UNSW Mathematics Society

MATH1131/1141 final exam workshop

Solutions

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Contents

Chapter 1

Algebra

1.1 Complex Numbers

Question 1

a) The conjugate of a complex number can be found by flipping the sign of the imaginary part. So we get

$$z + \overline{w} = (-1 + i) + (3 - 4i) = 2 - 3i.$$

b) We shall make use of the property

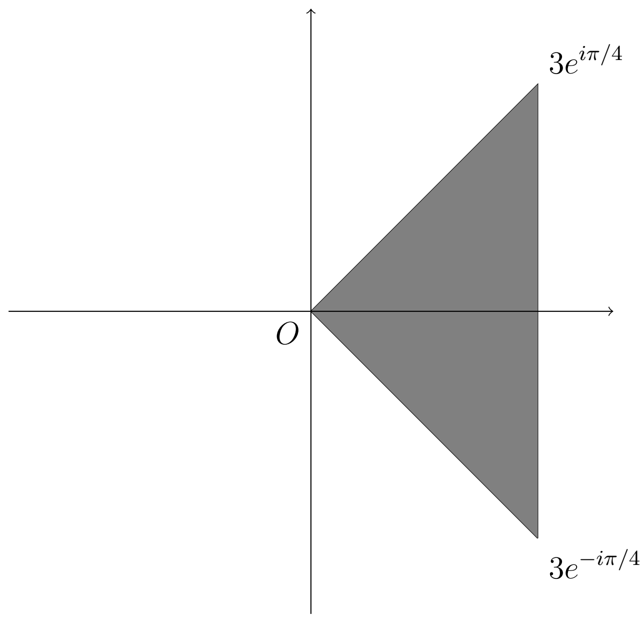
$$w\overline{w} = |w|^2$$

to force a real number in the denominator. So multiplying both sides by the conjugate of w gives

$$\frac{z}{w} = \frac{z\overline{w}}{|w|^2} = \frac{(-1+i)(3+4i)}{3^2+4^2} = \frac{1}{25}\,(1+7i)\,.$$

Question 2

a)



b) Looking at the graph, we see that the complex number with the biggest imaginary part is the positive complex number $3e^{i\pi/4}$. Using Euler's formula, we get

$$w = 3e^{i\pi/4} = \frac{3}{\sqrt{2}}\left(1+i\right).$$

Question 3

a) Multiplying and foiling out the complex number gives

$$\begin{aligned} zw &= (1+i)(\sqrt{3}+i) \\ &= \left(\sqrt{3}-1\right) + i\left(1+\sqrt{3}\right). \end{aligned}$$

b) Exploit the property

$$\arg(zw) = \arg(z) + \arg(w).$$

So

$$\arg(zw) = \frac{\pi}{4} + \frac{\pi}{6} = \frac{10\pi}{24} = \frac{5\pi}{12}.$$

So the principal argument is also

$$\operatorname{Arg}(zw) = \frac{5\pi}{12}.$$

c) We shall convert our result into its polar form. To do this, we note that the modulus is

$$|zw| = \sqrt{\left(\sqrt{3}-1\right)^2 + \left(1+\sqrt{3}\right)^2} = 2\sqrt{2}.$$

From part b), we found that the argument was $\frac{5\pi}{12}$. So writing everything in polar form, we have

$$zw = 2\sqrt{2}e^{i(5\pi/12)} = 2\sqrt{2}\left(\cos\left(\frac{5\pi}{12}\right) + i\sin\left(\frac{5\pi}{12}\right)\right).$$

Hence, comparing the real parts of zw, we get

$$2\sqrt{2}\cos\left(\frac{5\pi}{12}\right) = \sqrt{3}-1 \implies \cos\left(\frac{5\pi}{12}\right) = \frac{-1+\sqrt{3}}{2\sqrt{2}}.$$

Question 4

a) $|z| = \left|\sqrt{2}-\sqrt{2}i\right| = \sqrt{\sqrt{2}^2 + (-\sqrt{2})^2} = 2.$

b) $\operatorname{Arg}(z) = -\tan^{-1}\left(\frac{\sqrt{2}}{\sqrt{2}}\right) = -\tan^{-1}(1) = -\frac{\pi}{4}.$

c) By the previous parts, $z = |z|\, e^{i \operatorname{Arg} z} = 2e^{-i\pi/4}$. Taking the 6th power of z:

$$z^6 = 2^6 e^{-6i\pi/4} = 64e^{i\pi/2} = 64i.$$

Question 5

a) $p(2i) = (2i)^7 + 4(2i)^5 - (2i)^2 - 4 = -128i + 128i - (-4) - 4 = 0.$

b) By the conjugate root theorem, since $p(z)$ has real coefficients and $2i$ is a root then $\overline{2i} = -2i$ is also a root. Hence by the factor theorem, $(z-2i)$ and $(z+2i)$ are both factors of $p(z)$. So $(z-2i)(z+2i) = z^2+4$ is a factor of $p(z)$.

c) $p(z) = z^5(z^2+4) - (z^2+4)$ so

$$\frac{p(z)}{z^2+4} = z^5 - 1.$$

Hence the other 5 roots of $p(z)$ are the 5th roots of unity, which are

$$1, \quad e^{2\pi i/5}, \quad e^{-2\pi i/5}, \quad e^{4\pi i/5}, \quad e^{-4\pi i/5}.$$

Question 6

Given that $|w-z| = |w+z|$ so dividing by $|z|$:

$$\left|\frac{w}{z} - 1\right| = \left|\frac{w}{z} + 1\right|.$$

We can now do this either algebraically or geometrically.

Algebraic approach: Let $\dfrac{w}{z} = x + iy$. Then

$$\begin{aligned}|x + iy - 1| &= |x + iy + 1|\\(x-1)^2 + y^2 &= (x+1)^2 + y^2\\(x-1)^2 &= (x+1)^2\\x &= 0.\end{aligned}$$

So $\mathrm{Re}\left(\dfrac{w}{z}\right) = 0$, i.e. $\dfrac{w}{z}$ is purely imaginary.

Geometric approach: $\left|\dfrac{w}{z} - 1\right|$ is the distance between $\dfrac{w}{z}$ and 1, and $\left|\dfrac{w}{z} + 1\right|$ is the distance between $\dfrac{w}{z}$ and -1. If we want these distances to be equal, we parametrically define a line between 1 and -1 which is the set of all points with equal distance to these two points. This line will be $x = 0$ i.e. the imaginary axis. So $\dfrac{w}{z}$ is purely imaginary.

Question 7

Consider the expansion of $(\cos\theta + i\sin\theta)^5$. On the one hand, we use De Moivre's theorem so that

$$(\cos\theta + i\sin\theta) = \cos(5\theta) + i\sin(5\theta). \tag{1}$$

On the other hand, we can expand using the Binomial theorem so that

$$\begin{aligned}(\cos\theta + i\sin\theta)^5 &= \cos^5\theta + 5\cos^4\theta(i\sin\theta) + 10\cos^3\theta(i\sin\theta)^2 + 10\cos^2\theta(i\sin\theta)^3\\&\quad + 5\cos\theta(i\sin\theta)^4 + (i\sin\theta)^5\\&= \left(\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta\right)\\&\quad + i\left(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta\right). \qquad (2)\end{aligned}$$

So comparing the imaginary parts of (1) and (2) gives

$$\sin(5\theta) = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta.$$

Finally, to write $\cos^4\theta$ and $\cos^2\theta$ in terms of sin only, we use the Pythagorean identity twice:

$$\cos^2\theta = 1 - \sin^2\theta, \quad \cos^4\theta = 1 - 2\sin^2\theta + \sin^4\theta.$$

Substituting the two identities and simplifying, we finally get

$$\sin(5\theta) = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta.$$

1.2 Vector Geometry

Question 1

Let $A = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ be the pivot point. Then, a direction vector would be the vector from A to the point $B = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$, which becomes

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}.$$

Similarly, another direction vector would be the vector from A to the point $C = \begin{pmatrix} -2 \\ 1 \\ -5 \end{pmatrix}$, which becomes

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} -2 \\ 1 \\ -5 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \\ -4 \end{pmatrix} = -\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}.$$

Thus, a parametric representation could be

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}, \quad \lambda, \mu \in \mathbb{R}.$$

Question 2

a) From the given equation of the line,

$$x = z - 1 \qquad \text{and} \qquad y = -z + 5.$$

So we can write $z = \lambda$, and

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \lambda - 1 \\ -\lambda + 5 \\ \lambda \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 5 \\ 0 \end{pmatrix}.$$

b) The distance between our line and the origin is given by

$$d(\mathbf{x}, \mathbf{0}) = \|\mathbf{x} - \mathbf{0}\| = \|\mathbf{x}\|,$$

which is a function of λ. To minimise d is the same as minimising d^2, which is easier to differentiate, so we work with the square distance instead.

$$d^2 = \left\| \begin{pmatrix} \lambda - 1 \\ -\lambda + 5 \\ \lambda \end{pmatrix} \right\|^2 = (\lambda - 1)^2 + (\lambda - 5)^2 + \lambda^2 = 3\lambda^2 - 12\lambda + 26,$$

so

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(d^2\right) = 6\lambda - 12.$$

Hence the minimum value of d is at $\lambda = 2$, i.e. the point $\mathbf{x}_0$ on the line closest to the origin is

$$\mathbf{x}_0 = 2\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}.$$

Question 3

a) A set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ is an orthonormal set in $\mathbb{R}^n$ if

1) $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k$ are mutually orthogonal i.e. $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ for all $i \neq j$, and

2) $\|\mathbf{u}_i\| = 1$ for all $i \leq k$.

b) We define $M = (\mathbf{v}_1 \mid \mathbf{v}_2 \mid ... \mid \mathbf{v}_n)$ as the matrix with orthonormal columns. Then $M^T M$ will have entries equal to $(\mathbf{v}_i^T \mathbf{v}_j)$ where i is the row and j is the column. However $\mathbf{v}_i^T \mathbf{v}_j = \mathbf{v}_i \cdot \mathbf{v}_j$ which is 0 for all $i \neq j$, and 1 for all $i = j$ (by part (a)). Hence $M^T M$ has zeros everywhere except for the main diagonal, which is all 1, i.e. $M^T M = I$. Then

$$\begin{aligned} \det\left(M^T M\right) &= \det(I) \\ \det(M)^2 &= 1 \qquad\qquad (\det\left(M^T\right) = \det(M)) \\ \det(M) &= \pm 1. \end{aligned}$$

Question 4

Suppose we have a vector $\mathbf{n} \in \mathbb{R}^3$. We can rearrange the plane equation as follows:

$$\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}\right) \cdot \mathbf{n} = \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \cdot \mathbf{n} + \mu \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \cdot \mathbf{n}.$$

Setting

$$\mathbf{n} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ -3 \end{pmatrix},$$

then the RHS of our plane equation becomes 0. Hence

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 5 \\ -1 \\ -3 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ -1 \\ -3 \end{pmatrix}$$

$$5x - y - 3z = 9.$$

Question 5

a) Let $\lambda = 2$, then

$$\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}.$$

Hence the point B lies on line ℓ_1.

b) When the two lines intersect, we have

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 6 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$$

$$\lambda \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \\ 3 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} -1 & -3 & -3 \\ 2 & 0 & 6 \\ 1 & -1 & 3 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 1 & -1 & 3 \end{array}\right).$$

Hence we need $\lambda = 3, \mu = 0$. So point A has coordinates $(-2, 6, 4)$.

c) First we calculate $\overrightarrow{AB}$:

$$\overrightarrow{AB} = \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} - \begin{pmatrix} -2 \\ 6 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}.$$

The projection of $\overrightarrow{AB}$ onto the line ℓ_2 is equivalent to projecting $\overrightarrow{AB}$ onto the direction vector of ℓ_2, $\mathbf{v} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$. So by the projection formula,

$$\text{proj}_{\mathbf{v}}\left(\overrightarrow{AB}\right) = \frac{\overrightarrow{AB} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v} = \frac{1}{5}\mathbf{v} = \begin{pmatrix} 3/5 \\ 0 \\ 1/5 \end{pmatrix}.$$

Question 6

We are required to show that $(\mathbf{u} - \mathbf{v}) \cdot \mathbf{w} = 0$.

From our projections formula, we have

$$\begin{aligned} &\implies \frac{\mathbf{u} \cdot \mathbf{w}}{||\mathbf{w}||^2}\mathbf{w} = \frac{\mathbf{v} \cdot \mathbf{w}}{||\mathbf{w}||^2}\mathbf{w} \\ &\iff \mathbf{u} \cdot \mathbf{w} = \mathbf{v} \cdot \mathbf{w} \\ &\iff (\mathbf{u} - \mathbf{v}) \cdot \mathbf{w} = 0. \end{aligned}$$

Question 7

We aim to show that $(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = 0$.

Since the magnitude of $\mathbf{u}$ and $\mathbf{v}$ are the same, then we have

$$\begin{aligned}
&\implies ||\mathbf{u}||^2 = ||\mathbf{v}||^2 \\
&\iff ||\mathbf{u}||^2 - ||\mathbf{v}||^2 = 0 \\
&\iff \mathbf{u} \cdot \mathbf{u} - \mathbf{v} \cdot \mathbf{v} = 0 \\
&\iff (\mathbf{u} \cdot \mathbf{u}) + (\mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u}) - \mathbf{v} \cdot \mathbf{v} = 0 \\
&\iff (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = 0.
\end{aligned}$$

1.3 Matrices

Question 1

a)

$$PQ^T = \begin{pmatrix} 1 & 4 \\ 3 & 5 \\ 0 & 7 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix} = \begin{pmatrix} 23 & 18 \\ 34 & 26 \\ 35 & 28 \end{pmatrix}.$$

b) Since PQ^T is a 3×2 matrix and $QP^T = \left(PQ^T\right)^T$, then QP^T is a 2×3 matrix. Also we know that P is a 3×2 matrix, so PQP^T must be a 3×3 matrix.

Question 2

a) Take $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Then $A + B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\det(A + B) = 1$. But $\det(A) = \det(B) = 0$ and $\det(A) + \det(B) = 0 \neq \det(A + B)$.

b) Take $B = A^{-1}$. Then $AB = AA^{-1} = I$, and we have

$$\begin{aligned}
\det(I) = \det(AA^{-1}) &= \det(A)\det\left(A^{-1}\right) \\
1 &= \det(A)\det\left(A^{-1}\right) \\
\det\left(A^{-1}\right) &= 1/\det(A) \\
&= \det(A)^{-1}.
\end{aligned}$$

Question 3

From Question 2, we know that in general

$$\det\left(A^{-1}\right) = \det(A)^{-1}. \tag{1}$$

So we know that $\det(A) = \det\left(A^{-1}\right)^{-1}$. Finally, we observe that since A have only integer entries, then its determinant must also be an integer. Thus, it follows that $\det\left(A^{-1}\right)$ must be either -1 or 1. And so, it immediately follows from (1), that $\det(A) = \pm 1$.

Question 4

a) Consider the matrix $A = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$. Then $\overline{Q}^T = \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix}$. Then it is easy to check that $\overline{A}^T A = I$, i.e. A is unitary.

b) Suppose that the determinant of Q is some complex number $re^{i\theta}$. Since the sum and product of complex conjugates is equal to the conjugate of the sum and product of those complex numbers, we can take the determinant of $\overline{Q}^T Q$ as follows:

$$\det\left(\overline{Q}^T Q\right) = \det\left(\overline{Q}^T\right)\det(Q) = \det\left(\overline{Q}\right)\det(Q) = \overline{\det(Q)}\det(Q) = |\det(Q)|^2.$$

Hence if Q is unitary then $\det\left(\overline{Q}^T Q\right) = |\det(Q)|^2 = \det(I) = 1$. Writing Q as $re^{i\theta}$ then $r = 1$, so $Q = e^{i\theta}$ for some real θ.

Question 5

a) It is easy to check that $\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$ is nilpotent.

b) Suppose Q is a nilpotent matrix. Then $Q^2 = \mathbf{0}$ and so

$$\begin{aligned} \det\left(Q^2\right) &= \det(\mathbf{0}) \\ \det(Q)^2 &= 0 \\ \det(Q) &= 0. \end{aligned}$$

Hence Q is not invertible, i.e. nilpotent matrices are not invertible.

c) Since S and Q are commutative then $SQ = QS$. Also given that S is invertible, so $Q = SQS^{-1}$. Hence

$$S^{-1}Q = S^{-1}\left(SQS^{-1}\right) = \left(S^{-1}S\right)\left(QS^{-1}\right) = I\left(QS^{-1}\right) = QS^{-1}.$$

d) Suppose we have $k \in \mathbb{Z}$ such that

$$(S+Q)\left(S^{-1} - S^{-k}Q\right) = I.$$

Expanding the LHS:

$$QS^{-1} - S^{-k+1}Q - QS^{-k}Q = \mathbf{0}.$$

Now we multiply by Q on the left since Q is nilpotent:

$$Q^2S^{-1} - QS^{-k+1}Q - Q^2S^{-k}Q = \mathbf{0}.$$

Since $Q^2 = \mathbf{0}$ then

$$QS^{-k+1}Q = \mathbf{0}.$$

Here, Q is not invertible since Q is nilpotent. So we need $S^{-k+1} = I$, i.e. $k = 1$. Hence

$$(S+Q)\left(S^{-1} - S^{-1}Q\right) = I,$$

so $S+Q$ is invertible.

Question 6

We begin by expanding the transpose and inverse operators:

$$\left(P^{-1}Q\right)^T \left(Q^T P\right)^{-1} = Q^T \left(P^{-1}\right)^T P^{-1} \left(Q^T\right)^{-1}. \tag{$*$}$$

Recall that the transpose and inverse operators are commutative, so

$$\left(P^{-1}\right)^T P^{-1} = \left(P^T\right)^{-1} P^{-1} = \left(PP^T\right)^{-1}.$$

It is implicitly given that P is invertible so we can say that $PP^T = I$ since P is orthogonal. Hence

$$\left(P^{-1}\right)^T P^{-1} = I.$$

Substituting back into $(*)$,

$$\left(P^{-1}Q\right)^T \left(Q^T P\right)^{-1} = Q^T \left(Q^T\right)^{-1}.$$

However $Q^T \left(Q^T\right)^{-1} = Q^T \left(Q^{-1}\right)^T = \left(Q^{-1}Q\right)^T = I$, so

$$\left(P^{-1}Q\right)^T \left(Q^T P\right)^{-1} = I.$$

Question 7

Multiplying both sides by A^{-1} on the right yields

$$A^2A^{-1} = (2A + I)\,A^{-1}.$$

Simplifying both sides, we have

$$\begin{aligned} A^1 &= 2AA^{-1} + A^{-1} \\ \therefore A^{-1} &= A - 2I. \end{aligned}$$

Chapter 2

Calculus

2.1 Limits

Question 1

a) As $x \to \infty$, $\dfrac{\sin(x)}{x^2}$ and $\dfrac{\cos(x)}{x^2}$ both go to 0. Hence we divide the numerator and denominator by x^2:

$$\lim_{x\to\infty} \frac{6x^2 + \sin(x)}{4x^2 + \cos(x)} = \lim_{x\to\infty} \frac{6 + \frac{\sin(x)}{x^2}}{4 + \frac{\cos(x)}{x^2}} = \frac{6}{4} = \frac{3}{2}.$$

b) The numerator and denominator approach 0 as $x \to 0$, and the same can be said about the derivatives of the numerator and denominator. So we apply L'Hopital's rule twice:

$$\lim_{x\to 0} \frac{e^{2x} - 2x - 1}{4x^2} = \lim_{x\to 0} \frac{2e^{2x} - 2}{8x} = \lim_{x\to 0} \frac{4e^{2x}}{8} = \frac{1}{2}.$$

Question 2

Rewrite the limit as

$$\lim_{x\to\infty}\left(\sqrt{x^2+4x}-x\right)=\lim_{x\to\infty}\frac{\left(\sqrt{x^2+4x}-x\right)\left(\sqrt{x^2+4x}+x\right)}{\sqrt{x^2+4x}+x}.$$

The numerator is a difference of squares, so

$$\lim_{x\to\infty}\left(\sqrt{x^2+4x}-x\right)=\lim_{x\to\infty}\frac{4x}{\sqrt{x^2+4x}+x}.$$

Observe that $\dfrac{\sqrt{x^2+4x}+x}{x}=\sqrt{1+\dfrac{4}{x}}+1$ goes to 2 as $x\to\infty$, so we divide our numerator and denominator by x:

$$\lim_{x\to\infty}\left(\sqrt{x^2+4x}-x\right)=\lim_{x\to\infty}\frac{4}{\sqrt{1+4/x}+1}=\frac{4}{2}=2.$$

Question 3

We split the question into two separate cases of a.
Case 1: $a=0$. Then

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x=\lim_{x\to\infty}(1+0)^x=1.$$

Case 2: $a\neq 0$. Then we make a substitution $u=\dfrac{x}{a}$:

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x=\lim_{x\to\infty}\left(\left(1+\frac{1}{x/a}\right)^{x/a}\right)^a.$$

As $x\to\infty$, $u\to\infty$ as well so we have

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x=\lim_{u\to\infty}\left(\left(1+\frac{1}{u}\right)^u\right)^a=\left(\lim_{u\to\infty}\left(1+\frac{1}{u}\right)^u\right)^a.$$

The inner limit is Euler's number e so

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x = e^a.$$

In fact, in any case of $a \in \mathbb{R}$,

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x = e^a.$$

Question 4

Claim:

$$\lim_{x\to\infty}\frac{x^2-2}{x^2+3} = 1.$$

Proof: Let $M \in \mathbb{R}$, and suppose that $x > M \geq 0$. Then $x^2+3 > M^2+3$, or

$$\frac{1}{x^2+3} < \frac{1}{M^2+3}.$$

Multiplying both sides by 5:

$$\frac{5}{x^2+3} < \frac{5}{M^2+3}.$$

However the LHS can be written as

$$\frac{5}{x^2+3} = \left|\frac{x^2-2}{x^2+3} - 1\right|,$$

so

$$\left|\frac{x^2-2}{x^2+3} - 1\right| < \frac{5}{M^2+3}.$$

Setting $M = \sqrt{\frac{5}{\epsilon} - 3}$ then

$$\left|\frac{x^2-2}{x^2+3} - 1\right| < \epsilon.$$

Hence we can state that for each $\epsilon > 0$ there is M such that

$$x > M \Rightarrow \left| \frac{x^2 - 2}{x^2 + 3} - 1 \right| < \epsilon,$$

and we are done.

Question 5

Let $M \in \mathbb{R}$, and suppose that $x > M \geq 0$. Then $e^{-x} < e^{-M}$ and $e^x + e^{-x} > e^M$, or

$$\frac{1}{e^x + e^{-x}} < e^{-M}.$$

Hence

$$\frac{2e^{-x}}{e^x + e^{-x}} < 2e^{-M}e^{-M} = 2e^{-2M}.$$

However the LHS can be written as

$$\frac{2e^{-x}}{e^x + e^{-x}} = \left| \frac{-e^{-x}}{\cosh x} \right| = \left| \frac{e^x}{\cosh x} - 2 \right|$$

so

$$\left| \frac{e^x}{\cosh x} - 2 \right| < 2e^{-2M}.$$

Setting $M = \max \left\{ 0, \ln \sqrt{\frac{2}{\epsilon}} \right\}$ then

$$\left| \frac{e^x}{\cosh x} - 2 \right| < \epsilon.$$

Hence we can state that for each $\epsilon > 0$ there is M such that

$$x > M \Rightarrow \left| \frac{e^x}{\cosh x} - 2 \right| < \epsilon,$$

and we are done.

Question 6

First note that $-1 \leq \sin(x) \leq 1$, so multiplying the double-sided inequality by e^{-x} gives

$$-e^{-x} \leq e^{-x}\sin(x) \leq e^{-x}.$$

However

$$\lim_{x\to\infty} e^{-x} = 0,$$

so by the pinching theorem we have

$$\lim_{x\to\infty} e^{-x}\sin(x).$$

Question 7

Consider the following limit:

$$\lim_{x\to c} \frac{f'(x) - f'(c)}{2(x-c)}.$$

The numerator and denominator both approach 0 as $x \to c$, so we can apply L'Hopital's rule provided the following limit exists:

$$\lim_{x\to c} \frac{\frac{d}{dx}\left(f'(x) - f'(c)\right)}{\frac{d}{dx}\left(2(x-c)\right)} = \lim_{x\to c} \frac{f''(x)}{2}.$$

This limit does exist, and it equals $\frac{f''(c)}{2}$. Then by L'Hopital's rule we have

$$\lim_{x\to c} \frac{f'(x) - f'(c)}{2(x-c)} = \frac{f''(c)}{2}. \tag{$*$}$$

Now consider the limit

$$\lim_{x\to c} \frac{f(x) - f(c) - f'(c)(x-c)}{(x-c)^2}.$$

The numerator and denominator both go to 0 as $x \to c$, and the limit

$$\lim_{x\to c} \frac{\frac{d}{dx}\left(f(x) - f(c) - f'(c)(x-c)\right)}{\frac{d}{dx}\left((x-c)^2\right)} = \lim_{x\to c} \frac{f'(x) - f'(c)}{2(x-c)}$$

exists by $(*)$. Hence we can apply L'Hopital's rule:

$$\lim_{x\to c} \frac{f(x) - f(c) - f'(c)(x-c)}{(x-c)^2} = \lim_{x\to c} \frac{f'(x) - f'(c)}{2(x-c)} = \frac{f''(c)}{2}$$

i.e.

$$\lim_{x\to c} q(x) = \frac{f''(c)}{2}.$$

2.2 Differentiation

Question 1

a) First we show that f is continuous at $x = 0$. The limit as x approaches 0 from the left is

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (0) = 0,$$

and the limit as x approaches 0 from the right is

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2 = 0.$$

Hence

$$\lim_{x\to 0} f(x) = 0,$$

and since $f(0) = 0$ then

$$\lim_{x\to 0} f(x) = f(0).$$

So f is continuous at $x = 0$. If f is differentiable at $x = 0$ then the limit

$$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

must exist. We will prove the limit exists by finding the left and right limits, and showing they are equal. First consider the left limit. Here $h < 0$ so $f(h) = 0$, and

$$\lim_{h\to 0^-} \frac{f(h) - f(0)}{h} = \lim_{h\to 0^-} \frac{0-0}{h} = 0.$$

Now consider the right limit. Here $h > 0$ so $f(h) = h^2$, and

$$\lim_{h\to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h\to 0^+} \frac{h^2-0}{h} = \lim_{h\to 0^+} h = 0.$$

Since the left and right limits exist and are equal, then

$$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

exists. Furthermore,

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = 0.$$

b) Since f is differentiable on the intervals $[0,\infty)$ and $(-\infty, 0)$, we can differentiate each piece of f separately:

$$f'(x) = \begin{cases} 2x & x \geq 0 \\ 0 & x < 0. \end{cases}$$

Question 2

For $x \neq 0$ we know that f is differentiable since polynomials and trigonometric functions are differentiable. So our only concern is when $x = 0$. We need f to be continuous at this point, i.e.

$$\lim_{x\to 0} f(x) = f(0).$$

Calculating the left and right sided limits:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \left(x^2 + ax + b\right) = b, \qquad \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \cos 2x = 1.$$

For the limit to exist, we need $b = 1$. Note that $f(0) = 1$ so f is in fact continuous at $x = 0$ if $b = 1$. Now we want f to also be differentiable at $x = 0$, hence we need the limit

$$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

to exist. Calculating the left and right sided limits:

$$\lim_{h\to 0^-} \frac{f(h) - f(0)}{h} = \lim_{h\to 0^-} \frac{(h^2 + ah + 1) - 1}{h} = a,$$

$$\lim_{h\to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h\to 0^+} \frac{\cos 2h - 1}{h} = 2 \lim_{h\to 0^+} \frac{\cos 2h - 1}{2h} = 0.$$

For the limit to exist, we need $a = 0$. Hence f is differentiable if $a = 0$ and $b = 1$.

Question 3

Taking the natural logarithm of y, we have

$$\ln y = 2x \ln(\cosh x).$$

Now we differentiate both sides by x:

$$\begin{aligned} \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= 2\ln(\cosh x) + 2x \tanh x \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= (\cosh x)^{2x} \left(2\ln(\cosh x) + 2x\tanh x\right). \end{aligned}$$

Question 4

a) f is continuous on the closed interval $[0,1]$ and

$$f(0) = -1 \leq 0 \leq f(1) = 6 - \cos 1.$$

Then by the Intermediate Value Theorem, there exists $c \in (a,b)$ such that $f(c) = 0$. Since $c > 0$ then f has at least one positive root.

b) We know from part (a) that f has at least one root. Now we show that f must have at most one root, by taking the derivative of f:

$$f'(x) = 3x^2 + 5 + \sin x.$$

Since $\sin x$ is at least -1 then $f'(x) \geq 4 > 0$, and so f is monotonic increasing. Then f can only have at most one root, hence f has exactly one root.

Question 5

Suppose we have $a < b$ and $f(x) = \tan^{-1} x$. f is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b), so by the Mean Value Theorem:

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

for some $c \in (a,b)$. However $f'(c) = \dfrac{1}{1+c^2}$ so

$$0 < f'(c) \leq 1.$$

Therefore

$$0 < \frac{\tan^{-1} b - \tan^{-1} a}{b - a} \leq 1$$

and multiplying by $b - a$:

$$0 < \tan^{-1} b - \tan^{-1} a \leq b - a.$$

Question 6

Let $f : [0, x] \to \mathbb{R}$ be the function defined by $f(t) = \ln(1+t)$. Since $x > 0$ then f is continuous on $[0, x]$ and differentiable on $(0, x)$. So by the Mean Value Theorem there exists $c \in (0, x)$ such that

$$\frac{f(x) - f(0)}{x - 0} = f'(c).$$

Evaluating the function values and the derivative:

$$\frac{\ln(1+x)}{x} = \frac{1}{1+c}.$$

Since $c < x$ then $\dfrac{1}{1+c} > \dfrac{1}{1+x}$ so

$$\frac{\ln(1+x)}{x} > \frac{1}{1+x}.$$

Multiplying both sides by x:

$$\ln(1+x) > \frac{x}{1+x}.$$

Question 7

a) $f'(x) = 1 + \dfrac{1}{x^2} > 0$ so f is monotonic increasing.

b) Since f is monotonic increasing on the interval $(1, \infty)$ then g has domain $(f(1), f(\infty)) = (0, \infty)$.

c) If, for $x > 1$, $x - \dfrac{1}{x} = \dfrac{3}{2}$ then $x = 2$. Hence $g(\frac{3}{2}) = 2$. Then by the Inverse Function theorem:

$$g'\left(\frac{3}{2}\right) = \frac{1}{f'\left(g\left(\frac{3}{2}\right)\right)}.$$

$f'(g(\frac{3}{2})) = f'(2) = \dfrac{5}{4}$ so

$$g'\left(\frac{3}{2}\right) = \frac{1}{5/4} = \frac{4}{5}.$$

Question 8

a) First we evaluate the derivative of f:

$$f'(x) = 2x - 2x\sin(x^2).$$

When $f'(x) = 0$, $x = 0$ or $\sin(x^2) = 1$ so $x = \sqrt{\dfrac{4k+1}{2}\pi}$, $k = 0, 1$. The second derivative of f is

$$f''(x) = 2 - 2\sin(x^2) - 4x^2\cos(x^2).$$

Since $f''(0) = 2 > 0$ then f has a local minimum at $x = 0$. In fact, $x = 0$ is the absolute minimum of f. Now we consider points near $\sqrt{\dfrac{\pi}{2}}$:

$$f\left(\sqrt{\frac{\pi}{2}} + 0.01\right) = 0.0008 > 0, \quad \text{and} \quad f\left(\sqrt{\frac{\pi}{2}} - 0.01\right) = 0.00077 > 0,$$

so f has an inflection point at $x = \sqrt{\dfrac{\pi}{2}}$. Similarly with $\sqrt{\dfrac{5\pi}{2}}$,

$$f\left(\sqrt{\frac{5\pi}{2}} + 0.01\right) = 0.0089 > 0, \quad \text{and} \quad f\left(\sqrt{\frac{5\pi}{2}} - 0.01\right) = 0.0087 > 0,$$

so f has an inflection point at $x = \sqrt{\dfrac{5\pi}{2}}$. Also, f has absolute maximum at $x = 2\sqrt{\pi}$.

b) Since f is monotonic non-decreasing on $(0, 2\sqrt{\pi}]$ as proven in (a), then f is both injective and surjective. Hence f is invertible, and f^{-1} has domain $(0, 4\pi + 1]$.

Also,

$$f\left(\sqrt{\frac{5\pi}{2}}\right) = \frac{5\pi}{2}$$

so

$$f^{-1}\left(\frac{5\pi}{2}\right) = \sqrt{\frac{5\pi}{2}}.$$

c) f^{-1} is differentiable on its domain, except the points at which f has an inflection. We calculated the inflection points of f in (a), so now we find the values of f at these points:

$$f\left(\sqrt{\frac{\pi}{2}}\right) = \frac{\pi}{2}, \quad \text{and} \quad f\left(\sqrt{\frac{5\pi}{2}}\right) = \frac{5\pi}{2}.$$

So f^{-1} is differentiable on $\left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{5\pi}{2}\right) \cup \left(\frac{5\pi}{2}, 4\pi + 1\right]$.

2.3 Integration

Question 1

a) Use the substitution $u = \ln x$, then

$$\int \frac{\mathrm{d}x}{x\left(1 + (\ln x)^2\right)} = \int \frac{\mathrm{d}u}{1 + u^2} = \tan^{-1}(u) + C = \tan^{-1}(\ln(x)) + C.$$

b) Use integration by parts with $u = x$, $\mathrm{d}v = \sinh(2x)\,\mathrm{d}x$:

$$\begin{aligned}\int x \sinh(2x)\,\mathrm{d}x &= \frac{1}{2}x\cosh(2x) - \frac{1}{2}\int \cosh(2x)\,\mathrm{d}x \\ &= \frac{1}{2}x\cosh(2x) - \frac{1}{4}\sinh(2x) + C.\end{aligned}$$

c) Use the substitution $u = 3 + x^3$, then

$$\int x^2\sqrt{3+x^3}\,\mathrm{d}x = \int \frac{1}{3}\sqrt{u}\,\mathrm{d}u = \frac{2}{9}u^{3/2} + C = \frac{2}{9}\left(3+x^3\right)^{3/2} + C.$$

d) Use the substitution $x = \sinh(u)$, then

$$\begin{aligned}\int \sqrt{1+x^2}\,\mathrm{d}x &= \int \sqrt{1+(\sinh(u))^2}\cosh(u)\,\mathrm{d}u = \int \cosh^2(u)\,\mathrm{d}u \\ &= \frac{1}{2}\int \left(\cosh(2u)+1\right)\,\mathrm{d}u = \frac{1}{4}\sinh(2u) + \frac{u}{2} + C \\ &= \frac{1}{2}x\sqrt{1+x^2} + \frac{1}{2}\ln\left(x+\sqrt{1+x^2}\right) + C.\end{aligned}$$

Question 2

First we split the integral into two integrals:

$$\int_x^{x^2} \cosh\left(\sqrt{t}\right)\,\mathrm{d}t = \int_0^{x^2} \cosh\left(\sqrt{t}\right)\,\mathrm{d}t - \int_0^{x} \cosh\left(\sqrt{t}\right)\,\mathrm{d}t.$$

By the fundamental theorem of calculus,

$$\frac{d}{dx}\int_0^{x^2} \cosh\left(\sqrt{t}\right)\,\mathrm{d}t = 2x\cosh(x) \quad \text{and} \quad \frac{d}{dx}\int_0^{x} \cosh\left(\sqrt{t}\right)\,\mathrm{d}t = \cosh\left(\sqrt{x}\right).$$

Then we have

$$\frac{d}{dx}\int_x^{x^2} \cosh\left(\sqrt{t}\right)\,\mathrm{d}t = 2x\cosh(x) - \cosh\left(\sqrt{x}\right).$$

Question 3

a) Let $n > 0$, then by integration by parts for $u = f(x)$, $dv = \sin nx \, \mathrm{d}x$:

$$\int_a^b f(x) \sin nx \, \mathrm{d}x = \frac{K(n)}{n} + \frac{1}{n} \int_a^b f'(x) \cos nx \, \mathrm{d}x,$$

where $K(n) = f(a) \cos(na) - f(b) \cos(nb)$.

b) Since $|f'(x) \cos nx| \leq |f'(x)| \leq L$ then

$$\left| \int_a^b f'(x) \cos nx \, \mathrm{d}x \right| \leq \int_a^b L \, \mathrm{d}x = (b-a)L.$$

c) Taking the absolute value of our integral and applying the triangle inequality:

$$\begin{aligned} \left| \int_a^b f(x) \sin nx \, \mathrm{d}x \right| &= \left| \frac{K(n)}{n} + \frac{1}{n} \int_a^b f'(x) \cos nx \, \mathrm{d}x \right| \\ &\leq \left| \frac{K(n)}{n} \right| + \frac{1}{n} \left| \int_a^b f'(x) \cos nx \, \mathrm{d}x \right| \\ &\leq \frac{|K(n)|}{n} + \frac{(b-a)L}{n}. \end{aligned}$$

Clearly,

$$\lim_{n \to \infty} \frac{(b-a)L}{n} = 0.$$

Also we can find an upper bound of $K(n)$:

$$|K(n)| = |f(a) \cos(na) - f(b) \cos(nb)| \leq |f(a)| + |f(b)|.$$

Therefore

$$\lim_{n \to \infty} \frac{K(n)}{n} = 0.$$

Hence

$$\lim_{n \to \infty} \frac{|K(n)|}{n} + \frac{(b-a)L}{n} = 0,$$

and so

$$\lim_{n\to\infty}\int_a^b f(x)\sin nx\,\mathrm{d}x = 0.$$

Question 4

a) First we split the integral into two parts:

$$\int_0^\infty \frac{\mathrm{d}x}{x^2+e^x} = \int_0^1 \frac{\mathrm{d}x}{x^2+e^x} + \int_1^\infty \frac{\mathrm{d}x}{x^2+e^x}.$$

Clearly the first integral is finite, so we only need to test the second integral. Since $x^2+e^x > x^2$ then

$$\int_1^\infty \frac{\mathrm{d}x}{x^2+e^x} \le \int_1^\infty \frac{1}{x^2}\,\mathrm{d}x.$$

The larger integral converges by the p-test, so the smaller integral converges by the comparison test.

b) Since $\ln x < x$ then $x + \ln x < 2x$, and so

$$\int_e^\infty \frac{\mathrm{d}x}{x+\ln x} \ge \frac{1}{2}\int_e^\infty \frac{1}{x}\,\mathrm{d}x.$$

The smaller integral diverges by the p-test, so the larger integral diverges by the comparison test.

c) Since $e^{2x}+\cos^2 x \ge e^{2x}$ then

$$\int_0^\infty \frac{\mathrm{d}x}{e^{2x}+\cos^2 x} \le \int_0^\infty e^{-2x}\,\mathrm{d}x.$$

The larger integral is computable and finite, so by the comparison test the smaller integral converges.

d) This integral is computable and hence finite (use a substitution of $u = x^2$).

e) Since $\sqrt{1+x^6} \geq \sqrt{x^6} = x^3$ then

$$\int_1^\infty \frac{1}{\sqrt{1+x^6}}\,\mathrm{d}x \leq \int_1^\infty \frac{1}{x^3}\,\mathrm{d}x.$$

The larger integral converges by p-test, so the smaller integral converges by the comparison test.

Question 5

a) By taking the derivative of f we can see that f is monotonic decreasing on the interval $[0,1]$. Hence the lower Riemann sum will be given by

$$L_p(f) = \sum_{k=1}^{n} \frac{1}{n} f\left(\frac{k}{n}\right) = \sum_{k=1}^{n} \frac{1}{n}\left(\frac{1}{1+k/n}\right) = \sum_{k=1}^{n} \frac{1}{n+k}.$$

b) By the definition of the Riemann integral, if we assume that the limits of the upper and lower Riemann sums are equal then

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k} = \int_0^1 \frac{1}{1+x}\,\mathrm{d}x.$$

Evaluating the integral:

$$\int_0^1 \frac{1}{1+x}\,\mathrm{d}x = \left[\ln(1+x)\right]_0^1 = \ln 2.$$

Hence

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k} = \ln 2.$$