(Higher) Mathematics 1A

Calculus

Presented by: Joanna Lin, Raymond Li

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Limits

Finding Limits with Basic Operators

Basic Rules with Operators

If $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to\infty} g(x)$ are finite real numbers, then the following rules hold:

$$\lim_{x\to\infty}[f(x)+g(x)] = \lim_{x\to\infty} f(x) + \lim_{x\to\infty} g(x)$$

$$\lim_{x\to\infty}[f(x)-g(x)] = \lim_{x\to\infty} f(x) - \lim_{x\to\infty} g(x)$$

$$\lim_{x\to\infty}[f(x)\times g(x)] = \lim_{x\to\infty} f(x) \times \lim_{x\to\infty} g(x)$$

$$\lim_{x\to\infty}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim\limits_{x\to\infty} f(x)}{\lim\limits_{x\to\infty} g(x)}, \ \lim_{x\to\infty} g(x) \neq 0$$

Note for limits at a point: If $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$ are finite real numbers, then simply replace the above ∞'s with a's and the same rules hold.

Finding Limits at Infinity with the Pinching Theorem

The Pinching Theorem

Suppose that f, g and h are defined on the interval (b, ∞), and

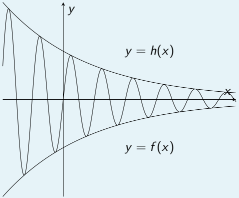
$$f(x) \leq g(x) \leq h(x)$$

if

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} h(x) = L$$

then

$$\lim_{x \to \infty} g(x) = L.$$



Intuitive graphical visualisation

MATH1131 June 2013 Q4(ii)

Use the pinching theorem to evaluate $\lim_{x \to \infty} e^{-x} \sin(x)$.

Limits of Indeterminate Form: '$\frac{\infty}{\infty}$', '$\frac{0}{0}$' or '$\infty - \infty$'

Dividing by the Fastest Growing Term

To calculate limits of the form $\lim_{x\to\infty} \dfrac{f(x)}{g(x)}$, $f(x)$ and $g(x)$ tend to infinity as $x \to \infty$, we divide the numerator and denominator by the fastest growing term in the denominator, which is usually the highest power of variable.

Rationalising the Numerator/Denominator

To calculate limits of the form $\lim_{x\to\infty} \left[\sqrt{f(x)} - \sqrt{g(x)}\right]$, we can multiply the expression by $\frac{\sqrt{f(x)}+\sqrt{g(x)}}{\sqrt{f(x)}+\sqrt{g(x)}}$. The signs between the square roots are reversed as to obtain the difference of two squares in the numerator.

L'Hôpital's Rule (Chapter 5)

L'Hôpital's Rule

Suppose f and g are both differentiable functions, a is a real number, and that, as $x \to a$ or ∞, one of the following conditions hold:

1. $f(x) \to 0$ and $g(x) \to 0$ or
2. $f(x) \to \infty$ and $g(x) \to \infty$

If $\lim_{x\to a} \dfrac{f'(x)}{g'(x)}$ exists then $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \lim_{x\to a} \dfrac{f'(x)}{g'(x)}$.

Tip with L'Hopital's rule

If $\lim_{x\to a} \dfrac{f'(x)}{g'(x)}$ is of indeterminate form, you can apply the rule many times until you find the limit (if it exists). Don't confuse limits of indeterminate form and limits that do not exist.

Finding Limits of Indeterminate Form

MATH1131 June 2012 Q1(i)(Modified)

Evaluate the limit:

$$\lim_{x\to\infty} \frac{10x^2 + 3x + \sin x}{5x^2 + 3x - 2}.$$

MATH1131 June 2011 Q1(iv)

Evaluate the limit

$$\lim_{x\to\infty} \frac{1}{x - \sqrt{x^2 - 6x - 4}}.$$

MATH1131 November 2010 Q3(i)

Evaluate the limit

$$\lim_{x\to 0} \frac{x^2 e^x}{1 - \cos \pi x}.$$

Defining Limits at a Infinity

Definition of Limit at Infinity

If L is a real number and f is a real valued function defined on an interval (b, ∞), then

$$\lim_{x \to \infty} f(x) = L$$

means that there exists an M for every positive ϵ such that, if $x > M$, then $|f(x) - L| < \epsilon$.

In other words, you prove that the limit is L if you find an **expression for** M **in terms of** ϵ, where M is defined for every positive ϵ.

Applying Definition of Limits at Infinity

Value of Limit vs Definition of Limit

This definition doesn't tell you the value of the limit itself. You can only find the value of the limit using the methods shown previously. You then can prove that this value is indeed the limit using the definition.

In the following example, the value of the limit is already given, leaving only the proof.

MATH 1141 June 2014

Use the ϵ-M definition of the limit to prove that

$$\lim_{x \to \infty} \frac{e^x}{\cosh x} = 2.$$

Recall that $\cosh x = \dfrac{e^x + e^{-x}}{2}$.

Determining Whether a Limit at a Point Exists

Required Conditions for the Existence of a Two-sided Limit

If both the left-hand limit $\lim\limits_{x \to a^-} f(x)$ and right-hand limit $\lim\limits_{x \to a^+} f(x)$ exists and is equal to the same number L, then the two-sided-limit $\lim\limits_{x \to a} f(x)$ exists and is equal to L.

That is, if $\lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x) = L$, then $\lim\limits_{x \to a} f(x) = L$.

MATH1131 June 2015 Q1 (ii)(c) (Modified)

The function f is defined by $f(x) = \begin{cases} 3 - x & 0 \leq x < 1 \\ (x-2)^2 + 1 & 1 \leq x \leq 3. \end{cases}$

Does $\lim\limits_{x \to 1} f(x)$ exist? Give brief reasons for your answer.

Recall the conditions for the existence of the two-sided limit.

Defining Continuity at a Point

Definition of Continuity at a Point

Suppose that f is defined on some open interval containing the point a. If $\lim_{x \to a} f(x) = f(a)$, then f is continuous at a; otherwise, f is discontinuous at a.

Note: The converse is also true: if f is continuous at a, then $\lim_{x \to a} f(x) = f(a)$.

Also **note**: $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = \lim_{x \to a} f(x) = f(a)$

'Continuous Everywhere'

If $f : \mathbb{R} \to \mathbb{R}$ is continuous at every point a in $\mathbb{R}$, then we say that f is continuous everywhere.

Note: In this definition, the domain is **all real numbers**, and so it has the property that its graph can be drawn on the Cartesian plane as an unbroken curve.

Applying Continuity at a Point

MATH1131 June 2014 Q1(iv)

A function g is defined by

$$g(x) = \begin{cases} \dfrac{|x^2 - 16|}{x-4} & \text{if } x \neq 4 \\ \alpha & \text{if } x = 4. \end{cases}$$

By considering the left and right hand limits at $x = 4$, show that no value of α can make g continuous at the point $x = 4$.

Properties of Continuous Functions

Combining Continuous Functions

Recall from the previous section that a function is defined as continuous at $x = a$ if $\lim_{x \to a} f(x) = f(a)$. From this, we can derive the following properties, using the existing properties of limits.

Dealing with Basic Operators (again)

$$\lim_{x \to a}[f(x) + g(x)] = f(a) + g(a)$$

$$\lim_{x \to a}[f(x) - g(x)] = f(a) - g(a)$$

$$\lim_{x \to a}[f(x) \times g(x)] = f(a) \times g(a)$$

$$\lim_{x \to a}\left[\frac{f(x)}{g(x)}\right] = \frac{f(a)}{g(a)}, \lim_{x \to a} g(a) \neq 0$$

Note: This also indicates that, if two continuous functions are added, subtracted, multiplied or divided, then the resulting function is also continuous.

Defining Continuity on Intervals

Continuity on Open Intervals

Suppose that f is a real-valued function defined on an open interval (a, b). f is a continuous on (a, b) if f is continuous at every point in the interval (a, b).

Continuity on Closed Intervals

Suppose that f is a real-valued function defined on a closed interval $[a, b]$. Then:

f is continuous at the endpoint a if $\lim\limits_{x \to a^+} f(x) = f(a)$.

f is continuous at the endpoint b if $\lim\limits_{x \to b^-} f(x) = f(b)$.

f is continuous on the closed interval $[a, b]$ if f is continuous on the open interval (a, b) and at the endpoints a and b.

Continuous Function vs Continuity Over an Interval

There's a Difference

Saying that a function is continuous means that it is continuous at every defined point in its domain. This doesn't necessarily mean that it is defined at every point in the interval.

For example, the function f, where $f(x) = \frac{1}{x}$, is continuous.

However, continuity on an interval requires that the function is **defined everywhere on the interval**.

The Intermediate Value Theorem

Intermediate Value Theorem (IVT)

Suppose f is continuous on the interval $[a, b]$. If $f(a) < z < f(b)$, then there is at least one real number c in $[a, b]$ such that $f(c) = z$.

An Issue with the IVT

The Intermediate Value Theorem does not guarantee that the value of c is unique.
When a problem requires proof that the value of c is unique, you must resort to other methods as well.

The Sign of a Derivative (Chapter 5)

Increasing/Decreasing Functions

Suppose f is continuous on $[a, b]$ and differentiable on (a, b).

1. If $f'(x) > 0$ for all x in (a, b) then f is increasing on $[a, b]$
2. If $f'(x) < 0$ for all x in (a, b) then f is decreasing on $[a, b]$
3. If $f'(x) = 0$ for all x in (a, b) then f is constant on $[a, b]$

The conditions for f cannot be relaxed

For example, even if you prove that $f'(x) = 0$ for all values in the domain of f, this doesn't mean that the function is the same constant value on the whole domain.
It means that f is constant over an interval on which it is continuous and differentiable, but different intervals may give different constant values. Consider $\tan^{-1}\frac{1}{x} + \tan^{-1}x$.

Applications of the IVT

Proving that Solutions Exist

Show that the equation

$$e^{-7x} = -2\cos(16x)$$

has a unique solution for $x \in [0, \frac{\pi}{16}]$.

To solve:
Firstly, we will use the IVT to prove that the solution exists. Then, we prove that this solution must be unique. To do this, we can prove that the function is either monotonically increasing or decreasing.

The Maximum-Minimum Theorem

Local vs Global Maxima/Minima

Global Maximum: $f(c)$ is the global/absolute maximum value of f on $[a, b]$ if $f(c) \geq f(x)$ for all x in $[a, b]$.
Global Minimum: $f(d)$ is the global/absolute minimum value of f on $[a, b]$ if $f(d) \leq f(x)$ for all x in $[a, b]$.

Maximum-Minimum Theorem

If f is **continuous** over the **closed** interval $[a, b]$, then **it is guaranteed that** f attains a minimum and maximum value on $[a, b]$. That is, there exists points c and d in $[a, b]$ such that

$$f(c) \leq f(x) \leq f(d)$$

for all $x \in [a, b]$

Applying the Maximum-Minimum Theorem

MATH1131 June 2012: Q2(vi)

Consider the three functions:

$$f : \mathbb{R} \to \mathbb{R} \qquad f(x) = \frac{x^2}{1+x^2}$$

$$g : (0,3) \to \mathbb{R} \qquad g(x) = (x-1)^2$$

$$h : [1,5] \to \mathbb{R} \qquad h(x) = \sqrt{1 + \ln x + \sin x \cos x}$$

Only one of these functions has a maximum value on its given domain. Which one is it? Give reasons for your answer.

To justify: Ensure that the function satisfies the criteria for applying the maximum-minimum theorem: the function must be **continuous** and the given domain/interval must be **closed**.

Differentiable Functions

Defining 'Differentiable'

Differentiation by First Principles

The derivative, or gradient function f' of function f, is given by:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Definition of 'Differentiable'

Suppose f is defined on some open interval containing x. f is differentiable at x if $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ exists.

In other words, you must prove, for the value of x in question, that

$$\lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h}$$

By this definition of differentiability, it follows that the function is also continuous at x.

Determining Whether Function is Differentiable at a Point

MATH1141 June 2011 Q2(iv)

Consider the function f defined by

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

1. Given that $\lim_{x \to \infty} xe^{-x} = 0$, evaluate the limit
$$\lim_{h \to 0} \frac{e^{-\frac{1}{h^2}}}{h}.$$
2. Using the definition of a derivative, determine whether f is differentiable at $x = 0$.

Much Easier Approach

Theorem for Piecewise-defined Functions

If a is a fixed real number and the function f is defined by

$$f(x) = \begin{cases} p(x) & x \geq a \\ q(x) & x < a \end{cases}$$

where $p(x)$ and $q(x)$ are continuous and differentiable in some open interval containing a. Then, if f is continuous at a and $p'(a) = q'(a)$, then f is differentiable at $x = a$.

Proceed with caution

If the question says 'using the definition of a derivative' (or something similar), do not use this approach.

Determining Conditions for Differentiability

MATH1131 June 2012 Q4(i) (Modified)

Given that the function h, defined by

$$h(x) = \begin{cases} e^{3x}, & x \leq 0 \\ q(x), & x > 0 \end{cases}$$

is differentiable at $x = 0$, and q is a monic quadratic function, find the expression for $q(x)$.

To solve: Use the definition of differentiability or theorem on previous slide.

When Using First Principles...

Look out for the 'split' formula at $x = 0$, and ensure that you are substituting values into the correct expression when using first principles to find $q(x)$.

Rules For Differentiation

We should all be familiar with these by now:

Rules for Differentiation

If f and g are differentiable at x, then $f+g$, $f-g$ and fg are also differentiable at x. If $g(x) \neq 0$, then $\frac{f}{g}$ is also differentiable at x.

$$(f \pm g)'(x) = f'(x) \pm g'(x)$$

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x) \qquad \text{(Product Rule)}$$

$$\left(\frac{f}{g}\right)(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} \qquad \text{(Quotient Rule)}$$

If g is differentiable at x and f is differentiable at $g(x)$, then $f \circ g$ is differentiable at x

$$(f \circ g)'(x) = f'(g(x))g'(x) \qquad \text{(Chain Rule)}$$

Implicit Differentiation

Implicit differentiation utilises all the rules of differentiation we have covered so far, and is best demonstrated by an example.

MATH1131 June 2013 Q2(iii)

Find the equation of the tangent at the origin to the curve implicitly defined by

$$e^x + \sin(y) = xy + 1$$

Defining Local Maxima/Minima

Definition of Local Maxima/Minima

Suppose that f is defined on an interval I. Point c in I is a

1. **local minimum point** if there is a positive number h such that $f(c) \leq f(x)$ whenever $x \in (c-h, c+h)$ and $x \in I$. In other words, $f(c)$ is the smallest value within the vicinity of c.
2. **local maximum point** if there is a postive number h such that $f(c) \geq f(x)$ whenever $x \in (c-h, c+h)$ and $x \in I$. In other words, $f(c)$ is the largest value within the vicinity of c.

Finding Local Maxima/Minima

Those are the definitions, but to find these local maxima/minima, we will use the following theory:

Theory for Finding Local Maxima/Minima

Suppose f is defined on (a,b) and has a local maximum/minimum at $c \in (a, b)$. If f is differentiable at c then $f'(c) = 0$.

The Converse Isn't Necessarily True

If $f'(c) = 0$, then it doesn't necessarily mean that a local maximum/minimum point is achieved! We must perform additional tests, demonstrated in the next example.

Using the Second Derivative (Chapter 5)

Classifying Nature of Stationary Points

Suppose that a function f can be differentiated twice (twice differentiable) on (a, b) and that $f'(c) = 0$, where $c \in (a, b)$.

1. If $f''(c) > 0$ (the curve representing function has upward concavity) then c is a local minimum;
2. If $f''(c) < 0$ (the curve representing function has downward concavity) then c is a local maximum.

When $f'(c) = f''(c) = 0$

In this case, you must determine the nature of the stationary point by examining the sign of the sign table. Consider $f(x) = x^4$ at $x = 0$.

How to Find Global Maxima/Minima (Chapter 5)

Critical Points

Suppose that f is defined on $[a, b]$. Point c in $[a, b]$ is a critical point for f on [a,b] if:

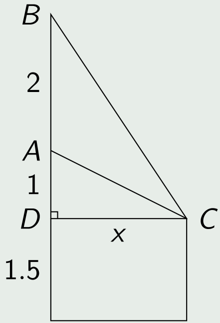
1. c is an endpoint a or b of the interval $[a, b]$,
2. f is not differentiable at c (e.g. a cusp), or
3. f is differentiable at c and $f'(c) = 0$.

To find a global maximum/minimum, you simply test the values at these points.

Applying Maxima/Minima

MATH1131 June 2011 Q4(iv) (Modified)

A statue 2 metres high stands on a pillar 2.5 metres high. A person, whose eye is 1.5m above the ground, stands at a distance x metres from the base of the pillar. The diagram shows the above information, with the person's eye being at C. Let $\angle BCA = \theta$ and $\angle ACD = \phi$.



1. Prove that $\dfrac{d}{dt}(\cot^{-1} t) = -\dfrac{1}{1+t^2}$.
2. Show that $\theta = \cot^{-1} \dfrac{x}{3} - \cot^{-1} x$.
3. Hence, find the distance x that maximise the angle θ.

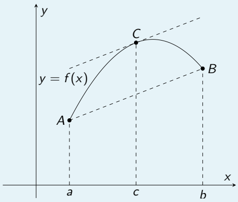
The Mean Value Theorem and Applications

The Mean Value Theorem

The Mean Value Theorem

If f is continuous on $[a, b]$ and differentiable on (a, b), then there is at least on real number c in (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$



This means that you can find a tangent in this interval with a gradient that is equal to the chord joining the two endpoints.

MATH1131 June 2014 Q4(v)(b)

Suppose $-1 < x < y < 1$. By applying the Mean Value Theorem to the function $f(t) = \sin^{-1} t$ on the interval $[x, y]$, prove that

$$\sin^{-1} y - \sin^{-1} x \geq y - x.$$

Inverse Functions (1141)

One-to-one functions

Definition

A function f is one-to-one if

$$f(x) = f(y) \text{ implies } x = y \text{ whenever } x, y \in \mathrm{Dom}(f)$$

Usually the easiest way to prove a function is one-to-one is to show $f'(x) \geq 0$ or $f'(x) \leq 0$ for all $x \in \mathrm{Dom}(f)$

Inverse

Definition

If f is a one-to-one function, then there exists a function g such that

$$f(g(x)) = x \quad \text{for all } x \in \text{Dom}(g)$$

$$g(f(x)) = x \quad \text{for all } x \in \text{Dom}(f)$$

where $\text{Dom}(f) = \text{Range}(g)$ and $\text{Range}(f) = \text{Dom}(g)$
g is the inverse function of f.

Example

1131 2011 Q2(vi)

The function f has domain [0,1] and is defined by $f(x) = e^x + ax$, where a is a positive constant.

1. Prove that 2 is in the range of f.
2. Prove that f has an inverse function f^{-1}.
3. Find the domain of f^{-1}.

Solution

1. Since $f(0) = 1$ and $f(1) = e + a > 2$, by IVT, there exists $x \in [0, 1]$ such that $f(x) = 2$ i.e. 2 is in the range of f
2. $f'(x) = e^x + a > 0$ for all $x \in [0, 1]$. Since f is increasing on the domain, the inverse function f^{-1} exists.
3. Since f is an increasing function on $[0, 1]$,
   $\text{Range}(f) = [f(0), f(1)] = [1, e + a]$
   $\text{Dom}\left(f^{-1}\right) = \text{Range}(f) = [1, e + a]$

Derivative of the Inverse

Derivative of the Inverse

If g is the inverse of f then the derivative of g is given by

$$g'(x) = \frac{1}{f'(g(x))}$$

Example

1141 2012 Q2(i)

Consider the function $f : (0, 2\sqrt{\pi}] \to \mathbb{R}$ defined by

$$f(x) = x^2 + \cos(x^2)$$

1. Find all the critical points of f and determine their nature.
2. Explain why f is invertible, state the domain of f^{-1} and find $f^{-1}(5\pi/2)$.
3. Where is f^{-1} differentiable?

Solution

1. $f'(x) = 2x - 2x\sin\left(x^2\right)$
   Setting $f'(x) = 0$ for critical points
   $\left(\sqrt{\frac{\pi}{2}}, \frac{\pi}{2}\right), \left(\sqrt{\frac{5\pi}{2}}, \frac{5\pi}{2}\right)$
   Both critical points are horizontal inflection points.
2. Since f is monotonically increasing f is invertible.
   $Dom(f^{-1}) = Range(f) = (f(0), f(2\sqrt{\pi})] = (1, 4\pi + 1]$
   $f^{-1}(5\pi/2) = \sqrt{5\pi/2}$
3. Derivative exists when $f'(f^{-1}(x))$ exists and is nonzero.
   Hence differentiable on $(1, 4\pi + 1)$ except at $x = \frac{\pi}{2}, \frac{5\pi}{2}$

Curve Sketching

Parametrically defined curves

Parametric Equations

The coordinates of each point on a curve can be represented as a function of the parameter t.

$$(x(t), y(t)) \text{ for } t \in I$$

1. For simple examples, the parameter can be eliminated to obtain an explicit equation $y = f(x)$. This cannot always be done.
2. Parametrisations are not unique. For example, (t, t) and $(2t, 2t)$ where $t \in \mathbb{R}$ both describe the line $y = x$
3. $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

Polar Curves

Polar Curves

A curve can be represented as $r = f(\theta)$ where $r = \sqrt{x^2 + y^2}$ and $\tan\theta = \frac{y}{x}$ where $\theta \in [0, 2\pi)$. Equivalently, $x = r\cos\theta, y = r\sin\theta$

Example

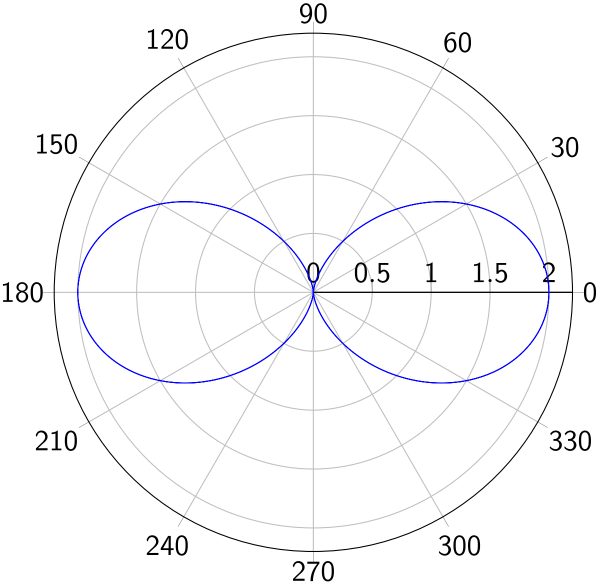
1131 2013 Q2(iv)

Sketch the polar curve whose equation in polar coordinates is given by $r = 1 + \cos 2\theta$

Solution

The plot can be parametrised as
$(r\cos\theta, r\sin\theta) = ((1+\cos 2\theta)\cos\theta, (1+\cos 2\theta)\sin\theta)$
Key features:
$\theta = 0, r = 2$
$\theta = \frac{\pi}{4}, r = 1$
$\theta = \frac{\pi}{2}, r = 0$
Other parts of the graph can be determined using symmetry of cosine function and periodic nature.

Polar Plot



Integration

Fundamental Theorem of Calculus

Theorem

First fundamental theorem of calculus: If f is continuous on $[a, b]$ then for all $x \in (a, b)$

$$F(x) = \int_a^x f(t)\,\mathrm{d}t$$

is continuous on $[a, b]$, differentiable on (a, b) and

$$F'(x) = f(x)$$

Second fundamental theorem of calculus: Suppose that f is continuous on $[a, b]$. If F is an antiderivative of F on $[a, b]$ then

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$

Example

1131 2011 Q4(ii)(b)

Find $\dfrac{d}{dx}\displaystyle\int_0^{\sinh x}\frac{\cos t}{1+t^2}\,\mathrm{d}t$

Let $u = \sinh x$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{\sinh x}\frac{\cos t}{1+t^2}\,\mathrm{d}t &= \frac{d}{dx}\int_0^{u}\frac{\cos t}{1+t^2}\,\mathrm{d}t \\
&= \frac{\mathrm{d}}{\mathrm{d}u}\left(\int_0^{u}\frac{\cos t}{1+t^2}\,\mathrm{d}t\right)\frac{\mathrm{d}u}{\mathrm{d}x} \\
&= \frac{\cos u}{1+u^2}\cdot\frac{\mathrm{d}u}{\mathrm{d}x} \\
&= \frac{\cos(\sinh x)}{1+\sinh^2 x}\cosh x \\
&= \frac{\cos(\sinh x)}{\cosh^2 x}\cosh x \\
&= \frac{\cos(\sinh x)}{\cosh x}
\end{aligned}$$

Integration by Parts

Integration by Parts

$$\int u \, dv = uv - \int v \, du$$

Choosing u and v

u should be chosen as the expression that becomes simpler when differentiated and v becomes simpler when integrated.

1131 2014 Q2(iii)

$$\int_0^{\frac{\pi}{3}} x \sin(2x) \, dx$$

Solution

Choose $u = x$ and $\mathrm{d}v = \sin(2x)\mathrm{d}x$
$\mathrm{d}u = \mathrm{d}x$ and $v = -\frac{1}{2}\cos(2x)$

$$\begin{aligned}
I &= \int_0^{\frac{\pi}{3}} x\sin(2x)\,\mathrm{d}x \\
&= \left[-\frac{1}{2}x\cos(2x)\right]_0^{\pi/3} + \frac{1}{2}\int_0^{\frac{\pi}{3}} \cos(2x)\,\mathrm{d}x \\
&= \frac{\pi}{12} + \frac{1}{4}\left[\sin(2x)\right]_0^{\pi/3} \\
&= \frac{\pi}{12} + \frac{\sqrt{3}}{8}
\end{aligned}$$

Improper Integrals

Definition

If there is a real number L such that

$$\lim_{R\to\infty}\int_a^R f(x)\,\mathrm{d}x = L$$

then $\int_a^\infty f(x)\,\mathrm{d}x$ is convergent and $\int_a^\infty f(x)\,\mathrm{d}x = L$. If such a limit does not exist, the integral is divergent.

If both limits of integration are infinity, the integral must first be broken up into two improper integrals.

$$\int_{-\infty}^\infty f(x)\,\mathrm{d}x = \int_0^\infty f(x)\,\mathrm{d}x + \int_{-\infty}^0 f(x)\,\mathrm{d}x$$

$\int_{-\infty}^\infty f(x)\,\mathrm{d}x$ converges if both improper integrals converge.

Comparison tests

The comparison test

Suppose f and g are integrable functions and $0 \leq f(x) \leq g(x)$ whenever $x > a$.

$$\text{If } \int_a^\infty g(x)\,dx \text{ converges, then } \int_a^\infty f(x)\,dx \text{ converges}$$

$$\text{If } \int_a^\infty f(x)\,dx \text{ diverges, then } \int_a^\infty g(x)\,dx \text{ diverges}$$

p integral

Usually comparisons are made with the integral $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ which converges if $p > 1$ and diverges for $p \leq 1$

Limit form of Comparison test

Theorem

Suppose f and g are non-negative and bounded on $[a, \infty]$. If

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = L$$

and $0 < L < \infty$ then $\int_a^\infty f(x)\,\mathrm{d}x$ and $\int_a^\infty g(x)\,\mathrm{d}x$ either both converge or both diverge.

Example

1131 2010 Q4 (ii)

Determine whether the improper integral

$$K = \int_1^\infty \frac{1 + \sin x}{3x^2}\,\mathrm{d}x$$

converges or diverges.

Solution

$$\begin{aligned} K &= \int_1^\infty \frac{1+\sin x}{3x^2}\,\mathrm{d}x \\ &< \int_1^\infty \frac{1+1}{3x^2}\,\mathrm{d}x \\ &= \frac{2}{3}\int_1^\infty \frac{1}{x^2}\,\mathrm{d}x \end{aligned}$$

which converges. Hence by the comparison test, K also converges.

Logarithmic and Exponential Functions

Logarithmic differentiation

Method

When differentiating expressions with a function of x as the exponent can be simplified by taking the logarithm.

1131 2010 Q3 (v)

Use logarithmic differentiation to calculate $\frac{dy}{dx}$ for $y = (\sin x)^x$

Solution

$\ln y = \ln(\sin x)^x = x \ln(\sin x)$

Differentiating both sides with respect to x

$\dfrac{1}{y}\dfrac{\mathrm{d}y}{\mathrm{d}x} = \ln(\sin x) + x\dfrac{\cos x}{\sin x}$

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = y\left(\ln(\sin x) + x\dfrac{\cos x}{\sin x}\right) = (\sin x)^x \cdot \left(\ln(\sin x) + x\dfrac{\cos x}{\sin x}\right)$

Indeterminate forms with powers

Other indeterminate forms

Limits of the form $0^0, 0^\infty$ and ∞^0 can found by first manipulating the expression with the exponential and logarithmic functions

$$a^b = e^{\log(a^b)} = e^{b\log(a)}$$

followed by the use of l'Hopital's Rule.

Hyperbolic Functions

Hyperbolic functions

Definitions

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\text{cosech } x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

$$\text{sech } x = \frac{2}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

$$\coth x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

Properties

Derivative of hyperbolic functions

$$\frac{d}{dx}\sinh x = \cosh x$$

$$\frac{d}{dx}\cosh x = \sinh x$$

$$\frac{d}{dx}\tanh x = \text{sech}^2 x$$

Identities

$$\cosh^2 x - \sinh^2 x = 1$$

Other identities can be proven from definition of the hyperbolic trig functions.

Example

1131 2011 Q4(i)(b)

Use the definition of $\cosh x$ to prove that

$$4\cosh^3 x = \cosh 3x + 3\cosh x$$

Solution

$$
\begin{aligned}
4\cosh^3 x &= 4\left(\frac{e^x + e^{-x}}{2}\right)^3 \\
&= 4\left(\frac{e^{3x} + 3e^x + 3e^{-x} + e^{-3x}}{8}\right) \\
&= \frac{e^{3x} + e^{-3x}}{2} + 3\frac{e^x + e^{-x}}{2} \\
&= \cosh 3x + 3\cosh x
\end{aligned}
$$