Probability Space : $(\Omega, \mathcal{F}, \mathbb{P})$

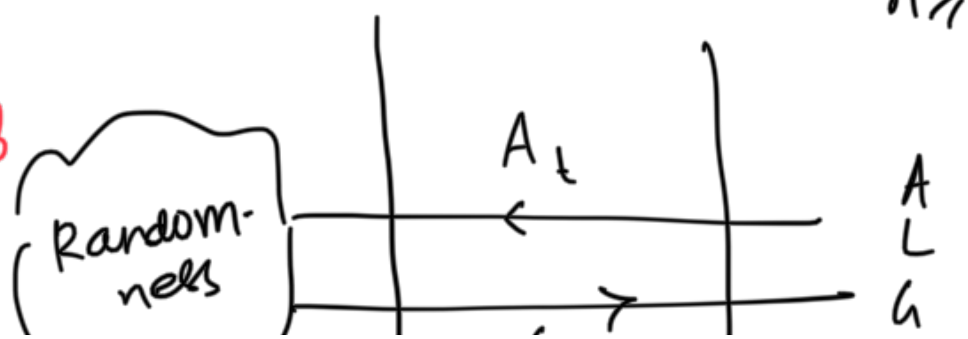
- Given say a joint distribution $F_{X_1 \cdots X_n}(x_1, \cdots, x_n)$ there exists $(\Omega, \mathcal{F}, \mathbb{P})$ on which random variables reside and follow $F_{X_1 \cdots X_n}$

- Example of $X_1, X_2, \cdots$ independent random variables on $([0,1], \mathcal{B}([0,1]), \text{length measure})$

Convergence of sequence of random variables

$\{X_n\}, n = 0, 1, 2 \cdots \qquad \{X_n\}_{n \geq 0}$

We have taken care of



$X \sim P$

search this side $(\Omega, \mathcal{F}, \mathbb{P})$ | Environment | X_t | 0 | A_t

- Almost Sure (a.s.) Pointwise convergence

Let $\{X_n\}_{n \geq 0}$ be a sequence of real valued random variables. We say they converge almost surely

$$X_n \xrightarrow{a.s.} X$$

$$\text{Prob}\Big(\underbrace{\lim_{n\to\infty} X_n = X}\Big) = 1$$

$\text{Prob}\left(\lim_{n\to\infty} X_n = X\right) = 1$ translates to

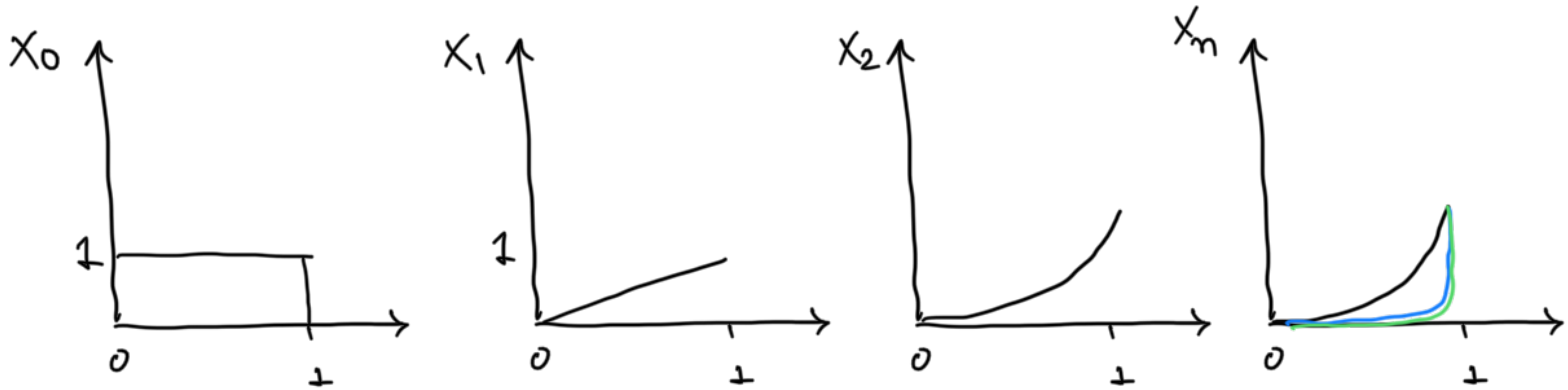
$$\mathbb{P}\left(\{\omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\}\right) = 1$$

$\{X_n\}_{n \geq 0} \rightarrow$ sequence of random variables

$\{X_n(\omega)\}_{n \geq 0} \rightarrow$ sequence of real numbers

Example 1: $\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}([0,1])$

$\mathbb{P}([a,b]) = b - a$



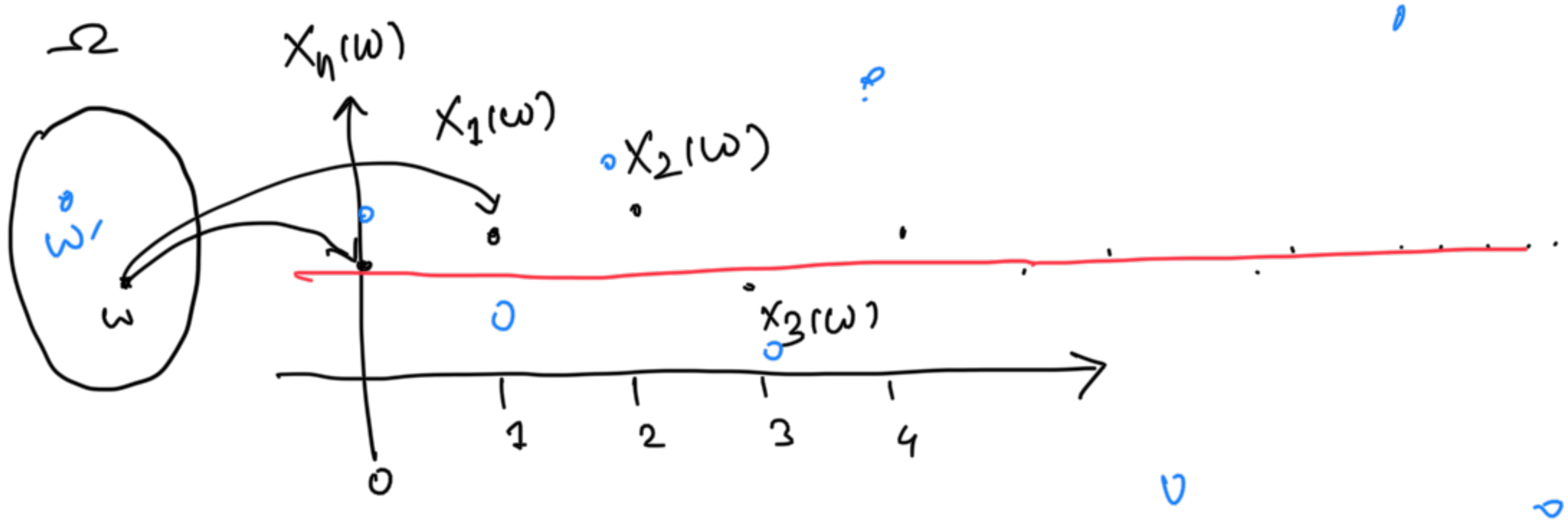
$X_n(\omega) = \omega^n$, $n \geq 0$?

$X_n \xrightarrow{a.s.}$?

$X_n \rightarrow X$

$X(\omega) = 1$, $\omega = 1$
$= 0$ otherwise

$$X_n \xrightarrow{a.s.} 0$$



$\{X_n(\omega)\}_{n\geq 0} \to$ sequence of reals

* for each ω, $X_n(\omega) \to X(\omega)$

* collect all such ω in $\{\omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\}$

* measure this set

$$\text{Prob}\left(\lim X_n = X\right) = 1$$

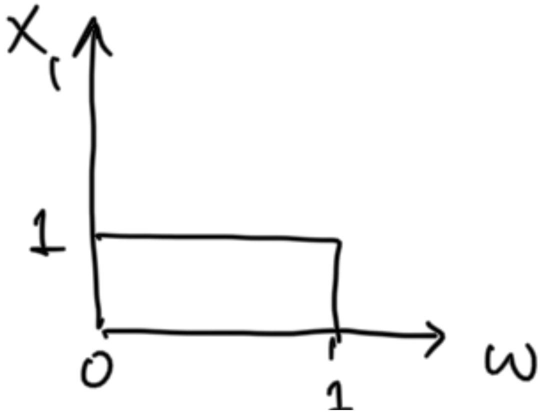
$n \to \infty$

$$\mathbb{P}\left(\{\underbrace{w: \quad X_n \not\to X}_{\text{disagreement set}}\}\right) = 0$$

In example 1: $\{w: X_n \not\to 0\} = \{1\}$

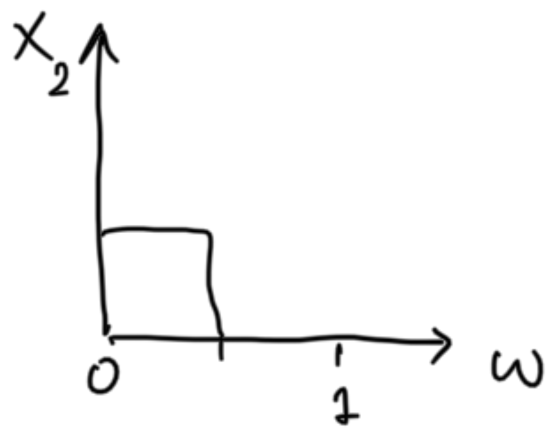
$$\mathbb{P}(\{1\}) = 0$$

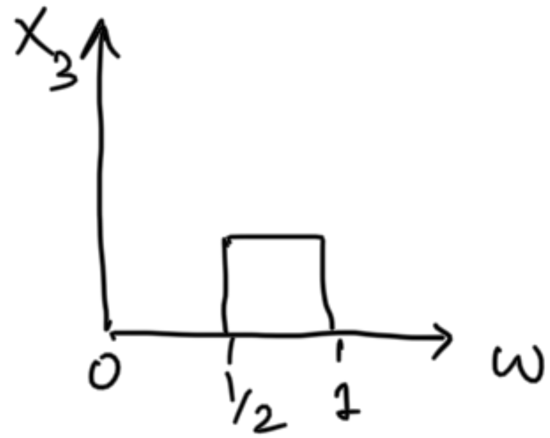
Example 2: Shrinking Moving Rectangles



$X_n \xrightarrow{p.} 0$

$\mathbb{P}(|X_n - X| > \varepsilon) \to 0$

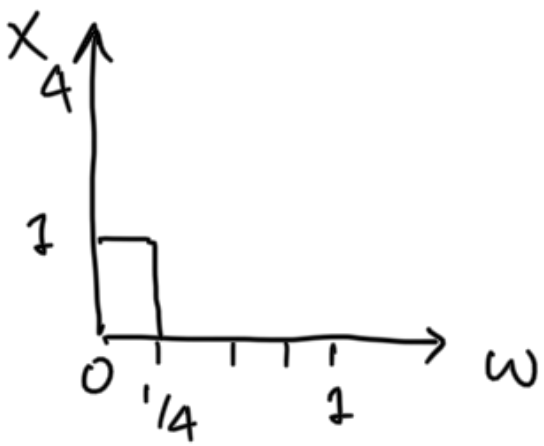


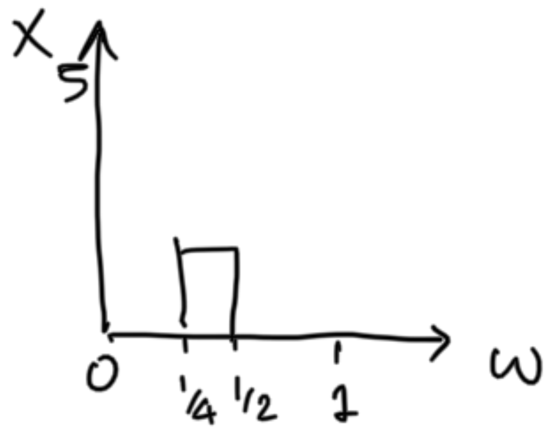


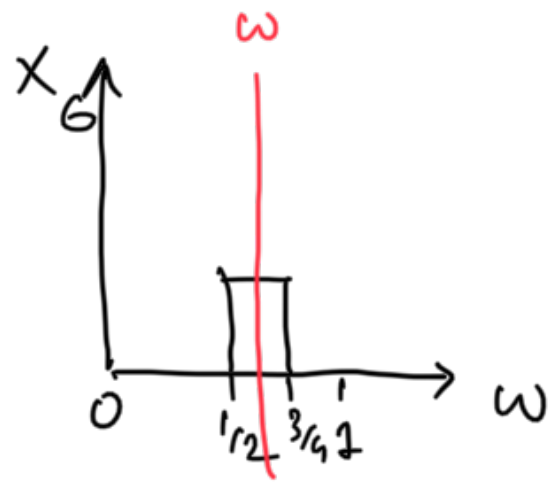
if $n = 2^k + i$

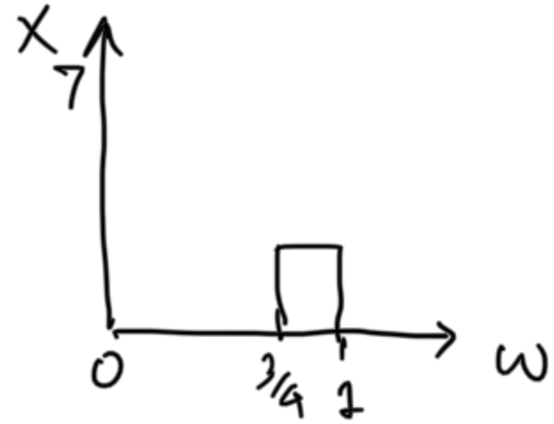
then

$\mathbb{P}(|X_n - X| > \varepsilon) = \frac{1}{2^k}$









x_8 to x_{15} by dividing $\frac{1}{8}$ intervals

$k = 0, 1, 2, \dots$

$i = 0, 1, \dots, 2^k - 1$

$X_{2^k+i}(\omega) = 1 \quad , \omega \in \left[\frac{i}{2^k}, \frac{i+1}{2^k}\right]$

$$2^i+1 \qquad\qquad -2^k \qquad 2^k-1$$

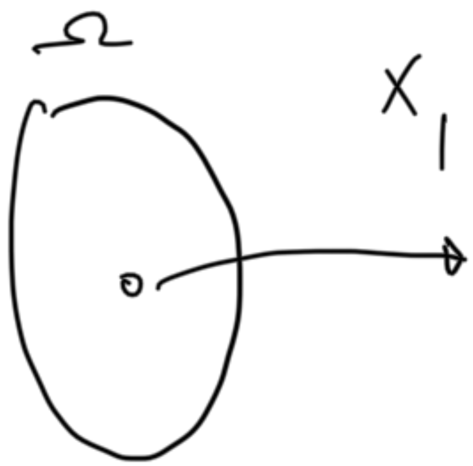
$$= 0$$

$k = 0, \quad i = 0 \text{ to } 0 \qquad X_{2^0+0} = X_1$

$k = 1, \quad i = 0, 1 \qquad X_{2^1+0} = X_2, \quad X_{2^1+1} = X_3$

$$X_2 \downarrow \left[\frac{0}{2}, \frac{1}{2}\right] \qquad X_3: \left[\frac{1}{2}, 1\right]$$

$$X_n \overset{a.s.}{\not\to} 0 \ ?$$



$X_1(\omega) \quad X_2(\omega) \quad X_3(\omega) \quad X_4(\omega) \quad \cdots \quad X_7(\omega)$

$1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0$

$$\mathbb{P}(\{\omega : X_n \not\to 0\}) = \mathbb{P}([0, 1])$$

$= 1$

- Convergene in Probability

A sequence $\{X_n\}_{n \geq 0}$ of real valued random variables are said to converge to X in probability

$$X_n \xrightarrow{p.} X$$

for any $\varepsilon > 0$, $\text{Prob}(|X_n - X| > \varepsilon) \to 0$ as $n \to \infty$

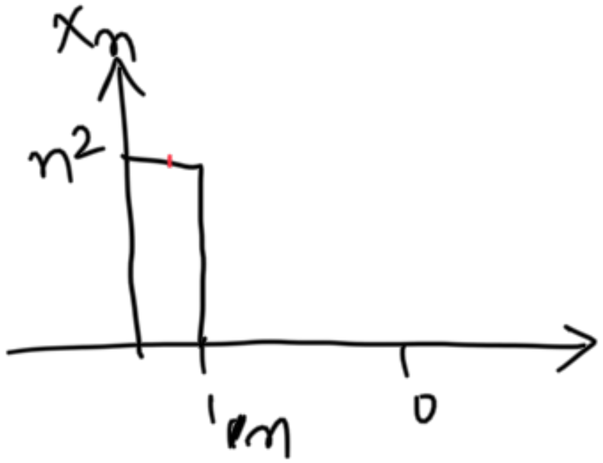
$$\lim_{n \to \infty} \mathbb{P}\left(\{w : |X_n(\omega) - X(\omega)| > \varepsilon\}\right) \to 0$$

- Mean Squared

$$X_n \xrightarrow{m.s.} X$$

$$\mathbb{E}\left[|X_n - X|^2\right] \to 0 \quad \text{as } n \to \infty$$

Example



$$X_n(\omega) = n^2, \quad \omega \in [0, \tfrac{1}{n}]$$
$$= 0$$

$$X_n \xrightarrow{p.} 0$$
$$X_n \not\xrightarrow{m.s.} 0$$

$$\mathbb{E}[|X_n - 0|^2] = \frac{1}{n} \cdot n^4 = n^3 \not\to 0 \quad \text{as } n \to \infty$$

$$\mathbb{P}[|X_n - 0| > \epsilon] = \frac{1}{n} \to 0 \quad \text{as } n \to \infty$$

- Convergence in distribution

$$X_n \xrightarrow{d.} X$$

$$\lim_{n \to \infty} F_{X_n}(x) = F_X(x)$$ at all continuity points of $F_X(x)$

$\{x_n\}$ i.i.d