UNSW Mathematics Society presents

MATH1031 Revision Seminar

Life Sciences

Mathematics For Life Sciences

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Matrix Algebra

Introduction to Matrices

What is a matrix?

Matrices are very useful little structures that allow us to express a number of equations in a very simple manner.

We can use them to turn tables or linear equations into simple rectangles or squares which we can manipulate to draw out some pretty cool information.

Creating Matrices

Making a matrix

Suppose we want to measure the ecological diversity of an ecosystem containing aardvarks, birds and crabs.
There are three distinct regions creatively called A, B and C.
On Thursday, we observe the following:

1. 3 aardvarks, 2 birds and 4 crabs in region A.
2. 9 aardvarks, 0 birds and 2 crabs in region B.
3. e aardvarks, $\frac{2}{3}$s of a bird, and half a crab in region C.

Creating Matrices

Plopping points into our matrix

To record our data, we can just get rid of the animal names (so long as we keep track of which column is for which animal), and put each region into its own row, then plop everything into a matrix! It's like magic:

$$\begin{pmatrix} 3 & 2 & 4 \\ 9 & 0 & 2 \\ e & \frac{2}{3} & \frac{1}{2} \end{pmatrix}$$

How do we describe matrix size?

Order

The **order** of a matrix is a descriptor of its size. The number of rows comes first, then the columns. For example, a matrix with 5 columns and 6 rows is a 6 x 5 matrix (and hence has 30 different number entries).

Matrix Addition

Only matrices that are of the same order can be added. Thus, you cannot add a 2 x 3 matrix to a 3 x 2 matrix.
When adding matrices, ANY matrices that are of the same order can be added to each other.

Adding Matrices

Matrix Elements

Since matrices are two dimensional objects, we can use something reminiscent of a coordinate system to identify specific elements.

We do this with the following notation: $\mathbf{A}_{mn}$ where $\mathbf{A}$ is our matrix, m is our mth row from the top, and n is our nth column from the left.

For example, in our previous 3 x 3 matrix, $\mathbf{A}_{13}$ is the value 4.

When adding matrices, we always add corresponding elements, so in matrices $\mathbf{A}$ and $\mathbf{B}$ we add elements $\mathbf{A}_{ab}$ and $\mathbf{B}_{ab}$.

Adding Matrices

Addition

$$\begin{pmatrix} 3 & 2 & 4 \\ 9 & 0 & 2 \\ e & \frac{2}{3} & \frac{1}{2} \end{pmatrix} + \begin{pmatrix} 5 & 0 & 1 \\ 4 & \pi & 4 \\ 1 & \frac{1}{3} & \frac{3}{2} \end{pmatrix} = \begin{pmatrix} 8 & 2 & 5 \\ 13 & \pi & 6 \\ e+1 & 1 & 2 \end{pmatrix}$$

Subtraction is the exact same concept, but instead of adding, we subtract (duh).

Scalar Multiplication

Multiplying Tings

When we multiply a matrix by some scalar, what we're actually doing is multiplying each entry in that matrix by that scalar.

Product of a matrix and scalar

$$4\begin{pmatrix} 3 & 2 & 8 & -3 \\ -8 & 1 & 3 & 9 \\ 4 & 0 & -3 & 5 \end{pmatrix} = \begin{pmatrix} 12 & 8 & 32 & -12 \\ -32 & 4 & 12 & 36 \\ 16 & 0 & -12 & 20 \end{pmatrix}$$

Order

The order of a matrix does not matter when multiplying by a scalar. A matrix of ANY order can have its entries multiplied by a scalar.

Matrix Multiplication

Multiplying two matrices

Two matrices $\mathbf{A}_{nm}$ and $\mathbf{B}_{ab}$ only have a defined product of $\mathbf{AB}$ if $m = a$. That is, the number of columns in the left matrix are equal to the number of rows in the right matrix.

To find the product $\mathbf{AB}$ (if it exists), we multiply the row entries of $\mathbf{A}$ with the column entries of $\mathbf{B}$, then sum these as one result.

Commutativity

Unlike regular multiplication with numbers, matrix multiplication is not commutative. That is, in general, $\mathbf{AB} \neq \mathbf{BA}$.

An Invalid Example

An invalid example

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix}$$

Note columns($\mathbf{A}$) $= 3$, but rows($\mathbf{B}$) $= 2$, so $\mathbf{AB}$ is not defined.

A valid example

An invalid example

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & -4 \\ 5 & 6 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & -2 & 3 & 4 \\ -5 & 6 & -7 & 8 \end{pmatrix}$$

Columns($\mathbf{A}$) $= 2$, and rows($\mathbf{B}$) $= 2$, so $\mathbf{AB}$ is just:

$$\begin{pmatrix} 1(1)+2(-5) & 1(-2)+2(6) & 1(3)+2(-7) & 1(4)+2(8) \\ 3(1)-4(-5) & 3(-2)-4(6) & 3(3)-4(-7) & 3(4)-4(8) \\ 5(1)+6(-5) & 5(-2)+6(6) & 5(3)+6(-7) & 5(4)+6(8) \end{pmatrix}$$

$$= \begin{pmatrix} -9 & 10 & -11 & 20 \\ 23 & -30 & 37 & -20 \\ -25 & 26 & -27 & 68 \end{pmatrix}$$

Special Matrices

The Zero Matrix

The Zero Matrix

A **zero** matrix is just a matrix where all entries are equal to zero.
They are analogous to the number "0".
Any matrix multiplied by this matrix will produce another zero matrix.

Making the Zero Matrix

Consider

$$\begin{pmatrix} 3 & 3 & 3 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Neither of the matrices being multiplied is a zero matrix, but their product is!

The Identity Matrix

What is it?

The identity matrix is always a square matrix (rows = columns) which has a leading diagonal of 1s. It has the special property $IA = AI = A$, where A is an appropriately sized matrix.

Some Identity Matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Careful!

You can't just multiply any matrix with an identity matrix, it still needs to be of an appropriate size!

The Inverse Matrix

?xirtaM esrevnI na s'tahW

When a matrix is multiplied by its inverse, the result is the identity matrix. That is, $AA^{-1} = A^{-1}A = I$.

Inverses In Action

Consider the product

$$AB = \begin{pmatrix} 1 & 1 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 5 & -1 \\ -4 & 1 \end{pmatrix} = \begin{pmatrix} 5 & -1 \\ -4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Since the product is the identity matrix, A and B are inverses of each other (so A can be written as B^{-1}) and vice versa.

Do all matrices have an inverse?

Only square matrices can have an inverse. However, not all square matrices have inverses.

Finding Inverses

Calculating the Inverse

This is usually done through Gaussian Elimination or black magic. Neither of those are within the scope of this course, and you can only be asked to find the inverse of 2 x 2 matrices, which is done using a simple formula.

Inverse Formula

Given a 2 x 2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, its inverse can be calculated as

$\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Clearly, if $ad - bc = 0$, the scalar will be undefined and hence there will be no inverse.

The Transpose

Definition

Every matrix A has a transpose A^T whose rows are the same as the columns of A. That is, every entry A_{mn} turns into A^T_{nm}. Bit like an acrostic poem.

A basic example

$$A = \begin{pmatrix} 1 & 8 & 9 & 3 \\ 3 & 2 & 7 & 4 \end{pmatrix}, A^T = \begin{pmatrix} 1 & 3 \\ 8 & 2 \\ 9 & 7 \\ 3 & 4 \end{pmatrix}.$$

Functions

Fantastic Functions

Fundamentals of Functions

Usually, functions have an independent variable and a dependent variable, typically (but not always) denoted as x and y, respectively. Functions ALWAYS have a unique y value for each x in their domain.

For Example

Functions can look like:

$$y = f(x)$$
$$f(x) = 5x + 10x^3$$
$$y = 5x + 10x^3$$

Restricting our Values

Interval Notation

If we're considering a specific domain of numbers (most commonly the reals, $\mathbb{R}$), then we may want to restrict our values to a specific interval. There are several ways to do this:

Interval Notation

$$-5 \leq x \leq 5 \text{ is the same as } x \in [-5, 5]$$
$$2 \leq t < 4 \text{ is the same as } t \in [2, 4)$$
$$8 < x < 12 \text{ is the same as } x \in (8, 12)$$
$$t \leq 0 \text{ is the same as } t \in (-\infty, 0]$$

Key Terms

REMEMBER!

Domain is the set of values in which we define our function. This can include natural restrictions of certain functions such as the square root or reciprocals. For example, the natural domain of $\frac{1}{\sqrt{x-68.9}}$ would be $x \in (68.9, \infty)$ since the denominator must be both positive and non-zero. Basically, the domain determines what we can input into our functions.

Range is the set of values in which our functions spits out values. In other words, it is the set of outputs of our functions.

Scaling the y-value

Quick Maffs

We can scale some function vertically

$$y = cf(x)$$

using some constant c. This can do several non-trivial things depending on c's value:

1. If $c > 1$ the graph is stretched vertically.
2. If $0 < c < 1$ the graph is shrinked vertically.
3. If c happens to be negative, both of the above still apply, but the graph is also flipped vertically.

Scaling the x-value

Quicker Maffs

We can scale some function horizontally

$$y = f(cx)$$

using some constant c. This can do several non-trivial things depending on c's value:

1. If $c > 1$ the graph is shrinked horizontally.
2. If $0 < c < 1$ the graph is stretched horizontally.
3. If c happens to be negative, both of the above still apply, but the graph is also flipped horizontally.

Shifting the x- and y-values

Quickest Maffs

We can shift some function vertically

$$y = f(x) + c$$

using some constant c. This will shift the graph up by c units. Alternatively, we can shift some function horizontally

$$y = f(x - c)$$

using some constant c. This will shift the graph right by c units.

Piecewise or Piecemeal Functions

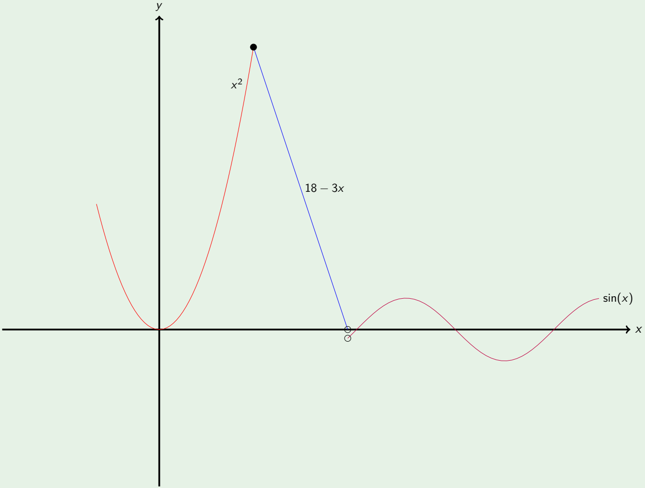
What is a PIECEMEAL/PIECEWISE fUnCtIoN?

These functions have different definitions in different intervals. A real life example would be tax brackets. The absolute value function can also be defined as a piecewise function (though this isn't always necessary).
A typical piecewise function may be defined in the following way:

$$f(x) = \begin{cases} x^2 & \text{for } x \leq 3 \\ 18 - 3x & \text{for } 3 < x < 6 \\ \sin(x) & \text{for } x \geq 6 \end{cases}$$

Piecewise or Piecemeal Functions

What it looks like



Periodic Functions

DefinitionDefinitionDefinition

Periodic functions repeat values at certain intervals. Common examples include the sine, cosine and tangent functions (the basic trigonometric functions).

More formally, we can define periodic functions as

$$f(x) = f(x+t) \text{ for all } x \text{ in the domain.}$$

where t is some constant.

If we were to shift the graph left or right by t, it would repeat itself (assuming an unbounded domain).

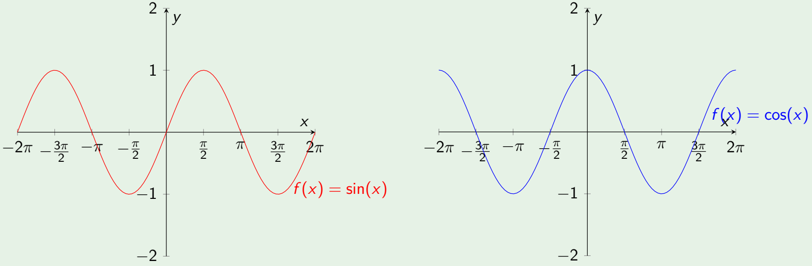
The smallest t that allows this to happen is known as the **period** of the function.

Sine, cosine and periodicity

sin(x) and cos(x)

$f(x) = \sin(x)$ and $g(x) = \cos(x)$ are both periodic functions with a period of 2π. Observe the pictures below, where we can see that these functions repeat every 2π radians:

Periodicity



Radians

Converting between radians and degrees

Remember, 2π radians is the equivalent of $360°$.
If we have an angle that is r radians, we can convert it to degrees using the following formula:

$$\theta = \frac{180 \times r}{\pi}$$

And vice versa:

$$r = \frac{\theta \times \pi}{180}$$

Where θ is the same angle but in degrees.

Exact values

Exact Values at Special Angles

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{(3)}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined

Compound Angles

The Formulae

Given some angle $A + B$, we can expand our trigonometric functions according to some very special rules known as the compound angle formulae.

Le Formulae

$$\sin(A \pm B) = \sin(A)\cos(B) \pm \sin(B)\cos(A)$$

$$\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$$

$$\tan(A \pm B) = \frac{1 \pm \tan(A)\tan(B)}{\tan(A) \mp \tan(B)}$$

Auxiliary Angle Formulae

The Formulae

We can rewrite the sum of two trig. functions (e.g. $3\sin(x) + 5\cos(x)$) as one trig. function.

Le Formulae

$$a\sin(x) \pm b\cos(x) = R\sin(x \pm \alpha)$$

$$a\cos(x) \pm b\sin(x) = R\cos(x \mp \alpha)$$

Where $R = \sqrt{a^2 + b^2}$ and $\alpha = \tan^{-1}(\frac{b}{a})$.

Logarithms

What are logarithms?

When we have two numbers, a and b, we might ask the question "how can I raise a to some power to get b?" And our answer comes in the form of the logarithm functions. Formally, our answer is written as $\log_a(b) = x$, where $a^x = b$.

Special Properties of the Logarithm

Some weird log properties

Some properties that this class of functions have are:

1. $\log(ab) = \log(a) + \log(b)$
2. $\log(a^b) = b\log(a)$
3. $\log(\frac{1}{a}) = -\log(a)$
4. $\log(\frac{a}{b}) = \log(a) - \log(b)$

The Natural Logarithm

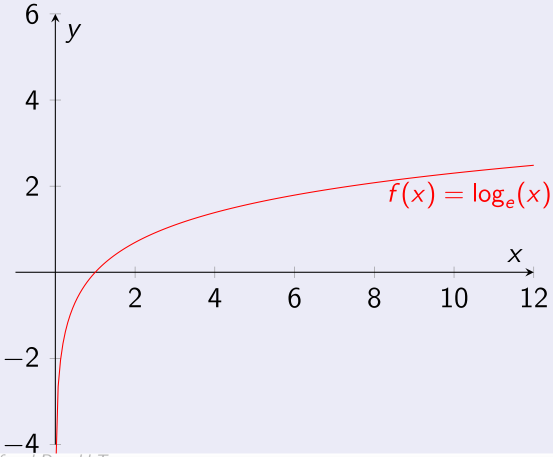
What is the natural logarithm?

This ($\ln(x)$ or $\log_e(x)$) is a special logarithm with base e. It has the exact same property as other logarithms, but it'll be popping up much, much more frequently.
Formally, this function is the area of the curve $\frac{1}{t}$ from 1 to x.

Special Properties of the Logarithm

Gawking at curves

$\log_e(x)$ has some special properties which we can see if we graph it.



Special Properties of the Natural Logarithm

Gawking at curves

Now that we've admired its curves, we can hopefully see that:

1. $\log_e(e) = 1$
2. Domain: $x \in (0, \infty)$
3. Range: $f(x) \in (-\infty, \infty)$
4. $\log_e(x) \to \infty$ as $x \to \infty$ (but really, really slowly!)
5. $\log_e(x) \to -\infty$ as $x \to 0^+$ (really, really fast!)

With the exception of the first property, all other properties apply to logarithms of bases other than e since all logarithms are just multiples of each other.

Special Properties of the Logarithm

Dat Asymptote

log(0) is not defined since no number can be raised to any power to produce 0. While we can get really close to it, $x = 0$ is ultimately an asymptote of the log functions.

The Exponential Functions

What is it?

Exponents are the power that we raise something to. They work somewhat inversely to logarithms. For example, in e^x, x is our exponent.

The Exponential Functions

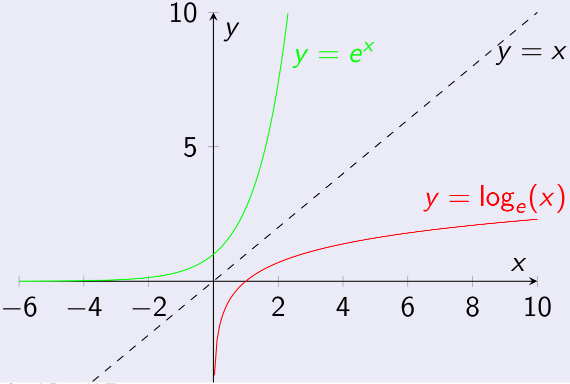
Properties of exponents

1. $x^{a/b} =^b \sqrt{x^a}$
2. $x^{-a} = \frac{1}{x^a}$
3. $x^a x^b = x^{a+b}$
4. $\frac{x^a}{x^b} = x^{a-b}$
5. $(x^p)^q = x^{pq}$
6. $(xy)^p = x^p y^p$

Exp and log

Inverses

Log functions and exponential functions are inverses of each other (provided they have the same base). Hence, $log_a(x)$ is the inverse of a^x. What this means is that $log_a(a^x) = a^{log_a(x)} = x$.



Exp, log, and inequalities

Inequalities

Recalling the graph from the previous slide, we can see that the logarithm and exponential functions always increase as x increases. Because of this nice property, we can log both sides or take both sides to the power of some number, and preserve the inequality for almost all values of x.

An Example

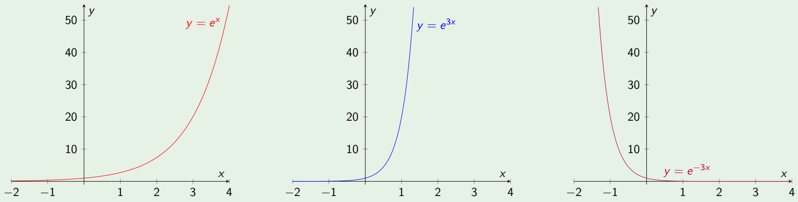
$$\begin{aligned} 4 < x < 5 &\implies e^4 < e^x < e^5 \text{ and} \\ &\implies \ln(4) < \ln(x) < \ln(5) \end{aligned}$$

Shifting and stretching exp functions

Exp

Shifting and stretching the exponential function is much like shifting any other kind of function.

Graphing



Limits

Limit at a point

Definition

A limit, loosely speaking, is the value which a function approaches at a specific point or otherwise. It does not have to be the actual value of the function at that point.

An Example Limit

Consider the limit of a function $\frac{x-3}{x^3-27}$ as x approaches 3. We can denote this as

$$\lim_{x \to 3} \frac{x-3}{x^3-27}.$$

If we substitute in $x = 3$, we get $\frac{0}{0}$ which is epic and not what we want.

Limit at a point continued

Factorising

To solve our issue here, we can just factorise, simplify then substitute as follows:

$$\begin{aligned}\lim_{x\to 3}\frac{x-3}{x^3-27} &= \lim_{x\to 3}\frac{x-3}{(x-3)(x^2+3x+9)}\\ &= \lim_{x\to 3}\frac{1}{(x^2+3x+9)}\\ &= \frac{1}{27}.\end{aligned}$$

And we've gotten our answer.

To Infinity and Beyond

Limits to Infinity

Several behaviours could occur here:

1. $f(x) \to \pm\infty$ as $x \to \infty$
2. $f(x) \to \pm\infty$ as $x \to -\infty$
3. $f(x)$ could have no limit (for example, periodic functions such as sine and cosine).
4. $f(x)$ approaches a concrete value (for example, in the case of asymptotes). Note that this does not mean the function will eventually touch its asymptote, just that it will get very close to it.

Substituting Infinity

Wait what

Substituting infinity is not possible, since it is not a number. For rational functions (polynomial divided by polynomial), we divide the top and bottom by the highest power of x, then argue that anything with an x in the bottom becomes 0 as x becomes bigger.

An Example

$$\begin{aligned}\lim_{x\to\infty}\frac{x^3-4x}{3x^3+5x^2+2x+3} &= \lim_{x\to\infty}\frac{1-\frac{4}{x^2}}{3+\frac{5}{x}+\frac{2}{x^2}+\frac{3}{x^3}} \\ &= \frac{1}{3}\end{aligned}$$

One-Sided Limits

Notation

1. Taking the limit as we approach a point from the left (negative direction): $\lim_{x \to a^-} f(x)$
2. Taking the limit as a approach a point from the right (positive direction): $\lim_{x \to a^+} f(x)$

These limits are crucial to considering the overall limit.

One-Sided Limits

Notation

Our overall limit only exists if and only if:

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$$

for some real number L. That is,

$$\lim_{x \to a} f(x) = L$$

as well. If our condition here is not satisfied, L does not exist.

Continuity

Continuity at a Point

Continuity is a property of curves. A curve is continuous at a point a if we can draw the curve without lifting our pen once we reach $x = a$. In other words,

$$\lim_{x \to a} f(x) = f(a).$$

Continuous Functions

Elementary Functions

Continuous everywhere

1. Polynomials
2. $\sin(x)$
3. $\cos(x)$
4. e^x

Continuous only in their domain

1. Rational functions
2. $\tan(x)$
3. $\ln(x)$

Limit Properties

Limit Properties

1. $\lim\limits_{x\to a}(f(x)+g(x)) = \lim\limits_{x\to a} f(x) + \lim\limits_{x\to a} g(x)$
2. $\lim\limits_{x\to a}(cf(x)) = c\lim\limits_{x\to a}(f(x))$
3. $\lim\limits_{x\to a}(f(x)g(x)) = \lim\limits_{x\to a} f(x) \lim\limits_{x\to a} g(x)$
4. $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$ PROVIDED $\lim\limits_{x\to a} g(x)$ is not equal to 0
5. $\lim\limits_{x\to a}(f(x))^n = (\lim\limits_{x\to a} f(x))^n$

The same properties hold for $\lim\limits_{x\to a^\pm}$ and $\lim\limits_{x\to\pm\infty}$.

Calculus

The Derivative

Definition

The derivative the rate of change of a function. In terms of curves, we can think of it as the gradient of the tangent at a specific point.

Calculating the Derivative

Product Rule

Used when two (or more) functions are multiplied by each other, like $x^2 \sin(x)$.

$$y = f(x)g(x)$$
$$y' = f'(x)g(x) + f(x)g'(x)$$

"Left D-Right, Right D-Left" - Useful "rhyme".

Calculating the Derivative

Quotient Rule

Used when one function is divided by another, like $\frac{x}{e^x}$.

$$y = \frac{f(x)}{g(x)}$$

$$y' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

Calculating the Derivative

Chain Rule

Used when one function is composed inside another, like $ln(x^2)$.

$$y = f(g(x))$$
$$y' = g'(x)f(g(x))$$

Using the First Derivative

Up and down

1. If $f'(a) > 0$, the graph is *increasing* when $x = a$.
2. If $f'(a) < 0$, the graph is *decreasing* when $x = a$.
3. If $f'(a) = 0$, the graph is *stationary* when $x = a$. $x = a$ is referred to as a *turning point.*

Using the Second Derivative

Up and down

The second derivative is just the derivative of the first derivative. That is,

$$y'' = \frac{d}{dx}(y').$$

1. If $f''(a) > 0$, the graph is *concave up* when $x = a$.
2. If $f''(a) < 0$, the graph is *concave down* when $x = a$.
3. If $f''(a) = 0$, there is no concavity when $x = a$.

Derivatives

A quick refresher of standard derivatives

1. $\frac{d}{dx}(e^{f(x)}) = f'(x)e^{f(x)}$
2. $\frac{d}{dx}(\ln(f(x)) = \frac{f'(x)}{f(x)}$
3. $\frac{d}{dx}(\sin(f(x)) = f'(x)\sin(f(x))$
4. $\frac{d}{dx}(\cos(f(x)) = f'(x)\cos(f(x))$
5. $\frac{d}{dx}(\sin(f(x)) = f'(x)\sin(f(x))$
6. $\frac{d}{dx}(\tan(f(x)) = f'(x)\sec^2(f(x))$
7. $\frac{d}{dx}(\sec(f(x)) = f'(x)\sec(f(x))\tan(f(x))$

Sketching

Rules for Sketching

Curve sketching can be a very systematic process. You can't memorise what every curve looks like since there's an infinite number, so keep this process in mind:

1. Find the domain of the function and any discontinuities.
2. Determine symmetry (is it an odd or an even function?)
3. Find the x- and y-intercept(s) (let $y/x = 0$)
4. Find vertical asymptotes (typically found at discontinuities)

Sketching

Rules for Sketching continued

1. Find horizontal asymptotes (usually found as $x \to \pm\infty$)
2. First Derivative: Find any turning points and their nature by finding the zeroes of $f'(x)$, then using a table or the second derivative to determine whether the turning point is concave up, down or otherwise.
3. Unique features: Remember to pay attention to periodicity or any other weird features.

The Maximum-Minimum Theorem

If a function $f(x)$ is **continuous** on a closed interval, $[a, b]$ where both $f(a)$ and $f(b)$ are both defined, then the function has both an absolute maximum and an absolute minimum in the interval.

Careful!

Absolute max/min does not mean both values are positive. It's just fancy speak for "smallest value" and "largest value". Another important thing to remember is that your interval **must** be continuous.

Maxima and Minima on Closed Intervals

Fantastic Maxima and Minima and Where to Find Them

Maxima and minima can be found in:

1. The endpoints of the interval.
2. Critical points (such as turning points or points where the derivative is undefined).

So we want to check the value of $f(x)$ at each respective point to find the maximum (the largest value we find) and the minimum (the smallest value we find).

Maxima and Minima on Open Intervals

Maxima and Minima

On open intervals, finding maxima and minima remains much the same, but we consider the limiting behaviour of the function as it reaches its "endpoints" instead of evaluating its value at the endpoints.

Modelling

Modelling with Exponentials

Modelling with Exponentials

In nature, lots of phenomena such as population growth or radioactive decay have rates of change proportional to their size. That is

$$\frac{dy}{dt} = ky$$

For some constant k. The solutions to this differential equation are all in the form of:

$$y = Ae^{kt}$$

For some constant A. We see our best friend e again!

Modelling with Exponentials

2018 S1 Q2ii

Not all populations grow exponentially. Consider the population of trout in a pond, with initially 100 trout and twice that number after one year. The number of trout will be limited by the availability of resources such as food supply and spawning habitat. A small population P of trout might grow exponentially if the pond is large and the food abundant, but the growth rate will decline as the population increases and the availability of resources declines. We can use the logistic equation to model population growth in a resource limited environment:

$$\frac{dP}{dt} = k(1 - \frac{P}{N})P$$

Modelling with Exponentials

2018 S1 Q2ii

a) Let's first consider an exponential growth model:

$$\frac{dP}{dt} = kP.$$

II) Using the exponential model, estimate the population after 10 years and discuss the behaviour of P as t tends to infinity. We know that our solution to this equation is in the form:

$$P(t) = Ae^{kt}$$

Since the initial population is 100, we also know that:

$$P(0) = Ae^{0}$$
$$100 = A$$

Modelling with Exponentials

2018 S1 Q2ii

And, since the population after a year is double 100:

$$\begin{aligned} P(1) &= 100e^{k} \\ 200 &= 100e^{k} \\ \ln(2) &= k. \end{aligned}$$

So

$$P(t) = 100e^{\ln(2)t}$$

and

$$P(t) = 100 \cdot 2^{t}.$$

Modelling with Exponentials

2018 S1 Q2ii

Now, as $t \to \infty$, we can clearly see that $P(t) \to \infty$ as well. Further,

$$\begin{aligned} P(10) &= 100 \cdot 2^{10} \\ &= 102400 \end{aligned}$$

Modelling with the Trigonometric Functions

Modelling with Trigonometry

Some phenomena are periodic, and so we call on our best friends in the trig department to help us out instead of the exponential function. For example, when we need to model tidal patterns or celestial orbits.

Modelling with the Trigonometric Functions

2019 T1 Q2i

During hyperventilation (rapid, shallow breathing) the volume of air in a person's lungs at a given time t may be approximated by

$$V = 0.30 + 0.18\cos(9.6t)$$

where V is the volume in litres and t the time in seconds, from the onset of the attack.

Modelling with the Trigonometric Functions

2019 T1 Q2i

b) Find the maximum and minimum volume of air in the lungs during hyperventilation. We can argue that since

$$-1 \leq \cos(9.6t) \leq 1$$

that

$$-0.18 \leq 0.18\cos(9.6t) \leq 0.18.$$

So the max is clearly 0.48L and the min is clearly 0.12L.

Modelling with the Trigonometric Functions

2019 T1 Q2i

d) When is the volume changing at a rate of 1.728 litres per second for the first time?

First, we find the derivative of our function with respect to t:

$$\frac{dV}{dt} = -1.728\sin(9.6t)$$

Then let it equal 1.728 and solve:

$$\begin{aligned} 1.728 &= -1.728\sin(9.6t) \\ \sin(9.6t) &= -1 \\ 9.6t &= \frac{3\pi}{2} \\ t &\approx 0.491 \text{ seconds (to 3 decimal places).} \end{aligned}$$

Lines of Best Fit Using Least Squares

Lines of Best Fit

1. Used to extrapolate data and make inferences (but this is risky).
2. Straight lines can be expressed in the form $y = mx + b$ where m is the gradient ($\frac{rise}{run} = \frac{y_1 - y_2}{x_1 - x_2}$) and b is the y-intercept.
3. Can estimate data within a known range - Interpolation.

Lines of Best Fit

Lines of Best Fit - Multiple Points

When there are more than two points we want to fit a single line to, we place the points in a matrix.

2019 T1 Q1i

The following table displays the percentage p of a glucose transporting protein on a cell surface as a function of time. The time, t, is measured in minutes, with t = 0 corresponding to the time when insulin is applied to the cell.

t	1	2	4	6
p	20	26	36	37

Let $Q = \ln(37.65 - p)$

Lines of Best Fit

2019 T1 Q1i

a) Using the experimental data, construct a new table of t versus Q.

t	1	2	4	6
Q	2.87	2.46	0.50	-0.43

Let $Q = \ln(37.65 - p)$

Lines of Best Fit

2019 T1 Q1i

b) Using your table of t versus Q from part a), find the least squares line of best fit $Q = mt + b$. Find m and b correct to 2 decimal places. We have the following equations by substituting each Q and t as appropriate:

$$\begin{aligned} 2.871 &= m + b \\ 2.455 &= 2m + b \\ 0.501 &= 4m + b \\ -0.431 &= 6m + b \end{aligned}$$

Lines of Best Fit

2019 T1 Q1i

Which we can plop into matrix form as below:

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 4 & 1 \\ 6 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 2.871 \\ 2.455 \\ 0.501 \\ -0.431 \end{pmatrix}$$

Left multiply by the transpose of the coefficient matrix:

$$\begin{pmatrix} 1 & 2 & 4 & 6 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 4 & 1 \\ 6 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 1 & 2 & 4 & 6 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2.871 \\ 2.455 \\ 0.501 \\ -0.431 \end{pmatrix}$$

Lines of Best Fit

2019 T1 Q1i

We get:

$$\begin{pmatrix} 57 & 13 \\ 13 & 4 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 7.199 \\ 5.396 \end{pmatrix}$$

Find the inverse, then left multiply by it:

$$\frac{1}{59} \begin{pmatrix} 4 & -13 \\ -13 & 57 \end{pmatrix} \begin{pmatrix} 57 & 13 \\ 13 & 4 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \frac{1}{59} \begin{pmatrix} 4 & -13 \\ -13 & 57 \end{pmatrix} \begin{pmatrix} 7.199 \\ 5.396 \end{pmatrix}$$

$$\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} -0.701 \\ 3.627 \end{pmatrix}$$

Hence, our line of best fit is approximately

$$Q = -0.7t + 3.63.$$

Lines of Best Fit

2019 T1 Q1i

c) Remembering that $Q = \ln(37.65 - p)$, use your result in (b) to show that the relation between p and t can be written as

$$p = A - Be^{-kt}$$

where A, B and k are constants. Identify the values of A, B and k from your line of best fit.

Lines of Best Fit

2019 T1 Q1i

From our expression of Q in b), we can see that

$$\begin{aligned} p &= 37.65 - e^{Q} \\ &= 37.65 - e^{-0.7t+3.63} \\ &= 37.65 - 3.63e^{0.7t} \end{aligned}$$

Hence $A = 37.65$, $B = 3.63$, $k = 0.7$ (these are approximate values since we did a lot of rounding).

Quadratic of Best Fit

Quadratics?

Lines are not always an accurate depiction of natural models. For example, a projectile travels in a parabola instead of a line.

When modelling with quadratics, we substitute our points into

$$y = ax^2 + bx + c$$

instead of a line. We then follow the steps as in the previous example to obtain our quadratic of best fit.

Exponential of Best Fit

Exponential Relationships

Sometimes, we may end up with a line of best fit that fits the equation

$$y = ae^{kx}.$$

We can take the log of both sides and rearrange to get a "linear" equation in the form of

$$Y = kx + b$$

and then we solve for k and b as normal, remembering to write our equation in terms of y instead of Y at the end.

3D Algebra

Three Dimensional Space

Planes

In a 3-dimensional, with axes x_1, x_2, x_3 (or x, y and z), we can express structures like planes as

$$ax_1 + bx_2 + cx_3 = d.$$

where $a, b,$ and c are some real numbers (including 0!). To find our axis intercepts here, we just let the other two variables equal zero. For example, our x_1-intercept here would be

$$x_1 = \frac{d}{a}, x_2 = 0, x_3 = 0.$$

The other intercepts are found through a similar process.

Distances

Distance Between Two Points

As with 2D space, we apply Pythagoras' theorem (although in a slightly different format) to find the distances between two points

$$A = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

That is, $\text{dist}_{AB} = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2}$

Midpoints

Midpoints

Given our two points A and B from before, we can find their midpoint in the same way we find the midpoint in two dimensional spaces.

$$mid(A,B) = \begin{pmatrix} \frac{a_1+b_1}{2} \\ \frac{a_2+b_2}{2} \\ \frac{a_3+b_3}{2} \end{pmatrix}.$$

Intersections

Intersections of Planes

In 3D, two planes can intersect in:

1. A plane (implying they are the same plane)
2. A line
3. Not at all

Somewhat analogous to how lines intersect in 2D.

Intersections

Intersections of Planes

In 3D, the intersection of three planes can result in:

1. A plane (implying they are the same plane) - Infinite solutions
2. A line - Infinite solutions
3. A point - A single solution
4. Each plane intersects with two other planes forming a triangular prism shape or two planes are parallel, and one plane intersects both - No solution since all three have no common points of intersection
5. All three planes are parallel to each other - No solutions

Somewhat analogous to how lines intersect in 2D.

Linear Equations

RE Form

Row Echelon Form

Also known as Echelon Form. Occurs when the first non-zero number in each row (the pivot element) is in a column to the right of the row above it (does not need to be immediately to the right).

Solving Equations

Row Operations

Usually, finding the inverse of a matrix can yield a solution to a system of linear equations. However, this isn't always possible, such as when dealing with non-square matrices. Further, not all square matrices have inverses. When solving a system of equations encoded in a matrix, we can:

1. Swap rows
2. Add multiples of rows to other rows
3. Multiply or divide rows by constants

Using a combination of these to row reduce (AKA Gaussian Elimination), we can convert any matrix into row echelon form and solve relevant equations.

Solving Equations

The How Tos

1. Put your system of equations into a matrix in the form of $A\mathbf{x} = B$ where $\mathbf{x}$ is your vector of variables.
2. Convert your matrix to augmented form.
3. Perform Gaussian Elimination (AKA row reduction).
4. If non-leading columns exist (i.e. columns without a pivot element), substitute in an arbitrary parameter.
5. Perform back-substitution and voila.

Solving Equations

T3 2019 Q1ii

ii) A restaurant serves three types of salad: Healthy, Green and Tasty. The recipe for a serve of the three types of salad involves mixing a whole number of prepacked dried fruits, mixed vegetables and meat. For each serve of salad, the recipe is shown below.
Healthy: 1 pack of dried fruit, 2 packs of mixed vegetables and 2 packs of meat are mixed;
Green: 1 pack of dried fruit, 5 packs of mixed vegetables and 0 packs of meat are mixed;
Tasty: 2 packs of dried fruit, 1 pack of mixed vegetable and 6 packs of meat are mixed.
On a certain day, the restaurant had 126 packs of dried fruit, 198 packs of mixed vegetables and 288 packs of meat. The manager decided to use all these ingredients for the day to make exactly x serves of Healthy, y serves of Green and z serves of Tasty.

Solving Equations

T3 2019 Q1ii

a) Write a system of linear equations in x, y and z on the above information.
We organise the rows according to the individual ingredients to get:

$$\begin{aligned} x + y + 2z &= 126 \\ 2x + 5y + z &= 198 \\ 2x + 0y + 6z &= 288 \end{aligned}$$

Solving Equations

T3 2019 Q1ii

b) Solve the system of equations and find all non-negative integer solutions.

We now put our equations into an augmented matrix and row reduce:

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 126 \\ 2 & 5 & 1 & 198 \\ 2 & 0 & 6 & 288 \end{array}\right) \xrightarrow[R_2-2R_1]{R_3-2R_1} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 126 \\ 0 & 3 & -3 & -54 \\ 0 & -2 & 2 & 36 \end{array}\right)$$

$$\xrightarrow{R_3+\frac{2}{3}R_1} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 126 \\ 0 & 3 & -3 & -54 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Now for some back-substitution!

Solving Equations

T3 2019 Q1ii

Here is our row reduced matrix:

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 126 \\ 0 & 3 & -3 & -54 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Since z is the only variable in a non-leading column, we let it equal some arbitrary parameter t. Hence, we can see that

$$y = t - 18$$

and

$$x = 144 - 3t$$

Solving Equations

T3 2019 Q1ii

Thus, our solution vector, which is given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

can be expressed as

$$\begin{pmatrix} 144 - 3t \\ t - 18 \\ t \end{pmatrix} = \begin{pmatrix} 144 \\ -18 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$$

However, keep in mind that we need $x, y, z \geq 0$ since these are quantities measuring real life objects, which implies that $t \geq 18$.

Solving Equations

T3 2019 Q1ii

c) The manager believes that Tasty is the most popular. Find the solution which maximises the number of serves of Tasty. Referring back to our solution vector

$$\begin{pmatrix} 144 \\ -18 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$$

We want as many of z as possible while keeping $x, y \geq 0$. By inspection, we can see this occurs when $t = 48$. When $t = 48$, we have 48 Tasty salads, 30 Green salads, and 0 Healthy salads. This is our solution.

Matrix Transformations

What is a matrix transformation?

Definition

We can interpret a 2×2 matrix transformation as shifting the unit vectors $\hat{i}$ and $\hat{j}$ by each column of the 2×2 matrix! And now, every vector under this transformation will now be expressed in terms of these 'new' unit vectors.

Essentially, when we apply the transformation matrix on a vector, we are just stretching and shifting it in some manner. The resulting vector after applying a transformation is called the **image** of the vector.

Example

Transformation

Apply the transformation of $\begin{pmatrix} 3 & 2 \\ -2 & 1 \end{pmatrix}$ onto the vectors $\begin{pmatrix} 5 \\ 7 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. What are their images?

Example

Transformation

Apply the transformation of $\begin{pmatrix} 3 & 2 \\ -2 & 1 \end{pmatrix}$ onto the vectors $\begin{pmatrix} 5 \\ 7 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. What are their images?

Matrix multiply them!

$$\begin{pmatrix} 3 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 29 \\ -3 \end{pmatrix}$$

Example

Transformation

Apply the transformation of $\begin{pmatrix} 3 & 2 \\ -2 & 1 \end{pmatrix}$ onto the vectors $\begin{pmatrix} 5 \\ 7 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. What are their images?

$$\begin{pmatrix} 3 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \end{pmatrix}$$

Example

Question

A right-angled triangle T has verticies at $A(1,1)$, $B(2,1)$ and $C(2,4)$. When T is transformed by the matrix

$$\mathbf{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

the image is T'. Find the coordinates of the vertices of T' and describe the transformation represented by $\mathbf{P}$.

What is a Markov Process?

Definition

A Markov Process is one where the probability of the next event is only dependent on the state of the previous event. When predicting future states in Markov processes, there is no difference between knowing the entire history of the process, and just knowing the state achieved currently.

Markov Processes

Modelling using matrices

Since the 'rules' for transitioning to the next state is fixed, we can describe each state-to-state transition with a matrix (size depending on the problem).

Thus, we can represent our current state by using a single column vector (also of variable size) - and to find the next state, we just multiply our column vector and the transition matrix together!

Checking if everything is right...

1. Each column in a transition matrix must sum to 1 (total probabilities!)
2. To find the previous state to our current one, we simply multiply the *inverse* of the transition matrix. In other words, $\mathbf{T}^{-1}\mathbf{v}$.

Long Term and Steady State Solutions

Definition

1. For some/many Markov processes, the distribution will eventually become constant after many transitions (the limiting distribution). These are the *long term solutions*.
2. A steady state solution is one where repeated applications of the transition matrix will not do anything.

Example

Question

A fast-food restaurant owner is considering the effect on his business (McDonalds) by a new fast-food restaurant that opened down the road (KFC). Currently, McDonalds has 80% of the market and KFC has 20%.
After analysis over the past week, they discovered the probabilities for the customers switching the fast-food restaurant they stop at each week:

- If they stop at McDonalds, there is a 0.75 chance they stay, and a 0.25 chance they switch to KFC.
- If they stop at KFC, there is a 0.45 chance they stay at KFC and a 0.55 chance they switch to McDonalds.

What is the expected market share for McDonalds and KFC after another two weeks have passed? What is the long-run prediction for their market shares?

Difference Equations

What are they?

Definition

A difference equation is where terms of a sequence are defined recursively, i.e. in terms of previous values. For example,

$$y_{n+1} = 6y_n + 3$$

is a simple difference equation.

Difference Equations

Defining them more

1. The *order* of a difference equation, is the difference between the highest subscript and lowest (how far apart are they! the one in the previous slide had an order of 1).
2. A homogeneous difference equation is one where there is no constant. If you move all the terms with subscripts to one side, the other will be 0. A non-homogeneous equation is where there is a constant on the side without the subscripts.

Something we already know...

1. A first-order homogeneous difference equation is just a geometric progression!
2. This course is on *first and second order* **homogeneous** difference equations! You will not be dealing with constant values.

Difference Equations

Second order

For second order difference equations, given the equation

$$ax_n + bx_{n-1} = cx_{n-2} = 0,$$

we can obtain the auxiliary equation

$$a\lambda^2 + b\lambda + c = 0.$$

Now, the general solution will be $x_n = A\lambda_1^n + B\lambda_2^n$, where λ_1, λ_2 are the solutions to the auxiliary equation above. (Note: they need to be distinct).

As for first order, we pretty much already know $x_n = C(a^n)$, where the equation is $x_n - ax_{n-1} = 0$.

Difference Equations

Questions

- $x_n - 3x_{n-1} = 0, \quad x_1 = 2.$
- $x_n - 5x_{n-1} + 6x_{n-2} = 0, \quad x_0 = 1, x_1 = 0.$

Differentiation Revisited

Implicit Differentiation

Definition

- $\frac{d}{dx}(f(y)) = \frac{d}{dy}(f(y))\frac{dy}{dx}$.
- Used to differentiate messy equations that we cannot use our traditional methods on e.g. $x^2 + 3x + 2y^2 + 3y = 0$.
- Move all the variables to one side, if we are finding $\frac{dy}{dx}$, differentiate the x normally and every time you differentiate y, multiply it by $\frac{dy}{dx}$. Then, express the entire thing in terms of $\frac{dy}{dx}$.

Parametric Differentiation

Definition

- When x and y are not in terms of each other, but by a parameter e.g. t.
- In this case, we would have something like $x = f(t), y = g(t)$.
- We can use parametric differentiation to find $\frac{dy}{dx}$ by expressing it as $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$.

Implicit Differentiation Example

Question

- Use implicit differentiation to find $\frac{dy}{dx}$ for $x^3y = 27$, and the gradient at the point $(3, 1)$.
- Find $\frac{dy}{dx}$ when $x = t^3 - t$ and $y = 4 - t^2$.

Related Rates Example

Question

The area A of a circle is increasing at a constant rate of 1.5cm^2 per second. Find the rate at which the radius r of the circle is increasing when the area of the circle is 2cm^2.

Integration

Integration

Applications

- Integration is the reverse of differentiation. When you differentiate the integral of $f(x)$, you will get $f(x)$.
- The definite integral of something is the area under a curve between its bounds.
- Deriving quantities from its rate of change
- Calculating mass of objects of non-uniform density

Integration

Integration Formulae

- $\int x^n dx = \frac{x^{n+1}}{n+1}$
- $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)}$
- $\int \frac{1}{x} dx = \ln|x|$
- $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)|$
- $\int e^{ax} dx = \frac{1}{a}e^{ax}$
- $\int \sin(ax) dx = -\frac{1}{a}\cos(ax)$
- $\int \cos(ax) dx = \frac{1}{a} sin(ax)$
- $\int sec^2(ax) dx = \frac{1}{a}\tan(ax)$

Integration

Don't forget...

Remember to add $+C$ at the end of every indefinite integral!

Integration

More rules

- In general, you can factorise constants outside the integral
- If you have terms added together, you can integrate each of them separately
- Evaluating definite integrals: $\int_a^b f(x)dx = F(b) - F(a)$, where F stands for the integral.
- You can split up integrals: $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

Integration

Accumulation of a quantity

If you have a quantity $Q(t)$ that varies at a rate of $R(t) = \frac{dQ}{dt}$ in the interval $[a, b]$, the amount accumulated/growth is calculated by

$$\int_a^b R(t)dt = Q(b) - Q(a)$$

Integration

Average of a function

The average value A of $f(x)$ over an interval $[a, b]$ is given by

$$A = \frac{1}{b-a}\int_a^b f(x)dx.$$

This is also the formula to find average velocity!

Integration

Density of a function

If $f(x)$ gives the density of a quantity for $x \in [a, b]$, then the total mass of that quantity in those regions is given by

$$M = \int_a^b f(x)dx.$$

Integration

Question

The temperature of a soup is increasing at a rate of $r(t) = 30e^{-0.3t}$ degrees Celsius per minute (where t is the time in minutes). At time $t = 0$, the temperature of the soup is 23 degrees Celsius. Find the amount at which the temperature increases between $t = 1$ and $t = 5$ minutes.

Integration

Integration by Substitution

- Substitution (and soon, by parts) is used to integrate functions which are not considered 'standard'.
- This is a very important and powerful technique!
- We make a ‘substitution’ into an integral we don’t know how to solve, to simplify it into something we already know.
- $\displaystyle\int f(g(x))g'(x)dx = \int f(u)\frac{du}{dx}dx = \int f(u)du$
- Choose a function to be u, differentiate u and substitute everything in - and solve the integral! You know you’ve done a correct substitution when the result is something simpler...

Integration

Question

$$\int \sin x \sec^3 x dx$$

Integration

Never forget...

Don't forget to substitute everything back in once you're done!!!

Integration by Parts

Definition

- Integration by parts is essentially a way of 'reversing' a product rule differentiation.
- Also a *very* important technique.
- In this case, we have to choose one function to be u, and one function to be dv.
- Apply the formula: $\displaystyle\int udv = uv - \int vdu$.

Integration

Warning

You may have to apply integration by parts multiple times! Just make sure each iteration of by parts is getting you somewhere, and it doesn't loop back to the same equation!

Integration

Question

$$\int x \sec^2 2x dx$$

Differential Equations

Differential Equations

Definition

- It relates a function $y = f(x)$ to its derivatives!
- E.g. $\frac{dy}{dx} - \frac{2}{x}y - 1 = 0$
- The solutions to these equations are *functions*. You sub these functions into the equations and it should all hold!
- Similar to difference equations, but instead of expressing in terms of previous terms, you express in terms of derivatives!

Differential Equations

Terminology

- The order of a differential equation is the order of the highest derivative that appears! E.g the order is 2 if we have a 2nd derivative as the highest.
- It is linear if all the y's are linear!
- $y'' + 5y' + 2y = 10x$ is a 2nd order linear DE.

Differential Equations

How to solve a first order separable DE

- There are many types of DEs, and the way to solve different DEs depends on its type!
- First, we focus on first order separable DEs
- These are DEs where you can 'separate' the x's and the y's by moving them to each side of the equation, and then integrating.

Differential Equations

Question

Solve $\frac{dy}{dx} = xe^y$.

Differential Equations

How to solve a linear first order DE

- What if we can't separate the DE?
- First order linear DE: where y and y' are linear, but coefficients are in terms of x.
- They are in the form $\frac{dy}{dx} + p(x)y = q(x)$.
- The general solution to this is:

$$y = \frac{1}{R(x)} \int R(x)q(x)dx,$$

where $R(x) = e^{\int p(x)dx}$.

Differential Equations

Solve $\frac{dT}{dt} = k(T - T_m)$, given the initial conditions that $T(0) = T_0$, where k, T_m and T_0 are constants.

Differential Equations

Homogeneous Second Order Differential Equations

- These are in the form $ay'' + by' + cy = 0$, where $a, b, c \in \mathbb{R}$.
- They are homogeneous as the RHS will equal 0 (no constants).
- Requires 2 initial conditions, or there will be just constants.

Differential Equations

Homogeneous Second Order Differential Equations

- We have $ay'' + by' + cy = 0$ where $a, b, c \in \mathbb{R}$.
- Solve the equation $a\lambda^2 + b\lambda + c = 0$. Let the solutions be λ_1, λ_2.
- Then your final solution will be:
  - if $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1 \neq \lambda_2$, then $y = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$.
  - if $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1 = \lambda_2$, then $y = (A + Bx)e^{\lambda_1 x}$.
  - if $\lambda_1, \lambda_2 \in \mathbb{C}$, where $\lambda_1 = a + ib, \lambda_2 = a - ib$, then $y = e^{ax}(A\cos(bx) + B\sin(bx))$.

Example

Question

Solve the differential equation

$$y'' + y' - 6y = 0.$$

Non-Homogeneous Differential Equations

Non-homogeneous case

- Now, we learn how to solve case where

$$ay'' + by' + cy = f(x), \quad a, b, c \in \mathbb{R}.$$

- The solution is different depending on values of $f(x)$
- First, find the homogeneous solution y_h.
- Next, we find the *particular* solution by 'guessing' solutions.
- The general solution will be $y = y_h + y_p$, where y_p is the particular solution.

Differential Equations

How to find the particular solution?

- $y'' - 5y' + 6y = f(x)$
- We can find the particular solution by 'guessing' what sort of function will fit in the RHS. For example, if on the RHS we have $-6x + 23$, it is clear that y_p could be a degree one polynomial.
- It cannot be degree two, because the y term will mean that the LHS will have a x of degree 2, which will never equal the RHS of degree 1. Thus, we can conclude our $y_p = ax + b$, and sub that in!
- Then, solve! Techniques may vary, but in this case we would equate coefficients for equal powers of x.

Differential Equations

How to find the particular solution?

- $y'' - 5y' + 6y = f(x)$
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- It cannot be degree two, because the y term will mean that the LHS will have a x of degree 2, which will never equal the RHS of degree 1. Thus, we can conclude our $y_p = ax + b$, and sub that in!
- Then, solve! Techniques may vary, but in this case we would equate coefficients for equal powers of x.

Non-Homogeneous Differential Equations

Common Guesses

- If RHS is a polynomial, your guess will be a polynomial of the same degree e.g. $ax^3 + bx^2 + cx + d$.
- If RHS is an exponential, e.g. $4e^{2x}$, your guess will be a similar exponential, but with a constant in front e.g. αe^{2x}.
- If RHS is a sin or cos function, e.g. $3 \sin 4x$ your guess will be $\alpha sin(4x) + \beta \cos(4x)$. Be *very* careful to remember this one!
- There are some where you cannot guess, e.g. $\frac{1}{x}, ln(x), \sqrt{x}, etc.$

Non-Homogeneous Differential Equations

Question

Solve the differential equation

$$y'' + y' - 6y = 36x.$$

Systems of First Order Linear Differential Equations

Definition

- We can have systems of differential equations, where we have multiple equations and have to find a solution that satisfies all.
- Method: Similar to solving a normal simultaneous equation, we rearrange, differentiate and substitute to obtain one equation (hopefully a differential equation we can solve).
- We solve for one variable, e.g. y, and substitute this back into our original system of equations to obtain the solution for x.

Systems of Differential Equations

Example

Solve the system of differential equations:

$$\frac{dx}{dt} = x + 9y$$
$$\frac{dy}{dt} = -x - 5y$$

given that $x(0) = 1, y(0) = -1$.

Methods of Approximation

Newton's Method and Bisection of the Interval

What is it

- Usually, we try to stick to exact values.
- However, there are a lot of situations where we cannot find exact solutions!
- Newton's Method and the Bisection of the Interval Method is used to find approximated solutions to an equation.

Bisection Method

Definition

- This uses the Intermediate Value Theorem: if f is a continuous function on an interval $[a, b]$ with $f(a)$ and $f(b)$ differing in sign, then there is at least one solution to $f(x) = 0$ in $[a, b]$.
- Bisection of the interval: If the Intermediate Value Theorem is satisfied, we can approximate a solution to $f(x) = 0$ by repeatedly halving the interval where the sign change occurs.

Newton's Method

Definition

- Let $f(x)$ be a differentiable function. If we know a value x_0 is close to a root $f(x) = 0$, then we can repeatedly apply $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ to obtain a solution that is closer and closer to the true value of $f(x) = 0$.

Careful!

Be very careful! If the point you choose is too far away, you may end up getting a wrong approximation! So make sure it is close to the place where you vaguely know the root is.

Approximation Example

Example

Approximate a solution for

$$f(x) = x\cos(x) - x^2$$

using the Bisection of the interval method for $x = [0.5, 2]$, and Newton's method letting $x_0 = 1$.

Maclaurin Series

Maclaurin Series

Reasons

- Sometimes, there are functions which are very hard to deal with. On the other hand, polynomials have always been very *easy* to deal with.
- The idea of Maclaurin/Taylor Series is to approximate functions *with* polynomials!

Maclaurin Series

Definition

The Maclaurin series of a function $f(x)$ is given by

$$f(x) = f(0) + f'(0)\frac{x}{1} + f''(0)\frac{x^2}{2!} + \cdots + f^n\frac{x^n}{n!} + \ldots$$

where f^n is the nth derivative of $f(x)$.

Maclaurin Series

Examples

Some common examples of Maclaurin series include (memorise?)

- $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$
- $e^x = 1 + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$
- $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$
- $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$

Maclaurin Series

Question

Find the Maclaurin series for $y = \tan^{-1} x$ up to the term for x^3.

Partial Differentiation

Partial Differentiation

Definition

If $z = f(x, y)$, $\frac{\partial z}{\partial x}$ is the rate of change in the x direction!
Similarly, $\frac{\partial z}{\partial y}$ is the rate of change of z in the y direction.

- To find $\frac{\partial z}{\partial y}$, simply differentiate the function with respect to y, treating *all other variables as constants.*

Partial Differentiation

Example

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for $f(x, y) = e^{-x} \sin(x + y)$.

Local Maxima

How to find

If we are given $z = f(x, y)$, a function in two variables - how do we find where it attains a maximum and minimum? At any point which has both $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ at (a, b) (there is a stationary point), we will consider

$$D = \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 .$$

We simply evaluate this, and depending on the result we will know what sort of stationary point it is.

- $D < 0$ then there is a saddle point at (a, b).
- $D > 0$ and $\frac{\partial^2 f}{\partial x^2} > 0$ there is a local minimum at (a, b).
- $D > 0$ and $\frac{\partial^2 f}{\partial x^2} < 0$ there is a local maximum at (a, b).

Local Max Example

Example

Find the critical points of the function $f(x, y) = 2x^2 + 2xy + 2y^2 - 6x$, stating whether it is a maximum, minimum or saddle point.

That's all folks!

Aaaand that's everything! Hopefully everyone learnt a lot today and I hope everyone gets the marks they deserve!
Good luck in your final exams!
- From the entire Mathsoc team