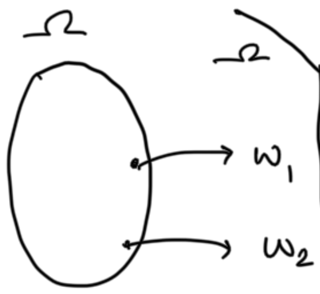
Sequence of random variables

Tape Model



Ω	X_1	S_1	X_2	S_2		X_n	S_n	X
ω_1	$X_1(\omega_1)$	$S_1(\omega_1)$	$X_2(\omega_1)$	$S_2(\omega_1)$	$\dots$	$X_n(\omega_1)$	$S_n(\omega_1)$	$X(\omega_1)$
ω_2	$X_1(\omega_2)$	$S_1(\omega_2)$	$X_2(\omega_2)$	$S_2(\omega_2)$	$\dots$	$X_n(\omega_2)$	$S_n(\omega_2)$	$X(\omega_2)$

* Nature picks $\omega \in \Omega$ at random (only randomness)

  (ω_1 or ω_2 etc are realisations)

* $X_1(\omega_1), X_2(\omega_1), \dots, X_n(\omega_1)$ are already (pre determined) written down in a tape and nature just reads out the entries of this tape.

- Let us pick $\{X_n\}_{n \geq 0}$ i.i.d

  * $\mathbb{E}[X_1] = \mathbb{E}[X_2] = \dots = \mathbb{E}[X_n] = \mu$
  * $Var[X_1] = Var[X_2] = \dots = Var[X_n] = \sigma^2$

We are interested in the sum

$$S_n = X_1 + \dots + X_n$$

$$S_n(\omega) = X_1(\omega) + \dots + X_n(\omega)$$

Strong Law of Large Numbers

$$\frac{S_n}{n} = \frac{X_1 + \dots + X_n}{n} \xrightarrow{a.s.} \mu$$

Weak Law of Large Numbers

$$\frac{S_n}{n} \xrightarrow{p} \mu$$

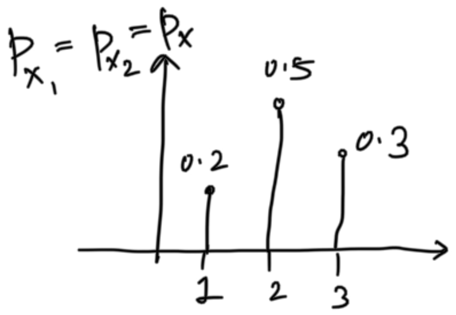
Central Limit Theorem

$$\frac{S_n - n\mu}{\sqrt{n}} = \frac{(X_1-\mu) + (X_2-\mu) + \cdots + (X_n-\mu)}{\sqrt{n}}$$

$$\xrightarrow{d.} \mathcal{N}(0, \sigma^2)$$

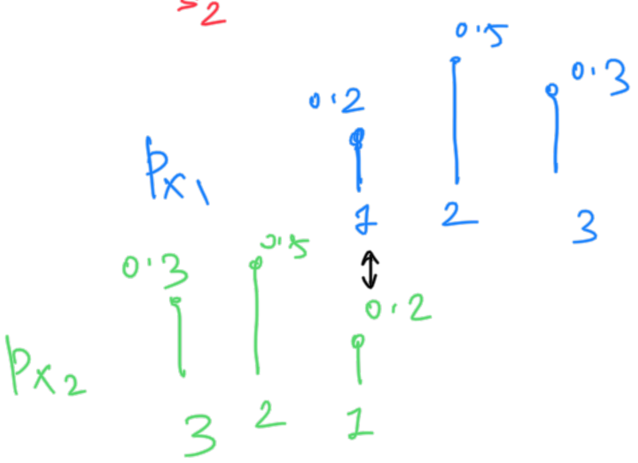
Example 2:

Look at distribution of $S_2 = X_1 + X_2$

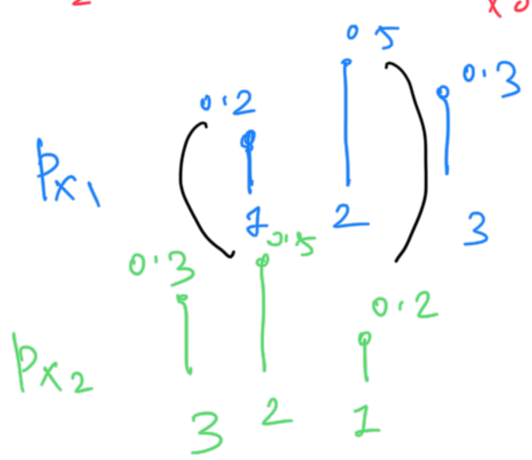


$$p_{S_2}(x) = p_{X_1+X_2}(x) = \sum_{k=-\infty}^{\infty} p_{X_1}(k)\, p_{X_2}(x-k)$$

$p_{S_2}(2) = 0.2 \times 0.2$ ← less mass in the extremes



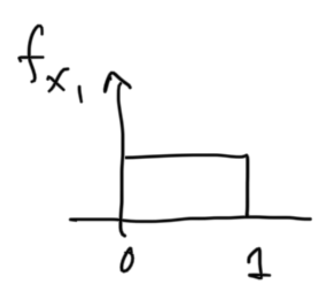
$p_{S_2}(3) = 0.2 \times 0.5 + 0.5 \times 0.2$

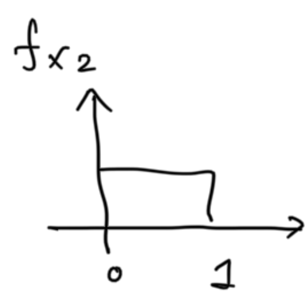


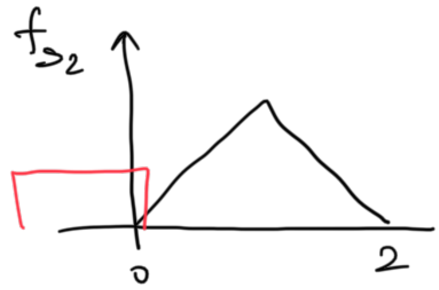
$p_{S_2}(4) = 0.3 \times 0.2 + 0.5 \times 0.5 + 0.2 \times 0.3$ } more mass in center

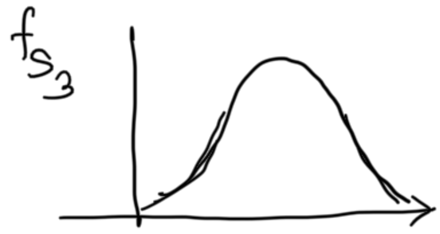
$p_{S_2}(6) = 0.3 \times 0.3$ → less mass in extremes.

Example: $X_1, X_2, \ldots$ are iid uniform $(0,1)$

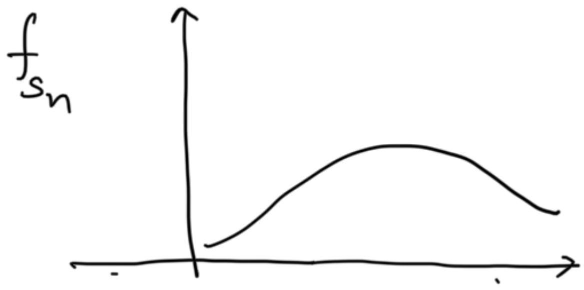






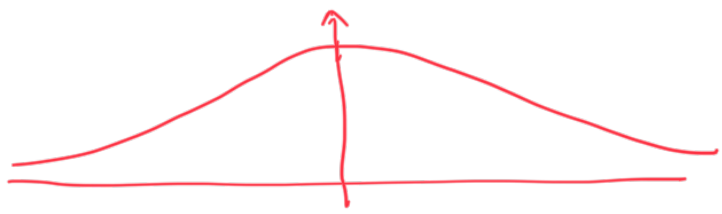


$\vdots$



If we mean center and divide by $\sqrt{n}$

$S_n - n\mu$ $\qquad\qquad$ $\dfrac{S_n - n\mu}{\sqrt{n}}$



* Repeat for $i = 1, 2, \ldots$ many times (N)

- start ith experiment on day 0 (ω_i)
- record after n days $\quad Y(i) = \dfrac{S_n(\omega_i) - n\mu}{\sqrt{n}}$

Plot histogram of $Y(1), \ldots, Y(N)$

this histogram will look like $N(0, \sigma^2)$

Weak Law of Large Numbers

$$\frac{S_n}{n} \xrightarrow{p.} \mu$$

$\forall$ any $\varepsilon > 0$, $\quad \mathbb{P}\left(\left|\frac{S_n}{n} - \mu\right| > \varepsilon\right) \to 0$ as $n \to \infty$

$$\mathbb{P}\left(\left|\frac{S_n}{n} - \mu\right| > \varepsilon\right) = \mathbb{P}\left(\left|\frac{S_n - n\mu}{n}\right| > \varepsilon\right)$$

$$= \mathbb{P}\left(\left|\frac{S_n - n\mu}{\sqrt{n}}\right| > \sqrt{n}\,\varepsilon\right)$$

$$= \mathbb{P}\left(-\sqrt{n}\,\varepsilon < \frac{S_n - n\mu}{\sqrt{n}} < \sqrt{n}\,\varepsilon\right)$$

$\xrightarrow{d.} N(0, \sigma^2)$

(approximation) $\lesssim \displaystyle\int_{\sqrt{n}\,\varepsilon}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{\left(-\frac{1}{2}\frac{x^2}{\sigma^2}\right)}\, dx$ $\longrightarrow$ not the actual density of $\frac{S_n - n\mu}{\sqrt{n}}$

$$+$$

$$\int_{-\infty}^{-\sqrt{n}\,\varepsilon} \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{\left(-\frac{1}{2}\frac{x^2}{\sigma^2}\right)}\, dx$$

$$\lesssim \frac{2}{\sqrt{2\pi\sigma^2}} \int_{\sqrt{n}\,\varepsilon}^{\infty} e^{-\frac{1}{2}\frac{x^2}{\sigma^2}}\, dx$$

$$\to > 1$$

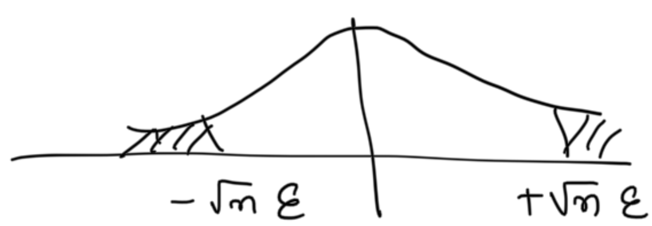
$$\le \frac{2}{\sqrt{2\pi\sigma^2}} \int_{\underline{\sqrt{n}\,\varepsilon}}^{\infty} \left(\frac{x}{\sqrt{n}\,\varepsilon}\right) e^{-\frac{1}{2}\frac{x^2}{\sigma^2}}\, dx$$

$$= \frac{2\sigma}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{n}\,\varepsilon} \int_{\sqrt{n}\,\varepsilon}^{\infty} d\left(\frac{x^2}{2\sigma^2}\right) e^{-\frac{1}{2}\frac{x^2}{\sigma^2}}\, dx$$

$$= \frac{2\sigma}{\sqrt{2\pi}\,\sqrt{n}\,\varepsilon}\, e^{\left(-\frac{1}{2}\frac{n\varepsilon^2}{\sigma^2}\right)}$$

not formally true

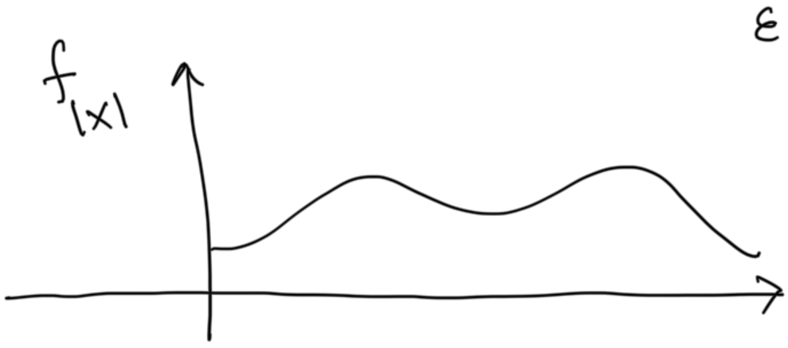
$$\mathbb{P}\left(\left|\frac{S_n}{n} - \mu\right| > \varepsilon\right) \lesssim \sqrt{\frac{2\sigma^2}{\pi n \varepsilon^2}}\; e^{-\left(\frac{n\varepsilon^2}{2\sigma^2}\right)} \qquad (+)$$



Markov and chebyshev inequalities

Markov

$$\mathbb{P}(|X| \ge \varepsilon) \le \frac{\mathbb{E}[|X|]}{\varepsilon}$$



$$\mathbb{E}[|X|] = \int_0^{\infty} x\, f_{|X|}(x)\, dx$$

$$= \int_0^{\varepsilon} x\, f_{|X|}(x)\, dx + \int_{\varepsilon}^{\infty} x\, f_{|X|}(x)\, dx$$

leave out

$$\geq \int_{\varepsilon}^{\infty} \varepsilon \, f_{|X|}(x) \, dx$$

$$= \varepsilon \int_{\varepsilon}^{\infty} f_{|X|}(x) \, dx$$

$$= \varepsilon \, \mathbb{P}(|X| \geq \varepsilon)$$

$$\mathbb{E}[|X|] \geq \varepsilon \, P(|X| \geq \varepsilon)$$

$$\mathbb{P}(|X| \geq \varepsilon) \leq \frac{\mathbb{E}[|X|]}{\varepsilon}$$

Chebyshev Inequality:

$$P(|X - \mathbb{E}[X]| \geq \varepsilon) \leq \frac{\mathrm{Var}[X]}{\varepsilon^2}$$

$$= \mathbb{P}(|X - \mathbb{E}[X]| \geq \varepsilon) = \mathbb{P}(|X - \mathbb{E}[X]|^2 \geq \varepsilon^2)$$

$$\leq \frac{\mathbb{E}[|X - \mathbb{E}[X]|^2]}{\varepsilon^2}$$

$$= \frac{\mathrm{Var}(X)}{\varepsilon^2}$$

$$\mathrm{Var}\left(\frac{S_n}{n}\right) = \frac{\sigma^2}{n}$$

$$\mathbb{P}\left(\left|\frac{S_n}{n} - \mu\right| \geq \varepsilon\right) \leq \frac{\sigma^2}{n\varepsilon^2} \qquad (*)$$

slower compared to $\sqrt{\dfrac{2\sigma^2}{\pi n \varepsilon^2}} \, e^{-\left(\frac{n\varepsilon^2}{2\sigma^2}\right)}$

$$\mathbb{P}\left(\left|\frac{S_n}{n} - \mu\right| \geq \varepsilon\right) \to 0, \text{ as } n \to \infty$$

because $\frac{\sigma^2}{n\varepsilon^2} \to 0$ as $n \to \infty$

- I start a random experiment (ω_i)
- $$Y_n(i) = \frac{S_n(\omega_i)}{n}$$

Multi - Armed Bandit Problem.

Arm 1 . . . Arm k

P_1 P_k