UNSW Mathematics Society Presents

MATH1131/1141 Workshop

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Overview I

1. Algebra
   - Introduction to vectors
   - Vector geometry
   - Complex numbers
   - Linear Equations and Matrices
   - Matrices

2. Calculus
   - Sets, Inequalities and Functions
   - Limits
   - Properties of Continuous Functions
   - Differentiable Functions
   - Mean Value Theorem

1. Algebra

Planes

Definition

In $\mathbb{R}^3$, the parametric form of a plane is

$$\mathbf{x} = \mathbf{a} + \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2, \text{for} \lambda_1, \lambda_2 \in \mathbb{R}.$$

Question 1

Find the parametric form of $\mathrm{x}_2 + 6x_3 = -1$.

Solution. We set $x_1 = \lambda_1$ and $x_3 = \lambda_2$. Then solving for x_2 gives $x_2 = -1 - 6\lambda_2$. Therefore, the parametric vector form of the equation is

$$\begin{aligned}
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} \lambda_1 \\ -1 - 6\lambda_2 \\ \lambda_2 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ -6 \\ 1 \end{pmatrix} \quad \text{where } \lambda_1, \lambda_2 \in \mathbb{R}.
\end{aligned}$$

Planes

Question 2 (from MATH1131 2015 Paper 1vi)

Consider the three points A, B, C in $\in \mathbb{R}^3$ with position vectors

$$\begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}$$

Find a parametric vector form for the plane Π that passes through points A, B, and C.

Planes

Solution. Let $\mathbf{OA}$ be the position vector of the plane Π, now find $\mathbf{AB}$ and $\mathbf{AC}$ which are direction vectors of the plane Π, this gives,

$$\mathbf{AB} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$$

$$\mathbf{AC} = \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix}$$

Therefore, by definition, the parametric form of the plane Π is

$$\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} + \lambda_2 \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix} \text{ for } \lambda_1, \lambda_2 \in \mathbb{R}.$$

Application of the cross product

Definition

The **cross product** of two vectors $a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ in $\mathbb{R}^3$ is

$$a \times b = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

Application of the cross product

Question 3

Consider the plane P with parametric vector form

$$x = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} \text{ where } \lambda_1, \lambda_2 \in \mathbb{R}.$$

Is vector c= $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ orthogonal to P?

Application of the cross product

Solution. the normal to plane P is $\mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} \times \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -4 \\ -8 \\ -4 \end{pmatrix}$.

Since $\mathbf{n} = -4 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ which is -4 times the vector c. Hence, c is parallel to the normal of plane p, which implies that c is orthogonal to the plane P.

Distance between a point and a line

Question 4

Find the shortest distance between the point (11, 2, -1) and the line of intersection of the planes

$$x \cdot \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = 0 \text{ and } x = \lambda_1 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ 1 \\ -3 \end{pmatrix}.$$

Distance between a point and a line

Solution. Subs $x = \begin{pmatrix} 2\lambda_1 + 3\lambda_2 \\ \lambda_1 + \lambda_2 \\ 2\lambda_1 - 3\lambda_2 \end{pmatrix}$ into the $x \cdot \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = 0$.

This gives,

$$\begin{pmatrix} 2\lambda_1 + 3\lambda_2 \\ \lambda_1 + \lambda_2 \\ 2\lambda_1 - 3\lambda_2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = 0.$$

$$\begin{aligned} 2\lambda_1 + 3\lambda_2 - \lambda_1 - \lambda_2 + 6\lambda_1 - 9\lambda_2 &= 0 \\ 7\lambda_1 &= 7\lambda_2 \\ \lambda_1 &= \lambda_2 \end{aligned}$$

Question 4 continued

Therefore, the lines of the intersection is $x = \lambda_1 \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} + \lambda_1 \begin{pmatrix} 3 \\ 1 \\ -3 \end{pmatrix} = \lambda_1 \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}$. Let $\overrightarrow{OA}$ be the point (11,2,-1) and $\overrightarrow{v} = \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}$, $\overrightarrow{OA} \cdot \overrightarrow{v} = 60$ and $\overrightarrow{v} \cdot \overrightarrow{v} = 30$. The shortest distance is then

$$\begin{aligned} |\overrightarrow{OA} - proj_{\overrightarrow{v}} \overrightarrow{OA}| = \left| \overrightarrow{OA} - \frac{\mathbf{OA} \cdot \mathbf{v}}{|\mathbf{v}|^2} \mathbf{v} \right| &= \left| \begin{pmatrix} 11 \\ 2 \\ -1 \end{pmatrix} - \frac{60}{30} \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix} \right| \\ &= \left| \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right| \\ &= \sqrt{6}. \end{aligned}$$

Distance between two lines

Question 5 (from 2019T3 MATH1141 Paper)

Let

$$x = \lambda_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \lambda_1 \in \mathbb{R} \text{ and } x = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \lambda_2 \in \mathbb{R}.$$

i) show that the lines are non-parallel and non-intersecting.
ii)Compute the distance between the two lines.

Distance between two lines

Solution.

i) Clearly, the direction vectors of the two lines are not multiples of each other. Therefore, the two lines are non-parallel.

To check if the two lines intersect, we equate the x_1, x_2 and x_3 components. That is,

$$\lambda_1 = 1 + \lambda_2 \tag{1}$$

$$0 = 1 \tag{2}$$

$$\lambda = 0 \tag{3}$$

Distance between two lines

However, from (2), we know that $LHS \neq RHS$. So there is no solution, and hence the 2 lines are non-intersecting.

ii) Here we can take $\mathbf{n} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$, and $\mathbf{a_1} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$,

$\mathbf{a_2} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, so $\mathbf{a_1} - \mathbf{a_2} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$.

The distance between the lines is

$$\begin{aligned} |proj_{\mathbf{n}} \mathbf{a_1} - \mathbf{a_2}| = \left| \frac{\mathbf{n} \cdot (\mathbf{a_1} - \mathbf{a_2})}{|\mathbf{n}|^2} \mathbf{n} \right| &= \left| \frac{\mathbf{n} \cdot (\mathbf{a_1} - \mathbf{a_2})}{|\mathbf{n}|} \right| \\ &= \left| \frac{-2}{2} \right| \\ &= 1. \end{aligned}$$

Complex Numbers

Question 6 (from MATH1141 2019T1 Paper 2iii)

a) Use de Moivre's theorem to express $\sin(5x)$ as a polynomial in $\sin(x)$, that is, find the polynomial P(z) with real coefficients that satisfies the equation

$$P(\sin(x)) = \sin(5x).$$

b) By examining the roots of $P(z)$, find the exact values of $\sin(\frac{2\pi}{5})$ and $\sin(\frac{4\pi}{5})$. Your answer should be expressed in terms of square roots and rational number.

Complex Numbers

Theorem

De Moivre's Theorem *For any real number θ and integer n* $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$.

Solution. a) Consider the expansion of $(\cos x + i\sin x)^5$. We can first apply the De Moivre's theorem to get

$$(\cos(x) + i\sin(x))^5 = \cos(5x) + i\sin(5x).$$

We can also apply binomial theorem to the LHS such that,

$$\begin{aligned}(\cos(x) + i\sin(x))^5 &= \cos^5 x + 5\cos^4 xi\sin(x) + 10\cos^3(x)(i\sin(x))^2 \\ &+10\cos^2 x(i\sin(x))^3 + 5\cos(x)(i\sin(x))^4 + (i\sin(x))^5\end{aligned}$$

Complex Numbers

Continued from previous page

$$(\cos(x) + i\sin(x))^5 = (\cos^5 x - 10\cos^3 x \sin^2 x + 5\cos x \sin^4 x) \\ + i(5\cos^4 x \sin x - 10\cos^2 x \sin^3 x + \sin^5 x)$$

Now, by equating the imaginary parts from both expansions of $(\cos x + i\sin x)^5$, we get

$$\sin(5x) = 5\cos^4 x \sin x - 10\cos^2 x \sin^3 x + \sin^5 x.$$

To express $\sin(5x)$ as a polynomial in $\sin(x)$, substitute $\cos^2 x = 1 - \sin^2 x$ and $\cos^4 x = 1 - 2\sin^2 x + \sin^4 x$ into the equation, which gives

$$\sin(5x) = 16\sin^5 x - 20\sin^3 x + 5\sin x$$

Solution b). let $\sin(5x) = 0$ and $z = \sin(x)$.
From part a), we get $16z^5 - 20z^3 + 5z = 0$
Now, solve for x,

$$5x = n\pi, \text{where n} = 0, 1, 2, 3, 4$$
$$x = 0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}$$

Therefore, the solutions are $z = 0, \sin(\frac{\pi}{5}), \sin(\frac{2\pi}{5}), \sin(\frac{3\pi}{5}), \sin(\frac{4\pi}{5})$. But since $\sin(\frac{\pi}{5}) = \sin(\pi - \frac{4\pi}{5}) = \sin(\frac{4\pi}{5})$ and similarly $\sin(\frac{3\pi}{5}) = \sin(\frac{2\pi}{5})$, we know that $\sin(\frac{2\pi}{5})$ and $\sin(\frac{4\pi}{5})$ are double roots.

Complex Numbers

As $16z^5 - 20z^3 + 5z = z(16z^4 - 20z^2 + 5) = 0$,
we could eliminate the solution $z = 0$, and hence the solutions to $16z^4 - 20z^2 + 5 = 0$ are double roots $\sin(\frac{2\pi}{5})$ and $\sin(\frac{4\pi}{5})$.
Now, find z^2 by applying the quadratic formula,

$$z^2 = \frac{20 \pm \sqrt{(-20)^2 - 4 \cdot 16 \cdot 5}}{32} = \frac{20 \pm \sqrt{80}}{32}$$

Hence $z = \sqrt{\frac{5 \pm \sqrt{5}}{8}}$ and since $\sin(\frac{2\pi}{5})$ $>$$\sin(\frac{4\pi}{5})$,

$$sin(\frac{2\pi}{5}) = \sqrt{\frac{5 + \sqrt{5}}{8}}$$
$$sin(\frac{4\pi}{5}) = \sqrt{\frac{5 - \sqrt{5}}{8}}.$$

Complex numbers

Question 7 (from MATH1141 2013 paper 3ii)

Let $p(z) = z^4 - z^3 - z^2 - z + 2$. Denote the roots of p by a_1, a_2, a_3, a_4 where a_1 is an integer.

a. Find the value of a_1.
b. Given that at least one of the roots of p is not real, deduce how many of the roots are real.
c. By considering the sum of the roots, or otherwise, prove that at least one of the roots has negative real part.
d. Prove that $|a_j| > \frac{1}{2}$ for $j = 1, 2, 3, 4$.

Solution.

a. Since $p(1) = 1 - 1 - 1 - 1 + 2 = 0$, we know that a root of p is 1. Therefore, a_1=1.

b. Since complex roots come in conjugate pair and p has real coefficients, p cannot have one non-real root and 3 real roots or three-non real roots and one real root or 4 non-real roots (given that a_1=1). Therefore, there are two complex roots and two real roots.

c. Assume that a_2 is the second real root, and a_3 and a_4 are conjugate complex roots. Since $a_1 + a_2 + a_3 + a_4 = 1(sum of roots)$ and a_1=1, hence

$$a_2 + a_3 + a_4 = a_2 + 2x = 0$$

where we express x as the real parts of a_3 and a_4. Since 0 is not a root of $p(z)$, we know that that either a_2<0 or x<0. Therefore we can prove that at least one of the roots have a negative real part.

d. Suppose a is a root which satisifies $|a| \leqslant \frac{1}{2}$ then from $p(a)$=0, we get $a^4 - a^3 - a^2 - a = -2$, this gives

$$|a^4 - a^3 - a^2 - a| = 2 \leqslant |a^4| + |a^3| + |a^2| + |a| \leqslant \frac{1}{2^4} + \frac{1}{2^3} + \frac{1}{2^2} + \frac{1}{2} = \frac{15}{16}$$

This is a contradiction. Therefore, $|a_j| > \frac{1}{2}$ for $j = 1, 2, 3, 4$

Reminder

Solubility from row-echelon form

After transforming the augmented matrix for a system of linear equations into row-echelon form $(U|\mathbf{y})$,

1. The system has **no solution** if and only if the right hand column is a leading column.
2. The system has a **unique solution** if and only if every column on the left is a leading column.
3. The system has **infinite** solutions otherwise.

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Question

For some values of the real parameters a, b, c and d, the curve $ax^2 + by^2 + cx + dy = 1$ passes through the points $A(1,1)$, $B(2,3)$ and $C(0,1)$.

1. Explain why the following equations can be used to determine the values of a, b, c and d for which the curve passes through the points.

$$\begin{array}{ccccccccc} a & + & b & + & c & + & d & = & 1 \\ 4a & + & 9b & + & 2c & + & 3d & = & 1 \\ & & b & & & + & d & = & 1. \end{array}$$

Continued

Question

2. Use Gaussian Elimination to solve the system of linear equations in part 1.
3. Are there zero, one, or infinitely many curves of the form $ax^2 + by^2 + cx + dy = 1$ which pass through the points A, B and C?
4. Using your answer from part 2, find the parabola of the form $y = \alpha x^2 + \beta x + \gamma$ which passes through A, B and C.

Solution

1. If the curve passes through $A(1,1)$, $B(2,3)$ and $C(0,1)$, then the coordinates of the points must satisfy the equation. We obtain the set of linear equations by substituting the coordinates of A, B and C into the equation $ax^2 + by^2 + cx + dy = 1$.
2. We can represent the system of linear equations as

$$
\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 4 & 9 & 2 & 3 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{array}\right) \xrightarrow{R_2 = R_2 - 4R_1} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 5 & -2 & -1 & -3 \\ 0 & 1 & 0 & 1 & 1 \end{array}\right)
$$

$$
\xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 5 & -2 & -1 & -3 \end{array}\right)
$$

$$
\xrightarrow{R_3 = R_3 - 5R_2} \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & -2 & -6 & -8 \end{array}\right)
$$

Solution (cont.)

3. There are infinitely many solutions to the system of linear equations, and so there are infinitely many curves of the form $ax^2 + by^2 + cx + dy = 1$ passing through A, B and C.
4. Let $d = 1$. From the matrix, we have

$$\begin{aligned} -2c - 6d &= -8, \\ b + d &= 1 \text{ and} \\ a + b + c + d &= 1 \end{aligned}$$

from which we can deduce, using back substitution, that $c = 1$, $b = 0$ and $a = -1$.
That is, one curve passing through A, B and C is given by $(-1)x^2 + (0)y^2 + (1)x + (1)y = 1$. Hence, the parabola passing through is given by the equation $y = x^2 - x + 1$.

Using the following Maple session, or otherwise, answer the questions below.

```
> with(LinearAlgebra):
> A := < < m, 1, 2 > | < 1, m, 1> | < 1, 1, 4*m > >:
> b := < -m^3-5*m^2-5*m+10, -m^2, -m >:
> M := < A | b >;
```

$$M := \begin{bmatrix} m & 1 & 1 & -m^3-5m^2-5m+10 \\ 1 & m & 1 & -m^2 \\ 2 & 1 & 4m & -m \end{bmatrix}$$

```
> M1 := RowOperation(M, [2, 1]):
> M2 := RowOperation(M1, [2, 1], -m):
> M3 := RowOperation(M2, [3, 1], -2);
```

$$M3 := \begin{bmatrix} 1 & m & 1 & -m^2 \\ 0 & 1-m^2 & 1-m & -5(m+2)(m-1) \\ 0 & 1-2m & 4m-2 & 2m^2-m \end{bmatrix}$$

```
> M4 := simplify(RowOperation(M3, 3, 1/(2*m - 1))):
> M5 := simplify(RowOperation(M4, 2, 1/(1 - m))):
> M6 := RowOperation(M5, [2, 3]):
> M7 := RowOperation(M6, [3, 2], m + 1);
```

$$M7 := \begin{bmatrix} 1 & m & 1 & -m^2 \\ 0 & -1 & 2 & m \\ 0 & 0 & 2m+3 & m^2+6m+10 \end{bmatrix}$$

1141 2020 T1 (cont.)

Question

1. For which real values of m, if any, does the system have no solution?
2. The system has infinitely many solutions when $m = 1$. For which other real value or values of m does the system have infinitely many solutions?
3. For which real value of values of m, if any, does the system have a unique solution?
4. For $m = 1$, the system has solution of the form $\mathbf{x} = \mathbf{a} + \lambda\mathbf{v}$. Find vectors $\mathbf{a}$ and $\mathbf{v}$.

Solution

1. When $m = -\dfrac{3}{2}$ *(the rightmost column becomes a leading column).*
2. When $m = \dfrac{1}{2}$ *(we test this value because we multiplied a row by $\dfrac{1}{2m-1}$, which means the system of linear equations represented by M_7 and M_3 differ when $m = \dfrac{1}{2}$).*
3. All $m \in \mathbb{R}$ where $m \neq 1, \dfrac{1}{2}, -\dfrac{3}{2}$.
4. Letting $m = 1$, we obtain, from M_3, the matrix

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 1 \end{array}\right).$$

4. From the matrix, we have

$$-x_2 + 2x_3 = 1,$$
$$x_1 + x_2 + x_3 = -1.$$

Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ represent the solution. We parameterise the variable x_3 corresponding to the non-leading column. That is, we let $x_3 = \lambda$. Then, $x_2 = 2\lambda - 1$ and $x_1 = -3\lambda$. Hence,

$$\mathbf{x} = \begin{pmatrix} -3\lambda \\ 2\lambda - 1 \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}.$$

That is, $\mathbf{a} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$.

Reminders

Calculating the Determinant of a 3×3 Matrix

Let $A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$

1. $\det(A) = a_1(b_2c_3 - c_2b_3) - a_2(b_1c_3 - c_1b_3) + a_3(b_1c_2 - c_1b_2)$.
2. Suppose that the matrix A consisted of the row vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. The determinant of the matrix is equal to $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$. That is, their scalar triple product.

Reminders (cont.)

Determinants and Solubility

Let A be an $n \times n$ matrix.

1. If $\det(A) \neq 0$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution and the solution is unique for all $\mathbf{b} \in \mathbb{R}^n$.
2. If $\det(A) = 0$, the equation $A\mathbf{x} = \mathbf{b}$ either has no solution or an infinite number of solutions for a given $\mathbf{b}$.

Reminders (cont.)

Some Properties of Determinants

Suppose that A and B are two $n \times n$ matrices. Then,

1. $\det(AB) = \det(A)\det(B)$.
2. A is an invertible matrix if and only if $\det(A) \neq 0$.
3. If a row (or column) of A is multiplied by a scalar, then the value of $\det(A)$ is multiplied by the same scalar. That is, if the matrix B is obtained from the matrix A by multiplying a row (or column) of A by the scalar λ, then $\det(B) = \lambda \det(A)$.

There are many more properties of determinants that are very useful to know.

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Examples

Consider the $n \times n$ matrix B and vector $\mathbf{y} \in \mathbb{R}^n$ such that $B\mathbf{y} \neq \mathbf{0}$ and $B^2\mathbf{y} = \mathbf{0}$.

1. Find a non-zero solution $\mathbf{x} \in \mathbb{R}^n$ to $B\mathbf{x} = \mathbf{0}$.
2. What can be said about $\det(B)$? Give reasons for your answer.
3. Show that the linear system $B^2\mathbf{x} = \mathbf{0}$ has infinitely many solutions.

Solution

1. We note that $B^2\mathbf{y} = B(B\mathbf{y})$. Since $B^2\mathbf{y} = \mathbf{0}$, a non zero solution to $B\mathbf{x} = \mathbf{0}$ would be $B\mathbf{y}$.
2. For a homogeneous system such as $B\mathbf{x} = \mathbf{0}$, the determinant of the matrix B could only be non-zero if the only solution to the system was the zero vector. Clearly, there is a non-zero solution, and so $\det(B) = 0$.
3. We can write $B^2\mathbf{x} = B(B\mathbf{x})$. Hence, if $B\mathbf{x} = \mathbf{0}$, then $B^2\mathbf{x} = \mathbf{0}$, so all solutions of $B\mathbf{x} = \mathbf{0}$ are also solutions of $B^2\mathbf{x} = \mathbf{0}$.
   Now, $\det(B^2) = \det(B)^2 = 0$ so we know that $B^2\mathbf{x} = \mathbf{0}$ either has no solution or an infinite number of solutions.
   Since we know that the equation $B\mathbf{x} = \mathbf{0}$ has more than one solution, the equation $B^2\mathbf{x} = \mathbf{0}$ can't have no solutions – it must have an infinite number of solutions.

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Examples

1. Show that

$$\det\begin{pmatrix}1 & t_1 & t_1^2\\ 1 & t_2 & t_2^2\\ 1 & t_3 & t_3^2\end{pmatrix} = (t_1 - t_2)(t_2 - t_3)(t_3 - t_1).$$

2. Suppose that t_1, t_2, t_3 are distinct real numbers. Prove that for any $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$, there is exactly one polynomial $p(t)$ of degree ≤ 2 with $p(t_i) = \alpha_i$, $i = 1, 2, 3$.

Solution

1. Let A $= \begin{pmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ 1 & t_3 & t_3^2 \end{pmatrix}$. Then,

$$\begin{aligned}
\det(A) &= (t_2t_3^2 - t_2^2t_3) - t_1(t_3^2 - t_2^2) + t_1^2(t_3 - t_2) \\
&= \left(-t_3t_2 + t_1(t_3 + t_2) - t_1^2)\right)(t_2 - t_3) \\
&= (t_1(t_3 - t_1) - t_1(t_3 - t_1))\,(t_2 - t_3) \\
&= (t_1 - t_2)(t_2 - t_3)(t_3 - t_1).
\end{aligned}$$

2. Since t_1, t_2, t_3 are distinct, the $t_1 - t_2, t_2 - t_3, t_3 - t_1$ are non-zero and so $\det(A) \neq 0$. Hence, there is always a unique solution of $\mathbf{x}$ to the equation $A\mathbf{x} = \mathbf{b}$ for any $\mathbf{b} \in \mathbb{R}^n$.

Solution (cont.)

Consider some polynomial p of degree ≤ 2, given by $p(t) = at^2 + bt + c$, where $a, b, c \in \mathbb{R}$. Suppose that $p(t_i) = \alpha_i$ for $i = 1, 2, 3$. Then, we can represent the equations formed as

$$\begin{pmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ 1 & t_3 & t_3^2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$$

From the previous result, we can deduce that the set of coefficients a, b, c satisfying the equation above exists and is unique.
Hence, there is exactly one polynomial of degree ≤ 2 such that $p(t_i) = \alpha_i$ for $i = 1, 2, 3$, as required.

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Question

Let $\mathbf{a}$, $\mathbf{b}$ be two non-zero and non-parallel vectors in $\mathbb{R}^3$.

1. Let A be the matrix with rows $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$. Show that $\det(A) = |\mathbf{a} \times \mathbf{b}|^2$ and hence determine whether or not A is invertible.
2. Let $\mathbf{v} \in \mathbb{R}^3$ be such that $\mathbf{v} \times \mathbf{a} = \mathbf{b}$, $\mathbf{v} \cdot \mathbf{a} = |\mathbf{a}|$, where $\times$ and $\cdot$ represent the cross product and scalar product, respectively. Write $\mathbf{v} = \lambda_1 \mathbf{a} + \lambda_2 \mathbf{b} + \lambda_3 \mathbf{a} \times \mathbf{b}$, $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$. Taking appropriate scalar or cross products of $\mathbf{v}$, or otherwise, find $\lambda_1, \lambda_2, \lambda_3$ and thus find the formula of $\mathbf{v}$ as a linear combination of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$.

Solution

1. The determinant is the scalar triple produce of the 3 row vectors. That is

$$\begin{aligned}\det(A) &= \mathbf{a} \cdot (\mathbf{b} \times (\mathbf{a} \times \mathbf{b})) \\ &= (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{b}) \\ &= |\mathbf{a} \times \mathbf{b}|^2.\end{aligned}$$

   Since $\mathbf{a}$ and $\mathbf{b}$ are non-zero and non-parallel, $|\mathbf{a} \times \mathbf{b}|$ is non-zero, and so $\det(A)$ is non-zero. That is, the matrix A is invertible.

2. Since $\mathbf{v} \times \mathbf{a} = \mathbf{b}$, we observe that $\mathbf{a}$ and $\mathbf{b}$ are orthogonal. This means that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$ are orthogonal, and so $\mathbf{v}$ can be decomposed into orthogonal component vectors by taking the projection of $\mathbf{v}$ onto each of these vectors. Then, $\mathbf{v}$ can be written as the sum of these projections, which will also be a linear combination of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$.

Solution (cont.)

We also note that, since $\mathbf{v}$ and $\mathbf{b}$ are also orthogonal, the projection of $\mathbf{v}$ onto $\mathbf{b}$ is $\mathbf{0}$. Hence, $\lambda_2 = 0$. We can write

$$\begin{aligned}\mathbf{v} &= \mathrm{proj}_{\mathbf{a}}\mathbf{v} + \mathrm{proj}_{\mathbf{a}\times\mathbf{b}}\mathbf{v}\\ &= \frac{\mathbf{v}\cdot\mathbf{a}}{|\mathbf{a}|^2}\mathbf{a} + \frac{\mathbf{v}\cdot(\mathbf{a}\times\mathbf{b})}{|\mathbf{a}\times\mathbf{b}|^2}\mathbf{a}\times\mathbf{b}.\end{aligned}$$

Hence,

$$\lambda_1 = \frac{\mathbf{v}\cdot\mathbf{a}}{|\mathbf{a}|^2} = \frac{|\mathbf{a}|}{|\mathbf{a}|^2} = \frac{1}{|\mathbf{a}|}$$

and

$$\lambda_3 = \frac{(\mathbf{v}\times\mathbf{a})\cdot\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|^2} = \frac{\mathbf{b}\cdot\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|^2} = \frac{|\mathbf{b}|^2}{|\mathbf{a}\times\mathbf{b}|^2}.$$

We can thus conclude that

$$\mathbf{v} = \frac{1}{|\mathbf{a}|}\mathbf{a} + \frac{|\mathbf{b}|^2}{|\mathbf{a}\times\mathbf{b}|^2}\mathbf{a}\times\mathbf{b}.$$

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Examples

In this question, A and B denote invertible $n \times n$ matrices such that $AB = -BA$.

1. Show that n must be even.
2. Suppose that $n = 2$ and that

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

   Show that B^2 is a multiple of the 2×2 identity matrix.

3. Suppose A is as in part 2 and furthermore that B is invertible. Show that there are infinitely many matrices C of the form

$$C = \lambda A + \mu B$$

   such that $C^2 = I$.

Solution

1. We first note that

$$\det(AB) = \det(A)\det(B)$$

and

$$\det(-BA) = (-1)^n \det(BA) = (-1)^n \det(B)\det(A).$$

Since $AB = -BA$, we have

$$\det(A)\det(B) = (-1)^n \det(B)\det(A). \tag{1}$$

Since both A and B are invertible, $\det(A)$ and $\det(B)$ are non-zero, and so $\det(A)\det(B) \neq 0$.
Hence, dividing both sides of (1) by $\det(A)\det(B)$, we see that $1 = (-1)^n$, and so n is even.

Solution (cont.)

2. Suppose $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$. Now

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ -b_{21} & -b_{22} \end{pmatrix}$$

and

$$BA = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} b_{11} & -b_{12} \\ b_{21} & -b_{22} \end{pmatrix}$$

Since $AB = -BA$, we have $b_{11} = -b_{11}$ and $-b_{22} = b_{22}$ and so $b_{11}, b_{22} = 0$. We can write B as $B = \begin{pmatrix} 0 & b_{12} \\ b_{21} & 0 \end{pmatrix}$ and so $B^2 = \begin{pmatrix} b_{21}b_{12} & 0 \\ 0 & b_{21}b_{12} \end{pmatrix}$ which is a multiple of the identity matrix.

Solution (cont.)

3. For any real constants λ and μ, using the distributive property of matrix multiplication

$$\begin{aligned}(\lambda A + \mu B)^2 &= \lambda A \lambda A + \lambda A \mu B + \mu B \lambda A + \mu B \mu B \\ &= \lambda^2 A^2 + \lambda\mu(AB + BA) + \mu^2 B^2 \\ &= \lambda^2 I + \lambda\mu \mathbf{0} + \mu^2 \gamma I \qquad \text{(for some constant } \gamma\text{)} \\ &= (\lambda^2 + \gamma\mu^2) I.\end{aligned}$$

For any γ, there are infinite combinations of λ and μ for which $\lambda^2 + \gamma\mu^2 = 1$. Hence, there are infinitely many matrices C of the form $\lambda A + \mu B$ for which $C^2 = I$.

2. Calculus

Question 1

a) Prove that $f(x) = 1 + x + x^2$ is positive for all real numbers x.

b) By considering cases (or otherwise) prove that $1 + x + x^2 + x^3 + x^4$ is always positive.

c) Generalise the above results.

Solution

a) $f(x) = 1 + x + x^2 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} > 0$

b) Let $f(x) = 1 + x + x^2 + x^3 + x^4$

- When $|x| = 1$:
  $f(1) = 5 > 0$
  $f(-1) = 1 > 0$
- When $|x| < 1$:
  $f(x) = \frac{1}{1-x} > 0$
- When $|x| > 1$:
  $f(x) = \frac{x^5-1}{x-1} > 0$
  - When x is positive, denominator and numerator are both positive.
  - When x is negative, denominator and numerator are both negative.

c) $1 + x + x^2 + ... + x^n$ (same argument as part b)

Definition (ϵ-M Definition of Limit at Infinity)

Suppose that L is a real number and f is a real-valued function defined on some interval (b, ∞). We say that $\lim_{x \to \infty} f(x) = L$ if for every positive real number ϵ, there is a real number M such that if $x > M$ then $|f(x) - L| < \epsilon$.

Question 2 *(MATH1141 Exam, June 2014)*

Use the ϵ-M definition of the limit to prove that:

$$\lim_{x \to \infty} \frac{e^x}{\cosh x} = 2.$$

Solution

Let $L = 2$. Then

$$
\begin{aligned}
|f(x) - L| &= \left| \frac{e^x}{\cosh x} - 2 \right| \\
&= \frac{e^x}{\cosh x} \left| 1 - 2\frac{\cosh x}{e^x} \right| \\
&= \frac{e^x}{\cosh x} \left| 1 - 1 - e^{-2x} \right| && \because \cosh x = \frac{1}{2}(e^x + e^{-x}) \\
&= \frac{e^x}{\cosh x} | - e^{-2x} | \\
&= \frac{e^{-x}}{\cosh x} \\
&\le e^{-x} && \because \cosh x \ge 1
\end{aligned}
$$

Solution (Con't)
Let $\epsilon > 0$. Then $e^{-x} < \epsilon$ if and only if $x > -\ln \epsilon = \ln \epsilon^{-1}$.
Let $M = \ln \epsilon^{-1}$. Thus we have shown that for all $x > M$ then there exists an $\epsilon > 0$ such that $|f(x) - L| < \epsilon$, as required.

Question 3 *(MATH1131 Exam, June 2011)*

Evaluate the limit

$$\lim_{x \to \infty} \frac{1}{x - \sqrt{x^2 - 6x - 4}}.$$

Solution
Rationalising the denominator,

$$\begin{aligned}
&\lim_{x\to\infty} \frac{x+\sqrt{x^2-6x-4}}{(x+\sqrt{x^2-6x-4})(x-\sqrt{x^2-6x-4})} \\
&= \lim_{x\to\infty} \frac{x+\sqrt{x^2-6x-4}}{x^2-(x^2-6x-4)} \\
&= \lim_{x\to\infty} \frac{x+\sqrt{x^2-6x-4}}{6x+4}
\end{aligned}$$

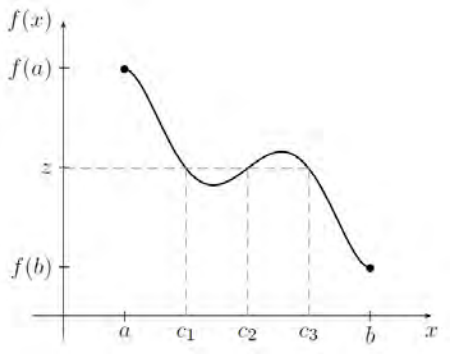
Divide by the highest power of x in the denominator to obtain:

$$\lim_{x\to\infty} \frac{1+\sqrt{1-\frac{6}{x}-\frac{4}{x^2}}}{6+\frac{4}{x}} = \frac{1}{3}.$$

Properties of Continuous Functions

Theorem (The Intermediate Value Theorem)

Suppose that f is continuous on the closed interval $[a, b]$. If z lies between $f(a)$ and $f(b)$ then there is at least one real number c in $[a, b]$ such that $f(c) = z$.



Properties of Continuous Functions

Question 4 *(MATH1131 Exam, November 2010)*

Let $f(x) = x^3 + \sqrt{3}x - 5$ for all real x.

a) Use the **Intermediate Value Theorem** to prove that f has at least one positive real root.

b) By considering f', or otherwise, show that f has only one real root.

Solution

a) f is continuous on the closed interval $[0, 2]$ and $f(0) = -5 < 0$, while $f(2) = 3 + 2\sqrt{3} > 0$. Hence by the **Intermediate Value Theorem**, f has at least one positive real root in the interval $[0, 2]$.

b) Since $f'(x) = 3x^2 + \sqrt{3} > 0$, the function f is increasing. Hence f has exactly one real positive root.

Question 5 *(MATH1141 Exam, June 2011)*

Consider the function f defined by

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

a) Given that $\lim\limits_{x \to \infty} xe^{-x} = 0$, evaluate the limit

$$\lim_{h \to 0} \frac{e^{-\frac{1}{h^2}}}{h}.$$

b) Using the definition of a derivative, determine whether f is differentiable at $x = 0$.

Solution

a)

$$\begin{aligned}\lim_{h\to 0}\frac{e^{-\frac{1}{h^2}}}{h} &= \lim_{h\to 0} h\frac{1}{h^2}e^{-\frac{1}{h^2}}\\ &= \left[\lim_{h\to 0} h\right]\left[\lim_{h\to 0}\frac{1}{h^2}e^{-\frac{1}{h^2}}\right]\end{aligned}$$

Let $x=\frac{1}{h^2}$, when $h\to 0$, $x\to\infty$,

$$\begin{aligned}&\therefore \left[\lim_{h\to 0} h\right]\left[\lim_{h\to 0}\frac{1}{h^2}e^{-\frac{1}{h^2}}\right]\\ &= \left[\lim_{h\to 0} h\right]\left[\lim_{x\to\infty} xe^{-x}\right]\\ &= (0)(0)\\ &= 0\end{aligned}$$

Differentiable Functions

b)

$$\because \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^{-\frac{1}{x^2}} = 0$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} e^{-\frac{1}{x^2}} = 0$$

$$\therefore f(x) \text{ is continuous at } x = 0.$$

The definition of the derivative at $x = 0$ is

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h\to 0} \frac{e^{-\frac{1}{h^2}} - 0}{h} \\ &= 0 \qquad \text{(from part a)} \end{aligned}$$

$$\therefore f(x) \text{ is differentiable at } x = 0.$$

Differentiable Functions

Question 6 *(MATH1141 Exam, June 2011)*

Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^3 & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases}$$

a) Explain why f is differentiable everywhere and determine $f'(x)$.

b) Explain why the function g defined by $g(x) = f'(x)$ is continuous at $x = 0$.

c) Use the definition of the derivative to determine whether g is differentiable at $x = 0$.

Solution

a) Away from 0 f is differentiable.
$\because \lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = 0.$
$\therefore f$ is continuous at $x = 0$.

$\because$ the derivatives of the two constituent functions are equal at $x = 0$
$\therefore$ by the **Split Function Theorem**, f is differentiable at $x = 0$ and hence everywhere.

$$f'(x) = \begin{cases} 3x^2 & \text{if } x < 0 \\ 2x & \text{if } x \geq 0 \end{cases}$$

Differentiable Functions

b) $\because \lim_{x \to 0^+} f'(x) = \lim_{x \to 0^-} f'(x) = 0$
$\therefore f'$ is continuous at $x = 0$.

c) $\lim_{h \to 0^-} \frac{f'(0+h) - f'(0)}{h} = \lim_{h \to 0^-} \frac{3h^2}{h} = 0$
$\lim_{h \to 0^+} \frac{f'(0+h) - f'(0)}{h} = \lim_{h \to 0^+} \frac{2h}{h} = 2$
$\therefore f'$ is not differentiable at $x = 0$.

Mean Value Theorem

Theorem (Mean Value Theorem)

Suppose that f is continuous on $[a, b]$ and differentiable on (a, b). There exists at least one real number c in (a, b) such that:

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Question 7 *(MATH1141 Exam, June 2015)*

Assume that a differentiable function f on $\mathbb{R}$ is such that $f'(x) \leq 1$ for all $x \in \mathbb{R}$. Given that $f(2) = 2$, show that $f(x) \geq x$ for all $x \leq 2$.

Mean Value Theorem

Solution

Let x be a real number, $x \leq 2$. The function f satisfies the requirements of the **Mean Value Theorem** on $[x, 2]$ so

$$\frac{f(2) - f(x)}{2 - x} = f'(c)$$

for some $c \in (2, x)$. Hence

$$\begin{aligned} \frac{2 - f(x)}{2 - x} &\leq 1 \\ 2 - f(x) &\leq 2 - x \\ f(x) &\geq x. \end{aligned}$$

Mean Value Theorem

Theorem (l'Hôpital's Rule)

Suppose that f and g are both differentiable functions and $a \in \mathbb{R}$. Suppose also that either 1 of the 2 following conditions hold:

- $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$;
- $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to a$;

If

$$\lim_{x \to a} \frac{f'(x)}{g'(x)}$$

exists then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Mean Value Theorem

Question 8 *(MATH1141 Exam, June 2015)*

Let f be a continuous function on $\mathbb{R}$ and

$$g(x) = \frac{\int_0^x f(t)dt - xf(0)}{x^2}$$

Use L'Hôpital's rule to show that if $f'(0)$ exists then

$$\lim_{x \to 0} g(x) = \frac{f'(0)}{2}.$$

Mean Value Theorem

Solution

$$\lim_{x\to 0}\left[\int_0^x f(t)dt - xf(0)\right] = 0 \qquad \lim_{x\to 0}(x^2) = 0.$$

Then by L'hôpital's Theorem

$$\begin{aligned} g(x) &= \lim_{x\to 0}\frac{\frac{d}{dx}\left[\int_0^x f(t)dt - xf(0)\right]}{\frac{d}{dx}(x^2)} \\ &= \lim_{x\to 0}\frac{f(x)-f(0)}{2x} \\ &= \lim_{x\to 0}\frac{f'(x)}{2} \\ &= \frac{f'(0)}{2} \end{aligned}$$

Inverse Functions

Theorem (Inverse Functions and One-to-oneness)

Suppose that f is a one-to-one function, then the inverse of f will be an unique function, f^{-1}, where the range(f^{-1}) = domain(f)

Theorem (Inverse Function Theorem)

Suppose that I is an open interval, $f : I \to \mathbb{R}$ is differentiable and $f'(x) \neq 0$ for all x in I. Then:

- f is one-to-one and has an inverse function, g: range(f) $\to$ domain(f)
- g is differentiable at all points in range(f) and
- the derivative of g is given by the formula

$$g'(x) = \frac{1}{f'(g(x))}$$

for all x in range (f)

Inverse Functions

Alternative form of the Inverse Function Theorem
For a particular value of x (i.e. c), we have that:

$$(f^{-1})'(c) = \frac{1}{f'(f^{-1}(c))}$$

Question 1 *(MATH1131 Exam, June 2011)*

The function f has domain $[0, 1]$ and is defined by $f(x) = e^x + ax$ where a is a postive constant.

a) Prove that 2 is in the range of f.

b) Prove that f has an inverse function f^{-1}

c) Find the domain of f^{-1}

d) If the constant a was 1, find $(f^{-1})'(e + 1)$

Solution

a)

1. f is a continuous function
2. $f(0) = 1$
3. $f(1) = e + a$, and as a is a positive constant, $f(1) > e$

Therefore, via the Intermediate Value Theorem, 2 lies in the range of f as 2 lies between 1 and e

b)

Taking the derivative of $f(x) = e^x + ax$, we get $f'(x) = e^x + a$. As $f'(x)$ is always positive, the f is monotonically increasing. Thus, as f is continuous and monotonically increasing, it is a one-to-one function and thus has an inverse function

Solution Continued

c)

As previously calculated in part a), $f(0) = 1$ and $f(1) = e + a$. Thus as the domain of f^{-1} is equivalent to the range of f, the answer is $[1, e + a]$.

d)

If the constant a is 1, then $f(x) = e^x + x$
Via the Alternate form of the Inverse Function Theorem, we have that in this case $c = e + 1$.
Thus the corresponding $f^{-1}(c) = f^{-1}(e + 1) = 1$.
And as $f'(x) = e^x + 1$,
Then, $\frac{1}{f'(f^{-1}(c))} = \frac{1}{e^1+1} = \frac{1}{e+1}$

Question 2

a)

$$sin^{-1}(cos(\frac{3\pi}{4}))$$

b)

$$sin(cos^{-1}(\frac{3}{5}))$$

Curve Sketching

Definition (Oblique Asymptotes)

Suppose that a and b are real numbers and that $a \neq 0$. We say that a straight line, given by the equation

$$y = ax + b,$$

is an oblique asymptote for a function f if

$$\lim_{n \to \infty} (f(x) - (ax + b)) = 0$$

Curve Sketching

Question 1

Sketch $\frac{3x^2-4}{x+2}$

Solution

1. Find intersections with the axes:
   When $x = 0$, $y = -2$.
   When $y = 0$, $x = \frac{\pm 2}{\sqrt{3}}$
2. Find vertical asymptote:
   This occurs when the denominator equals 0.
   Thus, the vertical asymptote exists at $x = -2$

Solution Continued

3. Find oblique asymptote:

   $\frac{3x^2-4}{x+2} = \frac{3x^2+6x-6x-12+8}{x+2}$

   Then, we have $3x-6+\frac{8}{x+2}$,
   Thus as we take $x \to \infty$, the oblique asymptote becomes $3x-6$

4. Then graph

Definition (Polar Coordinates)

Let P be every point in a plane
The pair of parameters (r, θ) defines the distance of the point P from the origin and the angle between OP and the positive horizontal axis respectively.

The Cartesian coordinates are defined as

$$x = rcos(\theta), y = rsin(\theta)$$

Curve Sketching

Question 2 *(MATH1131 Exam, November 2010)*

A curve in $\mathbb{R}^2$ is given in polar coordinates as

$$r = 6sin(\theta)$$

where $0 \leq \theta \leq \frac{\pi}{2}$

a) Express the equation of the curve using Cartesian coordinates and state the range of x and the range of y.

b) Hence, or otherwise, sketch the curve in the xy-plane

Solution

a) $(x, y) = (6sin(\theta)cos(\theta), 6sin(\theta)sin(\theta)),$

So, $x = 6sin(\theta)cos(\theta), y = 6sin^2(\theta),$

So, $sin(\theta) = \frac{x}{6cos(\theta)},$

Subbing into y we get, $y = \frac{6x^2}{36cos^2(\theta)} = \frac{x^2}{6(1-sin^2(\theta))},$

And as $y = 6sin^2(\theta)$, we have that $sin^2(\theta) = \frac{y}{6},$

We have that $y = \frac{x^2}{1-\frac{y}{6}},$

After rearranging, we obtain $x^2 + y^2 - 6y = 0.$

By completing the square we get, $x^2 + (y - 3)^2 = 9.$

Solution Continued

a) As θ ranges between 0 and $\frac{\pi}{2}$, only the first quadrant section of the graph exists,
Thus, the range of x is $[0, 3]$ and the range of y is $[0, 6]$

b) Then graph

Integration

Theorem (The First Fundamental Theorem of Calculus)

If f is continuous function defined on $[a, b]$, then the function $F : [a, b] \to \mathbb{R}$, defined by

$$F(x) = \int_a^x f(x)\,dx$$

is continuous on $[a, b]$, differentiable on (a, b) and has derivative F' given by

$$F'(x) = f(x)$$

for all x in (a, b).

Integration

Theorem (The Second Fundamental Theorem of Calculus)

Suppose that f is a continuous function on $[a, b]$. If F is an antiderivative of f on $[a, b]$ then,

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

Question 1 *(MATH1131 Exam, Semester 1 2014)*

Use the Fundamental Theorem of Calculus to find

$$\frac{d}{dx}\int_{x^2}^{x^3} cos(\frac{1}{t})\,dt.$$

Solution

Let's first define the anti-derivative of $cos(\frac{1}{t})$ as $F(t)$ such that $F'(t) = cos(\frac{1}{t})$ (First Fundamental Theorem of Calculus).

Via the Second Fundamental Theorem of Calculus, we have that $\int_{x^2}^{x^3} cos(\frac{1}{t})\, dt = F(x^3) - F(x^2)$.

So now we have to evaluate $\frac{d}{dx}[F(x^3) - F(x^2)]$.

Using the Chain Rule, we have that $\frac{d}{dx}[F(x^3) - F(x^2)] = 3x^2 F'(x^3) - 2xF'(x^2)$.

Now, as $F'(t) = cos(\frac{1}{t})$, we conclude that $\frac{d}{dx} \int_{x^2}^{x^3} cos(\frac{1}{t})\, dt = 3x^2 cos(\frac{1}{x^3}) - 2x cos(\frac{1}{x^2})$

Integration by Parts

$$\int uv' = uv - \int vu'.$$

Question 2

Evaluate $\int e^x sinx\, dx$

Integration

Solution
First define: $I = \int e^x sinx\, dx$

Then, preparing for integration by parts we have that:

$$u = sinx \rightarrow u' = cosx$$

$$v' = e^x \rightarrow v = e^x$$

So, $I = e^x sinx - \int e^x cosx\, dx$.

Preparing for integration by parts a second time we have that:

$$u = cosx \rightarrow u' = -sinx$$

$$v' = e^x \rightarrow v = e^x$$

So, $I = e^x sinx - (e^x cosx - \int e^x(-sinx))\, dx$.

Solution continued

As $I = \int e^x sinx\, dx$,

$I = e^x sinx - (e^x cosx + I)$.

So, $I = e^x sinx - e^x cosx - I$.

So, $2I = e^x(sinx - cosx)$.

So ultimately we have that, $I = \frac{e^x(sinx - cosx)}{2} + C$.

Integration

The comparison test

Suppose that f and g are integrable functions and that $0 \leq f(x) \leq g(x)$ whenever $x > a$.

(i) If $\int_a^\infty g(x)\,dx$ converges then $\int_a^\infty f(x)\,dx$ converges.

(ii) If $\int_a^\infty f(x)\,dx$ diverges then $\int_a^\infty g(x)\,dx$ diverges.

Question 3

(MATH1131 Exam, Semester 1 2014)
Determine, with reasons, whether the improper integral

$$K = \int_0^\infty \frac{dx}{e^{2x} + cos^2 x}$$

converges or diverges.

Solution

In this question, our $f(x) = \frac{1}{e^{2x}+cos^2x}$ and we have to determine an appropiate $g(x)$.

As cos^2x lies between 0 and 1, $\frac{1}{e^{2x}+cos^2x} \leq \frac{1}{e^{2x}}$,

It would be appropriate to choose $g(x) = \frac{1}{e^{2x}}$.

We can then use the p-convergence test on $\int_0^\infty g(x)\,dx$ to check whether it converges.

As $e^{2x} > x^1$ for $1 \leq x$, we have that $\int_0^\infty g(x)\,dx$ does indeed converge.

So ultimately, as $\int_0^\infty g(x)\,dx$ converges, then K converges as well via the comparison test.

Log and Exponentials

Question 1 *(MATH1131 Exam, November 2010)*

Use logarithmic differentiation to calculate $\frac{dy}{dx}$ for $y = (sinx)^x$

Solution

By logging both sides, we get $ln(y) = xln(sinx)$.

Then by implicit differentiation, we get $\frac{1}{y}\frac{dy}{dx} = ln(sinx)(1) + (x)(\frac{cosx}{sinx})$.

Thus, $\frac{dy}{dx} = yln(sinx) + yxcotx$.

And as $y = (sinx)^x$,

$\frac{dy}{dx} = (sinx)^x(ln(sinx) + xcotx)$

Hyperbolic Functions

Definition (Hyperbolic Cosine)

The hyperbolic cosine function $cosh : \mathbb{R} \to \mathbb{R}$ is defined by the formula

$$coshx = \frac{1}{2}(e^x + e^{-x}) \quad \forall x \in \mathbb{R}$$

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Question 1 *(MATH1131 Exam, June 2011)*

a) Give the definition of $cosh x$

b) Use the definition to prove that

$$4cosh^3 x = cosh 3x + 3cosh x$$

Solution

By the definition of $cosh x$, we have that $4cosh^3 x = 4(\frac{1}{2}(e^x + e^{-x}))^3$

So, $LHS = 4(\frac{1}{8}(e^{3x} + 3e^{2x}e^{-x} + 3e^x e^{-2x} + e^{-3x}))$

$LHS = \frac{1}{2}(e^{3x} + 3e^x + 3e^{-x} + e^{-3x})$

$LHS = \frac{1}{2}[(e^{3x} + e^{-3x}) + 3(e^x + e^{-x})]$

$LHS = \frac{1}{2}(e^{3x} + e^{-3x}) + 3(\frac{1}{2}(e^x + e^{-x}))$

$LHS = cosh 3x + 3cosh x$

Thus $LHS = RHS$