UNSW Mathematics Society presents

MATH2501/2601 Revision Seminar

(Higher) Linear Algebra

Seminar II / II

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Credit to Rui Tong and Kabir Agarwal's 2019 slides

Eigenvalues and Eigenvectors

Eigenvectors, Eigenvalues and Eigenspaces

Definitions

Let V be a vector space and $T : V \mapsto V$ a linear transformation. If λ is a scalar and $\mathbf{v}$ a non-zero vector in V such that $T(\mathbf{v}) = \lambda\mathbf{v}$, then λ is an eigenvalue of T and $\mathbf{v}$ is an eigenvector of T corresponding to λ. The set of eigenvalues of T is the spectrum of T.

Invariance

Definition

Let $T : V \mapsto V$ be a linear transformation. A subspace U of V is said to be T-invariant if $T(U) \subseteq U$, where:

$$T(U) = \{T(\mathbf{u}) | \mathbf{u} \in U\}.$$

Properties

Basic properties of eigenspaces

Let $T : V \mapsto V$ be linear.

1. The eigenvalues of T are λ such that $T(\mathbf{v}) = \lambda\mathbf{v}$.
2. The eigenspace corresponding to λ is given by $E_\lambda = \ker(\lambda I - T)$.
3. Eigenspaces are T-invariant
4. If λ and μ are eigenvalues of T and $\lambda \neq \mu$, then $E_\lambda \cap E_\mu = \{\mathbf{0}\}$.
5. If V is finite-dimensional, a basis B of V consists of eigenvectors of T if and only if the matrix of T with respect to B is diagonal.

More properties

More properties

1. A matrix $A \in M_{nn}(\mathbb{F})$ is diagonalisable if and only if it has n linearly independent eigenvectors associated with it.
2. Distinct eigenvalues correspond to linearly independent eigenvectors.
3. $\mathbf{v}$ is an eigenvector of T with eigenvalue λ if and only if $[\mathbf{v}]_B$ (where $T : V \mapsto V$, and B is a basis of V) is an eigenvector of T with eigenvalue λ.

Definition

The eigenvalues of a matrix A are given by the solutions to the characteristic polynomial, the polynomial obtained by solving $\det(A - \lambda I) = 0$.

Multiplicities

AM-GM Inequality Re-mastered

The algebraic multiplicity of an eigenvalue λ is the multiplicity of the root $z = \lambda$ for the characteristic equation $\det(A - \lambda I) = 0$. The geometric multiplicity is the dimension of the eigenspace associated with λ, that is, $\dim(\ker(A - \lambda I)) = \text{GM}(\lambda)$. The relationship between these two can be described as $\text{GM}(\lambda) \leq \text{AM}(\lambda)$.

Corollary

A matrix A is diagonalisable if for every eigenvalue λ_i, we have $\text{GM}(\lambda_i) = \text{AM}(\lambda_i)$.

Examples of eigenvalues, eigenvectors and diagonalisation

Example

Find all the eigenvalues and eigenvectors of the following matrices:

1. $\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$
2. $\begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix}$
3. $\begin{pmatrix} 1 & 4 \\ -1 & 1 \end{pmatrix}$
4. $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix}$

Solutions

Note that in any matrix, the sum of the eigenvalues is the trace of the matrix, and the product of eigenvalues is the determinant.

- 1) Let the eigenvalues be λ_1, λ_2. Then $\lambda_1 + \lambda_2 = 5, \lambda_1\lambda_2 = 6$. Therefore, $\lambda_1, \lambda_2 = 2, 3$ by inspection. The eigenvectors are given by the kernel of $A - \lambda I$ for each eigenvalue. So the eigenvectors are given by: For $\lambda = 2$: $\ker \begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix} = \text{span} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$
  For $\lambda = 3$: $\ker \begin{pmatrix} -2 & 2 \\ -1 & 1 \end{pmatrix} = \text{span} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
  So for $\lambda_1 = 2, \mathbf{v}_1 = (2, 1)$, and $\lambda_2 = 3, \mathbf{v}_2 = (1, 1)$.
- 2) Similarly, we obtain the eigenvalues to be $\lambda_1 = 1, \lambda_2 = 6$ upon using the same idea. The eigenvectors are then given by the following kernels respectively:

$$\ker \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} = \text{span} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Solutions

- 3) Similarly, we obtain the eigenvalues to be $\lambda_1 = 1 - 2i, \lambda_2 = 1 + 2i$. The eigenvectors are of the form:

$$\ker \begin{pmatrix} 2i & 4 \\ -1 & 2i \end{pmatrix} = \operatorname{span} \begin{pmatrix} 2i \\ 1 \end{pmatrix}$$

$$\ker \begin{pmatrix} -2i & 4 \\ -1 & -2i \end{pmatrix} = \operatorname{span} \begin{pmatrix} -2i \\ 1 \end{pmatrix}$$

- 4) The characteristic polynomial of this matrix is going to be $\det(A - \lambda I) = (2 - \lambda)\det \begin{pmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{pmatrix} = (2 - \lambda)((3 - \lambda_2)^2 - 1)$. Solving for λ, we obtain $\lambda = 2, 2, 4$. The corresponding eigenvectors are $\lambda_1 = (1, 0, 0), (0, -1, 1)$ for $\lambda = 2$, and $\lambda = 4$ means that the eigenvector is just $(0, 1, 1)$.

Examples on AM, GM

Example

Find the algebraic and geometric multiplicity of each of the above matrices.

Solutions

1. $\text{AM}(2) = 1$, $\text{AM}(3) = 1$. The corresponding GMs are $\text{GM}(2) = 1$, $\text{GM}(3) = 1$.
2. Same as above.
3. Same as above.
4. $\text{AM}(2) = 2$, $\text{AM}(4) = 1$. $\text{GM}(2) = 2$, $\text{GM}(4) = 1$.

Examples on AM, GM

Example

For each of the following matrices, use the given additional information to find all eigenvalues and eigenvectors *without calculating the characteristic polynomial*. Also write down the algebraic and geometric multiplicities of each eigenvalue.

1. $C = \begin{pmatrix} 2 & -5 & -5 \\ -4 & 8 & 4 \\ 4 & -11 & -7 \end{pmatrix}$, given that 2 and -3 are eigenvalues.
2. $D = \begin{pmatrix} 1 & 4 & 2 \\ 2 & 1 & -2 \\ -3 & 4 & 6 \end{pmatrix}$, given that $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ are eigenvectors.

Solutions

1. $\lambda_1 = 2, \lambda_2 = -3$. Since trace is equal to the sum of eigenvalues, we have $\lambda_3 = (2 + 8 + -7) - (2 + -3) = 4$. The arithmetic multiplicity of each eigenvalue is 1, and since geometric multiplicity is positive and less than or equal to 1, the GM of each eigenvalue must be 1.
2. The eigenvalue associated with $(1, 0, 1)$ is given by $\lambda = 3$, and for $(2, -1, 2)$ is given by 1. Therefore, the last eigenvalue must be 4 because of the trace of the matrix. Taking the kernel of 4, we obtain:

$$\begin{pmatrix} -3 & 4 & 2 \\ 2 & -3 & 2 \\ -3 & 4 & 2 \end{pmatrix}$$

which has a kernel of $(-2, -2, 1)$.

More theorems

Conditions for diagonalisability

Let $T : V \mapsto V$ be a linear map on a finite dimensional vector space V. Then the following are equivalent:

1. T is diagonalizable
2. There is a basis for V consisting of the eigenvectors of T.
3. V is the direct sum of the eigenspaces of each of the eigenvalues.
4. The sum of geometric multiplicities of distinct eigenvalues is the dimension of V.

Examples

Example [2501 Eigenvalues Q8]

Let V be a vector space and $\{v_1, v_2, v_3\}$ a basis for V. Let T be a linear map from V to V such that:

$$T(\mathbf{v}_1) = 2\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3, \quad T(\mathbf{v}_2) = 2\mathbf{v}_2. \quad T(\mathbf{v}_3) = \mathbf{v}_2 + \mathbf{v}_3.$$

Is there a basis B for V such that the matrix of T with respect to B is diagonal? Explain.

Solutions

With respect to the basis, the linear map T can be written as:

$$T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = (2a)\mathbf{v}_1 + (a+3b+c)\mathbf{v}_2 + (a+c)\mathbf{v}_3 = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

The characteristic polynomial is given by $(2-\lambda)(2-\lambda)(1-\lambda) = 0$. Thus the eigenvalues are given by $\lambda = 1, 2, 2$. Considering the kernel of $A - 2I$, we get:

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

which has a dimension of 2. Therefore sum of GM is GM(1)+GM(2) $= 1+2 = 3$. Diagonalisable.

Spectral Theorem

Theorem: SPECTRAL THEOREM

Let $A \in M_{n\times n}(\mathbb{R})$ be a real symmetric matrix. Then:

1. All the eigenvalues are real.
2. Eigenvectors corresponding to distinct eigenvalues are orthogonal
3. There exists an orthogonal matrix Q such that $Q^{-1}AQ$ is the diagonal matrix corresponding to distinct eigenvalues.
4. A has n orthogonal, real eigenvalues.

Examples of Diagonalisation

Example [Lecture Slides]

Diagonalise the following matrix given that the characteristic polynomial is $p(\lambda) = (\lambda - 3)(\lambda^2 - 1)$:

$$\begin{pmatrix} -1 & -12 & 0 \\ 2 & 5 & 4 \\ 0 & 4 & -1 \end{pmatrix}$$

Solutions

The eigenvalues are given by $\lambda = -1, 3, 1$. Finding the eigenvectors for each of the eigenvalues, we have: For $\lambda = -1$:

$$\begin{pmatrix} 0 & -12 & 0 \\ 2 & 6 & 4 \\ 0 & 4 & 0 \end{pmatrix}$$

which has a kernel spanned by the vector $(-2, 0, 1)$. For the eigenvalue $\lambda = 3$, we obtain:

$$\begin{pmatrix} -4 & -12 & 0 \\ 2 & 2 & 4 \\ 0 & 4 & -4 \end{pmatrix}$$

which has a kernel of $(-3, 1, 1)$. For the eigenvalue $\lambda = 1$:

$$\begin{pmatrix} -2 & -12 & 0 \\ 2 & 4 & 4 \\ 0 & 4 & -2 \end{pmatrix}$$

Solutions

So the matrix can be diagonlised as follows:

$$\begin{pmatrix} -2 & -3 & -3 \\ 0 & 1 & \frac{1}{2} \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -2 & -3 & -3 \\ 0 & 1 & \frac{1}{2} \\ 1 & 1 & 1 \end{pmatrix}^{-1}$$

Normal Operators (MATH2601)

Definition

A linear transformation on an inner product space is normal if and only if the maps commute with their adjoints.

Theorem

1. If T is normal, then $||T\mathbf{v}|| = ||T^*\mathbf{v}||$ for all $\mathbf{v} \in V$.
2. If T is normal, then $T - \alpha\text{id}$ is normal for any $\alpha \in \mathbb{F}$.
3. The eigenspace of T with eigenvalue λ is the same as the eigenspace of T^* with eigenvalue $\bar{\lambda}$.
4. If T is normal, the 2 eigenspaces corresponding to distinct eigenvalues are orthogonal to each other.

Conic Sections and quadrics

Consider a quadratic equation of the form $ax^2 + 2bxy + cy^2 = k$ for some constant k. Then we can reframe this problem as a matrix equation:

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = k$$

You can check this by expansion of the matrix equation.

Graphing and identifying conics

Begin by diagonalising the real symmetric matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix} = A$ so as to obtain QDQ^T [This just follows from Spectral Theorem]. Let $\mathbf{X} = Q^T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix}$. This allows us to write the form:

$$\begin{pmatrix} X & Y \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = k \implies \lambda_1 X^2 + \lambda_2 Y^2 = k$$

WHICH IS A CONIC!!! We already know that Q consists of the eigenvectors, so the eigenvectors describe the axes of symmetry of the conic and becomes easy to construct from there.

Examples

Example

Sketch the curve $5x^2 + 4xy + 8y^2 = 36$ including all important features and points.

Solutions

Rewriting the equation as the following expression:

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 5 & 2 \\ 2 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 36$$

Computing the eigenvalues of the matrix $\begin{pmatrix} 5 & 2 \\ 2 & 8 \end{pmatrix}$, we have $\lambda = 4, 9$. The eigenspaces are given by the span $(-2, 1)$ (for $\lambda = 4$) and the span of $(1, 2)$ (for $\lambda = 9$). Therefore, the new matrix form of the equation will be:

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = 36$$

Solutions

Creating an orthogonal matrix out of this, we write:

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix} \begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 36$$

Letting $X = \frac{1}{\sqrt{5}}(-2x + y)$ and $Y = \frac{1}{\sqrt{5}}(x + 2y)$, we obtain the equation $4X^2 + 9Y^2 = 36$. So we take the graph of this ellipse with intercepts at $(X = \pm 3, 0)$ and $(0, Y = \pm 2)$. Then we rotate the axes X, Y until they match the new axes given by $X = 0$ and $Y = 0$. So the axes of the ellipse are $y = 2x$ (along which we go 3 units) and $y = -\frac{1}{2}x$ (along which we go 2 units). Note that these will also give the closest and furthest points along the ellipse.

Rotations and reflections

Orthogonal matrices are special matrices with determinant such that $\det Q = \pm 1$. This is equivalent to saying that the eigenvalues each have modulus of 1.

Rotations and reflections

Consider an orthogonal matrix R form $\mathbb{R}^2$ to $\mathbb{R}^2$. Then we can always write R as:

$$R(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

where $e^{i\alpha}$ is an eigenvalue of the linear map T. This describes a ROTATION by an angle α about the origin.

Rotations and reflections

Consider a matrix R to be a 3×3 orthogonal matrix so that it's columns are an orthonormal basis for $\mathbb{R}^3$. Then R is similar to one of the following 2 matrices:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}$$

Angles and axes of reflection

The angles of rotation should not be difficult to work out. You can evaluate these by determining the trace, because the matrix map of T is similar to R described above (based on the diagonalisation procedure). The axis of rotation/reflection is given by the eigenvector corresponding the ± 1 eigenvalue.

Examples of Orthogonal maps

Example [Lecture Slides]

Give a geometric description of the following matrices:

$$A = \frac{1}{9}\begin{pmatrix} 4 & 7 & -4 \\ 1 & 4 & 8 \\ 8 & -4 & 1 \end{pmatrix}, \text{Given spectrum of } A \text{ is } \lambda = 1, -i, i$$

$$B = \frac{1}{9}\begin{pmatrix} 4 & -7 & -4 \\ 1 & -4 & 8 \\ 8 & 4 & 1 \end{pmatrix}, \text{Given one eigenvalue } B \text{ is } \lambda = -1, \det(B) = -1$$

Solutions

First check A is orthogonal: the easiest way to do this is to use dot products of columns. Then check the determinant, which is 1. The axis for that is $(2, 2, 1)$. Thus it's a rotation. The angle of rotation is given by $2\cos\alpha + 1 = 1 \implies \alpha = \frac{\pi}{2}$ using the idea of trace.

Likewise, check B is orthogonal. Then check the determinant, which is -1. The eigenvector corresponding to this is $(-1, -3, 2)$. Check the trace, trace is $1/9$. That means, other eigenvalues will not include 1 as $|\lambda| = 1$. So there is a reflection occurring about a plane, and the plane of reflection will be
$(-1, -3, 2) \cdot \mathbf{x} = 0 \implies -x_1 - 3x_2 + 2x_3 = 0$. The angle of rotation about the axis is given by
$2\cos\alpha - 1 = \frac{1}{9} \implies \alpha = \cos^{-1}\frac{5}{9}$.

Singular Values (MATH2601 only section)

Definition 1: Singular Values

A singular value of a $m \times n$ matrix A is the square root of an eigenvalue of A^*A.

Recall: A^*A denotes the adjoint of A.

Definition 2: Singular Value Decomposition

A SVD for an $m \times n$ matrix A is of the form $A = U\Sigma V^*$ where

- U is an $m \times m$ unitary matrix.
- V is an $n \times n$ unitary matrix.
- Σ has entries
  - $\sigma_{ii} > 0$. (These are determined by the singular values.)
  - $\sigma_{ij} = 0$ for all $i \neq j$.

SVD Algorithm

Algorithm 1: Finding a SVD

1. Find all eigenvalues λ_i of A^*A and write in descending order. Also find their associated eigenvectors of unit length $\mathbf{v}_i$.
2. Find an orthonormal set of eigenvectors for A^*A.
   - Automatically occurs when all eigenvalues are distinct, which will usually be the case. Otherwise require Gram-Schmidt for any eigenspace with dimension strictly greater than 1.
3. Compute $\mathbf{u}_i = \frac{1}{\sqrt{\lambda_i}} A\mathbf{v}_i$ for each non-zero eigenvalue.
4. State U and V from the vectors found, Σ from the singular values.

Lemma 2: Used to speed up step 1

- A^*A and AA^* share the same non-zero eigenvalues.
- If $\text{rank}(A) = r$, then A^*A has r non-zero eigenvalues. All other eigenvalues are 0.

SVD Example

Example: MATH2601 2017 Q2 c)

For the matrix $A = \begin{pmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{pmatrix}$

1. Find the eigenvalues of AA^*.
2. Explain why the eigenvalues in part 1 are also eigenvalues of A^*A, and state any other eigenvalues of A^*A.
3. Find all eigenvectors of A^*A.
4. Find a singular value decomposition for A.

SVD Example

Part 1: We compute

$$AA^* = \begin{pmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 11 & 1 \\ 1 & 11 \end{pmatrix}.$$

This matrix has two eigenvalues, and they sum to $\text{tr}(AA^*) = 22$ and multiply to $\det(AA^*) = 120$. By inspection, $\lambda_1 = 12$ and $\lambda_2 = 10$.

SVD Example

Part 2: Quoted word for word from the answers...

"We know that A^*A and AA^* have the same non–zero eigenvalues, so 12 and 10 are eigenvalues of A^*A.

Also, all eigenvalues of A^*A are real and non–negative, so its third eigenvalue is 0."

SVD Example

Part 3: We compute

$$A^*A = \begin{pmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{pmatrix}$$

For $\lambda = 12$:

$$A^*A - 12I = \begin{pmatrix} -2 & 0 & 2 \\ 0 & -2 & 4 \\ 2 & 4 & -10 \end{pmatrix}.$$

Looking at row 1, arbitrarily set first component to 1, and then the third component is 1. Equating row 2, the second component is 2.

$$\therefore \boxed{\mathbf{v}_1 = t \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}}.$$

SVD Example

Part 3: We compute

$$A^*A = \begin{pmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{pmatrix}$$

For $\lambda = 10$:

$$A^*A - 10I = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 4 \\ 2 & 4 & -8 \end{pmatrix}.$$

Rows 1 and 2 force the third component to be 0. Looking at row 3, it is easier to set the second component to 1, and then the first component will be -2.

$$\therefore \boxed{\mathbf{v}_2 = t \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}}.$$

SVD Example

Part 3: We compute

$$A^*A = \begin{pmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{pmatrix}$$

For $\lambda = 0$, looking at A^*A itself, there really are many possibilities we can go about it. But I follow the answers, which arbitrarily set the first component to -1. See if you can then show that

$$\boxed{\mathbf{v}_3 = t \begin{pmatrix} -1 \\ -2 \\ 5 \end{pmatrix}}.$$

Note: In each case, $t \in \mathbb{R}$.

SVD Example

Part 4: In each case, choose the value of t that normalises the eigenvectors:

$$\mathbf{v}_1 = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\2\\1\end{pmatrix} \qquad (\lambda_1 = 12)$$

$$\mathbf{v}_2 = \frac{1}{\sqrt{5}}\begin{pmatrix}-2\\1\\0\end{pmatrix} \qquad (\lambda_2 = 10)$$

$$\mathbf{v}_3 = \frac{1}{\sqrt{30}}\begin{pmatrix}-1\\-2\\5\end{pmatrix} \qquad (\lambda_3 = 0)$$

SVD Example

Part 4: Compute $\mathbf{u}_i = \frac{1}{\sqrt{\lambda_i}} A\mathbf{v}_i$ for each non-zero eigenvector:

$$\mathbf{u}_1 = \frac{1}{\sqrt{12}} \begin{pmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{pmatrix} \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\mathbf{u}_2 = \frac{1}{\sqrt{10}} \begin{pmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

SVD Example

Part 4: We conclude that a SVD for A is $A = U\Sigma V^*$, where

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} \sqrt{12} & 0 & 0 \\ 0 & \sqrt{10} & 0 \end{pmatrix}$$

$$V = \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{30}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & \frac{5}{\sqrt{30}} \end{pmatrix}$$

Canonical Jordan Form

Jordan Blocks

Definition 3: Jordan blocks

The $k \times k$ Jordan block for λ is the matrix

$$J_k(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ 0 & 0 & \lambda & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & \lambda \end{pmatrix} \in \mathrm{M}_{k\times k}(\mathbb{C}).$$

That is, put λ on every entry along the main diagonal, and a 1 immediately above each λ wherever possible.
It can be proved that every matrix can be decomposed into PJP^{-1}, where P is the matching eigenvector matrix, and J is a matrix of corresponding Jordan blocks joined together by direct sums.

Jordan Blocks

Quick examples:

$$J_3(-4) = \begin{pmatrix} -4 & 1 & 0 \\ 0 & -4 & 1 \\ 0 & 0 & -4 \end{pmatrix}$$

$$J_4(0) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Powers of Jordan Forms

Find the pattern.

$$J_1(\lambda)^n = \begin{pmatrix} \lambda^n \end{pmatrix}$$

$$J_2(\lambda)^n = \begin{pmatrix} \lambda^n & \binom{n}{1}\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}$$

$$J_3(\lambda)^n = \begin{pmatrix} \lambda^n & \binom{n}{1}\lambda^{n-1} & \binom{n}{2}\lambda^{n-2} \\ 0 & \lambda^n & \binom{n}{1}\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{pmatrix}$$

$$J_4(\lambda)^n = \begin{pmatrix} \lambda^n & \binom{n}{1}\lambda^{n-1} & \binom{n}{2}\lambda^{n-2} & \binom{n}{3}\lambda^{n-3} \\ 0 & \lambda^n & \binom{n}{1}\lambda^{n-1} & \binom{n}{2}\lambda^{n-2} \\ 0 & 0 & \lambda^n & \binom{n}{1}\lambda^{n-1} \\ 0 & 0 & 0 & \lambda^n \end{pmatrix}$$

Powers of Jordan Forms

Lemma 3: Computing powers of Jordan forms

1. Start with λ^n on every diagonal entry.
2. Put $\binom{n}{1}\lambda^{n-1}$ wherever you can immediately above λ^n
3. Put $\binom{n}{2}\lambda^{n-2}$ wherever you can immediately above $\binom{n}{1}\lambda^{n-1}$
4. Keep doing this, increasing the binomial coefficient and decreasing the power on λ.

Note: Not *quite* the above. If you ever bump into $\binom{n}{n}$, that's the last diagonal you fill. Just put 0's everywhere else above.

Matrix Direct Sums

Definition 4: Direct sums of matrices

The direct sum of matrices $A_1, A_2, \ldots, A_n$ is the matrix formed by putting these matrices on the diagonals and zeroes everywhere else.

$$A_1 \oplus A_2 \oplus \cdots \oplus A_n = \begin{pmatrix} A_1 & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & A_2 & \ldots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & A_n \end{pmatrix}$$

In MATH2501 and MATH2601, we only worry about this with Jordan blocks.

Matrix Direct Sums

Quick example:

$$J_2(3) \oplus J_4(5) = \begin{pmatrix} 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{pmatrix}$$

Matrix Direct Sums

Lemma 4: Powers on direct sums of Jordan blocks

The power of a Jordan form is the direct sum of powers on each individual block. I.e.,

$$(J_1 \oplus \cdots \oplus J_m)^n = J_1^n \oplus \cdots \oplus J_m^n.$$

Example:

$$[J_2(3) \oplus J_1(2)]^n = \begin{pmatrix} 3^n & \binom{n}{1}3^{n-1} & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & 2^n \end{pmatrix}$$

The Generalised Eigenvector

Definition 5: Generalised Eigenvector

A generalised eigenvector corresponding to eigenvalue λ is a non-zero vector $\mathbf{v}$ satisfying the property $\boxed{(A-\lambda I)^k\mathbf{v}=\mathbf{0}}$, for some $k \geq 1$.

This differs from the (usual) eigenvector in the sense that those must satisfy $(A-\lambda I)\mathbf{v}=\mathbf{0}$, i.e. we *must* take $k=1$.

The Generalised Eigenvector

Example 2: MATH2601 2016 Q4 c)

Let $C = \begin{pmatrix} 9 & 7 & -3 \\ -2 & 2 & 1 \\ 2 & 5 & 2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$.

Show that for the matrix C, $\mathbf{v}$ is a generalised eigenvector corresponding to $\lambda = 5$.

The Generalised Eigenvector

We compute that

$$C - 5I = \begin{pmatrix} 4 & 7 & -3 \\ -2 & -3 & 1 \\ 2 & 5 & -3 \end{pmatrix}$$

and continuously left-multiplying to $\mathbf{v}$,

$$(C - 5I)\mathbf{v} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$$(C - 5I)^2\mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

so $\mathbf{v} \in GE_5$.

Generalised Eigenspaces

Definition 6: Generalised Eigenspace

The generalised eigenspace of λ, denoted GE_λ, is the set of all *generalised* eigenvectors corresponding to λ.

$$GE_\lambda = \{\mathbf{v} \in \mathbb{C}^n \mid (A - \lambda I)^k \mathbf{v} = \mathbf{0} \text{ for some } k \geq 1\}$$

Lemma 4: Alternate representation of GE_λ

$$GE_\lambda = \ker(A - \lambda I) \cup \ker(A - \lambda I)^2 \cup \ker(A - \lambda I)^3 \cup \dots$$

Computing Jordan Forms

Definition 7: Jordan matrix

A Jordan matrix J is a direct sum of Jordan blocks.

Lemma 5: Uniqueness

Every matrix A has one unique Jordan matrix, up to some permutation (arrangement) of the Jordan blocks.

Computing Jordan Forms

Theorem 1: Useful properties in computing Jordan forms

Let $\dim\ker(A-\lambda I)^k$, i.e. $\text{nullity}(A-\lambda I)^k = d_k$. Set $d_0 = 0$. Then

1. $\ker(A-\lambda I) \subseteq \ker(A-\lambda I)^2 \subseteq \dots$
2. $d_0 \leq d_1 \leq d_2 \leq d_3 \leq \dots$
3. $d_1 - d_0 \geq d_2 - d_1 \geq d_3 - d_2 \geq \dots$

That is to say, the nullities must *not decrease*, but the *difference* in nullities must *not INcrease*.

Remark: Multiplicity

As a corollary, the algebraic multiplicity of an eigenvalue λ equals to $\dim GE_\lambda$. This allows us to not compute $(A-\lambda I)^k$ forever - we stop when $\text{nullity}(A-\lambda I)^k = \text{AM}$.

Computing Jordan Forms

We use Jordan chains to find the matrices P and J, such that $A = PJP^{-1}$. For an eigenvalue λ with algebraic multiplicity k, we need to start with some vector $\mathbf{v}_1$ such that on multiplication, we have

$$\mathbf{v}_1 \xrightarrow{A-\lambda I} \mathbf{v}_2 \xrightarrow{A-\lambda I} \dots \xrightarrow{A-\lambda I} \mathbf{v}_k \xrightarrow{A-\lambda I} \mathbf{0}.$$

We then include (note the reverse order!)

$$\begin{pmatrix} \mathbf{v}_k & \dots & \mathbf{v}_2 & \mathbf{v}_1 \end{pmatrix}$$

to P. This corresponds to *one* Jordan block $J_k(\lambda)$ in the direct sum for the Jordan matrix J of A.

Computing Jordan Forms

(Or maybe your lecturer taught things the other way around.) We consider this chain

$$\mathbf{v}_k \xrightarrow{A-\lambda I} \mathbf{v}_{k-1} \xrightarrow{A-\lambda I} \dots \xrightarrow{A-\lambda I} \mathbf{v}_1 \xrightarrow{A-\lambda I} \mathbf{0}.$$

and we include this to P instead.

$$\begin{pmatrix} \mathbf{v}_1 & \dots & \mathbf{v}_{k-1} & \mathbf{v}_k \end{pmatrix}$$

We still use the Jordan block $J_k(\lambda)$.

Computing Jordan Forms: Example 1

Example: MATH2601 2016 Q4 c)

Let $C = \begin{pmatrix} 9 & 7 & -3 \\ -2 & 2 & 1 \\ 2 & 5 & 2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$.

1. Calculate $(C - 5I)\mathbf{v}$ and $(C - 5I)^2\mathbf{v}$. (Done earlier)
2. Without using any matrix calculations, write down all the eigenvalues of C and their algebraic and geometric multiplicities. Give reasons for your answers.
3. (Not originally in the exam:) Find an invertible matrix P and a Jordan matrix J such that $C = PJP^{-1}$.

Computing Jordan Forms: Example 1

Part 2: The trace is usually helpful, because it is the sum of the eigenvalues.

$$\text{tr}(C) = 9 + 2 + 2 = 13$$

From part 1, 5 is an eigenvalue of C with algebraic multiplicity *at least 2*. The third eigenvalue λ_3 satisfies

$$5 + 5 + \lambda_3 = 13 \implies \lambda_3 = 3.$$

Computing Jordan Forms: Example 1

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Which is, of course, the only remaining eigenvalue and hence must have AM $= 1$. So we have:

- Eigenvalue 5: AM $= 2$, GM $= 1$
- Eigenvalue 3: AM $= 1$, GM $= 1$

Note: I haven't justified the GM's! Try doing that yourself!

Computing Jordan Forms: Example 1

Part 3: Row reducing $C - 3I$,

$$C - 3I = \begin{pmatrix} 6 & 7 & -3 \\ -2 & -1 & 1 \\ 2 & 5 & -1 \end{pmatrix}$$
$$\sim \begin{pmatrix} 0 & -12 & 0 \\ -2 & -1 & 1 \\ 0 & 4 & 0 \end{pmatrix}$$
$$\sim \begin{pmatrix} -2 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

so we can take a corresponding eigenvector $\mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$.

Computing Jordan Forms: Example 1

So our chains are:

$$\begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} \xrightarrow{C-5I} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \xrightarrow{C-5I} \mathbf{0}$$

$$\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \xrightarrow{C-3I} \mathbf{0}$$

and hence $A = PJP^{-1}$ where

$$J = \begin{pmatrix} 5 & 1 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix} \text{ and } P = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & -3 & 2 \end{pmatrix}$$

Computing Jordan Forms: Example 2 (time permitting...)

Example: MATH2601 2017 Q3 a)

Let $A = \begin{pmatrix} 3 & 1 & -2 \\ 2 & 6 & -7 \\ 2 & 2 & -2 \end{pmatrix}$. We are **given** that

$$GE_2 = \text{span}\left\{ \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\} \text{ and } GE_3 = \text{span}\left\{ \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} \right\}.$$

1. Find the Jordan chain for $\lambda = 2$ starting with (0,1,1).
2. Without any calculation, write down the geometric multiplicity of $\lambda = 2$, giving reasons for your answer.
3. Find a Jordan form J and invertible matrix P for A, such that $A = PJP^{-1}$.

Computing Jordan Forms: Example 2 (time permitting...)

Part 1:

$$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \xrightarrow{A-2I} \begin{pmatrix} -1 \\ -3 \\ -2 \end{pmatrix} \xrightarrow{A-2I} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Part 2: The algebraic multiplicity is 2, since we have another distinct eigenvalue, so GM ≤ 2. But GM $\neq 2$ since we have a chain of length 2, so GM $= 1$.

Computing Jordan Forms: Example 2 (time permitting...)

Part 3: $A = PJP^{-1}$ where

$$P = \begin{pmatrix} -1 & 0 & 1 \\ -3 & 1 & 4 \\ -2 & 1 & 2 \end{pmatrix}$$

$$J = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Computing Jordan Forms: Example 2 (time permitting...)

Example: MATH2601 2017 Q3 a)

4. Find $\mathbf{v} \in GE_2$ and $\mathbf{w} \in GE_3$ such that $\mathbf{v} + \mathbf{w} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

Theorem 2: $\mathbb{C}^n$ and the generalised eigenspaces

The direct sum of generalised eigenspaces of any $A \in \mathrm{M}_{n\times n}$ span $\mathbb{C}^n$.

Computing Jordan Forms: Example 2 (time permitting...)

Hence we just need to express $\begin{pmatrix}2\\1\\0\end{pmatrix}$ as a linear combination of

$\begin{pmatrix}1\\3\\2\end{pmatrix}$, $\begin{pmatrix}0\\1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\4\\2\end{pmatrix}$.

You can have fun with the row reduction... I'll just state the final answer:

$$\begin{pmatrix}2\\1\\0\end{pmatrix} = 3\begin{pmatrix}1\\3\\2\end{pmatrix} - 4\begin{pmatrix}0\\1\\1\end{pmatrix} - \begin{pmatrix}1\\4\\2\end{pmatrix}$$
$$= \begin{pmatrix}3\\5\\2\end{pmatrix} + \begin{pmatrix}-1\\-4\\-2\end{pmatrix}$$

Remark: Similarity Invariants

Theorem 3: Jordan forms are the complete similarity invariant

Two matrices A and B are similar, i.e. $A = PBP^{-1}$ for some invertible matrix P, if and only if they have the same Jordan forms. (At least, to within a different arrangement of direct sums.)

Jordan forms given nullities

The Jordan matrix J can sometimes be found with less information if we don't need to find P.

Example: MATH2601 2016 Q4 b)

Let B be a 10×10 matrix and let λ be a scalar. Suppose it is known that

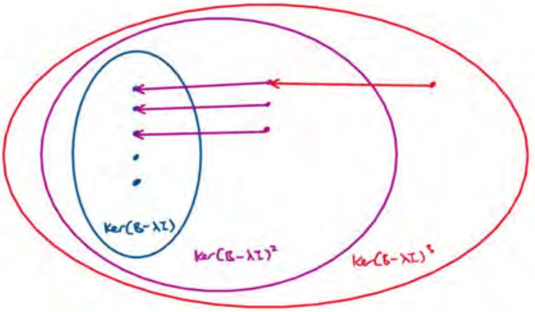
$$\begin{aligned} \text{nullity}(B - \lambda I) &= 5, \\ \text{nullity}(B - \lambda I)^2 &= 8, \\ \text{nullity}(B - \lambda I)^3 &= 9. \end{aligned}$$

Find all possible Jordan forms of B.

Idea: Our Jordan chains can be drawn on an onion diagram.

Jordan forms given nullities

There are 5 eigenvectors in $\ker(B - \lambda I)$. The idea is that there are $8 - 5 = 3$ *more* generalised eigenvectors in $\ker(B - \lambda I)^2$. This is because we know that $\ker(B - \lambda I) \subseteq \ker(B - \lambda I)^2$.



Similarly, there is another $9 - 8 = 1$ in $\ker(B - \lambda I)^3$.

Jordan forms given nullities

We've addressed 9 of the 10 eigenvalues for B. There is only one more left to go.

Case 1: The tenth eigenvalue is NOT λ.
Then it must be some other value $\mu \neq \lambda$. It can only correspond to one eigenvector, so we include $J_1(\mu)$ to the direct sum.

The Jordan chains for λ have lengths 3, 2, 2, 1 and 1, so therefore (up to some permutation),

$$J = J_3(\lambda) \oplus J_2(\lambda) \oplus J_2(\lambda) \oplus J_1(\lambda) \oplus J_1(\lambda) \oplus J_1(\mu).$$

(again, up to some permutation of the Jordan blocks).

Jordan forms given nullities

We've addressed 9 of the 10 eigenvalues for B. There is only one more left to go.

Case 2: The tenth eigenvalue IS also λ.
Problem: We cannot add it in $\ker(B - \lambda I)$, $\ker(B - \lambda I)^2$ or $\ker(B - \lambda I)^3$ without screwing up the nullities!

Recall that the difference is nullities is non-increasing. This means that the last generalised eigenvector must be in $\ker(B - \lambda I)^4$. Our original chain of length 3 also becomes chain of length 4. So we get

$$J = J_4(\lambda) \oplus J_2(\lambda) \oplus J_2(\lambda) \oplus J_1(\lambda) \oplus J_1(\lambda)$$

(again, up to some permutation of the Jordan blocks).

Jordan forms given nullities

Remark: Why $\ker(B - \lambda I)^4$? For completeness sake, here's a quick contradiction.

Suppose, say, the remaining generalised eigenvector was in $\ker(B - \lambda I)^5$ but *not* in $\ker(B - \lambda I)^4$. Then $\ker(B - \lambda I)^4$ must in fact be equal to $\ker(B - \lambda I)^3$, so $d_4 = d_3$, i.e. $d_4 - d_3 = 0$. Yet $d_5 - d_4 = 1$. Therefore $d_5 - d_4 > d_4 - d_3$, which cannot happen.

Invalid nullities

The property $d_1 - d_0 \geq d_2 - d_1 \geq d_3 - d_2 \geq \dots$ helps determine things that are impossible.

Example: David Angell's MATH2601 notes

Let A be a matrix with eigenvalue λ. Explain why this is not possible:

$$\begin{aligned}
\text{nullity}(A - \lambda I) &= 5, \\
\text{nullity}(A - \lambda I)^2 &= 8, \\
\text{nullity}(A - \lambda I)^3 &= 9, \\
\text{nullity}(A - \lambda I)^4 &= 12, \\
\text{nullity}(A - \lambda I)^k &= 12 \text{ for all } k > 4.
\end{aligned}$$

Answer: $d_4 - d_3 = 3 > 1 = d_3 - d_2$, which can't happen.

From Jordan forms back to nullities

Example: Peter Brown's MATH2501 notes

If A is similar to $J = J_2(-4) \oplus J_2(-4) \oplus J_2(-4) \oplus J_3(5) \oplus J_1(5)$.
find

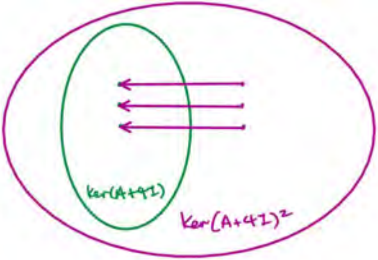
$$\text{nullity}(A+4I)^k \text{ and } \text{nullity}(A-5I)^k$$

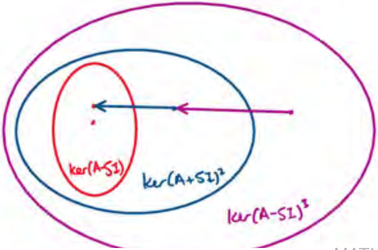
for all $k \geq 1$.

Solution: Go backwards!

From Jordan forms back to nullities

We know the lengths of the chains...





From Jordan forms back to nullities

So we see that:

- nullity$(A + 4I) = 3$
- nullity$(A + 4I)^k = 6$ for all $k \geq 2$
- nullity$(A - 5I) = 2$
- nullity$(A - 5I)^2 = 3$
- nullity$(A - 5I)^k = 4$ for all $k \geq 3$

Matrix Exponentials

Matrix Exponential

Definition 11: Exponential of a matrix

The matrix exponential $\exp(tA)$ is defined as

$$\exp(tA) = e^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k.$$

Computing matrix exponentials

We will illustrate the ideas...

Lemma 7: Properties of matrix exponentials

1. If $A = PBP^{-1}$, then $\exp(A) = P\exp(B)P^{-1}$.
2. If $A = A_1 \oplus \cdots \oplus A_n$, then $\exp(A) = \exp(A_1) \oplus \cdots \oplus \exp(A_n)$
3. $$\exp\left(tJ_k(\lambda)\right) = e^{\lambda t} \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \dots & \frac{t^{k-1}}{(k-1)!} \\ 0 & 1 & t & & \\ 0 & 0 & 1 & \ddots & \\ & \vdots & \ddots & & \\ 0 & 0 & 0 & \dots & t \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

Also nice to note is that if $AB = BA$, then $\exp(A)\exp(B) = \exp(A + B)$.

Computing matrix exponentials

Example for a Jordan block:

$$\exp\left(tJ_5(2)\right) = e^{2t}\begin{pmatrix} 1 & t & \frac{t^2}{2!} & \frac{t^3}{3!} & \frac{t^4}{4!} \\ 0 & 1 & t & \frac{t^2}{2!} & \frac{t^3}{3!} \\ 0 & 0 & 1 & t & \frac{t^2}{2!} \\ 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

We're really filling the matrix in with terms from the power series of e^t, but then leaving a usual exponential in front.

Computing matrix exponentials

Example: Not really an example...

Consider $C = \begin{pmatrix} 9 & 7 & -3 \\ -2 & 2 & 1 \\ 2 & 5 & 2 \end{pmatrix}$ from earlier. We want $\exp(tC)$.

We have

$$P = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & -3 & 2 \end{pmatrix} \text{ and } J = \begin{pmatrix} 5 & 1 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Computing matrix exponentials

The earlier results show that we can do powers of Jordan blocks *one at a time*. So we obtain

$$\exp(J) = \begin{pmatrix} e^{5t} & te^{5t} & 0 \\ 0 & e^{5t} & 0 \\ 0 & 0 & e^{3t} \end{pmatrix}$$

and hence

$$\exp(C) = P \begin{pmatrix} e^{5t} & te^{5t} & 0 \\ 0 & e^{5t} & 0 \\ 0 & 0 & e^{3t} \end{pmatrix} P^{-1}$$

A huge pain, as you can see.

So you probably won't be asked to do *that* in an exam. But you may be asked something else.

The 'Columns' technique

Theorem 6: Matrix Exponential times Generalised Eigenvector

If we have the Jordan chain

$$\mathbf{v}_1 \xrightarrow{A-\lambda I} \mathbf{v_2} \xrightarrow{A-\lambda I} \mathbf{v}_3 \xrightarrow{A-\lambda I} \dots \xrightarrow{A-\lambda I} \mathbf{v}_k$$

then

$$\exp(tA)\mathbf{v}_1 = e^{\lambda t}\left(\mathbf{v}_1 + t\mathbf{v}_2 + \frac{t^2}{2}\mathbf{v}_3 + \cdots + \frac{t^{k-1}}{(k-1)!}\mathbf{v}_k\right)$$

The 'Columns' technique

This does come with a caveat in that $\mathbf{v}_1$ must be a generalised eigenvector corresponding to λ.

(Otherwise, we have to decompose it into a linear combination of generalised eigenvectors first.)

Solving Homogeneous systems of DEs

More often than not, we just need to compute $\exp(tA)\mathbf{v}$ for some vector $\mathbf{v}$, instead of the actual matrix exponential itself.

Theorem 7: Solution to a homogeneous system

The solution to $\frac{d\mathbf{y}}{dt} = A\mathbf{y}$ with initial condition $\mathbf{y}(0) = \mathbf{c}$ is

$$\mathbf{y} = \exp(tA)\mathbf{c}.$$

Solving Homogeneous systems of DEs

Example: MATH2601 2016 Q4 c)

Recall for $C = \begin{pmatrix} 9 & 7 & -3 \\ -2 & 2 & 1 \\ 2 & 5 & 2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$ we have the chain

$$\begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} \xrightarrow{C-5I} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \xrightarrow{C-5I} \mathbf{0}.$$

Use this to solve the initial value problem $\mathbf{y}' = C\mathbf{y}$, $\mathbf{y}(0) = \mathbf{v}$.

Solving Homogeneous systems of DEs

The solution is

$$\mathbf{y} = \exp(tA)\begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}.$$

So using the columns method,

$$\begin{aligned}\mathbf{y} &= e^{5t}\left[\begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}\right] \\ &= e^{5t}\begin{pmatrix} 1-t \\ -2+t \\ -3+t \end{pmatrix}\end{aligned}$$

Solving Non-homogeneous systems of DEs

Example: MATH2601 2016 Q4 c)

Find a particular solution of $\mathbf{y}' = C\mathbf{y} + te^{5t}\mathbf{w}$, where
$C = \begin{pmatrix} 9 & 7 & -3 \\ -2 & 2 & 1 \\ 2 & 5 & 2 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} -8 \\ 2 \\ -4 \end{pmatrix}$, given that $\mathbf{w}$ is a generalised eigenvector of C.

Subbing $\mathbf{y} = e^{tC}\mathbf{z}$ gives

$$Ce^{tC}\mathbf{z} + e^{tC}\mathbf{z}' = Ce^{tC}\mathbf{z} + te^{5t}\mathbf{w}$$

$$\mathbf{z}' = te^{5t}e^{-tC}\mathbf{w}$$

Solving Non-homogeneous systems of DEs

We need to construct a Jordan chain starting at $\mathbf{w}$ first:

$$\begin{pmatrix} -8 \\ 2 \\ -4 \end{pmatrix} \xrightarrow{C-5I} \begin{pmatrix} -6 \\ 6 \\ 6 \end{pmatrix} \xrightarrow{C-5I} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

so therefore

$$e^{tC}\mathbf{w} = e^{5t}\left[\begin{pmatrix} -8 \\ 2 \\ -4 \end{pmatrix} + t\begin{pmatrix} -6 \\ 6 \\ 6 \end{pmatrix}\right].$$

But observe how we want the negative exponent e^{-tC}! This means what we're really interested in is

$$e^{-tC}\mathbf{w} = e^{-5t}\left[\begin{pmatrix} -8 \\ 2 \\ -4 \end{pmatrix} - t\begin{pmatrix} -6 \\ 6 \\ 6 \end{pmatrix}\right]$$

Solving Non-homogeneous systems of DEs

Therefore

$$\mathbf{z}' = te^{5t}e^{-tC}\mathbf{w} = t\begin{pmatrix}-8\\2\\-4\end{pmatrix} - t^2\begin{pmatrix}-6\\6\\6\end{pmatrix}$$

so upon integrating,

$$\mathbf{z} = \frac{t^2}{2}\begin{pmatrix}-8\\2\\-4\end{pmatrix} - \frac{t^3}{3}\begin{pmatrix}-6\\6\\6\end{pmatrix} + \mathbf{c}$$

Question: How to deal with that constant of integration?

Solving Non-homogeneous systems of DEs

In general, you can only deal with it when you know what $\mathbf{y}(0)$ is, i.e. you have an initial value for the original systems of DEs. When that's the case, you let $\mathbf{z}(0) = \mathbf{y}(0)$ to find it.

Here we don't, so we just proceed as usual.

$$\mathbf{y}_P = e^{tC}\mathbf{z} = \frac{t^2}{2}e^{tC}\begin{pmatrix} -8 \\ 2 \\ -4 \end{pmatrix} - \frac{t^3}{3}e^{tC}\begin{pmatrix} -6 \\ 6 \\ 6 \end{pmatrix} + e^{tC}\mathbf{c}.$$

Solving Non-homogeneous systems of DEs

To finish this off, we can recycle our Jordan chain from earlier:

$$\begin{pmatrix} -8 \\ 2 \\ -4 \end{pmatrix} \xrightarrow{C-5I} \begin{pmatrix} -6 \\ 6 \\ 6 \end{pmatrix} \xrightarrow{C-5I} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This gives us

$$\mathbf{y}_P = \frac{t^2}{2} e^{5t} \left[\begin{pmatrix} -8 \\ 2 \\ -4 \end{pmatrix} - t \begin{pmatrix} -6 \\ 6 \\ 6 \end{pmatrix} \right] - \frac{t^3}{3} e^{5t} \begin{pmatrix} -6 \\ 6 \\ 6 \end{pmatrix} + e^{tC} \mathbf{c}$$

Solving Non-homogeneous systems of DEs

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$$\begin{pmatrix} -8 \\ 2 \\ -4 \end{pmatrix} \xrightarrow{C-5I} \begin{pmatrix} -6 \\ 6 \\ 6 \end{pmatrix} \xrightarrow{C-5I} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This gives us

$$\mathbf{y}_P = \frac{t^2}{2}e^{5t}\left[\begin{pmatrix} -8 \\ 2 \\ -4 \end{pmatrix} - t\begin{pmatrix} -6 \\ 6 \\ 6 \end{pmatrix}\right] - \frac{t^3}{3}e^{5t}\begin{pmatrix} -6 \\ 6 \\ 6 \end{pmatrix} + e^{tC}\mathbf{c}$$

The remainder is trivial and is left as an exercise to the audience.

Solving Non-homogeneous systems of DEs

Note: The harsh reality is that if we knew what $\mathbf{c}$ was, that would potentially be *another* Jordan chain we need to deal with. Fingers crossed you don't have to do that in your exam.