MATH1231/1241 Final Exam Revision Session

UNSW Mathematics Society

UNSW Sydney

Introduction

Today, we will cover

- Functions of several variables
- Integration techniques
- Ordinary differential equations
- Taylor series
- Averages, arc length, speed and surface area.

We will go through sample exam questions from these topics, showing how to approach such questions for the final exam.

Functions of several variables – Some key formulas

- $z = z_0 + F_x(x_0, y_0)(x - x_0) + F_y(x_0, y_0)(y - y_0)$ (tangent plane to a smooth surface $z = F(x, y)$ at the point (x_0, y_0, z_0))
- A normal to smooth surface $F(x, y, z) = C$ (where C is a constant) at $\mathbf{x}_0 := (x_0, y_0, z_0)$: $\mathbf{n} := \nabla F(\mathbf{x}_0)$ (gradient of F evaluated at $\mathbf{x}_0$)
- Total differential approximation: $\Delta F \approx \frac{\partial F}{\partial x}\Delta x + \frac{\partial F}{\partial y}\Delta y$, where $F \equiv F(x, y)$ is a differentiable function
- $\Delta F \lesssim \left|\frac{\partial F}{\partial x}\right| |\Delta x| + \left|\frac{\partial F}{\partial y}\right| |\Delta y|$ (approximate upper bound for "error" in F)
- $\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\partial F}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial F}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$ (chain rule when $F \equiv F(x, y)$, $x \equiv x(t)$, $y \equiv y(t)$)
- $\dfrac{\Delta z}{z}$ (relative change/error in a quantity z; convert to a percentage to get "percentage change/error")

Integration techniques

General tips

- Be familiar with all the integration methods shown in the Calculus course pack notes (Chapter 2), e.g. calculating trigonometric integrals, reduction formulas, various substitutions, partial fractions.
- Know your trig. and hyperbolic identities.
- Section 2.3 (page 37 of physical copy, page 51 of PDF in electronic copy) of Calculus course pack notes contains useful table for which trig. or hyperbolic substitutions to use in which situation.
- For indefinite integrals, don't forget the "$+C$"!

Ordinary differential equations – General tips

- To solve $ay'' + by' + cy = f(x)$, you should **first** solve the homogeneous equation (0 on the RHS), and then find a particular solution y_P. The general solution is the particular solution plus the general solution to the homogeneous equation.
- Calculus course pack notes (page 85 of physical copy, page 99 of PDF in electronic copy) contain useful table to help guess the form of y_P. Also, many helpful worked examples here: `http://tutorial.math.lamar.edu/Classes/DE/UndeterminedCoefficients.aspx`
- If the RHS function $f(x)$ has a term in common with the homogeneous solution, multiply your usual guess for y_P by x. (This is why it is important to first know the homogeneous solution.)
- Multiply your guess for y_P by x as many times an necessary to no longer have terms in common with the homogeneous solution (but no more times than strictly necessary).
- For something like $ay'' + by' + cy = f_1(x) + f_2(x)$, the guess for y_P is a guess for the $f_1(x)$ part plus a guess for the $f_2(x)$ part (superposition principle) – e.g. see Problem 2 of the ordinary differential equations problems of this presentation.

Functions of several variables – Problem 1

[MATH1231 2012 S2 Q2 i)]

For a gas confined in a container, the ideal gas law states that the pressure P is related to the volume V and temperature T by

$$P = k\frac{T}{V},$$

where k is a positive constant.

a) Find $\frac{\partial P}{\partial V}$ and $\frac{\partial P}{\partial T}$.

b) The volume V is increased by 4% and the temperature T is decreased by 3%. Use the total differential approximation to estimate the percentage increase or decrease in the pressure P.

Functions of several variables – Problem 2

[MATH1241 2014 S2 Q4 i)]
Show that the equation of the tangent plane to the paraboloid S given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$$

at the point $P\,(x_0, y_0, z_0)$ is

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = \frac{z + z_0}{2c}.$$

Functions of several variables – Problem 3

[MATH1231 2015 S2 Q1 iii)]
Suppose that $z = a^2 + b^3 + c^4$, where

$$\begin{aligned} a &= u - v + w, \\ b &= u + v - w, \\ c &= uvw. \end{aligned}$$

Use the chain rule to find $\dfrac{\partial z}{\partial u}$ at the point $(u, v, w) = (1, 0, 1)$.

Integration techniques – Problem 1

[MATH1231 2014 S2 Q1 ii)]
Evaluate the integral

$$I = \int_0^1 \frac{x^2}{\sqrt{4-x^2}}\,\mathrm{d}x.$$

Integration techniques – Problem 2

[MATH1241 2013 S2 Q1 iv) b)]
Evaluate the integral

$$J = \int_0^{\frac{\pi}{2}} \sin^3\theta \cos^3\theta \, \mathrm{d}\theta.$$

Integration techniques – Problem 3

[MATH1241 2016 S2 Q4 i)]

Find a reduction formula for $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^n x \, \mathrm{d}x$ and use it to show that

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^4 x \, \mathrm{d}x = \frac{\pi}{4} - \frac{2}{3}.$$

(Note that $1 + \cot^2 x = \csc^2 x$ and $\frac{\mathrm{d}}{\mathrm{d}x} \cot x = -\csc^2 x$.)

Ordinary differential equations – Problem 1

[MATH1241 2015 S2 Q4 ii)]
The charge, $Q(t)$, in a certain circuit satisfies the differential equation

$$Q'' + Q' - 6Q = 0, \quad \text{with} \quad Q(0) = 3.$$

For which values, if any, of $Q'(0)$ will the charge tend to 0 as t tends to infinity?

Ordinary differential equations – Problem 2

Find the general solution to

$$y'' + 4y' + 4y = 6e^{-2x} + 3x + 1.$$

Ordinary differential equations – Problem 3

[based on MATH1231 2014 S2 Q2 i)]
Solve the initial value problem (defined for $x > 0$)

$$y' + \left(2 + \frac{1}{x}\right) y = \frac{2}{x}, \quad y(1) = 0.$$

Ordinary differential equations – Problem 4

[MATH1241 2014 S2 Q4 iv)]
Suppose that y satisfies the initial value problem

$$y' + y^2 = \cos x, \quad \text{with} \quad y(0) = 0.$$

Using implicit differentiation, or otherwise, find the first two non-zero terms of the Maclaurin series of y. (You may assume without proof that $y(x)$ has a Maclaurin series.)

Taylor series – Problem 1

[MATH1231 2013 S2 Q4 ii)]
The Maclaurin series for $\sin x$ is

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}, \quad x \in \mathbb{R}.$$

Do not prove this result.

a) Write down the Maclaurin series for $\sin\left(x^2\right)$.

b) Find the Maclaurin series for $\cos x$.

c) Hence derive the identity

$$\frac{1}{\sqrt{2}} = \sum_{k=0}^{\infty} (-1)^k \frac{\pi^{2k}}{2^{4k}(2k)!}.$$

Problem 1 - (a)

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}, \quad x \in \mathbb{R}$$

a) Replace x with x^2 in the given Maclaurin Series

$$\sin(x^2) = \sum_{k=0}^{\infty} (-1)^k \frac{(x^2)^{2k+1}}{(2k+1)!}$$

$$= \sum_{h=0}^{\infty} (-1)^k \frac{x^{4k+2}}{(2k+1)!} \quad x \in \mathbb{R}$$

Problem 1 - (b)

b) Differentiate both sides.

$$\frac{d}{dx}(\sin x) = \frac{d}{dx}\left(\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}\right) \quad x \in \mathbb{R}$$

$$\Longrightarrow \cos x = \sum_{k=0}^{\infty} \frac{d}{dx}\left((-1)^k \frac{x^{2k+1}}{(2k+1)!}\right)$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{(2k+1)x^{2k}}{(2k+1)!}$$

$$\Longrightarrow \cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

Problem 1 - (c)

c) Sub $x = \frac{\pi}{4}$

$$\cos\left(\frac{\pi}{4}\right) = \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{\pi}{4}\right)^{2k}}{(2k)!}$$

$$\Rightarrow \frac{1}{\sqrt{2}} = \sum_{k=0}^{\infty} (-1)^k \frac{\pi^{2k}}{4^{2k}(2k)!}$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{\pi^{2k}}{2^{4k}(2k)!}$$

Taylor series – Problem 2

Determine whether each of the following series converges or diverges, stating any tests you use.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos 2n}{\sqrt{n^3+1}}$ [MATH1231 2012 S2 Q4 ii)]

b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^4}{n!}$ [MATH1231 2012 S2 Q4 ii)]

c) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n(\ln n)^2}$ [MATH1231 2016 S2 Q2 ii) b)]

Problem 2 - (a)

a) For all $n \in \mathbb{Z}^+$

$$0 \le \left| \frac{\cos 2n}{\sqrt{n^3+1}} \right| = \frac{|\cos 2n|}{\sqrt{n^3+1}}$$

$$\le \frac{1}{\sqrt{n^3+1}}$$

$$\le \frac{1}{n^{\frac{3}{2}}}$$

$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ converges by the p-test.

Hence, by comparison test,

$\sum_{n=1}^{\infty} \left| \frac{\cos 2n}{\sqrt{n^3+1}} \right|$ converges, which implies

$\sum_{n=1}^{\infty} \frac{\cos 2n}{\sqrt{n^3+1}}$ converges. (abs. convergence implies convergence)

Problem 2 - (b)

b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^4}{n!}$

Ratio Test:

$$\lim_{n\to\infty} \left| \frac{\frac{(n+1)^4}{(n+1)!}}{\frac{n^4}{n!}} \right| = \lim_{n\to\infty} \left(\frac{(n+1)^4}{n^4} \cdot \frac{n!}{(n+1)!} \right)$$

$$= \lim_{n\to\infty} \frac{(n+1)^3}{n^4}$$

$$= 0 < 1.$$

Hence, $\displaystyle\sum_{n=1}^{\infty} \frac{n^4}{n!}$ converges by the ratio test.

Problem 2 - (c)

c) let $f(x) = \dfrac{1}{x(\ln x)^2}$

$f(x)$ continuous & positive for all $x \geq 3$.

$f(x)$ is decreasing for $x \geq 3$.
(notice or via derivative)

Hence we can apply integral test.

$$\int_3^\infty f(x)\,dx = \int_3^\infty \frac{1}{x(\ln x)^2}\,dx$$

$$= \int_{\ln 3}^\infty \frac{1}{u^2}\,du$$

let $u = \ln x$
$du = \dfrac{dx}{x}$

$$= \left[\frac{-1}{u}\right]_{\ln 3}^\infty$$

$$= \frac{1}{\ln 3}$$

Since integral converges, integral test

Taylor series – Problem 3

[MATH1241 2015 S2 Q4 vi)]
[The root test] Let $\{a_n\}$ be a sequence of positive terms such that for some constant $r < 1$, we have $\sqrt[n]{a_n} \to r$ as $n \to \infty$.

a) Explain why this implies that there is a constant $R < 1$ and an integer N such that $a_n < R^n$ for all $n > N$.

b) Hence or otherwise prove that $\sum_{n=1}^{\infty} a_n$ converges.

Problem 3 - (a)

$\{a_n\}$ $r<1$ $\sqrt[n]{a_n} \to r$ as $n \to \infty$

a) Def. of limit:

every $\varepsilon > 0$, exists some integer N such that

$$\left|\sqrt[n]{a_n} - r\right| < \varepsilon$$

for all $n > N$.

Implies for every $\varepsilon > 0$, exists some integer N such that

$$\left|\sqrt[n]{a_n} - r\right| < \varepsilon$$

for all $n > N$.

Problem 3 - (a) continued

Taking ε to be some fixed pos. num.

such that $r+\varepsilon < 1$ i.e. $\varepsilon < 1-r$

e.g. $\varepsilon = \frac{1-r}{2}$

Exists some integer N such that

$$\sqrt[n]{a_n} < r + \varepsilon < 1. \text{ for all } n > N$$

Defining R to be (fixed) number $r+\varepsilon$ so that R is a constant less than 1, we have then there exists some integer N such that

$$\sqrt[n]{a_n} < R \quad \text{i.e. } a_n < R^n$$

for all $n > N$ as required.

Problem 3 - (b)

b) Sufficient to show $\sum_{n=N+1}^{\infty} a_n$ converges.

We have

$$0 < a_n < R^n \text{ for all } n > N$$

Since $R<1$ (& $R>0$) for all $n>N$,

$R > \sqrt[n]{a_n} > 0$.

$\sum_{n=N+1}^{\infty} R^n$ converges (GM , common ratio <1)

$\therefore$ By Comparison Test,

$\sum_{n=N+1}^{\infty} a_n$ & $\sum_{n=1}^{\infty} a_n$ converges.

Averages, arc length, speed and surface area – Problem 1

[MATH1231 2013 S2 Q2 ii)]
The curve C is given parametrically by $x = t^3, y = 2t^2$. Find the arc length of the curve C between $t = 0$ and $t = 1$.

Problem 1

$$x = t^3 \qquad y = 2t^2$$

$$\ell = \int_0^1 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

$$= \int_0^1 \sqrt{(3t^2)^2 + (4t)^2}\, dt$$

$$= \int_0^1 \sqrt{9t^4 + 16t^2}\, dt$$

$$= \int_0^1 t\sqrt{9t^2 + 16}\, dt$$

$$= \int_{16}^{25} \frac{1}{18}\sqrt{u}\, du \qquad \text{Let } u = 9t^2 + 16, \quad du = 18t\, dt$$

$$= \frac{1}{18}\left[\frac{2}{3}u\sqrt{u}\right]_{16}^{25}$$

$$= \frac{1}{27}(25 \times 5 - 16 \times 4)$$

$$= \frac{125 - 64}{27} = \frac{61}{27}$$

Averages, arc length, speed and surface area – Problem 2

Find the average value of the function

$$f(x) = \sin^2 x$$

on the interval $[0, \pi]$.

Problem 2

$$f(x) = \sin^2 x$$

$$\bar{f} = \frac{1}{\pi - 0} \int_0^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi} \int_0^{\pi} \sin^2 x \, dx$$

$$= \frac{1}{\pi} \int_0^{\pi} \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) dx$$

$$= \frac{1}{\pi} \left(\frac{\pi}{2} - 0\right)$$

$$= \frac{1}{2}.$$

Averages, arc length, speed and surface area – Problem 3

Determine the surface area of the solid obtained by rotating the curve

$$y = \sqrt[3]{x}, \quad 1 \leq y \leq 2$$

about the y-axis.

Problem 3

$$y = \sqrt[3]{x}, \quad 1 \le y \le 2$$

$$SA = \int_1^2 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \qquad x = y^3$$

$$= \int_1^2 2\pi y^3 \sqrt{1 + (3y^2)^2}\, dy$$

$$= 2\pi \int_1^2 y^3 \sqrt{1 + 9y^4}\, dy$$

$$= 2\pi \times \frac{1}{36} \int_{10}^{145} \sqrt{u}\, du \qquad \text{Let } u = 1 + 9y^4, \quad du = 36y^3\, dy$$

$$= \frac{\pi}{27}\left(145^{\frac{3}{2}} - 10^{\frac{3}{2}}\right).$$

Taylor series – General tips

- Know all your series convergence tests and when they apply (e.g., k-th term test can only tell you that a series does not converge; integral test assumes $f(x)$ is positive, integrable, and **decreasing**; for the comparison test, you must make sure the terms of the series are non-negative).
- Ratio test lets you work out radius of convergence of a power series (e.g. see Example 4.7.7 of Calculus course pack notes (page 147 of physical copy, page 161 of electronic PDF copy))
- Given a series $\sum_{k\in\mathbb{N}} a_k$ that you are asked to test for convergence, first check whether $\lim_{k\to\infty} a_k = 0$, since this is easy to do. If $\lim_{k\to\infty} a_k \neq 0$, then the series is automatically divergent (k-th term test). Otherwise, we cannot immediately conclude anything and more work needs to be done.
- Comparing a_k to something like $\frac{1}{k^p}$ by using the limit form of the comparison test (together with the p-test) is a common approach.
- For alternating series, a common test to use is the alternating series test. Make sure to check and state that the hypotheses of this hold if using it.
- Remember, **if** $\sum_{k\in\mathbb{N}} |a_k|$ converges, then $\sum_{k\in\mathbb{N}} a_k$ converges (but not the other way round in general). This can help deal with series that involve a trigonometric term (like Problem 2(a) in the "Taylor series" problems section of these slides).

Averages, arc length, speed and surface area – Some key formulas

- $\overline{f} = \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x$ (average value of $f(x)$ on $[a,b]$)
- $\ell = \int_{t_0}^{t_1}\sqrt{(x'(t))^2 + (y'(t))^2}\,\mathrm{d}t$ (arc length of parametric curve $(x(t), y(t))$, for $t \in [t_0, t_1]$)
- $\ell = \int_a^b \sqrt{1 + (f'(x))^2}\,\mathrm{d}x \equiv \int_a^b \sqrt{1 + (y')^2}\,\mathrm{d}x$ (arc length of graph of $y = f(x)$ for $a \leq x \leq b$)
- $\ell = \int_{\theta_0}^{\theta_1}\sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta \equiv \int_{\theta_0}^{\theta_1}\sqrt{r^2 + (r')^2}\,\mathrm{d}\theta$ (arc length of polar curve $r = f(\theta)$, for $\theta_0 \leq \theta \leq \theta_1$)
- $v(t) = \sqrt{(x'(t))^2 + (y'(t))^2} \equiv \|\mathbf{x}'(t)\|$ (speed at time t of particle with position $\mathbf{x}(t) = (x(t), y(t))$)
- Know **all** the surface area formulas in the course pack! (See section 5.4 of Calculus course pack notes, starting page 181 of physical copy, page 195 of PDF in electronic copy)

Good luck!

ANY QUESTIONS?
GOOD LUCK FOR YOUR FINAL EXAMS!
MATHSOC WISHES YOU ALL THE BEST IN YOUR FUTURE STUDY.