Markov Inequality : $\mathbb{P}(|X| \geq t) \leq \frac{\mathbb{E}[|X|^m]}{t^m}$

increasing

decreasing

$$\frac{1}{t^m} \int_0^\infty x^m f_{|X|}(x)\,dx$$

fight between these two terms.

Does not work for heavy taile $f_X(x) = \frac{1}{\pi(1+x^2)}$

increases

$$x \cdot \frac{1}{c(1+x^2)}$$

decreases but not rapidly enough

$$\sum_{n \geq 0} \frac{1}{n} = \infty \qquad \sum_{n \geq 0} \frac{1}{n^2} < \infty$$

(C1) we want smaller deviation of sample mean from true mean, but what we have in $\mathbb{P}(|x| \geq t)$ is that 't' ...

in $\mathbb{P}(|X|>t)$ is that t goes farther and farther away.

(C2) $$\mathbb{P}(|X|<t) \leq \frac{\mathbb{E}[|X|^m]}{t^m}$$ ← does the tail really fall at t^m rate?

(C3) Do we always gain by increasing m (or) does $\mathbb{E}[|X|^m]$ not blow up?

Chebyshev

$$\mathbb{P}\left(\left|\frac{S_n}{n} - \mu\right| \geq \varepsilon\right) = \mathbb{P}(|S_n - n\mu| \geq n\varepsilon)$$

(C1) gets resolved, so as $n \to \infty$, we are indeed looking at increasing deviation of S_n from $n\mu$.

$$\mathbb{P}(|S_n - n\mu| \geq n\varepsilon) \leq \mathbb{E}[|S_n - n\mu|^m]$$

$$\mathbb{P}(|S_n - n\mu| \geq n\varepsilon) \leq \frac{\mathbb{E}[|S_n - n\mu|^m]}{(n\varepsilon)^m}$$

Do we get $(n\varepsilon)^m$ rate ?

Pick $m = 2$

$$\mathbb{E}[(S_n - n\mu)^2]$$

$$= \mathbb{E}[[(X_1 - \mu) + \cdots + (X_n - \mu)][(X_1 - \mu) + \cdots + (X_n - \mu)]]$$

$$= \sum_{i,j} \mathbb{E}[(X_i - \mu)(X_j - \mu)] = \sum_i \mathbb{E}[(X_i - \mu)^2] + \sum_{i \neq j} \mathbb{E}[\underbrace{(X_i - \mu)(X_j - \mu)}_{0}]$$

because of independence

$$\forall\ i \neq j, \quad \mathbb{E}[(X_i - \mu)(X_j - \mu)] = \mathbb{E}[(X_i - \mu)] \cdot \mathbb{E}[(X_j - \mu)] = 0 \cdot 0 = 0$$

$$\mathbb{E}[(X_i - \mu)] = \mathbb{E}[X_i] - \mathbb{E}[\mu] = \mu - \mu$$

$= 0$

$$\mathbb{E}\left[(S_n - n\mu)^2 \right] = n\, \mathbb{E}\left[(X_i - \mu)^2 \right]$$

$$= n\sigma^2$$

$$\mathbb{P}\left(\left| \frac{S_n}{n} - \mu \right| \geq \varepsilon \right) \leq \frac{n\sigma^2}{n^2 \varepsilon^2} = \frac{\sigma^2}{n\varepsilon^2}$$

Moral is we do not get $\frac{1}{n^2}$ rate but only $\frac{1}{n}$ rate

Can we improve by increasing m?

Pick $m = 4$ $\quad \mathbb{E}\left[(S_m - n\mu)^4 \right]$

$$\mathbb{E}\Big[[(X_1-\mu) + \cdots + (X_n-\mu)][(X_1-\mu) + \cdots + (X_n-\mu)][(X_1-\mu) + \cdots + (X_n-\mu)]$$

$[(x_1-\mu)+\cdots+(x_n-\mu)]$

$$= \sum_{i,j,k,\ell} \mathbb{E}\left[(x_i-\mu)(x_j-\mu)(x_k-\mu)(x_\ell-\mu)\right]$$

* $i \neq j \neq k \neq \ell$, $\mathbb{E}\left[(x_i-\mu)(x_j-\mu)(x_k-\mu)(x_\ell-\mu)\right]$

$$= \mathbb{E}[x_i-\mu]\,\mathbb{E}[x_j-\mu]\,\mathbb{E}[x_k-\mu]\,\mathbb{E}[x_\ell-\mu]$$

$$= 0$$

* $\mathbb{E}\left[(x_i-\mu)^2(x_k-\mu)(x_\ell-\mu)\right] = 0$
* $\mathbb{E}\left[(x_i-\mu)^3(x_\ell-\mu)\right] = 0$
* $\mathbb{E}\left[(x_i-\mu)^2(x_\ell-\mu)^2\right] = \mathbb{E}[(x_i-\mu)^2]\cdot\mathbb{E}[(x_\ell-\mu)^2]$

$$= \sigma^2\cdot\sigma^2$$

* $\mathbb{E}\left[(x_i-\mu)^4\right] \neq 0$

$$\mathbb{E}[(S_n - n\mu)^4] = n\,\mathbb{E}[(X_i-\mu)^4] + \sum_{i,j} \mathbb{E}[(X_i-\mu)^2]\cdot\mathbb{E}[(X_j-\mu)^2]$$

$$\qquad\qquad {}^4C_2 \quad {}^nC_2 \quad \overset{=}{\sigma^4}$$

$$\cancel{\frac{4\times 3}{1\times 2}} \quad \frac{(n)(n-1)}{1\times \cancel{2}} \quad \sigma^4$$

$$\mathbb{P}\left(\left|\frac{S_n}{n}-\mu\right| \geq \varepsilon\right) \leq \frac{n\,\mathbb{E}[(X_i-\mu)^4] + 3n(n-1)\sigma^4}{n^4\varepsilon^4}$$

$$\leq \frac{3\sigma^2 + \mathbb{E}[(X_i-\mu)^4]/n}{n^2\varepsilon^4}$$

$m = 2$

$$\frac{\sigma^2}{n\varepsilon^2}$$

$3\sigma^2 + \mathbb{E}[(X_i-\mu)^4]/n$

$m = 4$ $\Big|$ $\dfrac{3\sigma^2 + \mathbb{E}[(X_1-\mu)^4]/n}{n^2 \varepsilon^4}$

Pick $\varepsilon = 0.1$, $\sigma = 1$, for bound to be non-vacuous

we need at least $\dfrac{1}{n \times (0.1)^2} < 1$

$n\varepsilon^2 > 1$

$n > 100$

$m = 4$ we need at least $\dfrac{3}{n^2 \times (0.1)^4} < 1$

$n > \sqrt{3} \cdot 100$

Comparing denominators

$n\varepsilon^2$ vs $(n\varepsilon^2)^2$ ✓ always better.

Moral : Increasing $m = 4$ from $m = 2$ helps.