UNSW Mathematics Society

Engineering Mathematics 2D/2E

Seminar II / II

Presented by: Oscar Kamensky, Janzen Choi

Term 1, 2020

Seminar Overview

Part V: ODEs

Introduction to ODEs

ODEs are functions defined by the relation between themselves and their derivatives. As such, it is common for their derivatives to in some way resemble the original function. The function types that allow this most easily are sine/cosine functions and exponentials.

Examples

- $y' + y = 0.$
- $\dfrac{dy}{dx} + y = 0.$
- $y'' + y' + y = 0.$

Introduction to ODEs

Types of ODEs

1. **First order ODE**: highest derivative order is 1. Three forms:
   - *separable*
   - *linear*
   - *substitution*
2. **Second order ODE**: highest derivative order is 2. Two forms:
   - *homogeneous*
   - *non-homogeneous*

Introduction to ODEs

Types of ODEs

1. **First order ODE**: highest derivative order is 1. Three forms:
    - *separable*
    - *linear*
    - *substitution*
2. **Second order ODE**: highest derivative order is 2. Two forms:
    - *homogeneous*
    - *non-homogeneous*

Form of ODE

$$y' = P(x)Q(y).$$

Introduction to ODEs

Types of ODEs

1. **First order ODE**: highest derivative order is 1. Three forms:
   - *separable*
   - *linear*
   - *substitution*
2. **Second order ODE**: highest derivative order is 2. Two forms:
   - *homogeneous*
   - *non-homogeneous*

Form of ODE

$$y' + P(x)y = Q(x).$$

Introduction to ODEs

Types of ODEs

1. **First order ODE**: highest derivative order is 1. Three forms:
   - *separable*
   - *linear*
   - *substitution*
2. **Second order ODE**: highest derivative order is 2. Two forms:
   - *homogeneous*
   - *non-homogeneous*

Form of ODE

$$P(x)y' = Q(x) + G(y).$$

Introduction to ODEs

Types of ODEs

1. **First order ODE**: highest derivative order is 1. Three forms:
   - *separable*
   - *linear*
   - *substitution*
2. **Second order ODE**: highest derivative order is 2. Two forms:
   - *homogeneous*
   - *non-homogeneous*

Form of ODE

$$ay'' + by' + cy = 0.$$

Introduction to ODEs

Types of ODEs

1. **First order ODE**: highest derivative order is 1. Three forms:
   - *separable*
   - *linear*
   - *substitution*
2. **Second order ODE**: highest derivative order is 2. Two forms:
   - *homogeneous*
   - *non-homogeneous*

Form of ODE

$$ay'' + by' + cy' = f(x),$$
$$y'' + P(x)y' + Q(x)y = g(x).$$

First order ODE I: Separable ODE

Method of solution

1. Separate variables so both sides contain one variable.
2. Integrate both sides with respect to said variable.
3. Rearrange for y as a function of x (or any equivalent form).

Example

Solve $\sec^2(x)\tan(y) + \dfrac{dy}{dx}\sec^2(y)\tan(x) = 0$.

First order ODE II: Linear ODE

Method of solution

$$\frac{dy}{dx} + P(x)y = Q(x).$$

1. Identify expressions for $P(x)$ and $Q(x)$.
2. Create the **integrating factor** given by

$$R(x) = e^{\int P(x)\,dx}$$

3. Solution is of the form

$$y(x) = \frac{1}{R(x)} \int R(x)Q(x)\,dx.$$

First order ODE III: Substitution ODE

Method of solution

$$P(x)\frac{dy}{dx} = Q(x) + G(y).$$

1. Substitution is given in the question (use $v = f(x, y)$).
2. ODE becomes
$$\frac{dy}{dx} = \frac{Q(x)}{P(x)} + \frac{G(y)}{P(x)}.$$
3. Manipulate v to get expressions similar to $Q(x)/P(x)$ and $G(y)/P(x)$.
4. Result becomes either a separable or linear ODE.

First order ODE III: Substitution ODE

Example: (16s2, Q2a)

Solve

$$\frac{dy}{dx} = (x+y)^2$$

using the substitution $v = y + x$.

Second order ODE I: Homogeneous

Method of solution

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0.$$

1. Obtain a **characteristic polynomial**: $a\lambda^2 + b\lambda + c = 0$.
2. Solve the quadratic for values of λ.
   - If the values of λ are **distinct**, the solution is of the form $y(x) = c_1 \cdot e^{\lambda_1 x} + c_2 \cdot e^{\lambda_2 x}$.
   - If the value of λ is **repeated**, the solution is of the form $y(x) = c_1 \cdot e^{\lambda x} + c_2 \cdot xe^{\lambda x}$.
   - If the values of λ are **complex** (let $\lambda = \alpha + i\beta$), then the solution is of the form $y(x) = e^{\alpha x}\left[c_1 \cos(\beta x) + c_2 \sin(\beta x)\right]$.

Second order ODE I: Homogeneous

Example: (paper)

Compare the similar cases

1. $y'' - 2y' + 3y = 0.$
2. $y'' - 2y' + y = 0.$
3. $y'' - 2y' - y = 0.$

Second order ODE II: Non-homogeneous (undetermined coefficients)

Method of solution

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x).$$

1. Solve the homogeneous differential equation to find the form of the **homogeneous solution** $y_H(x)$.
2. Guess a suitable form for $y_p(x)$.
   - If $f(x)$ has an *exponential term*, include it in the assumed form. For example: if $f(x) = 3e^{3x}$, then guess Ae^{3x}.
   - If $f(x)$ has a *polynomial term*, include **all lower degree terms** of the polynomial.
   - If $f(x)$ has a *sinusoidal* term, include a sum of **both** sin and cos with the same frequency but different amplitudes.

Second order ODE II: Non-homogeneous (undetermined coefficients)

Method of solution

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x).$$

3. Use your particular solution guess and substitute it into the differential equation and equate it to $f(x)$.
4. Find common terms between each side and obtain a system of equations in terms of the undetermined coefficients.
5. Solution is the sum of its homogeneous and particular solution: $y(x) = y_H(x) + y_p(x)$.

Second order ODE II: Non-homogeneous (undetermined coefficients)

Example: (16S2, Q2b)

Solve

$$y'' + 3y' + 2y = e^{-2t} + 4t^2 + 2$$

by method of undetermined coefficients and describe the long-term steady state solution.

Second order ODE III: Non-homogeneous (variation of parameters)

Method of solution

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x).$$

1. Solve the homogeneous differential equation to find the form of the **homogeneous solution** $y_H(x)$.
   - The homogeneous solution will be of the form

$$y_H(x) = c_1 \cdot y_1(x) + c_2 \cdot y_2(x).$$

2. Evaluate the **Wronskian**, which is of the form

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}.$$

Second order ODE III: Non-homogeneous (variation of parameters)

Method of solution

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x).$$

3. The **particular solution** is then of the form

$$y_p(x) = -y_1(x)\int \frac{y_2(x)f(x)}{W(x)}\,dx + y_2(x)\int \frac{y_1(x)f(x)}{W(x)}\,dx.$$

4. The solution is the sum of its homogeneous and particular solution: $y(x) = y_H(x) + y_p(x)$.

Second order ODE III: Non-homogeneous (variation of parameters)

Example: (paper)

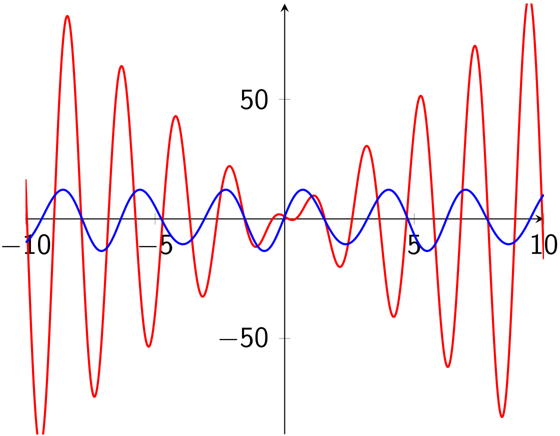
Use the method of variation of parameters to find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = \frac{e^x}{x^3}.$$

Resonance of solutions

- Sometimes, the components of the particular solution will have an overlap with components of the homogenous solution. Were this to happen in a physical system, the amplitude of vibration would increase to uncontainable levels and thus destroy itself (e.g. Tacoma Bridge).
- To accurately show this, in instances where the frequency of the particular solution would be the same as part of the homogeneous solution, we instead multiply the particular solution by x (or t, if f is in terms of that instead), so that the solution is unbounded (i.e: amplitude of oscillation increases with time).

Resonance of solutions



Part VI: Laplace transforms

Introduction to Laplace Transforms

A **Laplace transform** is a method of transferring a function in a *real* domain to a *complex* domain. For MATH2018/2019, it is usually defined as transforming from the **time** domain to the **frequency** domain.

The Laplace transform of any function $f(t)$ is given by:

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\, dt = F(s).$$

- Very useful for discontinuous graphs, including step functions.
- The Laplace transformation is **linear**, so the additive and distributive properties hold.

$\implies \quad \mathcal{L}(\lambda f(t) + \mu g(t)) = \lambda \mathcal{L}(f(t)) + \mu \mathcal{L}(g(t)).$

Introduction to Laplace Transforms

Note: they will provide you a sheet in the exams that contains all of the fundamental Laplace transformations; it's best to focus on how variations on those input change the output!

Laplace Transforms and derivatives of functions

If the Laplace Transformation of a function $f(t)$ exists and is given by $F(s)$, then

$$\mathcal{L}(f'(t)) = sF(s) - f(0)$$
$$\mathcal{L}(f''(t)) = s^2F(s) - sf(0) - f'(0)$$

Step function

- The **Heaviside** (or unit step) function corresponds to a function that is 0 for $t < 0$ and 1 for $t > 0$.
- The exact value at $t = 0$ is a little disputed but generally accepted to be $\frac{1}{2}$.
- Incredibly useful in inverse Laplace transformations and shifting functions in the time domain.

The Laplace transformation of the unit step function is

$$\mathcal{L}(u(t-a)) = \frac{e^{-cs}}{s}.$$

Inverse Laplace transformations I

Converse to Laplace transforms, if we are given a function in terms of s, we can deduce what the original function in terms of t was.

- Best to look for components in the function that resemble transformations in your table and look for instances of the shifting theorems:
  - **First shifting theorem**
  - **Second shifting theorem**

Inverse Laplace transformations II (First shifting theorem)

First shifting theorem

$$\mathcal{L}(e^{-at}f(t)) = F(s+a).$$

- If all given terms of s can be expressed in the form of $(s \pm a)$ in the initial function, it indicates we can use the **first shifting theorem**.
- We can rewrite each of those terms simply as s, complete the inverse transformation of this modified function, then multiply resultant function in terms of t by $e^{\mp at}$ to obtain the true inverse transformation.

Inverse Laplace transformations II (First shifting theorem)

Example: (08S1, Q2aii)

Find $\mathcal{L}^{-1}\left(\dfrac{s+1}{s^2+4s+5}\right)$.

Inverse Laplace transformations III (Second shifting theorem)

Second shifting theorem

$$\mathcal{L}\{f(t-c)\cdot u(t-c)\} = e^{-cs}F(s).$$

- If we are given a function that can be expressed as $e^{\pm cs}F(s)$, it indicates that we can use the **second shifting theorem**.
- We can remove the multiplying factor e^{-cs} from it, then complete the inverse transformation of the modified function.
- Multiply the function in terms of t by $u(t-c)$, then replace every other instance of t in the function by $t \mp c$ to obtain the true inverse transformation.

Inverse Laplace transformations II (Second shifting theorem)

Example: (08S2, Q2aiii)

Find $\mathcal{L}^{-1}\left(\dfrac{e^{-2s}}{3s^4}\right)$.

Miscellaneous transformation tips I

- If you have an instance where one of the terms in the function indicates that a theorem could be used, you can add and subtract or multiply and divide terms to make it fit, as long as you ensure the end result is the same as the original.

Example:

1. $F(s) = \dfrac{s}{(s+1)^2+4}$.
2. $P(s) = e^{-3s}\left(\dfrac{s}{(s+1)^2+4}\right)$.

Miscellaneous transformation tips II

$$F(s) = \frac{s}{(s+1)^2+4}$$
$$= \left(\frac{s}{(s+1)^2+4} + \frac{1}{(s+1)^2+4}\right) - \frac{1}{(s+1)^2+4}.$$

From the **first shifting theorem**, consider a function of similar form $G(s+1) = F(s)$. Thus $G(s) = \frac{s}{s^2+4} - \frac{1}{s^2+4}$. Set

$$G(s) = \frac{s}{s^2+2^2} - \frac{1}{2}\left(\frac{2}{s^2+2^2}\right).$$

Thus $\mathcal{L}^{-1}\left[G(s)\right] = \cos(2t) - \frac{1}{2}\sin(2t)$. Thus

$$\mathcal{L}^{-1}\left[F(s)\right] = e^{-1\cdot t}\left[\cos(2t) - \frac{1}{2}\sin(2t)\right].$$

Miscellaneous transformation tips III

$$P(s) = e^{-3s}\left(\frac{s}{(s+1)^2+4}\right).$$

This is of the form $P(s) = e^{-cs} \cdot F(s)$ where $F(s) = \dfrac{s}{(s+1)^2+4}$.
Thus, we apply the **second shifting theorem**, removing the exponential term, and find the inverse transform of $F(s)$. Note that

$$\mathcal{L}^{-1}\left[F(s)\right] = e^{-t}\left[\cos(2t) - \frac{1}{2}\sin(2t)\right].$$

Add in the result of applying the **second shifting theorem**, which is

$$\begin{aligned}\mathcal{L}^{-1}\left[P(s)\right] &= u(t-c)\cdot f(t-c) \\ &= u(t-c)\cdot e^{-(t-c)}\left[\cos(2(t-c)) - \frac{1}{2}\sin(2(t-c))\right].\end{aligned}$$

A final example

a) Find

i) $\mathcal{L}(te^{-t}\sin 3t)$.

ii) $\mathcal{L}^{-1}\left\{\dfrac{s+1}{s^2+4s+5}+\dfrac{e^{-2s}}{3s^4}\right\}$.

b) The function $f(t)$ is defined for $t \geq 0$ by

$$f(t)=\begin{cases}1 & 0 \leq t \leq 1\\ t-2 & 1 < t \leq 2\\ 0 & t > 2.\end{cases}$$

Express $f(t)$ in terms of the Heaviside function and hence or otherwise find $\mathcal{L}(f(t))$, the Laplace transform of $f(t)$.

A final example

c) Use the Laplace transform method to solve the differential equation

$$y'' - 4y' + 4y = e^{2t}, \quad t > 0$$

subject to the initial condition $y(0) = 1,\ y'(0) = 0.$

Part VII: Fourier series

Introduction to Fourier series

The **Fourier transform** describes the process of approximating a **periodic function** using a series of trigonometric functions. Fourier transforms are defined by the following equations.

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\, dx.$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\, dx.$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\, dx.$$

n represents the **number of iterations** for the series, and L represents the **half period** of the function.

Even and Odd integrals

The **Fourier transformation process** evidently involves the evaluation of *integrals* – the process can be made simpler by using the arithmetic properties of even and odd integrals.

Properties of even and odd integrals

$$\int_{-L}^{L} f(x)\,dx = \begin{cases} 2\displaystyle\int_{0}^{L} f(x)\,dx & \text{if } f(x) \text{ is even} \\ 0 & \text{if } f(x) \text{ is odd} \end{cases}$$

- If a function f is **even**, then $f(x) = f(-x)$.
- If a function f is **odd**, then $f(x) = -f(-x)$.

Function arithmetic I

The integrals are often comprised of sums/differences or products/quotients of different functions.

The sum/differences of two functions:

- The sum/difference of **two even** functions is **even**.
- The sum/difference of **two odd** functions is **odd**.
- The sum/difference of an **even** and **odd** function is **neither** even nor odd.

$+/-$	even	odd
even	even	neither
odd	neither	odd

Function arithmetic II

The integrals are often comprised of sums/differences or products/quotients of different functions.

The product/quotient of two functions:

- The product/quotient of **two even** functions is **even**.
- The product/quotient of **two odd** functions is **even**.
- The product/quotient of an **even** and **odd** function is **odd**.

$\times/\div$	even	odd
even	even	odd
odd	odd	even

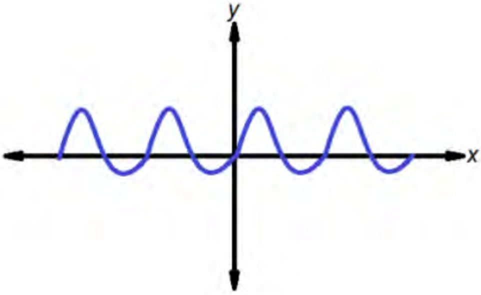
Periodic Extension I

Since the Fourier transform can only be used for periodic functions, non-periodic functions must be artificially extended via periodic extension. There are three ways that a function can be periodically extended:

- Normal periodic extension.
- Even periodic extension.
- Odd periodic extension.

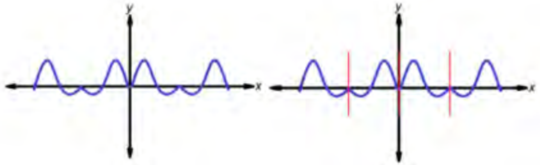
Periodic Extension II (Normal periodic extension)

Normal periodic extension will cause the function to repeat every L distance such that $f(x) = f(x + L)$. In such a case, f is **neither** odd nor even.



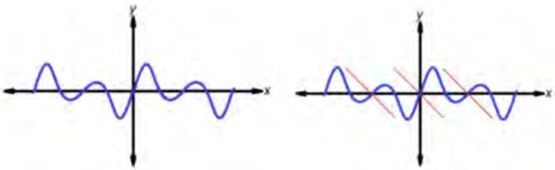
Periodic Extension III (Even periodic extension)

Even periodic extension will cause the function to reflect against the vertical plane every L distance, such that $f(x) = f(L - x)$. In such a case, f is **even**.



Periodic Extension IV (Odd periodic extension)

Odd periodic extension will cause the function to reflect against a diagonal plane every L distance, such that $f(x) = -f(L-x)$. In such a case, f is **odd**.



Applying periodic extensions I

The cos and sin functions are **even** and **odd** respectively.

Fourier series for even periodic extensions (f is even)

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{L}\int_0^L f(x)\,dx,$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx,$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx = 0.$$

Applying periodic extensions II

The cos and sin functions are **even** and **odd** respectively.

Fourier series for odd periodic extensions (f is odd)

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx = 0,$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) = 0,$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right)\, dx = \frac{2}{L} \int_{0}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right)\, dx.$$

Convergence value I

To find the **value** that the Fourier series converges to for a certain value of $x = x_0$, use:

$$\text{value} = \frac{f(x_0^+) + f(x_0^-)}{2}.$$

Note: x_0^+ denotes a value *slightly* greater than x_0 and x_0^- denotes a value *slightly* smaller than x_0. The values of $f(x_0^+)$ and $f(x_0^-)$ can usually be found by either

- looking at the **function definition** or
- looking at the **graph of the function**.

Example: MATH2019 14S1 Q3ii

Normal periodic extension

Let

$$f(x) = \begin{cases} 1 & 0 < x < \pi \\ 0 & \pi \leq x \leq 2\pi \\ f(x + 2\pi) & \text{otherwise} \end{cases}$$

- Sketch the graph of $y = f(x)$ over the interval $-2\pi \leq x \leq 2\pi$.

Example: MATH2019 tutorial question 109

Even and Odd periodic extension

For the function g given by

$$g(x) = \begin{cases} 1, & 0 < x < 1 \\ 4 - 2x, & 1 \leq x \leq 2, \end{cases}$$

- sketch over $(-10, 10)$ the graph of the function represented by the half-range Fourier **cosine** series.
- sketch over $(-10, 10)$ the graph of the function represented by the half-range Fourier **sine** series.

Example: MATH2019 15S1 Q4b

Periodic extensions and convergence

Describe the piecewise continuous function f by

$$f(x) = \begin{cases} 1, & 0 \leq x \leq \frac{\pi}{2} \\ 0, & \frac{\pi}{2} \leq x < \pi \end{cases}$$

- Show that the Fourier cosine series of f is given by

$$f(x) = \frac{1}{2} + \sum_{k=0}^{\infty} \frac{2(-1)^k}{\pi(2k+1)} \cos\left[(2k+1)\pi\right].$$

- To what value will the Fourier series converge at $x = \dfrac{\pi}{2}$?

A final example (MATH2019 10S2, Q3b)

The function f is given by

$$f(x) = \begin{cases} -x & \text{for } -\pi \le x \le 0 \\ x & \text{for } 0 \le x \le \pi \end{cases}$$

with $f(x + 2\pi) = f(x)$ for all x.

1. Make a sketch of this function for $-4\pi \le x \le 4\pi$.
2. Is $f(x)$ odd, even or neither?
3. Find the Fourier series of $f(x)$.
4. By considering the value at $x = \pi$ in your answer for the Fourier series in iii), find the sum of the series

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$

Part VIII: PDEs

Introduction to Partial Differential Equations

Partial Differential Equations (PDEs) are differential equations involving *multiple variables*.

Initial and boundary conditions

- PDEs involve (temporal) initial conditions.
  $\implies \quad t = 0.$
- PDEs also involve (spatial) boundary conditions.
  $\implies \quad x = 0$ and $x = L$ where $0 \leq x \leq L$.

- MATH2018/2019 only deals with second order PDEs with two variables
  $\implies \quad$ (e.g. $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$)

D'Alembert's Solution I

One dimensional-wave PDE

$$\textbf{PDE}: \quad \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

- D'Alembert's solution is a **general solution** for one-dimensional wave equations, expressed as

$$u(x,t) = \phi(x+ct) + \psi(x-ct)$$

  where ϕ and ψ are **arbitrary functions**.
- Will be asked to express the equation in arbitrary functions if we are not given any initial/boundary conditions.

D'Alembert's Solution II

One dimensional-wave PDE

$$\textbf{PDE}: \quad \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

- Given the initial/boundary conditions, D'Alembert's solution can be expressed as

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\, d\xi$$

  where f represents the initial displacement and g represents the initial velocity of the system.

- Corresponding conditions:

$\Longrightarrow \quad f(x) = u(x,0).$

$\Longrightarrow \quad g(x) = \dfrac{d}{dx}u(x,0).$

Method of solution – Separation of Variables I

$$\textbf{PDE}: \quad \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

- Assume that the solution is separable; that is

$$u(x,t) = F(x) \cdot G(t).$$

- Substitute the solution into the original PDE; we get

$$F(x) \cdot G''(t) = c^2 F''(x) G(t).$$

- Rearrange the expression and equate it to a constant k;

$$\frac{G''(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)} = k.$$

Hence, $F''(x) - kF(x) = 0$ and $G''(t) - kc^2 G(t) = 0$.

Method of solution – Separation of Variables II

$$\textbf{PDE}: \quad \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

- To find a suitable value of k, substitute different values of k into $F''(x) - kF(x) = 0$.
  - Case 1: $k > 0$; set $k = p^2$ for some $p > 0$.
  - Case 2: $k = 0$.
  - Case 3: $k < 0$; set $k = -p^2$ for some $p > 0$.
- For each case, find the characteristic equation and solve for associated λ.

$$\implies \lambda^2 - k = 0$$
$$\implies \lambda = \pm\sqrt{k}.$$

Method of solution – Separation of Variables III

$$\textbf{PDE}: \quad \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

- Depending on the case, F will have a different particular solution (A and B are **constant coefficients**):
    - Case 1: $F(x) = A \cdot e^{px} + B \cdot e^{-px}$.
    - Case 2: $F(x) = Ax + B$.
    - Case 3: $F(x) = A \cdot \cos(px) + B \cdot \sin(px)$.
- Consider the cases until we reach a non trivial (nonzero) solution for $F(x)$ – call it $F_n(x)$.
    - To determine the triviality, apply the boundary conditions.
    - If $F(x) \neq 0$, then the solution is **nontrivial** – replace p with an expression of n and set $F_n(x) = F(x)$.

Method of solution – Separation of Variables IV

$$\textbf{PDE}: \quad \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

- Once a nontrivial solution for $F(x)$ is found, use that value of k to find the characteristic equation of $G''(t) - kc^2 G(t) = 0$ to acquire a solution for $G(t)$. Replace p with an expression of n and set $G_n(t) = G(t)$.
- After acquiring $F_n(x)$ and $G_n(t)$, express $u_n(x,t)$ as

$$u_n(x,t) = F_n(x) G_n(t).$$

Hence, the solution of the PDE can be expressed as

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} F_n(x) G_n(t).$$

Procedures for Other PDEs

- For PDEs in the form $\dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ (e.g. 1D Heat equation):
    - Instead of solving $G''(t) - kc^2 G(t) = 0$ for $G_n(t)$, solve $G'(t) - kc^2 G(t) = 0$.
    - Solution is usually an **exponent**.
- For PDEs in the form $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ (e.g. Laplace systems):
    - Rearrange the PDE into the form $\dfrac{\partial^2 u}{\partial x^2} = -\dfrac{\partial^2 u}{\partial y^2}$ and set $c^2 = -1$.

Example: 19T1, Q114

D'Alembert's Solution

Consider the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} - 4\frac{\partial^2 u}{\partial y^2} = 0.$$

- Use D'Alembert's method to find a solution in terms of arbitrary functions.
- Determine the particular solution satisfying $u(x, 0) = 0$ and $u_y(x, 0) = 8\sin(2x)$.

Example: 08S1 Q4b

The steady-state distribution of heat in a slab of width L is given by

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0, \quad \text{for } 0 < x < L,\ 0 < y < \infty$$

$$\begin{aligned} &U(0, y) = U(L, y) = 0, && \text{for } 0 < y < \infty \\ &U \text{ bounded} && \text{as } y \to +\infty \\ &U(x, 0) = f(x), && \text{for } 0 \le x \le L. \end{aligned}$$

Use the method of separation of variables to find the general solution $U(x, y)$, where any unknown constants are related to $U(x, 0) = f(x)$. You must explicitly consider all possibilities for the separation constant in your working.

Example: 16S2 Q4ii)

A stretched wire satisfies the wave equation

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2},$$

where $u(x,t)$ is the displacement of the wire. The ends of the wire are held fixed so that

$$u(0,t) = u(\pi,t) = 0, \quad \text{for all } t.$$

- Assuming a solution of the form $u(x,t) = F(x)G(t)$ show that

$$\frac{G''(t)}{4G(t)} = \frac{F''(x)}{F(x)} = k$$

  for some constant k.

Example: 16S2 Q4ii)

A stretched wire satisfies the wave equation

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2},$$

where $u(x,t)$ is the displacement of the wire. The ends of the wire are held fixed so that

$$u(0,t) = u(\pi,t) = 0, \quad \text{for all } t.$$

- Apply the boundary conditions to show that possible solutions for $F(x)$ are

$$F_n(x) = B_n \sin(nx)$$

  where B_n are constants and $n = 1, 2, 3\cdots$. You must consider all possible values of k.

Example: 16S2 Q4ii)

A stretched wire satisfies the wave equation

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2},$$

where $u(x,t)$ is the displacement of the wire. The ends of the wire are held fixed so that

$$u(0,t) = u(\pi,t) = 0, \quad \text{for all } t.$$

- Find all possible solutions $G_n(t)$ for $G(t)$.
- If the initial displacement and velocity of the wire are

$$u(x,0) = 3\sin(x) + 4\sin(3x), \quad \text{and} \quad u_t(x,0) = 0,$$

find the general solution $u(x,t)$.