MathSoc Second Year Calculus Revision Session 2019 T1 Solutions

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We cannot guarantee that our answers are correct - please notify us of any errors or typos at unswmathsoc@gmail.com, or on our Facebook page. There are sometimes multiple methods of solving the same question. Remember that in the real class test, you will be expected to explain your steps and working out.

Seminar 1 First Half (Curves to Extrema)

Example 1.

To find the level curves of a function, we must "cut" the function with a plane. In this case, our plane is the yz-plane (as we are varying x), which is described by $x = c$ for varying c. So, to cut our surface, we can simply substitute into the surface equation to yield

$$\begin{aligned} & c^2 + y^2 + z^2 = 1 \\ \iff \quad & y^2 + z^2 = 1 - c^2. \end{aligned}$$

So this is a circle in the yz-plane with radius $\sqrt{1 - c^2}$. Clearly this has no solution if $|c| > 1$, and as we vary c from -1 to 1 the radius of the circle increases from 0 to 1 and back to 0. We can simply plot these circles for varying values of c to get the desired plot.

Example 2.

To prove a limit does not exist in multiple variables, we try to construct two different paths leading to the limit value that have different values. Of course, if you can show that there is a path such that the limit simply doesn't exist, then there's no need for a second path, however this is generally not possible.

The simplest paths are $x = 0$ and $y = 0$, so we try these, giving

$$\lim_{\substack{(x,y)\to(0,0)\\x=0}} \frac{x}{x+y^2} = \lim_{y\to 0} \frac{0}{0+y^2} = \lim_{x\to 0} 0 = 0,$$
$$\lim_{\substack{(x,y)\to(0,0)\\y=0}} \frac{x}{x+y^2} = \lim_{x\to 0} \frac{x}{x+0^2} = \lim_{y\to 0} 1 = 1.$$

So, approaching from these paths results in differing limits, and thus the limit of the function at zero cannot exist. There are many paths to take, but it's generally best to keep it simple. Another path you could try is $x = y^2$, which results in a limit of $\frac{1}{2}$.

Example 3.

To show differentiability, we fall back on the definition, and simply substitute in the values we are given. Note that row vectors were used sometimes in place of column vectors simply due to formatting considerations.

$$\begin{aligned}
\lim_{\mathbf{x}\to\mathbf{a}} \frac{\|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a}) - \mathbf{L}(\mathbf{x}-\mathbf{a})\|}{\|\mathbf{x}-\mathbf{a}\|} &= \lim_{(x,y)\to(1,0)} \frac{\left\| \begin{pmatrix} x^2+y \\ x+y^2 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x-1 \\ y \end{pmatrix} \right\|}{\|(x,y)-(1,0)\|} \\
&= \lim_{(x,y)\to(1,0)} \frac{\left\| \begin{pmatrix} x^2+y-1 \\ x+y^2-1 \end{pmatrix} - \begin{pmatrix} 2x-2+y \\ x-1 \end{pmatrix} \right\|}{\sqrt{(x-1)^2+y^2}} \\
&= \lim_{(x,y)\to(1,0)} \frac{\left\|(x^2-2x+1, y^2)\right\|}{\sqrt{(x-1)^2+y^2}} \\
&= \lim_{(x,y)\to(1,0)} \sqrt{\frac{(x-1)^4+y^4}{(x-1)^2+y^2}} \\
&\le \lim_{(x,y)\to(1,0)} \sqrt{\frac{(x-1)^4+2(x-1)^2y^2+y^4}{(x-1)^2+y^2}} \\
&= \lim_{(x,y)\to(1,0)} \sqrt{(x-1)^2+y^2} \\
&= 0.
\end{aligned}$$

Now clearly the limit is greater than or equal to zero (as it is the limit of a norm), so we can apply Squeeze Theorem to show that the limit in question is, in fact, equal to zero. Thus, $\mathbf{f}$ has derivative D at $\mathbf{a}$ by definition, as required.

Example 4.

This question is a simple application of chain rule. To begin with, we draw the following chain diagram:

$$\begin{tikzcd}
 & u \arrow[r, dashed] \arrow[ddr, "\frac{\partial u}{\partial y}"] & x \\
f \arrow[ur, "\frac{\partial f}{\partial u}"] \arrow[dr, "\frac{\partial f}{\partial v}"'] & & \\
 & v \arrow[uur, dashed] \arrow[r, "\frac{\partial v}{\partial y}"'] & y
\end{tikzcd}$$

Since we only care about finding the partial derivative with respect to y, we can ignore the x branch in the diagram, and follow the two paths to y. So we calculate the four partial derivatives we need:

$$\frac{\partial f}{\partial u} = 2u, \qquad \frac{\partial f}{\partial v} = 2v;$$
$$\frac{\partial u}{\partial y} = 1, \qquad \frac{\partial v}{\partial y} = -2y.$$

Following each path in the chain diagram and summing the product of every path, this gives us

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} \\
&= (2u)(1) + (2v)(-2y) \\
&= 2(x^2 + y) - 4(z - y^2)y \\
&= 2x^2 + 2y + 4y^3 - 4yz.
\end{aligned}$$

Take note of how the answer is in terms of x, y, z. Make sure to expand out all terms (in this case, u and v).

Example 5.

As an application of multivariable chain rule we must first find the Jacobian of both $\mathbf{f}$ and $\mathbf{g}$. These are

$$J_{\mathbf{x}}(\mathbf{f}) = \begin{pmatrix} 2x & -2y & 0 \\ 2y & 2x & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad J_{\mathbf{x}}(\mathbf{g}) = \begin{pmatrix} 1 & 0 & 2z \\ \frac{1}{z} & 0 & -\frac{x}{z^2} \end{pmatrix}.$$

Now, we wish to find $J_{\mathbf{a}}(\mathbf{f} \circ \mathbf{g}) = J_{\mathbf{f}(\mathbf{a})}(\mathbf{g}) J_{\mathbf{a}}(\mathbf{f})$, where $\mathbf{a} = (2,1,2)$ and so $\mathbf{f}(\mathbf{a}) = (3,4,2)$. Calculating this yields

$$\begin{aligned} J_{\mathbf{a}}(\mathbf{f} \circ \mathbf{g}) &= \begin{pmatrix} 1 & 0 & 2(2) \\ \frac{1}{2} & 0 & -\frac{3}{2^2} \end{pmatrix} \begin{pmatrix} 2(2) & -2(1) & 0 \\ 2(1) & 2(2) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 4 \\ \frac{1}{2} & 0 & -\frac{3}{4} \end{pmatrix} \begin{pmatrix} 4 & -2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 4 & -2 & 4 \\ 2 & -1 & -\frac{3}{4} \end{pmatrix}, \end{aligned}$$

taking special care to evaluate the Jacobians at the correct points.

Example 6.

A simple calculation yields

$$J\mathbf{f} = \begin{pmatrix} 2x & 1 \\ 1 & 2y \end{pmatrix}.$$

So, the determinant is $|J\mathbf{f}| = (2x)(2y) - (1)(1) = 4xy - 1$. So, by the Inverse Function Theorem, $\mathbf{f}$ is invertible wherever $J\mathbf{f}$ is invertible. That is, when $4xy - 1 \neq 0$.

Now, to find the Jacobian of the inverse function $\mathbf{f}^{-1}$ at $(1,-1)$, we can use Inverse Function Theorem to find that

$$\begin{aligned} J_{\mathbf{f}(1,-1)}\mathbf{f}^{-1} &= \left(J_{(1,-1)}\mathbf{f}\right)^{-1} \\ &= \begin{pmatrix} 2(1) & 1 \\ 1 & 2(-1) \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}^{-1} \\ &= -\frac{1}{5} \begin{pmatrix} -2 & -1 \\ -1 & 2 \end{pmatrix}. \end{aligned}$$

Example 7.

We are given two implicit equations,

$$
\begin{aligned}
g_1(x,y,u,v) &= e^{xyu} + yuv + x - 3 = 0,\\
g_2(x,y,u,v) &= \ln yv + xu^3v - x^3u = 0.
\end{aligned}
$$

So, we let

$$
\mathbf{g}(x,y,u,v) = \begin{pmatrix} e^{xyu} + yuv + x - 3 \\ \ln yv + xu^3v - x^3u \end{pmatrix} = \mathbf{0}.
$$

That is, we express our system of equations as a vector. Then, we calculate the derivative of $\mathbf{g}$,

$$
D\mathbf{g} = \left(\begin{array}{cc|cc} yue^{xyu}+1 & xue^{xyu}+uv & xye^{xyu}+yv & yu \\ u^3v - 3x^2u & \frac{1}{y} & -x^3 & \frac{1}{v} + xu^3 \end{array}\right).
$$

Since we're looking to show there is a solution for $u(x,y)$ and $v(x,y)$, we partition the matrix to isolate the part of the derivative involving u and v. Then, we can evaluate this part of the derivative at our point $(0,1,2,1)$ to give us

$$
\begin{pmatrix} xye^{xyu}+yv & yu \\ -x^3 & \frac{1}{v}+xu^3 \end{pmatrix} = \begin{pmatrix} 0+(1)(1) & (1)(2) \\ 0 & \frac{1}{1}+0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.
$$

This matrix has determinant 1, and is thus invertible. So, by the Implicit Function Theorem, the system has a unique solution $(u,v) = (u(x,y), v(x,y))$ near $(0,1,2,1)$.

Example 8.

h describes a surface, so to find the direction of steepest descent, we need to find the direction of maximal change. To do this, we calculate the gradient

$$\nabla h(x, y) = \left(-\frac{4x^3}{10^8}, -\frac{2y}{10^2}\right).$$

Evaluating this at the point $(100, 1)$ gives

$$\nabla h(100, 1) = \left(-\frac{4(100)^3}{10^8}, -\frac{2(1)}{10^2}\right) = \left(-\frac{4}{100}, -\frac{2}{100}\right).$$

Now, since we only need the direction, we can (and are encouraged to) simplify this, giving us the direction $(-2, -1)$. However, since we want steepest **descent**, and decreasing x or y will increase h (at the point we're considering), we instead take the direction to be $(2, 1)$.

No, to calculate the slope, we first normalise the direction to $\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$ (as we require a **unit** vector), and find the slope is

$$\nabla h(100, 1) \cdot \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} -\frac{1}{25} \\ -\frac{1}{50} \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} = -\frac{1}{10\sqrt{5}}.$$

A quick sanity check: we are looking for the slope in the direction of steepest descent, so we should get something negative, which we do.

Example 9.

To find the tangent plane of a surface, we must first find its gradient (as it is normal to the surface). Since we're given a surface represented as a graph of a function, we first write it as $z = f(x, y)$, and find that $x^2 - y^2 - z = 0$. We let this be $g(x, y, z)$, so that

$$\nabla g(x, y, z) = (2x, -2y, -1).$$

Then, we want to find the tangent plane at $(x, y) = (1, 2)$, and we find that $z = 1^2 - 2^2 = -3$ at this point, so we evaluate the gradient at $(1, 2, -3)$:

$$\nabla g(1, 2, -3) = (2(1), -2(2), -1) = (2, -4, -1).$$

Notice how the value of z doesn't actually matter here. This is because the surface was defined in terms of a graph. If it wasn't, then z would come into play (e.g. a sphere). We do, however, need to calculate the point anyway, since our surface is given by

$$\begin{aligned}
& & \nabla g(1, 2, -3) \cdot \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} \right) &= 0 \\
&\iff & \begin{pmatrix} 2 \\ -4 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} x - 1 \\ y - 2 \\ z + 3 \end{pmatrix} &= 0 \\
&\iff & 2x - 2 - 4y + 8 - z - 3 &= 0 \\
&\iff & 2x - 4y - z &= -3
\end{aligned}$$

in Cartesian form.

Example 10.

Just as with the previous example, we find the gradient (remembering that even in 2D this is **normal** to the curve) of $f(x,y) = x^2 + y^3 - 5$, which is

$$\nabla f(x,y) = (2x, 3y^2).$$

At our point $(2,1)$, this is

$$\nabla f(2,1) = (2(2), 3(1)^2) = (4,3).$$

Thus, our normal line is the line through the point $(2,1)$ in the direction $(4,3)$. So we have

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 3 \end{pmatrix}.$$

So, to get the line in Cartsian form,

$$\begin{aligned} & \begin{cases} x = 2 + 4\lambda \\ y = 1 + 3\lambda \end{cases} \\ \iff \quad & \begin{cases} 3x = 6 + 12\lambda & \text{multiplying by 3} \\ 4y = 4 + 12\lambda & \text{multiplying by 4} \end{cases} \\ \implies \quad & 4y - 3x = 4 - 6 \quad \text{taking their difference} \\ \iff \quad & 3x - 4y - 2 = 0 \end{aligned}$$

Example 11.

To approximate a function, we find its tangent plane. In this case, we can use one of our previous examples to find the tangent plane

$$2x - 4y - z = -3 \implies z = 2x - 4y + 3,$$
$$f(1.05, 1.9) \approx z = 2(1.05) - 4(1.9) + 3 = -2.5.$$

Then, to find the error, we need the actual function value, which is

$$f(1.05, 1.9) = (1.05)^2 - (1.9)^2 = -2.5075.$$

Thus, our absolute error is $|-2.5075 - (-2.5)| = 0.0075$, and our relative error is

$$\left| \frac{0.0075}{-2.5075} \right| \approx 0.003 = 0.03\%.$$

Remember that the absolute error is the difference between the approximation and actual value, and the relative error is the ratio between the absolute error and the actual value (i.e. what percentage of the real value the error makes up).

Example 12.

The differentials $\mathrm{d}x = 0.04$ and $\mathrm{d}y = 0.02$ are given in the question (the maximum amount of error in the measured value). Then we calculate the partial derivatives of T:

$$T_x = \cosh y, \qquad T_y = x \sinh y.$$

Then, at our measured values, we have

$$\begin{aligned} T_x(2, \ln 2) &= \cosh(\ln 2) &&= \frac{e^{\ln 2} + e^{-\ln 2}}{2} &&= \frac{5}{4}, \\ T_y(2, \ln 2) &= 2\sinh(\ln 2) &&= 2\frac{e^{\ln 2} - e^{-\ln 2}}{2} &&= \frac{3}{2}. \end{aligned}$$

Thus,

$$\begin{aligned} \mathrm{d}T &= T_x\,\mathrm{d}x + T_y\,\mathrm{d}y \\ &= \frac{5}{4}(0.04) + \frac{3}{2}(0.02) \\ &= \frac{8}{100}. \end{aligned}$$

So the maximum error in the measured value of T is $\frac{8}{100}$.
Notice how the formula for differentials is just like that of the multivariable chain rule.

Example 13.

To find the fifth degree Taylor polynomial of a function, we must first find all derivatives up to order five. To skip some working with mixed partial derivatives, we note that $f(x, y) = \sin(x + y)$ is infinitely continuously differentiable (that is, all partial derivatives exist and are continuous), so all mixed partial derivatives will commute. So, we calculate

$$f = \sin(x + y) \qquad f_{x_i} = \cos(x + y) \qquad f_{x_i x_j} = -\sin(x + y) = -f.$$

We can already see a cyclic structure, so the third order derivatives will be $-f_{x_i}$, fourth order will be f, and fifth will be f_{x_i}. Now evaluating all these at $(0, 0)$ gives us

$$f = 0, f_{x_i} = 1, f_{x_i x_j} = 0, f_{x_i x_j x_k} = -1, f_{x_i x_j x_k x_l} = 0, f_{x_i x_j x_k x_l x_m} = 1,$$

which we can substitute into the Taylor series expansion, to yield

$$\begin{aligned} P(x, y) &= x + y + \frac{1}{3!}\left(-x^3 - 3x^2y - 3xy^2 - y^3\right) \\ &+ \frac{1}{5!}\left(x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5\right). \end{aligned}$$

Note that the zeroth, second, and fourth order terms are gone, as the partial derivatives are zero, and the third order terms are all negative, as the third order partial derivative is -1. Don't confuse the binomial coefficients (the 3s, 5s, and 10s) for the partial derivatives.

Example 14.

To find the remainder term, we get the next term in a Taylor expansion, and evaluate it at some point between (x, y) and the point we're expanding around (we don't choose this point, it exists, but we don't know what it is, usually). So for a second degree Taylor polynomial, the form is

$$R(x, y) = \frac{1}{3!}\left(f_{xxx}(\mathbf{z})x^3 + 3f_{xxy}(\mathbf{z})x^2y + 3f_{xyy}(\mathbf{z})xy^2 + f_{yyy}(\mathbf{z})y^3\right),$$

where $\mathbf{z}$ is some point on the line between $(0, 0)$ (the point we're expanding around), and (x, y).

Example 15.

To find the critical points of f, we find

$$\nabla f(x, y) = \left(2(2x)(x^2 + y^2)^2 - (2x), 2(2y)(x^2 + y^2)^2 - (-2y)\right) = \mathbf{0}.$$

The function is differentiable everywhere, so we only consider $\nabla f(x, y) = \mathbf{0}$. That is,

$$\begin{cases} 2x\left(2(x^2 + y^2) - 1\right) = 0 \\ 2y\left(2(x^2 + y^2) + 1\right) = 0 \end{cases}.$$

Clearly, $2(x^2 + y^2) + 1 \neq 0$, so $y = 0$. Then $2x(2x^2 - 1) = 0$, so $x = 0, \pm\frac{1}{\sqrt{2}}$. Now, we have

$$f_{xy}(x, y) = 8xy, \qquad f_{xx}(x, y) = 12x^2 + 4y^2 - 2, \qquad f_{yy}(x, y) = 4x^2 + 12y^2 + 2.$$

To find the discriminant at each point, we calculate f_{xy}, f_{xx}, and f_{yy} for each point.

$$\begin{aligned}
f_{xy}(0, 0) &= 8(0)(0) &&= 0, \\
f_{xx}(0, 0) &= 12(0)^2 + 4(0)^2 - 2 &&= -2, \\
f_{yy}(0, 0) &= 4(0)^2 + 12(0)^2 + 2 &&= 2, \\
f_{xy}\left(\pm\frac{1}{\sqrt{2}}, 0\right) &= 8\left(\pm\frac{1}{\sqrt{2}}\right)(0) &&= 0, \\
f_{xx}\left(\pm\frac{1}{\sqrt{2}}, 0\right) &= 12\frac{1}{2} + 4(0)^2 - 2 &&= 4, \\
f_{yy}\left(\pm\frac{1}{\sqrt{2}}, 0\right) &= 4\frac{1}{2} + 12(0)^2 + 2 &&= 4.
\end{aligned}$$

Thus,

$$D(0, 0) = 0^2 - (-2)(2) = 4, \qquad D\left(\pm\frac{1}{\sqrt{2}}, 0\right) = 0^2 - (4)(4) = -16.$$

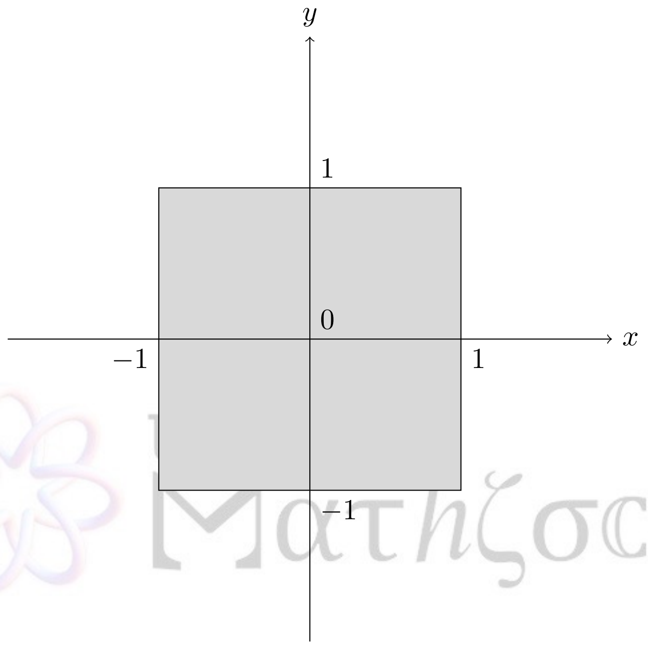
Since $D(0, 0) > 0$, $(0, 0)$ is a saddle point.
Since $D\left(\pm\frac{1}{\sqrt{2}}, 0\right) < 0$ and $f_{xx}\left(\pm\frac{1}{\sqrt{2}}, 0\right) > 0$, $\left(\pm\frac{1}{\sqrt{2}}, 0\right)$ are local minimums.

Example 16.

To find the maximum value of a function, we find the critical points. In this case, we also have a boundary to consider. So, for critical points, we have

$$\nabla f(x, y) = (2x + y, x) = \mathbf{0}.$$

Clearly, this has one solution, $(0, 0)$. Again, since the function is differentiable everywhere, we don't have to consider points where the gradient doesn't exist.
Since we only care about the maximum, we don't have to classify $(0, 0)$ and instead just evaluate f, as $f(0, 0) = 0$. Now, the set we're considering is:



So, the boundary can be broken into the four sides of the square, and we calculate the maximum of the function on these four sides:

(a) Along $x = 1$ we have $f(1, y) = y$, which has a max of 1 at $(1, 1)$;

(b) Along $x = -1$ we have $f(-1, y) = 2 - y$, which has a max of 3 at $(-1, -1)$;

(c) Along $y = 1$ we have $f(x, 1) = x^2$, which has a max of 1 as $(\pm 1, 1)$;

(d) Along $y = -1$ we have $f(x, -1) = x^2 - 2x$, which has a max of 3 at $(-1, -1)$.

So, out of all the points on the boundary, and all critical points in the set, the point $(-1, -1)$ gives us the largest value of 3. That is, the maximum value of f on the set is 3.

Example 17.

To use Sylvester's Criterion, we first have to calculate the Hessian of the function. So,

$$\begin{aligned}
f_x &= 4x^3 - 4yz, & f_y &= 4y^3 - 4xz, & f_z &= 4z^3 - 4xy; \\
f_{xx} &= 12x^2, & f_{yy} &= 12y^2, & f_{zz} &= 12z^2; \\
f_{xy} &= -4z, & f_{yz} &= -4x, & f_{xz} &= -4y.
\end{aligned}$$

Notice that we can skip evaluating the other mixed partial derivatives, as the partial derivatives are continuous and thus commute. So, putting these in a matrix, we find

$$H(f)(x, y, z) = \begin{pmatrix} 12x^2 & -4z & -4y \\ -4z & 12y^2 & -4x \\ -4y & -4x & 12z^2 \end{pmatrix}.$$

At the point we're considering, we have

$$H(f)(1, 1, 1) = \begin{pmatrix} 12 & -4 & -4 \\ -4 & 12 & -4 \\ -4 & -4 & 12 \end{pmatrix}.$$

So, to use this, we find the submatrix determinants.

$$\Delta_1 = 12,$$

$$\Delta_2 = \begin{vmatrix} 12 & -4 \\ -4 & 12 \end{vmatrix} = 144 - 16 = 128,$$

$$\begin{aligned}
\Delta_3 &= \begin{vmatrix} 12 & -4 & -4 \\ -4 & 12 & -4 \\ -4 & -4 & 12 \end{vmatrix} = \frac{4^3}{3^2} \begin{vmatrix} 3 & -1 & -1 \\ -3 & 9 & -3 \\ -3 & -3 & 9 \end{vmatrix} = \frac{4^3}{3^2} \begin{vmatrix} 3 & -1 & -1 \\ 0 & 8 & -4 \\ 0 & -4 & 8 \end{vmatrix} \\
&= \frac{4^3 \cdot 2}{3^2} \begin{vmatrix} 3 & -1 & -1 \\ 0 & 4 & -2 \\ 0 & -4 & 8 \end{vmatrix} = \frac{4^3 \cdot 2}{3^2} \begin{vmatrix} 3 & -1 & -1 \\ 0 & 4 & -2 \\ 0 & 0 & 6 \end{vmatrix} = \frac{4^3 \cdot 2}{3^2} (3 \cdot 4 \cdot 6) = 1024.
\end{aligned}$$

Since all submatrix determinants are (strictly) positive, the Hessian is positive definite, so the stationary point $(1, 1, 1)$ is a local minimum.

Example 18.

Since this is a function in $\mathbb{R}^{\nvDash}$, we rearrange the equation to obtain $f(x,y,z) = x+y+2z$, with the constraint being $x^2+y^2+z^2-3 = c(x,y,z)$.

Therefore, $\nabla f = \lambda \nabla c \implies (1,1,2)^T = \lambda(2x,2y,2z)^T$. This means, upon equating the components, $2x\lambda = 1, 2y\lambda = 1, 2z\lambda = 2$.

So substituting $x = y = \frac{1}{2\lambda}, z = \frac{1}{\lambda}$ into the constraint, we can solve for λ:

$$\left(\frac{1}{2\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 = 3 \implies \lambda = \sqrt{\frac{2}{3}}$$

The constrained stationary point is $(\frac{\sqrt{3}}{2\sqrt{2}}, \frac{\sqrt{3}}{2\sqrt{2}}, \sqrt{\frac{3}{2}})$

Example 19.

The constraint becomes $c(x,y) = 3x^2+y^2-6$. So we equate $\nabla f = \lambda \nabla c$. This yiels:

$$(y,x) = \lambda(6x,2y) \implies y = 6x\lambda, x = 2y\lambda$$

Solving these equations simultaneously, we get $y = 6(2y\lambda)\lambda \implies y = 12y\lambda^2$ Therefore, either $y = 0, \lambda^2 = \frac{1}{12}$. $y = 0 \implies x = 0$, and $\lambda = \pm\frac{1}{2\sqrt{3}} \implies y = \pm\sqrt{3}x$. Obviously, $(0,0)$ does not work because it doesn't lie on the constraint. So we are looking for the intersection point of $y = \pm\sqrt{3}x$ and $3x^2+y^2 = 6$. This solves to $x = 1, y = \pm\sqrt{3}$ or $x = -1, y = \pm\sqrt{3}$

Example 20.

$$\begin{aligned}
\int_{x=-5}^{x=4}\int_{y=0}^{y=3}(2x-4y^3)dydx &= \int_{x=-5}^{y=4}\left[2xy-y^4\right]_{y=0}^{y=3}dx \\
&= \int_{x=-5}^{x=4}\big((6x-81)-(0-0)\big)dx \\
&= \int_{x=-5}^{x=4}(6x-81)dx \\
&= \left[3x^2-81x\right]_{x=-5}^{x=4} \\
&= -756
\end{aligned}$$

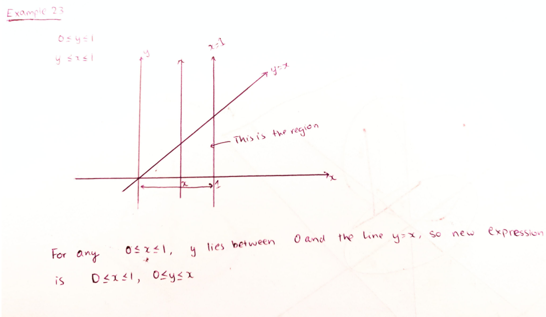
Example 21.

$$\begin{aligned}
\int_{y=0}^{y=1}\int_{x=-1}^{x=2}xe^{xy}dxdy &= \int_{x=-1}^{x=2}\int_{y=0}^{y=1}xe^{xy}dydx \\
&= \int_{x=-1}^{x=2}\left[e^{xy}\right]_{y=0}^{y=1}dx \\
&= \int_{x=-1}^{x=2}(e^x-e^0)dx \\
&= \left[e^x-x\right]_{x=-1}^{x=2} \\
&= e^2-2-(e^{-1}-(-1)) \\
&= e^2-e^{-1}-3
\end{aligned}$$

Example 22.

$$\begin{aligned}
\int_{y=1}^{y=2}\int_{x=y}^{x=y^3}e^{\frac{x}{y}}dxdy &= \int_{y=1}^{y=2}\left[ye^{\frac{x}{y}}\right]_{x=y}^{x=y^3}dy \\
&= \int_{y=1}^{y=2}ye^{y^2}-ye^1dy \\
&= \int_{y=1}^{y=2}(ye^{y^2}-ey)dy \\
&= \left[\frac{1}{2}e^{y^2}-\frac{e}{2}y^2\right]_{y=1}^{y=2} \\
&= \frac{1}{2}e^4-\frac{e}{2}\cdot 4-\Big(\frac{1}{2}e-\frac{e}{2}\Big) \\
&= \frac{1}{2}e^4-2e
\end{aligned}$$

Example 23.

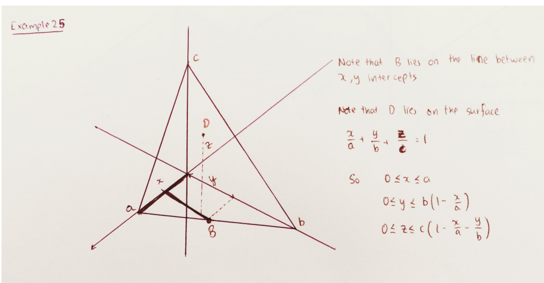


$$\begin{aligned}
\int_0^1 \int_{y^2}^1 2\sqrt{x}e^{x^2}\,dx\,dy &= \int_{x=0}^{x=1} \int_{y=0}^{y=\sqrt{x}} 2\sqrt{x}e^{x^2}\,dydx \\
&= \int_{x=0}^{x=1} 2\sqrt{x}e^{x^2}\,[y]_{y=0}^{y=\sqrt{x}}dx \\
&= \int_0^1 2xe^{x^2}\,dx \\
&= \left[e^{x^2}\right]_0^1 \\
&= e-1
\end{aligned}$$

Example 24.

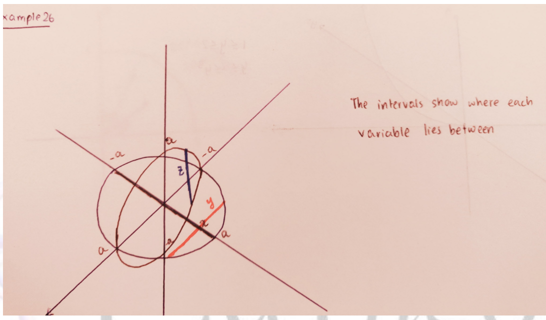
$$\begin{aligned}
\int_0^1 \int_y^1 \sin(x^2)dx\,dy &= \int_{x=0}^{x=1} \int_{y=0}^{y=x} \sin(x^2)dydx \\
&= \int_{x=0}^{x=1} \sin(x^2)\,[y]_0^x dx \\
&= \int_0^1 x\sin(x^2)dx \\
&= \frac{1}{2}\sin 1
\end{aligned}$$

Example 25.



$$
\begin{aligned}
\iiint_S dV &= \int_{x=0}^{x=a} \int_{y=0}^{y=b\left(1-\frac{x}{a}\right)} \int_{z=0}^{z=c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dz\, dy\, dx \\
&= c\int_{x=0}^{x=a} \int_{y=0}^{y=b\left(1-\frac{x}{a}\right)} 1-\frac{x}{a}-\frac{y}{b} dy\, dx \\
&= c\int_{x=0}^{x=a} \left[\left(1-\frac{x}{a}\right)y-\frac{1}{2b}y^2\right]_{y=0}^{y=b\left(1-\frac{x}{a}\right)} dx \\
&= c\int_{x=0}^{x=a} b\left(1-\frac{x}{a}\right)^2-\frac{b}{2}\left(1-\frac{x}{a}\right)^2 dx \\
&= \frac{1}{2}bc\int_{x=0}^{x=a} \left(1-\frac{x}{a}\right)^2 dx \\
&= \frac{abc}{6}
\end{aligned}
$$

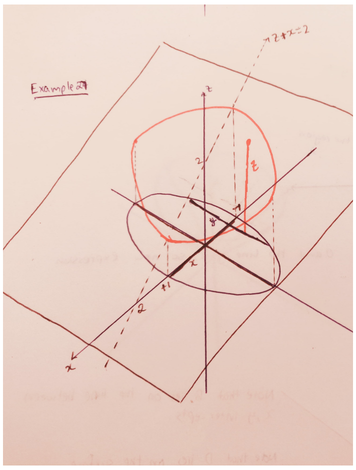
Example 26.



The region in consideration is the circle $x^2+y^2=a^2$ in the xy-plane. So we are measuring the integral:

$$
\begin{aligned}
\iiint_S dV &= \int_{x=-a}^{x=a} \int_{y=-\sqrt{a^2-x^2}}^{y=\sqrt{a^2-x^2}} \int_{z=-\sqrt{a^2-y^2}}^{z=\sqrt{a^2-y^2}} dz\, dy\, dx \\
&= \int_{y=-a}^{y=a} \int_{x=-\sqrt{a^2-y^2}}^{x=\sqrt{a^2-y^2}} 2\sqrt{a^2-y^2}dx\, dy \qquad \text{(changing order of integration)} \\
&= \int_{y=-a}^{y=a} 2\sqrt{a^2-y^2}\,[x]_{x=-\sqrt{a^2-y^2}}^{x=\sqrt{a^2-y^2}} dy \\
&= \int_{y=-a}^{y=a} 4(a^2-y^2)dy \\
&= \frac{16}{3}a^3
\end{aligned}
$$

Example 27.



$$
\begin{aligned}
\iiint_S x^2 dV &= \int_{x=-1}^{x=1} \int_{y=-\sqrt{4-4x^2}}^{y=\sqrt{4-4x^2}} \int_{z=0}^{z=2-x} x^2 dz\, dy\, dx \\
&= \int_{x=-1}^{x=1} \int_{y=-\sqrt{4-4x^2}}^{y=\sqrt{4-4x^2}} x^2(2-x) dy\, dx \\
&= \int_{-1}^{1} x^2(2-x) \cdot 2\sqrt{4-4x^2} dx \\
&= 4\int_{-1}^{1} x^2(2-x)\sqrt{1-x^2} dx \\
&= 4\int_{-1}^{1} 2x^2\sqrt{1-x^2} dx - 4\int_{-1}^{1} x^3\sqrt{1-x^2} dx
\end{aligned}
$$

The first integral can be solved using $x = \sin(u)$, the second integral is just 0 because the function is odd.

This yields:

$$
\begin{aligned}
4\int_{-1}^{1} 2x^2\sqrt{1-x^2}dx &= 8\int_{-\frac{pi}{2}}^{\frac{\pi}{2}} \sin^2(u)\sqrt{1-\sin^2(u)}\cos(u)du \\
&= 8\int_{-\frac{pi}{2}}^{\frac{\pi}{2}} \sin^2(u)\cos^2(u)du \\
&= 2\int_{-\frac{pi}{2}}^{\frac{\pi}{2}} (\sin(2u))^2du \\
&= 2\int_{-\frac{pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}(1-\cos(4u))du \\
&= \pi - \frac{1}{4}\Big[\sin(u)\Big]_{-\frac{pi}{2}}^{\frac{\pi}{2}} \\
&= \pi - \frac{1}{4}\Big(1-(-1)\big) \\
&= \pi - \frac{1}{2}
\end{aligned}
$$

Example 28.

$$
\begin{aligned}
\iint_R x^2y^3dA &= \int_{\theta=0}^{\theta=\frac{\pi}{2}}\int_{r=0}^{r=1}(r\cos(\theta))^2(r\sin(\theta))^3rdr\,d\theta \\
&= \int_{\theta=0}^{\theta=\frac{\pi}{2}} r^6\cos^2(\theta)\sin^3(\theta)dr\,d\theta \\
&= \int_{\theta=0}^{\theta=\frac{\pi}{2}} \frac{1}{7}\cos^2(\theta)\sin^2\theta\sin(\theta)d\theta \\
&= \frac{1}{7}\int_{1}^{0} u^2(1-u^2)\cdot -du \\
&= \frac{1}{7}\int_{0}^{1} u^2-u^4du \\
&= \frac{2}{105}
\end{aligned}
$$

Example 29.

$$
\begin{aligned}
\iint_R dA &= \int_{\theta=0}^{\theta=\frac{3\pi}{4}}\int_{r=0}^{r=3-3\cos\theta} rdr\,d\theta \\
&= \int_{0}^{\frac{3\pi}{4}} \frac{1}{2}(3-3\cos\theta)^2d\theta \\
&= \text{la-da-daa-da-da-de-di-da-di-day} \\
&= \frac{81\pi}{16} - \frac{9}{8}(4\sqrt{2}+1)
\end{aligned}
$$

Example 30.

$$\begin{aligned}\iiint_S dV &= \int_{r=0}^{r=\sqrt{3}} \int_{\theta=0}^{\theta=2\pi} \int_{z=2-\sqrt{4-x^2-y^2}}^{z=2+\sqrt{4-r^2}} r dz\, d\theta\, dr \\ &= \int_{r=0}^{r=\sqrt{3}} \int_{\theta=0}^{\theta=2\pi} \int_{z=2-\sqrt{4-r^2}}^{z=2+\sqrt{4-r^2}} r dz\, d\theta\, dr\end{aligned}$$

The outer integral for the bounds on r are because we only consider how far away from the original in the xy plane we travel, θ ranges from 0 to 2π because we can go around in a full circle in the valid region for r, and z ranges from the lower sphere to the upper sphere.

Example 31.

$$\iiint_S dV = \int_{r=2}^{r=3} \int_{\theta=0}^{\theta=2\pi} \int_{z=0}^{z=r-2} r dz\, d\theta\, dr$$

To work out the bounds, $\theta \in [0, 2\pi]$ because we can rotate around the full circle. z must lie between 0 and $r-2$ because $1 \leq z \leq \sqrt{x^2+y^2} - 2 \leq 2$ and this also gives the bounds for r, which are $r \in [2, 3]$ to allow z to lie over the range $0 \leq z \leq 1$.

Example 32.

$$\iiint_S dV = \int_{\theta=0}^{\theta=2\pi} \int_{\varphi=0}^{\varphi=\frac{\pi}{4}} \int_{\rho=0}^{\rho=a} \rho^2 \sin(\varphi) d\rho\, d\varphi\, d\theta$$

To work out the bounds in this case, $\theta \in [0, 2\pi]$ since we can rotate a full circle in the x, y plane. $\varphi \in [0, \frac{\pi}{4}]$ so that we can consider the appropriate section of the sphere [*The shape of the solid is an ice-cream cone, so the largest angle away from the z-axis it can get is* $\frac{\pi}{4}$. $\rho \in [0, a]$ because the furthest distance from the origin the surface can get is a.

Example 33.

$$\iiint_S dV = \int_{\theta=0}^{\theta=2\pi} \int_{\varphi=0}^{\varphi=\pi} \int_{\rho=0}^{\rho=\frac{1}{\sin(\varphi)}} \rho^2 \sin(\varphi) d\rho\, d\varphi\, d\theta$$

In this case, $\theta \in [0, 2\pi]$ because we can go in any region in the xy plane, $\varphi \in [0, \pi]$ because z can be both positive and negative so we have the upper and lower half of $\mathbb{R}^3$ to consider. Now for the bounds of ρ, we must consider it as a function of φ or θ. After substituting $x = \rho\cos(\theta)\sin(\varphi), y = \rho\sin(\theta)\sin(\varphi), z = \rho\cos(\varphi)$, we can rearrange and solve for ρ and we get $\rho = 0, \operatorname{cosec}(\varphi)$. These provide the bounds for ρ.

Example 34.

We shall consider the substitution $u = x^2 - 2y^2, v = xy$. Then the Jacobian of this substitution is:

$$\det \frac{\partial(u,v)}{\partial(x,y)} = \det \begin{bmatrix} 2x & y \\ -4y & x \end{bmatrix} = 2x \cdot x - (-4y) \cdot y = 2x^2 + 4y^2$$

$$\begin{aligned} \iint_\Omega (x^2 - 2y^2)x^2y^2(2x^2+4y^2)dx\,dy &= \iint_{\Omega'} uv^2(2x^2+4y^2) \cdot \frac{1}{2x^2+4y^2} du\,dv \\ &= \int_{u=1}^{u=3} \int_{v=1}^{v=2} uv^2 du\,dv \end{aligned}$$

Example 35.

Suppose we re-arrange the given equations: $\frac{x^2+y^2}{x} = a, a'$ and $\frac{x^2+y^2}{y} = b, b'$. Then we use the subtitutions $u = \frac{x^2+y^2}{x}, v = \frac{x^2+y^2}{y}$. Thus, we have:

$$\begin{aligned} \det \begin{bmatrix} 1 - \frac{y^2}{x^2} & \frac{2x}{y} \\ \frac{2y}{x} & -\frac{x^2}{y^2} + 1 \end{bmatrix} &= \Big(1 - \frac{y^2}{x^2}\Big)\Big(1 - \frac{x^2}{y^2}\Big) - \Big(\frac{2y}{x}\Big)\Big(\frac{2x}{y}\Big) \\ &= 2 - \frac{x^2}{y^2} - \frac{y^2}{x^2} - (4) \\ &= -\frac{x^2}{y^2} - 2 - \frac{y^2}{x^2} \\ &= -\Big(\frac{x}{y} + \frac{y}{x}\Big)^2 \\ &= -\Big(\frac{x^2+y^2}{xy}\Big)^2 \end{aligned}$$

Hence we have: $-\Big(\frac{x^2+y^2}{xy}\Big)^2 = \frac{uv}{xy}$

Thus the integral becomes:

$$\iint_{\Omega'} \frac{1}{xy} dx\,dy = \iint_\Omega \frac{1}{xy}\frac{xy}{uv} du\,dv = \int_{v=b}^{v=b'} \int_{u=a}^{u=a'} \frac{1}{uv} du\,dv$$

Example 36.

Beginning with the given integral, we differentiate both sides with respect to a

$$\frac{d}{da}\int_{-\infty}^{\infty}\frac{\cos(3x)}{x^2+a^2}dx = \frac{d}{da}\frac{\pi}{a}e^{-3a}$$

$$\int_{-\infty}^{\infty}\frac{\partial}{\partial a}\frac{\cos(3x)}{x^2+a^2}dx = -\frac{-\pi e^{-3x}(3x+1)}{x^2}$$

$$\int_{-\infty}^{\infty}-\frac{2a\cos(3x)}{(x^2+a^2)^2}dx = -\frac{\pi e^{-3a}(3a+1)}{a^2}$$

Substituting $a = 2$

$$\int_{-\infty}^{\infty}-4\frac{\cos(3x)}{(x^2+4)^2}dx = -\frac{7\pi e^{-6}}{4}$$

$$\int_{-\infty}^{\infty}\frac{\cos(3x)}{(x^2+4)^2}dx = \frac{7\pi e^{-6}}{16}$$

Example 37.

(a) By Leibniz Rule:

$$\begin{aligned}\frac{dF}{dx} &= \frac{d}{dx}\int_0^{u(x)} f(v(x),y)dy \\ &= \int_0^{u(x)} \frac{\partial}{\partial x}f(v(x),y)dy + f(v(x),u(x))\frac{d}{dx}u(x) - f(v(x),0)\frac{d}{dx}(0) \\ &= \int_0^{u(x)} v'(x)\frac{\partial f(v(x),y)}{\partial x}dy + f(v(x),u(x))u'(x)\end{aligned}$$

(b) Let $u(x) = x, v(x) = x$. Then upon substitution into the expression from a), we have:

$$\begin{aligned}\frac{d}{dx}\int_0^x \frac{\sin(xy)}{y}dy &= \int_0^x 1\cdot y\cdot\frac{\cos(xy)}{y} + \frac{\sin(x^2)}{x}\cdot 1 \\ &= \int_0^x \cos(xy)dy + \frac{\sin(x^2)}{x} \\ &= \Big[\frac{\sin(xy)}{x}\Big]_0^x + \frac{\sin(x^2)}{x} \\ &= 2\frac{\sin(x^2)}{x}\end{aligned}$$

Seminar 2 First Half (Vector Calculus)

1. (a)

$$\begin{aligned}\nabla \cdot \mathbf{F} &= (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \cdot (x, y, z) \\ &= 1 + 1 + 1 \\ &= 3\end{aligned}$$

(b)

$$\begin{aligned}\nabla \times \mathbf{F} &= (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \times (x, y, z) \\ &= \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{bmatrix} \\ &= \hat{i} \det \begin{bmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z \end{bmatrix} - \hat{j} \det \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ x & z \end{bmatrix} + \hat{k} \det \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ x & y \end{bmatrix} \\ &= 0\hat{i} - 0\hat{j} + 0\hat{k}\end{aligned}$$

(c) First we compute $\mathbf{a} \times \mathbf{F}$

$$\begin{aligned}\mathbf{a} \times \mathbf{F} &= (a_1, a_2, a_3) \times (x, y, z) \\ &= \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{bmatrix} \\ &= \hat{i} \det \begin{bmatrix} a_2 & a_3 \\ y & z \end{bmatrix} - \hat{j} \det \begin{bmatrix} a_1 & a_3 \\ x & z \end{bmatrix} + \hat{k} \det \begin{bmatrix} a_1 & a_2 \\ x & y \end{bmatrix} \\ &= (a_2 z - a_3 y)\hat{i} - (a_1 z - a_3 x)\hat{j} + (a_1 y - a_2 x)\hat{k}\end{aligned}$$

To compute the curl, we now compute:

$$\begin{aligned}\det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_2 z - a_3 y & a_3 x - a_1 z & a_1 y - a_2 x \end{bmatrix} &= \hat{i} \det \begin{bmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z \end{bmatrix} - \hat{j} \det \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ x & z \end{bmatrix} + \hat{k} \det \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ x & y \end{bmatrix} \\ &= 2a_1\hat{i} + 2a_2\hat{j} + 2a_3\hat{k} \\ &= 2\mathbf{a}\end{aligned}$$

(d) Based on the norm, we get $\|F\| = \sqrt{x^2 + y^2 + z^2}^{\frac{1}{2}}$. Therefore we need to compute:

$$\begin{aligned}
\nabla \frac{1}{\|\mathbf{F}\|^3} &= \nabla (x^2 + y^2 + z^2)^{-\frac{3}{2}} \\
&= \frac{-3}{2}\left(\frac{2x}{(x^2 + y^2 + z^2)^{\frac{-5}{2}}}, \frac{2y}{(x^2 + y^2 + z^2)^{\frac{-5}{2}}}, \frac{2z}{(x^2 + y^2 + z^2)^{\frac{-5}{2}}}\right) \\
&= \left(\frac{-3x}{(x^2 + y^2 + z^2)^{-\frac{5}{2}}}, \frac{-3y}{(x^2 + y^2 + z^2)^{-\frac{5}{2}}}, \frac{-3z}{(x^2 + y^2 + z^2)^{-\frac{5}{2}}}\right)
\end{aligned}$$

2. (a) The curve is given by $\vec{r}(t) = (\cos(t), \sin(t), t)$. Then $\vec{r}'(t) = (-\sin(t), \cos(t), 1)$

$$\begin{aligned}
\int_C (x + y + z) ds &= \int_{t=0}^{t=2\pi} (\cos(t) + \sin(t) + t) \left\| \vec{r'(t)} \right\| dt \\
&= \int_0^{2\pi} (\cos(t) + \sin(t) + t) \sqrt{\cos^2(t) + \sin^2(t) + 1} dt \\
&= \int_0^{2\pi} \sqrt{2}(\cos(t) + \sin(t) + t) dt \\
&= 2\pi\sqrt{2}
\end{aligned}$$

(b) We let $\vec{r}(t) = (1, 2, t^2)$. Then $\vec{r}'(t) = (0, 0, 2t)$.

$$\begin{aligned}
\int_C e^{\sqrt{z}} ds &= \int_{t=0}^{t=1} e^{\sqrt{t^2}} \sqrt{0^2 + 0^2 + (2t)^2} dt \\
&= \int_0^1 e^{|t|} \, |2t| \, dt \\
&= \int_0^1 2te^t dt \\
&= 2\Big[(t-1)e^t\Big]_0^1 \\
&= 2
\end{aligned}$$

3. (a) To parameterise the surface, we let the 2 parameters be x, z, therefore $y = \frac{1}{3}(4 - 2x - z)$. Therefore, $\vec{s}(x, z) = (x, \frac{4}{3} - \frac{2}{3}x - \frac{1}{3}z$. The associated domain is given by $x \in [1, 2], z \in [2, 4]$

   (b) We parameterise the surface using $u = x+y+z$ and $v = x-y$. Then we have $u+v = 2x + z = 4 - 3y$ upon re-arranging the given function. Therefore, $y = \frac{1}{3}(4 - u - v)$.

   Similarly, we have $u - v = 2y + z$. Back substituting, we obtain $z = u - v - \frac{2}{3}(4 - u - v) \implies z = -\frac{8}{3} + \frac{5}{3}u - \frac{1}{3}v$. Substituting into $u + v = 2x + z$, we get $x = \frac{1}{2}(u + v - (-\frac{8}{3} + \frac{5}{3}u - \frac{1}{3}v)) = \frac{4}{3} - \frac{1}{3}u + \frac{2}{3}v$. Hence, the surface can be described as:

$$\vec{s}(u, v) = \Big(\frac{4}{3} - \frac{1}{3}u + \frac{2}{3}v, \frac{1}{3}(4 - u - v), -\frac{8}{3} + \frac{5}{3}u - \frac{1}{3}v\Big)$$

   This has the domain $u \in [0, 7], v \in [2, 4]$

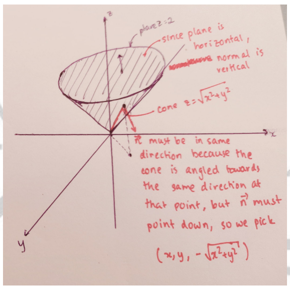
   (c) We use a cylindrical co-ordinate representation, with $x = r\cos\theta, y = r\sin\theta, z = 4 - 2r\cos\theta - 3r\sin\theta$ and thus the parametric representation is given by $(r\cos\theta, r\sin\theta, 4 - 2r\cos\theta - 3r\sin\theta)$. This has the domain $r \in [0, 2], \theta \in \Big[-\frac{pi}{2}, \frac{pi}{2}\Big]$ since we need the right half of the xy plane and the region is a circle of radius 2.

4. (a) The boundary is given by a conical surface which has the equation $z = \sqrt{x^2+y^2}$. Consider a cylindrical co-ordinate parameterisation. Then $x = u\cos(v), y = u\sin(v)$ and substituting, we get $z = u$ since $u \geq 0$. The parameteric representation of the top circle is given as $(0, 0, 2)$. Therefore, the overall surface is given by:

$$\mathbf{x}(u,v) = \begin{cases} (u\cos(v), u\sin(v), u) & 0 \leq z < 2, u \in [0,2), v \in [0, 2\pi] \\ (u\cos(v), u\sin(v), 2) & z = 2, u = 2, v \in [0, 2\pi] \end{cases}$$

(b) As seen from the diagram, the unit normal clearly has to be directed upwards at $z = 2$, so the vector will be given by $\hat{\mathbf{n}} = (0, 0, 1)$.

For the curved surface of the cone, the unit normal will have to point in the same direction as the corresponding vector in the xy plane, but in the vertically opposite direction. So if we move to the position $(u\cos(v), u\sin(v), u)$, the normal vector is given by $(u\cos(v), u\sin(v), -u)$. To unitise this, we must divide by the length of this vector, which is $\sqrt{u^2\cos^2(v) + u^2\sin^2(v) + u^2} = u\sqrt{2}$. Hence, the unit normal is given by $\left(\frac{1}{\sqrt{2}}\cos(v), \frac{1}{\sqrt{2}}\sin(v), \frac{1}{\sqrt{2}}\right)$



(c) For the top of the cone, the surface vector is given by $g_1(u,v) = (u\cos(v), u\sin(v), 2)$. Hence, we have $\frac{\partial g_1}{\partial u} = (\cos(v), \sin(v), 0)$, $\frac{\partial g_1}{\partial v} = (-u\sin(v), u\cos(v), 0)$. Computing the cross product, we have:

$$\det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos(v) & \sin(v) & 0 \\ -u\sin(v) & u\cos(v) & 0 \end{bmatrix} = 0\hat{i} - 0\hat{j} + u\hat{k}$$

This vector has length u, so the unit normal would be $\frac{1}{u}(0, 0, u) = (0, 0, 1)$.

Similarly, for the lateral boundary surface, we have the vector $g_2(u,v) = (u\cos(v), u\sin(v), u) \implies \frac{\partial g_2}{\partial u} = (\cos(v), \sin(v), 1)$, $\frac{\partial g_2}{\partial v} = (-u\sin(v), u\cos(v), 0)$. Thus, we have:

$$\det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos(v) & \sin(v) & 1 \\ -u\sin(v) & u\cos(v) & 0 \end{bmatrix} = u\cos(v)\hat{i} - (-u)\sin(v)\hat{j} + u\hat{k} = (u\cos(v), u\sin(v), u)$$

This normalizes to be $\left(\frac{1}{\sqrt{2}}\cos(v), \frac{1}{\sqrt{2}}\sin(v), \frac{1}{\sqrt{2}}\right)$.

(d) To compute the surface area of the cone, we evaluate:

$$
\begin{aligned}
\iint_{\partial\Omega} dS &= \iint_{\Omega'} \|\mathbf{n}\|\, dA \\
&\qquad (\Omega \text{ is the solid, } \Omega' \text{ is the region describing restrictions on } u, v) \\
&= \int_{v=0}^{v=2\pi} \int_{u=0}^{u=2} \sqrt{(\cos(v))^2 + (\sin(v))^2 + 1^2} du\, dv \\
&+ \int_{v=0}^{v=2\pi} \int_{u=0}^{u=2} \sqrt{(u\cos(v))^2 + (-u\sin(v))^2 + 0^2} du\, dv \\
&\qquad\qquad\qquad\qquad (\text{There are 2 distinct surfaces}) \\
&= \int_{v=0}^{v=2\pi} \int_{u=0}^{u=2} \sqrt{2} du\, dv + \int_{v=0}^{v=2\pi} \int_{u=0}^{u=2} \sqrt{1} du\, dv \\
&= 4\pi\sqrt{2} + 4\pi \\
&= 4\pi(\sqrt{2} + 1)
\end{aligned}
$$

5. (a) First parameterising the line between $(0,0,-1),(0,0,1)$, we get:

$$\mathbf{r}(t) = (1-t)(0,0,-1) + t(0,0,1) = (0,0,2t-1), t \in [0,1]$$

This further implies that:

$$\mathbf{r}'(t) = (0,0,2)$$

Therefore:

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_{t=0}^{t=1} (0^2 + 0^2 + (2t-1)^2, -(2t-1), (0+1)) \cdot (0,0,2)dt \\ &= \int_0^1 2dt \\ &= 2\end{aligned}$$

(b) We now parameterise the curve $y^2 + z^2 = 1$ in the plane of $x = 0$, and thus we have $\mathbf{r}(t) = (0, \sin(t), \cos(t)), t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Therefore, we evaluate:

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (0^2 + \cos^2(t) + \sin^2(t), -\sin(t), (\cos(t)+1)) \cdot (0, -\sin(t), \cos(t))dt \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (0 + \sin^2(t) + \cos^2(t) + \cos(t))dt \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + \cos(t))dt \\ &= \pi + 2\end{aligned}$$

So the vector field is not conservative because the integrals across 2 difference paths are not equal to each other. [NOTE: This is generally what you do for disproving statements - provide counterexamples.]

6.

$$
\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{S} &= \iint_R (\sinh(x)\hat{\imath} + \cosh(y)\hat{k}) \cdot (1\hat{\imath} + 2y\hat{\jmath} - 1\hat{k}) dA \\
&\qquad\qquad (\nabla f = \nabla(x + y^2 - z) \text{ is normal to the surface}) \\
&= \int_{x=0}^{x=1} \int_{y=0}^{y=x} \sinh(x) - \cosh(y) dy\, dx \\
&= \int_{x=0}^{x=1} x\sinh(x) - \sinh(x) + \sinh(0) dx \\
&= \Big[x\cosh(x) - \sinh(x) - \cosh(x) \Big]_{x=0}^{x=1} \\
&\qquad\qquad (\text{Integrating by parts with } u = x, v' = \sinh(x)) \\
&= \cosh(1) - \sinh(1) - \cosh(1) - (0 - \sinh(0) - \cosh(0)) \\
&= 1 - \sinh(1)
\end{aligned}
$$

7.

$$
\begin{aligned}
\iint_S \mathbf{F} \cdot \hat{n} dS &= \iint_R \mathbf{F} \cdot \mathbf{n} dA \\
&= \iint_R (y, -x, z) \cdot (2x, 2y, 1) dA \\
&= \iint_R z dA \\
&= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} (4 - r^2) r dr\, d\theta \\
&\qquad (\text{Using polar co-ordinates, because the region of integration is circular}) \\
&= 2\pi \Big[2r^2 - \frac{1}{4} r^4 \Big]_{r=0}^{r=2} \\
&= 2\pi \times 4 \\
&= 8\pi
\end{aligned}
$$

8. Since the vector field $\mathbf{F} = (x^2 - 2xy, x^2 + 3)$ is clearly differentiable over $\mathbb{R}^2$, we can use Green's Theorem:

$$\begin{aligned}
\oint_C (x^2 - 2xy)dx + (x^2+3)dy &= \iint_R \frac{\partial}{\partial x}(x^2+3) - \frac{\partial}{\partial y}(x^2 - 2xy)dA \\
&= \int_{x=0}^{x=2} \int_{y=-\sqrt{8x}}^{y=\sqrt{8x}} 2x - (-2x)dy\, dx \\
&= \int_{x=0}^{x=2} 4x \cdot 2\sqrt{8x}dx \\
&= 16\sqrt{2} \int_0^2 x^{\frac{3}{2}}\, dx \\
&= 16\sqrt{2}\frac{2}{5}\left[x^{\frac{5}{2}}\right]_0^2 \\
&= \frac{32\sqrt{2}}{5} \cdot 4\sqrt{2} \\
&= \frac{256}{5}
\end{aligned}$$

9.

$$\begin{aligned}
\frac{1}{2}\oint_{\partial\Omega}(xdy - ydx) &= \frac{1}{2}\iint_R \frac{\partial}{\partial x}x - \frac{\partial}{\partial y}(-y)dA \\
&\qquad \text{(Be careful of what the integral was, } xdy - ydx\text{, not } xdx - ydy\text{)} \\
&= \frac{1}{2}\iint_R 2dA \\
&= \iint_R dA \\
&= \text{area}(\Omega) && \text{(By definition)}
\end{aligned}$$

10. Since the vector field is differentiable (because it consists only of polynomial functions) over W and W is a closed and bounded box, we have the following by Divergence Theorem,

$$\begin{aligned}
\iint_{\partial W} \mathbf{F} \cdot d\mathbf{S} &= \iiint_W \nabla \cdot F dV \\
&= \int_{x=0}^{x=1} \int_{y=0}^{y=2} \int_{z=0}^{z=4} \frac{\partial}{\partial x} xy + \frac{\partial}{\partial y} z^3 + \frac{\partial}{\partial z} y^2 dV \\
&= \int_{x=0}^{x=1} \int_{y=0}^{y=2} \int_{z=0}^{z=4} y dV \\
&= 1 \cdot 4 \cdot \int_0^2 y dy \\
&= 4 \cdot 2 \\
&= 8
\end{aligned}$$

11. Since the vector field is differentiable on all of $\mathbb{R}^3$, and Ω is a closed and bounded cylinder given by the equations $z = 0, z = 1, x^2 + y^2 = 1$, we can apply Divergence Theorem.

$$\begin{aligned}
\iint_S \mathbf{F} \cdot \hat{\mathbf{n}} ds &= \iint_S \mathbf{F} \cdot d\mathbf{s} \\
&= \iiint_\Omega \nabla \cdot \mathbf{F} \cdot dV \\
&= \int_{z=0}^{z=1} \int_{x=-1}^{x=1} \int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} \nabla \cdot (x, -y, z^2 - 1) dV \\
&= \int_{z=0}^{z=1} \int_{x=-1}^{x=1} \int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} (1 - 1 + 2z) dy\, dx\, dz \\
&= \int_{z=0}^{z=1} \int_{x=-1}^{x=1} \int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} 2z dy\, dx\, dz \\
&= \int_{z=0}^{z=1} \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} 2zr dr\, d\theta\, dz \\
&= \int_{z=0}^{z=1} \int_{\theta=0}^{\theta=2\pi} z \Big[r^2\Big]_{r=0}^{r=1} d\theta\, dz \\
&= 2\pi \cdot \int_0^1 z dz \\
&= \pi
\end{aligned}$$

12. Method 1: Brute Force

First, we compute $\nabla \times \mathbf{F}$

$$\det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ 2x-y & -yx^2 & -y^2z \end{bmatrix} = \hat{i}\det\begin{bmatrix} \partial_y & \partial_z \\ -yx^2 & -y^2z \end{bmatrix} - \hat{j}\det\begin{bmatrix} \partial_x & \partial_z \\ 2x-y & -y^2z \end{bmatrix}$$
$$+ \hat{k}\det\begin{bmatrix} \partial_x & \partial_y \\ 2x-y & -yx^2 \end{bmatrix}$$
$$= \hat{i}(-2yz-0) - \hat{j}(0) + \hat{k}(-2xy-(-1))$$
$$= (-2yz, 0, 1-2xy)$$

We shall use Spherical co-ordinates to evaluate this integral. Let $x = \cos(\theta)\sin(\varphi), y = \sin(\theta)\sin(\varphi), z = \cos(\varphi)$ since $\rho = 1$. Then $\mathbf{r}(\theta,\varphi) = (\cos(\theta)\sin(\varphi), \sin(\theta)\sin(\varphi), \cos(\varphi))$.

$$\mathbf{r}_\theta = (-\sin(\theta)\sin(\varphi), \cos(\theta)\sin(\varphi), 0)$$

$$\mathbf{r}_\varphi = (\cos(\theta)\cos(\varphi), \sin(\theta)\cos(\varphi), -\sin(\varphi))$$

Thus, we set

$$\mathbf{n} = \mathbf{r}_\theta \times \mathbf{r}_\varphi = (-\cos(\theta)\sin^2(\varphi), -\sin(\theta)\sin^2(\varphi), -\sin(\varphi)\cos(\varphi))$$

But this is an inward pointing normal vector (*test by plugging in a value,* $\varphi = \frac{\pi}{2} \implies \mathbf{n} = (-\cos(\theta), -\sin(\theta), 0)$ *which points towards the origin*), so we reset

$$\mathbf{n} = (\cos(\theta)\sin^2(\varphi), \sin(\theta)\sin^2(\varphi), \sin(\varphi)\cos(\varphi))$$

$$
\begin{aligned}
\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{s} &= \iint_{S'} (-2yz, 0, 1-2xy) \cdot \mathbf{n} dA \\
&\qquad\qquad \text{(Where } S' \text{ is the area projected by S onto the } xy \text{ plane)} \\
&= \iint_{S'} (-2yz, 0, 1-2xy) \cdot \mathbf{n} dA \\
&= \iint_{S'} \begin{bmatrix} -2\sin(\theta)\sin(\varphi)\cos(\varphi) \\ 0 \\ 1 - 2\sin(\theta)\cos(\theta)\sin^2(\varphi) \end{bmatrix} \cdot \begin{bmatrix} \cos(\theta)\sin^2(\varphi) \\ \sin(\theta)\sin^2(\varphi) \\ \sin(\varphi)\cos(\varphi) \end{bmatrix} dA \\
&= \int_{\varphi=0}^{\varphi=\frac{\pi}{2}} \int_{\theta=0}^{\theta=2\pi} -4\sin(\theta)\cos(\theta)\cos(\varphi)\sin^3(\varphi) + \sin(\varphi)\cos(\varphi) d\theta\, d\varphi \\
&= \int_{\varphi=0}^{\varphi=\frac{\pi}{2}} 0 + \frac{1}{2}\sin(2\varphi) \cdot 2\pi d\varphi \\
&\qquad\qquad (\sin(\theta)\cos(\theta) = \tfrac{1}{2}\sin(2\theta) \text{ has odd symmetry about } \pi) \\
&= \pi \int_0^{\frac{\pi}{2}} \sin(2\varphi) d\varphi \\
&= \pi
\end{aligned}
$$

NOTE: You don't need the Jacobian because the change of variables is done originally in the parameterisation, and the normal vector already takes this into account. So don't multiply again by $\rho^2 \sin(\varphi)$. S' already describes the new region in terms of the parameters θ, φ.

Method 2: Stokes' Theorem

Since the surface is closed and bounded, and the vector field is clearly continuous and differentiable over $\mathbb{R}^3$, we can apply Stokes' Theorem.

The boundary curve of S is given by $\mathbf{r}(t) = (\cos(t), \sin(t), 0)$ since it is a circle of radius 1 in the plane $z = 0$.

Therefore, $\mathbf{r}'(t) = (-\sin(t), \cos(t), 0)$

$$\begin{aligned}
\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{s} &= \int_C (2x - y, -yx^2, -yz^2) \cdot d\mathbf{r} \\
&\qquad \text{(Where } \mathbf{r} \text{ parameterises } C\text{, the curve bounding } S\text{)} \\
&= \int_{t=0}^{t=2\pi} (2\cos(t) - \sin(t), -\sin(t)\cos^2(t), 0) \cdot (-\sin(t), \cos(t), 0) dt \\
&= \int_0^{2\pi} (-\sin(t)(2\cos(t) - \sin(t)) - \sin(t)\cos^3(t)) dt \\
&= \int_0^{2\pi} -\sin(2t) + \sin^2(t) - \sin(t)\cos^3(t) dt \\
&= \int_0^{2\pi} \sin^2(t) dt \quad \text{(The other 2 functions have odd symmetry about } \pi\text{)} \\
&= \pi
\end{aligned}$$

13. Let S denote the surface of the plane $z = y + 1$ such that $x^2 + y^2 \leq 1$ Since the surface in consideration is closed and bounded, and the vector field is continuous and differentiable (as it consists of continuous elementary functions), we may apply Stokes' Theorem.

Parameterise $\mathbf{s}(u, v) = (u\cos(v), u\sin(v), u\sin(v) + 1), u \in [0, 1], v \in [0, 2\pi]$.

Then we have:

$$\begin{aligned}
\mathbf{n} &= \mathbf{s}_u \times \mathbf{s}_v \\
&= (\cos(v), \sin(v), \sin(v)) \times (-u\sin(v), u\cos(v), u\cos(v)) \\
&= \begin{bmatrix} \sin(v)u\cos(v) - \sin(v)u\cos(v) \\ -(u\cos(v)\cos(v) - -u\sin(v)\sin(v)) \\ u\cos(v)\cos(v) - (-u\sin(v)\sin(v)) \end{bmatrix} \\
&= (0, -u, u)
\end{aligned}$$

$$
\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{s} &= \int_{\partial S} \nabla \times (4z + x^2, -2x + 3y^5, 2x^2 + 5\sin(z)) \cdot d\mathbf{s} \\
&= \iint_{\S'} (\partial_y(2x^2 + 5\sin(z)) - \partial_z(-2x + 3y^5), -\partial_x(2x^2 + 5\sin(z)) + \partial_z(4z + x^2), \partial_x(-2x + 3y^5) - \\
&= \iint_{S'} (0, -4x + 4, -2) \cdot \mathbf{n} dA \\
&= \int_{v=0}^{v=2\pi} \int_{u=0}^{u=1} (0, -4u\cos(v) + 4, -2) \cdot (0, -u, u) du\, dv \\
&= \int_{v=0}^{v=2\pi} \int_{u=0}^{u=1} (4u\cos(v) - 4u - 2u) du\, dv \\
&= 2\pi \int_0^1 -6u du \\
&= -6\pi
\end{aligned}
$$

14. (a) To verify that it forms a right handed system, we must prove the basis vectors in each direction correspond to the next quantity in line.

$$\mathbf{x} = ((a + w\cos(\varphi))\cos(\theta), (a + w\cos(\psi))\sin(\theta), w\sin(\varphi))$$

Then, we must have:

$$\mathbf{b}_w = \frac{\partial}{\partial w}\mathbf{x} = (\cos(\psi)\cos(\theta), \cos(\psi)\sin(\theta), \sin(\psi))$$

$$\mathbf{b}_\theta = \frac{\partial}{\partial \theta}\mathbf{x} = (-(a + w\cos(\psi))\sin(\theta), (a + w\cos(\psi))\cos(\theta), 0)$$

$$\mathbf{b}_\psi = \frac{\partial}{\partial \psi}\mathbf{x} = (-w\sin(\psi)\cos(\theta), -w\sin(\psi)\sin(\theta), w\cos(\psi))$$

The scaling factor in each case is the norm of each vector. Hence:

$$h_w = \|\mathbf{b}_w\| = 1, h_\theta = a + w\cos(\psi), h_\psi = w$$

So the standard basis vectors in the toroidal co-ordinate system is given by:

$$\mathbf{e}_w = \begin{bmatrix} \cos(\psi)\cos(\theta) \\ \cos(\psi)\sin(\theta) \\ \sin(\psi) \end{bmatrix}, \mathbf{e}_\theta = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \\ 0 \end{bmatrix}, \mathbf{e}_\psi = \begin{bmatrix} -\sin(\psi)\cos(\theta) \\ -\sin(\psi)\sin(\theta) \\ \cos(\psi) \end{bmatrix}$$

To verify it is right handed, we must prove that

$$\mathbf{e}_w \times \mathbf{e}_\theta = \mathbf{e}_\psi, \mathbf{e}_\theta \times \mathbf{e}_\psi = \mathbf{e}_w, \mathbf{e}_\psi \times \mathbf{e}_w = \mathbf{e}_\theta]$$

(b) When $\theta = \frac{\pi}{2}, w = b$, we obtain a circle given by $y = a + b\cos(\psi), z = b\sin(\psi)$ in the plane $x = 0$.

(c) When $\theta = \psi, w = b$, we obtain $x = (a + b\cos(\theta))\cos(\theta), y = (a + b\cos(\theta))\sin(\theta), z = b\sin(\theta)$.

To get a good grasp of what the shape looks like, we plug in some values. $\theta = 0, \theta = 2\pi \implies x = a+b, y = 0, z = 0$ so it starts and ends at the same point. If we fix θ, we obtain a circle, so means by varying ψ, we obtain some kind of circle perpendicular to the circle when θ is fixed. So if both vary at the same time, it traces out a spiral.

(d) Computing the arc lengths involves the same formulas:

Curve 1: $(w, \theta, \psi) = (b, \frac{\pi}{2}, t)$ where $t \in [0, 2\pi]$

$$\begin{aligned}
\int_C ds &= \int_{t=0}^{t=2\pi} \sqrt{1^2\Big(\frac{dw}{dt}\Big)^2 + (1 + \frac{1}{2}\cos(t))^2\Big(\frac{d\theta}{dt}\Big)^2 + w^2\Big(\frac{d\psi}{dt}\Big)^2} dt \\
&= \int_0^{2\pi} \sqrt{0 + 0 + b^2} dt \\
&= 2\pi \cdot b \\
&= \pi
\end{aligned}$$

Curve 2: $(w, \theta, \psi) = (b, t, t)$ where $t \in [0, 2\pi]$

$$
\begin{aligned}
\int_C ds &= \int_{t=0}^{t=2\pi} \sqrt{1^2\left(\frac{dw}{dt}\right)^2 + (1+\frac{1}{2}\cos(t))^2\left(\frac{d\theta}{dt}\right)^2 + w^2\left(\frac{d\psi}{dt}\right)^2}dt \\
&= \int_0^{2\pi} \sqrt{0 + (a + b\cos(t))^2 + b^2}dt \\
&= \int_0^{2\pi} \sqrt{\frac{1}{4} + (1+\frac{1}{2}\cos(t))^2}dt \\
&= \int_0^{2\pi} \frac{1}{2}\sqrt{1 + (2+\cos(t))^2}dt \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}\left[\sqrt{1+(2+\cos(t))^2} + \sqrt{1+(2-\cos(t))^2}\right]dt \\
&\qquad\qquad \text{(Due to symmetry of cos, with } \cos(\pi - x) = -\cos(x)\text{)} \\
&= \int_0^{\frac{\pi}{2}} \left[\sqrt{1+(2+\cos(t))^2} + \sqrt{1+(2-\cos(t))^2}\right]dt \qquad \approx 7.10 \\
&\qquad\qquad (\cos(-x) = \cos(x))
\end{aligned}
$$

Seminar 2 Second Half (Fourier Series and Analysis)

Example 1.

From the definition of f, we know it is 2-periodic, so $L = 1$. Further, it is odd, so we can ignore the coefficients of the even functions. That is, $a_0[f] = a_k[f] = 0$. Then, we calculate the coefficients using the standard formula.

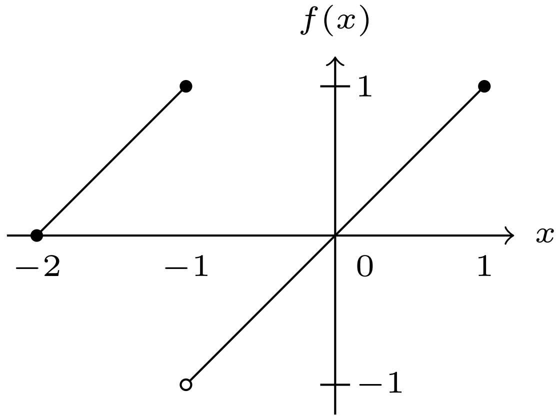
$$\begin{aligned}
b_k[f] &= \frac{2}{L}\int_0^L f(x)\sin\frac{k\pi x}{L}\mathrm{d}x \\
&= 2\int_0^1 x\sin k\pi x\mathrm{d}x \\
&= 2\left(\left[x\frac{-\cos k\pi x}{k\pi}\right]_0^1 - \int_0^1 \frac{-\cos k\pi x}{k\pi}\mathrm{d}x\right) \\
&= -2\frac{\cos k\pi}{k\pi} + \frac{2}{k\pi}\left[\frac{\sin k\pi x}{k\pi}\right]_0^1 \\
&= 2\frac{(-1)^{k+1}}{k\pi},
\end{aligned}$$

as $\sin k\pi = 0$ and $\cos k\pi = (-1)^k$ for all $k \in \mathbb{Z}$. Then, writing the series, we have

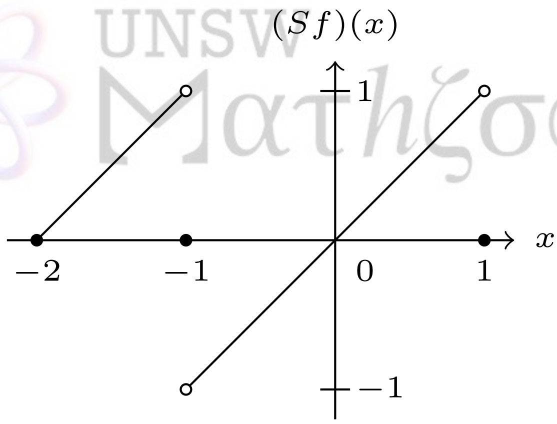
$$(Sf)(x) = \sum_{k=1}^{\infty} 2\frac{(-1)^{k+1}}{k\pi}\sin k\pi x = \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\sin k\pi x.$$

Example 2.

To draw the graph of Sf for $-2 \leq x \leq 1$ we must first consider the graph of f over the same domain:



So the Fourier series graph will be identical everywhere except the jump discontinuities at $x = -1$ and $x = 1$ (extend the graph a bit more to see the discontinuity at $x = 1$). At these discontinuities, the Fourier series will approach the average of the function coming from both sides. So, in this case, it will approach $\frac{1+-1}{2} = 0$. Putting this all together in a graph, we get:



Take special note of the point at $x = 1$. Although we don't draw anything after $x = 1$ we need to make sure to draw the single point at $x = 1$.

Example 3.

Proving pointwise convergence is simple. Simply evaluating the limit we find

$$\lim_{\substack{n\to\infty \\ x\in[0,1)}} x^n = 0.$$

Thus, the function sequence converges to 0 on $[0, 1)$.

To disprove uniform convergence, we can start by assuming the condition of uniform convergence is satisfied, and derive a contradiction. In this case, we can take $\epsilon = \frac{1}{2}$, and state

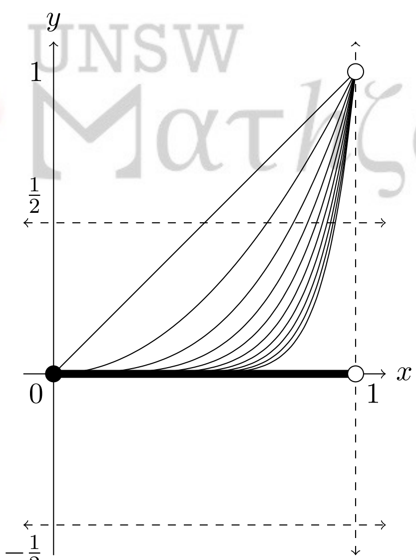
$$|f_n(x) - f(x)| = x^n < \frac{1}{2}$$

for sufficiently large $n \geq N$ and every $x \in [0, 1)$. But then we must have

$$x < \frac{1}{\sqrt[n]{2}} \leq \frac{1}{\sqrt[N]{2}}.$$

So the condition is only satisfied for $x < \frac{1}{\sqrt[N]{2}} < 1$. But we assumed the condition was true for all $x \in [0, 1)$, so we derive a contradiction, and thus the function sequence does not converge uniformly on $[0, 1)$.

To explain why we chose $\epsilon = \frac{1}{2}$, refer to the following graph:



The solid black line along the x-axis is the function f we're taking as the sequence limit. In this case, it's just the zero function. The horizontal dashed lines are $f + \frac{1}{2}$ and $f - \frac{1}{2}$ (if f curved, they would too). Now each of the other graphs are $f_n(x) = x^n$ for varying $n \geq 1$. As we can see, no matter how large n is, there is always some point on f_n that is outside of the box, we just have to take some value closer and closer to $x = 1$. We could take $\epsilon \in (0, 1)$ for the contradiction, $\epsilon = \frac{1}{2}$ is just a nice number.

Geometrically, we see that a sequence of functions converges uniformly if we can take $f + \epsilon$ and $f - \epsilon$ and "squish" the sequence between them, as $\epsilon \to 0^+$.

Example 4.

We can use Weierstrass M-test to prove series converge uniformly. Applying the test, we have

$$\left|\frac{\cos nx}{n^2+x^2}\right| \leq \frac{1}{n^2+x^2} \leq \frac{1}{n^2}.$$

Then, by p-test, we know the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges. Thus, by Weierstass M-test, the function

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n^2+x^2}$$

converges uniformly for all $x \in \mathbb{R}$.

Example 5.

Again, applying Weierstrass M-test, we find

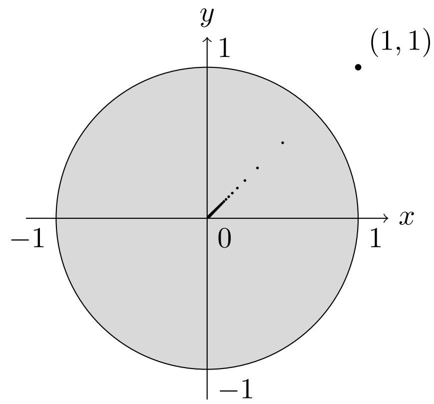
$$\sum_{k \text{ odd}} \left|\frac{-4}{k^2\pi^2}\right| + |0| \leq \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2},$$

which converges by p-test, so the corresponding Fourier series converges uniformly to f (as it's the Fourier series of f) on $\mathbb{R}$.

Now, if a sequence of continuous functions converges uniformly to a function f, then f must be continuous. Since all the partial Fourier series $S_n f$ are continuous, this means f must be continuous also. This is true for any uniformly convergent Fourier series.

Example 6.

We can draw the set we're considering:



Now, we're considering the origin, and the point $(1, 1)$:

(a) The origin is an interior point, as there is a ball around it (namely $B(\mathbf{0}, 1)$) that is contained entirely within the set.
It is not a boundary point, as not every ball around it contains both point inside and outside of the set. In general, interior and boundary are exclusive (a point can be only one).
It is a limit point, because we can form a sequence entirely within the set that has limit $(0, 0)$. For example, $\left\{\left(\frac{1}{n}, \frac{1}{n}\right)\right\}_{n=1}^{\infty}$, which is drawn on the diagram.

(b) The point $\mathbf{a} = (1, 1)$ is not an interior point, because every ball around $\mathbf{a}$ contains points outside the set. Specifically, for every ball $B(\mathbf{a}, \epsilon)$, the point $\left(1, 1 + \frac{\epsilon}{2}\right) \in B(\mathbf{a}, \epsilon)$ is not inside the set.
It is a boundary point, because every ball around it contains elements both inside and outside the set. As above, we have a point outside the set, and the point $\mathbf{a} \in B(\mathbf{a}, \epsilon)$ is part of the set.
Finally, it is not a limit point, because there is no sequence that is contained entirely within the set with limit $\mathbf{a}$ that doesn't eventually become the constant sequence. To have such a sequence, you need a dense set of points around $\mathbf{a}$ (informally).

Example 7.

To prove a set is open, we need to show that there is a ball around every point contained in the set. So, since we're working in $\mathbf{R}$, this means we need to show there is an open interval around every $x \in (a, b)$, say $(x-\epsilon, x+\epsilon)$, contained in (a, b). So, we let $\epsilon = \min\{x-a, b-x\}$. That is, the distance from x to each endpoint. Now,

$$\begin{aligned} & x - a \geq \min\{x-a, b-x\} = \epsilon \\ \iff \quad & x - \epsilon \geq a. \end{aligned}$$

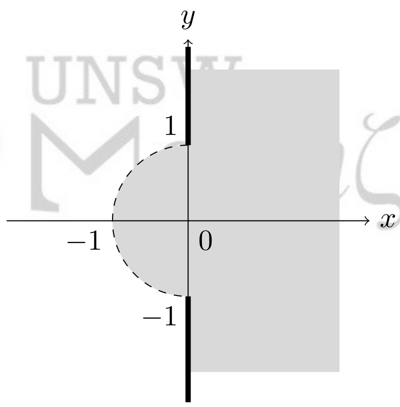
Similarly,

$$\begin{aligned} & b - x \geq \min\{x-a, b-x\} = \epsilon \\ \iff \quad & b \geq x + \epsilon. \end{aligned}$$

Putting this together, $(x-\epsilon, x+\epsilon) \subseteq (a, b)$ as required. So, we have found an ϵ for every element $x \in (a, b)$ such that $B(x, \epsilon) \subseteq (a, b)$.

Example 8.

The set in consideration is:



The points on the y-axis are included in the set, in a bolder line to differentiate from the axes. Now, this set is clearly not open, since some boundary points are contained in the set (all of the y-axis below -1 and above 1), and every ball around one of these points, say $(0, 2)$ contains points outside of the set.
Similarly, it isn't closed, because it doesn't contain all of its boundary points.
In general, if a set contains some, but not all of its boundary points, then it is neither open nor closed. In $\mathbb{R}^n$ the only sets that are **both** open and closed are $\mathbb{R}^n$ itself and the empty set $\emptyset$ (if you were curious).

Example 9.

To apply the definition of the limit for this limit, we need to express $|f(x,y)-1|$ as an inequality in terms of something we can relate to $\sqrt{x^2+y^2}<\delta$. To this effect, we find

$$\begin{aligned}\left|\frac{x^2+y^2+x^2y^2}{x^2+y^2}-1\right| &= \left|\frac{x^2y^2}{x^2+y^2}\right| \\ &= \frac{x^2y^2}{x^2+y^2} \\ &\le \frac{x^4+2x^2y^2+y^4}{x^2+y^2} \\ &= \frac{(x^2+y^2)^2}{x^2+y^2} \\ &= x^2+y^2.\end{aligned}$$

Notice that we needed to get rid of the awkward term in the numerator, so we force positive terms into the numerator to cancel with the denominator and give us something we can work with. Be careful when doing this, as the inequality doesn't hold if you force in terms like xy, which could be negative.

Now that we have something similar to $\sqrt{x^2+y^2}$, just as with single variable limits, we can choose our $\delta=\sqrt{\epsilon}$. Then, if $\sqrt{x^2+y^2}<\delta$, we have $|f(x)-1|\le x^2+y^2<\delta^2=\epsilon$. Thus, we have found a δ for every $\epsilon>0$ that satisfies the limit condition.

Example 10.

To find the limit of a vector-valued function, we can consider each component separately as a scalar function. So, the first component is the same as that from our previous question, but a simpler way to evaluate its limit using Pinching theorem is establishing the inequality

$$1 \leq \frac{x^2+y^2+x^2y^2}{x^2+y^2} \leq 1 + \frac{x^4+2x^2y^2+y^4}{x^2+y^2} = 1 + x^2 + y^2.$$

We have done the same thing, forcing in positive terms, but in this case we aren't aiming to get an expression we can relate to $\sqrt{x^2+y^2}$, but instead trying to cancel the denominator (as it makes the limit indeterminate). Now as $(x, y) \to \mathbf{0}$, the first component gets "squeezed" between two functions both approaching 1, so the limit must be 1.
To deal with the second component, we have to first establish a few inequalities. Notice that the order of the numerator is 3 and the order of the denominator is 2. So we expect the limit, if it exists, to be 0 (higher order polynomial on the top). Of course, it may not exist, so we prove it does. Because the numerator is of order 3, we can try to prove (though it's not obvious by any means) that

$$\left|\frac{xy}{x^2-xy+y^2}\right| \leq 1.$$

So we need to prove $|x^2-xy+y^2| \geq |xy|$. If you've done four unit maths in the HSC, this may look familiar. In fact, this can be shown by proving a slightly stronger inequality:

$$\begin{aligned} & (|x|-|y|)^2 \geq 0 \\ \iff \quad & x^2 - 2|xy| + y^2 \geq 0 \\ \iff \quad & x^2 - |xy| + y^2 \geq |xy|. \end{aligned}$$

Now, $xy \leq |xy|$ for all $x, y \in \mathbb{R}$, so $-xy \geq -|xy|$, and applying this to the previous inequality gives us

$$|x^2-xy+y^2| \geq x^2-xy+y^2 \geq x^2-|xy|+y^2 \geq |xy|.$$

This proves our original inequality, and we can use that to finally show that

$$0 \leq \left|\frac{xy(x+y)}{x^2-xy+y^2}\right| \leq |x+y|\,.$$

Since both sides tend to 0 as we approach $\mathbf{0}$, the limit of the second component is 0, and so the whole limit is $(1, 0)$. This is certainly one of the more difficult limits questions.

Example 11.

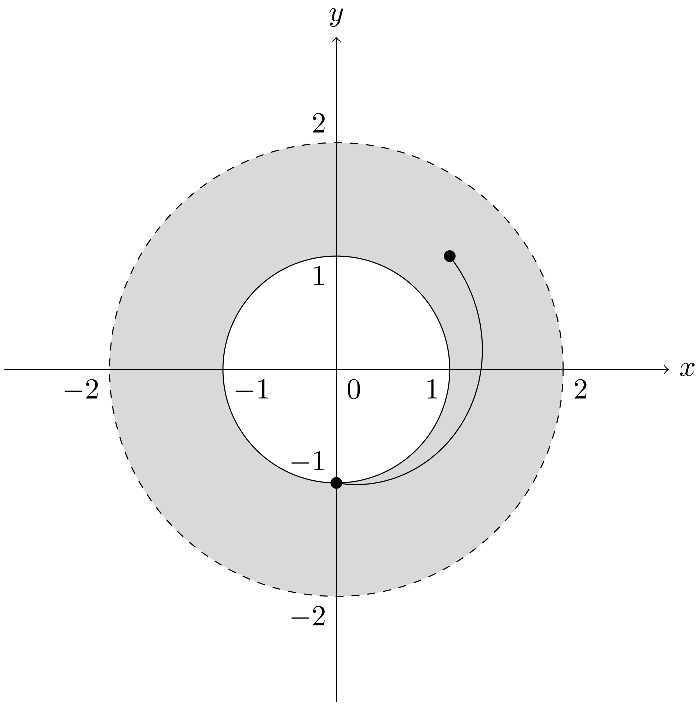
Notice that $\mathbb{R}^2$ is open, so every point in $\mathbb{R}^2$ is a limit point. This means we need to show the limit at every point is equal to the function value. Now, for every $(x, y) \neq \mathbf{0}$, the limit is clearly the function value, as it isn't in an indeterminate form. However, at $\mathbf{0}$ the limit may not exist. To computer the limit of the first component, notice the numerator has order 3, whereas the denominator has order 2, so we expect the limit to be 0 if it exists. So, we have

$$0 \leq \left| \frac{x^3}{x^2 + y^2} \right| = \frac{|x|x^2}{x^2 + y^2} \leq \frac{|x|x^2 + |x|y^2}{x^2 + y^2} = |x|.$$

So, by Squeeze Theorem, the first component's limit is 0 at $\mathbf{0}$. By symmetry, the second component also approaches 0, and thus the function limit is $\mathbf{0}$, which is equal to the function value at $\mathbf{0}$. Since at every limit point, the limit exists and is equal to the function value, we conclude that $\mathbf{f}$ is continuous on all $\mathbb{R}^2$.

Example 12.

As always, we draw the set:



Now just looking at the set we can see that it is path connected. To prove it, we construct a path between two arbitrary points. Since it's radially symmetric, we use polar coordinates. That is, taking two arbitrary points $\mathbf{x}_1 = (r_1 \cos\theta_1, r_1 \sin\theta_1), \mathbf{x}_2 = (r_2 \cos\theta_2, r_2 \sin\theta_2) \in \Omega$, where $1 \le r_1 \le r_2 < 2$ without loss of generality. Now one such path could be a sort of spiral, where we change the radius and angle uniformly throughout the path. One such path is shown on the diagram. To do this, we define two functions

$$r(t) = r_1 + t(r_2 - r_1),$$
$$\theta(t) = \theta_1 + t(\theta_2 - \theta_1).$$

Then, as t varies from 0 to 1, $r(t)$ varies from r_1 to r_2, and $\theta(t)$ varies from θ_1 to θ_2. Using these as polar coordinates for our path, we define

$$\phi(t) = (r(t)\cos\theta(t), r(t)\sin\theta(t)),$$

which is clearly continuous. Now, we have to confirm it's a valid path:

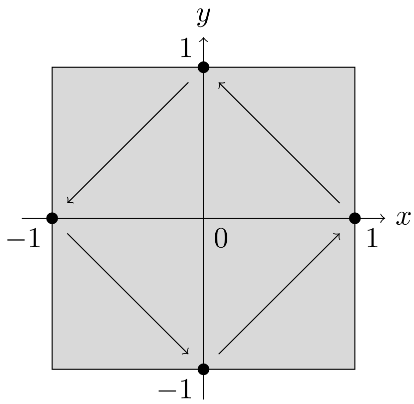
(a) For $t \in [0, 1]$, we have $1 \le r_1 \le r_1 + t(r_2 - r_1) \le r_2 < 1$, so $\phi(t) \in \Omega$.

(b) $\phi(0) = \mathbf{x}_1$ and $\phi(1) = \mathbf{x}_2$.

So, the function ϕ is a continuous path between any two points in the set lying entirely within the set. Since such a function exists, Ω is path connected.
Since the set contains only some of its boundary, it is not closed, and thus not compact.

Example 13.

Again, we draw the set, and the sequence:

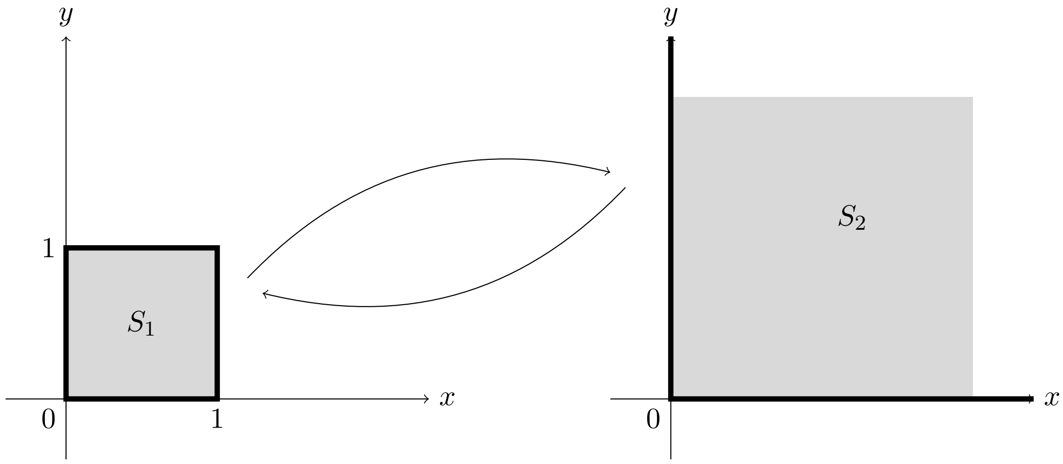


Now, since the sequence cycles the four points, it cannot converge (analogous to boundedly divergent). However, since the set is bounded and closed, we can state that there is a subsequence that does converge to an element in the set by Bolzano-Weierstass. One such subsequence could be the constant sequence $\{(1,0)\}_{i=1}^{\infty}$, or any other eventually constant sequence.

Note that although sequences we deal with generally cannot be constant (like in the definition of a limit point), there is no such restriction for Bolzano-Weierstrass.

Example 14.

Here we have two sets:



We see that S_1 is compact as it is both closed and bounded. If there were a continuous function mapping S_1 onto S_2, then the image of S_1 would need to be compact as well. However, since we're looking for a function mapping **onto** S_2, the image of S_1 would be S_2. Since S_2 is not compact, as it is not bounded, there cannot be such a function. Note that since both sets are path connected, we can only apply the compact part of the image theorems.

S_2 is only path connected, so we cannot apply the theorem on compactness. Since both sets are path connected, there is no contradiction, and there may be a function from S_2 onto S_1. We don't know if there is such a function, as the image theorems only give us one-way implications, however we can check if there is such a function by creating one. We want to compress an infinitely large region into a small box, so we look for functions that are bounded. One such familiar function is sin. Since $0 \leq |\sin x| \leq 1$, and we want our components to be within that range, we can try the function

$$\mathbf{f}(x, y) = (|\sin x|, |\sin y|) .$$

It's not difficult to show that this function maps S_2 onto S_1, and since all its components are continuous, the function itself is continuous. Thus, there is a continuous function from S_2 onto S_1. This is not the only function, but is a simple one.