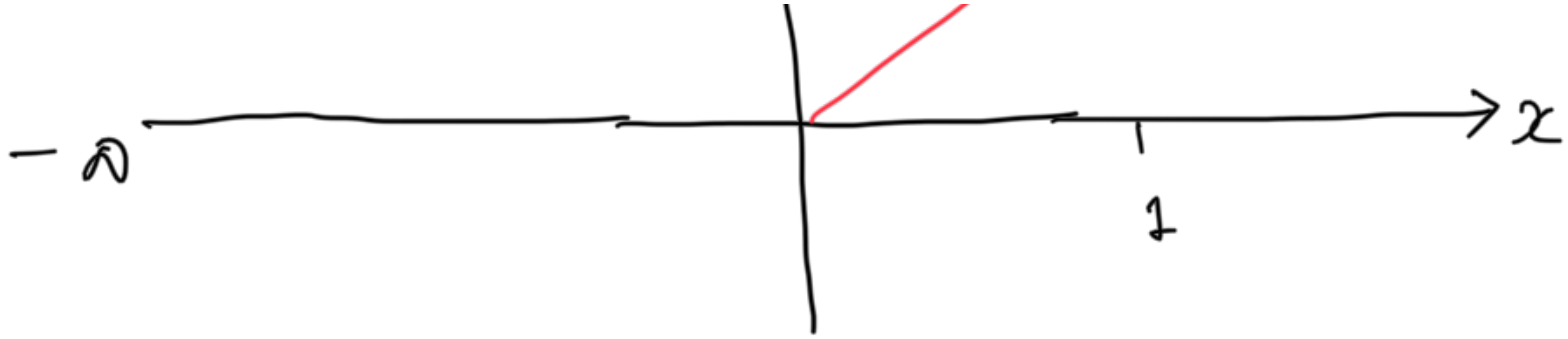
$$\mathbb{P}((a,b)) = b-a = \mathbb{P}([a,b])$$




$X(\omega) = \omega$

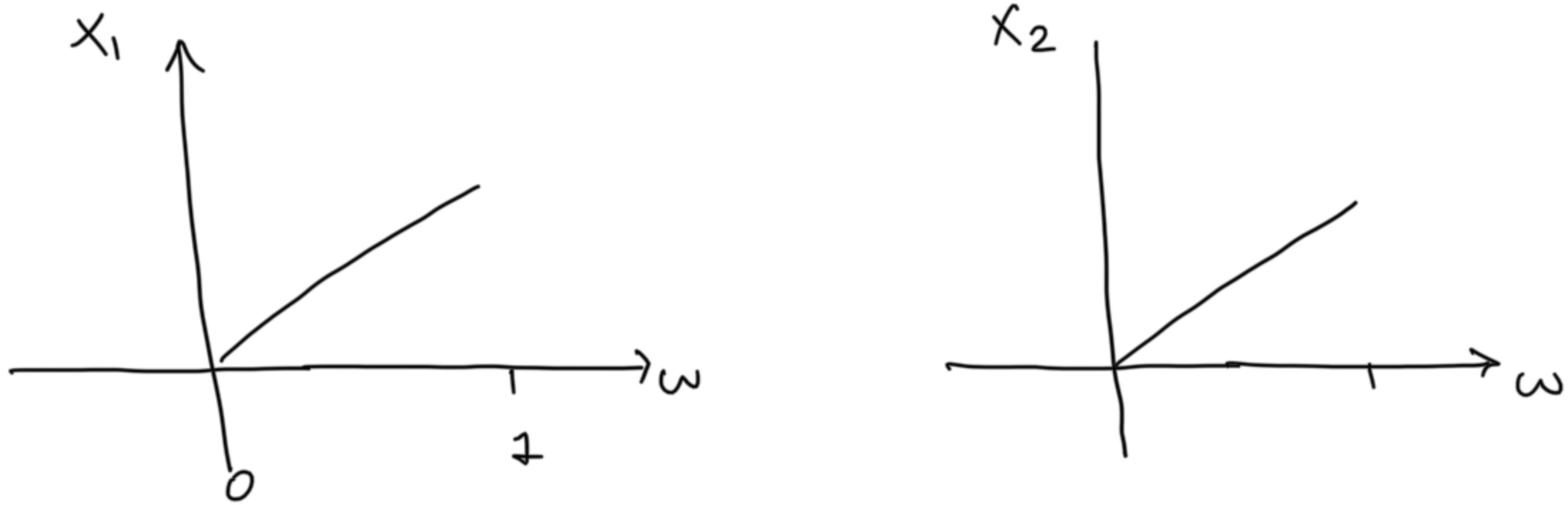
distribution → $F_X(x)$ → $dF_X = f_X$ density



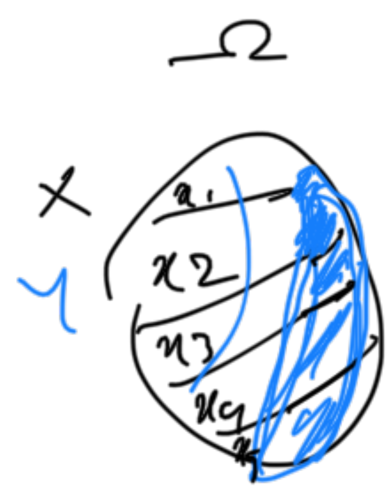


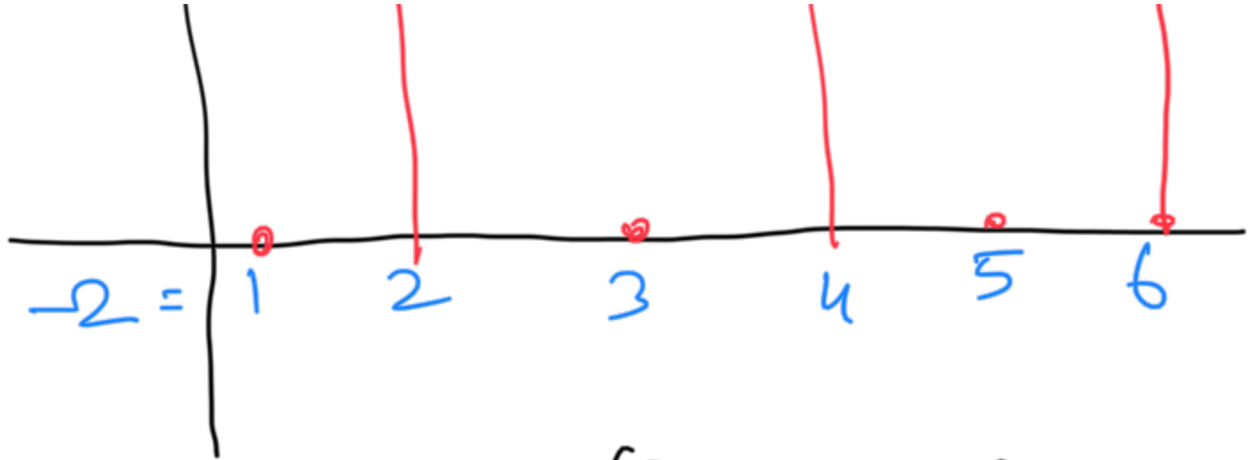

$$F_{X_1}(x) = \text{Prob}(X_1 < x)$$

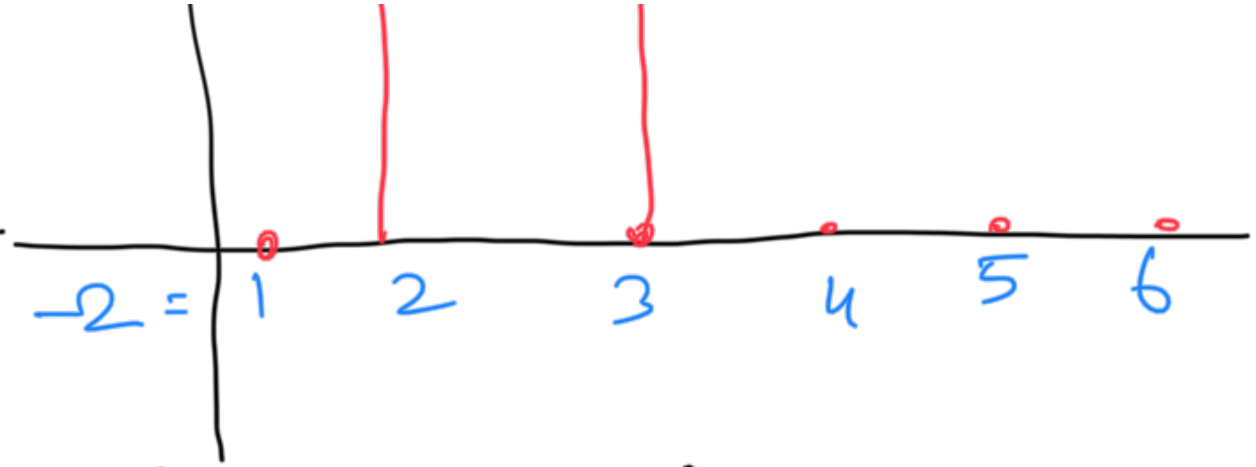
$$= \mathbb{P}(\{\omega : \underline{\underline{X(\omega) < x}}\})$$




cannot have this

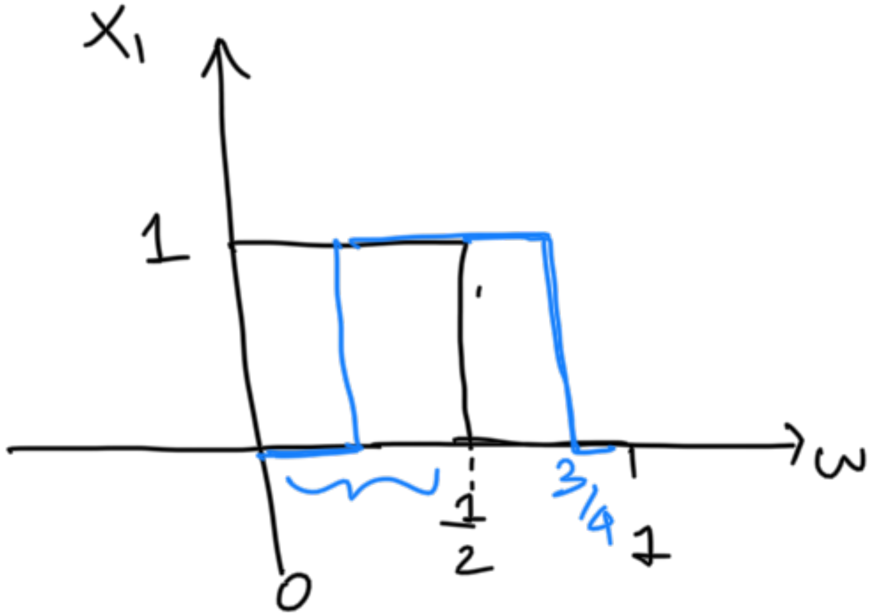



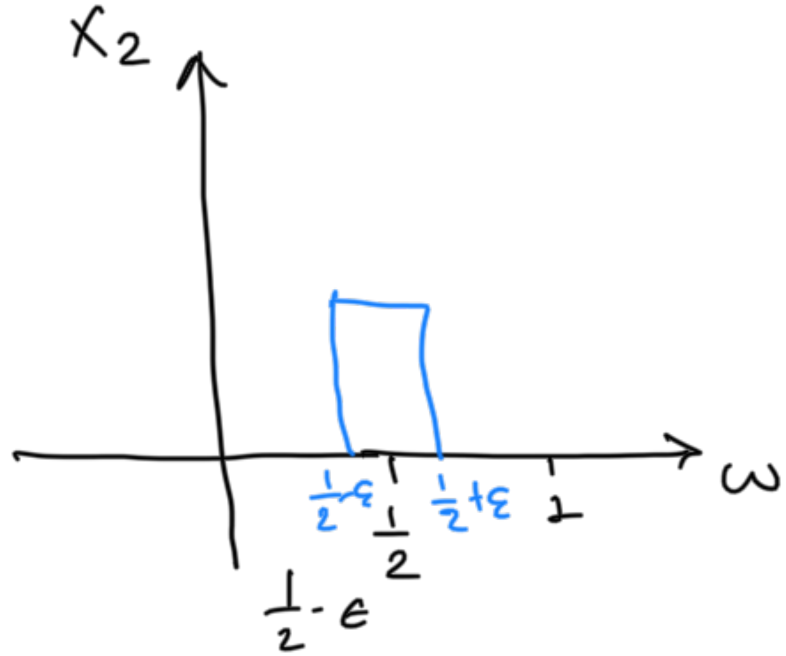



$$\mathbb{P}(\{1,3,5\} \cap \{2,3\}) = \mathbb{P}(\{3\}) = \frac{1}{6}$$

$$\mathbb{P}(\{1,3,5\}) \times \mathbb{P}(\{2,3\}) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$$




$$X_1(\omega) = 1 \quad , \omega \in [0, \tfrac{1}{2})$$

$$= 0 \quad , \omega \in [\tfrac{1}{2}, 1]$$

$$X_2(\omega) = 1 \quad , \quad \omega \in [0, \tfrac{1}{4}] \cup [\tfrac{1}{2}, \tfrac{3}{4}]$$
$$= 0 \quad , \quad \omega \in (\tfrac{1}{4}, \tfrac{1}{2}) \cup (3/4, 1]$$