MATH1131/1141 Revision

Calculus

Abdellah Islam

UNSW MathSoc

29 Apr 2019

Introduction

Content reviewed today:

- Limits
- Continuity
- Differentiation
- Inverse Functions
- Curve Sketching
- Integration
- Hyperbolic Functions

• Exam Preparation - Focus on exam-style questions!
• Looking at exam tricks and common mistakes by students.

Definition of Limit

Limit at a point (Not Assessed!!)

$\lim\limits_{x \to a} f(x) = L$ if for each $\epsilon > 0$, there exists $\delta > 0$ such that if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.

Limit to infinity

$\lim\limits_{x \to \infty} f(x) = L$ if for each $\epsilon > 0$ we have some $M > 0$ such that when $x > M$, $|f(x) - L| < \epsilon$

Proving Limits to Infinity

Example

Prove by definition that

$$\lim_{x\to\infty} \frac{x^2 - 2}{x^2 + 3} = 1$$

(1131 2012 S1 Q4.iii)

Techniques for Computations

To evaluate any limit to infinity:

Form of Limit	**Technique**	**Sample**
$\frac{f(x)}{g(x)}$	Divide top and bottom by highest power on x	$\frac{x^2+x-1}{2x^2-3x-100}$
$\sqrt{f(x)} - \sqrt{g(x)}$	Multiply top and bottom by $\sqrt{f(x)} + \sqrt{g(x)}$	$\sqrt{x^2+x} - x$
$f(x)\sin x$ or $f(x)\cos x$	*Pinching theorem*	$\frac{\cos x}{x}$

Pinching Theorem

Theorem

If $f(x) \leq g(x) \leq h(x)$ as $x \to \infty$, and

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} h(x) = L,$$

then

$$\lim_{x\to\infty} g(x) = L$$

.

- Also works for limits to a point.
- Extremely flexible theorem, but usually reserved for limits involving $\sin x$ and $\cos x$.

L'Hopital's Rule

Definition

Let f and g be differentiable functions such that as $x \to a$, either

- $f(x) \to \infty$ and $g(x) \to \infty$, or
- $f(x) \to 0$ and $g(x) \to 0$.

If $\lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)} \to L$ then $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} \to L$.

Notes & Requirements

- $x \to a$ can be replaced with $x \to \pm\infty$ as well
- For the sake of L'Hopital's rule, L can be $\pm\infty$
- If $\dfrac{f'(x)}{g'(x)}$ does not approach a real number or infinity, then we CANNOT apply L'Hopital's Rule.

L'Hopital's Rule

Example

Evaluate

$$\lim_{x \to \pi} \frac{x - \pi}{\sin 3x}$$

(1131 2015 S2 Q1.i.c)

L'Hopital's Rule

Example

Evaluate

$$\lim_{x \to 0^+} x^x$$

Shows a useful technique: Other indeterminate forms include

$$0 \times \infty, \quad \infty - \infty, \quad 1^\infty, \quad 0^0 \quad \infty^0$$

but they can potentially be *transformed* into $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

L'Hopital's Rule

WARNING

Do not always state that you can apply L'Hopital's Rule if you're not sure the limit of the derivatives exist!! For example,

$$\lim_{x \to \infty} \frac{x + \sin x}{x - \sin x}$$

Definition of Continuity

At a Point

We say that a function f is continuous at a point $x = a$ if

- $f(a)$ is defined,
- $\lim_{x \to a} f(x)$ exists, and
- $\lim_{x \to a} f(x) = f(a)$

On an Interval

We say a function f is continuous on an interval if f is continuous at each point in that interval.

Intermediate Value Theorem

Theorem

Suppose we have a ***continuous*** function f on the ***closed*** interval $[a, b]$. If $f(a) \leq L \leq f(b)$ then there is at least one $c \in [a, b]$ such that $f(c) = L$.

NOTES

- Common Mistake: You MUST state that the interval is closed, and the function is continuous on the interval!!
- Very useful when finding zeros of a function.

Intermediate Value Theorem

Example

Show that the equation

$$e^x = x + 2$$

has a solution in the interval $[0, 2]$.

(1131 2014 S2 Q1.v.b)

Max-Min Theorem

Theorem

If f is ***continuous*** on a ***closed*** interval $[a, b]$ then f attains its maximum and minimum values on this interval. Hence, there exists $c, d \in [a, b]$ such that $f(c) \leq f(x) \leq f(d)$ for all $x \in [a, b]$.

NOTE

If the interval is not closed, or f is not continuous on the interval (or both) then f may or may not attain its maximum and minimum values on the interval.

Differentiability

Definition of Differentiability

If a function f is defined at a point a, then we say f is differentiable at a if $\lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ exists. We call this limit $f'(a)$, which is the derivative of f at a.

NOTE

Differentiability $\Rightarrow$ Continuity

i.e. If f is differentiable at a, then f is continuous at a. However the converse is not always true!

Piece-wise Differentiability

Example

Find a and b such that the function

$$f(x) = \begin{cases} x^2 + ax + b & x < 0 \\ \cos 2x & x \geq 0 \end{cases}$$

is differentiable.

(1131 2015 S2 Q2.i)

Mean Value Theorem

Theorem

Suppose a function f is ***continuous*** on the ***closed*** interval $[a, b]$ and ***differentiable*** on the ***open*** interval (a, b). Then there is at least one $c \in (a, b)$ such that $\dfrac{f(b) - f(a)}{b - a} = f'(c)$.

NOTE

Again, you MUST state that the function is continuous on the closed interval and differentiable on the open interval before using the theorem!

Mean Value Theorem

Example

Suppose that $-1 < x < y < 1$. Prove that

$$\sin^{-1} y - \sin^{-1} x \geq y - x$$

by applying the Mean Value Theorem on the function $f(t) = \sin^{-1} t$ on the interval $[x, y]$.

(1131 2014 S1 Q4.v.b)

Mean Value Theorem

Example

Suppose $f : [0,2] \to [0,8]$ is continuous and differentiable on its domain.

(a) By considering the function $g(x) = f(x) - x^3$, prove that there is a real number $\xi \in [0,2]$ such that $f(\xi) = \xi^3$, stating any theorems you use.

(b) Now suppose that $f(0) = 0$ and $f(2) = 8$. Explain why $f'(\eta) = 4$ for some real $\eta \in (0,2)$, stating any theorems you use.

(1141 2013 S1 Q4.iv)

Inverse Function Theorem

Theorem

Suppose that a function $f : I \to \mathbb{R}$ is differentiable on an open interval I and $f'(x) \neq 0$ for all $x \in I$. Then by the Inverse Function Theorem:

- f is one-to-one and has an inverse function $g : Range(f) \to I$,
- g is differentiable on all points in its domain,
- $g'(x) = \dfrac{1}{f'(g(x))}$ for all x in $Range(f)$.

Inverse Functions

Example

Let $g(x) = 3x - \cos 2x - 1$ for all $x \in \mathbb{R}$. Explain why g has a differentiable inverse function $h = g^{-1}$, and calculate $h'(-2)$.

(1141 2014 S1 Q3.i)

Polar Curves

Polar Coordinates

- $x = r\cos\theta$, $y = r\sin\theta$
- $r = \sqrt{x^2 + y^2}$
- $\tan\theta = \dfrac{y}{x}$, $x \neq 0$

Example

Sketch the polar curve $r = 1 - \cos\theta$ for $0 \leq \theta < 2\pi$.

(1131 2016 S1 Q2.ii)

Symmetry

Example

Consider the polar curve $r = 1 + \cos 2\theta$.

(a) Prove that the curve is symmetric about the x-axis and also about the y-axis.

(b) Sketch the curve (Not required to find derivative).

(1141 2012 S1 Q4.i)

10 minute Break

Let f be continuous and differentiable on $\mathbb{R}$, and

$$|f'(x)| \leq \sqrt{x}$$

for all x. Find the limit

$$\lim_{x\to\infty} \frac{f(x+1) - f(x)}{x}$$

Riemann Sums

Upper & Lower Sums

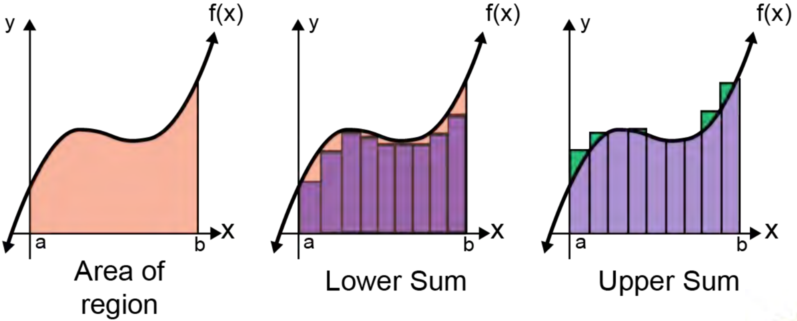
Let f be a function continuous on the interval $[a, b]$ with maximum value $\overline{f}_k$ and minimum value $\underline{f}_k$ on the subinterval $[a_{k-1}, a_k]$.
Suppose we have a partition P_n of $[a, b]$ such that $P_n = \{a_0, a_1, ..., a_n\}$, where $a_0 = a$ and $a_n = b$. Then the Upper Riemann Sum $\overline{S}_{P_n}(f)$ and Lower Riemann Sum $\underline{S}_{P_n}(f)$ are given by:

- $\overline{S}_{P_n}(f) = \sum_{k=1}^{n} (a_k - a_{k-1})\overline{f}_k$
- $\underline{S}_{P_n}(f) = \sum_{k=1}^{n} (a_k - a_{k-1})\underline{f}_k$

REMEMBER

$\overline{S}_{P_n}(f)$ is the sum of largest rectangles in each subinterval, and $\underline{S}_{P_n}(f)$ is the sum of smallest rectangles in each subinterval.

Graphical interpretation of Riemann sums



Definite Integral

Definition

If a function f has upper Riemann sum $\overline{S}_{P_n}(f)$ and lower Riemann sum $\underline{S}_{P_n}(f)$ on an interval $[a, b]$, and there exists unique real number I such that

$$\underline{S}_{P_n}(f) \leq I \leq \overline{S}_{P_n}(f)$$

for all n.Then we call I the definite integral of f from a to b,

$$I = \int_a^b f(x)dx$$

Limit of Riemann sums

If $\lim_{n\to\infty} \underline{S}_{P_n}(f) = \lim_{n\to\infty} \overline{S}_{P_n}(f) = L$ then $I = L$.

Application of Riemann sum

Example

(a) Calculate the upper Riemann sum of the function $f(x) = x^2$ for the partition $P_n = \left\{0, \frac{1}{n}, \frac{2}{n}, ..., 1\right\}$ of the interval $[0, 1]$, where n is a positive integer.

(b) Find the value of the definite integral

$$\int_0^1 x^2 dx$$

.

(1131 2015 S2 Q2.vi)

First Fundamental Theorem of Calculus

Let the function $f : [a, b] \to \mathbb{R}$ be continuous on its domain. Then the function $F : [a, b] \to \mathbb{R}$ defined by

$$F(x) = \int_a^x f(t)dt$$

is continuous on $[a, b]$, differentiable on (a, b), and has derivative given by

$$F'(x) = f(x)$$

for all $x \in (a, b)$.

First Fundamental Theorem of Calculus

Example

Use the First Fundamental Theorem of Calculus to find

$$\frac{d}{dx}\left(\int_{\cos x}^{\sin x} e^{1-t^2}\,dt\right)$$

(1131 2015 S2 Q2.iv)

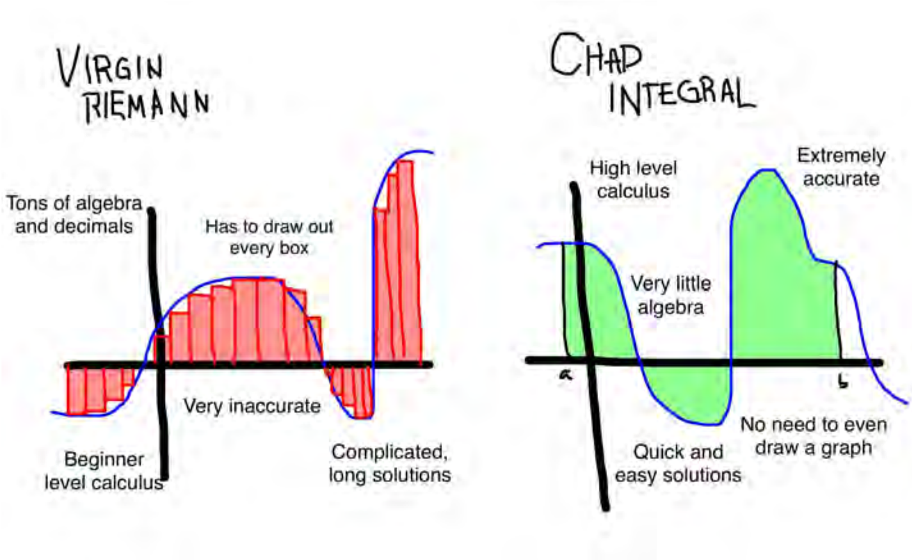
Second Fundamental Theorem of Calculus

Let the function $f : [a, b] \to \mathbb{R}$ be continuous on its domain. If the function F is an antiderivative of f on $[a, b]$, then

$$\int_a^b f(t)dt = F(b) - F(a)$$

This theorem is extremely powerful when it comes to calculating definite integrals compared to Riemann sums. This is literally how we evaluated definite integrals in high school!

Integration>Riemann Sum



Integration Techniques

If the integral is not possible via basic integration, then use a substitution! Remember to include the derivative of substitution as well.

Substitution

Let $x = g(u)$. Then

$$\int f(x)dx = \int f(g(u))g'(u)du$$

If the integral is made up of two different functions, one of which has an easily found anti-derivative, try integrating by parts!

By Parts

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g'(x)dx$$

Integration Techniques

Example

Find:

(a)

$$\int x^2\sqrt{3+x^3}\,dx$$

(b)

$$\int xe^{3x}\,dx$$

(1131 2016 S1 Q3.i)

Integration Techniques

Example

Find using integration by substitution:

$$\int x\sqrt{x+1}dx$$

Find using integration by substitution:

$$\int \frac{1}{(1+x^2)^{\frac{3}{2}}}dx$$

Find using integration by parts:

$$\int e^x \cos(x) dx$$

(Important techniques for 1131/1141)

Integration Techniques

WARNING

Don't always go straight to integration by parts simply because it requires less thinking. Many integrals become much harder when tackled with integration by parts. For example,

$$\int xe^{x^2}\,dx$$

Improper Integrals

Improper Integral

$$\int_a^{\infty} f(x)dx = \lim_{R\to\infty} \int_a^R f(x)dx$$

Common Mistakes

- You MUST take the limit as $R \to \infty$
- If we have an integral of the form $\int_{-\infty}^{\infty} f(x)dx$ then we must separate the interval:

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_0^{\infty} f(x)dx$$

- Do not always start integrating immediately. Check to see if the integral converges or diverges.

Convergence/Divergence Tests

p-test

The integral

$$\int_1^\infty \frac{1}{x^p}\,dx$$

converges if $p > 1$ and diverges if $p \leq 1$.
Not so useful by itself, but can be used along with the next test...

Convergence/Divergence Tests

Comparison Test

Suppose that, for continuous functions f and g, $0 \leq f(x) \leq g(x)$ whenever $x > a$. Then:

- If $\int_a^\infty g(x)dx$ converges then $\int_a^\infty f(x)dx$ converges
- If $\int_a^\infty f(x)dx$ diverges then $\int_a^\infty g(x)dx$ diverges

NOTE

- Easy to compare functions to $\dfrac{1}{x^p}$, so combine this test with p-test.
- Easy to remember: If the bigger function converges, smaller functions should converge too. If the smaller function diverges, larger functions should diverge too.

Convergence/Divergence Tests

Example

Does the following improper integral converge? If so, find its value. If not, show that it diverges.

$$\int_e^\infty \frac{dx}{x + \ln x}$$

(1131 2014 S2 Q1.ii)

Convergence/Divergence Tests

Limit Comparison Test

Let f and g be non-negative continuous functions on $[a, \infty)$ such that

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = L$$

where $L > 0$. Then either:

- $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ both converge, or
- $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ both diverge

Convergence/Divergence Tests

Example

Determine whether or not the following improper integral converges.Give reasons for your answer.

$$\int_2^\infty \frac{x^2+\sqrt{x}}{x^{\frac{8}{3}}-x^2-1}\,dx$$

(1131 2015 S2 Q1.v)

Hyperbolic Trig

$\cosh x$

$$\cosh x = \frac{e^x + e^{-x}}{2}, x \in \mathbb{R}$$

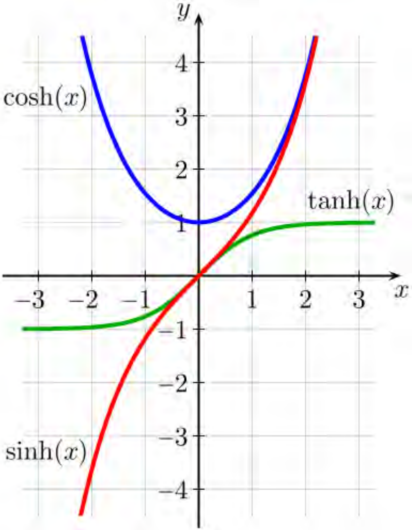
$\sinh x$

$$\sinh x = \frac{e^x - e^{-x}}{2}, x \in \mathbb{R}$$

$\tanh x$

$$\tanh x = \frac{\sinh x}{\cosh x}, x \in \mathbb{R}$$

Graphs



Hyperbolic Trig

Identities

$$\cosh^2 x - \sinh^2 x = 1$$

Other results can be deduced from here by dividing by either $\cosh^2 x$ or $\sinh^2 x$.

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$

Convenient Mnemonic

Consider the classic trigonometric identities involving $\cos x$ and $\sin x$. By replacing $\cos x$ with $\cosh x$ and $\sin x$ with $i \sinh x$, we arrive at the hyperbolic trig identities.

Hyperbolic Trig

Derivatives

$$\frac{d}{dx}\cosh x = \sinh x$$

$$\frac{d}{dx}\sinh x = \cosh x$$

$$\frac{d}{dx}\tanh x = \operatorname{sech}^2 x$$

Hyperbolic Trig

Example

(a) Give the definitions of $\sinh x$ and $\cosh x$ in terms of the exponential function.
(b) Use your definitions to prove that $\sinh(2x) = 2\sinh x \cosh x$

(1131 2014 S1 Q2.ii)

Hyperbolic Trig

Example

(a) Express $\tanh x$ in terms of exponentials.

(b) Sketch the graph $y = \tanh x$.

(c) Find

$$\lim_{x\to\infty} \frac{1 - \tanh x}{e^{-2x}}$$

(d) Explain why the improper integral converges.

$$\int_0^\infty (1 - \tanh x)dx$$

(e) Compute

$$\int_0^\infty (1 - \tanh x)dx$$

(1141 2012 S1 Q4.ii)

Inverse Hyperbolic Trig

$\sinh^{-1} x$

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right), x \in \mathbb{R}$$

$\cosh^{-1} x$

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right), x \in [1, \infty)$$

$\tanh^{-1} x$

$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), x \in (-1, 1)$$

Inverse Hyperbolic Trig

Derivatives

$$\frac{d}{dx}\sinh^{-1}x = \frac{1}{\sqrt{x^2+1}}$$

$$\frac{d}{dx}\cosh^{-1}x = \frac{1}{\sqrt{x^2-1}}$$

$$\frac{d}{dx}\tanh^{-1}x = \frac{1}{1-x^2}$$

Inverse Hyperbolic Trig

Example

Sketch, on one set of axes, the graphs of $y = \cosh x$ and $y = \cosh^{-1} x$

(1131 2016 S1 Q3.i)

Questions involving several topics

Example

(a) Carefully state the first fundamental theorem of calculus.

(b) For $\alpha > 0$ and $n > 0$, determine whether the improper integral

$$\int_0^\infty u e^{\alpha u^n} \, du$$

converges or diverges. Give reasons for your answer.

(1141 2016 S1 Q3.iv)

Questions involving several topics

Example

(c) Using L'Hopital's Rule, find, without integration, $\lim\limits_{x\to\infty} f(x)$, where

$$f(x) = \frac{\left(\displaystyle\int_0^x ue^{3u^2}\,du\right)^2}{\displaystyle\int_0^x ue^{6u^2}\,du}$$

(d) Show that the function f is an even function, that is, $f(-x) = f(x)$.

(1141 2016 S1 Q3.iv)

Questions involving several topics

Example

Use the ϵ-M definition of the limit to prove that

$$\lim_{x \to \infty} \frac{e^x}{\cosh x} = 2$$

(1141 2014 S1 Q3.iii)