$\Omega = \{1, 2, \ldots, 12\}$ $\mathcal{F} = 2^{\Omega}$, $\boxed{\mathbb{P}(A) = \frac{|A|}{12}}$

a) Roll of a dice and toss of a coin, but coin
(independent)

is biased, Prob(H) = 0·6

$X_1(1) = X_1(7) = 1$
$X_1(2) = X_1(8) = 2$
$\vdots$
$X_1(6) = X_1(12) = 6$

$X_2(1) = X_2(2) = \cdots = X_2(6) = H$
$X_2(7) = X_2(8) = \cdots = X_2(12) = T$

$$\Omega = \{\overset{H}{1, 2, 3, 4, 5, 6}, \underbrace{\overset{T}{7, 8, 9, 10, 11, 12}}\}$$

$$\mathbb{P}(\{1\}) = \frac{1}{12} = \frac{0\cdot5}{6}$$

$$\mathbb{P}(\{1\}) = \mathbb{P}(\{2\}) = \cdots = \mathbb{P}(\{6\}) = \frac{0\cdot6}{6}$$

$$\mathbb{P}(\{7\}) = \mathbb{P}(\{8\}) = \cdots = \mathbb{P}(\{12\}) = \frac{0\cdot4}{6}$$

$$
\begin{aligned}
\text{Prob}(X_1 = 1) &= \mathbb{P}(\{\omega : X_1(\omega) = 1\}) \\
&= \mathbb{P}(\{1\} \cup \{7\}) \\
&= \mathbb{P}(1) + \mathbb{P}(7) \\
&= \frac{0.6}{6} + \frac{0.4}{6} \\
&= \frac{1}{6}
\end{aligned}
$$

$\text{Prob}(X_1 = 1) = \cdots = \text{Prob}(X_1 = 5) = 0.1$

$\text{Prob}(X_1 = 6) = 0.5$

$\text{Prob}(H) = 0.7$

Case I
(independent)

Case II
(dependent)

Ω

have weights / area / measure
probability

$\in \mathcal{F}$ $\quad \mathbb{P} : \mathcal{F} \to [0, 1]$

Can you give two independent events in roll of a single dice

Moral I: Ω is abstract, $X : \Omega \to \mathbb{R}$

Prob(X belongs to blah)

$$\mathbb{P}\left(X^{-1}(\;) \right), \quad X^{-1}(\;) \in \mathcal{F}$$

Moral II: $\mathcal{F} \neq 2^{\Omega}$ ($\mathcal{F} = 2^{\Omega}$ is always not possible)

Property: Rationals can be enumerated

q \ p	1	2	3	4
1	$\frac{1}{1}$ (1)	$\frac{2}{1}$ (3)	$\frac{3}{1}$ (6)	$\frac{4}{1}$
2	$\frac{1}{2}$ (2)	$\frac{2}{2}$ (5)	$\frac{3}{2}$	$\frac{4}{2}$

3	$\frac{1}{3}$	(n)	$\frac{2}{3}$	$\frac{3}{3}$	$\frac{4}{3}$
4	$\frac{1}{4}$		$\frac{2}{4}$	$\frac{3}{4}$	$\frac{4}{4}$

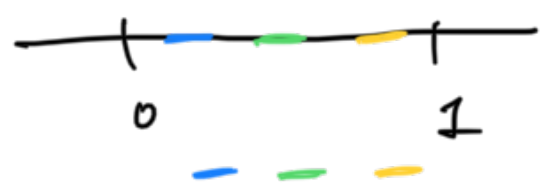
we can say n^{th} rational $1/323$

Take uniform random variable

- $\Omega = [0, 1]$

length measure Lebesgue
- $\mathbb{P}([a,b]) = b - a$, $\forall\ a,b \in [0,1]$

translation invariance
- $\mathbb{P}(S+\alpha) = \mathbb{P}(S)$



S

$S' = S + 0.5$

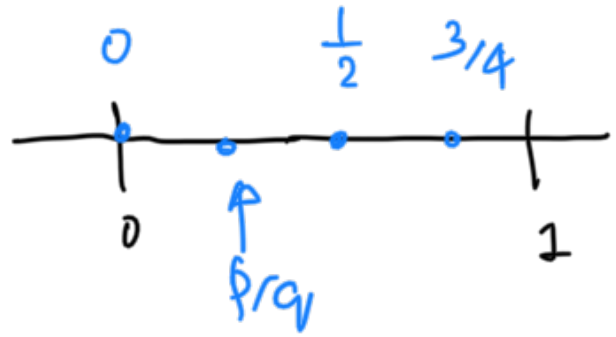
$S' = S + \alpha$

* Pick $s \in S$

* $s' \leftarrow s + \alpha$ if $s' \in [0, 1]$
* $s' \leftarrow s + \alpha - 1$

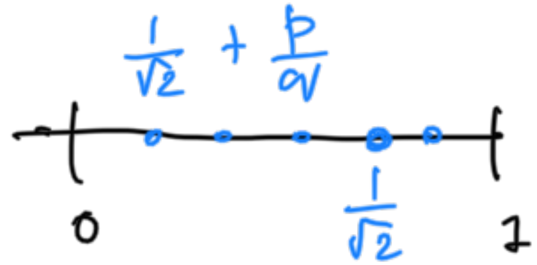
- $A_x = \{ y : y - x \text{ is rational}, y \in [0, 1] \}$

$A_0 = \{ y : y - 0 \text{ is rational}, y \in [0, 1] \}$
$= \{ y : y \text{ is rational} \}$



- $A_{\frac{1}{2}} = A_0$

- $A_{\frac{1}{\sqrt{2}}} = \{ y : y - \frac{1}{\sqrt{2}} \text{ is rational}, y \in [0, 1] \}$



→ irrational x + all rational

$q + x \; , x \in [0, 1]$ points

$$A_{\frac{\pi}{4}} , \quad A_{\frac{\pi}{6}} , \quad A_{\frac{e}{4}}$$

- $$A_{\frac{\pi}{4}} = A_{\frac{\pi}{4} + \frac{1}{2}} = A_{\frac{\pi}{4} + \frac{1}{4}} \cdots$$

$$A_{x + \frac{p}{q}} = A_x$$

		B_i		B_R
A_0	0	$\frac{1}{2}$	$\frac{3}{4}$	p/q
$A_{\frac{1}{\sqrt{2}}}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}} + \frac{1}{2}$	$\frac{1}{\sqrt{2}} + \frac{3}{4}$	$\frac{1}{\sqrt{2}} + p/q$
$\vdots$				
$A_{\frac{\pi}{4}}$	$\frac{\pi}{4}$	$\frac{\pi}{4} + \frac{1}{2}$	$\frac{\pi}{4} + \frac{3}{4}$	$\frac{\pi}{4} + p/q$

- Pick i^{th} element from each A_x and form set B_i

- $\bigcup_{i=1}^{\infty} B_i = [0, 1]$

- $B_i \cap B_k = \phi, \quad i \neq k$

- $\mathbb{P}(B_j) = \mathbb{P}(B_k) = M$ (Translation invariance)

- $\mathbb{P}(\bigcup_{i=1}^{\infty} B_i) = \mathbb{P}(\Omega) = 1$

countable additivity

$$\mathbb{P}(\bigcup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} \mathbb{P}(B_i) = \mathbb{P}(\Omega) = 1$$

$$M \cdot \infty = 1$$

$M \geq 0$

if $M > 0 \qquad M \cdot \infty = \infty \neq 1$

$\quad M = 0 \qquad 0 \cdot \infty = 0 \neq 1$

$$B_i \subseteq 2^{[0,1]}$$

$(\Omega, 2^{\Omega}, \mathbb{P})$ ✗

$(\Omega, \mathcal{F}, \mathbb{P})$ ✓

set/shapes for which
we can consistently assign area/measure/
probability