UNSW Mathematics Society Presents

MATH1081 Workshop

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Overview I

1. Sets, functions and sequences

2. Graphs

3. Integers, Modular Arithmetic and Relations

4. Logic and Proofs

5. Enumeration and Probability

1. Sets, functions and sequences

2019 T2 Q3 (iv)

Let $f : A \to B$ and $g : B \to C$ be functions, and consider the statement: "If g is injective and $g \circ f$ is surjective, then f is surjective."

(a) What's wrong with the following proof?
(b) Prove that this statement is true.

Injective Function Definition

If a function $f : A \to B$ is injective, then for any x_1 and x_2 in A, $f(x_1) = f(x_2)$ implies that $x_1 = x_2$.
In other words, if the outputs are equal then the inputs are equal.

Surjective Function Definition

If a function $f : A \to B$ is surjective, then for any y in B, there exists x in A such that $f(x) = y$.
In other words, every element in the codomain is a function value, i.e. f of something in A.

(a) What's wrong with this proof?

Let

$$f : \mathbb{Z} \to \mathbb{R}, \text{ where } f(x) = x,$$

$$g : \mathbb{R} \to \mathbb{R}, \text{ where } g(x) = x^3.$$

Then g is injective, and also $(g \circ f)(x) = g(f(x)) = g(x)$ and so $g \circ f = g$, which is surjective. However, f is not surjective. This counterexample proves that the statement is false.

(b) The Actual Proof

Suppose that $f : A \to B$ and $g : B \to C$ are functions, and that g is injective and $g \circ f$ is surjective. We need to show that f is surjective.

Injective Function Definition

If a function $f : A \to B$ is injective, then for any x_1 and x_2 in A, $f(x_1) = f(x_2)$ implies that $x_1 = x_2$.
In other words, if the outputs are equal then the inputs are equal.

Surjective Function Definition

If a function $f : A \to B$ is surjective, then for any y in B, there exists x in A such that $f(x) = y$.
In other words, every element in the codomain is a function value, i.e. f of something in A.

(b) Continued...

Our Supposed Facts:

(1) $f : A \to B$ and $g : B \to C$,
(2) g is injective, so if $g(y_1) = g(y_2)$ then $y_1 = y_2$,
(3) $g \circ f$ is surjective, so for any z in C, there is an x in A so that $g(f(x)) = z$.

To show that f is surjective, using the definition before, we have to show that for any y in B, there is an x in A such that $f(x) = y$.
So, let's suppose that y is any element in B.
Fact (1): Since g is a function from B to C, $g(y)$ is in C.
Fact (3): $g(y)$ is in C, so there is an x in A such that $g(f(x)) = g(y)$.
Fact (2): $g(f(x)) = g(y)$, so $f(x) = y$. So we have found an x in A such that $f(x) = y$.
Since y was any element in B, we just showed that for any y in B, there exists x in A such that $f(x) = y$. Therefore, f is surjective.

2018 S2 Q1 (i)

Let A be a set with 5 elements.

(a) Find $|P(A)|$.
(b) ...
(c) Is the function $f : P(A) \to P(A)$ given by $f(S) = S^c$ surjective? Prove your answer.

(a) Size of a power set

Power Set Definition

A power set $P(A)$ is the set of all subsets of A.

The cardinality (number of elements) of $P(A)$ is $2^{|A|}$. Why?
Think of making a subset of A. For each element of A, we can decide to either include it or not include it in our subset. So there are $2^{|A|}$ different ways of building a subset of A.

(c) Proof

A reminder of the definition:

Surjective Function Definition

If a function $f : A \to B$ is surjective, then for any y in B, there exists x in A such that $f(x) = y$.
In other words, every element in the codomain is a function value, i.e. f of something in A.

We need to decide first: Is $f : P(A) \to P(A)$ given by $f(S) = S^c$ a surjective function? In other words, can we get every element of $P(A)$ as a function value?
Hint: If $C = f(B)$, then what must B be in terms of C? Is B in $P(A)$?
(For this question, if S is a subset of A then $S^c = A - S$, the set of all elements in A that are not in S.)

(c) continued...

We prove that f is indeed surjective.
Suppose that S is any element in $P(A)$. Then $S^c = A - S$ is another subset of A, and so S^c is in $P(A)$ as well.
Also, $f(S^c) = (S^c)^c = S$, by the double complement law.
So we have found that there is a set S^c in $P(A)$ so that $f(S^c) = S$ (for any S in $P(A)$).
Thus, by definition, f is surjective.

2017 S2 Q1 (i)

Questions

(a) Let $X = \{1, 2\}$ and $Y = \{1, 2, 3\}$. List $X \times X$ and $X \times Y$.
(b) Prove that if $A \subseteq B$ then $A \times A \subseteq A \times B$.
(c) Suppose that $A \subseteq B$ and $|A| = m$ and $|B| = n$. Find

$$|A \times B - A \times A|.$$

(a) Cartesian Product

Definition

Let A and B be sets. Then the Cartesian product $A \times B$ is defined by

$$A \times B = \{(a, b) : a \in A, b \in B\}.$$

So $X \times X = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$.

(b) Proof

Prove that if $A \subseteq B$ then $A \times A \subseteq A \times B$.
Reminder: $X \times Y = \{(x, y) : x \in X, y \in Y\}$.

Subset proof method

To prove something like $X \subseteq Y$, we need to show that if x is in X, then x is also in Y. So, we start by writing "Let x be an element in X", and try to show that $x \in Y$.

Suppose that $A \subseteq B$.
Suppose a is an element of $A \times A$. Then $a = (a_1, a_2)$, where a_1 and a_2 are in A.
But A is a subset of B, and so a_2 must also be in B.
Thus, since a_1 is in A and a_2 is in B, we have that (a_1, a_2) (which is a) is in $A \times B$.
Therefore, any element a in $A \times A$ is also in $A \times B$, and so $A \times A \subseteq A \times B$.

(c)

Suppose that $A \subseteq B$ and $|A| = m$ and $|B| = n$. Find

$$|A \times B - A \times A|.$$

From (b), we found that if A is a subset of B then $A \times A \subseteq A \times B$.
In other words, every element in $A \times A$ is also in $A \times B$.
So what can we say about the size of A $\times B - A \times A$?

2017 S2 Q1 (ii)

Use the laws of set algebra to simplify

$$A^c - (A \cap B^c).$$

Show working and give a reason for each step.

$$\begin{aligned}
A^c - (A \cap B^c) &= A^c \cap (A \cap B^c)^c && \text{(Difference Law)} \\
&= A^c \cap (A^c \cup (B^c)^c) && \text{(De Morgan's Law)} \\
&= A^c \cap (A^c \cup B) && \text{(Double Complement Law)} \\
&= A^c && \text{(Absorption Law)}
\end{aligned}$$

2017 S2 Q1 (iii)

For each $i = 1, 2, 3, ...$, let $A_i = [i-1, i+1]$, the closed interval of the real line from $i-1$ to $i+1$. Determine $\bigcup_{i=1}^{\infty} A_i$, giving reasons.

Definition

$\bigcup_{i=1}^{n} A_i$ is a short way of writing $A_1 \cup A_2 \cup ... \cup A_n$.

Let's try drawing some of these sets A_i...

2017 S2 Q1 (iv) (a) (alternate version)

Prove that if a function $f : X \to Y$ has an inverse then it is injective (one-to-one).

Inverse Functions

Let f and g be functions, where $f : X \to Y$ and $g : Y \to X$. If $g(f(x)) = x$ for every x in X and $f(g(y)) = y$ for every y in Y, then we say that f is invertible and has an inverse function g.

To show that a function $f : X \to Y$ is injective, we have to show that if $f(x_1) = f(x_2)$, then $x_1 = x_2$ (where x_1 and x_2 are in X).
We are also going to need this (which might seem obvious and useless):

A Basic Fact About Functions

If $f : X \to Y$ is a function, then for every x in X, there is exactly one y in Y where $y = f(x)$. Another way of saying this is that if the inputs are equal then the outputs must be equal.

(a) Proof

Suppose $f : X \to Y$ has an inverse, say $g : Y \to X$. Then $g(f(x)) = x$ for every $x \in X$, and $f(g(y)) = y$ for every $y \in Y$ (Fact 1).

We need to show that f is injective, that is, if x_1 and x_2 are in X and $f(x_1) = f(x_2)$, then $x_1 = x_2$.

Thus, suppose that x_1 and x_2 are in X, and that $f(x_1) = f(x_2)$. Then $g(f(x_1)) = x_1$, and $g(f(x_2)) = x_2$, by Fact 1.

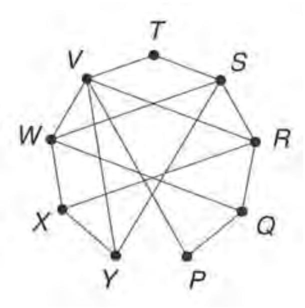
But $f(x_1) = f(x_2)$, and so since g is a function (if the inputs are equal the the outputs are equal), we must have that $g(f(x_1)) = g(f(x_2))$.

So $x_1 = x_2$, and hence we have shown that if x_1 and x_2 are in X and $f(x_1) = f(x_2)$, then $x_1 = x_2$.

Therefore, f must be an injective function.

2. Graphs

2019 T2 Q1 (ii)



(a) Does G contain an Euler path? Give reasons.
(b) Show that G is bipartite.
(c) Prove that G does not contain a Hamilton circuit.

(a) Euler Path

Euler's Path Theorem

A connected graph G has an Euler path if and only if there are exactly 2 vertices of odd degree.

In G, there are 4 vertices X, Y, V and Q which have odd degree, so by Euler's Path Theorem, there is no Euler path.

(b) Bipartite Graph

Bipartite Graph Definition

A graph G is bipartite if there are two disjoint sets V_1 and V_2 such that $V_1 \cup V_2 = V$ and every edge of G connects a vertex in V_1 to a vertex in V_2.

In loose terms, a graph is bipartite if we can split the vertices into 2 groups so that every edge makes a "bridge" between the two groups. So, to show that G is bipartite, we just need to find a way to split the vertices in this way.

(c) Hamilton Circuit

Hamilton Circuit Definition

A Hamilton circuit of a graph G is a circuit that passes through all vertices of G exactly once.

Proof by Contradiction...
Remember from (b) that G is bipartite...

2019 T1 Q3 (iii)

Let G be a connected, simple, bipartite graph with vertex partition $V = V_1 \cup V_2$, where $|V_1| = |V_2| = n$, and suppose that every vertex has degree 4.

(a) Determine the number of edges of G, with explanation.
(b) Prove that G is not planar.

(a) Handshakes

The Handshaking Lemma

For any graph with vertices V and edges E:

$$2|E| = \sum_{v \in V} deg(v).$$

Why? Think of how the "total degree sum" changes when we add one edge.

Using the Handshaking Lemma, since there are a total of $2n$ vertices, and each vertex has degree 4, the sum of vertex degrees is $8n$, and so there are $4n$ edges.

(b) Planar Graphs

Planar Graph Definition

A graph is planar if you can draw it (in 2D!) without crossing edges.

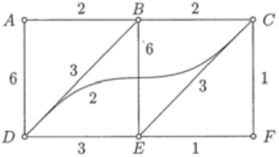
A Useful Theorem

If G is a connected planar graph with e edges and $v >= 3$ vertices, then
(1) $e <= 3v - 6$,
(2) $e <= 2v - 4$ (if G has no cycles of length 3).

We can use the contrapositive of the above theorem, which is:
"if $e > 2v - 4$ and G is a connected graph that has no cycles of length 3, the G is not planar."
Now we know that $e = 4n$ and $v = 2n$, so $e > 2v - 4$. We also know that G is connected. Also, G has no cycles of length 3. (Why? G is bipartite, so if you start at any vertex and travel across 3 different edges, can you end up at where you started?)
Therefore, G is not planar.

2019 T1 Q3 (iv)

Consider this weighted graph:



(a) Use Dijkstra's Algorithm to find a spanning tree that gives the shortest paths from A to every other vertex of the graph. Make a table showing the details of the steps taken when applying the algorithm.
(b) Is the spanning tree from (a) a minimum spanning tree? Explain your answer.

(a) Dijkstra's Algorithm

Dijkstra's Algorithm

1. Set $T = \{a\}$.
2. Let A be the edges with one vertex v not in T and the other in T.
3. Choose an edge e from A that gives a shortest path from a to any v.
4. Add e to T.
5. Continue until T contains all vertices of G.
6. Then T is a minimal a-path spanning tree for G.

Make a table with 2 columns, one for "candidate" edges at each step and another for the "selected" edge at each step.

(b) Minimum Spanning Tree

To show whether the spanning tree in the previous question is a minimum spanning tree, we need to find a minimum spanning tree and compare their weights (sum of edge weights).

Kruskal's Algorithm (in simple terms)

To get a minimum spanning tree for a graph G:

1. Start with an empty tree T.
2. Find all edges of G with the minimum weight.
3. From the edges in the previous step, put as many of them as possible in T so that they do not cause a cycle in T.
4. Repeat steps 2 and 3 with edges in G that have the next minimum weight, until T has contains all vertices in G.

3. Integers, Modular Arithmetic and Relations

2019 T1 Q2 (iii)

a) Find the smallest positive integer n such that $7^n \equiv 1 \pmod{57}$.
b) Hence or otherwise, calculate

$$\left(\sum_{n=0}^{1000} 7^n\right) \pmod{57}$$

2019 T1 Q2 (iii) a

a) Find the smallest positive integer n such that $7^n \equiv 1 \pmod{57}$.

We can first observe $n = 0$

$$7^0 \equiv 1 \pmod{57},$$

but... n has to be a positive integer
Test other numbers!

$$7^1 \equiv 7 \pmod{57},$$

$$7^2 \equiv 49 \equiv -8 \pmod{57},$$

$$7^3 \equiv -56 \equiv 1 \pmod{57},$$

so the smallest positive integer n satisfying $7^n \equiv 1 \pmod{57}$ is 3.

2019 T1 Q2 (iii) b

b) Hence or otherwise, calculate

$$\left(\sum_{n=0}^{1000} 7^n\right) \pmod{57}$$

Now

$$\left(\sum_{n=0}^{1000} 7^n\right) \bmod 57$$

$$\equiv (7^0 + 7^1 + 7^2 + 7^3 + ... + 7^{997} + 7^{998} + 7^{999} + 7^{1000}) \bmod 57$$

2019 T1 Q2 (iii) b cont.

Since $7^0 \equiv 7^3 \equiv 1$, then we can see the pattern repeats every 3 powers and so for some integer k,

$$7^0 \equiv 7^3 \equiv 7^6 \equiv \ldots \equiv 7^{3k} \equiv 1,$$

$$7^1 \equiv 7^4 \equiv 7^7 \equiv \ldots \equiv 7^{3k+1} \equiv 7,$$

$$7^2 \equiv 7^5 \equiv 7^8 \equiv \ldots \equiv 7^{3k+2} \equiv -8.$$

Hence,

$$(7^0 + 7^1 + 7^2 + 7^3 + 7^4 + 7^5 + \ldots + 7^{996} + 7^{997} + 7^{998} + 7^{999} + 7^{1000}) \bmod 57$$

$$\equiv (1 + 7 + (-8)) + (1 + 7 + (-8)) + \ldots + (1 + 7 + (-8)) + 1 + 7) \bmod 57.$$

$$\equiv 0 + \ldots. + 0 + 1 + 7 \equiv 8.$$

Therefore, $\sum_{n=0}^{1000} 7^n \equiv 8 \pmod{57}$

2018 T2 Q1 (iv)

Let a, b, q, r be integers such that $a = bq + r$, where a and b are not zero. Prove that $\gcd(b, r) \leq \gcd(a, b)$.

Proof. Suppose a, b, q, r are integers such that $a = bq + r$, where a and b are not zero. Let $d_0 = \gcd(b, r)$ and $d = \gcd(a, b)$.
Since $d_0 | b$ and $d_0 | r$, that means $d_0 | a$ as per $a = bq + r$.

Thus, we can see that d_0 is a common divisor of a and b but d is the greatest common divisor of a and b, so $d_0 \leq d$, that is
$\gcd(b, r) \leq \gcd(a, b)$ $\square$

Extension. Prove that $\gcd(a, b) \leq \gcd(b, r)$.

2016 T2 Q2 (i)

Solve the following congruences, or explain why they have no solution
a) $14x \equiv 7 \pmod{21}$
b) $14x \equiv 5 \pmod{21}$
c) $14x \equiv 0 \pmod{21}$

2016 T2 Q2 (i)

The Bézout Property

Consider the equation

$$ax + by = c,$$

where a, b, and c are integers, with a and b not both zero. Then
(i) if $c = \gcd(a, b)$, then the equation has an integer solution;
(ii) if c is a multiple of $\gcd(a, b)$, then the equation has an integer solution;
(iii) if c is not a multiple of $\gcd(a, b)$, then the equation has no integer solution.

To check solubility, we refer to The Bézout Property.

2016 T2 Q2 (i)

Solve the following congruences, or explain why they have no solution
a) $14x \equiv 7 \pmod{21}$
b) $14x \equiv 5 \pmod{21}$
c) $14x \equiv 0 \pmod{21}$

By definition, $14x \equiv n \pmod{21} \Leftrightarrow 14x = 21k + n$ for some integers n and k and so,

$$14(1) + 21(-k) = n$$

$$ax + by = c$$

has a solution only if n is a multiple of the gcd of 21 and 7.
Therefore, (a) and (c) have solutions but (b) does not.

2016 T2 Q2 (i) a)

Solve the following congruences, or explain why they have no solution
a) $14x \equiv 7 \pmod{21}$

Start by dividing both sides by 7,

$$14x \equiv 7 \pmod{21},$$

$$2x \equiv 1 \pmod{3}.$$

Now, the numbers are small enough such that we can guess the solution.
Guessing $x = 2$ works!

$$2(2) = 4 \equiv 1 \pmod{3}.$$

2016 T2 Q2 (i) b

Solve the following congruences, or explain why they have no solution
b) $14x \equiv 5 \pmod{21}$

No solution.

2016 T2 Q2 (i) c

Solve the following congruences, or explain why they have no solution
c) $14x \equiv 0 \pmod{21}$

Like in (a), we can divide both sides by 7,

$$14x \equiv 0 \pmod{21}$$

$$2x \equiv 0 \pmod{3}$$

Again, the solutions here can easily be guessed. And so $x = 0$ is the solution to $2x \equiv 0 \pmod{3}$ in modulo 3.

2019 T2 Q2 (i)

Solve the congruence $39x \equiv 15 \pmod{87}$ Give your answer as
a) a congruence to the smallest possible modulus;
b) a congruence modulo 87

2019 T2 Q2 (i) a

Solve the congruence $39x \equiv 15 \pmod{87}$. Give your answer as
a) a congruence to the smallest possible modulus;

By inspection, 3 divides into all the numbers in the equation, so dividing both sides by 3 gives,

$$13x \equiv 5 \pmod{29}.$$

Now to find the gcd of 13 and 29, we must use the Euclidean Algorithm.

$$\begin{aligned} 29 &= 2(13) + 3, \text{i.e. } \gcd(29{,}13) = \gcd(13{,}3); \\ 13 &= 4(3) + 1, \text{i.e. } \gcd(13{,}3) = \gcd(3{,}1); \\ 3 &= 3(1). \end{aligned}$$

Thus the greatest common divisor of 29 and 13 is 1. By Bezout's Theorem, there exists a solution to the congruence as $1|7$.

2019 T2 Q2 (i) a cont.

Now we use the Extended Euclidean Algorithm to find a solution x to the congruence. We want

$$ax + by = c$$

This gives,

$$1 = 13 - 4(3)$$

Substituting $3 = 29 - 2(13)$ gives

$$1 = 13 - 4(29 - 2(13))$$

$$1 = 13 - 4(29) + 8(13)$$

$$9(13) - 4(29) = 1$$

Thus in modulo 29,

$$9(13) \equiv 1$$

and so

$$45(13) \equiv 16(13) \equiv 5 \pmod{29},$$

2019 T2 Q2 (i) b

Solve the congruence $39x \equiv 15 \pmod{87}$ Give your answer as
b) a congruence modulo 87

We know that in modulo 29, $x = 16$ is a solution and

$$x = 16 \equiv 16 + 29 \equiv 16 + 2(29) \pmod{29},$$

$$x = 16 \equiv 45 \equiv 74.$$

However, in modulo 87, these give rise to new solutions so the set of solutions to the congruence in modulo 83 are

$$\{16, 45, 74\}$$

2016 T2 Q2 (ii)

Let $\sim$ be the relation on the set of integers $\mathbb{Z}$ defined by

$$a \sim b \text{ if and only if } a^2 \equiv b^2 (\text{mod } 4)$$

(a) Show that $\sim$ is an equivalence relation.
(b) Find the equivalence classes of $\sim$.

2016 T2 Q2 (ii) a

Let $\sim$ be the relation on the set of integers $\mathbb{Z}$ defined by

$$a \sim b \text{ if and only if } a^2 \equiv b^2 \pmod 4$$

(a) Show that $\sim$ is an equivalence relation.

Reflexivity. (We are trying to show that for any integer a, $a \sim a$, that is, showing $a^2 \equiv a^2 \pmod 4$ to be true as this implies $a \sim a$.)
Proof. Let a be any integer. Consider

$$a \equiv a \pmod 4$$

so

$$a^2 \equiv a^2 \pmod 4$$

as $a^n \equiv b^n \pmod m$ for all $n \geq 0$.
Since $a^2 \equiv a^2 \pmod 4$, $a \sim a$ by definition of the relation. Therefore the relation given by $\sim$ is reflexive.

2016 T2 Q2 (ii) a cont.

Symmetry (We are trying to show that if $a \sim b$ for some integers a and b, $a \sim a$, then $b \sim a$)

Proof. Suppose that $a \sim b$ for some integers a and b, that is, $a^2 \equiv b^2 \pmod 4$. Then by definition,

$$a^2 = 4k + b^2$$

for some integer k. Rearranging terms,

$$b^2 = a^2 - 4k.$$

Since k is an integer,

$$b^2 \equiv a^2 \pmod 4.$$

This implies that $b \sim a$. Therefore, since $a \sim b$ implies $b \sim a$, $\sim$ is symmetric.

2016 T2 Q2 (ii) a cont.

Transitivity (We are trying to show that if $a \sim b$ and $b \sim c$ for some integers a, b and c, then $a \sim c$.)

Proof. Suppose that $a \sim b$ and $b \sim c$ for some integers a, b and c, that is, $a^2 \equiv b^2 \pmod 4$ and $b^2 \equiv c^2 \pmod 4$. Then by definition,

$$a^2 = 4k_1 + b^2 \text{ and } b^2 = 4k_2 + c^2$$

for some integer k_1 and k_2.

Substituting the second equation into the first gives

$$a^2 = 4k_1 + 4k_2 + c^2,$$

$$a^2 = 4(k_1 + k_2) + c^2.$$

Since $k_1 + k_2$ is an integer, then $a^2 \equiv c^2 \pmod 4$.

Hence, if $a \sim b$ and $b \sim c$ for some integers a, b and c, then $a \sim c$ and the relation $\sim$ is transitive.

2016 T2 Q2 (ii) a cont.

Conclusion. Therefore, the relation given by $\sim$ is reflexive, symmetric and transitive, and so, the relation is an equivalence relation. $\square$

2016 T2 Q2 (ii) b

Let $\sim$ be the relation on the set of integers $\mathbb{Z}$ defined by

$$a \sim b \text{ if and only if } a^2 \equiv b^2 \pmod 4$$

(b) Find the equivalence classes of $\sim$.

Recall. Let $\sim$ be an equivalence relation on a set A. For any element $a \in A$, the *equivalence class* of a with respect to $\sim$, denotes by $[a]$, is the set

$$[a] = x \in A | x \sim a.$$

2016 T2 Q2 (ii) b cont.

Now as we are dealing with modulo 4, we only need to check the elements of the set $0, 1, 2, 3$ as **all integers** are represented by these numbers in modulo 4.

When $a = 0$, $a^2 \equiv 0 \pmod 4$ and so all integers which are equivalent to 0 in modulo 4 i.e. ...-8, 0, 4, 8, 12... (all multiples of 4) are part of the equivalence class [0].

Similarly, when $a = 1$, $a^2 \equiv 1 \pmod 4$ and so all integers which are equivalent to 1 in modulo 4 i.e. ...-3, 1, 5, 9, 13... are part of the equivalence class [1].

2016 T2 Q2 (ii) b cont.

However, when $a = 2$, $a^2 = 4 \equiv 0 \pmod 4$ so every integer equivalent to 2 in modulo 4 i.e $... - 6, -2, 2, 6...$ is a subset of the equivalence class $[0]$.
Similarly, when $a = 3$, $a^2 = 9 \equiv 1 \pmod 4$ so every integer equivalent to 3 in modulo 4 i.e $... - 5, -1, 3, 7...$ is a subset of the equivalence class $[1]$. Therefore, the equivalence classes in $\sim$ are given by

$$[0] = \{-4, -2, 0, 2, 4, 6, 8...\} = \{2k | k \in \mathbb{Z}\}$$

$$[1] = \{-3, -1, 1, 3, 5, 7, 9...\} = \{2k + 1 | k \in \mathbb{Z}\}.$$

2019 T1 Q2 (ii)

Let

$$S_n = \{(x_1, x_2, x_3) \in \mathbb{Z}^3\} : x_1 + 2x_2 + 3x_3 = n \text{ and } x_1, x_2, x_3 \geq 0$$

and define a relation $\preceq$ on S_n by

$$(x_1, x_2, x_3) \preceq (y_1, y_2, y_3) \Leftrightarrow x_2 \leq y_2 \text{ and } x_3 \leq y_3.$$

a) Draw the Hasse diagram for $\preceq$ when $n = 5$.
b) Prove that $\preceq$ is a partial order on S_n for each positive integer n.

2019 T1 Q2 (ii) a)

Let

$$S_n = \{(x_1, x_2, x_3) \in \mathbb{Z}^3\} : x_1 + 2x_2 + 3x_3 = n \text{ and } x_1, x_2, x_3 \geq 0$$

a) Draw the Hasse diagram for $\preceq$ when $n = 5$.

$$x_1 + 2x_2 + 3x_3 = n$$

$$0 + 2(1) + 3(1) = 5$$

$$1 + 2(2) + 3(0) = 5$$

$$2 + 2(0) + 3(1) = 5$$

$$3 + 2(1) + 3(0) = 5$$

2019 T1 Q2 (ii) a)

The relation $\preceq$ on S_n is defined by

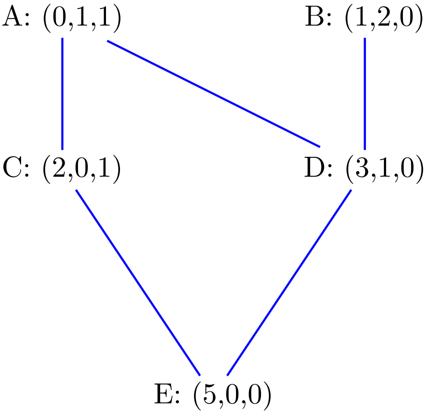
$$(x_1, x_2, x_3) \preceq (y_1, y_2, y_3) \Leftrightarrow x_2 \leq y_2 \text{ and } x_3 \leq y_3.$$

Therefore, $S_5 = \{(0,1,1),(1,2,0),(2,0,1),(3,1,0),(5,0,0)\}$

$Node$	x_1	x_2	x_3
A	0	1	1
B	1	2	0
C	2	0	1
D	3	1	0
E	5	0	0

By inspection, $E \preceq \{A, B, C, D\}$, $D \preceq \{A, B\}$, $C \preceq \{A\}$

Hasse Diagram



2019 T1 Q2 (ii) b)

Let

$$S_n = \{(x_1, x_2, x_3) \in \mathbb{Z}^3\} : x_1 + 2x_2 + 3x_3 = n \text{ and } x_1, x_2, x_3 \geq 0$$

and define a relation $\preceq$ on S_n by

$$(x_1, x_2, x_3) \preceq (y_1, y_2, y_3) \Leftrightarrow x_2 \leq y_2 \text{ and } x_3 \leq y_3.$$

b) Prove that $\preceq$ is a partial order on S_n for each positive integer n.

Reflexivity. Suppose $x_1 + 2x_2 + 3x_3 = n$ for some integers $x_1, x_2, x_3 \leq 0$.
Since $x_2 \leq x_2$ and $x_3 \leq x_3$, and $(x_1, x_2, x_3) \in S_n$,
$(x_1, x_2, x_3) \preceq (x_1, x_2, x_3)$.
Therefore the relation $\preceq$ is reflexive.

2019 T1 Q2 (ii) b)

Asymmetry. Suppose $(x_1, x_2, x_3) \preceq (y_1, y_2, y_3)$ and $(y_1, y_2, y_3) \preceq (x_1, x_2, x_3)$ for some integers $x_1, x_2, x_3, y_1, y_2, y_3 \leq 0$.

Then by definition $x_2 \leq y_2$ and $x_3 \leq y_3$ but also $y_2 \leq x_2$ and $y_3 \leq x_3$. From this, we can conclude that $x_2 = y_2$ and $x_3 = y_3$ and from $x_1 + 2x_2 + 3x_3 = n = y_1 + 2y_2 + 3y_3$, we get

$$x_1 + 2x_2 + 3x_3 = y_1 + 2x_2 + 3x_3$$

$$x_1 = y_1.$$

Therefore $(x_1, x_2, x_3) = (y_1, y_2, y_3)$ and the relation is asymmetric.

2019 T1 Q2 (ii) b)

Transivity. Suppose $(x_1, x_2, x_3) \preceq (y_1, y_2, y_3)$ and $(y_1, y_2, y_3) \preceq (z_1, z_2, z_3)$ for some integers $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3 \leq 0$.

Then by definition $x_2 \leq y_2$ and $x_3 \leq y_3$ but also $y_2 \leq z_2$ and $y_3 \leq z_3$.

Due to the *transitivity* of the relation $\leq$, $x_2 \leq z_2$ and $x_3 \leq z_3$, and since $(x_1, x_2, x_3), (z_1, z_2, z_3) \in S_n$, $(x_1, x_2, x_3) \preceq (z_1, z_2, z_3)$.

Therefore the relation is transitive. As it is reflexive, asymmetric and transitive, the relation is a partial order.

2016 T2 Q1 (iv) a)

Let m be the product of all primes between 10 and 20. Let n be the product of all the numbers between 20 and 30 (inclusive).
a) What is the $\gcd(m, n)$
b) What is the highest power of 10 that divides n?

Now $m = 11 \times 13 \times 17 \times 19$, and $n = 20 \times 21 \times 22 \times ... \times 29 \times 30$.

2016 T1 Q1 (iv) a)

Let m be the product of all primes between 10 and 20. Let n be the product of all the numbers between 20 and 30 (inclusive).
a) What is the $\gcd(m, n)$

Now the gcd comes from the product of numbers that divide BOTH m and n so it is only feasible to test the which primes of m divide n.

Now $m = 11 \times 13 \times 17 \times 19$, and
$n = 20 \times 21 \times 22 \times ... \times 26 \times 27 \times 28 \times 29 \times 30$.
We can see that 11 divides n due to the factor of 22. Similarly 13 divides n due to the factor of 26 but neither 17 nor 19 divide n as the numbers 17, 19 or any of their multiples are not in the product.

Therefore, $\gcd(m, n) = 11 \times 13 = 143$.

2016 T2 Q1 (iv) b)

Let m be the product of all primes between 10 and 20. Let n be the product of all the numbers between 20 and 30 (inclusive).
b) What is the highest power of 10 that divides n?

To find the highest power of 10, we look for sets of the primes 2,5. Now, we will have an abundance of 2's so we look at how many 5's we have.

$$n = 20 \times 21 \times 22 \times 23 \times 24 \times 25 \times 26 \times 27 \times 28 \times 29 \times 30$$

$$n = 5 \times 4 \times 21 \times 22 \times 23 \times 24 \times 5 \times 5 \times 26 \times 27 \times 28 \times 29 \times 5 \times 6$$

$$n = 5^4 \times 4 \times 21 \times 22 \times 23 \times 24 \times 26 \times 27 \times 28 \times 29 \times 6$$

We can get one 5 from each 20 and 30 but we can get two 5's from 25 which gives a total of four 5's.

2016 T2 Q1 (iv) b)

Now we can extract four 2's to give

$$n = 5^4 \times 2 \times 2 \times 21 \times 11 \times 2 \times 23 \times 2 \times 12 \times 26 \times 27 \times 28 \times 29 \times 6$$

$$n = 5^4 \times 2^4 \times 21 \times 11 \times 23 \times 12 \times 26 \times 27 \times 28 \times 29 \times 6$$

$$n = 10^4 \times 21 \times 11 \times 23 \times 12 \times 26 \times 27 \times 28 \times 29 \times 6$$

This means we can write $n = 10^4 \times N$ where N is an integer that does not have 10 as a factor.
Therefore the highest power of 10 that divides 10 is the fourth power.

The End!

Good luck for your exams! :)

4. Logic and Proofs

2020 T3 Q5

An idiosyncratic professor grades four correspondingly idiosyncratic students Ximena, Yosef, Zoe and William according to the following rubric.

- Either Yosef or Ximena will pass, but not both.
- William will pass if Zoe passes.
- Zoe will fail or William will fail.
- It is certain that Ximena will pass because they are top of the class.

Based on this information, for each student, determine whether they will pass or fail, or that there is not enough information.

2020 T3 Q5 Solution

- Either Yosef or Ximena will pass, but not both.
- William will pass if Zoe passes.
- Zoe will fail or William will fail.
- Ximena will pass.

Clearly,

2020 T2 Q5

For a sequence of integers $\{a_n\}_{n=1}^{\infty}$, we say that a_n is eventually even if

$$\exists N \in \mathbb{Z}\ \forall n > N\ a_n \text{ is even.}$$

(a) Write symbolically the statement "a_n is not eventually even", simplifying your answer so that the negation symbols are not used.
(b) Prove that the sequence given by

$$a_n = 837 + \left\lfloor \frac{10n + 471}{n} \right\rfloor$$

is not eventually even.

2020 T2 Q5 Solution

a_n is eventually even means $\exists N \in \mathbb{Z}\ \forall n > N\ \ a_n$ is even. Write symbolically "a_n is not eventually even".

$$\forall N \in \mathbb{Z}\ \exists n > N\ \ a_n \text{ is odd.}$$

Prove that $a_n = 837 + \left\lfloor \frac{10n+471}{n} \right\rfloor$ is not eventually even.

For a given $N \in \mathbb{Z}$ we choose $n = \max\{N+1, 472\}$.

2020 T1 Q8

Consider the statement:
Given a positive integer x, if x is a perfect square, then for all prime integers p, there exists an even integer such that p^a divides x and p^{a+1} does not divide x.
The contrapositive of the above statement is

x__________ a perfect square if __________ $p \in \mathbb{Z}$, __________ $a \in \mathbb{Z}$

p^a__________ x__________ p^{a+1}__________ x.

2020 T1 Q9

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be "eventually increasing" if

$$\exists x \in \mathbb{R}\ \forall y_1 \in \mathbb{R}\ \forall y_2 \in \mathbb{R}\ [((y_1 > x) \wedge (y_2 > y_1))) \to (f(y_2) > f(y_1))].$$

Prove that the function $f(x) = 7 + 5\cos x$ is *not* eventually increasing.

For a given $x \in \mathbb{R}$ we choose $y_1 = x + \pi$ and $y_2 = x + 3\pi$.

2019 T2 Q3 (ii)

Consider the following argument. “If I buy a new car then I will have to give up eating out and seeing movies. If I have to give up eating out then I won’t give up seeing movies. Therefore, I won’t buy a new car.”
(a) Express the above argument in symbol form using logical connectives. Make sure you carefully define any notation you introduce.
(b) Show that the argument is logically valid.

2019 T2 Q3 (ii)(b) Solution

From part (a) the argument could be written as

$$(c \to (e \land m)) \land (e \to \sim m) \implies \sim c.$$

Show that this is logically valid.

2018 S2 Q3 (iii) Solution

Prove by mathematical induction that for all integers $n \geq 2$,

$$1 + \tfrac{1}{4} + \tfrac{1}{9} + \ldots + \tfrac{1}{n^2} < 2 - \tfrac{1}{n}.$$

For the base case, let $n = 2$. Then $1 + \frac{1}{4} = \frac{5}{4} < \frac{3}{2} = 2 - \frac{1}{2}$ as required.
For the inductive hypothesis, we assume the inequality for some $n = k$:

$$1 + \tfrac{1}{4} + \tfrac{1}{9} + \ldots + \tfrac{1}{k^2} < 2 - \tfrac{1}{k}.$$

Then

$$1 + \tfrac{1}{4} + \tfrac{1}{9} + \ldots + \tfrac{1}{k^2} + \tfrac{1}{(k+1)^2} < 2 - \tfrac{1}{k} + \tfrac{1}{(k+1)^2}.$$

2017 T1 Q3 (i)

Show, using truth tables, that $(p \to q) \vee (\sim p \to r)$ is a tautology.

8 cases? Seems like a pain.
Instead, we do 2.

p	$p \to q$	$\sim p \to r$	$(p \to q) \vee (\sim p \to r)$
T	q	T	T
F	T	r	T

Thus $(p \to q) \vee (\sim p \to r)$ is always true.

2017 T1 Q3 (ii)

Show, using standard logical equivalences, that $(q \vee \sim r) \to p$ is logically equivalent to $(r \vee p) \wedge (q \to p)$.

2017 T1 Q3 (iii)

Prove that $\sqrt{13}$ is irrational.

Assume for the sake of contradiction that $\sqrt{13}$ is rational. Then, by definition, we can write $\sqrt{13} = \frac{a}{b}$ where a and b are coprime integers.

This gives $a^2 = 13b^2$ which means $13|a^2$, so $13|a$. Then we can write $a = 13k$, where $k \in \mathbb{Z}$.

This gives $(13k)^2 = 13b^2$, so $b^2 = 13k^2$ which means $13|b^2$, so $13|b$.

We have shown that 13 is a factor of both a and b, which contradicts the coprimality of a and b.

Thus $\sqrt{13}$ is not rational.

Lemma: For an integer n, if $13|n^2$, then $13|n$.
Remark: Intuitive, but annoying to make rigorous. 13 cases will work.
Proof: Assume $13|n^2$ but $13\nmid n$. Write $13x + ny = 1$ by Bezout's. Then $13nx + n^2y = n$, so $13(nx + ky) = n$ for some $k \in \mathbb{Z}$. Contradiction.

2017 T1 Q3 (vi)

A subset S of $\mathbb{R}$ is said to be open if

$$\forall x_0 \in S \ \exists \delta > 0 \text{ such that } |x - x_0| < \delta \Rightarrow x \in S.$$

(a) Write down the negation of the open condition given above.
(b) Use your answer to (a) to show that the interval $[0, 1]$ is not open.

2020 T1 Q3 (vi) Solution

S is open means $\forall x_0 \in S\ \exists \delta > 0$ such that $|x - x_0| < \delta \Rightarrow x \in S$. Write symbolically "$S$ is not open".

$$\exists x_0 \in S\ \forall \delta > 0\ \exists x \in \mathbb{R} \text{ such that } |x - x_0| < \delta \text{ and } x \notin S$$

Hence show that $[0, 1]$ is not open.

We choose $x_0 = 1$ and $x = 1 + \frac{\delta}{2}$.

Isaiah's Question

(a) Given that $x \geq \sin x$ for $x \geq 0$, show that $\cos x \geq 1 - \frac{1}{2}x^2$ for all x.
(b) Prove by induction that for $n \in \mathbb{Z}^+$, $(\cos x)^n \geq 1 - \frac{1}{2}nx^2$ for all x.

Isaiah's Question

(a) Given that $x \geq \sin x$ for $x \geq 0$, show that $\cos x \geq 1 - \frac{1}{2}x^2$ for all x.
(b) Prove by induction that for $n \in \mathbb{Z}^+$, $(\cos x)^n \geq 1 - \frac{1}{2}nx^2$ for all x.

5. Enumeration and Probability

2019 T2 Q2 (iv)

Consider the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 100,$$

where x_1, x_2, x_3, x_4, x_5 are to be non-negative integers.
a) How many solutions has this equation altogether?
b) How many solutions has this equation in which all of x_1, x_2, x_3, x_4, x_5 are congruent to 2 modulo 3?
c) How many solutions has this equation in which none of x_1, x_2, x_3, x_4, x_5 are congruent to 2 modulo 3?

Counting Partitions

Suppose we want to partition a indistinguishable objects between b groups, where each of these b groups can have 0 objects or more. The number of ways of doing this is $C(a + b - 1, b - 1)$.

2019 T2 Q2 (iv)

Consider the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 100,$$

where x_1, x_2, x_3, x_4, x_5 are to be non-negative integers.
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(c) How many solutions has this equation in which none of x_1, x_2, x_3, x_4, x_5 are congruent to 2 modulo 3?

(a)

$$C(104, 4).$$

2019 T2 Q2 (iv)

Consider the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 100,$$

where x_1, x_2, x_3, x_4, x_5 are to be non-negative integers.
(a) How many solutions has this equation altogether?
(b) How many solutions has this equation in which all of x_1, x_2, x_3, x_4, x_5 are congruent to 2 modulo 3?
(c) How many solutions has this equation in which none of x_1, x_2, x_3, x_4, x_5 are congruent to 2 modulo 3?

(b) We want to encode this information in each number, so for each x_i, we can write $x_i = 3k_i + 2$ for k_i a non-negative integer. That is, $(3k_1 + 2) + (3k_2 + 2) + ... + (3k_5 + 2) = 100$.
We then obtain $k_1 + k_2 + ... + k_5 = 30$.

$$C(34, 4).$$

2019 T2 Q2 (iv)

Consider the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 100,$$

where x_1, x_2, x_3, x_4, x_5 are to be non-negative integers.
(a) How many solutions has this equation altogether?
(b) How many solutions has this equation in which all of x_1, x_2, x_3, x_4, x_5 are congruent to 2 modulo 3?
(c) How many solutions has this equation in which none of x_1, x_2, x_3, x_4, x_5 are congruent to 2 modulo 3?

(c) Now we know that for each x_i, $x_i = 3k_i + r_i$, where $r_i \in \{0, 1\}$ and k_i is a non-negative integer. First we can consider r_i.
$r_1 + r_2 + ... + r_5 = 1$ or $r_1 + r_2 + ... + r_5 = 4$. We can actually consider these cases separately, and then start thinking about k_i.

2019 T2 Q2 (iv)

Case 1: $r_1 + r_2 + ... + r_5 = 1$.

2019 T2 Q2 (iv)

Case 2: $r_1 + r_2 + ... + r_5 = 4$.

Example 2

There are 4 separate groups of students with 4 students each. We want to pick 6 students from this group of 16, such that at least one group has at least 3 representatives in the final group of 6. How many such groups of 6 students are there?

At least 1 group needs to have at least 3 representatives in the group of 6. So a good way to count the number of combinations is to cycle through the four groups (Group A, Group B, Group C, Group D). That is to say, we can first calculate the number of ways for Group A to have at least 3 representatives, then count the number of ways for Group B to have at least 3 representatives, and so on.

However, in counting problems, when we count separate groups and add them together, we need to be wary of double counting!

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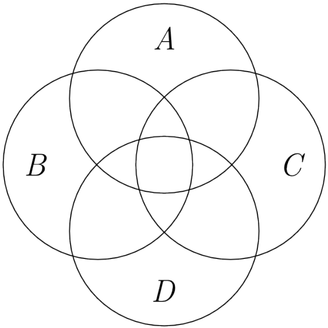

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The Inclusion-Exclusion Principle

$$\begin{aligned}|A_1 \cup A_2 \cup \ldots \cup A_n| = {} & |A_1| + |A_2| + \ldots + |A_n| \\ & - |A_1 \cap A_2| - |A_1 \cap A_3| - \ldots - |A_{n-1} \cap A_n| \\ & + |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + \ldots + |A_{n-2} \cap A_{n-1} \cap A_n| + \ldots \\ & \ldots \\ & + (-1)^{n+1}|A_1 \cap A_2 \cap \ldots \cap A_n|\end{aligned}$$

For instance, in this case, $|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D|$
$-|A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| +$
$|A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| - |A \cap B \cap C \cap D|$.

Example 2

There are 4 separate groups of students with 4 students each. We want to pick 6 students from this group of 16, such that at least one group has at least 3 representatives in the final group of 6. How many such groups of 6 students are there?

So we start by calculating the size of each set: how many ways are there to pick at least three representatives from, say, Group A?

$$C(4,3) \times C(12,3) + C(4,4) \times C(12,2) = 946.$$

All our sets are the same, so, altogether,

$$|A| + |B| + |C| + |D| = 4 \times 946 = 3784.$$

Example 2

There are 4 separate groups of students with 4 students each. We want to pick 6 students from this group of 16, such that at least one group has at least 3 representatives in the final group of 6. How many such groups of 6 students are there?

Now we can start "excluding."

$$|A \cap B| = C(4,3) \times C(4,3) = 16.$$

All such groups are the same, so we have

$$|A \cap B| + |A \cap C| + ... + |C \cap D| = C(4,2) \times 16 = 96.$$

What about $|A \cap B \cap C|$? We need three separate groups to have at least 3 students each in the group, which would amount to 9 students in total. Obviously, this is not possible, so we have our answer:

$$3784 - 96 = 3688.$$

2016 S1 Q4 (iii)

A hand of 10 cards is dealt from a standard pack of 52 cards.
(a) Find the probability that the hand contains exactly 3 cards of the same value and the remaining 7 cards from the remaining suit.
(b) How many hands are there that contain exactly 3 cards in at least one suit?

2016 S1 Q4 (iii)

A hand of 10 cards is dealt from a standard pack of 52 cards.
(a) Find the probability that the hand contains exactly 3 cards of the same value and the remaining 7 cards from the remaining suit.
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2016 S1 Q4 (v)

The English alphabet has 26 letters, of which 5 are vowels and 21 are consonants. We will write all our words using upper case (capital) letters. Repetition of letters in words is allowed.
Find the number of 20 letter words (strings) using the English alphabet:
(a) without any restrictions
(b) containing exactly 3 vowels
(c) containing the subword "MATHS"

(a)

$$26^{20}.$$

(b)

$$C(20,3) \times 5^3 \times 21^{17}.$$

2016 S1 Q4 (v)

Find the number of 20 letter words (strings) using the English alphabet:
(c) containing the subword "MATHS"

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(c)

2013 S2 Q4 (v)

When ascending a flight of stairs, an elf can take 1 stair in one stride or 3 stairs in one stride. Let a_n be the number of different ways for the elf to ascend an n-stair staircase.
(a) Find a_1, a_2 and a_3.
(b) Obtain a recurrence relation for a_n. Give a brief reason. (You do NOT need to solve this recurrence relation.)
c) Find the value of a_6.

(a)

$$a_1 = 1.$$

$$a_2 = 1.$$

$$a_3 = 2.$$

2013 S2 Q4 (v)

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(a) Find a_1, a_2 and a_3.
(b) Obtain a recurrence relation for a_n. Give a brief reason. (You do NOT need to solve this recurrence relation.)
c) Find the value of a_6.

(b) Let's consider what an arbitrary path up the staircase would be. We might have

$$1, 3, 3, 1, 3, ..., 3.$$

To construct a recurrence relation, we want to think of ways to break each valid answer down into smaller parts.

$$a_n = a_{n-3} + a_{n-1}.$$

2013 S2 Q4 (v)

When ascending a flight of stairs, an elf can take 1 stair in one stride or 3 stairs in one stride. Let a_n be the number of different ways for the elf to ascend an n-stair staircase.
(a) Find a_1, a_2 and a_3.
(b) Obtain a recurrence relation for a_n. Give a brief reason. (You do NOT need to solve this recurrence relation.)
c) Find the value of a_6.

(c)

$$a_6 = 6.$$

2014 S2 Q4 (iv)

Strings of the digits 1 and 2 are to be constructed having sum n. The strings may be of any length, and order is regarded as important. For example, if $n = 10$, then three different possible strings are 1112212, 1122121 and 22222.

(a) If the number of 2s in the string is exactly k, how many digits are in the string? How many strings are possible in this case?

(b) Let a_n be the total number of strings with sum n. Explain why $a_n = a_{n-1} + a_{n-2}$, and find the values of a_1 and a_2.

(c) Using the above results, or otherwise, prove that

$$F_{n+1} = \Sigma_{k=0}^{\lfloor \frac{n}{2} \rfloor} C(n-k, k),$$

where F_m denotes the mth Fibonacci number, with $F_1 = 1, F_2 = 1$.

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2014 S2 Q4 (iv)

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(b) Let a_n be the total number of strings with sum n. Explain why $a_n = a_{n-1} + a_{n-2}$, and find the values of a_1 and a_2.

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(c) Using the above results, or otherwise, prove that

$$F_{n+1} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} C(n-k, k),$$

where F_m denotes the mth Fibonacci number, with $F_1 = 1, F_2 = 1$.

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