Second Year Mathematics Revision

Linear Algebra - Part 1

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Example 1

Example 1 [MATH2601 Problem Set 1 Q2]

For what values of λ does the following equation have:

1. no solutions
2. infinite number of solutions
3. a unique solution

$$x + 2y + \lambda z = 1$$

$$-x + \lambda y - z = 0$$

$$\lambda x - 4y + \lambda z = -1$$

Invertibility

A square matrix is invertible if and only if it has a non-zero determinant.

Example 2

Which of the following matrices are invertible?

1. $\begin{bmatrix} 2 & 7 \\ 1 & 4 \end{bmatrix}$
2. $\begin{bmatrix} 4 & 2 \\ 10 & 5 \end{bmatrix}$

To find an inverse of a matrix A, create an augmented matrix $(A|I)$ and row-reduce both sides till the LHS becomes the identity matrix I.

Vector Space

Definition 1: Vector Spaces

A **vector space** is denoted by $(V, +, \cdot, \mathbb{F})$. We have a set of vectors V, 2 operations denoted by $+, \cdot$ and a set of scalars $\mathbb{F}$. The operations can be anything, just so long as the following conditions are satisfied:

1. *Associativity of addition*
2. *Commutativity of addition*
3. *Zero Element*
4. *Additive Inverse*
5. *"Associativity" of multiplication*
6. *Multiplicative identity*
7. *Scalar distributivity*
8. *Vector distributivity*

Properties of vector spaces

Corollaries

1. $0\mathbf{v} = \mathbf{0}$
2. $\alpha\mathbf{0} = \mathbf{0}$
3. $(-1)\mathbf{v} = -\mathbf{v}$
4. $\alpha\mathbf{v} = \mathbf{0} \iff \alpha = 0 \text{ OR } \mathbf{v} = \mathbf{0}$

Examples

Example 3 [MATH2601 Problems Part 1 Q26]

State which of the following are vector spaces

1. $S = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 0\}$
2. $S = \{(x, y, z) \in \mathbb{R}^3 : xyz = 0\}$
3. $S = \{(x, y, z) \in \mathbb{R}^3 : \sin(x + y + z) = 0\}$

A handy trick to recognize if something is a vector space is if the set only involves linear terms.

To show a set does not form a vector space, you must find a counter-example.

Subspaces

Lemma 1: Subspace Lemma

Consider a vector space V over the field $\mathbb{F}$. A set $W \subseteq V$ is a subspace over the same field if and only if:

1. W is non empty
2. W is **closed** under addition
3. W is closed under scalar multiplication

Examples

Example 4 [MATH2601 Problems Part 1 Q29]

Are the following subspaces of $\mathbb{R}^3$

1. $S_1 = \{\mathbf{x} : x_1^2 = x_2^3\}$
2. $S_2 = \{\mathbf{x} : 2x_1 - 3x_2 + 3x_3 = 13\}$
3. $S_3 = \{\mathbf{x} : \mathbf{x} = t_1\mathbf{u}_1 + t_2\mathbf{u}_2,\ \text{for fixed } u_1, u_2 \in \mathbb{R}^3, t_1, t_2 \in \mathbb{R}\}$

Linear Combinations

Definition 2: Linear Combination

Consider a vector space V over $\mathbf{F}$. A **linear combination** of vectors $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n \in V$ is any vector $\mathbf{v}$ that can be written as:

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + ... + \alpha_n \mathbf{v}_n$$

where $\alpha_i \in \mathbf{F}$ for all $i = 1, ..., n$.

Span

Definition 3: Span

Let V be a vector space over $\mathbb{F}$, and let S be a subset of vectors in V. Then the span of S is the set of all linear combinations of vectors in S. We say that S spans V if $\text{span}(S) = V$.

Linear Independence

Definition 4: Independence

A set $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ is **linearly independently** if we have the following:

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + ... + \alpha_n\mathbf{v}_n = \mathbf{0} \iff \alpha_i = 0, i = 1, 2, ..., n$$

Note that if S was a **linearly dependent** set, then $\mathbf{v}_n$ could be expressible as a linear combination of $S' = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_{n-1}\}$

Basis

Definition 5: Basis

A **basis** for a vector space V is a linearly independent subset of V which spans V.

Theorem 1

Let V be a vector space over $\mathbb{F}$, and suppose V has a finite spanning set.

1. If S is a finite spanning set for V, then S contains a basis for V.
2. If T is a linearly independent subset of V, then there is a basis of V that contains T.
3. Any 2 bases of V have the same number of elements.

Independent and Spanning Sets

Lemmas

1. Any subset of linearly independent set is also linearly independent
2. If $\mathbf{v} \in \text{span}(S)$, then $S \cup \{\mathbf{v}\}$ is linearly dependent. The converse holds, but only if S is linearly independent.
3. If $S_1 \subseteq S_2$ or $S_1 \subseteq \text{span}(S_2)$, then $\text{span}(S_1) \subseteq \text{span}(S_2)$.
4. $\text{span}(S \cup \{\mathbf{v}\}) = \text{span}(S) \iff \mathbf{v} \in S$

A set $\mathbb{S}$ is said to **span** a vector space if any vector in the vector space can be written as a linear combination of the vectors in S.

Examples

Example 5 [MATH2601 Problems Part 1 Q36][MATH2501 Problems Q20]

Show that the set $\{t^2 + t^3, 1 + t + t^2 + t^3, 1 + t^2 + 2t^3\}$ is a linearly independent set.

Example 6 [MATH2601 Vector Spaces 39]

Let V be a vector space of all symmetric 2×2 matrices A with real entries, that is, A satisfies $A^T = A$. Show that:

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 4 & -1 \\ -1 & 5 \end{pmatrix} \right\}$$

Is a basis for V.

Co-ordinate vectors

Definition

The co-ordinate vector of a vector $\mathbf{v}$ with respect to a basis B is a vector of the coefficients so that the linear combination of the vectors in B in the order prescribed.

Examples

Example 7

Find the co-ordinate vector of

1. $\mathbf{b} = (0, -5, 8)$ with respect to the basis $B = \{(1, 0, 4), (-1, 1, 3), (2, -3, 6)\}$
2. $p(t) = 8 - 9t + 3t^2 + 5t^3$ with respect to the basis
$$B = \{-1 + 2t, 1 - 3t + t^3, t + t^2 - 2t^3, 1 + 2t^2 + t^3\}$$
3. $\mathbf{b} = (b_1, b_2)$ with respect to $B = \{(2, 3), (3, 5)\}$.

Example 8 [Example 6 Continued]

Find the coordinates of $\begin{pmatrix} 4 & -11 \\ -11 & -7 \end{pmatrix}$ with respect to this basis.

Dimension

Theorem

1. If a vector space V admits a finite spanning set S, then S contains a basis.
2. Any spanning set of a vector space contains a basis, and each basis must contain the same number of elements.

Dimensions

The Exchange Lemma

Suppose we have a finite dimensional spanning set S and a finite dimensional linearly independent set T. Then there exists a finite set S' such that $T \subseteq S'$, and $|S'| = |S|$.

All this says is that you can keep exchanging vectors in a linearly dependent set that is spanning for vectors that are linearly independent and still span the set. This implies that linearly independent sets are no larger than spanning sets.

Dimensions

Definition

The <u>dimension</u> of a vector space V is the size of a basis set of V.

Lemmas

1. Any spanning set of V has dimension at least thatof V.
2. Dimension of a linearly independent set is at most that of V.
3. If $\text{span}(S) = V$ OR S is a linearly independent set and $|S| = n$, then S is a basis.
4. B is a basis for V if every element of V can be written as a unique linear combination of vectors in B.

Direct Sums

Definition

The sum $S + T$ of 2 subspaces is defined as $S + T = \{\mathbf{a} + \mathbf{b} : \mathbf{a} \in S, \mathbf{b} \in T\}$. If we have $S \cap T = \{\mathbf{0}\}$, then we call the sum a direct sum.

Proposition

The sum of the subspaces S and T is direct iff if any vector $\mathbf{x} \in S + T$ can be written uniquely as $\mathbf{a} + \mathbf{b}$.

Dimensions and direct sums

Theorem

Suppose that S and T are finite dimensional subspaces of a vector space V. Then $\dim(S) + \dim(T) = \dim(S + T) + \dim(S \cap T)$. As a corollary, if $S \cap T = \{\mathbf{0}\}$, then $\dim(S) + \dim(T) = \dim(S \oplus T)$.

Complementary Subspace

Definition

Let V be a finite dimensional vector space and $X \subseteq V$. Then there is a subspace Y for which $V = X \oplus Y$.

The most obvious example is that of the orthogonal complement, which will be discussed shortly. The external direct sum basically involves no notion of a subspace. So if we have 2 **vector spaces** over the same field, then the external direct sum is the direct sum of the 2 spaces with the associated rules that they have.

Examples

Example 9 [MATH2601 Vector Spaces 43]

Suppose that W and X are subspaces of $\mathbb{R}^8$.

1. Show that if $\dim W = 3$, $\dim x = 5$ and $W + X = \mathbb{R}^8$, then $W \cap X = \{0\}$.
2. Show that i f $\dim W = \dim X = 5$, then $W \cap X$ has a linearly independent set with at least 2 elements.
3. Find an example of W and X with $\dim W = \dim X = 5$ and $\dim(W \cap X) = 3$.

Linear Transformations and Maps

Definition

Suppose that V and W are vector spaces. Then $T : V \mapsto W$ is a linear map/transformation iff $T(\lambda \mathbf{u} + \mu \mathbf{v}) = \lambda T(\mathbf{u}) + \mu T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in V, \lambda, \mu \in \mathbb{F}$.

Note: A lot of the linear transformations we cover do have matrix representations, that is, we can write them as $T(\mathbf{x}) = A\mathbf{x}$ where A is a $m \times n$ matrix.

Corollary

Consider two linear maps S, T from V to W. Then $S + T$, λS and $S \circ T$ are all linear maps.

Kernel and Images

Definition

Let $T : V \mapsto W$ be a linear transformation. Then the kernel of T is the set $\ker(T) = \{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{0}\}$. [Essentially, the set of roots of the function.] The image of T is the set of all possible outputs of T. $\operatorname{im}(T) = \{\mathbf{w} = T(\mathbf{v}) \text{ for each} \mathbf{v} \in V\}$.

Rank and nullity

The rank of a map T is the dimension of its image, and the nullity is the dimension of the kernel.

Kernal and Images

RANK-NULLITY THEOREM

If V is a finite dimensional vector space over $\mathbb{F}$ and $T : V \mapsto W$, then we have:

$$\text{rank}(T) + \text{nullity}(T) = \dim(V)$$

Bijectivity

1. A linear map is injective iff $\text{nullity}(T) = 0$.
2. A linear map is surjective if $\text{rank}(T) = \dim(V)$.
3. T is invertible.

In this case, T^{-1} (provided that the inverse exists) is also linear.

Finding bases

Example 10

1. Find the conditions on b_1, b_2, b_3 such that the vector $\mathbf{b} = (b_1, b_2, b_3)$ is in the column space of the matrix:

$$A = \begin{pmatrix} 1 & -3 & 3 \\ 2 & -5 & 4 \\ 2 & -9 & 12 \end{pmatrix}$$

and the basis of the image of A.

2. The kernel and image for the following matrices:

$$A = \begin{pmatrix} 2 & 3 & 1 \\ 1 & -2 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 5 & 0 \\ 1 & -2 & -9 \\ -1 & 1 & 7 \end{pmatrix}$$

Isomorphisms

Isomorphic maps

A bijective linear map $T : V \mapsto W$ is an isomoprhism of the vector spaces V, W. The two spaces are therefore called isomorphic.

Theorem

Finite dimension vector spaces are isomoprhic iff they have the same dimension.

Inverse maps

If $T : V \mapsto W$ is linear and invertible, the matrix of T^{-1} is the inverse of the matrix T.

Matrix of a linear map

Theorem

Let V, W be two finite dimensional vector spaces over $\mathbb{F}$. Suppose that V has a basis $\mathcal{B}$ and W a basis $\mathcal{C}$. If T is linear, then there is a unique $A \in M_{p,q}(\mathbb{F})$ such that $[T(\mathbf{v})]_{\mathcal{C}} = A[\mathbf{v}]_{\mathcal{B}}$

Basically, there exists a matrix A which maps a co-ordinate vector with respect to one basis to the co-ordinate vector of the image with respect to the other basis.

The matrix map of a composition of 2 linear maps is just the product of the matrices used to represent each linear transformation.

Matrix Representation Examples

Example 11 [MATH2601 Linear Transforms 52]

Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is linear and has matrix $\begin{pmatrix} 4 & 9 \\ 1 & 1 \end{pmatrix}$ with respect to the standard basis of $\mathbb{R}^2$. What is the matrix of T with respect to the basis $B = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -3 \\ 2 \end{pmatrix} \right\}$?

Matrix Representation Examples

Example 12

Let:

$$v_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \quad Q = \begin{pmatrix} -1 & -2 & 2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{pmatrix}$$

Defined subspaces $v = \text{span}\{v_1\}$ and $W = \text{span}\{v_2, v_3\}$ of $\mathbb{R}^3$.

1. Show that $\mathbb{R}^3 = V \oplus W$.
2. Show that V and W are invariant under the map T with $T(x) = Qx$.
3. Find the matrix of T in part **b** with respect to the basis $\{v_1, v_2, v_3\}$ of $\mathbb{R}^3$.

Matrix Representation Examples

Obviously we can come up with examples where we want to map a weird basis to another weird basis. So we shall use some nice tools called commutative diagrams.

Definition

If vector space V has 2 bases $\mathcal{B}, \mathcal{C}$, the matrix $[\mathrm{id}]_{\mathcal{C}}^{\mathcal{B}}[\mathbf{v}]_{\mathcal{B}}$

Commutative diagrams

Definition

A commutative diagram helps outline the process of finding what the matrices for the maps are.

Examples

Example 13

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Find a formula for $T(x_1, x_2)$, given that:

1. $T(1, 0) = (3, 4)$ and $T(0, 1) = (4, 9)$.
2. $T(4, 7) = (3, 4)$ and $T(3, 5) = (4, 9)$.
3. $T(5, 7) = (3, 4)$ and $T(2, 7) = (4, 9)$.

Examples

Example 14

Given that the linear function $T : \mathbb{R}^2 \to \mathbb{R}^2$ has matrix A with respect to the standard basis of $\mathbb{R}^2$, find the matrix of T with respect to the basis B, if:

1. $A = \begin{pmatrix} -2 & 1 \\ 5 & 2 \end{pmatrix}$ and $B = \{(1,5),(1,6)\}$.
2. $A = \begin{pmatrix} 4 & 9 \\ 1 & 1 \end{pmatrix}$ and $B = \{(1,-1),(3,2)\}$.
3. $A = \begin{pmatrix} 6 & 11 \\ 12 & -1 \end{pmatrix}$ and $B = \{(1,4),(1,3)\}$.

Normal Form

Theorem

Let $T : V \mapsto W$ be a matrix with $\dim(V) = p$, $\dim(W) = q$ and $\text{rank}(T) = p$. Then there are bases $\mathcal{B}$ of V and $\mathcal{C}$ of W such that the matrix representation N is the direct sum of an $r \times r$ matrix map and the 0 map.

Corollary

As a result, there is an $r \times r$ matrix A and invertible matrices Q, P such that the above matrix map takes the form $Q^{-1}AP$. Such a map is the normal form for a map.

Isomorphisms Again

Theorem

Let $T : V \mapsto W$ be a linear map between finite dimensional spaces and A the matrix representation of T. Then T is invertible iff A is invertible.

Similarity

Definition

2 matrices A and B are similar is $B = P^{-1}AP$ for some matrix P that is invertible.

A property that is identical for similar matrices is called a similarity invariant. Similarity invariants include: trace, rank, nullity, determinant.

Examples

Example 15 [2501 Q9a)]

Show that if A and B are matrices so that AB and BA are both defined then AB and BA have the same trice. Hence show that $S^{-1}MS$ and M have the same trace.

Dual bases [MATH2601 ONLY]

Definition

The dual space of a vector space is the set of linear functionals or covectors. Such linear functionals/covectors are linear maps from a vector to a field of scalars.

Definition

The dual basis to the basis of a vector space $\mathcal{B} = \{\mathbf{v}_1, ..., \mathbf{v}_n\}$ is a set of maps $\mathcal{B}^* = \{\omega^1, ..., \omega^n\}$ so that $\omega^i(\mathbf{v}_j) = 1$ if $i = j$ and 0 otherwise.

Dual bases

Example 16

For each of the following bases of vector space V, find the dual basis of V^*:

1. $\left\{ \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\}$ of $\mathbb{R}^2$.
2. $\{1, 1+t, 1+t+t^2\}$ of $\mathcal{P}_2(\mathbb{R})$.

Multilinear maps and tensors [MATH2601 ONLY]

Definition

A map T is a multilinear map if it is linear map for each argument the map has. More specifically, it is called symmetric if it retains its value after swapping arguments, and alternating if it switches the sign after swapping the elements.

Definition

A tensor of type (r, s) maps $r + s$ arguments (r from the dual of a vector space V^* and s from the vector space V) to an element in the field $\mathbb{F}$ over which V existed. r is then then contravariant order and s the covariant order.

Orthogonal Complements

Definition 1

Let W be a subspace of a vector space V. Then $W^{\perp}$ is also a subspace, called the **orthogonal complement** of W in V.

$$W^{\perp} = \{\mathbf{v} \in V | \mathbf{v} \cdot \mathbf{w} = 0, \text{ for all } \mathbf{w} \in W\}$$

Example 17

Find the orthogonal complement of the set of vectors defined by $\mathbf{S} = \{(1, -1, 2, 0), (-2, 1, 0, 1)\}$

Projections

Definition

A **projection** of a vector onto another "plane" is the shadow

Consider a vector $\mathbf{v}$ and a plane $W = \text{span}\{\mathbf{w}_1, \mathbf{w}_2\}$. Then we can write: $\mathbf{w} = \alpha_1\mathbf{w}_1 + \alpha_2\mathbf{w}_2 + \mathbf{w}$. The first 2 terms refer to a linear combination in the space of W, and then $\mathbf{w}$ is the vector perpendicular to W such that $\mathbf{v}$ can be expressed as a sum of vectors in W and $\mathbf{w}$. The objective is to solve for α_1, α_2, as it is the linear combination of vectors that corresponds to the projection.

Examples

Example 18

1. Find the projection when the vector $(6, 1, -5)$ is projected onto the plane $W = \text{span}\{\mathbf{w}_1, \mathbf{w}_2\}$, where $\mathbf{w}_1 = (1, 2, 1), \mathbf{w}_2 = (-1, 1, 0)$.
2. Find the projection when the $\mathbf{v} = (8, 6, 3, 5)$ is projected onto $\{(1, 1, 0, -1), (-1, 1, 1, 2)\}$.

Orthogonality and Orthonormal vectors

Definition

A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ are said to be an **orthonormal basis** if the following holds true:

$$\mathbf{v}_i \cdot \mathbf{v}_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

2 vectors are **orthogonal** if they satisfy $\mathbf{v}_i \cdot \mathbf{v}_j = 0, i \neq j$.

Projections and orthonormal bases

Theorem

Suppose that $W \subseteq V$ is a subspace with an orthonormal basis $\mathbf{w}_1, \mathbf{w}_2, ..., \mathbf{w}_m$. Then the projection of $\mathbf{v} \in V$ onto W is given by:

$$\text{proj}_W \mathbf{v} = (\mathbf{v} \cdot \mathbf{w}_1)\mathbf{w}_1 + (\mathbf{v} \cdot \mathbf{w}_2)\mathbf{w}_2 + ... + (\mathbf{v} \cdot \mathbf{w}_m)\mathbf{w}_m$$

Note: $\|\mathbf{w}_i\| = 1$ so remember to normalise if you do this technique.

Inner Products

Definition

An **inner product** is a generalised dot product defined over a vector space V with scalar field $\mathbb{F}$. An inner product satisfies:

1. *Linearity* For all vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, and $\alpha, \beta \in \mathbb{F}$, $\langle \alpha\mathbf{u} + \beta\mathbf{v} | \mathbf{w} \rangle = \alpha\langle \mathbf{u}|\mathbf{w}\rangle + \beta\langle \mathbf{v}|\mathbf{w}\rangle$
2. *Conjugate symmetry* $\langle \mathbf{u}|\mathbf{v}\rangle = \overline{\langle \mathbf{v}|\mathbf{u}\rangle}$
3. *Non-negativity* $\langle \mathbf{v}|\mathbf{v}\rangle \geq 0$
4. *Positive definiteness* $\langle \mathbf{v}|\mathbf{v}\rangle = 0 \iff \mathbf{v} = \mathbf{0}$

Orthonormal Bases

Note that the same definition as above applies, replacing the dot product with the inner product.

Example 19 [Lecture Slides]

Show that in $M_{2,2}(\mathbb{R})$, the following set is an orthonormal basis with $\langle X, Y \rangle = \mathrm{trace}(X^*Y)$:

$$\left\{ \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right\}$$

Projections

Definition on Inner Product Spaces

Suppose that $W \subseteq V$ is a subspace with an orthonormal basis $\mathbf{w}_1, \dots \mathbf{w}_m$. Then the projection of $\mathbf{v} \in V$ onto W is:

$$\text{proj}_W \mathbf{v} = \sum_{k=1}^{m} \frac{\langle \mathbf{v} | \mathbf{w}_k \rangle}{\langle \mathbf{w}_k | \mathbf{w}_k \rangle} \mathbf{w}_k = \sum_{k=1}^{m} \langle \mathbf{v} | \mathbf{w}_k \rangle \mathbf{w}_k$$

The ideas of orthogonality are retained, but we now replace dot products with inner products which are more general functions that yield similar properties.

Gram-Schmidt Procedure

Definition : Gram-Schmidt

Consider a set of vectors $\{\mathbf{v}_i\}_{i=1}^{n}$ that form a basis for a vector space. The objective is to create an orthonormal basis based on these vectors. The following recurrence relation describes the procedure:

$$\mathbf{w}_1 = \frac{\mathbf{v}_1}{||\mathbf{v}_1||}, \mathbf{w}_k = \mathbf{v_k} - \text{the projections of } \mathbf{v}_k \text{ onto each preceding } \mathbf{w}_k$$

Examples

Example 20

Use the Gram-Schmidt process to take the linearly independent set of vectors $\{(1,3),(-1,2)\}$ from $\mathbb{R}^2$ with the standard dot product and transform them into

Example 21

Apply the same process to the set $\{1,x\}$ with respect to the inner product $\langle p|q\rangle = \int_0^1 p(x)q(x)dx$.

Cauchy-Schwarz Inequality

Cauchy-Schwarz Inequality

Cauchy-Schwarz Inequality states that

$$|\langle \mathbf{v}|\mathbf{w}\rangle|^2 \leq \langle \mathbf{v}|\mathbf{v}\rangle\langle \mathbf{w}|\mathbf{w}\rangle$$

Norms

Definition: Norms

A **norm** is a type of inner product (implicitly, of course) defined with the following properties

1. *Non-negativity* $||\mathbf{v}|| \geq 0$
2. *Positive definite* $||\mathbf{v}|| = 0 \iff \mathbf{v} = \mathbf{0}$
3. *Scalar multiplication*
4. *Triangle Inequality*

The norm in a vector space fundamentally means length, which is the square root of the dot product, and the inner product is a generalised function of a dot product.

Reconstruction formula

Reconstruction Formula

Let V be a vector space. Then there exists an orthonormal basis $\{\mathbf{e}_i\}_{i=1}^{n}$ and any vector $\mathbf{v}$ can be expressed as a linear combination of the orthonormal basis vectors.

Co-ordinate vectors and properties

Co-ordinate vectors

The **co-ordinate vector** refers to the vector of coefficients for the ordered basis. Considered the ordered basis $B = \{\mathbf{e}_i\}_{i=1}^n$. Then the co-ordinate vector is given by $[\mathbf{v}]_B = (\langle \mathbf{v}|\mathbf{e}_1\rangle, \langle \mathbf{v}|\mathbf{e}_2\rangle, ..., \langle \mathbf{v}|\mathbf{e}_n\rangle)^T$. Note that the co-ordinates are given in corresponding order to the basis vectors.

Theorem

Consider a basis $B = \{\mathbf{e}_i\}_{i=1}^n$ and an isomorphism $S : V \mapsto \mathbb{F}^n$ with $S(\mathbf{v}) = [\mathbf{v}]_B$. Then S and S^{-1} preserve inner products.

$$\langle S(\mathbf{u})|S(\mathbf{v})\rangle = \langle \mathbf{u}|\mathbf{v}\rangle$$

$$\langle S^{-1}(\mathbf{x})|S^{-1}(\mathbf{y})\rangle = \langle \mathbf{x}|\mathbf{y}\rangle$$

Important Properties

Properties of orthogonal complements

Consider a vector space V and a subspace $W \subseteq V$. Then we have the following:

1. $W^\perp$ is a subspace of V.
2. $W \cap W^\perp = \varnothing$
3. $V = W \oplus W^\perp$
4. $(W^\perp)^\perp = W$

Examples

Example 22

Let V be a finite-dimensional scalar product space and let W be a subspace of V. define:

$$W^{\perp} = \{v \in V | \langle v, w \rangle = 0 \quad \forall \quad w \in W\}.$$

Show that:

1. $W^{\perp}$ is a subspace of V.
2. $W \cap W^{\perp} = \{0\}$.
3. For each $v \in V$ there are unique vectors $w_1 \in W$ and $w_2 \in W^{\perp}$ such that $v = w_1 + w_2$.
4. $W^{\perp\perp} = W$.
5. $\dim W + \dim W^{\perp} = \dim V$.

Projections onto subspaces

Orthogonal projection

The **orthogonal projection** of $\mathbf{v}$ onto W is the unique vector $\mathbf{x} \in W$ for which $\mathbf{v} - \mathbf{x} \in W^{\perp}$. Thus:

$$\mathrm{proj}_W V \mapsto V, \mathbf{v} \mapsto \mathrm{proj}_W.$$

Projections onto subspaces

Lemmas

1. Let $\{\mathbf{e}_1, ..., \mathbf{e}_n\}$ be an orthonormal basis for W. Then for any $\mathbf{v} \in V$:
$$\text{proj}_W \mathbf{v} = \langle \mathbf{v} | \mathbf{e}_1 \rangle + ... + \langle \mathbf{v} | \mathbf{e}_n \rangle.$$
2. proj_W is linear.
3. The image of proj is W.
4. The kernel of proj is $W^\perp$.
5. If $\mathbf{v} \in V$ then $\mathbf{v} - \text{proj}_W \mathbf{v} \in W^\perp$.
6. Idempotency: Iterated composition of projection mappings is the same as applying one projection.
7. For all $\mathbf{v} \in V$, we have $\|\text{proj}_W \mathbf{v}\| \leq \|\mathbf{v}\|$
8. $\text{proj}_W \cdot + \text{proj}_{W^\perp} \cdot = \text{id}_V$

Adjoints

Linear functionals

Let V be a vector space over $\mathbb{F}$. A **linear functional** on V is a linear function from V to $\mathbb{F}$. Linear functionals are induce an inner product.

Adjoints

Definition : Adjoints

Consider 2 finite-dimensional finite-dimensional inner product spaces, and let $T : V \mapsto W$ be a linear transformation. The **adjoint** of T is the function $T^* : W \mapsto V$ with the property that:

$$\langle T(\mathbf{v})|\mathbf{w}\rangle = \langle \mathbf{v}|T^*(\mathbf{w})\rangle$$

Considering the matrix representation A of a linear map T has an adjoint equivalent to the mapping T^* where it's matrix representation is equivalent to the conjugate transpose of A. Here, we also take complex valued scalars and matrices.

More properties

Properties of adjoints

1. T^* is a linear transformation from W to V.
2. $(S+T)^* = S^* + T^*$
3. For any scalar α, we have $(\alpha T)^* = \bar{\alpha} T^*$
4. $(T^*)^* = T$.
5. The identity map on V is its own adjoint.
6. Suppose T is a transformation from V to W and S is a transformation from W to X, then $(S \circ T)^* = T^* \circ S^*$

Examples of adjoints

Example 23 [2601 Tutorial Problems]

Let $T : \mathbb{R}^2 \mapsto \mathbb{P}_1(\mathbb{R})$ be the transformation described by $T(v_1, v_2) = (v_1 + v_2) + (2v_2)t$, and the inner product on $\mathbb{P}_1$ as $\langle p|q\rangle = p(1)q(1) + p(2)q(2)$ and the standard inner product on $\mathbb{R}^2$.

Adjoints

Properties of adjoints, kernels and images

If $T : V \mapsto W$ is a linear transformation of finite-dimensional inner product spaces, then:

1. $\ker T^* = (\operatorname{im} T)^\perp$
2. $\operatorname{im} T^* = (\ker T)^\perp$
3. $\ker T = (\operatorname{im} T^*)^\perp$
4. $\operatorname{im} T = (\ker T^*)^\perp$

Invertibility

Theorem

Let V be a finite-dimensional vector space and suppose that S, T are linear transformations from V to V. Then $S^{-1} = T$ if and only if $S \circ T$ is the identity transformation.

Special Adjoints

1. Unitary if $T^* = T^{-1}$.
2. Isometry if $\|T(\mathbf{v})\| = \|\mathbf{v}\|$ for all $\mathbf{v} \in V$
3. Self-adjoint if $T^* = T$

Isometries

Properties of isometries

Let T be a linear map on a finite dimensional inner product space. Then the following are equivalent:

1. T is an isometry.
2. T preserves inner products, that is:

$$\langle T(\mathbf{v}) | T(\mathbf{w}) \rangle = \langle \mathbf{v} | \mathbf{w} \rangle$$

3. T is unitary, that is, $T^* T$ is the identity.
4. T^* is an isometry.
5. If the set $\{\mathbf{e}_1, ...\mathbf{e}_n\}$ is an orthonormal basis for V then so is $\{T(\mathbf{e}_1), ..., T(\mathbf{e}_n)\}$

Unitary and self-adjoint matrices

Analogy in real matrix system

In the set of matrices consisting of real elements, the corresponding words for self-adjoint and unitary are symmetric and orthogonal respectively.

Theorem

For any $n \times n$ matrix A over $\mathbb{R}$ or $\mathbb{C}$, the following are equivalent:

1. A is unitary
2. col$A$ and row$A$ are basis for $\mathbb{F}^n$

QR-Factorisation

QR Factorisation

Let A be a real and complex $m \times n$ matrix with linearly independent columns. Then there exists an orthogonal matrix Q and right triangular matrix R such that $A = QR$.

Algorithm for QR factorisation

Let $\mathbf{v}_i$ for $i = 1, 2, ..., q$ be the columns of A.

1. Employ Gram-Schmidt algorithm on $\mathbf{v}_k$ to work out the orthonormal vectors.
2. Re-write each $\mathbf{v}_k$ as a linear combination of the orthonormal vectors. Inner products make this easy because the formula would just be:

$$\mathbf{v}_k = \sum_{j=1}^{k-1} \langle \mathbf{q}_j | \mathbf{v}_k \rangle \mathbf{q}_j + ||\mathbf{w}_k|| \mathbf{q}_k$$

Note that this allows the construction:

$$A = \begin{pmatrix} \mathbf{q}_1 & ... & \mathbf{q}_q \end{pmatrix} \begin{pmatrix} \langle \mathbf{q}_1 | \mathbf{v}_1 \rangle & \langle \mathbf{q}_1 | \mathbf{v}_2 \rangle & ... & \langle \mathbf{q}_1 | \mathbf{v}_q \rangle \\ 0 & \langle \mathbf{q}_2 | \mathbf{v}_2 \rangle & ... & \langle \mathbf{q}_2 | \mathbf{v}_q \rangle \\ 0 & 0 & ... & ... \\ ... & & & \\ 0 & 0 & ... & \langle \mathbf{q}_q | \mathbf{v}_q \rangle \end{pmatrix} = QR$$

Examples of QR

Example 24

Find QR factorisations for each of the following:

1. $A = \begin{pmatrix} 5 & -4 \\ 12 & 6 \end{pmatrix}$
2. $B = \begin{pmatrix} -2 & 1 \\ 1 & 4 \\ -2 & -8 \end{pmatrix}$

Least Squares

Projection matrix

Suppose you are given the matrix equation $A\mathbf{x} = \mathbf{b}$, where A is a $p \times q$ matrix and $p > q$. $\mathbf{x}, \mathbf{b}$ are of the appropriate dimensions. Then in order to minimise the error (because the system may not have a solution, so we actually need to find the best solution we could) when fitting the linear map T with matrix A compared to $\mathbf{b}$, we must project the vector $A\mathbf{x} - \mathbf{b}$ onto the column space of A.

Using all the axioms, if $A\mathbf{x} - \mathbf{b} \in \mathrm{col}(A)^{\perp}$, then $A^*A\mathbf{x} = A^*\mathbf{b}$. This new matrix equation is called the normal equations.

Examples

Example 25

Find the least squares solution to the following equation:

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 11 \\ 3 \\ 4 \end{pmatrix}$$

Example 26

Find a quadratic of the form $y = a + bx + cx^2$ through the points $(-1, 7), (0, 4), (1, -2), (2, -6)$.

Examples

Example 27 [2501 Least Squares Q19]

Let u and v be two non-zero vectors in $\mathbb{R}^n$. Show that the vector:

$$w = \frac{1}{(||u|| + ||v||)}(||u||v + ||v||u)$$

bisects the angle between u and v.

Eigenvectors, Eigenvalues and Eigenspaces

Definitions

Let V be a vector space and $T : V \mapsto V$ a linear transformation. If λ is a scalar and $\mathbf{v}$ a non-zero vector in V such that $T(\mathbf{v}) = \lambda\mathbf{v}$, then λ is an eigenvalue of T and $\mathbf{v}$ is an eigenvector of T corresponding to λ. The set of eigenvalues of T is the spectrum of T.

Invariance

Definition

Let $T : V \mapsto V$ be a linear transformation. A subspace U of V is said to be T-invariant if $T(U) \subseteq U$, where:

$$T(U) = \{T(\mathbf{u}) | \mathbf{u} \in U\}.$$

Properties

Basic properties of eigenspaces

Let $T : V \mapsto V$ be linear.

1. The eigenvalues of T are λ such that $T(\mathbf{v}) = \lambda\mathbf{v}$.
2. The eigenspace corresponding to λ is given by $E_\lambda = \ker(\lambda I - T)$.
3. Eigenspaces are T-invariant
4. If λ and μ are eigenvalues of T and $\lambda \neq \mu$, then $E_\lambda \cap E_\mu = \{\mathbf{0}\}$.
5. If V is finite-dimensional, a basis B of V consists of eigenvectors of T if and only if the matrix of T with respect to B is diagonal.

More properties

More properties

1. A matrix $A \in M_{nn}(\mathbb{F})$ is diagonalisable if and only if it has n linearly independent eigenvectors associated with it.
2. Distinct eigenvalues correspond to linearly independent eigenvectors.
3. $\mathbf{v}$ is an eigenvector of T with eigenvalue λ if and only if $[\mathbf{v}]_B$ (where $T : V \mapsto V$, and B is a basis of V) is an eigenvector of T with eigenvalue λ.

Definition

The eigenvalues of a matrix A are given by the solutions to the characteristic polynomial, the polynomial obtained by solving $\det(A - \lambda I) = 0$.

Multiplicities

AM-GM Inequality Re-mastered

The algebraic multiplicity of an eigenvalue λ is the multiplicity of the root $z = \lambda$ for the characteristic equation $\det(A - \lambda I) = 0$. The geometric multiplicity is the dimension of the eigenspace associated with λ, that is, $\dim(\ker(A - \lambda I)) = \mathrm{GM}(\lambda)$. The relationship between these two can be described as $\mathrm{GM}(\lambda) \leq \mathrm{AM}(\lambda)$.

Corollary

A matrix A is diagonalisable if for every eigenvalue λ_i, we have $\mathrm{GM}(\lambda_i) = \mathrm{AM}(\lambda_i)$.

Examples of eigenvalues, eigenvectors and diagonalisation

Example 28

Find all the eigenvalues and eigenvectors of the following matrices:

1. $\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$
2. $\begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix}$
3. $\begin{pmatrix} 1 & 4 \\ -1 & 1 \end{pmatrix}$
4. $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix}$

Examples on AM, GM

Example 29

Find the algebraic and geometric multiplicity of each of the above matrices.

Example 30

For each of the following matrices, use the given additional information to find all eigenvalues and eigenvectors *without calculating the characteristic polynomial*. Also write down the algebraic and geometric multiplicities of each eigenvalue.

1. $C = \begin{pmatrix} 2 & -5 & -5 \\ -4 & 8 & 4 \\ 4 & -11 & -7 \end{pmatrix}$, given that 2 and -3 are eigenvalues.
2. $D = \begin{pmatrix} 1 & 4 & 2 \\ 2 & 1 & -2 \\ -3 & 4 & 6 \end{pmatrix}$, given that $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ are eigenvectors.

More theorems

Conditions for diagonalisability

Let $T : V \mapsto V$ be a linear map on a finite dimensional vector space V. Then the following are equivalent:

1. T is diagonalizable
2. There is a basis for V consisting of the eigenvectors of T.
3. V is the direct sum of the eigenspaces of each of the eigenvalues.
4. The sum of geometric multiplicities of distinct eigenvalues is the dimension of V.

Examples

Example 31 [2501 Eigenvalues Q8]

Let V be a vector space and $\{v_1, v_2, v_3\}$ a basis for V. Let T be a linear map from V to V such that:

$$T(\mathbf{v}_1) = 2\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3, \quad T(\mathbf{v}_2) = 2\mathbf{v}_2. \quad T(\mathbf{v}_3) = \mathbf{v}_2 + \mathbf{v}_3.$$

Is there a basis B for V such that the matrix of T with respect to B is diagonal? Explain.

Spectral Theorem

Theorem: SPECTRAL THEOREM

Let $A \in M_{n\times n}(\mathbb{R})$ be a real symmetric matrix. Then:

1. All the eigenvalues are real.
2. Eigenvectors corresponding to distinct eigenvalues are orthogonal
3. There exists an orthogonal matrix Q such that $Q^{-1}AQ$ is the diagonal matrix corresponding to distinct eigenvalues.
4. A has n orthogonal, real eigenvalues.

Examples of Diagonalisation

Example 32 [Lecture Slides]

Diagonalise the following matrix given that the characteristic polynomial is $p(\lambda) = (\lambda - 3)(\lambda^2 - 1)$:

$$\begin{pmatrix} -1 & -12 & 0 \\ 2 & 5 & 4 \\ 0 & 4 & -1 \end{pmatrix}$$

Normal Operators

Definition

A linear transformation on an inner product space is <u>normal</u> if and only if the maps commute with their adjoints.

Theorem

1. If T is normal, then $||T\mathbf{v}|| = ||T^*\mathbf{v}||$ for all $\mathbf{v} \in V$.
2. If T is normal, then $T - \alpha\text{id}$ is normal for any $\alpha \in \mathbb{F}$.
3. The eigenspace of T with eigenvalue λ is the same as the eigenspace of T^* with eigenvalue $\bar{\lambda}$.
4. If T is normal, the 2 eigenspaces corresponding to distinct eigenvalues are orthogonal to each other.

Conic Sections and quadrics

Consider a quadratic equation of the form $ax^2 + 2bxy + cy^2 = k$ for some constant k. Then we can reframe this problem as a matrix equation:

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = k$$

You can check this by expansion of the matrix equation.

Graphing and identifying conics

Begin by diagonalising the real symmetric matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix} = A$ so as to obtain $Q^T DQ$ [This just follows from Spectral Theorem]. Let $\mathbf{X} = Q\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix}$. This allows us to write the form:

$$\begin{pmatrix} X & Y \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} = k \implies \lambda_1 X^2 + \lambda_2 Y^2 = k$$

WHICH IS A CONIC!!! We already know that Q consists of the eigenvectors, so the eigenvectors describe the axes of symmetry of the conic and becomes easy to construct from there.

Examples

Example 33

Sketch the curve $5x^2 + 4xy + 8y^2 = 36$ including all important features and points.

Rotations and reflections

Orthogonal matrices are special matrices with determinant such that $\det Q = \pm 1$. This is equivalent to saying that the eigenvalues each have modulus of 1.

Rotations and reflections

Consider an orthogonal matrix R form $\mathbb{R}^2$ to $\mathbb{R}^2$. Then we can always write R as:

$$R(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

where $e^{i\alpha}$ is an eigenvalue of the linear map T. This describes a ROTATION by an angle α about the origin.

Rotations and reflections

Consider a matrix R to be a 3×3 orthogonal matrix so that it's columns are an orthonormal basis for $\mathbb{R}^3$. Then R is similar to one of the following 2 matrices:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}$$

Angles and axes of reflection

The angles of rotation should not be difficult to work out. You can evaluate these by determining the trace, because the matrix map of T is similar to R described above (based on the diagonalisation procedure). The axis of rotation/reflection is given by the eigenvector corresponding the ± 1 eigenvalue.

Examples of Orthogonal maps

Example 34 [Lecture Slides]

Give a geometric description of the following matrices:

$$A = \frac{1}{9}\begin{pmatrix} 4 & 7 & -4 \\ 1 & 4 & 8 \\ 8 & -4 & 1 \end{pmatrix}$$

$$B = \frac{1}{9}\begin{pmatrix} 4 & -7 & -4 \\ 1 & -4 & 8 \\ 8 & 4 & 1 \end{pmatrix}$$