Test I on Friday: Portion everything till Thursday

either $4 \times 5 = 20$ or $5 \times 4 = 20$

Concentration Inequalities

Markov : $\mathbb{P}(|X| > t) \leq \dfrac{\mathbb{E}[|X|^m]}{t^m}$

Chebyshev : $\mathbb{P}\left(\left|\dfrac{S_n}{n} - \mu\right| \geq \varepsilon\right) \leq \dfrac{\mathbb{E}[|S_n - n\mu|^m]}{(n\varepsilon)^m}$

$m = 2$: $\dfrac{\mathbb{E}[|S_n - n\mu|^2]}{(n\varepsilon)^2} = \dfrac{\sigma^2}{n\varepsilon^2}$

$m = 4$: $\dfrac{\mathbb{E}[|S_n - n\mu|^4]}{(n\varepsilon)^4} \leq \dfrac{3\sigma^4 + \mathbb{E}[(X_1 - \mu)^4]/n}{n^2 \varepsilon^4}$

Message: * higher m better the rate

$$\frac{\sigma^2}{n\varepsilon^2} \quad \text{vs} \quad \frac{3\sigma^4}{(n\varepsilon^2)^2}$$

Bound kick in roughly after $n\varepsilon^2 > 1$

* Take all possible m, and take minimum of the bounds

* Need to know a bound on

$$\mathbb{E}[|S_n - n\mu|^m]$$

Moment Generating Function

$$M_X(\lambda) = \mathbb{E}[e^{\lambda X}]$$

$$= \mathbb{E}\left[1 + \frac{\lambda X}{\lfloor 1} + \frac{\lambda^2 X^2}{\lfloor 2} + \frac{\lambda^3 X^3}{\lfloor 3} + \cdots\right]$$

* contains all the moments $X, X^2, X^3, \ldots$

Let us calculate $M_X(\lambda)$ for Gaussian r.v $N(0, \sigma^2)$

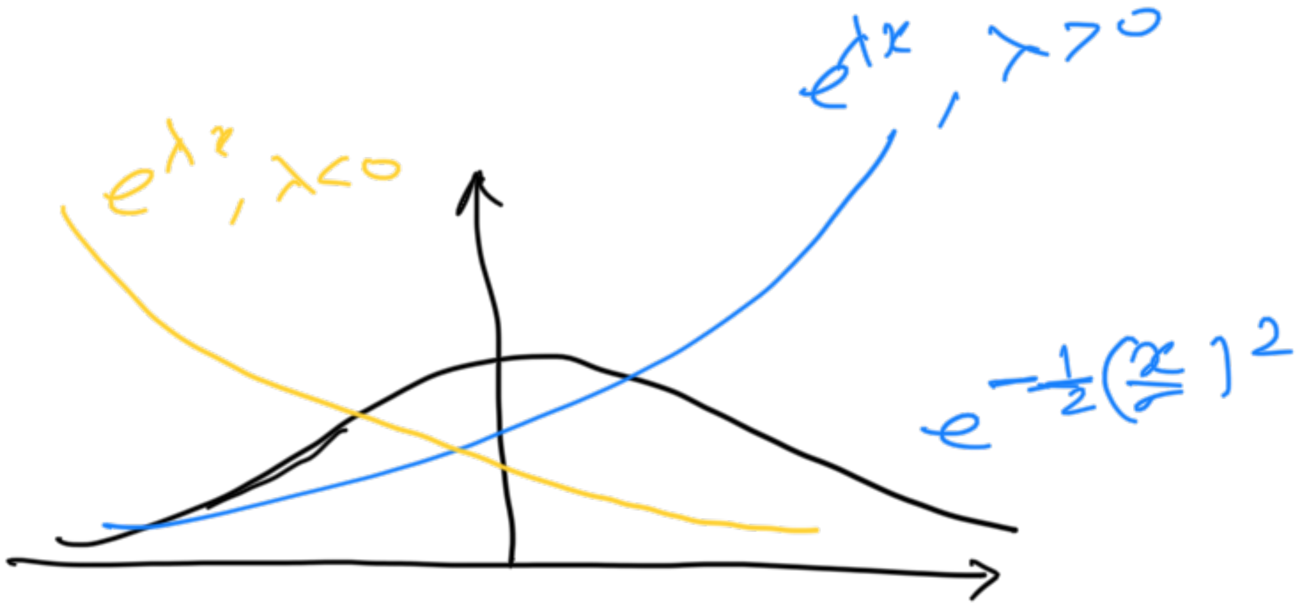
$$M_X(\lambda) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{\overset{\mathrm{II}}{\lambda x}} e^{-\frac{1}{2}\overset{\mathrm{I}}{\left(\frac{x}{\sigma}\right)^2}} dx$$

Complete the squares

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\overset{\mathrm{I}}{\frac{x^2}{\sigma^2}} - \overset{\mathrm{II}}{2\lambda x} + \lambda^2\sigma^2\right)} \cdot e^{\frac{\lambda^2\sigma^2}{2}} dx$$

$$= e^{(\lambda^2\sigma^2/2)} \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x - \lambda\sigma^2}{\sigma}\right)^2} dx$$

$$= e^{\left(\frac{\lambda^2\sigma^2}{2}\right)}$$



σ-SubGaussian Random Variable

$$M_X(\lambda) = \mathbb{E}[e^{\lambda x}] \underset{\text{sub}}{\leq} e^{(\lambda^2\sigma^2/2)}, \quad \forall\, \lambda \in \mathbb{R}$$

The tail falls as fast as a Gaussian $N(0,\sigma^2)$

Properties of SubGaussian Random Variables.

1) X is σ-subgaussian $\Rightarrow \mathbb{E}[X] = 0$

Proof :

$$\mathbb{E}[e^{\lambda x}] \leq e^{(\lambda^2\sigma^2/2)}$$ (expand both sides)

$$\mathbb{E}\left[1 + \frac{\lambda x}{1!} + \frac{\lambda^2 x^2}{2!} + \cdots \right] \leq 1 + \frac{\lambda^2\sigma^2}{2\, 1!} + \frac{\lambda^4\sigma^4}{2^2\, 2!} + \cdots$$

(pushing $\mathbb{E}$ inside)

$$\cancel{1} + \frac{\lambda\,\mathbb{E}[x]}{1!} + \frac{\lambda^2\,\mathbb{E}[x^2]}{2!} + \cdots \leq \cancel{1} + \frac{\lambda^2\sigma^2}{2\, 1!} + \frac{\lambda^4\sigma^4}{2^2\, 2!} + \cdots$$

For $\lambda > 0$, divide by sides by λ and let $\lambda \to 0_+$

$$\mathbb{E}[x] + \underbrace{\lambda\frac{\mathbb{E}[x^2]}{2!}}_{\to 0} + \underbrace{\lambda^2\frac{\mathbb{E}[x^3]}{3!}}_{\to 0} \leq \underbrace{\frac{\lambda\sigma^2}{2\, 1!}}_{\to 0} + \underbrace{\frac{\lambda^3\sigma^4}{2^2\, 2!}}_{\to 0} + \cdots$$

$$\mathbb{E}[x] \leq 0 \quad \text{— (1)}$$

$$\mathbb{E}[X] \leq 0 \qquad ①$$

For $\lambda < 0$, divide by λ and let $\lambda \to 0_-$

$$\mathbb{E}[X] + \underset{\to 0}{\lambda \frac{\mathbb{E}[X^2]}{\lfloor 2}} + \underset{\to 0}{\lambda^2 \frac{\mathbb{E}[X^3]}{\lfloor 3}} \geq \underset{\to 0}{\frac{\lambda \sigma^2}{2 \lfloor 1}} + \underset{\to 0}{\frac{\lambda^3 \sigma^4}{2^2 \lfloor 2}} + \cdots$$

$$\mathbb{E}[X] \geq 0 \quad - ②$$

$$① \;\&\; ② \Rightarrow \mathbb{E}[X] = 0$$

2) X is σ-subgaussian $\Rightarrow \mathrm{Var}[X] \leq \sigma^2$

$$= \mathbb{E}[X^2] \quad (\mathbb{E}[X] = 0)$$

$$\cancel{1} + \lambda \frac{\mathbb{E}[X]}{\lfloor 1} + \lambda^2 \frac{\mathbb{E}[X^2]}{\lfloor 2} + \cdots \leq \cancel{1} + \frac{\lambda^2 \sigma^2}{2 \lfloor 1} + \frac{\lambda^4 \sigma^4}{2^2 \lfloor 2} + \cdots$$

divide by λ^2

$$\frac{\mathbb{E}[x^2]}{L^2} \le \frac{\sigma^2}{2L'}$$

3) X is σ- subgaussian $\Rightarrow$ cX is $|c|\sigma$ - subgaussian

$Y = cX$

$$M_Y(\lambda) = M_{cX}(\lambda) = \mathbb{E}[e^{\lambda c x}] \le e^{\lambda^2 c^2 \sigma^2 / 2}$$

X is σ- subgaussian

4) If X_1 and X_2 are independent, σ_1- subgaussian

σ_2- subgaussian respectively, $X_1 + X_2$ is

$\sqrt{\sigma_1^2 + \sigma_2^2}$ - subgaussian

(because of independence)

$$\mathbb{E}[e^{\lambda(X_1 + X_2)}] = \mathbb{E}[e^{\lambda X_1}] \cdot \mathbb{E}[e^{\lambda X_2}]$$

$$\leq e^{\lambda^2 \sigma_1^2/2} \; e^{\lambda^2 \sigma_2^2/2}$$

$$= e^{\lambda^2 \left(\sqrt{\sigma_1^2 + \sigma_2^2}\right)^2/2}$$

Cramer - chernoff bound :

If X is σ- subgaussian

$$\mathbb{P}(X \geq \varepsilon) \leq e^{-(\varepsilon^2/2\sigma^2)}$$

Proof :

$$\mathbb{P}(X \geq \varepsilon) = \mathbb{P}(e^{\lambda X} \geq e^{\lambda \varepsilon})$$

$$\leq \frac{\mathbb{E}[e^{\lambda X}]}{e^{\lambda \varepsilon}} \quad \text{(Markov)}$$

$$\leq \frac{e^{\lambda^2 \sigma^2/2}}{} \quad \text{(SubGaussian)}$$

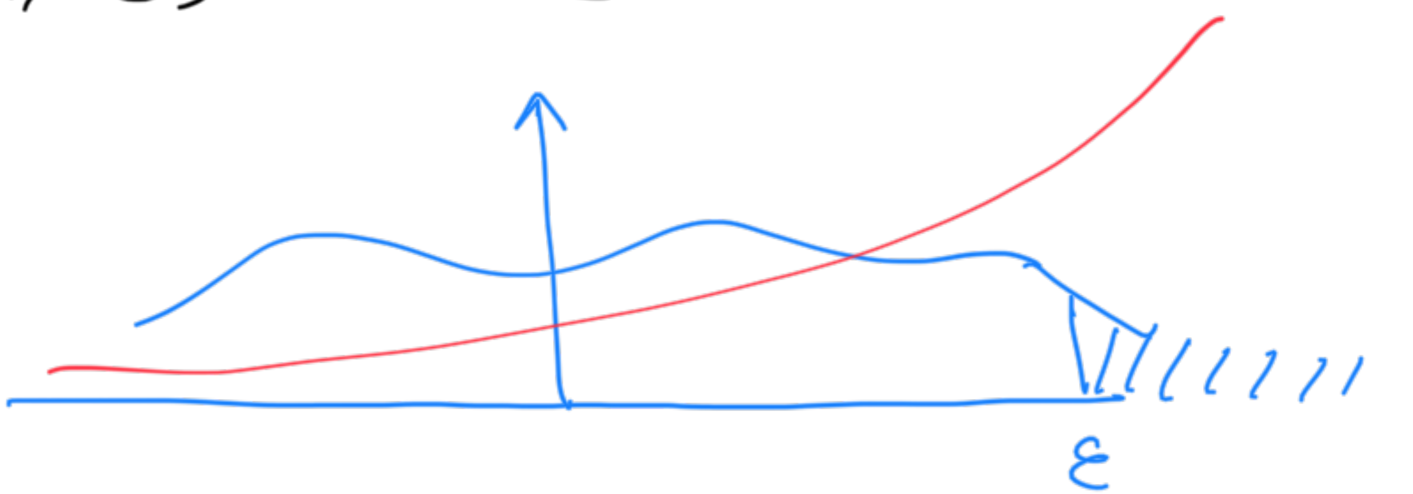
$$\overline{\qquad e^{\lambda \varepsilon}}$$

$$\mathbb{P}(X \geq \varepsilon) \leq \min_{\lambda} e^{\left(\frac{\lambda^2 \sigma^2}{2} - \lambda \varepsilon\right)}$$

I can find the right λ that works best

$$\frac{2\lambda\sigma^2}{2} - \varepsilon = 0 \implies \lambda_* = \frac{\varepsilon}{\sigma^2}$$

$$\mathbb{P}(X \geq \varepsilon) \leq e^{-\left(\frac{\varepsilon^2}{2\sigma^2}\right)}$$



$$\mathbb{P}(X \leq \varepsilon) \leq e^{-\left(\frac{\varepsilon^2}{2\sigma^2}\right)}$$

More:

$$\mathbb{E}[e^{\lambda X}] = 1 + \lambda \frac{\overset{\cancel{0}}{\mathbb{E}[X]}}{\underline{1}} + \lambda^2 \frac{\overset{\sigma}{\mathbb{E}[X^2]}}{\underline{2}} + \ldots$$

* leave all moments (just like Markov)

* have all moments (just like Markov)

* choose the best by tuning for λ

$$\mathbb{P}(|X| \geq \varepsilon) \leq 2 e^{-\left(\frac{\varepsilon^2}{2\sigma^2}\right)} \quad \text{— (SGTB)}$$

So far given a deviation ε, we are looking at the tail probability

Given a tail probability δ, we can look at

Confidence interval

$$\text{SGTB} \equiv X \in \left(-\sqrt{2\sigma^2 \log(2/\delta)}, +\sqrt{2\sigma^2 \log(2/\delta)}\right)$$

random variable X belongs to the confidence interval with probability $> 1 - \delta$

$$-\left(\frac{\varepsilon^2}{2\sigma^2}\right)$$

$$\delta = 2 e$$

$$e^{\varepsilon^2/2\sigma^2} = \frac{2}{\delta}$$

$$\frac{\varepsilon^2}{2\sigma^2} = \log\left(\frac{2}{\delta}\right) \qquad \log = \log_e$$

$$\varepsilon = \sqrt{2\sigma^2 \log\left(\frac{2}{\delta}\right)}$$