MATH1081 – Seminar solutions

[Relations & Graph Theory]

Written by: Karen Zhang

Updated on *May 1, 2020*

The following document are full worked solutions to the questions that was discussed in the revision seminar on April 29, 2020. The solutions were written by Karen Zhang and can be used as supplement resources while preparing for final exams. Please use this resource ethically. The following document is **NOT** endorsed by the School of Mathematics and Statistics and may be prone to errors; if you spot an error, please message us here. Happy studying!

Contents

Seminar solutions

Equivalence relations

(2016 S2, Q2ii)
Let $\sim$ be the relation on the set of integers $\mathbb{Z}$ be defined by

$$a \sim b \text{ if and only if } a^2 \equiv b^2 \pmod 4.$$

1. Show that $\sim$ is an equivalence relation.
2. Find the equivalence classes of $\sim$.

SOLUTION.

1. PROOF.
   To show $\sim$ is an equivalence relation, we must show that $\sim$ is **reflexive, symmetric** and **transitive** on the set of integers $\mathbb{Z}$.

   Reflexive. Let $a \in \mathbb{Z}$. We can see that $a^2 \equiv a^2 \pmod 4$, since $a^2 = a^2 + 4n$ for $n = 0 \in \mathbb{Z}$. Hence, $a \sim a$, and therefore $\sim$ is reflexive.

   Symmetric. Let $a, b \in \mathbb{Z}$. A relation is defined to be symmetric if, given that $a \sim b$, then $b \sim a$. First, we assume $a \sim b$. By definition, this means that $a^2 \equiv b^2 \pmod 4$. We can rewrite this as $a^2 - b^2 = 4n$, where $n \in \mathbb{Z}$.

   $$\begin{aligned} a^2 - b^2 &= 4n, \\ -(b^2 - a^2) &= 4n, \\ b^2 - a^2 &= 4(-n) \\ &= 4m, \text{where } m = -n \in \mathbb{Z}. \end{aligned}$$

   Hence, $b^2 \equiv a^2 \pmod 4$, and $b \sim a$. Therefore, $\sim$ is symmetric.

   Transitive. Let $a, b, c \in \mathbb{Z}$. A relation is defined to be transitive if, given that $a \sim b$ and $b \sim c$, then $a \sim c$. First, we assume the following,

   $$a^2 \equiv b^2 \pmod 4 \text{ (definition of } \sim\text{)}, \tag{1}$$
   $$b^2 \equiv c^2 \pmod 4 \text{ (definition of } \sim\text{)}. \tag{2}$$

   Equation (1) can be rewritten as $a^2 - b^2 = 4q$ and equation (2) can be rewritten as $b^2 - c^2 = 4r$, where both $q, r \in \mathbb{Z}$. By taking b^2 as subject in equation (2) and substituting this into equation (1), we have

$$\begin{aligned} a^2 - (4r + c^2) &= 4q, \\ a^2 - c^2 &= 4q + 4r \\ &= 4(q + r) \\ &= 4s, \text{ where } s = q + r \in \mathbb{Z}. \end{aligned}$$

Hence, $a^2 \equiv c^2 \pmod 4$, and $a \sim c$. Therefore, $\sim$ is transitive.

We have shown that the defined $\sim$ is reflexive, symmetric and transitive. Therefore, $\sim$ is an equivalence relation.

2. An equivalence class of b with respect to $\sim$ is the set

$$[b] = \{a \in \mathbb{Z} \mid a \sim b\}.$$

First, take the case $b = 0$. By definition, we have

$$\begin{aligned} [0] &= \{a \in \mathbb{Z} \mid a \sim 0\}, \\ &= \{a \in \mathbb{Z} \mid a^2 \equiv 0^2 \pmod 4\}, \\ &= \{a \in \mathbb{Z} \mid a^2 = 4n, \text{ where } n \in \mathbb{Z}\} \\ &= \{a \in \mathbb{Z} \mid a = \pm 2\sqrt{n}, \text{ where } n \in \mathbb{Z}\}, \\ &= \{\dots, -6, -4, -2, 0, 2, 4, 6, \dots\}. \end{aligned}$$

Now, you can see that we're missing all the cases where a is an odd number. So, take $b = 1$. We have

$$\begin{aligned} [1] &= \{a \in \mathbb{Z} \mid a \sim 1\}, \\ &= \{a \in \mathbb{Z} \mid a^2 \equiv 1^2 \pmod 4\}, \\ &= \{a \in \mathbb{Z} \mid a^2 = 4n + 1, \text{ where } n \in \mathbb{Z}\} \\ &= \{a \in \mathbb{Z} \mid a = \pm\sqrt{4n + 1}, \text{ where } n \in \mathbb{Z}\}, \\ &= \{\dots, -5, -3, -1, 0, 1, 3, 5, \dots\}. \end{aligned}$$

Hence, we can see that the equivalence classes $[0]$ and $[1]$ are pairwise disjoint and $[0] \cup [1] = \mathbb{Z}$. This means that every element of $\mathbb{Z}$ is in some equivalence class, and no element of $\mathbb{Z}$ is in two different classes. Therefore, the proof is complete. $\square$

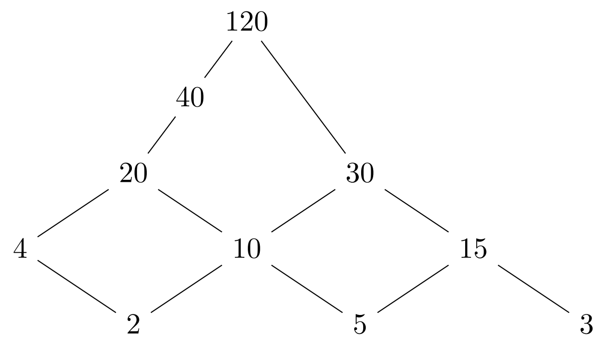
Hasse diagrams

Let $S = \{2, 3, 4, 5, 10, 15, 20, 30, 40, 120\}$.

1. Draw the Hasse diagram for $\{S, |\}$.
2. Find all
   (a) maximal elements,
   (b) minimal elements.
3. Find two elements of S that do not have a greatest lower bound and explain why they do not.

SOLUTION.

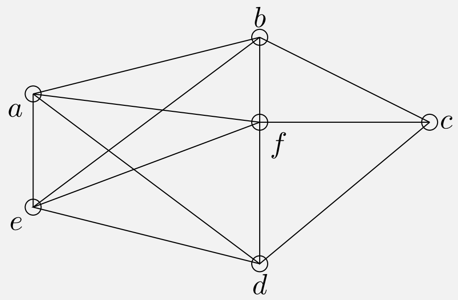
1. Hasse Diagram for partial order $\preceq = \{S, |\}$.



2. Maximal elements: $\{120\}$. Minimal elements $\{2, 3, 5\}$.
3. Since the elements 2 and 5 have no element below them, they do not have a greatest lower bound.

Euler, Hamilton and Bipartite Graphs

Consider the following graph G.



1. Does G have a Euler path? Explain your answer.
2. Does G have a Hamilton circuit? Explain your answer.
3. Is G bipartite? Explain your answer.

Solution.

1. A Euler path exists if there exists a path between two distinct vertices of odd degree such that each edge is used exactly once. Generally, you would look for two vertices of odd degree and every other vertex being even degree. In the diagram above, we can see the following:

$$\begin{aligned} &deg(c) = 3 \\ &deg(a) = deg(b) = deg(d) = deg(e) = 4 \\ &deg(f) = 5. \end{aligned}$$

   Therefore, a Euler path from c to f does exist. Note: It's always useful to give an example of a path e.g. $c, b, a, e, d, a, f, e, b, f, d, c, f$.

2. A Hamilton circuit is a circuit which contains each vertex of a graph exactly once. Although there is no simple way of determining if a graph contains a Hamilton circuit, we can use the sufficient condition for a Hamilton circuit: $\deg(v) \geq \frac{n}{2}$ where n is the number of vertices. In the graph above, we can see that

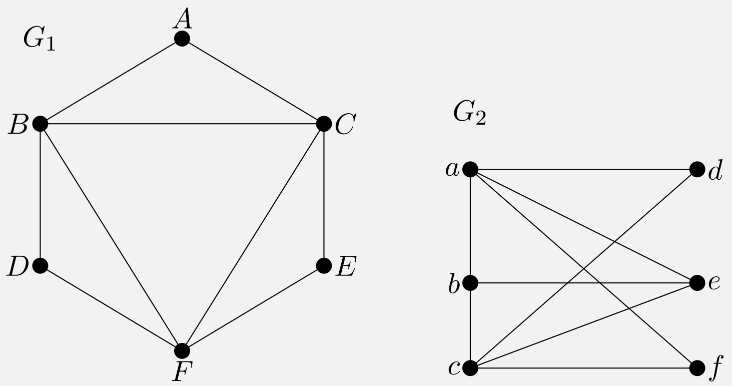
$$deg(v) \geq \frac{n}{2} = \frac{6}{2} = 3$$

   for every v in graph G. Hence, G contains a Hamilton circuit (e.g. c, b, a, e, d, f, c).

3. A graph is bipartite if and only if it contains no odd cycles. We can see that G contains numerous odd cycles such as the cycle (b, a, f, b), and hence cannot be bipartite.

Euler, Hamilton, Planar and Isomorphism

Consider the graphs G_1 and G_2.



1. Does G_1 contain a Euler circuit? Explain your answer.
2. Is G_2 planar? Explain your answer.
3. Are G_1 and G_2 isomorphic?. Explain your answer.
4. Does G_2 contain a Hamilton cycle? Explain your answer.

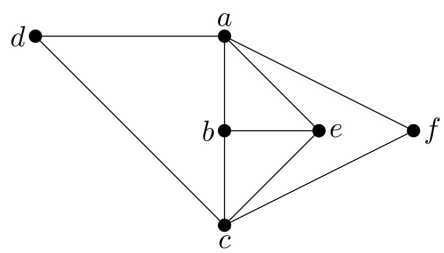
Solution.

1. The existence of a Euler circuit depends on whether G_1 contains vertices of even degree only. From the graph, we can see that

$$deg(A) = deg(D) = deg(E) = 2$$
$$deg(B) = deg(C) = deg(F) = 4.$$

Since there are no vertices of odd degree, G_1 contains a Euler circuit (e.g. $A, B, D, F, B, C, F, E, C, A$).

2. A graph is planar if it can be drawn isomorphically with no intersecting edges. For G_2, if the vertex d is placed to the left of vertex a, and vertex f to the right of vertex e, G_2 will have no intersecting edges. Therefore, G_2 is planar. (Note that in the exam, drawing a planar representation of a graph, if it exists, is enough as a proof).



3. To prove whether two graphs are isomorphic, simply showing that any one isomorphic invariant is false is enough as a proof. By comparing the individual vertex degrees of G_1 and G_2, we can see that G_1 contains 3 vertices of degree 4, while G_2 only contains 2 vertices of degree 4. Hence, the number of vertices of a given degree are not equal, and G_1 is not isomorphic to G_2.

4. Assume that G_2 contains a Hamilton cycle. In order for vertex f to be included, the circuit must contain edges af and cf. It also has to contain two of the following edges: ae, be, ce. However, it's impossible to visit e without visiting a or c again. Therefore, G_2 cannot contain a Hamilton cycle.

Euler's Formula for Planar Graphs

1. State Euler's formula for a connected planar graph having v vertices, r regions and e edges.
2. Show that if G is a connected planar simple graph with $v \geq 3$, then
$$e \leq 3v - 6.$$
3. Hence show that a connected planar simple graph with $v \geq 3$ has at least one vertex of degree less than or equal to 6.

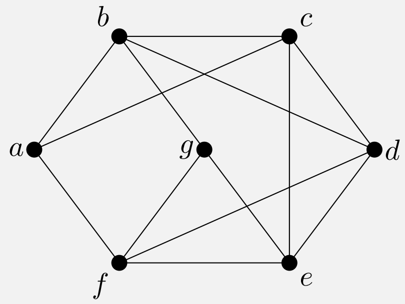
SOLUTION.

1. Euler's formula is given by: $r + v = e + 2$.

2. If G is a simple, connected planar graph, then no region has a degree of 1 or 2. This is because $deg(r) = 1$ only occurs if G contains loops, and $deg(r) = 2$ only occurs if G contains parallel edges. Therefore, the minimum $\sum deg(r) = 3r$. We also know that $\sum deg(r) = 2e$. Hence,
$$2e \geq 3r.$$
We know that $r + v = e + 2$ (Euler's formula), thus multiplying 3 to both sides of the equation gives
$$\begin{aligned} 3(e+2) &= 3(r+v), \\ &= 3r + 3v, \\ &\leq 2e + 3v. \end{aligned}$$
Therefore,
$$\begin{aligned} 3e + 6 &\leq 2e + 3v, \\ e &\leq 3v - 6. \end{aligned}$$

3. Assume that $deg(v) > 6$ for all vertices in the graph. This results in minimum $\sum deg(v) = 6v$, and thus $\sum deg(v) = 2e > 6v$. However, from part (2) we found that $2e \leq 6v - 12$ for a simple connected planar graph with $v \geq 3$. Hence, we have a contradiction and our original assumption that $deg(v) > 6$ for all vertices is false. Therefore, the graph contains at least one vertex of degree less than or equal to 6.

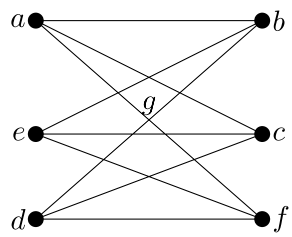
Kuratowski's Theorem



Show that the graph is NOT planar.

Solution.

We will be using Kuratowski's Theorem which states that graph is planar iff it has no subgraph K_5, $K_{3,3}$ or any graph homeomorphic to K_5 or $K_{3,3}$. First, obtain a subgraph of the above graph by deleting edges bc, de and fg. Next, redrawing the subgraph gives a graph homeomorphic to $K_{3,3}$. Hence, by Kuratowki's Theorem, the graph is not planar.



Tips: notice that the given graph has 7 vertices, however vertex g looks like it was added onto edge be, hinting at a homeomorphic subgraph. The remaining 6 vertices is a hint that the subgraph could be $K_{3,3}$.

Trees and Handshaking Lemma

Prove that the average vertex degree

$$\frac{1}{n}\sum_{v\in V(T)} d(v)$$

of a tree T on $|V(T)| = n$ vertices is strictly less than 2.

Solution.
The Handshaking Lemma states that

$$\sum_{v\in V(T)} d(v) = 2e$$

where e is the number of edges in the given graph. For a tree T with n vertices, we know that there are $n-1$ edges. Substituting this into the Handshaking Lemma gives

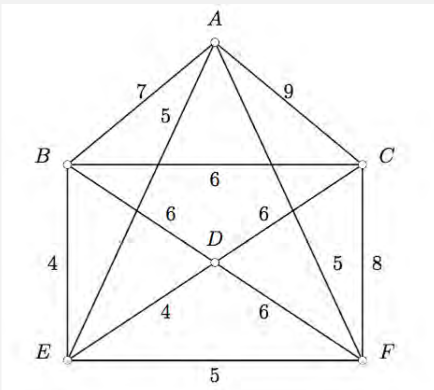
$$\begin{aligned}\sum_{v\in V(T)} d(v) &= 2(n-1)\\ &= 2n-2.\end{aligned}$$

Dividing both sides by n where $n \geq 2$, we have

$$\frac{1}{n}\sum_{v\in V(T)} d(v) = 2 - \frac{2}{n} < 2.$$

Therefore, the average vertex degree of a tree T is strictly less than 2.

Kruskal's Algorithm

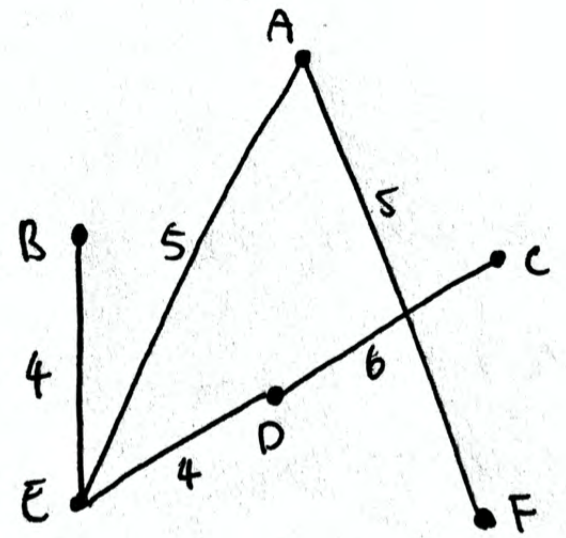


Use Kruskal's algorithm to construct a minimal spanning tree T for the following weighted graph. Make a table showing the details of each step.

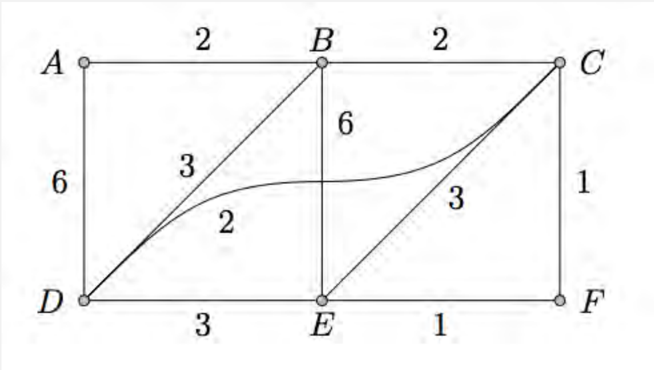
Solution.

Step	Edge	Weight	Used?
1	BE	4	Yes
2	ED	4	Yes
3	EA	5	Yes
4	AF	5	Yes
5	EF	5	No
6	BD	6	No
7	DF	6	No
8	DC	6	Yes
9	BC	6	No
10	AB	7	No
11	CF	8	No
12	AC	9	No

Total (minimal) weight $= 4+4+5+5+6 = 24$. Note that there can be more than one minimal spanning tree, depending on which edges you choose to add and omit. However, the total minimal weight should still be the same.



Dijkstra's Algorithm



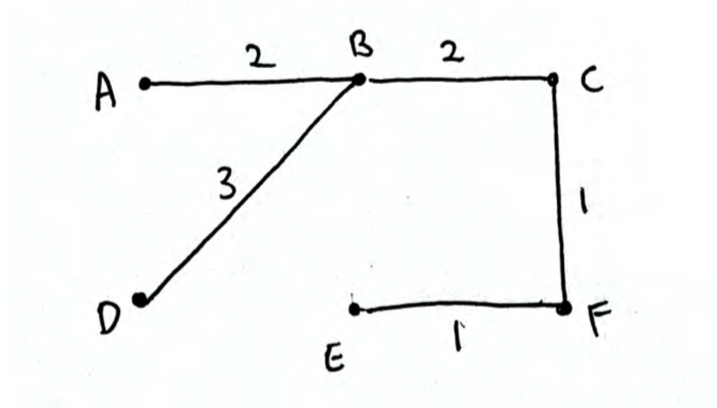
1. Use Dijkstra's algorithm to find a spanning tree that gives the shortest paths from A to every other vertex of the graph. Make a table showing the details of each step.
2. Is this spanning tree found in part 1 a minimal spanning tree? Explain your answer.

SOLUTION.

1.

Step	Available edges (total path weights)	Choice	Distance from A
1	$AB(2), AD(6)$	AB	$d(A,B) = 2$
2	$BD(5), BE(8), BC(4), AD(6)$	BC	$d(A,C) = 4$
3	$CE(7), CF(5), CD(6), AD(6), BD(5), BE(8)$	CF	$d(A,F) = 5$
4	$FE(6), CE(7), CD(6), AD(6), BD(5), BE(8)$	BD	$d(A,D) = 5$
5	$DE(8), FE(6), CE(7)$	FE	$d(A,E) = 6$

Note that in step 3, you could've chosen edge BD instead of CF. This doesn't change the spanning tree, however in some questions there is more than one spanning tree that gives the shortest path from a given point.



2. Note that the shortest A-path spanning tree gives shortest paths from A to other vertices only. If you apply the Kruskal's Algorithm for a minimal spanning tree,

Step	Edge	Weight	Used?
1	EF	1	Yes
2	CF	1	Yes
3	BC	2	Yes
4	CD	2	Yes
5	AB	2	Yes
6	BD	3	No
7	CE	3	No
8	DE	3	No
9	AD	6	No
10	BE	6	No

The total minimal weight $= 1 + 1 + 2 + 2 + 2 = 8$, while the total weight for the shortest A-path spanning tree $= 1 + 1 + 2 + 2 + 3 = 9$. Hence, this spanning tree is not minimal.