Probability Space

$(\Omega, \mathcal{F}, \mathbb{P})$

Goal

- Framework for Randomness
- Capture all Randomness at one place
- Assign Probabilities

- Ω (Sample space)

  Ω (capital omega)

All randomness happens in Ω $\equiv$

Step 1

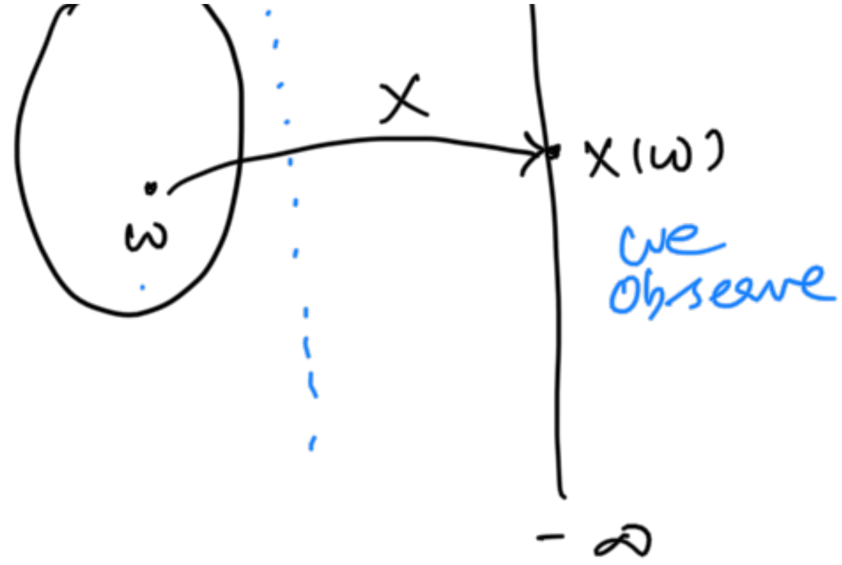
nature chooses $\omega \in \Omega$ (realisation)

- $X : \Omega \to \mathbb{R}$ is a real valued random variable

Ω

Step 2

- we don't get to see ω
- we see $X(\omega)$



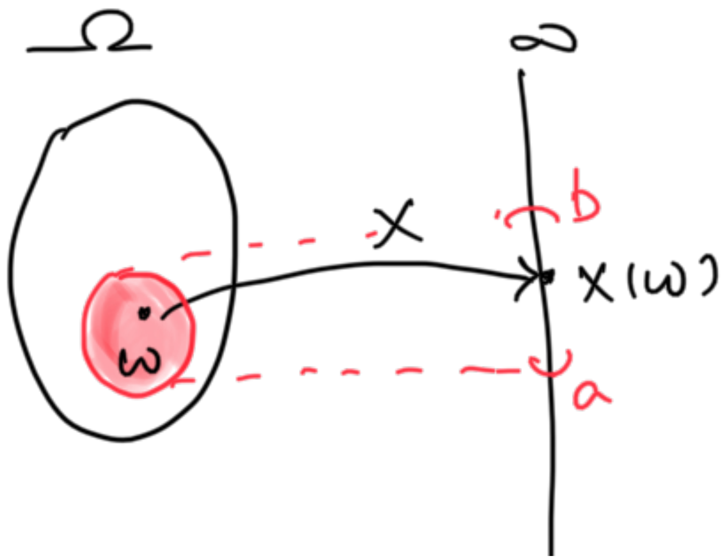
nature uses X and reveals us $X(\omega)$

Step 3

we are interested in

open set

$$\text{Prob}(a < X < b) = \text{Prob}(X \in (a,b))$$



$-\infty$

$$\omega = \{\omega : a < X(\omega) < b\}$$

$$\overset{\text{def}}{=} X^{-1}((a,b)) \quad (\text{pull back})$$

we know total area / measure / probability

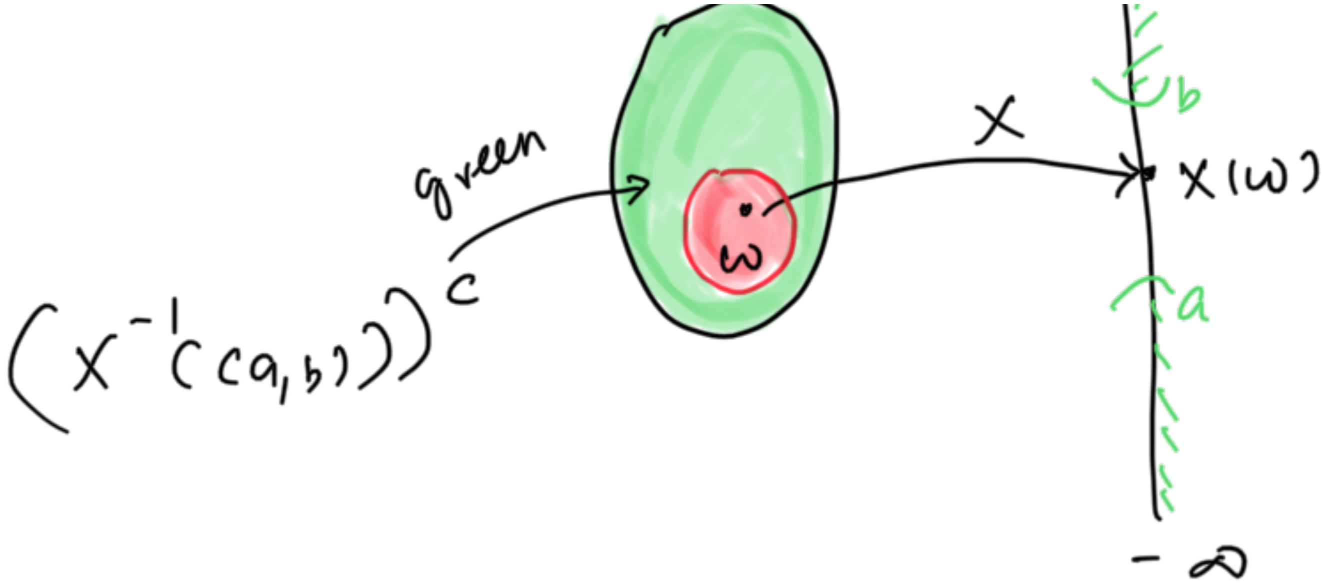
Prob$(\cdot)$ $\quad \mathbb{P}(\Omega) = 1$

we need area / measure / probability of

$$\text{Prob}(a < X < b) = \mathbb{P}(X^{-1}((a,b)))$$

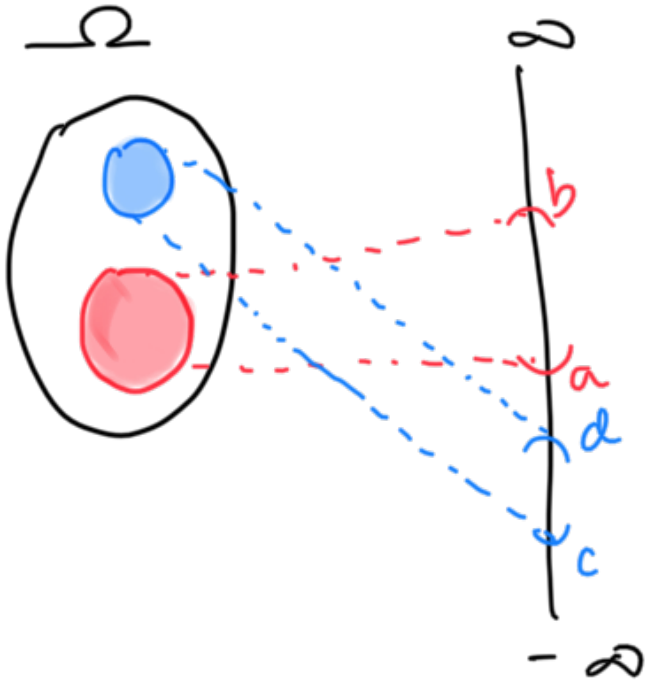
- we also need Prob $(X \notin (a,b))$

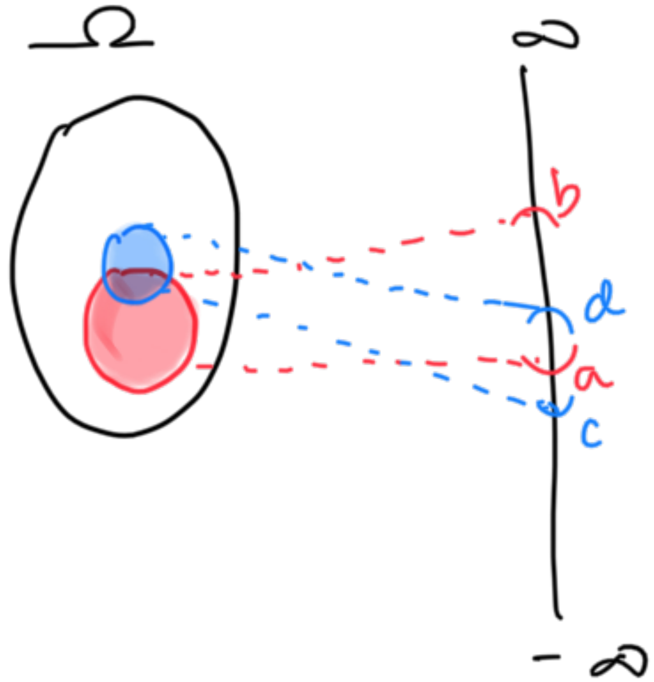
Ω $\quad \infty$



$$\mathbb{P}\left((X^{-1}((a,b)))^c \right)$$

- we also need Prob ($X \in (a,b)$ or $X \in (c,d)$)





Now we need to measure

$$\underline{\mathbb{P}}\left(\underbrace{X^{-1}((a,b))} \cup \underbrace{X^{-1}((c,d))} \right)$$

Step 3:

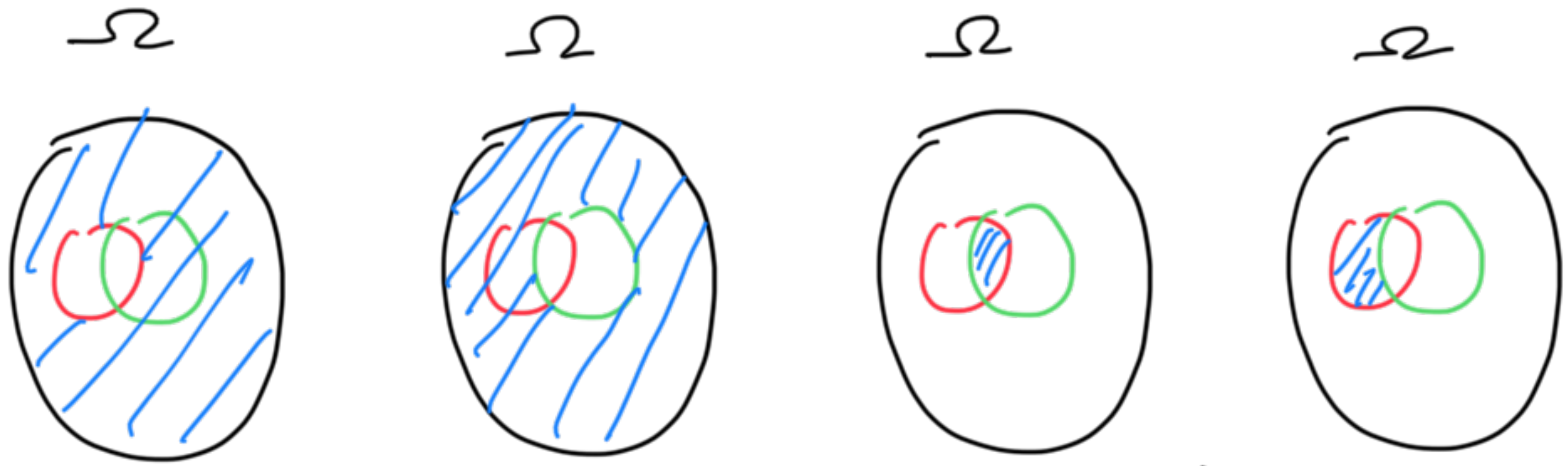
Collect all sets we want to measure in a bag

$\mathcal{F}$ (sigma algebra)

Bag of shapes/sets that we will measure

* $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$, (closed under complements)

* $A_1, A_2, \dots \in \mathcal{F} \Rightarrow \bigcup A_i \in \mathcal{F}$ (closed under countable union)



A_1^c $\quad$ A_2^c $\quad$ $(A_1^c \cup A_2^c)^c$ $\quad$ $(A_1^c \cup (A_1^c \cup A_2^c)^c)^c$

Moral: Pick a bunch of shapes / sets $\in \mathcal{F}$ if we break them into pieces / combine them to form a new shape / set then it $\in \mathcal{F}$

Step 4

Assign areal measure / probability

$$\mathbb{P} : \mathcal{F} \to [0,1]$$

* $\mathbb{P}(\Omega) = 1$, $\quad (\mathbb{P}(\phi) = 1 - \mathbb{P}(\Omega) = 0)$

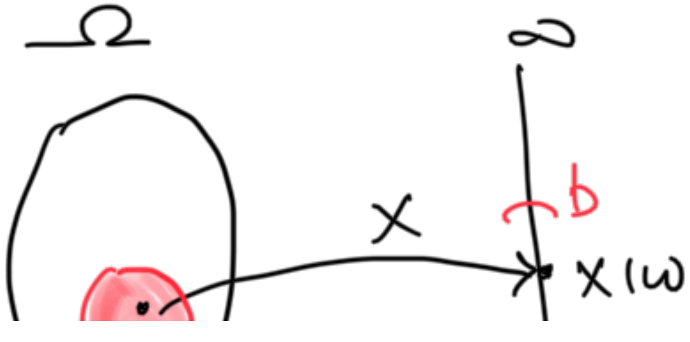
Finite additivity * $A_1 \cap A_2 = \phi \Rightarrow \mathbb{P}(A_1 \cup A_2) = \mathbb{P}(A_1) + \mathbb{P}(A_2)$

same idea for "countably infinite sets"

Countable additivity * $\{A_i\}_{i \geq 0}$, $A_i \cap A_j = \phi$, $i \neq j$

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

Step 5 (Generalising Step 3)



$\text{Prob}(a < X < b) = \mathbb{P}(X^{-1}((a,b)))$

$= \underbrace{X^{-1}((a,b))}_{\text{has to be part of } \mathcal{F}} \in \mathcal{F}$

* In step 3, we used X to get $X^{-1}((a,b))$

* We want our $(\Omega, \mathcal{F}, \mathbb{P})$ framework to talk about "all randomness", so we need to support many random variables $X, Y, Z \cdots$ so on.

* So instead of imposing condition on $\mathcal{F}$, we impose condition on X

$$X : \Omega \to \mathbb{R}$$

real valued r.v is $\mathcal{F}$-measurable if

$$X^{-1}((a,b)) \in \mathcal{F} \quad , \forall\, a,b \in \mathbb{R}$$

Finally : Sanity check

* Basic Question :

what is Prob($a < x < b$) ?

* Ans $= \mathbb{P}(\{\omega : a < X(\omega) < b\})$

$= \mathbb{P}(X^{-1}((a,b)))$

Moral : since $\mathbb{P} : \mathcal{F} \to [0,1]$

and because $X^{-1}((a,b)) \in \mathcal{F}$

we have got everything covered .

$$(\Omega, \mathcal{F}, \mathbb{P})$$

Some examples

Example 1 : Toss of coin $(\Omega, \mathcal{F}, \mathbb{P})$

(call this Ω_{coin}) $\Omega = \{H, T\}$, $\mathcal{F} = 2^{\Omega}$ (power set)

$= \{\Omega, \phi, \{H\}, \{T\}, \{H, T\}\}$

$$\mathbb{P}(A) = \frac{|A|}{2}$$

$$X(H) = H$$
$$X(T) = T$$

Example 2 : Roll of Dice

$\Omega = \{1, 2, \ldots, 6\}$ $\mathcal{F} = 2^{\Omega}$

$\Omega = \{1, 2, 3, 4, 5, 6\}$, $\mathcal{F} = 2^{\Omega}$

$= \{\phi, \Omega, \{1\}, \{2\}, \ldots, \{6\}$
$\{1,2\}, \ldots, \{6,6\}, \{1,2,3\} \ldots\}$

(call this Ω dice)

$$\mathbb{P}(A) = \frac{|A|}{6}$$

$X_1(1) = 1, \; X_1(2) = 2, \; \ldots, \; X_1(6) = 6$

Can we have a toss of a coin and roll of dice with this Ω_{dice}? X_1 dice / X_2 coin

Yes.

$X_1(1) = 1, \; X_1(2) = 2, \; \ldots, \; X_1(6) = 6$

$X_2(1) = X_2(2) = X_2(3) = H$
$X_2(4) = X_2(5) = X_2(6) = T$

Issue? We don't have independence

Question: Can I simulate toss of a coin with roll of dice ?

Example 3: Roll of a Dice and Toss of a coin

(independence)

(call this Ω_{dc}) $\Omega = \{(1,H), (2,H), (3,H), (4,H), (5,H), (6,H),$
$(1,T), (2,T), (3,T), (4,T), (5,T), (6,T)\}$

$\mathcal{F} = 2^{\Omega}$, $\mathbb{P}(A) = \frac{|A|}{12}$

Pick $\omega = (\omega_1, \omega_2) \in \Omega$

Roll of Dice : $X_1(\omega) = \omega_1$

Toss of coin : $X_2(\omega) = \omega_2$

Roll two dice $\Omega = \{(1,1), \ldots, (1,6)$

$\vdots$

$(6,1), \; \cdot \; \cdot \; \cdot \; , (6,6) \}$

$$X_1(\omega) = \omega_1$$

$$X_2(\omega) = H \quad \text{if} \quad 1 \le \omega_2 \le 3$$

$$= T \quad \text{if} \quad 4 \le \omega_2 \le 6$$

Example 4: Roll of a Dice and Toss of a coin

(different $(\Omega, \mathcal{F}, \mathbb{P})$ for Example 3)

(independence)

Ω_{num}

$$\Omega = \{1, 2, \cdots, 12\}, \quad \mathcal{F} = 2^{\Omega}, \quad \mathbb{P}(A) = \frac{|A|}{12}$$

Roll of Dice :

$$X_1(1) = X_1(7) = 1$$

$$X_1(2) = X_1(8) = 2$$

$$\vdots$$

$$X_1(6) = X_1(12) = 6$$

Toss of Coin : $X_2(1) = X_2(2) = \cdots = X_2(6) = H$

$X_2(7) = X_2(8) = \cdots = X_2(12) = T$

Toss of another coin : $X_3(1) = H, \quad X_3(2) = T$

$X_3(3) = H, \quad X_3(4) = T$

$\vdots$

$X_3(\text{odd}) = H, \quad X_3(\text{even}) = T$

check this

$$\boxed{P(A \cap B) = P(A) \cdot P(B)}$$

X_1 and X_2 are ?

X_1 and X_3 are ?

X_2 and X_3 are ?

$\text{Prob}(X_1 = 6, X_2 = T)$

$\mathbb{P}(\{\omega : X_1(\omega) = 6, X_2(\omega) = T\})$

$\frac{1}{6}$ $\mathbb{P}(\{\omega : X_1(\omega) = 6\})$

$\times$

$\frac{1}{2}$ $\mathbb{P}(\{\omega : X_2(\omega) T\})$

$$\mathbb{P}(\{\omega : X_1(\omega) = 6, X_2(\omega) = T\}$$

$$= \mathbb{P}(\{12\}) = \frac{|\{12\}|}{12}$$

$$= \frac{1}{12} = \frac{1}{6} \times \frac{1}{2}$$

Q1) Can we have roll of dice and toss of coin independent, but coin is biased, i.e., $\text{Prob}(H) = 0.6$?

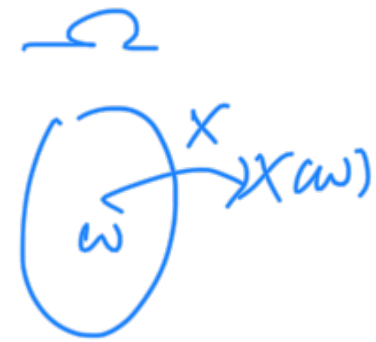
Q2) Can we have roll of dice and toss of coin independent, but dice is biased

$$\text{Prob}(1) = \text{Prob}(2) = \cdots = \text{Prob}(5) = 0.1$$

$$\text{Prob}(6) = 0.5$$

Moral I : Ω can be abstract

- Don't have to write Ω separately for each problem.



- Enough to know that "some" Ω is there

if we want coin : Ω_{coin}, Ω_{dc}, Ω_{num}

" dice : Ω_{dice}, Ω_{dc}, Ω_{num}

- use X to map

- we should just be able to answer Prob (X takes values)

Moral II: $\mathcal{F} \neq 2^{\Omega}$ ($\mathcal{F} = 2^{\Omega}$ always not possible)

We will now construct a set that is