MATH2121/MATH2221

Theory and Application of Differential Equations

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Table of Contents

Linearity

Defintion 1:

An operator L is linear if:

$$L(c_1 u_1 + c_2 u_2) = c_1 L u_1 + c_2 L u_2$$

Linear Differential Operators

Definition 2:

The **linear differential operator** L of **order m** is:

$$Lu(x) = \sum_{j=0}^{m} a_j(x) D^j u(x) = a_m D^m u + a_{m-1} D^{m-1} u + \dots + a_0 u,$$

where $D^j u = d^j u / dx^j$ (with $D^0 u = u$).

Definition 3:

The ODE $Lu = f$ is said to be **singular** with respect to an interval $[a, b]$ if the leading coefficient $a_m(x)$ vanishes for any $x \in [a, b]$.

Homogeneous/Inhomogeneous

Definiton 4:

ODEs of the form:

$$Lu = 0$$

are known as **homogeneous**. Those of the form:

$$Lu = f$$

are known as **inhomogenous** or non-homogeneous. In physical systems, the inhomogeneity is often described as a forcing term.

Note that homogenoeous solutions form a vector space: Given $u_1, u_2, ...u_k$ solutions of the linear homogeneous DE, $Lu = 0$ - since L is linear, the linear combination of the solutions is also a solution (i.e. 0).

Uniqueness of Solutions

Theorem 1:

For an ODE $Lu = f$ which is not singular with respect to $[a, b]$, with f continuous on $[a, b]$, the IVP below has a unique solution.

Theorem 2:

Assume that the linear, mth-order differential operator L is not singular on [a,b]. Then the set of all solutions to the homogeneous equation $Lu = 0$ on $[a, b]$ is a vector space of dimension m.

If $u_1, u_2, ..., u_m$ is any basis for the solution space of $Lu = 0$, then every solution can be written in a unique way as:

$$u(x) = c_1 u_1(x) + c_2 u_2(x) + ... + c_m u_m(x) \text{ for } a \leq x \leq b$$

which is called the **general solution** of the homogeneous equation on $[a, b]$.

Inhomogenous Problem

Consider the inhomogeneous equation $Lu = f$ on $[a, b]$, and fix a **particular solution** u_P.

For any solution u, the difference $u - u_P$ is a solution of the homogeneous equation, and so:

$$u(x) = u_P(x) + c_1 u_1(x) + ... + c_m u_m(x), \qquad a \leq x \leq b$$

is the **general solution** of the inhomogeneous equation $Lu = f$.

Reduction of Order

Theorem 3:

For $u = u_1(x) \neq 0$, a solution to the ODE

$$u'' + p(x)u' + q(x)u = 0,$$

on some interval I, then a second solution is:

$$u = u_1(x) \int \frac{1}{u_1^2 \exp(\int p dx)} dx$$

Constant Coefficients

Theorem 4:

For the constant-coefficient case, the general solution of the homogeneous equation $Lu = 0$ is:

$$u(x) = \sum_{q=1}^{r} \sum_{l=0}^{k_q - 1} c_{ql} x^l e^{\lambda_q x}$$

where the c_{ql} are arbitrary constants.

Solutions

Distinct Real Roots

The general solution is:

$$u(x) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \ldots + c_n e^{\lambda_n t}$$

where λ_i are solutions to the characteristic equation and $c_i \in \mathbb{R}$.

Repeated Real Roots

For roots $\lambda_1, \ldots, \lambda_j$ each with multiplicity n_j, the general solution is:

$$u(x) = \sum_{1}^{j} \sum_{0}^{n_j} x^i c_i e^{\lambda_j t}$$

Complex Roots

The general solution is similar to above but includes $\sin x$ and $\cos x$. Note that roots exist in conjugate pairs.

Example

MATH2221 2016 T2 (1i)

Find the general solution $u = u(x)$ to the ODE:

$$u'' + 4u' + 3u = 9x^2.$$

Linear Independence

Definition 5:

Let $u_1(x), u_2(x), ..., u_m(x)$ be functions defined on an interval $I \subset \mathbb{R}$. The functions $u_1, ..., u_,$ are called **linearly dependent** if there exist constants $a_1, a_2, ..., a_m$ **not all zero** such that:

$$a_1 u_1(x) + a_2 u_2(x) + ... + a_m u_m(X) = 0 \qquad \forall x \in I.$$

If the above equation only holds for:

$$a_i - 0, \qquad i = 1, 2, ..., m$$

then the functions are **linearly independent**.

Wronskians and Linear Independence

The Wronskian is a function that provides us with a way of testing whether a family of solutions to $Lu = 0$ is linearly independent amongst other uses.

Definition 6:

The **Wronskian** of the functions $u_1, u_2, ..., u_m$ is the $m \times m$ determinant:

$$W(x) = W(x; u_1, u_2, ..., u_m) = det\left[D^{i-1}u_j\right]$$

For instance, when $m = 3$ and $W(x)$ differentiable $m - 1$ times:

$$W(x) = \begin{vmatrix} u_1 & u_2 & u_3 \\ u_1' & u_2' & u_3' \\ u_1'' & u_2'' & u_3'' \end{vmatrix}$$

Linearly Dependent Functions

Lemma 1:

If $u_1, ..., u_m$ are linearly dependent over an interval $[a, b]$ then $W(x; u_1, ..., u_m) = 0$ for $a \leq x \leq b$.

Wronskian satisfies a first-order ODE

Lemma 2:

If $u_1, u_2, ..., u_,$ are solutions of $Lu = 0$ on the interval $[a, b]$ then their Wronskian satisfies:

$$a_m(x)W'(x) + a_{m-1}(x)W(x) = 0, \qquad a \leq x \leq b.$$

example

MATH2221 2014 T2 (2iiib)

Given the functions u_1, u_2, prove that if they are solutions to a second-order, homogeneous linear differentiatial equation:

$$a_2(x)u'' + a_1(x)u' + a_0(x)u = 0,$$

then their Wronskian W satisfies:

$$a_2(x)W' + a_1(x)W = 0.$$

Linear independence of solutions

Theorem 5:

Let $u_1, u_2, ..., u_m$ be solutions of a non-singular, linear, homogeneous mth-order ODE $Lu = 0$ on the interval $[a, b]$.

Either

$W(x) = 0$ for $a \leq x \leq b$ and the m solutions are linearly **dependent**,

or else:

$W(x) \neq 0$ for $a \leq x \leq b$and the m solutions are linearly **independent**.

Superposition of solutions

Let $L = p(D)$ be a linear differential operator of order m with constant coefficients.

Theorem 6:

Assume that $a_0 = p(0)$. For any integer $r \geq 0$, there exists a unique polynomial u_p of degree r such that $Lu_p = x^r$.

Exponential Solutions

Theorem 7:

Let $L = p(D)$ and $\mu = \mathbb{C}$. If $p(\mu) \neq 0$, then the function:

$$u_p(x) = \frac{e^{\mu x}}{p(\mu)}$$

Product of Polynomial and Exponential

Theorem 8:

Let $L = p(D)$ and assume that $p(\mu) \neq 0$. For any integer $r \geq 0$, there exists a unique polynomial v of degree r such that $u_p = v(x)e^{\mu x}$ satisfies $Lu_p = x^r e^{\mu x}$.

Annihilator method

The following is a method to derive a particular solution given $Lu = f$. If $f(x)$ is differentiable at least n times and:

$$[a_n D^n + a_{n-1} Dn - 1 + ... + a_1 D^1 + a_0] f(x) = 0$$

then $[a_n D^n + a_{n-1} D^{n-1} + ... + a_1 D^1 + a_0]$ **annihilates** f.

Polynomial solutions: the remaining case

Theorem 9:

Let $L = p(D)$ and assume $p(0) = p'(0) = \ldots = p^{(k-1)}(0) = 0$, but $p^{(k)}(0) \neq 0$ where $1 \leq k \leq m-1$. For any integer $r \geq 0$, there exists a unique polynomial v of degree r such that $u_p(x) = x^k v(x)$ satisfies $Lu_p = x^r$.

Example

MATH2221 2014 T2 (2ii)

Let:

$$p(z) = z^5 + 3z^4 + z^3 - z^2 - 4 = (z-1)(z+2)^2(z^2+1)$$

and $D = \frac{d}{dx}$.

1. Write down the general solution u_H of the fifth-order, linear homogeneous ODE $p(D)u = 0$.
2. Write down the **form** of a particular solution u_p to the in-homogeneous ODE:

$$p(D)u = e^{-2x} + x^2 + \cos x$$

You are **NOT** required to find the values of the undetermined coefficients.

Exponential times polynomial: remaining case

Lemma 3:

If $u(x) = w(x)e^{\mu x}$ then:

$$p(D)u = e^{\mu x} q(D)w \quad \text{where} \quad q(z) = \sum^{m} j = 0 p^{(j)}(\mu)\frac{z^j}{j!}.$$

Theorem 10:

Let $L = p(D)$ and assume $p(\mu) = p'(\mu) = \ldots = p^{(k-1)}(\mu) = 0$, but $p^{(k)}(\mu) \neq 0$, where $1 \leq k \leq m-1$. For any integer $r \geq 0$, there exists a unique polynomial v of degree r such that $u_p(x) = x^k v(x) e^{\mu x}$ satisfies $Lu_p = x^r e^{\mu x}$.

Variation of parameters

Consider a linear, second-order, inharmonious ODE **with leading coefficient 1:**

$$Lu = u''(x) + p(x)u(x) + q(x)u(x) = f(x)$$

If you stare very very hard, the two solutions are of the form:

$$v_1'(x) = \frac{-u_2(x)f(x)}{W(x)}; \qquad v_2'(x) = \frac{u_1(x)f(x)}{W(X)}.$$

where:

$$u(x) = v_1(x)u_1(x) + v_2(x)u_2(x).$$

Example

MATH2121 2018 T2 (1i)

Use the variation of parameters method to solve:

$$y'' - 2y' + y = e^x \cos x$$

Series Solution

Consider an initial-value problem:

$$Lu = (1 - x^2)u'' - 5xu' - 4u = 0, u(0) = 1, u'(0) = 2.$$

Looking for a solution in the form of a power series:

$$u(x) = \sum_{k=0}^{\infty} A_k x^k = A_0 + A_1 x + A_2 x^2 + ...$$

Formal calculation's show that:

$$Lu = \sum_{k=0}^{\infty} (k+2)[(k+1)A_{k+2} - (k+2)A_k]x^k,$$

where the A_0 and A_1 are derived from the initial conditions. Since Lu is identically zero iff the coefficient of x^k vanishes for every k, we obtain the **recurrence relation:**

$$A_{k+2} = \frac{k+2}{k+1}A_k \quad \text{for } k = 0, 1, 2, ...$$

General Case

Power series provide a flexible way to represent u when L has variable coefficients. In general, given $u'' + p(x)u' + q(x)u = 0$ where $p(x) = \frac{a_1(x)}{a_2(x)}$ and $q(x) = \frac{a_0(x)}{a_2(x)}$:

Assume that a_j is **analytic** at 0 for $0 \leq j \leq 2$, and that $a_2(0) \neq 0$. Then, p and q are analytic at 0, that is, they admit power series expansions:

$$p(z) = \sum_{k=0}^{\infty} p_k z^k \quad \text{and} \quad q(z) = \sum_{k=0}^{\infty} q_k z^k \quad \text{for } |z| < \rho$$

for some $\rho = 0$.

Formal Expansions

If:

$$u(z) = \sum_{k=0}^{\infty} A_k z^k$$

then we find that:

$$Lu(z) = (2A_2 + p_0 A_1 + q_0 A_0) + 6(A_3 + 2p_0 A_2 + p_1 A_1 + q_0 A_1 + q_1 A_0)z + \ldots$$

where, on the RHS, the coefficeint of z^{n-1} for a general $n \geq 1$ is:

$$(n+1)nA_n + \sum_{j=0}^{n-1} [(n-j)p_j A_{n-j} + q_j A_{n-1-j}]$$

Convergence Theorem

Given $u(0)$ and $u'(0)$, we put

$$A_0 = u(0); \qquad A_1 = u'(0),$$

and compute recursively:

$$A_{n+1} = \frac{-1}{n(n+1)} \sum_{j=0}^{n-1} [(n-k)p_j A_{n-j} + q_j A_{n-1-j}], \qquad n \geq 1.$$

Theorem 11:

If the coefficients $p(z)$ and $q(z)$ are analytic for $|z| < \rho$, then the formal power series for the solution $u(Z)$, constructed above, is also analytic for $|z| < \rho$.

Expansion about a point

Expanding around the initial conditions $x = c$, a simple change of the independent variable allows us to write:

$$u = \sum_{k=0}^{\infty} A_k (z-c)^k = \sum_{k=0}^{\infty} A_k Z^k \quad \text{where } Z = z - c$$

Then,

$$\frac{d^2 u}{dZ^2} + p(Z+c)\frac{du}{dZ} + q(Z+c)u = 0.$$

Example

MATH2121 2018 T2 (1iii)

We aim to construct a series solution to the ODE about the ordinary point $x_0 = 0$:

$$(1 - x^2)y'' - 2xy' + 20y = 0, y(0) = 1, y'(0) = 0,$$

of the form:

$$y(x) = \sum_{n=0}^{\infty} A_n x^n$$

1. Give the recurrence relation for the coefficients A_n,
2. Explain from the recurrence relation that one of the series will terminate yielding a polynomial solution, and the other does not.
3. Write down the polynomial solution.

Example

MATH2221 2015 T2 (1iii)

Consider the ODE:

$$(1+z^2)u'' - zu' - 3u = 0.$$

1. Find the recurrence relation satisfied by the coefficients A_k in any power series solution:

$$u = \sum_{k=0}^{\infty} A_k z^k.$$

2. Show that $A_5 = A_7 = A_9 = \ldots = 0$.
3. Hence find the solution for which $u(0) = 0$ and $u'(0) = 6$.

Singular/Cauchy-Euler ODEs

For singular ODEs, it suffices to consider the case when the leading coefficient vanishes at the origin.

A second-order **Cauchy-Euler ODE** has the form:

$$Lu = ax^2u'' + bxu' + cu = f(x),$$

where a, b and c are constants, with $a \neq 0$. This ODE is singular at $x = 0$. Noticing that:

$$Lx^r = [ar(r-1) + br + c]x^r,$$

we see that $u = x^r$ is a solution of the homogeneous equation ($f = 0$) iff:

$$ar(r-1) + br + c = 0.$$

Factorisation

Suppose $ar(r-1)+br+c = a(r-r1)(r-r2)$. If $r_1 \neq r_2$, then the general solution of the homogenous equation $Lu = 0$ is:

$$u(x) = C_1x^{r_1} + C_2x^{r_2}, \qquad x > 0.$$

Lemma 4:

If $r_1 = r_2$, then the general solution of the homogeneous Cauchy-Euler equation $Lu = 0$ is:

$$u(x) = C_1x^{r_1} + C_2x^{r_2} log(x), \qquad x > 0.$$

Example

MATH2121 2016 T2 (2i)

Find the general solution of the Cauchy-Euler ODE

$$2x^2y'' + 7xy' - 3y = 13x^{\frac{1}{4}} \quad \text{for } x > 0.$$

More general singular ODEs

A number of important applications lead to ODEs that can be written in the **Frobenious normal form**:

$$z^2 u'' + zP(z)u' + Q(z)u = 0,$$

where $P(z)$ and $Q(z)$ are analytic at $z = 0$:

$$P(z) = \sum_{k=0}^{\infty} P_k z^k; \qquad \sum_{k=0}^{\infty} Q_k z^k, \qquad |z| < \phi.$$

Here, in general, we cannot expect a solution $u(z)$ to be analytic at $z = 0$.

General case

For equations in the **Frobenious normal form**, formal manipulations show that $Lu(z)$ equals:

$$I(r)A_0z^r + \sum_{k=1}^{\infty}\left(I(k+r)A_k + \sum_{j=0}^{k-1}[(j+r)P_{k-j} + Q_{k-j}]A_j\right) z^{k+r}$$

so we define $A_0(r) = 1$ and:

$$A_k(r) = \frac{-1}{I(k+r)}\sum_{j=0}^{k-1}[(j+r)P_{k-j} + Q_{k-j}]A_j(r), \quad k \geq 1,$$

provided $I(k+r) \neq 0 \quad \forall k \geq 1.$

Bessel and Legendre Equations

The **Bessel equation with parameter** ν is:

$$z^2 u'' + zu' + (z^2 - v^2)u = 0.$$

This ODE is in Frobenious normal form, with indicial polynomial:

$$I(r) = (r + \nu)(r - \nu),$$

and we seek a series solution:

$$u(z) = \sum_{k=0}^{\infty} A_k z^{k+r}.$$

We assume $\Re(\nu) \geq 0$, so $r_1 = \nu$ and $r_2 = -\nu$.

Bessel Function

With the normalisation:

$$A_0 = \frac{1}{2^\nu \Gamma(1+\nu)}$$

the series solution is called the **Bessel function of order** ν and is denoted:

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(1+\nu)}\left[1 - \frac{(z/2)^\nu}{1+\nu} + \frac{(z/2)^4}{2!(1+\nu)(2+\nu)} - \ldots\right].$$

And from the functional equation $\Gamma(1+z) = z\Gamma(z)$:

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)k(z/2)^{2k+\nu}}{k!\Gamma(k+1+\nu)}$$

Bessel function of negative order

If ν is not an integer, then a second linearly independent, solution is:

$$J_{-\nu}(z) = \sum^{\infty} k = 0 \frac{(-1)^k (z/2)^{2k-\nu}}{k!\Gamma(k+1-\nu)}.$$

For an integer $\nu = n \in \mathbb{Z}$, since $\Gamma(n+1) = n!$, we have:

$$J_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k+n}}{k!(k+n)!}.$$

Also, since $\frac{1}{\Gamma(z)} = 0$ for $z = 0, -1, -2, \ldots$, we find that J_n and J_{-n} are linearly independent; in fact:

$$J_{-n}(z) = (-1)^n J_n(z).$$

Example

MATH2221 2015 T2 (2ii)

1. Use term-by-term differentiation to prove that for $\nu \in \mathbb{R}$ and $x > 0$:
$$\frac{d}{dx}(x^{\nu} J_{\nu}(x)) = x^{\nu} J_{\nu-1}(x).$$
2. Hence evaluate the definite integral:
$$I = \int_0^1 x^{\frac{7}{2}} J_{\frac{1}{2}}(x).$$

Legendre Equation

The **Legendre equation** with paramater ν is:

$$(1-z^2)u'' - 2zu' + \nu(\nu+1)u = 0.$$

This ODE is not singular at $z = 0$, so the solution has an ordinary Taylor series expansion:

$$u = \sum_{k=0}^{\infty} A_k z^k.$$

The A_k must satisfy:

$$(k+1)(k+2)A_{k+2} - [k(k+1) - \nu(\nu+1)]A_k = 0.$$

The recurrence relation is:

$$A_{k+2} = \frac{(k-\nu)(k+\nu+1)}{(k+1)(k+2)}A_k \quad \text{for } k \geq 0.$$

General Solution

We have:

$$u(z) = A_0 u_0(z) + A_1 u_1(z)$$

where:

$$u_0(z) = 1 - \frac{\nu(\nu+1)}{2!}z^2 + \frac{(\nu-2)\nu(\nu+1)(\nu+3)}{4!}z^{-}\dots$$

and:

$$u_1(z) = z - \frac{(\nu-1)(\nu-2)}{3!}z^3 + \frac{(\nu-3)(\nu-1)(\nu+2)(\nu+4)}{5!}z^5 - \dots$$

Suppose now that $\nu = n$ is a non-negative integer. If n is even, then the series for $u_0(z)$ terminates, whereas if n is odd, then the series for $u_1(z)$ terminates. The terminating solution is then called the **Legendre polynomial** of degree n and is denoted by $P_n(z)$ with the normalisation:

$$P_n(1) = 1.$$

Table of Contents

Dynamical Systems

State variables are natural variables which depending on a single independent variable. A **dynamical system** is a natural process described by these state variables. The state of a system at a given time is described by the values of the state variables at that instant.

Note that any nth order ODE can be writen as a system of **first order** ODEs (not vice versa):

$$\frac{d^n y}{dt^n} = g\left(y, \frac{dy}{dt}, ..., \frac{d^{n-1}y}{dt^{n-1}}\right)$$
$$\frac{dx}{dt} = f(x_1, x_2, ..., x_n)$$

Non-autonomous ODEs

Definition 7:

A system of ODEs of the form:

$$\frac{dx}{dt} = F(x)$$

is said to be **autonomous**.

Definition 8:

In a **non-autonomous system, F may depend explicitly on t:**

$$\frac{dx}{dt} = F(x, t).$$

Predator and Prey

There are many examples of different predator and prey relationships:

$$\frac{dx}{dt} = -ax + cxy$$
$$\frac{dy}{dt} = by - dxy$$

$$\frac{dx}{dt} = -ax - \alpha x^2 + cxy$$
$$\frac{dy}{dt} = by - \beta y^2 - dxy$$

$$\frac{dx}{dt} = ax + cxy$$
$$\frac{dy}{dt} = by + dxy$$

$$\frac{dx}{dt} = -ax + \frac{c}{m + ky}xy$$
$$\frac{dy}{dt} = by - \frac{d}{m + ky}xy$$

Lipschitz

Defition 9:

$f(x,t)$ is **Lipschitz** in x for $x \in D, t \in I,$ iff $\exists$ a number $L \in \mathbb{R}$ such that:

$$|f(x,t) - f(y,t)| \leq L|x-y|$$

for all $x, y \in D$ and all $t \in I$.

Theorem 12:

If f is Lipschitz, then f is uniformly continuous.

Lemma 5:

If $f(x,t)$ is differentiable in every component x_i of x then $f(x,t)$ satisfies a Lipschitz condition in x.

Existence and Uniqueness Theorem

The initial value problem defined by:

$$\frac{dx}{dt} = f(x,t), \quad (x,t) \in \mathbb{R}^{n+1}$$
$$x(t_0) = x_0$$

has a unique solution $x(t)$ over a time interval $|t - t_0| < \alpha$ if:

$$f(x,t) \text{ is continuous and Lipschitz}$$

i.e.

$$|f(x,t) - f(y,t)| \leq L|x - y|$$

on the domain:

$$D : |t - t_0| \leq a, \quad |x - x_0| \leq b, \quad a, b > 0$$

where:

$$\alpha = min\left(a, \frac{b}{M}\right), \quad M = max|f(x,t)| \text{ on } D$$

Lipschitz Vector Field

A vector field $F : S \to \mathbb{R}^N$ is Lipschitz on $S \subseteq \mathbb{R}^N$ if

$$||F(x) - F(y)|| \leq L||x - y|| \forall x, y \in S.$$

Here,

$$||x|| = \left(\sum^{N} +j - 1x_j^2 \right)^{\frac{1}{2}}$$

denotes the **Euclidean norm** of the vector $x \in \mathbb{R}^N$.

We say that $F(x, t)$ is **Lipschitz in x** if:

$$F(x, t) - F(y, t)|| \leq L||x - y||.$$

Notes

- The existence of solutions follow from continuity in x and t.
- The uniqueness of solutions follow from the Lipschitz condition in x.
- The theorem is a local existence theorem. It provides for the existence of solutions over a finite time interval.

Example

MATH3201

$$f(x,t) = 2\sqrt{x}, \quad x \in \mathbb{R}^1$$

Where is $f(x,t)$ Lipschitz?

MATH3201

$$f(x,t) = 2\sqrt{|x|} \quad x \in \mathbb{R}^2$$

Is f Lipschitz for $|x| > 0$?

Linear systems of ODEs

Definition 10:

We say that the $N \times N$, first-order system of ODEs:

$$\frac{dx}{dt} = F(x, t)$$

is **linear** if the RHS has the form:

$$F(x, t) = A(t)x + b(t)$$

for some $N \times N$ matrix-valued function $A(t) = [a_i j(t)]$ and a vector-valued function $b(t) = [b_i(t)]$.

Global Existence and Uniqueness

We have a stronger existence result in the linear case:

Theorem 13:

If $A(t)$ and $b(t)$ are continuous for $0 \leq t \leq T$, then the linear initial-value problem:

$$\frac{dx}{dt} = A(t)x + b(t) \quad \text{with } x(0) = x_0,$$

has a unique solution $x(t)$ for $0 \leq t \leq T$.

General Solution via Eigen-system

If $Av = \lambda v$ and we defined $x(t) = e^{\lambda t} v$, then:

$$\frac{dx}{dt} = \lambda e^{\lambda t} v = e^{\lambda t}(\lambda v) = e^{\lambda t}(Av) = A(e^{\lambda t} v) = Ax$$

that is, x is a solution of $\frac{dx}{dt} = Ax$. If $Av_j = \lambda_j v_j$ for $1 \leq j \leq N$, then the linear combination:

$$x(t) = \sum_{j=1}^{N} c_j e^{\lambda_j t} v_j$$

is also a solution because the ODE is linear and homogeneous. Provided the v_j are linearly independent, the above equation is the **general solution** because given any $x_o \in \mathbb{R}^N$, there exist unique c_j such that:

$$x(0) = \sum_{j=1}^{N} c_j v_j = x_0.$$

Exponential of a matrix

Let $x(t) = e^{tA}x_0$ where $A \in \mathbb{C}^{N\times N}$, from term by term differentiation, we obtain:

$$\frac{d}{dt}e^{tA} = Ae^{tA},$$

we have:

$$\frac{dx}{dt} = Ae^{tA}x_0 = Ax; \quad x(0) = Ix_0 = x_0$$

Disgonalising a matrix

Definition 11:

A square matrix $A \in C^{N\times N}$ is **diagonalisable** if there exists a non-singular matrix $Q \in \mathbb{C}^{N\times N}$ such that $Q^{-1}AQ$ is diagonal.

Theorem 14:

A square matrix $A \in \mathbb{C}^{N\times N}$ is diagonalisable if and only if there exists a basis $v_1, v_2, ..., v_n$ for $\mathbb{C}^N$ consisting of eigenvectors of A. Indeed, if:

$$Av_j = \lambda_j v_j \quad \text{for } j = 1, 2, ..., N,$$

and we put $A = [v_1 v_2 ... v_n]$, the $Q^{-1}AQ = \Lambda$ where:

$$\Lambda = \begin{pmatrix} \lambda_1 & & \\ & ... & \\ & & \lambda_N \end{pmatrix}$$

Matrix Powers

In general since $Q^{-1}AQ = \Lambda$, we can see by induction on k such that:

$$A^k = Q\Lambda^k Q^{-1} \quad \text{for } k = 0, 1, 2, ...$$

Polynomial of a Matrix

For any polynomial:

$$p(z) = c_0 + c_1 z + c_2 z^2 + \ldots + c_m z^m$$

and any square matrix A, we define:

$$p(A) = c_0 I + c_1 A + c_2 A^2 + \ldots + c_m A_m.$$

When A is diagonalisable, $A^k = Q\Lambda^k Q^{-1}$ so:

$$p(A) = Qp(\Lambda)Q^{-1}.$$

Lemma 6:

For any polynomial p and any diagonal matrix Λ:

$$p(\Lambda) = \begin{pmatrix} p(\lambda_1) & & & \\ & p(\lambda_2) & & \\ & & \dots & \\ & & & p(\lambda_N) \end{pmatrix}$$

Theorem 15:

If two polynomials p and q are equal on the spectrum of a diagonalisable matrix A, that is, if:

$$p(\lambda_j) = q(\lambda_j) \quad \text{for } j = 1, 2, \dots, N.$$

then $p(A) = q(A)$.

Exponential of a diagonalisable matrix

Theorem 16:

If $A = Q\Lambda Q^{-1}$ is diagnonalisable, then:

$$e^A = Qe^\Lambda Q^{-1}; \qquad e^\Lambda = p(\Lambda) = \begin{pmatrix} p(e^{\lambda_1}) & & & \\ & p(e^{\lambda_2}) & & \\ & & \ldots & \\ & & & p(e^{\lambda_N}) \end{pmatrix}$$

Example

MATH2121 2016 T2 (2iii)

For an $n \times n$ matrix A.

1. State the definition of e^A.
2. Show that if $Av = \lambda v$, then $e^A v = e^\lambda v$.

Stability and Equilibrium Points

Definition 12:

We say that $a \in \mathbb{R}^N$ is an **equilibrium point** for the dynamical system $\frac{dx}{dt} = F(x)$ if:

$$F(a) = 0.$$

In this case, the solution of:

$$\frac{dx}{dt} = F(x) \quad \forall t, \text{with } x(0) = a,$$

is just the constant function $x(t) = a$.

Stable Equilibrium

Definition 13:

An equilibrium point a is **stable** if for every $\epsilon > 0$, there exist $\gamma > 0$ such that whenever $||x_0 - a|| < \gamma$, the solution of:

$$\frac{dx}{dt} = F(x) \quad \text{for } t > 0, \text{ with } x(0) = x_0$$

satisfies:

$$||x(t) - a|| < \epsilon \quad \forall t > 0.$$

Asymptotic Stability

Definition 14:

Let D be an open subset of $\mathbb{R}^N$ that contains an equilibrium point a. We say that a is **asymptotically stable** in D if a is stable, and, whenever $x_0 \in D$, the solution of:

$$\frac{dx}{dt} = F(x) \quad \text{for } t > 0, \text{ with } x(0) = x_0$$

satisfies:

$$x(t) \to a \quad \text{as } t \to \infty.$$

In this case, D is called a **domain of attraction** for a.

Criteria for stability

Theorem 17:

Let A be a diagonalisable matrix with eigenvalues $\lambda_1, \lambda_2, ...\lambda_N$. The equilibrium point $a = -A^{-1}b$ is:

1. **Stable** if and only if $\Re(\lambda_j \leq 0)$ for all j.
2. **asymptotically stable** if and only if $\Re(\lambda_j < 0)$ for all j.

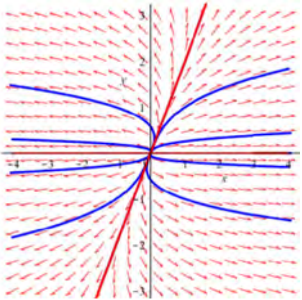
In the second case, the domain of attraction is the whole of $\mathbb{R}^N$.

Classification of 2D Linear Systems

Type	Eigenvalues	Eigenvectors	$X(t)$	Classification
1: $\boldsymbol{B} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$	$\lambda_1 \neq \lambda_2 \in \mathbb{R}$	$\boldsymbol{v}^{(1)}, \boldsymbol{v}^{(2)}$	$\left[\boldsymbol{v}^{(1)} e^{\lambda_1 t} \;\; \boldsymbol{v}^{(2)} e^{\lambda_2 t}\right]$	Improper Node
	$\lambda_1 < 0 < \lambda_2$	$\boldsymbol{v}^{(1)}, \boldsymbol{v}^{(2)}$	$\left[\boldsymbol{v}^{(1)} e^{\lambda_1 t} \;\; \boldsymbol{v}^{(2)} e^{\lambda_2 t}\right]$	Saddle Point
2: $\boldsymbol{B} = \begin{pmatrix} \lambda & \gamma \\ 0 & \lambda \end{pmatrix}$ $\lambda, \gamma \in \mathbb{R}$	$\lambda_1 = \lambda_2 = \lambda \in \mathbb{R}$ (multiplicity 2)	$\boldsymbol{v}^{(1)}, \boldsymbol{v}^{(2)}$ ($\boldsymbol{v}^{(2)}$generalised eigenvector)	$\left[\boldsymbol{v}^{(1)} e^{\lambda t} \;\; (\boldsymbol{v}^{(2)} + t\boldsymbol{v}^{(1)}) e^{\lambda t}\right]$	Deficient Node
	$\lambda_1 = \lambda_2 = \lambda$ (2D eigenspace)	$\boldsymbol{v}^{(1)}, \boldsymbol{v}^{(2)}$ any basis of $\mathbb{R}^2$	$\left[\boldsymbol{v}^{(1)} e^{\lambda_1 t} \;\; \boldsymbol{v}^{(2)} e^{\lambda_2 t}\right]$	Star (or proper) Node
3: $\boldsymbol{B} = \begin{pmatrix} \alpha & -\omega \\ \omega & \alpha \end{pmatrix}$ $\alpha, \omega \in \mathbb{R}$	$\lambda_1 = i\beta = \overline{\lambda_2}$ $(\beta \neq 0) \in \mathbb{R}$	$\boldsymbol{v}^{(2)} = \overline{\boldsymbol{v}^{(1)}}$	$\left[\mathbb{R}\mathrm{e}\left(\boldsymbol{v}^{(1)} e^{i\beta t}\right) \;\; \mathbb{I}\mathrm{m}\left(\boldsymbol{v}^{(2)} e^{i\beta t}\right)\right]$	Centre (or vortex)
	$\lambda_1 = \alpha + i\beta = \overline{\lambda_2}$ $(\alpha \neq 0, \beta \neq 0) \in \mathbb{R}$	$\boldsymbol{v}^{(2)} = \overline{\boldsymbol{v}^{(1)}}$	$\left[\mathbb{R}\mathrm{e}\left(\boldsymbol{v}^{(1)} e^{(\alpha+i\beta)t}\right) \;\; \mathbb{I}\mathrm{m}\left(\boldsymbol{v}^{(2)} e^{(\alpha+i\beta)t}\right)\right]$	Spiral point (or focus)

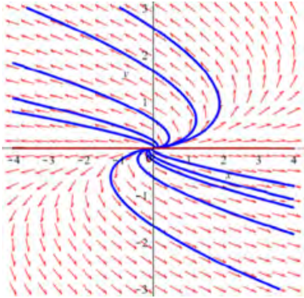
Improper Node

Here $\lambda_1 \neq \lambda_2 \in \mathbb{R}$. Which eigenline orbits tend to depends on the eigenvalues.



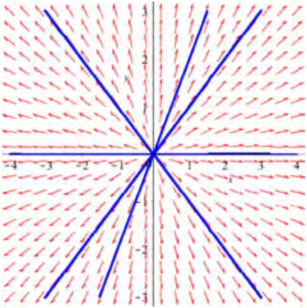
Deficient Node

Here $\lambda_1 = \lambda_2 \in \mathbb{R}$, and the eigenspace is deficient (out of the scope of the course).



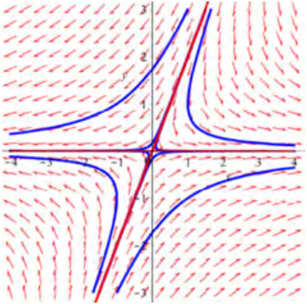
Star Node

Here $\lambda_1 = \lambda_2 \neq 0$ and all nonzero vectors are eigenvectors. Therefore, all orbits are either being attracted or repelled by the equilibrium points.



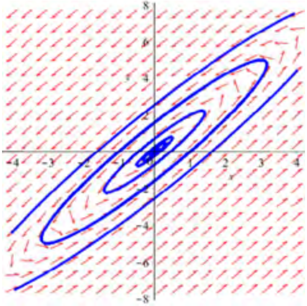
Saddle Point

Here, $\lambda_1 < 0 < \lambda_2$. This means λ_1 is attracting, whilst λ_2 is repelling.



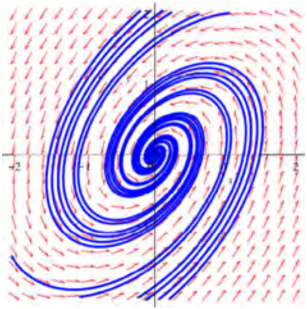
Centre

Here $\lambda_1 = i\beta = \lambda_2$, where $\beta \in \mathbb{R}$. If eigenvalues are purely imaginary, orbits are given by ellipses around the eigen-plane as $e^{it} = \cos(t) + i\sin(t)$.



Spiral

Here $\lambda_1 = \alpha + i\beta = \bar{\lambda}_2$. If eigenvalues are in conjugate pairs, real parts of the orbit will be defined by $e^{-\lambda t} = e^{-\Re(\lambda)t}(c_1 \cos(t) + c_2 \sin(t))$, which will spiral inward or outward, clockwise or anticlockwise depending on values of $\Re(\lambda)$ and $\Im(\lambda)$.



Example

MATH2121 2018 T2 (2iii)

Solve for $x(t)$ and $y(t)$ and determine the type and stability of the equilibrium point of the following system of differential equations:

$$\frac{dx}{dt} = x + y;$$
$$\frac{dy}{dt} = 2x.$$

Linearization

Suppose that x_0 is close to an equilibrium point a. If:

$$\frac{dx}{dt} = F(x) \forall t, \text{ with } x(0) = x_0,$$

then for small t the difference $y = x - a$ is a small and satisfies:

$$\frac{dy}{dt} = \frac{dx}{dt} = F(x) = F(a+y) \approx F(a) + F'(a)y.$$

This suggests that if $y_0 = x_0 - a$ and y is the solution of the **linear** dynamical system:

$$\frac{dx}{dy} = F(a) + F'(a)y \quad \forall t, \text{ with } y(0) = y_0,$$

then $x(t) \approx a + y(t)$ for small t. In particular, we can infer stability properties at an equilibrium point a from the eigenvalues of $A = F'(a)$.

First Integrals

Definition 15:

A function $G : \mathbb{R}^N \to \mathbb{R}$ is a **first integral** (constant of the motion) for a system of ODEs:

$$\frac{dx}{dt} = F(x)$$

if $G(x(t))$ is constant for every solution $x(t)$.

Geometrically: G is a first integral iff:

$$\bigtriangledown G(x) \perp F(x) \text{ for all } x.$$

Table of Contents

Inner Products and Norms

Definition 16

The **inner product** of two function f, g defined on $[a, b]$ is

$$\langle f, g\rangle = \int_a^b f(x)g(x)\,dx.$$

Definition 17

The **norm** of a function f defined on $[a, b]$ is

$$\|f\| = \sqrt{\langle f, f\rangle} = \sqrt{\int_a^b (f(x))^2\ dx}.$$

Inner Products and Norms

Example 1

Let $f(x) = \sin x$ and $g(x) = \cos x$ be defined on $[0, \pi]$. Find $\langle f, g\rangle$ and $\|f\|$.

We have

$$\begin{aligned}\langle f, g\rangle &= \int_0^\pi \sin x \cos x \, dx \\ &= \frac{1}{2}\int_0^\pi \sin 2x \, dx \\ &= \frac{1}{4}[-\cos 2x]_0^\pi \\ &= 0.\end{aligned}$$

$$\begin{aligned}\|f\| &= \sqrt{\int_0^\pi \sin^2 x \, dx} \\ &= \sqrt{\frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^\pi} \\ &= \sqrt{\frac{\pi}{2}}.\end{aligned}$$

Adjoint Operators

Definition 18

Let

$$Lu = a_2 u'' + a_1 u' + a_0 u$$

be a linear second-order differential operator. Then the **formal adjoint** L^* is

$$L^* v = (a_2 v)'' + (a_1 v)' + a_0 v.$$

Adjoint Operators

Example 2

Let

$$Lu = xu'' + e^x u' + (x^2 + 1)u.$$

Find the formal adjoint.

The formal adjoint is

$$\begin{aligned} L^* v &= (xv)'' + (e^x v)' + (x^2 + 1)v \\ &= (xv' + v)' + e^x v' + e^x v + (x^2 + 1)v \\ &= xv'' + 2v' + e^x v' + e^x v + (x^2 + 1)v \\ &= xv'' + (e^x + 2)v' + (e^x + x^2 + 1)v. \end{aligned}$$

Lagrange Identity

Theorem 18

Let L be a linear second-order operator on $[a, b]$, and define a_0, a_1, a_2 as previously. Then

$$\langle Lu, v\rangle = \langle u, L^* v\rangle + [P(u, v)]_a^b\,,$$

where P is the **bilinear concomitant**

$$P(u, v) = u'(a_2 v) - u(a_2 v)' + u(a_1 v).$$

Self-adjoint Operators

Definition 19

An operator L is **formally self-adjoint** if $L = L^*$.

Theorem 19

A second-order linear differential operator L is formally self-adjoint if and only if it can be written as

$$Lu = -(pu')' + qu$$

for some p, q.

Using the integrating factor $e^{\int \frac{a_1}{a_2}\,dx}$, any second-order linear operator can be written in self-adjoint form.

Self-adjoint Operators

Example 3

Write $Lu = xu'' + xu' - 2u = f$ in formally self-adjoint form.

The integrating factor is e^x, so

$$
\begin{aligned}
& & xu'' + xu' - 2u &= f \\
&\iff & u'' + u' - \frac{2}{x}u &= \frac{f}{x} \\
&\iff & e^x u'' + e^x u' - \frac{2e^x}{x}u &= \frac{e^x f}{x} \\
&\iff & \left(e^x u'\right)' - \frac{2e^x}{x}u &= \frac{e^x f}{x}.
\end{aligned}
$$

So, we have $Lu = -(pu')' + qu = g$, where

$$
p = -e^x, \qquad q = \frac{2e^x}{x}, \qquad g = -\frac{e^x f}{x}.
$$

Lagrange Identity for Self-adjoint Operators

Theorem 20

Let $Lu = -(pu')' + qu$ be formally self-adjoint. Then

$$\langle Lu, v\rangle - \langle u, Lv\rangle = \sum_{i=1}^{2} \left(B_i u R_i v - R_i u B_i v\right),$$

where

$$\begin{aligned} B_1 u &= b_{11} u'(a) + b_{10} u(a), \\ B_2 u &= b_{21} u'(b) + b_{20} u(b), \\ R_1 u &= \frac{p(a)u(a)}{b_{11}} = -\frac{p(a)u'(a)}{b_{10}}, \\ R_2 u &= -\frac{p(b)u(b)}{b_{21}} = \frac{p(b)u'(b)}{b_{20}}. \end{aligned}$$

Fredholm Alternative

Theorem 21 (Fredholm Alternative)

Consider the system

$$\begin{aligned} Lu &= f && \text{for } a < x < b, \\ b_{11}u' + b_{10}u &= \alpha_1 && \text{at } x = a, \\ b_{21}u' + b_{20}u &= \alpha_2 && \text{at } x = b. \end{aligned}$$

1. If the homogeneous system (that is, $f \equiv 0$ and $\alpha_1 = \alpha_2 = 0$) has only the trivial solution, then the inhomogeneous system has a unique solution for every choice of f, α_1, α_2.
2. Else, the inhomogeneous system has a solution if and only if

$$\langle f, v \rangle = \alpha_1 R_1 v + \alpha_2 R_2 v$$

for every solution v to the homogeneous system.

Fredholm Alternative

Example 4

Under what conditions does

$$Lu = u'' + u = f$$

where $u(0) = \alpha_1$, $u(\pi) = \alpha_2$ have a solution?

We can easily see that $v = A\sin x$ is a non-trivial solution to the homogeneous system. So, we apply the Fredholm alternative, to deduce that there is a solution if and only if

$$\langle f, v\rangle = A\int_0^\pi f(x)\sin x\,dx = A\alpha_1 + A\alpha_2.$$

That is,

$$\int_0^\pi f(x)\sin x\,dx = \alpha_1 + \alpha_2.$$

Table of Contents

Fourier Series

You may have already seen trigonometric Fourier series. If not:

Definition 20

Suppose $f : \mathbb{R} \to \mathbb{R}$ is bounded, piecewise continuous, and $2L$-periodic. Then the n^{th} **real Fourier polynomial** of f is

$$(S_n f)(x) = \frac{a_0[f]}{2} + \sum_{k=1}^{n} \left(a_k[f] \cos \frac{k\pi x}{L} + b_k[f] \sin \frac{k\pi x}{L} \right),$$

where

$$a_0[f] = \frac{1}{L} \int_{-L}^{L} f(x)\, dx,$$

and for $1 \leq k \leq n$,

$$a_k[f] = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{k\pi x}{L}\, dx, \quad b_k[f] = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{k\pi x}{L}\, dx.$$

Fourier Series

A couple of results help us when dealing with even or odd functions.

Theorem 22

If f is an even function, then

$$a_0[f] = \frac{2}{L}\int_0^L f(x)\,dx, \quad a_k[f] = \frac{2}{L}\int_0^L f(x)\cos\frac{k\pi x}{L}\,dx,$$

and $b_k[f] = 0$.
If f is an odd function, then

$$b_k[f] = \frac{2}{L}\int_0^L f(x)\sin\frac{k\pi x}{L}\,dx,$$

and $a_0[f] = a_k[f] = 0$.

Fourier Series

A **Fourier series** is the limit of the sequence $\{S_n f\}_{n=1}^{\infty}$ denoted Sf.

Example 5

Find the real Fourier series of f where $f(x) = x$ for $-1 < x \leq 1$, and requiring that $f(x+2) = f(x)$ for all $x \in \mathbb{R}$.

So f is 2-periodic, and odd. Thus $a_0[f] = a_k[f] = 0$. Then

$$b_k[f] = 2\int_0^1 x \sin k\pi x \, dx = 2\frac{\sin \pi k - \pi k \cos \pi k}{\pi^2 k^2}.$$

Now, $\sin \pi k = 0$ and $\cos \pi k = (-1)^k$ as k is an integer, so

$$(Sf)(x) = \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin k\pi x.$$

Fourier Series

At jump discontinuities, the Fourier series approaches the average of the function value either side of the discontinuity.

Theorem 23

Suppose f has a jump discontinuity at a. That is, the limits

$$f(a^+) = \lim_{x \to a^+} f(x), \quad f(a^-) = \lim_{x \to a^-} f(x)$$

both exist, but $f(a^+) \neq f(a^-)$. Then

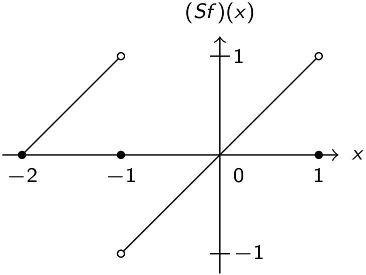
$$\lim_{n \to \infty} (S_n f)(a) = \frac{f(a^+) + f(a^-)}{2}.$$

This requires the one-sided derivatives existing at a. For the functions you deal with, this will almost always be the case.

Fourier Series

Example 6

For the previous example, draw the graph of $(Sf)(x)$ for $-2 \leq x \leq 1$



Wherever the function is continuous, we can simply draw f. At each jump discontinuity, the Fourier series approaches the average of the function, in this case 0.

Half-Range Expansions

We can extend functions that are defined on an interval $[0, L]$ to an interval $[-L, L]$ to find their Fourier series as follows.

Definition 21

Suppose f is defined on the interval $[0, L]$. Then the

1. **Odd extension** of f is defined to be

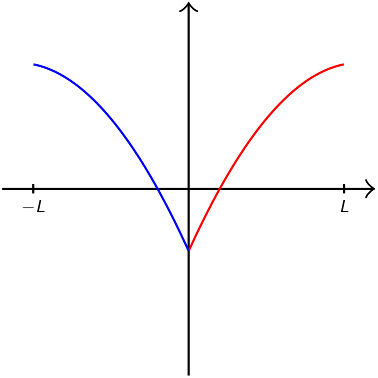
$$g(x) = \begin{cases} f(x), & 0 < x \leq L; \\ 0, & x = 0; \\ -f(-x), & -L \leq x < 0. \end{cases}$$

2. **Even extension** of f is defined to be

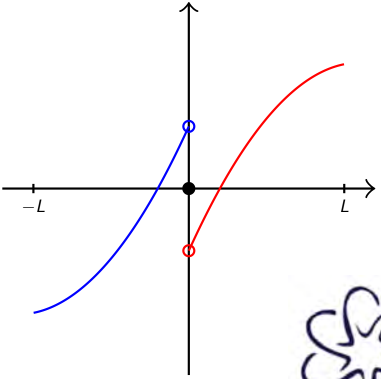
$$g(x) = \begin{cases} f(x), & 0 \leq x \leq L; \\ f(-x), & -L \leq x < 0. \end{cases}$$

Half-Range Expansions

Even Extension



Odd Extension



Half-Range Expansions

Example 7

Find the Fourier cosine series of f, where $f(x) = x$ for $0 < x \leq 1$.

To begin, we find the even expansion of f:

$$g(x) = \begin{cases} x, & 0 \leq x \leq 1; \\ -x, & -L \leq x < 0 \end{cases} = |x|.$$

So, we find

$$a_k[g] = 2\int_0^1 |x| \cos k\pi x dx = \begin{cases} \frac{-4}{k^2\pi^2}, & k \text{ odd}; \\ 0, & k \text{ even}. \end{cases}$$

Also, $a_0[g] = 1$, and $b_k[g] = 0$. Thus,

$$(Sg)(x) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{k \text{ odd}} \frac{\cos k\pi x}{k^2}.$$

Orthogonality of Functions

Definition 22

The **inner product** of two functions f and g, on the interval (a, b), and with respect to the weight function w, is

$$\langle f, g \rangle_w = \int_a^b w(x) f(x) g(x)\, dx,$$

where $w(x) > 0$ on (a, b).

Definition 23

Two functions f and g are **orthogonal** over (a, b) with respect to the weight w if

$$\langle f, g \rangle_w = 0.$$

If it is obvious from context, we can usually omit the interval.

Orthogonality of Sets of Functions

We can extend this notion to sets of functions.

Definition 24

A set of functions $\{f_1, f_2, \cdots\}$ is **orthogonal** on the interval (a, b) and with respect to the weight function w if for all $n \neq m$ we have

$$\langle f_n, f_m \rangle_w = 0.$$

That is, every function is orthogonal (with respect to w) to every other function.

Orthogonality of Sets of Functions

Example 8 (2018 MATH2221 Problem Set 8, 3i)

Show that

$$y_n(x) = \sin\left(\frac{n\pi x}{a}\right), \qquad n = 1, 2, 3, \cdots$$

is an orthogonal set on $[0, a]$ and with respect to $w(x) = 1$.

If $n \neq m$, we have

$$\begin{aligned}
\langle y_n, y_m \rangle_1 &= \int_0^a \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right)\,dx \\
&= \frac{1}{2}\int_0^a \cos\left(\frac{(n-m)\pi x}{a}\right) - \cos\left(\frac{(n+m)\pi x}{a}\right)\,dx \\
&= 0.
\end{aligned}$$

Thus, the set is orthogonal.

Norms

Finally, for notational convenience, we define the norm.

Definition 25

The **norm** of a function f over the interval (a, b) with respect to the weight function w is

$$\|f\|_w = \sqrt{\langle f, f\rangle_w} = \sqrt{\int_a^b w(x)\,(f(x))^2\,dx}.$$

Norms

Example 9

Using the orthogonal set from the previous example, find $\|y_n\|_1$.

$$\begin{aligned}\|y_n\|_1 &= \sqrt{\int_0^a \sin^2\left(\frac{n\pi x}{a}\right)\,dx}\\ &= \sqrt{\frac{1}{2}\int_0^a 1-\cos\left(\frac{2n\pi x}{a}\right)\,dx}\\ &= \sqrt{\frac{1}{2}\left[x-\frac{a}{2n\pi}\sin\left(\frac{2n\pi x}{a}\right)\right]_0^a}\\ &= \sqrt{\frac{a}{2}}.\end{aligned}$$

Generalised Fourier Series

With these definitions, we can finally define the generalised Fourier series of a function more generally.

Definition 26

Let $\{\phi_1, \phi_2, \cdots\}$ be an orthogonal set of functions over (a, b) with respect to w. The **generalised Fourier series** of a function f is

$$F(x) = \sum_{n=1}^{\infty} c_n \phi_n(x),$$

where

$$c_n = \frac{\langle f, \phi_n \rangle_w}{\|\phi_n\|_w^2}.$$

Those who have done linear algebra may recognise this as least-squares regression.

Completeness (MATH2221 only)

Completeness of an orthogonal set is important when using generalised Fourier series. A Fourier series using an incomplete set may not accurately represent the function.

Definition 27

An orthogonal set S is called **complete** if no non-trivial function $f \in L_2(a, b, w)$ (functions on (a, b) for which $\|f\|_w$ exists and is finite) is orthogonal to every function in S. That is, if, for every $\phi \in S$,

$$\langle f, \phi \rangle_w = 0,$$

then

$$\|f\|_w = 0.$$

This can be thought of as a set that forms a basis for $L_2(a, b, w)$.

Parseval's Identity

Theorem 24 (Parseval's Identity)

Let $\{\phi_1, \phi_2, \cdots\}$ be a complete orthogonal set. If A_k denotes the k^{th} Fourier coefficient of f, then

$$\|f\|_w^2 = \sum_{k=1}^{\infty} A_k^2 \|\phi_k\|_w^2.$$

For those in MATH2221, the converse holds. If Parseval's identity is true for every function $f \in L_2(a, b, w)$, then the orthogonal set is complete.

Parseval's Identity

Example 10

Find the Fourier coefficients for $f(x) = x^2$ over $(-\pi, \pi)$, and hence show

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

First, we find the coefficients. Since f is even, we have $b_k[f] = 0$, and

$$\begin{aligned} a_0[f] &= \frac{2}{\pi}\int_0^{\pi} x^2\,dx &&= \frac{2\pi^2}{3}, \\ a_k[f] &= \frac{2}{\pi}\int_0^{\pi} x^2 \cos kx\,dx &&= \frac{4(-1)^k}{k^2}. \end{aligned}$$

Parseval's Identity (cont.)

To apply Parseval's Identity, we find

$$\|1\|_1^2 = 2\int_0^\pi 1^2\,dx \qquad = 2\pi,$$
$$\|\cos k\pi\|_1^2 = 2\int_0^\pi \cos^2 kx\,dx = \pi.$$

Then we also calculate

$$\|f\|_1^2 = 2\int_0^1 x^4\,dx = \frac{2\pi^5}{5}.$$

Finally, using Parseval's Identity, we find

$$\frac{2\pi^5}{5} = \left(\frac{2\pi^2}{2\cdot 3}\right)^2 \cdot 2\pi + \sum_{k=1}^{\infty}\left(\frac{4(-1)^k}{k^2}\right)^2 \cdot \pi.$$

Rearranging this gives the result.

Sturm-Liouville Equations

Definition 28

Let p, q, r be all real-valued functions, with

$$p(x) > 0, \qquad r(x) > 0$$

over the interval (a, b). An ODE in the form

$$\left(p(x)u'\right)' + (\lambda r(x) - q(x))\, u = 0, \qquad a < x < b$$

is called a **Sturm-Liouville equation**.

Throughout this subsection, we will assume p, q, r satisfy the conditions in the definition above.

Eigenstuffs

If we let $Lu = -\left(p(x)u'\right)' + q(x)u$ be a self-adjoint operator, then the general Sturm-Liouville equation can be written as $Lu = \lambda ru$. Using this form, we can define a few terms.

Definition 29

Write $Lu = \lambda ru$ over (a, b). Then a non-trivial solution ϕ to this equation is said to be an **eigenfunction** of L with **eigenvalue** λ. Then (ϕ, λ) is referred to as an **eigenpair**.

Sturm-Liouville Equations

Example 11

Write Legendre's equation

$$(1 - x^2)u'' - 2xu' + \nu(\nu + 1)u = 0$$

as a Sturm-Liouville equation and identify the eigenvalue. Over what domain is this equation Sturm-Liouville?

We can write the equation as

$$\left((1 - x^2)u'\right)' + (\nu(\nu + 1) - 0)u = 0,$$

which, using the previous notation, has

$$p(x) = 1 - x^2, \qquad q(x) = 0, \qquad r(x) = 1,$$

and eignvalue $\lambda = \nu(\nu + 1)$.
We require $p(x) > 0$, which is true over the interval $(0, 1)$.

Regular Sturm-Liouville Eigenproblems

Definition 30

A **regular Sturm-Liouville eigenproblem** is of the form

$$\begin{aligned} Lu &= \lambda r u && \text{for } a < x < b, \\ B_1 u = b_{11} u' + b_{10} u &= 0 && \text{at } x = a, \\ B_2 u = b_{21} u' + b_{20} u &= 0 && \text{at } x = b, \end{aligned}$$

where $a, b, b_{10}, b_{11}, b_{20}, b_{21}$ are all (finite) reals, with

$$\begin{aligned} p(a) &\neq 0 \quad \text{and} & p(b) &\neq 0, \\ |b_{10}| + |b_{11}| &\neq 0 \quad \text{and} & |b_{20}| + |b_{21}| &\neq 0. \end{aligned}$$

Regular Sturm-Liouville Eigenproblems

Example 12

Consider the Sturm-Liouville problem $u'' + \lambda u = 0$ where $u(0) = u(1) = 0$. Find the eigenpairs.

This Sturm-Liouville eigenproblem is regular, defined on $(0, 1)$. We first try to find possible values of λ. Let $k > 0$, then

1. If $\lambda = 0$, we find $u = Ax + B$, which yields $u \equiv 0$ with the initial conditions.
2. If $\lambda = k^2 > 0$, we find $u = A \sin kx + B \cos kx$, and we finally get a non-trivial solution, $u = A \sin n\pi x$ for $n \in \mathbb{Z}^+$. This means $\lambda = n^2\pi^2$.
3. If $\lambda = -k^2 < 0$, we find $u = A \cosh kx + B \sinh kx$, which again yields $u \equiv 0$.

So, our eigenpairs are $(\sin n\pi x, n^2\pi^2)$.

Regular Sturm-Liouville Eigenproblems

There are a few properties of these kinds of problems that are important.

Theorem 25

The eigenvalues from a regular Sturm-Liouville eigenproblem are real.

Theorem 26

The eigenfunctions $\phi_1, \phi_2, \phi_3, \cdots$ from a regular Sturm-Liouville eigenproblem are orthogonal with respect to r if and only if their eigenvalues are distinct. Further, their eigenvalues satisfy (after reordering)

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots ,$$

with $\lambda_k \to \infty$ as $k \to \infty$.

Regular Sturm-Liouville Eigenproblems

Example 13

Show that the eigenfunctions from the previous example are orthogonal.

We can reason that the eigenvalues are all distinct, and they come from a regular Sturm-Liouville eigenproblem, but calculating the inner products, we find for $n \neq m$,

$$\begin{aligned}
\langle \phi_n, \phi_m \rangle &= \int_0^1 \sin n\pi x \sin m\pi x \, dx \\
&= \frac{1}{2} \int_0^1 \left(\cos{(n-m)\pi x} - \cos{(n+m)\pi x} \right) \, dx \\
&= \frac{1}{2} \left[\frac{\sin{(n-m)\pi x}}{(n-m)\pi} - \frac{\sin{(n+m)\pi x}}{(n+m)\pi x} \right]_0^1 \\
&= 0.
\end{aligned}$$

Singular Sturm-Liouville Eigenproblems

Theorem 27

Consider the singular Sturm-Liouville eigenproblem

$$x^2 u'' + xu' + \left(\lambda x^2 - \nu^2\right) u = 0 \quad \text{for } 0 < x < l,$$
$$u(x) \text{ bounded with } xu'(x) \to 0 \text{ as } x \to 0^+,$$
$$c_1 u' + c_0 u = 0 \text{ at } x = l,$$

where $c_0 c_1 \geq 0$. The solution eigenpairs are

$$\left(J_\nu(k_j x), k_j^2\right) \text{ for } j \geq 1,$$

where k_j is the j^{th} positive solution to

$$c_0 J_\nu(kl) + c_1 k J_\nu'(kl) = 0.$$

If $c_0 = \nu = 0$, then $(1, 0)$ is an additional eigenpair.

Singular Sturm-Liouville Eigenproblems

Example 14

Find all eigenpairs of

$$x^2u'' + xu' + (\lambda x^2 - 1)u = 0 \text{ for } 0 < x < 1,$$
$$u'(1) = 0,$$

where u is bounded at 0.

Let k_j be the j^{th} positive solution to

$$kJ_1'(k) = 0.$$

Then the eigenpairs are

$$(J_1(k_jx), k_j^2).$$

Table of Contents

Separation of Variables

To solve IBPs in two (or more) variables, it is usually helpful to assume the solution is made up of univariate parts, and solve each one independently. This generally results in more than one solution, which can be superimposed to find the general solution.

Separation of Variables

Example 15

Use the separation of variables $u(x,t) = X(x)T(t)$ to find two ODEs for X and Y for

$$\nabla^2 u = 0.$$

Let $u(x,t) = X(x)T(t)$, so that

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial t^2} = X''T + XT'' = 0.$$

This can be rearranged so that, for some constant p,

$$\frac{X''}{X} = -\frac{T''}{T} = p.$$

That is,

$$X'' - pX = 0, \qquad\qquad T'' + pT = 0.$$

Solving IBPs in 2D

There is a somewhat general method of solving IBPs you will find in this course.

1. Use the separation of variables $u = X(x)T(t)$ (or similar);
2. Determine the conditions on X and T implied by the boundary conditions, which usually result in a separation constant, p;
3. Either
   1. Solve the resulting ODEs for different cases of the separation constant, ($p < 0$, $p = 0$, $p > 0$ usually), or
   2. Solve the resulting ODEs using methods previously taught;
4. Superimpose the resulting solutions to find the general solution.

Solving IBPs in 2D

Example 16

Consider the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \text{ for } 0 \leq x \leq 1, t \geq 0,$$
$$u(0,t) = u(1,t) = 0,$$
$$u(x,0) = f(x),$$

where f is a piecewise continuous function. Find u.

Let $u(x,t) = X(x)T(t)$, so that

$$XT' = X''T \text{ for } 0 \leq x \leq 1, t \geq 0,$$
$$X(0) = X(1) = 0.$$

We can't yet use the last boundary condition.

Solving IBPs in 2D (cont.)

Rearranging, and with some suitable separation constant p, we have

$$X'' - pX = 0 \qquad\qquad T' - pT = 0.$$

Letting $k > 0$, we take three cases:

1. If $p = 0$, then $X'' = 0$, and we find $X \equiv 0$ using the boundary conditions, so we ignore this case.
2. If $p = k^2 > 0$, then $X'' - k^2 X = 0$, and we find that $X \equiv 0$ again, so we ignore this case.
3. If $p = -k^2 < 0$, then $X'' + k^2 X = 0$, and we find that $X = A \sin n\pi x$, where $n \in \mathbb{Z}^+$ (whence $p = -n^2\pi^2$).

So, $p = -n^2\pi^2$, and $X(x) = A \sin n\pi x$.

Solving IBPs in 2D (cont.)

Now we tackle T. Since we know p now, we have

$$T' + n^2\pi^2 T = 0$$

for some $n \in \mathbb{Z}^+$. This can be easily solved to give us

$$T(t) = Ae^{-n^2\pi^2 t}.$$

Putting this together with our solution for X, we get

$$u_n(x,t) = A_n e^{-n^2\pi^2 t} \sin n\pi x.$$

That is,

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-n^2\pi^2 t} \sin n\pi x,$$

for some choice of A_n's.

Solving IBPs in 2D (cont.)

Finally, we can use the initial condition, yielding

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin n\pi x = f(x).$$

This is precisely the Fourier sine series of f, so

$$A_n = 2\int_0^1 f(x) \sin n\pi x \, dx.$$

If we were given an explicit f, we could evaluate this to get the final solution for u.

Laplacian in Polar Coordinates

Sometimes we approach problems which are easier to express in polar coordinates. For example, problems defined on the unit disk, or which are radially symmetric.

Theorem 28

Suppose u is defined in polar coordinates (r, θ). Then

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

Laplacian in Polar Coordinates

Example 17

The temperature $u(r, \theta)$ of a circular surface obeys

$$\nabla^2 u = 0 \text{ for } 0 \leq r < 1,$$
$$u(1, \theta) = f(\theta),$$
$$u \text{ continuous.}$$

Since u is continuous, we require $u(r, \theta) = u(r, \theta + 2\pi)$. Now use the separation of variables $u(r, \theta) = R(r)H(\theta)$. Then,

$$H(\theta) = H(\theta + 2\pi),$$
$$r^2 R'' H + rR' H + RH'' = 0.$$

That is, for a suitable constant p,

$$r^2 R'' + rR' - pR = 0, \qquad H'' + pH = 0.$$

Laplacian in Polar Coordinates (cont.)

Since the ODE for R is Cauchy-Euler, we find H first (so we know p). Considering all three cases, the only possible periodic case is if H is constant ($p = 0$), or when $p > 0$, say $p = k^2$ where $k > 0$. This gives us

$$H = A \sin k\theta + B \cos k\theta.$$

Using our periodic condition, we find

$$B = A \sin 2k\pi + B \cos 2k\pi,$$

from which we deduce $k \in \mathbb{Z}^+$.
Thus, $H(\theta) = A \sin k\theta + B \cos k\theta$ and $p = k^2$ for $k \in \mathbb{N}$.

Laplacian in Polar Coordinates (cont.)

Now we have

$$r^2 R'' + rR' - k^2 R = 0.$$

This is Cauchy-Euler, with indicial equation

$$\lambda(\lambda - 1) + \lambda - k^2 = 0,$$

which has solutions $\lambda = \pm k$. If $k \neq 0$, we have

$$R(r) = Ar^k + Br^{-k}.$$

If $k = 0$, we have

$$R(r) = A + B \ln r.$$

Note, however, that we require R to be bounded at 0, so $B = 0$, and $R(r) = Ar^k$ for $k \in \mathbb{N}$.

Laplacian in Polar Coordinates (cont.)

Finally, we find the general solution for u,

$$u(r,\theta) = B_0 + \sum_{k=1}^{\infty} r^k (A_k \sin k\theta + B_k \cos k\theta),$$

for some choice of A_k and B_k. Since $u(1,\theta) = f(\theta)$, we have

$$u(1,\theta) = B_0 + \sum_{k=1}^{\infty} (A_k \sin k\theta + B_k \cos k\theta) = f(\theta).$$

So, the A_k and B_k are chosen to the the trigonometric Fourier coefficients of f.

Elliptic Operators

We can write a general partial differential operator (in $\mathbb{R}^d$) in the form

$$Lu = -\sum_{j=1}^{d}\sum_{k=1}^{d} a_{jk}(\mathbf{x})\partial_j\partial_k u + \sum_{k=1}^{d} b_k(\mathbf{x})\partial_k u + c(\mathbf{x})u.$$

If we let A be the matrix formed from second order coefficients (the a_{ij}s), then we can define what it means to be elliptic.

Definition 31

A second-order linear partial differential operator is called **elliptic** in Ω if there exists a $c > 0$ such that, for all $\mathbf{x} \in \Omega$ and $\mathbf{y} \in \mathbb{R}^d$,

$$\mathbf{y}^T A(\mathbf{x})\mathbf{y} \geq c\|\mathbf{y}\|^2.$$

Elliptic Operators

It is generally difficult to work with the previous definition of an elliptic operator, so we can apply the following theorem instead.

Theorem 29

Let the symmetric part of A be A'. Then let λ_k be the k^{th} eigenvalue of A'. The operator is then elliptic on Ω if and only if there exists a $c > 0$ such that, for all $\mathbf{x} \in \Omega$ and $1 \leq k \leq d$,

$$\lambda_k(\mathbf{x}) \geq c.$$

A Note on the Eigenvalue Condition

The condition $\lambda_k(\mathbf{x}) > 0$ is not sufficient, as it is possible for the eigenvalues to approach 0 on the set. We need them to be bounded away from 0.

Elliptic Operators

Example 18

Define the operator

$$L = -(2x+4)\partial_1^2 + 3\partial_1\partial_2 + \partial_2\partial_1 - (2x+1)\partial_2^2 + (x+2)\partial_1 - 3$$

with respect to x. Is this operator elliptic on $\Omega = (0,1)$?

The relevant matrix and its symmetric part are

$$A = \begin{pmatrix} 2x+4 & -3 \\ -1 & 2x+1 \end{pmatrix}, \qquad A' = \begin{pmatrix} 2x+4 & -2 \\ -2 & 2x+1 \end{pmatrix}.$$

A' has eigenvalues $2x$ and $2x+5$, but there is no $c > 0$ such that $2x \geq c$ for all $x \in \Omega$, so the operator isn't elliptic on Ω. It is, however, elliptic on every $[a, 1)$ for $0 < a < 1$.

Elliptic Eigenproblems

The theory behind elliptic eigenproblems is complex, and generally, you don't need to know it in too much detail for this course. Instead, we will go through solving an elliptic eigenproblem. The method is very similar to that of other IBPs in 2D, however, we now have an eigenvalue to worry about, which sometimes complicates things.

Elliptic Eigenproblems

Example 19

Let Ω be the unit square $0 < x < 1$ and $0 < y < 1$. Let

$$\nabla^2 u + \lambda u = 0 \text{ on } \Omega,$$
$$u(x,0) = u(x,1) = 0,$$
$$\frac{\partial u}{\partial x}(0,y) = \frac{\partial u}{\partial x}(1,y) = 0.$$

Find the eigenpairs of this system.

Letting $u(x,y) = X(x)Y(y)$, we get (for some p)

$$X'' + pX = 0 = Y'' + (\lambda - p)Y,$$
$$Y(0) = Y(1) = 0,$$
$$X'(0) = X'(1) = 0.$$

Elliptic Eigenproblems (cont.)

Since X seems easier, we solve it for three cases of p. Let $k > 0$.

1. If $p = 0$, then $X'' = 0$, and we find $X = A$ for some constant.
2. If $p = k^2 > 0$, then $X'' + k^2 X = 0$, so $X = A\cos kx + B\sin kx$ for some A, B. Using the initial conditions, we find $X = A\cos m\pi x$ where $m \in \mathbb{Z}^+$ (so $p = m^2\pi^2$).
3. If $p = -k^2 < 0$, then $X'' - k^2 X = 0$, so $X = A\cosh kx + B\sinh kx$ for some A, B. The initial conditions result in $X \equiv 0$, so we ignore this case.

So we have $X = A$ (whence $p = 0$), and $X = A\cos m\pi x$ (whence $p = m^2\pi^2$). These can be combined to yield

$$X = A\cos m\pi x$$

for $m = 0, 1, 2, \cdots$, and $p = m^2\pi^2$.

Elliptic Eigenproblems (cont.)

Now, we can attack Y. If we let $\mu = \lambda - p$, then we can write

$$Y'' + \mu Y = 0.$$

As with X, we consider three cases.

1. If $\mu = 0$, then $Y'' = 0$ and using the initial conditions, we get $Y \equiv 0$, so we ignore this case.
2. If $\mu = k^2$, then $Y'' + k^2 Y = 0$, so $Y = A \sin ky + B \cos ky$ for some A, B. Using the initial conditions, we find $Y = A \sin n\pi y$ where $n \in \mathbb{Z}^+$ (so $\mu = n^2\pi^2$).
3. If $\mu = -k^2 < 0$, then $Y'' - k^2 Y = 0$, so $Y = A \cosh ky + B \sinh ky$ for some A, B. The initial conditions result in $Y \equiv 0$, so we ignore this case.

So, we get $Y = A \sin n\pi y$ for $n = 1, 2, \cdots$, and $\mu = n^2\pi^2$.

Elliptic Eigenproblems (cont.)

Finally, we can return to our initial separation and put things together. The eigenvalues are $\lambda_{m,n} = \mu + p = (n^2 + m^2)\pi^2$, with corresponding eigenfunctions

$$\phi_{m,n}(x, y) = \cos m\pi x \sin n\pi y.$$

Superimposing these functions will give us the general solution.