MATH1231/1241 MathSoc Calculus Revision Session 2017 S2 Solutions

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1 Functions of several variables

Problem 1

[MATH1231 2012 S2 Q2 i)]
For a gas confined in a container, the ideal gas law states that the pressure P is related to the volume V and temperature T by

$$P = k\frac{T}{V},$$

where k is a positive constant.

a) Find $\frac{\partial P}{\partial V}$ and $\frac{\partial P}{\partial T}$.

b) The volume V is increased by 4% and the temperature T is decreased by 3%. Use the total differential approximation to estimate the percentage increase or decrease in the pressure P.

Solution 1

a) We have from simple differentiation that

$$\frac{\partial P}{\partial V} = -k\frac{T}{V^2} \quad \text{and} \quad \frac{\partial P}{\partial T} = \frac{k}{V}.$$

b) Manipulating our results from part a), we can clearly write

$$\begin{aligned}\frac{\partial P}{\partial V} &= -\frac{1}{V}\left(k\frac{T}{V}\right)\\ &= -P\cdot\frac{1}{V}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial P}{\partial T} &= \frac{1}{T}\left(k\frac{T}{V}\right)\\ &= \frac{P}{T}.\end{aligned}$$

Using the total differential approximation and the above results, we have

$$\begin{aligned}\Delta P &\approx \frac{\partial P}{\partial V}\Delta V + \frac{\partial P}{\partial T}\Delta T\\ &= -P\frac{\Delta V}{V} + P\frac{\Delta T}{T}\\ \Rightarrow \frac{\Delta P}{P} &\approx -\frac{\Delta V}{V} + \frac{\Delta T}{T}\end{aligned}$$

We are given that volume V is increased by 4% and the temperature T is decreased by 3%; that is, $\frac{\Delta V}{V} = 0.04$ and $\frac{\Delta T}{T} = -0.03$ (because the percentage change in a quantity z is $\frac{\Delta z}{z}$). Thus

$$\begin{aligned}\frac{\Delta P}{P} &\approx -0.04 - 0.03\\ &= -0.07.\end{aligned}$$

That is, we estimate that P decreases by 7%.

Problem 2

[MATH1241 2014 S2 Q4 i)]
Show that the equation of the tangent plane to the paraboloid S given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$$

at the point $P\left(x_0, y_0, z_0\right)$ is

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = \frac{z + z_0}{2c}.$$

Solution 2

We can write the equation of the paraboloid as

$$z = \frac{c}{a^2}x^2 + \frac{c}{b^2}y^2.$$

Taking the partial derivatives of z with respect to x and y and evaluating at $x = x_0, y = y_0$, we have

$$\frac{\partial z}{\partial x}(x_0, y_0) = \frac{2c}{a^2} \cdot x_0$$
$$\frac{\partial z}{\partial y}(x_0, y_0) = \frac{2c}{b^2} \cdot y_0.$$

Now, the equation of tangent plane at P is

$$\begin{aligned} z &= z_0 + \frac{\partial z}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial z}{\partial y}(x_0, y_0)(y - y_0) \\ &= z_0 + \frac{2c}{a^2} \cdot x_0(x - x_0) + \frac{2c}{b^2} \cdot y_0(y - y_0) \\ &= z_0 + 2c\left(\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2}\right) - 2c\left(\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2}\right). \qquad \text{(rearranging terms)} \end{aligned}$$

Since (x_0, y_0, z_0) lies on the paraboloid, we have $\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = \frac{z_0}{c}$. Substituting this above, we have that the equation of tangent plane at P is

$$\begin{aligned} z &= z_0 + 2c\left(\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2}\right) - 2z_0 \\ \Leftrightarrow z &= -z_0 + 2c\left(\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2}\right) \\ \Leftrightarrow \frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} &= \frac{z + z_0}{2c}, \end{aligned}$$

as required.

Problem 3

[MATH1231 2015 S2 Q1 iii)]
Suppose that $z = a^2 + b^3 + c^4$, where

$$\begin{aligned} a &= u - v + w, \\ b &= u + v - w, \\ c &= uvw. \end{aligned}$$

Use the chain rule to find $\frac{\partial z}{\partial u}$ at the point $(u, v, w) = (1, 0, 1)$.

Solution 3

Using the chain rule, taking a, b and c as functions of u, v and w, we have:

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial a}\frac{\partial a}{\partial u} + \frac{\partial z}{\partial b}\frac{\partial b}{\partial u} + \frac{\partial z}{\partial c}\frac{\partial c}{\partial u}$$

$$
\begin{aligned}
&= 2a \cdot 1 + 3b^2 \cdot 1 + 4c^3 \cdot vw \\
&= 2a + 3b^2 + 4c^3 vw.
\end{aligned}
$$

At $(u, v, w) = (1, 0, 1)$, we have $a = 2, b = 0, c = 0$ (using the formulas for a, b, c in terms of u, v, w). Thus at $(u, v, w) = (1, 0, 1)$, we have

$$
\begin{aligned}
\frac{\partial z}{\partial u} &= 2a + 3b^2 + 4c^3 vw \quad \text{with } a = 2, b = 0, c = 0 \\
&= 2 \times 2 + 0 + 0 \\
&= 4.
\end{aligned}
$$

2 Integration techniques

Problem 1

[MATH1231 2014 S2 Q1 ii)]
Evaluate the integral

$$
I = \int_0^1 \frac{x^2}{\sqrt{4 - x^2}} \, \mathrm{d}x.
$$

Solution 1

Substitute $x = 2\sin\theta$, $\mathrm{d}x = 2\cos\theta \, \mathrm{d}\theta$, for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. When $x = 0$, we have $2\sin\theta = 0$, so $\theta = 0$. When $x = 1$, we have $2\sin\theta = 1$, so $\sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$. Note that

$$
\begin{aligned}
\sqrt{4 - x^2} &= \sqrt{4 - 4\sin^2\theta} \\
&= 2\sqrt{1 - \sin^2\theta} \\
&= 2\sqrt{\cos^2\theta} && \text{(using the trig. identity } 1 - \sin^2\theta = \cos^2\theta) \\
&= 2|\cos\theta| \\
&= 2\cos\theta,
\end{aligned}
$$

since $\cos\theta > 0$ for $\theta \in \left(0, \frac{\pi}{6}\right)$, so $|\cos\theta| = \cos\theta$. Thus our integral becomes

$$
\begin{aligned}
I &= \int_0^{\frac{\pi}{6}} \frac{4\sin^2\theta}{2\cos\theta} \times 2\cos\theta \, \mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{6}} 4\sin^2\theta \, \mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{6}} 2\left(1 - \cos 2\theta\right) \mathrm{d}\theta && \text{(trig. identity: } \sin^2\theta = \tfrac{1}{2}\left(1 - \cos 2\theta\right)) \\
&= 2\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{6}} \\
&= 2\left(\frac{\pi}{6} - \frac{1}{2}\sin\frac{2\pi}{6}\right) \\
&= \frac{\pi}{3} - \frac{\sqrt{3}}{2}. && \left(\text{since } \sin\tfrac{2\pi}{6} = \sin\tfrac{\pi}{3} = \tfrac{\sqrt{3}}{2}\right)
\end{aligned}
$$

Problem 2

[MATH1241 2013 S2 Q1 iv) b)]
Evaluate the integral

$$J = \int_0^{\frac{\pi}{2}} \sin^3\theta \cos^3\theta \, \mathrm{d}\theta.$$

Solution 2

We have

$$\begin{aligned} J &= \int_0^{\frac{\pi}{2}} \sin^3\theta \cos^3\theta \, \mathrm{d}\theta \\ &= \int_0^{\frac{\pi}{2}} \sin^3\theta \cdot \cos^2\theta \cdot \cos\theta \, \mathrm{d}\theta \\ &= \int_0^{\frac{\pi}{2}} \sin^3\theta \left(1 - \sin^2\theta\right) \cdot \cos\theta \, \mathrm{d}\theta. \end{aligned}$$

Now, substitute $u = \sin\theta$, so $\mathrm{d}u = \cos\theta \, \mathrm{d}\theta$. When $\theta = 0, u = 0$ and when $\theta = \frac{\pi}{2}, u = 1$. Hence

$$\begin{aligned} J &= \int_0^1 u^3\left(1 - u^2\right) \mathrm{d}u \\ &= \int_0^1 \left(u^3 - u^5\right) \mathrm{d}u \\ &= \frac{1}{4} - \frac{1}{6} \\ &= \frac{1}{12}. \end{aligned}$$

Problem 3

[MATH1241 2016 S2 Q4 i)]

Find a reduction formula for $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^n x \, \mathrm{d}x$ and use it to show that

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^4 x \, \mathrm{d}x = \frac{\pi}{4} - \frac{2}{3}.$$

(Note that $1 + \cot^2 x = \csc^2 x$ and $\frac{\mathrm{d}}{\mathrm{d}x}\cot x = -\csc^2 x$.)

Solution 3

For $n = 0, 1, 2, \ldots$, let

$$I_n = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^n x \, \mathrm{d}x = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^2 x \cot^{n-2} x \, \mathrm{d}x.$$

Using the fact that $\cot^2 x + 1 = \csc^2 x$, we have for $n \geq 2$,

$$
\begin{aligned}
I_n &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\csc^2 x - 1)\cot^{n-2} x \,\mathrm{d}x \\
&= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \csc^2 x \cot^{n-2} x \,\mathrm{d}x - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^{n-2} x \,\mathrm{d}x \\
&= \left[-\frac{1}{n-1}\cot^{n-1} x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} - I_{n-2} \\
&\qquad \text{(Reverse Chain Rule on first integral, since } \tfrac{\mathrm{d}}{\mathrm{d}x}\cot x = -\csc^2 x\text{)} \\
\Rightarrow I_n &= \frac{1}{n-1} - I_{n-2},
\end{aligned}
$$

since $\cot\frac{\pi}{2} = 0$ and $\cot\frac{\pi}{4} = 1$. Now, to find I_4, it turns out we will also need to find I_0:

$$
\begin{aligned}
I_0 &= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^0 x \,\mathrm{d}x \\
&= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 1 \,\mathrm{d}x \\
&= \frac{\pi}{2} - \frac{\pi}{4} \\
\Rightarrow I_0 &= \frac{\pi}{4}.
\end{aligned}
$$

Now, we have

$$
\begin{aligned}
I_4 &= \frac{1}{4-1} - I_2 && \text{(using our reduction formula (in blue above))} \\
&= \frac{1}{3} - \left(\frac{1}{2-1} - I_0\right) && \text{(using the reduction formula again)} \\
&= \frac{1}{3} - \left(1 - \frac{\pi}{4}\right) \\
&= \frac{\pi}{4} - \frac{2}{3}. && \text{(answer)}
\end{aligned}
$$

3 Ordinary differential equations

Problem 1

[MATH1241 2015 S2 Q4 ii)]
The charge, $Q(t)$, in a certain circuit satisfies the differential equation

$$Q'' + Q' - 6Q = 0, \quad \text{with} \quad Q(0) = 3.$$

For which values, if any, of $Q'(0)$ will the charge tend to 0 as t tends to infinity?

Solution 1

The characteristic equation is

$$\lambda^2 + \lambda - 6 = 0 \Leftrightarrow (\lambda - 2)(\lambda + 3) = 0.$$

Hence the general solution is

$$Q(t) = Ae^{-3t} + Be^{2t}.$$

Using the initial condition $Q(0) = 3$, we have

$$\begin{aligned} Q(0) &= A + B = 3 \\ \Rightarrow B &= 3 - A \\ \Rightarrow Q(t) &= Ae^{-3t} + (3 - A)e^{2t} \\ \Rightarrow Q'(t) &= -3Ae^{-3t} + 2(3 - A)e^{2t} \\ \Rightarrow Q'(0) &= -3A + 6 - 2A = 6 - 5A. \end{aligned}$$

For $Q(t)$ to tend to zero as $t \to \infty$, it is necessary and sufficient that the coefficient of e^{2t} in the expression for $Q(t)$ is equal to zero, because $e^{2t} \to \infty$ as $t \to \infty$ and $e^{-3t} \to 0$ as $t \to \infty$. Thus $Q(t) \to 0$ iff $3 - A = 0$, that is, $A = 3$. But $Q'(0) = 6 - 5A$, so $A = 3$ iff $Q'(0) = 6 - 5 \times 3 = -9$. Thus the charge will tend to 0 as $t \to \infty$ iff $Q'(0) = -9$.

Problem 2

Find the general solution to

$$y'' + 4y' + 4y = 6e^{-2x} + 3x + 1. \qquad (*)$$

Solution 2

First solve the homogeneous equation

$$y'' + 4y' + 4y = 0.$$

The characteristic equation is

$$\lambda^2 + 4\lambda + 4 = 0 \Leftrightarrow (\lambda + 2)^2 = 0.$$

Hence we have a double root of -2. So the general solution to the *homogeneous* equation is

$$y_H = c_1 e^{-2x} + c_2 x e^{-2x},$$

where c_1, c_2 are arbitrary constants. Now we need to find a particular solution y_P to $(*)$. We try a solution of the form*

$$y_P = Ax^2 e^{-2x} + Bx + C.$$

* The $6e^{-2x}$ term in the RHS of $(*)$ implies we should try an Ax^2e^{-2x} term in our y_P (since -2 is a double root of the characteristic equation) and the $3x + 1$ term in the RHS of $(*)$ implies we should try a $Bx + C$ term in our y_P. We simply add these two guesses together to get our overall guess for y_P, thanks to what is known as the **superposition principle**.

Now, we have

$$\left(x^2e^{-2x}\right)' = 2xe^{-2x} - 2x^2e^{-2x} = \left(-2x^2 + 2x\right)e^{-2x},$$

and*

$$\left(x^2e^{-2x}\right)'' = 2e^{-2x} - 8xe^{-2x} + 4x^2e^{-2x} = \left(4x^2 - 8x + 2\right)e^{-2x}.$$

Therefore, since $y_P = Ax^2e^{-2x} + Bx + C$ and differentiation is a *linear operator*, we have

$$y_P' = A\left(x^2e^{-2x}\right)' + (Bx + C)' = A\left(-2x^2 + 2x\right)e^{-2x} + B$$

and

$$y_P'' = A\left(x^2e^{-2x}\right)'' + (Bx + C)'' = A\left(4x^2 - 8x + 2\right)e^{-2x}.$$

Substituting y_P into $(*)$, we have

$$\begin{aligned} y_P'' + 4y_P' + 4y_P &= 6e^{-2x} + 3x + 1 \\ \Rightarrow A\left(4x^2 - 8x + 2\right)e^{-2x} + 4\left(A\left(-2x^2 + 2x\right)e^{-2x} + B\right) + 4\left(Ax^2e^{-2x} + Bx + C\right) &= 6e^{-2x} + 3x + 1 \\ \Rightarrow 2Ae^{-2x} + 4Bx + (4B + 4C) &= 6e^{-2x} + 3x + 1. \\ &\text{(simplifying the LHS)} \end{aligned}$$

Equating coefficients of like terms, we have $2A = 6 \Rightarrow \boxed{A = 3}$, $4B = 3 \Rightarrow \boxed{B = \frac{3}{4}}$, and $4B+4C = 1 \Rightarrow 3 + 4C = 1 \Rightarrow \boxed{C = -\frac{1}{2}}$. So our particular solution is

$$y_P = Ax^2e^{-2x} + Bx + C = 3x^2e^{-2x} + \frac{3}{4}x - \frac{1}{2}.$$

So our general solution to the ODE is

$$\begin{aligned} y &= y_P + y_H \\ \text{i.e. } y &= 3x^2e^{-2x} + \frac{3}{4}x - \frac{1}{2} + c_1e^{-2x} + c_2xe^{-2x}, \end{aligned}$$

where c_1, c_2 are arbitrary constants.

Problem 3

[based on MATH1231 2014 S2 Q2 i)]
Solve the initial value problem (defined for $x > 0$)

$$y' + \left(2 + \frac{1}{x}\right)y = \frac{2}{x}, \quad y(1) = 0.$$

* This second derivative can be computed relatively easily by using the "second derivative of a product" formula $(uv)'' = u''v + 2u'v' + uv''$.

Solution 3

The integrating factor is

$$\begin{aligned} e^{\int \left(2+\frac{1}{x}\right)\,\mathrm{d}x} &= e^{2x+\ln x} && \left(\text{note } \textstyle\int \frac{1}{x}\,\mathrm{d}x = \ln|x| = \ln x \text{ since } x>0\right) \\ &= e^{2x}\cdot e^{\ln x} \\ &= xe^{2x}. \end{aligned}$$

Thus multiplying the ODE through by the integrating factor, the ODE becomes

$$\begin{aligned} \left(xe^{2x}y\right)' &= xe^{2x}\cdot\frac{2}{x} \\ \Rightarrow \left(xe^{2x}y\right)' &= 2e^{2x} \\ \Rightarrow xe^{2x}y &= e^{2x}+C \quad \text{for some constant } C && \text{(integrating)} \\ \Rightarrow y &= \frac{1+Ce^{-2x}}{x}. \end{aligned}$$

Using the initial condition, we have

$$\begin{aligned} y(1) &= 1+Ce^{-2} = 0 \\ \Rightarrow C &= -e^2 \\ \Rightarrow y &= \frac{1-e^2e^{-2x}}{x} \\ \text{i.e. } y &= \frac{1-e^{2-2x}}{x}. \end{aligned}$$

Problem 4

[MATH1241 2014 S2 Q4 iv)]
Suppose that y satisfies the initial value problem

$$y' + y^2 = \cos x, \quad \text{with} \quad y(0) = 0.$$

Using implicit differentiation, or otherwise, find the first two non-zero terms of the Maclaurin series of y. (You may assume without proof that $y(x)$ has a Maclaurin series.)

Solution 4

As we know, the Maclaurin series of $y(x)$ is

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y^{(3)}(0)}{3!}x^3 + \cdots.$$

So to find the first two non-zero terms of the Maclaurin series of y, we essentially need to find the first two non-zero terms of the sequence

$$y(0), y'(0), \frac{y''(0)}{2!}, \frac{y^{(3)}(0)}{3!}, \ldots. \qquad (*)$$

We are given $y(0) = 0$. Putting $x = 0$ into the given ODE, we have

$$\begin{aligned} y'(x) + (y(x))^2 &= \cos x \\ \Rightarrow y'(0) + y(0)^2 &= \cos 0 \\ \Rightarrow y'(0) &= 1, && \text{(since } y(0) = 0 \text{ and } \cos 0 = 1) \end{aligned}$$

so the first non-zero term of the sequence $(*)$ is $y'(0) = 1$. Now, differentiating both sides of the given ODE, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left(y' + y^2\right) &= \frac{\mathrm{d}}{\mathrm{d}x}(\cos x) \\ \Rightarrow y'' + 2yy' &= -\sin x && \text{(note } \tfrac{\mathrm{d}}{\mathrm{d}x}\left(y^2\right) = 2yy' \text{ by implicit differentiation)} \\ \Rightarrow y''(0) + 2y(0)y'(0) &= -\sin 0 && \text{(evaluating the above at } x = 0) \\ \Rightarrow y''(0) &= 0. && \text{(since } y(0) = 0 \text{ and } \sin 0 = 0) \end{aligned}$$

Differentiating once more, we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(y'' + 2yy') &= \frac{\mathrm{d}}{\mathrm{d}x}(-\sin x) \\ \Rightarrow y^{(3)} + 2(y')^2 + 2yy'' &= -\cos x && \text{(using product rule and implicit differentiation)} \\ \Rightarrow y^{(3)}(0) + 2y'(0)^2 + 2y(0)y''(0) &= -\cos 0 && \text{(evaluating the above at } x = 0) \\ \Rightarrow y^{(3)}(0) + 2 &= -1 && \text{(since } y'(0) = 1, y(0) = 0, \cos 0 = 1) \\ \Rightarrow y^{(3)}(0) &= -3. \end{aligned}$$

So since the Maclaurin series of y is

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y^{(3)}(0)}{3!}x^3 + \cdots,$$

the series written explicitly with the first two non-zero terms is

$$y(x) = x - \frac{1}{2}x^3 + \cdots,$$

by substituting the obtained values for $y(0), y'(0), y''(0), y^{(3)}(0)$ and simplifying.

4 Taylor series

Problem 1

[MATH1231 2013 S2 Q4 ii)]
The Maclaurin series for $\sin x$ is

$$\sin x = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!}, \quad x \in \mathbb{R}.$$

Do not prove this result.

a) Write down the Maclaurin series for $\sin(x^2)$.

b) Find the Maclaurin series for $\cos x$.

c) Hence derive the identity

$$\frac{1}{\sqrt{2}} = \sum_{k=0}^{\infty}(-1)^k \frac{\pi^{2k}}{2^{4k}(2k)!}.$$

Solution 1

a) Replace x with x^2 in the given Maclaurin series, so

$$\sin\left(x^2\right) = \sum_{k=0}^{\infty}(-1)^k \frac{\left(x^2\right)^{2k+1}}{(2k+1)!}, \quad x \in \mathbb{R},$$

i.e.

$$\sin\left(x^2\right) = \sum_{k=0}^{\infty}(-1)^k \frac{x^{4k+2}}{(2k+1)!}, \quad x \in \mathbb{R}.$$

The above is a Maclaurin series, so is the Maclaurin series for $\sin(x^2)$.

b) We are given that

$$\sin x = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!}, \quad x \in \mathbb{R}.$$

Differentiating both sides and recalling that a convergent power series can be differentiated term by term within its interval of convergence, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!}\right), \quad x \in \mathbb{R} \\
\Rightarrow \cos x &= \sum_{k=0}^{\infty}\frac{\mathrm{d}}{\mathrm{d}x}\left((-1)^k \frac{x^{2k+1}}{(2k+1)!}\right) && \text{(term by term differentiation)} \\
&= \sum_{k=0}^{\infty}(-1)^k \frac{(2k+1)x^{2k}}{(2k+1)!} \\
\Rightarrow \cos x &= \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!}, && \text{(since } (2k+1)! = (2k+1)(2k)! \text{ for every } k \in \mathbb{N})
\end{aligned}$$

so this is the Maclaurin series for $\cos x$.

c) Since the Maclaurin series for $\cos x$ converges to $\cos x$ at every $x \in \mathbb{R}$, we can substitute $x = \frac{\pi}{4}$ to get

$$\cos\left(\frac{\pi}{4}\right) = \sum_{k=0}^{\infty}(-1)^k \frac{\left(\frac{\pi}{4}\right)^{2k}}{(2k)!}$$

$$\Rightarrow \frac{1}{\sqrt{2}} = \sum_{k=0}^{\infty}(-1)^k \frac{\pi^{2k}}{4^{2k}(2k)!} \qquad \text{(since we know that } \cos\left(\tfrac{\pi}{4}\right) = \tfrac{1}{\sqrt{2}}\text{)}$$
$$= \sum_{k=0}^{\infty}(-1)^k \frac{\pi^{2k}}{2^{4k}(2k)!}, \qquad \text{(since } 4 = 2^2\text{)}$$

as required.

Problem 2

Determine whether each of the following series converges or diverges, stating any tests you use.

a) $\sum\limits_{n=1}^{\infty} \dfrac{\cos 2n}{\sqrt{n^3+1}}$ [MATH1231 2012 S2 Q4 ii)]

b) $\sum\limits_{n=1}^{\infty} \dfrac{n^4}{n!}$ [MATH1231 2012 S2 Q4 ii)]

c) $\sum\limits_{n=3}^{\infty} \dfrac{1}{n(\ln n)^2}$ [MATH1231 2016 S2 Q2 ii) b)]

Solution 2

a) **Converges.** We have for all $n \in \mathbb{Z}^+$

$$0 \le \left|\frac{\cos 2n}{\sqrt{n^3+1}}\right| = \frac{|\cos 2n|}{\sqrt{n^3+1}}$$
$$\le \frac{1}{\sqrt{n^3+1}} \qquad \text{(since } |\cos 2n| \le 1\text{)}$$
$$\le \frac{1}{n^{\frac{3}{2}}}, \qquad \text{(since } n^3+1 > n^3\text{)}$$

and $\sum\limits_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ converges by the p-test. Hence by the comparison test, $\sum\limits_{n=1}^{\infty} \left|\frac{\cos 2n}{\sqrt{n^3+1}}\right|$, which implies that $\sum\limits_{n=1}^{\infty} \frac{\cos 2n}{\sqrt{n^3+1}}$ converges*, since absolute convergence implies convergence.

b) **Converges.** We use the ratio test. Note that

$$\lim_{n\to\infty}\left|\frac{\frac{(n+1)^4}{(n+1)!}}{\frac{n^4}{n!}}\right| = \lim_{n\to\infty}\left(\frac{(n+1)^4}{n^4}\cdot\frac{n!}{(n+1)!}\right) = \lim_{n\to\infty}\left(\frac{(n+1)^4}{n^4}\cdot\frac{1}{n+1}\right)$$
$$= \lim_{n\to\infty}\frac{(n+1)^3}{n^4}$$
$$= 0 < 1.$$

* Recall that "absolute convergence implies convergence", which is Theorem 4.5.25 of the Calculus course pack notes (page 138 in the physical notes, which is page 152 of the electronic PDF)

Hence $\sum_{n=1}^{\infty} \frac{n^4}{n!}$ converges by the ratio test. (Note that $\lim_{n\to\infty} \frac{(n+1)^3}{n^4} = 0$ because the denominator is a polynomial with strictly greater degree than the polynomial that is the numerator.)

c) **Converges.** Let $f(x) = \frac{1}{x(\ln x)^2}$. Note that $f(x)$ is continuous and positive for all $x \geq 3$. Furthermore, $x \mapsto x(\ln x)^2$ is an increasing positive function for $x \geq 3$, since it is the product of two positive increasing functions for $x \geq 3$. Since $f(x) = \frac{1}{x(\ln x)^2}$, it follows that f is decreasing for $x \geq 3$. (Alternatively, observe that $f'(x) = -\frac{1}{x^2(\ln x)^2} - \frac{2}{x^2(\ln x)^3}$, which is negative for all $x \geq 3$, so $f(x)$ is decreasing on $[3, \infty)$.) Hence we can apply the integral test. We have

$$\begin{aligned}\int_3^\infty f(x)\,\mathrm{d}x &= \int_3^\infty \frac{1}{x(\ln x)^2}\,\mathrm{d}x \\ &= \int_{\ln 3}^\infty \frac{1}{u^2}\,\mathrm{d}u && \text{(substituting } u = \ln x,\ \mathrm{d}u = \tfrac{\mathrm{d}x}{x}\text{)} \\ &= \left[-\frac{1}{u}\right]_{\ln 3}^\infty \\ &= \frac{1}{\ln 3}.\end{aligned}$$

Since the integral converges, the integral test implies that $\sum_{n=3}^{\infty} f(n) \equiv \sum_{n=3}^{\infty} \frac{1}{n\,(\ln n)^2}$ converges.

Problem 3

[MATH1241 2015 S2 Q4 vi)]

(The root test) Let $\{a_n\}$ be a sequence of positive terms such that for some constant $r < 1$, we have $\sqrt[n]{a_n} \to r$ as $n \to \infty$.

a) Explain why this implies that there is a constant $R < 1$ and an integer N such that $a_n < R^n$ for all $n > N$.

b) Hence or otherwise prove that $\sum_{n=1}^{\infty} a_n$ converges.

Solution 3

a) From the definition of limits, we know that for **every** $\varepsilon > 0$, there exists some integer N such that

$$|\sqrt[n]{a_n} - r| < \varepsilon.$$

for all $n > N$. Since $\left|\sqrt[n]{a_n} - r\right| < \varepsilon$ implies that $\sqrt[n]{a_n} - r < \varepsilon$ (for $\varepsilon > 0$), i.e. $\sqrt[n]{a_n} < r + \varepsilon$, we have that for **every** $\varepsilon > 0$, there exists some integer N such that

$$\sqrt[n]{a_n} < r + \varepsilon$$

for all $n > N$.

In particular, by taking ε to be some **fixed** positive number (independent of n) that is less than $1 - r$, so that $r + \varepsilon < 1$ (we know such an ε exists because $1 - r > 0$, as $r < 1$; e.g. we can simply take $\varepsilon = \frac{1-r}{2}$), we know that there exists some integer N such that

$$\sqrt[n]{a_n} < r + \varepsilon < 1$$

for all $n > N$. Defining R to be the (fixed) number $r + \varepsilon$ so that R is a constant less than 1, we have that there exists some integer N such that

$$\sqrt[n]{a_n} < R, \quad \text{i.e.} \quad a_n < R^n$$

for all $n > N$, as required.

b) It suffices to show that $\sum\limits_{n=N+1}^{\infty} a_n$ converges (where the N is our N from part a)), because remember, the starting point of the sum does not affect its convergence status. We are given that the a_n are all positive, and we know from part a) that $a_n < R^n$ for all $n > N$. So we have

$$0 < a_n < R^n$$

for all $n > N$. Since $R < 1$ (and also $R > 0$ since from a), for all $n > N$, we have $R > \sqrt[n]{a_n} > 0$, since the a_n are positive), we know that $\sum\limits_{n=N+1}^{\infty} R^n$ converges (it is a geometric series with common ratio less than 1 in absolute value). Therefore, by the comparison test, $\sum\limits_{n=N+1}^{\infty} a_n$ converges, which proves the result.

5 Averages, arc length, speed and surface area

Problem 1

[MATH1231 2013 S2 Q2 ii)]
The curve C is given parametrically by $x = t^3, y = 2t^2$. Find the arc length of the curve C between $t = 0$ and $t = 1$.

Solution 1

Using the arc length formula for a parametric curve, we have that the arc length is

$$\begin{aligned}
\ell &= \int_0^1 \sqrt{(x'(t))^2 + (y'(t))^2}\,\mathrm{d}t \\
&= \int_0^1 \sqrt{(3t^2)^2 + (4t)^2}\,\mathrm{d}t && \text{(differentiating the formulas for } x(t) \text{ and } y(t)) \\
&= \int_0^1 \sqrt{9t^4 + 16t^2}\,\mathrm{d}t \\
&= \int_0^1 t\sqrt{9t^2 + 16}\,\mathrm{d}t && \text{(factoring out a } t^2 \text{ and noting } \sqrt{t^2} = t \text{ since } t \geq 0) \\
&= \int_{16}^{25} \frac{1}{18}\sqrt{u}\,\mathrm{d}u && \text{(substituting } u = 9t^2 + 16,\ \mathrm{d}u = 18t\,\mathrm{d}t, \text{ and adjusting the bounds as usual)} \\
&= \frac{1}{18} \times \frac{2}{3}\left[u\sqrt{u}\right]_{16}^{25} && \text{(since } \textstyle\int \sqrt{u}\,\mathrm{d}u = \frac{2}{3}u\sqrt{u}) \\
&= \frac{1}{27}(25 \times 5 - 16 \times 4)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{125-64}{27} \\
&= \frac{61}{27}.
\end{aligned}
$$

Problem 2

Find the average value of the function

$$f(x) = \sin^2 x$$

on the interval $[0, \pi]$.

Solution 2

Using the formula for the average value of a function, the answer $\overline{f}$ is

$$
\begin{aligned}
\overline{f} &= \frac{1}{\pi - 0} \int_0^\pi f(x)\,\mathrm{d}x \\
&= \frac{1}{\pi} \int_0^\pi \sin^2 x\,\mathrm{d}x \\
&= \frac{1}{\pi} \int_0^\pi \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)\,\mathrm{d}x && \left(\text{trig. identity } \sin^2 x = \tfrac{1}{2} - \tfrac{1}{2}\cos 2x\right) \\
&= \frac{1}{\pi} \left[\frac{x}{2} - \frac{1}{4}\sin 2x\right]_0^\pi \\
&= \frac{1}{\pi}\left(\frac{\pi}{2} - 0\right) && (\text{remember, } \sin 0 = \sin 2\pi = 0) \\
&= \frac{1}{2}.
\end{aligned}
$$

Problem 3

Determine the surface area of the solid obtained by rotating the curve

$$y = \sqrt[3]{x}, \quad 1 \le y \le 2$$

about the y-axis.

Solution 3

Using the formula for the surface area of a solid of revolution (where the rotated curve is of the form $x = f(y)$ for $1 \le y \le 2$ here), we have

$$
\begin{aligned}
\text{surface area} &= \int_1^2 2\pi x \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,\mathrm{d}y, \quad \text{where } x = y^3 \text{ (since } y = \sqrt[3]{x} \text{ on the curve)} \\
&= 2\pi \int_1^2 y^3 \sqrt{1 + (3y^2)^2}\,\mathrm{d}y && \left(\text{since } x = y^3 \Rightarrow \tfrac{\mathrm{d}x}{\mathrm{d}y} = 3y^2\right) \\
&= 2\pi \int_1^2 y^3 \sqrt{1 + 9y^4}\,\mathrm{d}y
\end{aligned}
$$

$$
\begin{aligned}
&= 2\pi \times \frac{1}{36}\int_{10}^{145} \sqrt{u}\,\mathrm{d}u && \text{(substituting } u = 1+9y^4,\ \mathrm{d}u = 36y^3\,\mathrm{d}y\text{)}\\
&= \frac{\pi}{18} \times \frac{2}{3}\left[u^{\frac{3}{2}}\right]_{10}^{145} && \text{(since } \textstyle\int \sqrt{u}\,\mathrm{d}u = \frac{2}{3}u^{\frac{3}{2}}\text{)}\\
&= \frac{\pi}{27}\left(145^{\frac{3}{2}} - 10^{\frac{3}{2}}\right). && \text{(answer)}
\end{aligned}
$$