Regret $R_n = \mathbb{E}\left[ \sum_{t=1}^{n} (\mu_* - X_t) \right]$

$$= \sum_{a \in A} \Delta_a \, \mathbb{E}[T_a(n)]$$

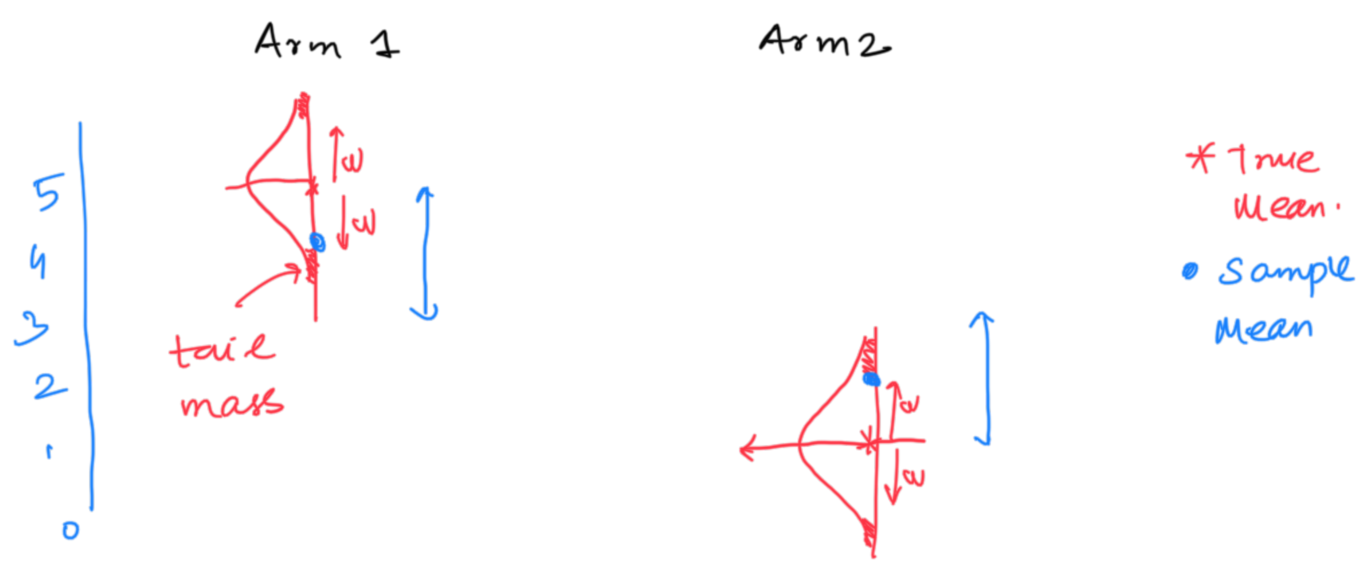
number of times arm a gets picked in n rounds

( a random variable)

Sequence of rewards → Algorithm → $A_t$

(random) ⟶ (random)

Overall aim is to keep $\mathbb{E}[T_a(n)]$ as low as possible




Message: In the high probability event

$\mathbb{P}(\,|\text{sample Mean} - \text{True Mean}| < \varepsilon\,) > 1 - \text{tail mass}$

Requirements from a tail bound

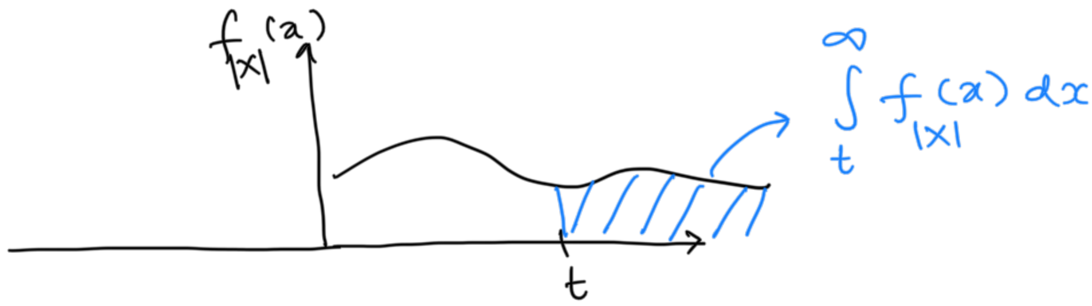
* Cannot assume knowledge of $P_1, \ldots, P_k$
* Can make certain structural assumptions such as availability of a bound on the variance, bound on the range,

# Markov Inequality

$$\mathbb{P}(|X| > t) \leq \frac{\mathbb{E}[X]}{t}$$

+ve random Variable

upper bound

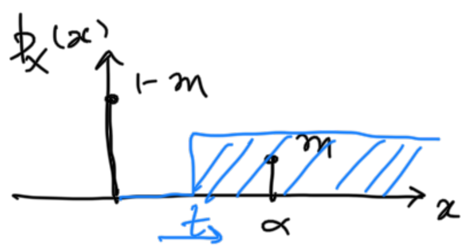



$$\int_t^\infty f_{|X|}(x)\,dx \leq \int_t^\infty \underbrace{\left(\frac{x}{t}\right)}_{>1} f_{|X|}(x)\,dx$$

$$\leq \frac{1}{t}\int_0^\infty x\, f_{|X|}(x)\,dx$$

$$= \frac{\mathbb{E}[X]}{t}$$

Example 1: Case where Markov inequality is tight




$X \sim p_X$

$$\mathbb{E}[X] = 0\cdot(1-m) + \alpha\cdot m$$

$$= \alpha\cdot m$$

$$\mathbb{P}[|X| > t] \leq \frac{\mathbb{E}[X]}{t}$$

$$= \frac{\alpha\cdot m}{t} \quad \text{— (1)}$$

Bound is tight for $t = \alpha$

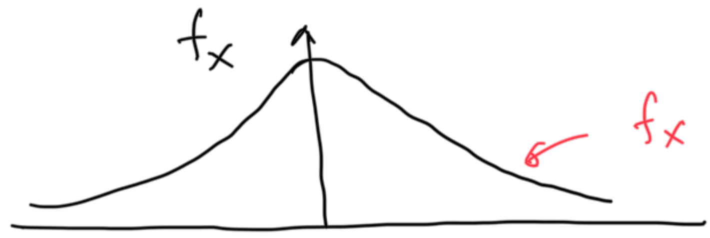
* Bound is loose for $t < \alpha$ $\rightarrow$ t is in the denominator

* Bound is loose for $t > \alpha$, because

$$\mathbb{P}[|X| > \alpha] = 0$$

Example 2: When we cannot apply Markov

Take Cauchy density $f_X(x) = \frac{1}{\pi(1+x^2)}$




$f_X$ does not even kill the growth of $x$

$$\mathbb{E}[|X|] = \int_0^\infty x \frac{2}{\pi(1+x^2)} dx$$

$$= \int_0^\infty \frac{d(1+x^2)}{\pi(1+x^2)}$$

$2x\,dx$

$$= \frac{1}{\pi}\left[\log(1+x^2)\right]_0^\infty = \infty$$

There is a structural reason : heavy tailed

General Fact I : $\sum_{n\geq 0} \frac{1}{n} = \infty$ (Harmonic Series)

Proof : $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$

$$\geq \frac{1}{2} + \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}}_{} + \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}}_{}$$

$$= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$$

$\{\frac{1}{n}\}$ itself is diminishing to 0

Fact II : $\sum_{n\geq 0} \frac{1}{n^2} < \infty$

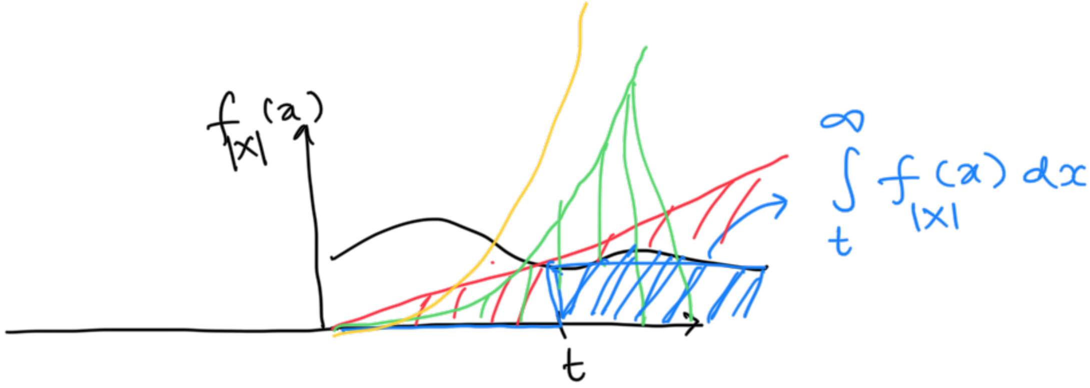
$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{8^2}$$

$$\leq 1 + 1 + \underbrace{\frac{1}{2^2} + \frac{1}{2^2}}_{} + \underbrace{\frac{1}{4^2} + \frac{1}{4^2} + \frac{1}{4^2} + \frac{1}{4^2}}_{}$$

$$= 1 + 1 + \frac{1}{2} + \frac{1}{4}$$

$$\leq 3$$

$$\sum_{n\geq 0} \frac{1}{n^2} = \frac{\pi^2}{6}$$




$$\int_t^\infty f_{|X|}(x)\, dx \leq \int_t^\infty \left(\frac{x}{t}\right)^2 f_{|X|}(x)\, dx$$

more steeper function

$$\frac{1}{t} \int_t^\infty x \, f_{|X|}(x)\, dx \quad \text{vs} \quad \frac{1}{t^m} \int_t^\infty \underline{x^m} f_{|X|}(x)\, dx$$

gained here.

hope is that this kills.

$$\mathbb{P}[\,|X| > t\,] \leq \frac{\mathbb{E}[\,|X|^m\,]}{t^m}$$

* L.H.S has the original probability

* R.H.S $\mathbb{E}[\,|X|^m\,] \rightarrow$ may be there is a hope to bound this by a constant
(reasonable for the algorithm to assume)

Markov Inequality in other words.

Consider a discrete positive random variable

$$X \sim p_X$$

$$\mathbb{E}[X] = \sum_{x \geq 0} x \cdot p_X(x)$$

$$= 0 \cdot p_X(0)$$

$$+ \; p_X(1)$$

$$+ \; p_X(2) + p_X(2)$$

$$+$$

$\mathbb{P}[X \geq 3]$

$$p_x(3) + p_x(3) + p_x(3)$$

$$+$$

$$p_x(4) + p_x(4) + p_x(4) + p_x(4)$$

$\mathbb{E}[X]$ contains $P[|X| \geq t]$ $t$ times

$$P[|X| \geq t] \leq \frac{\mathbb{E}[|X|]}{t}$$

$$P[|X| \geq t] \leq \mathbb{E}[|X|^m] / t^m$$

Apparent contradiction:

(C1)* Markov is for $\mathbb{P}(|X| \geq t) \rightarrow$ tail mass for larger and larger deviation away from origin. What we want is smaller deviations of sample mean from true mean.

(C2)* It seems we can always win by increasing $m$. Is it true?

(C3)* Do we always win to the extent of $t^m$, how about $\mathbb{E}[|X|^m]$?

Chebyshev Inequality

Markov inequality with $X = \frac{S_n}{n} - \mu$

$$\mathbb{P}\left[\left|\frac{S_n}{n} - \mu\right| \geq \varepsilon\right] = \mathbb{P}[|S_n - n\mu| \geq n\varepsilon]$$

sample mean true mean

Resolving (C1), small deviation of sample mean from true mean translates to large deviations of $S_n$ from $n\mu$

or $|S_n - n\mu|$ away from origin

[illegible] away from origin.