UNSW Mathematics Society Presents

MATH2221/2121 Seminar Part 1

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Overview I

1. Part I: Linear ODEs

Differential Operators

Definition 1: Linear differential operators

Define the linear differential operator L of order m to be

$$
\begin{aligned}
Lu(x) &= \sum_{j=0}^{m} a_j(x) \cdot D^j u(x) \\
&= a_m(x)D^m u(x) + a_{m-1}(x)D^{m-1}u(x) + \cdots + a_0(x)u(x),
\end{aligned}
$$

where $D^j u = \dfrac{d^j u}{du^j}$ and $D^0 u = u$.

Definition 2: Singular ODEs

An ODE is said to be singular with respect to $[a, b]$ if the leading coefficient vanishes for any $x \in [a, b]$. E.g. $xu' - u = 0$ is singular on $[0, 1]$.

Homogeneous and inhomogeneous ODEs

Definition 3: Homogeneous ODEs

An ODE is said to be homogeneous if the right hand side is 0. That is, we have a differential equation of the form

$$Lu = 0.$$

- **Example**: $u'' + u' + u = 0$.

Definition 4: Inhomogeneous ODEs

An ODE is said to be inhomogeneous if the right hand side is not 0. Then we can write the differential equation as

$$Lu = f.$$

- **Example**: $u'' + u' + u = \cos(x)$.

Initial-value problems (IVP)

Definition 5: Initial-value problems

Consider an m-th order differential equation

$$Lu = f, \quad \text{on } [a, b] \tag{1}$$

along with the values

$$u(a) = v_0, u'(a) = v_1, ..., u^{(m-1)}(a) = v_{m-1}. \tag{2}$$

The problem (1) with (2) is called an initial-value problem.

- **Example**: $u' + u = x, \quad u(0) = 0.$
- **Solution**: $u(x) = x - 1 + e^{-x}.$

Unique solutions

Theorem 1: Unique solution for IVP

If f is continuous on $[a, b]$ and the ODE $Lu = f$ is non-singular on $[a, b]$, then the IVP (1) and (2) has a unique solution.

Theorem 2: Homogeneous equation solution

If L is a linear m-th order differential operator and non-singular on $[a, b]$, then the set of all solutions to the homogenous equation $Lu = 0$ on $[a, b]$ forms a vector space of dimension m.

- **What does this mean?**: The solution space has a basis of dimension m, with elements $u_1, ..., u_m$. And so every solution to the homogeneous equation can be written as a linear combination of this basis:

$$u(x) = c_1 u_1(x) + ... + c_m u_m(x), \quad \forall x \in [a, b].$$

This is called the general solution.

Unique solutions

We can do the same for an inhomogeneous equation $Lu = f$ by fixing a particular solution u_P. Then for any solution u, $L(u - u_p) = f - f = 0$ and so we can write $u - u_p$ as a linear combination of the homogeneous equation basis:

$$u(x) - u_p(x) = c_1 u_1(x) + ... + c_m u_m(x), \quad \forall x \in [a, b].$$

Rearranging, we have the general solution for an inhomogeneous differential equation:

$$u(x) = u_P(x) + c_1 u_1(x) + ... + c_m u_m(x), \quad \forall x \in [a, b].$$

Solving an inhomogeneous DE

Example 1

Solve the second order ODE

$$2u'' + u' - u = 10\sin(x).$$

Characteristic equation $2t^2 + t - 1 = 0$. Solutions $t = -1, \frac{1}{2}$ so a basis for the homogeneous solution space is $\{e^{-x}, e^{x/2}\}$. Particular solution guess $u_P(x) = A\cos(x) + B\sin(x)$. Substituting gives $A = -1$ and $B = -3$, so $u_P(x) = -\cos(x) - 3\sin(x)$. We can write the general solution as

$$u(x) = c_1 e^{-x} + c_2 e^{x/2} - \cos(x) - 3\sin(x).$$

Linear Independence

Definition 6: Linear independence

Let $u_1(x), ..., u_m(x)$ be functions on some interval $I \subset \mathbb{R}$. We say that $u_1, ..., u_m$ are **linearly independent** if

$$a_1 u_1(x) + ... + a_m u_m(x) = 0, \quad \forall x \in I,$$

implies that the constants $a_1, ..., a_m$ are all 0. Otherwise, we say that that $u_1, ..., u_m$ are **linearly dependent**.

Wronskian

Definition 7: Wronskian

The **Wronskian** of the functions $u_1, ..., u_m$ is the $m \times m$ determinant

$$W(x) = W(x; u_1, ..., u_m) = \det(D^{i-1} u_j).$$

For example, a 3×3 Wronskian is

$$W(x) = W(x; u_1, ..., u_m) = \det \begin{pmatrix} u_1 & u_2 & u_3 \\ u_1' & u_2' & u_3' \\ u_1'' & u_2'' & u_3'' \end{pmatrix}.$$

Lemmas

Lemma 1

If $u_1, ..., u_m$ are linearly dependent over an interval $I \subset \mathbb{R}$ then $W(x; u_1, ..., u_m) = 0$ for all $x \in I$.

Lemma 2

If $u_1, ..., u_m$ are solutions to the ODE

$$a_m(x)u^{(m)}(x) + a_{m-1}(x)u^{(m-1)}(x) + ... + a_0(x)u(x) = 0$$

on an interval $I \subset \mathbb{R}$, then the Wronskian satisfies

$$a_m(x)W'(x) + a_{m-1}(x)W(x) = 0 \quad \forall x \in I.$$

Wronskian

Example 2: MATH2221 2014 T2 2.iii).b

Given the functions u_1, u_2, prove that if they are solutions to a second-order, homogeneous linear differential equation

$$a_2(x)u'' + a_1(x)u' + a_0(x)u = 0,$$

then the Wronskian W satisfies

$$a_2(x)W' + a_1(x)W = 0.$$

$W = u_1u_2' - u_1'u_2$, $W' = u_1u_2'' - u_1''u_2$.

$$\begin{aligned} a_2W' + a_1W &= a_2(u_1u_2'' - u_1''u_2) + a_1(u_1u_2' - u_1'u_2) \\ &= u_1(a_2u_2'' + a_1u_2') - u_2(a_2u_1'' + a_1u_1'). \end{aligned}$$

Wronskian

Add and subtract $a_0 u_1 u_2$:

$$a_2 W' + a_1 W = u_1(a_2 u_2'' + a_1 u_2' + a_0 u_2) - u_2(a_2 u_1'' + a_1 u_1' + a_0 u_1).$$

Since u_1 and u_2 are solutions to the ODE, the RHS is 0.

Linear Independence of solutions

Theorem 3: Linear independence

Let $u_1, ..., u_m$ be solutions to the non-singular, linear, homogeneous m-th order ODE $Lu = 0$ on the interval $[a, b]$. Then either

$$W(x) = 0 \text{ and the } m \text{ solutions are linearly dependent,}$$

or

$$W(x) \neq 0 \text{ and the } m \text{ solutions are linearly independent.}$$

Polynomial solution guess

Theorem 4

Let $L = p(D)$ be a linear differential operator of order m with constant coefficients. Assume that $p(0) \neq 0$. Then for any integer $r \geq 0$, there exists a unique polynomial u_P of degree r such that $Lu_P = x^r$.

This means that if our linear ODE has a polynomial on the RHS, we should guess a polynomial of the same degree for our particular solution.

Exponential solution guess

Theorem 5

Let $L = p(D)$ and $\mu \in \mathbb{C}$. If $p(\mu) \neq 0$, then the function

$$u_P(x) = \frac{e^{\mu x}}{p(\mu)}$$

satisfies $Lu_P = e^{\mu x}$.

This means that we should guess a multiple of $e^{\mu x}$ when the RHS of a linear ODE is $e^{\mu x}$ if it is not already a solution of the homeogeneous solution.

Polynomial + exponential

Theorem 6

Let $L = p(D)$ and assume $\mu \in \mathbb{C}$. If $p(\mu) \neq 0$ then for any integers $r \geq 0$, there exists a unique polynomial v of degree r such that

$$u_P(x) = v(x)e^{\mu x}$$

satisfies $Lu_p = x^r e^{\mu x}$.

So if the RHS of the inhomogeneous linear ODE is a polynomial times an exponential, and the exponential isn't in the homogeneous solution, then the guess for the particular solution should be a polynomial times exponential.

General solutions

Example 3: MATH2221 2014 T2 2.ii)

Let $p(z) = (z-1)(z+2)^2(z^2+1)$ and $D = \frac{d}{dx}$.

- Write down the general solution u_H of the 5-th order, linear homogeneous ODE $p(D)u = 0$.
- Write down the form of a particular solution u_P to the inhomogeneous ODE

$$p(D)u = e^{-2x} + x^2 + \cos(x).$$

The zeros of $p(z)$ are 1, -2 (multiplicity 2), i and $-i$. So the homogeneous ODE has general solution

$$u_H(x) = c_1 e^x + c_2 e^{-2x} + c_3 x e^{-2x} + c_4 \cos(x) + c_5 \sin(x).$$

General solutions

Now consider the inhomogeneous ODE. The solution form will need to contain a x^2e^{-2x} term (since all lower powers are in the homogeneous solution space), as well a second order polynomial, as well as $x\cos(x)$ and $x\sin(x)$ terms (since $\cos(x)$ and $\sin(x)$ are in the homogeneous solution space). Putting these together,

$$u_P(x) = a_1x^2e^{-2x} + (a_2x^2 + a_3x + a_4) + x(a_5\cos(x) + a_6\sin(x)).$$

Reduction of order

Theorem 7: Reduction of order

If we know a solution $u_1(x) \neq 0$ to the ODE

$$u'' + p(x)u' + q(x)u = 0$$

then we can find a second solution

$$u_2(x) = u_1(x) \int \frac{1}{u_1(x)^2 \exp\left(\int p(x)dx\right)} dx.$$

Reduction of order

Example 4

Find the general solution to

$$x^2y'' + 2xy' - 2y = 0$$

given that $y_1(x) = x$ is a solution.

We need to rewrite the ODE in the form from Theorem 3:

$$y'' + \frac{2}{x}y' - \frac{2}{x^2}y = 0.$$

Then $\exp\left(\int p(x)dx\right) = \exp\left(\int \frac{2}{x}dx\right) = \exp\left(2\ln(x)\right) = x^2$. Substituting into the reduction of order formula,

$$y_2(x) = x\int \frac{1}{x^4}dx = -\frac{1}{3x^2}.$$

Annihilator method (2221 only)

This method gives us a way to find particular solutions to an inhomogeneous differential equation. Start with an ODE

$$Lu = f(x).$$

We find a differential operator $A(D)$ which **annihilates** $f(x)$, meaning $A(D)f(x) = 0$. For example D^3 annihilates x^2. We apply this differential operator to both sides of our ODE:

$$A(D)Lu = 0.$$

A solution to this homogeneous differential equation will be a particular solution to the original differential equation.

Annihilator method (2221 only)

Example 5

Find a particular solution to the ODE

$$y'' - 2y' + y = e^x + \sin(x).$$

The LHS differential operator is $p(D) = D^2 - 2D + 1 = (D-1)^2$. The function e^x is annihilated by $(D-1)$. The function $\sin(x)$ is annihilated by (D^2+1). So the annihilator is $A(D) = (D-1)(D^2+1)$. Apply the annihilator to both sides:

$$A(D)p(D)y(x) = (D-1)^3(D^2+1)y(x) = 0.$$

Solutions to the characteristic equation here is 1 (multiplicity 3), i and $-i$. The particular solution is of the form

$$y_P(x) = c_1e^x + c_2xe^x + c_3x^2e^x + c_4\sin(x) + c_5\cos(x).$$

Annihilator method (2221 only)

The first two terms in this particular solution are contained in the homogeneous solution of the original ODE, so we discard them:

$$y_P(x) = c_3 x^2 e^x + c_4 \sin(x) + c_5 \cos(x).$$

Substitute into the original ODE, coefficients are $c_3 = \frac{1}{2}$, $c_4 = 0$ and $c_5 = \frac{1}{2}$. So our particular solution is

$$y_P(x) = \frac{1}{2} x^2 e^x + \frac{1}{2} \cos(x).$$

Variation of parameters

If we have a linear, 2nd order, inhomogeneous ODE with leading coefficient 1,

$$Lu = u'' + p(x)u' + q(x)u = f(x).$$

Let u_1, u_2 be a basis for the homogeneous solution space, and let $W(x)$ be the Wronskian $W(x; u_1, u_2)$.Then a particular solution to the inhomogeneous equation is

$$u(x) = v_1(x)u_1(x) + v_2(x)u_2(x),$$

where

$$v_1'(x) = \frac{-u_2(x)f(x)}{W(x)}, \quad \text{and} \quad v_2'(x) = \frac{u_1(x)f(x)}{W(x)}.$$

Variation of parameters

Example 6: MATH2121 2018 T2 1.i)

Use the variation of parameters method to solve

$$y'' - 3y' + 2y = e^x \sin(x).$$

Homogeneous solution has characteristic equation $t^2 - 3t + 2 = 0$ so $u_1(x) = e^x$ and $u_2(x) = e^{2x}$.The Wronskian is

$$W(x) = \det \begin{pmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{pmatrix} = e^{3x}.$$

Then

$$v_1'(x) = \frac{-e^{2x} e^x \sin(x)}{e^{3x}} = -\sin(x),$$

and

$$v_2'(x) = \frac{e^x e^x \sin(x)}{e^{3x}} = e^{-x} \sin(x).$$

Variation of parameters

Integrating both v_1 and v_2 we have

$$v_1(x) = \cos(x), \quad \text{and} \quad v_2(x) = -\frac{1}{2}e^{-x}(\sin(x) + \cos(x)).$$

Then a particular solution is

$$\begin{aligned} u(x) &= \cos(x)e^x - \frac{1}{2}e^{-x}(\sin(x) + \cos(x))e^{2x} \\ &= \frac{1}{2}e^x(\cos(x) - \sin(x)). \end{aligned}$$

Power series

consider a general second-order, linear, homogeneous ODE

$$Lu = a_2(x)u'' + a_1(x)u' + a_0(x)u = 0,$$

or equivalently

$$Lu = u'' + p(x)u' + q(x)u = 0$$

with $p(x) = \frac{a_1(x)}{a_2(x)}$ and $q(x) = \frac{a_0(x)}{a_2(x)}$. Assume that a_0, a_1, a_2 are analytic at 0 (ie. they have local convergent power series), and $a_2(0) \neq 0$. Then p and q are analytic at 0, so we can find power series expansions of both:

$$p(z) = \sum_{k=0}^{\infty} p_k z^k, \quad q(z) = \sum_{k=0}^{\infty} q_k z^k \quad \text{for } |z| < \rho,$$

where $\rho > 0$.

Power series

Theorem 8

If coefficients $p(z)$ and $q(z)$ are analytic for $|z| < \rho$, then the formal power series for the solution $u(z)$ constructed in the previous slide, is also analytic for $|z| < \rho$.

This means that if we find where $p(z)$ and $q(z)$ are both analytic, the power series solution $u(z)$ is also analytic in this region.

Power series

Example 7: MATH2121 2018 T2 1.iii)

We aim to construct a series solution to the ODE about the ordinary point $x_0 = 0$:

$$(1-x^2)y'' - 2xy' + 20y = 0, \quad y(0) = 1, y'(0) = 0,$$

of the form

$$y(x) = \sum_{n=0}^{\infty} A_n x^n.$$

- Give the recurrence relation for the coefficients A_n.
- Explain from the recurrence relation that one of the series will terminate yielding a polynomial solution, and the other does not.
- Write down the polynomial solution.

Power series

Note that

$$y'(x) = \sum_{n=1}^{\infty} nA_n x^{n-1}, \quad y''(x) = \sum_{n=2}^{\infty} n(n-1)A_n x^{n-2}.$$

Then we substitute into the ODE:

$$\begin{aligned}
Ly &= y'' + (-x^2y'' - 2xy' + 20y) \\
&= \sum_{n=2}^{\infty} n(n-1)A_n x^{n-2} + \sum_{n=2}^{\infty} -n(n-1)A_n x^n \\
&\quad - \sum_{n=1}^{\infty} 2nA_n x^n + \sum_{n=0}^{\infty} 20A_n x^n \\
&= \sum_{n=2}^{\infty} n(n-1)A_n x^{n-2} + \sum_{n=0}^{\infty} -n(n-1)A_n x^n \\
&\quad - \sum_{n=0}^{\infty} 2nA_n x^n + \sum_{n=0}^{\infty} 20A_n x^n.
\end{aligned}$$

Power series

We combine the last three sums as follows.

$$Ly = \sum_{n=2}^{\infty} n(n-1)A_n x^{n-2} + \sum_{n=0}^{\infty} (-n^2 - n + 20)A_n x^n.$$

In the first sum, we change n to $n+2$.

$$Ly = \sum_{n=0}^{\infty} (n+1)(n+2)A_{n+2} x^n + \sum_{n=0}^{\infty} (-n^2 - n + 20)A_n x^n.$$

We can finally combine both sums.

$$Ly = \sum_{n=0}^{\infty} \left[(n+1)(n+2)A_{n+2} - (n+5)(n-4)A_n\right] x^n.$$

Power series

Since $Ly = 0$, we need $(n+1)(n+2)A_{n+2} - (n+5)(n-4)A_n = 0$. Rearranging,

$$A_{n+2} = \frac{(n+5)(n-4)}{(n+1)(n+2)} A_n.$$

We have $A_0 = 1$ and $A_1 = 0$, so all odd terms are zero and the even terms terminate Since $A_6 = 0$. Hence the polynomial solution is

$$y(x) = 1 - 10x^2 + \frac{35}{3}x^4.$$

Power series

Example 8: MATH2221 2015 T2 1.iii)

Consider the ODE

$$(1+z^2)u'' - zu' - 3u = 0.$$

- Find the recurrence relation satisfied by the coefficients A_k in any power series solution:
$$u = \sum_{k=0}^{\infty} A_k z^k.$$
- Show that $A_5 = A_7 = A_9 = ... = 0$.
- Hence find the solution for which $u(0) = 0$, $u'(0) = 6$.

Power series

Note that

$$u'(z) = \sum_{k=1}^{\infty} kA_k z^{k-1}, \quad u''(z) = \sum_{k=2}^{\infty} k(k-1)A_k z^{k-2}.$$

Substitute into the ODE:

$$\begin{aligned}
Lu &= u'' + (z^2u'' - zu' - 3u) \\
&= \sum_{k=2}^{\infty} k(k-1)A_k z^{k-2} + \sum_{k=2}^{\infty} k(k-1)A_k z^k \\
&\qquad - \sum_{k=1}^{\infty} kA_k z^k - \sum_{k=0}^{\infty} A_k z^k \\
&= \sum_{k=2}^{\infty} k(k-1)A_k z^{k-2} + \sum_{k=0}^{\infty} k(k-1)A_k z^k \\
&\qquad - \sum_{k=0}^{\infty} kA_k z^k - \sum_{k=0}^{\infty} A_k z^k.
\end{aligned}$$

Power series

Combine the last three sums.

$$Lu = \sum_{k=2}^{\infty} k(k-1)A_k z^{k-2} + \sum_{k=0}^{\infty} (k^2 - 2k - 3)A_k z^k.$$

In the first sum, change k to $k+2$.

$$Lu = \sum_{k=0}^{\infty} (k+1)(k+2)A_{k+2} z^k + \sum_{k=0}^{\infty} (k^2 - 2k - 3)A_k z^k.$$

Finally, we can combine the sums.

$$Lu = \sum_{k=0}^{\infty} (k+1)\left[(k+2)A_{k+2} + (k-3)A_k\right] z^k.$$

Power series

Since $Lu = 0$, we need $(k+2)A_{k+2} + (k-3)A_k = 0$. Rearranging,

$$A_{k+2} = -\frac{k-3}{k+2}A_k.$$

Since $A_5 = -\frac{3-3}{3+2}A_3 = 0$, then all odd terms past A_5 are zero. Also $A_1 = 6$ and $A_3 = 4$.
The even terms start at $A_0 = 0$ so all even terms past this are zero.
Hence

$$u(z) = 6z + 4z^3.$$

Singular/Cauchy-Euler ODEs

For singular ODEs, we only need to check the case when the leading coefficient vanishes at the origin.
A second-order **Cauchy-Euler ODE** has the form

$$Lu = ax^2u'' + bxu' + cu = f(x),$$

where a, b, c are constants with $a \neq 0$. This is singular at $x = 0$.
Applying this differential operator L to x^r,

$$Lx^r = [ar(r-1) + br + c]\, x^r,$$

we can see that x^r is a solution to the homogeneous equation $Lu = 0$ iff

$$ar(r-1) + br + c = 0.$$

Singular/Cauchy-Euler ODEs

Lemma 3

Suppose there are distinct solutions r_1, r_2 to the equation $ar(r-1)+br+c=0$. That is, $r_1 \neq r_2$. Then the general solution of the homogeneous equation $Lu = 0$ is

$$u(x) = C_1 x^{r_1} + C_2 x^{r_2}, \quad x > 0.$$

Lemma 4

Suppose there is one solution r_1 to $ar(r-1)+br+c=0$. Then the general solution to the homogeneous equation $Lu = 0$ is

$$C_1 x^{r_1} + C_2 x^{r_1} \log(x), \quad x > 0.$$

For a particular solution to the inhomogeneous Cauchy-Euler equation

$$ax^2u'' + bxu' + cu = x^r,$$

we can use the particular solution guess $u(x) = \alpha x^r$.

Cauchy-Euler ODEs

Example 9: MATH2121 2016 T2 2.i)

Find the general solution of the Cauchy-Euler ODE

$$2x^2y'' + 7xy' + 3y = 13x^{1/4}, \quad x > 0.$$

We want to solve the equation $ar(r-1) + br + c = 0$, where $a = 2$, $b = 7$ and $c = 3$. That is, $2r^2 + 5r + 3 = 0$. Solutions are $r_1 = -1$ and $r_2 = -3/2$. Using Lemma 3, we have general homogeneous solution

$$y_H(x) = C_1x^{-1} + C_2x^{-3/2}.$$

A particular solution can be found by applying variation of parameters, giving us

$$y_P(x) = -8x^{1/4}.$$

So $y(x) = C_1x^{-1} + C_2x^{-3/2} - 8x^{1/4}$.

Cauchy-Euler ODEs - Complex Roots

Example

Find the general solution of the Cauchy-Euler ODE

$$x^2u'' - xu' + 2u = 0, \quad x > 0.$$

We want to solve the equation $ar(r-1) + br + c = 0$, where $a = 1$, $b = -1$ and $c = 2$. That is, $r^2 - 2r + 2 = 0$. Solutions are $r_1 = 1 + i$ and $r_2 = 1 - i$. Using Lemma 3, we have general homogeneous solution

$$u_H(x) = C_1x^{1+i} + C_2x^{1-i}.$$

But we want real solutions, and this is clearly not real! However, we can use the fact that

$$\begin{aligned} x^{1+i} = \exp(\ln(x^{1+i})) &= \exp((1+i)\ln x) \\ &= xe^{i\ln x} \\ &= x(\cos(\ln x) + i\sin(\ln x)). \end{aligned}$$

Cauchy-Euler ODEs - Complex Roots - Continued

Similarly, we have $x^{1-i} = x(\cos(\ln x) - i\sin(\ln x))$. Using these facts, we can get real solutions that form a basis for the solution space, by taking $1/2(x^{1+i} + x^{i-1}) = x\cos(\ln x)$ and $1/2(x^{1+i} - x^{i-1}) = x\sin(\ln x)$. Thus our general solution is

$$u_H(x) = B_1 x\cos(\ln x) + B_2 x\sin(\ln x)$$

for any constants B_1, B_2.

Frobenious normal form

A frequent form of ODE that appears in many applications can be written in **Frobenious normal form**:

$$z^2u'' + zP(z)u' + Q(z)u = 0,$$

where $P(z)$ and $Q(z)$ are analytic at 0. Let $P_0 = P(0)$ and $Q_0 = Q(0)$, and define a series F as

$$F(z;r) = z^r \sum_{k=0}^{\infty} A_k(r)z^k.$$

Consider the equation $r(r-1) + P_0r + Q_0 = 0$ with solutions r_1 and r_2.

Lemma 5

If $r_1 \neq r_2$, then $f(z;r_1)$ is a solution to the Frobenious normal form ODE. If $r_1 - r_2$ is **not** a whole number, then a second linearly independent solution is $F(z;r_2)$.

Bessel function

The **Bessel equation with parameter** ν is:

$$z^2u'' + zu' + (z^2 - \nu^2)u = 0.$$

This ODE is in Frobenious normal form, with indicial polynomial:

$$I(r) = (r + \nu)(r - \nu),$$

and we seek a series solution:

$$u(z) = \sum_{k=0}^{\infty} A_k z^{k+r}.$$

We assume $\mathrm{Re}(\nu) \geq 0$, so $r_1 = \nu$ and $r_2 = -\nu$.

Bessel function

With the normalisation:

$$A_0 = \frac{1}{2^\nu \Gamma(1+\nu)}$$

the series solution is called the **Bessel function of order** ν and is denoted:

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(1+\nu)} \left[1 - \frac{(z/2)^\nu}{1+\nu} + \frac{(z/2)^4}{2!(1+\nu)(2+\nu)} - ... \right].$$

And from the functional equation $\Gamma(1+z) = z\Gamma(z)$:

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)k(z/2)^{2k+\nu}}{k!\Gamma(k+1+\nu)}$$

Bessel function

If ν is not an integer, then a second linearly independent, solution is:

$$J_{-\nu}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k-\nu}}{k!\Gamma(k+1-\nu)}.$$

For an integer $\nu = n \in \mathbb{Z}$, since $\Gamma(n+1) = n!$, we have:

$$J_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k+n}}{k!(k+n)!}.$$

Also, since $\frac{1}{\Gamma(z)} = 0$ for $z = 0, -1, -2, \ldots$, we find that J_n and J_{-n} are linearly dependent; in fact:

$$J_{-n}(z) = (-1)^n J_n(z).$$

Bessel function

Example 10: MATH2221 2015 T2 2.ii)

1. Use term-by-term differentiation to prove that for $\nu \in \mathbb{R}$ and $x > 0$:
$$\frac{d}{dx}(x^\nu J_\nu(x)) = x^\nu J_{\nu-1}(x).$$
2. Hence evaluate the definite integral:
$$I = \int_0^1 x^{\frac{7}{2}} J_{\frac{1}{2}}(x) dx.$$

Bessel function

Note that

$$x^\nu J_\nu(x) = \sum_{k=0}^{\infty} x^\nu \frac{(-1)^k (x/2)^{2k+\nu}}{k!\Gamma(k+1+\nu)}.$$

Moving the x^ν into the fraction,

$$x^\nu J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x)^{2k+2\nu}}{2^{2k+\nu} k!\Gamma(k+1+\nu)}.$$

Take one term in this sum and differentiate with respect to x:

$$\frac{d}{dx}\left(\frac{(-1)^k (x)^{2k+2\nu}}{2^{2k+\nu} k!\Gamma(k+1+\nu)}\right) = \frac{(-1)^k (2k+2\nu) x^{2k+2\nu-1}}{2^{2k+\nu} k!\Gamma(k+1+\nu)}.$$

Bessel function

Since $\Gamma(k+1+\nu) = (k+\nu)\Gamma(k+1+\nu)$, then

$$\frac{d}{dx}\left(\frac{(-1)^k (x)^{2k+2\nu}}{2^{2k+\nu} k!\Gamma(k+1+\nu)}\right) = \frac{(-1)^k (2k+2\nu) x^{2k+2\nu-1}}{2^{2k+\nu} k!(k+\nu)\Gamma(k+\nu)}.$$

However there is a factor of $2(k+\nu)$ in both the numerator and denominator. So

$$\frac{d}{dx}\left(\frac{(-1)^k (x)^{2k+2\nu}}{2^{2k+\nu} k!\Gamma(k+1+\nu)}\right) = \frac{(-1)^k (x)^{2k+2\nu-1}}{2^{2k+\nu-1} k!\Gamma(k+\nu)}.$$

Finally, factor out a factor x^ν:

$$\frac{d}{dx}\left(\frac{(-1)^k (x)^{2k+2\nu}}{2^{2k+\nu} k!\Gamma(k+1+\nu)}\right) = x^\nu \frac{(-1)^k (x/2)^{2k+\nu-1}}{k!\Gamma(k+\nu)}.$$

Bessel function

Since the derivative is linear,

$$\frac{d}{dx}\left(x^{\nu}J_{\nu}(x)\right)=\sum_{k=0}^{\infty}\frac{d}{dx}\left(\frac{(-1)^{k}(x)^{2k+2\nu}}{2^{2k+\nu}k!\Gamma(k+1+\nu)}\right).$$

Substituting in the derivative we found,

$$\frac{d}{dx}\left(x^{\nu}J_{\nu}(x)\right)=x^{\nu}\sum_{k=0}^{\infty}\frac{(-1)^{k}(x/2)^{2k+\nu-1}}{k!\Gamma(k+\nu)}.$$

The RHS here is just $x^{\nu}J_{\nu-1}(x)$, so we are done. To find the integral $\int_0^1 x^{\frac{7}{2}}J_{\frac{1}{2}}(x)dx$, we separate and use integration by parts:

$$\int_0^1 x^2\cdot x^{\frac{3}{2}}J_{\frac{1}{2}}(x)dx$$

where $u'=x^{\frac{3}{2}}J_{\frac{1}{2}}(x)$ and $v=x^2$.

Bessel function

Applying integration by parts:

$$I = [x^2 . x^{\frac{3}{2}} J_{\frac{3}{2}}(x)]_0^1 - \int_0^1 2x \cdot x^{\frac{3}{2}} J_{\frac{3}{2}}(x) dx$$

$$= J_{\frac{3}{2}}(1) - 2 \int_0^1 x^{\frac{5}{2}} J_{\frac{3}{2}}(x) dx.$$

However by our previous result, $\frac{d}{dx}(x^\nu J_\nu(x)) = x^\nu J_{\nu-1}(x)$. So the antiderivative of $x^{5/2} J_{3/2}$ is $x^{5/2} J_{5/2}$, hence

$$= J_{\frac{3}{2}}(1) - 2[x^{\frac{5}{2}} J_{\frac{5}{2}}(x)]_0^1$$

$$= J_{\frac{3}{2}}(1) - 2J_{\frac{5}{2}}(1).$$

Legendre equation (2221 only)

The **Legendre equation** with paramater ν is:

$$(1-z^2)u'' - 2zu' + \nu(\nu+1)u = 0.$$

This ODE is not singular at $z = 0$, so the solution has an ordinary Taylor series expansion:

$$u = \sum_{k=0}^{\infty} A_k z^k.$$

The A_k must satisfy:

$$(k+1)(k+2)A_{k+2} - [k(k+1) - \nu(\nu+1)]A_k = 0.$$

The recurrence relation is:

$$A_{k+2} = \frac{(k-\nu)(k+\nu+1)}{(k+1)(k+2)} A_k \quad \text{for } k \geq 0.$$

Legendre equation (2221 only)

We have:

$$u(z) = A_0 u_0(z) + A_1 u_1(z)$$

where:

$$u_0(z) = 1 - \frac{\nu(\nu+1)}{2!} z^2 + \frac{(\nu-2)\nu(\nu+1)(\nu+3)}{4!} z^{-} \ldots$$

and:

$$u_1(z) = z - \frac{(\nu-1)(\nu-2)}{3!} z^3 + \frac{(\nu-3)(\nu-1)(\nu+2)(\nu+4)}{5!} z^5 - \ldots$$

Suppose now that $\nu = n$ is a non-negative integer. If n is even, then the series for $u_0(z)$ terminates, whereas if n is odd, then the series for $u_1(z)$ terminates. The terminating solution is then called the **Legendre polynomial** of degree n and is denoted by $P_n(z)$ with the normalisation:

$$P_n(1) = 1.$$

2. Dynamical Systems

Dynamical Systems

State variables are natural variables which depend on a single independent variable. A **dynamical system** is a natural process described by these state variables. The state of a system at a given time is described by the values of the state variables at that instant.

Note that any nth order ODE can be writen as a system of **first order** ODEs (not vice versa):

$$\frac{d^n y}{dt^n} = g\left(y, \frac{dy}{dt}, ..., \frac{d^{n-1}y}{dt^{n-1}}\right)$$
$$\frac{dx}{dt} = f(x_1, x_2, ..., x_n)$$

Non-autonomous ODEs

Definition 8

A system of ODEs of the form:

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x})$$

is said to be **autonomous**.

Definition 9

In a non-autonomous system, $\mathbf{F}$ may depend explicitly on t:

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}, t).$$

Lipschitz (2221 only)

Definition 10

The number $L \in \mathbb{R}$ is a Lipschitz constant for a function $f : [a, b] \to \mathbb{R}$ if

$$|f(x) - f(y)| \leq L\,|x - y| \quad \forall x, y \in [a, b].$$

We say that the function f is Lipschitz if a Lipschitz constant for f exists.

Theorem 9

If f is Lipschitz, then f is uniformly continuous.

Lemma 6

If $f : I \to \mathbb{R}$ is differentiable and f' is continuous on I, then f is Lipschitz.

Lipschitz (2221 only)

Example 11

$$f(x) = 2\sqrt{x}, \quad x \geq 0$$

Where is f Lipschitz?

Since $f'(x) = \dfrac{1}{\sqrt{x}}$ is continuous on $x > 0$, f is Lipschitz on any closed interval $[a, b]$ such that $0 < a < b$ (Lemma 6).
Suppose $a > 0$ and consider some interval $[0, a]$ containing 0. Let $x \in (0, a]$. Then,

$$\left| \frac{f(x) - f(0)}{x - 0} \right| = \frac{1}{\sqrt{x}} \to \infty \text{ as } x \to 0.$$

Hence, a Lipschitz constant does not exist for f on a non-empty closed interval containing 0.

Lipschitz (2221 only)

Example 11

Find a Lipschitz constant for the function $f : [0, \pi] \to \mathbb{R}$ defined by $f(x) = x \cos x$.

First note that

$$\begin{aligned} f'(x) &= \cos x - x \sin x \\ &\le |\cos x - x \sin x| \\ &\le |\cos x| + |x \sin x| && \text{(by triangle inequality)} \\ &\le 1 + |x| \le 1 + \pi \quad \text{for } x \in [0, \pi]. \end{aligned}$$

For any $x, y \in [0, \pi]$ where $x < y$, there exists $c \in (x, y)$ such that

$$|f(y) - f(x)| = |f'(c)||y - x| \le (1 + \pi)|y - x|$$

by the Mean Value Theorem. Hence, $1 + \pi$ is a Lipschitz constant.

Lipschitz Vector Field (2221 only)

We extend the definition of Lipschitz to vector fields.

Definition 11

A vector field $\mathbf{F} : S \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz on S if

$$||\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{y})|| \leq L \, ||\mathbf{x} - \mathbf{y}|| \quad \forall \, \mathbf{x}, \mathbf{y} \in S.$$

Here,

$$||\mathbf{x}|| = \left(\sum_{j=1}^{m} x_j^2 \right)^{1/2}$$

denotes the Euclidean norm of the vector $\mathbf{x} \in \mathbb{R}^m$.

We say that $\mathbf{F}(\mathbf{x}, t)$ is Lipschitz in $\mathbf{x}$ if, for all t:

$$||\mathbf{F}(\mathbf{x}, t) - \mathbf{F}(\mathbf{y}, t)|| \leq L \, ||\mathbf{x} - \mathbf{y}|| \, , \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^m.$$

Existence and Uniqueness Theorem (2221 only)

We want to find solutions to a non-autonomous system. The following theorem guarantees a unique solution for a non-autonomous system, under certain conditions.

Theorem 10

Let $r > 0, \tau > 0$ and suppose $S = \{(\mathbf{x}, t) : ||\mathbf{x} - \mathbf{x}_0|| \le r, |t - t_0| \le \tau\}$. If $\mathbf{F}(\mathbf{x}, t)$ is Lipschitz and $||\mathbf{F}(\mathbf{x}, t)|| \le M$ for $(\mathbf{x}, t) \in S$ then the initial value problem defined by

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}, t), \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

has a unique solution $\mathbf{x}(t)$ over a time interval $|t - t_0| < \min(r/M, \tau)$.

In fact, $\mathbf{x}(t)$ is continuous and differentiable.

Notes

- The existence of solutions follow from continuity in x and t.
- The uniqueness of solutions follow from the Lipschitz condition in x.
- The theorem is a local existence theorem. It provides for the existence of solutions over a finite time interval.
- Outside of the time interval specified by the theorem, there may or may not be a unique solution.

Linear systems of ODEs

Definition 12

We say that the $n \times n$, first-order system of ODEs:

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}, t)$$

is linear if the RHS has the form:

$$\mathbf{F}(\mathbf{x}, t) = A(t)\mathbf{x} + \mathbf{b}(t)$$

for some $n \times n$ matrix-valued function $A(t) = [a_{i,j}(t)]$ and a vector-valued function $\mathbf{b}(t) = [b_i(t)]$.

The linear first-order system is autonomous when A and $\mathbf{b}$ are constant.

Global Existence and Uniqueness

We have a stronger existence result in the linear case:

Theorem 11

If $A(t)$ and $\mathbf{b}(t)$ are continuous for $0 \leq t \leq T$, then the linear initial-value problem

$$\frac{d\mathbf{x}}{dt} = A(t)\mathbf{x} + \mathbf{b}(t) \quad \text{with } \mathbf{x}(0) = \mathbf{x}_0,$$

has a unique solution $\mathbf{x}(t)$ for $0 \leq t \leq T$.

A special case

It is much easier to work with the special case when $A(t) = A$ is a constant $n \times n$ matrix and $\mathbf{b}(t) = \mathbf{0}$:

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}.$$

The general solution to this system is

$$\mathbf{x}(t) = \sum_{i=1}^{n} c_i e^{\lambda_i t} \mathbf{v}_i,$$

where λ_i is the i-th eigenvalue with corresponding eigenvector $\mathbf{v}_i$ and $c_1, ..., c_n$ are constants.

Initial-valued system

Recall that for an $n \times n$ complex matrix A, we define the exponential of a matrix as

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots .$$

Consider the initial-valued system

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0.$$

Then we can write the solution as a matrix exponential,

$$\mathbf{x}(t) = e^{tA}\mathbf{x}_0.$$

Example

Example 12: MATH2121 2016 T2 2.iii)

For an $n \times n$ matrix A.

1. State the definition of e^A.
2. Show that if $A\mathbf{v} = \lambda\mathbf{v}$, then $e^A\mathbf{v} = e^\lambda\mathbf{v}$.

The definition of the matrix exponential is

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots .$$

If $A\mathbf{v} = \lambda\mathbf{v}$ then

$$\begin{aligned} e^A\mathbf{v} &= \left(\sum_{k=0}^{\infty} \frac{1}{k!} A^k \right) \mathbf{v} \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \left(A^k \mathbf{v} \right). \end{aligned}$$

Example

But we can find $A^k\mathbf{v}$:

$$A^k\mathbf{v} = A^{k-1}\lambda\mathbf{v} = \cdots = \lambda^k\mathbf{v}.$$

Hence

$$\begin{aligned} e^A\mathbf{v} &= \sum_{k=0}^{\infty} \frac{1}{k!}\left(A^k\mathbf{v}\right) \\ &= \left(\sum_{k=0}^{\infty} \frac{\lambda^k}{k!}\right)\mathbf{v}. \end{aligned}$$

This is the exponential function Taylor series at the point λ, so

$$e^A\mathbf{v} = e^{\lambda}\mathbf{v}.$$

Sanity Check

Matrix Exponential and Linear Combination of Eigenvectors

Using a similar method as what we've done above, it can be shown that $e^{tA}\mathbf{v} = e^{\lambda t}\mathbf{v}$ where $\mathbf{v}$ is an eigenvector for eigenvalue λ of the matrix A. The above result confirms that the solution obtained by writing $\mathbf{x}(t) = e^{tA}\mathbf{x_0}$ is the same as writing $\mathbf{x}(t) = \sum_{i=1}^{n} c_i e^{\lambda_i t}\mathbf{v}_i$, where $\mathbf{v}_i$ is the eigenvector for eigenvalue λ_i of A because

$$e^{tA}\mathbf{x_0} = e^{tA}\mathbf{x}(0) = e^{tA}\sum_{i=1}^{n} c_i\mathbf{v}_i = \sum_{i=1}^{n} c_i e^{tA}\mathbf{v}_i = \sum_{i=1}^{n} c_i e^{\lambda_i t}\mathbf{v}_i.$$

Calculating the Matrix Exponential

We now want to find a practical way of calculating e^{tA}, which requires finding A^k for $k \in \mathbb{N}$. To do this efficiently, we have a few methods and properties to look to.

- If A is a nilpotent matrix – that is, $A^n = \mathbf{0}$ for some $n \in \mathbb{Z}^+$ – we can sum all the terms in the Taylor series expansion up to the smallest such n.
- If A is diagonalisable, we can first diagonalise the matrix A. We explore the method of diagonalisation in more detail.
- If A can be written as a sum of two matrices B and C such that $BC = CB$, then we can write $e^A = e^B e^C$. We can use the above two methods to find e^B and e^C.

Diagonalising a matrix

Definition 13

An $n \times n$ complex matrix A is diagonalisable if there exists a non-singular matrix $n \times n$ matrix M such that $M^{-1}AM$ is diagonal.

Theorem 12

An $n \times n$ complex matrix A is diagonalisable if and only if there exists a basis $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$ for $\mathbb{C}^n$, where $\mathbf{v}_1, ..., \mathbf{v}_n$ are eigenvectors of A. In fact the columns of M are the eigenvectors of A, $M = (\mathbf{v}_1 | \cdots | \mathbf{v}_n)$, and $M^{-1}AM = \Lambda$ where:

$$\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

for eigenvalues λ_i corresponding to eigenvector $\mathbf{v}_i$.

Matrix Powers

In general since $M^{-1}AM = \Lambda$, we can efficiently calculate A^k:

$$A^k = \overbrace{M\Lambda M^{-1} \cdot M\Lambda M^{-1} \cdots M\Lambda M^{-1}}^{k \text{ times}} = M\Lambda^k M^{-1}.$$

This is better because

$$\Lambda^k = \begin{pmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{pmatrix}.$$

In fact, due to the Taylor series of the exponential function, we can simplify the solution to the initial-valued system further.

Exponential of a diagonalisable matrix

Theorem 13

If $A = M\Lambda M^{-1}$ is diagonalisable, then:

$$e^A = Me^\Lambda M^{-1} \quad \text{and} \quad e^\Lambda = \begin{pmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{pmatrix}.$$

We can now find the exponential of tA:

$$e^{tA} = Me^{t\Lambda} M^{-1}, \quad \text{and} \quad e^{t\Lambda} = \begin{pmatrix} e^{t\lambda_1} & & \\ & \ddots & \\ & & e^{t\lambda_n} \end{pmatrix}.$$

Equilibrium points

Definition 14

We say that $\mathbf{a} \in \mathbb{R}^n$ is an equilibrium point for the dynamical system $\dfrac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x})$ if

$$\mathbf{F}(\mathbf{a}) = \mathbf{0}.$$

Suppose $\mathbf{a}$ is an equilibrium point for the system $\dfrac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x})$. Consider the initial-valued system

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}), \quad \mathbf{x}(0) = \mathbf{a}.$$

Then the solution is the constant function $\mathbf{x}(t) = \mathbf{a}$.

Stable Equilibrium

Definition 15

An equilibrium point $\mathbf{a}$ is stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that whenever $||\mathbf{x}_0 - \mathbf{a}|| < \delta$, the solution of

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}), \quad \mathbf{x}(0) = \mathbf{x}_0$$

satisfies

$$||\mathbf{x}(t) - \mathbf{a}|| < \epsilon \quad \forall\, t > 0.$$

Intuitively: if a solution starts close enough to the stable equilibrium point, then they will remain close to the stable equilibrium point.

Asymptotic Stability

This is a stronger form of stability, on a particular subset of $\mathbb{R}^n$.

Definition 16

Let N be an open subset of $\mathbb{R}^n$ that contains an equilibrium point $\mathbf{a}$. We say that $\mathbf{a}$ is asymptotically stable in N if $\mathbf{a}$ is stable, and whenever $\mathbf{x}_0 \in N$ the solution of

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}) \quad \mathbf{x}(0) = \mathbf{x}_0$$

satisfies

$$\mathbf{x}(t) \to \mathbf{a} \quad \text{as} \quad t \to \infty.$$

We call N a domain of attraction for $\mathbf{a}$.

Intuitively: not only do the solutions stay close to the stable equilibrium point, but they also approach the equilibrium point as t goes to infinity.

Linear constant case

Consider the linear system

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} + \mathbf{b}, \quad \mathbf{x}(0) = \mathbf{x}_0,$$

where $\det(A) \neq 0$. Then the unique equilibrium point of this system is

$$\mathbf{a} = -A^{-1}\mathbf{b},$$

and the solution to the system is

$$\mathbf{x}(t) = \mathbf{a} + e^{tA}\left(\mathbf{x}_0 - \mathbf{a}\right).$$

Linear constant case

Theorem 14

Consider the previous linear constant coefficient system. Let $\lambda_1, ..., \lambda_n$ be the eigenvalues of A. The equilibrium point $\mathbf{a} = -A^{-1}\mathbf{b}$ is:

1. Stable if and only if $\mathrm{Re}(\lambda_j) \leq 0$ for all j.
2. asymptotically stable if and only if $\mathrm{Re}(\lambda_j) < 0$ for all j.

In the second case, the domain of attraction is the whole of $\mathbb{R}^n$.

Classification of 2D Linear Systems

Type	Eigenvalues	Eigenvectors	$X(t)$	Classification
1: $\boldsymbol{B} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$	$\lambda_1 \neq \lambda_2 \in \mathbb{R}$	$\boldsymbol{v}^{(1)}, \boldsymbol{v}^{(2)}$	$\left[\boldsymbol{v}^{(1)} e^{\lambda_1 t} \;\; \boldsymbol{v}^{(2)} e^{\lambda_2 t}\right]$	Improper Node
	$\lambda_1 < 0 < \lambda_2$	$\boldsymbol{v}^{(1)}, \boldsymbol{v}^{(2)}$	$\left[\boldsymbol{v}^{(1)} e^{\lambda_1 t} \;\; \boldsymbol{v}^{(2)} e^{\lambda_2 t}\right]$	Saddle Point
2: $\boldsymbol{B} = \begin{pmatrix} \lambda & \gamma \\ 0 & \lambda \end{pmatrix}$ $\lambda, \gamma \in \mathbb{R}$	$\lambda_1 = \lambda_2 = \lambda \in \mathbb{R}$ (multiplicity 2)	$\boldsymbol{v}^{(1)}, \boldsymbol{v}^{(2)}$ ($\boldsymbol{v}^{(2)}$generalised eigenvector)	$\left[\boldsymbol{v}^{(1)} e^{\lambda t} \;\; (\boldsymbol{v}^{(2)} + t\boldsymbol{v}^{(1)}) e^{\lambda t}\right]$	Deficient Node
	$\lambda_1 = \lambda_2 = \lambda$ (2D eigenspace)	$\boldsymbol{v}^{(1)}, \boldsymbol{v}^{(2)}$ any basis of $\mathbb{R}^2$	$\left[\boldsymbol{v}^{(1)} e^{\lambda_1 t} \;\; \boldsymbol{v}^{(2)} e^{\lambda_2 t}\right]$	Star (or proper) Node
3: $\boldsymbol{B} = \begin{pmatrix} \alpha & -\omega \\ \omega & \alpha \end{pmatrix}$ $\alpha, \omega \in \mathbb{R}$	$\lambda_1 = i\beta = \overline{\lambda_2}$ $(\beta \neq 0) \in \mathbb{R}$	$\boldsymbol{v}^{(2)} = \overline{\boldsymbol{v}^{(1)}}$	$\left[\mathbb{R}\mathrm{e}\left(\boldsymbol{v}^{(1)} e^{i\beta t}\right) \;\; \mathbb{I}\mathrm{m}\left(\boldsymbol{v}^{(2)} e^{i\beta t}\right)\right]$	Centre (or vortex)
	$\lambda_1 = \alpha + i\beta = \overline{\lambda_2}$ $(\alpha \neq 0, \beta \neq 0) \in \mathbb{R}$	$\boldsymbol{v}^{(2)} = \overline{\boldsymbol{v}^{(1)}}$	$\left[\mathbb{R}\mathrm{e}\left(\boldsymbol{v}^{(1)} e^{(\alpha+i\beta)t}\right) \;\; \mathbb{I}\mathrm{m}\left(\boldsymbol{v}^{(2)} e^{(\alpha+i\beta)t}\right)\right]$	Spiral point (or focus)

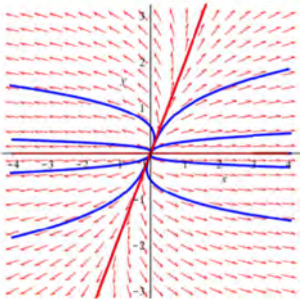
Improper Node

In the case where $0 < \lambda_1 < \lambda_2$,

1. Draw lines representing the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ of λ_1 and λ_2.
2. Construct trajectories starting from the equilibrium point.
3. Initially, the trajectory is close to $\mathbf{v}_1$, but approaches the direction of $\mathbf{v}_2$ quickly.
4. All trajectories are directed away from the equilibrium point.

When $\lambda_1 = 1, \mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\lambda_2 = 2, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, solutions have the form

$$\mathbf{x} = Ae^t \begin{pmatrix} 1 \\ 2 \end{pmatrix} + Be^{2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$



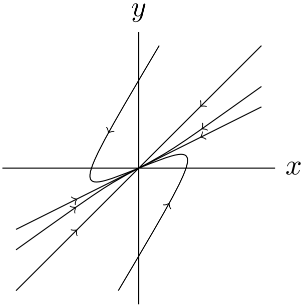
Improper Node

In the case where $\lambda_1 < \lambda_2 < 0$,

1. Draw lines representing the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ of λ_1 and λ_2.
2. Construct trajectories starting far from the equilibrium point.
3. The trajectory starts close to the direction of $\mathbf{v}_1$, but approaches $\mathbf{v}_2$ as it approaches the equilibrium point.

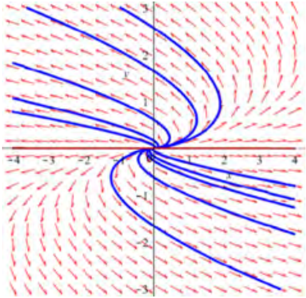
When $\lambda_1 = -3, \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\lambda_2 = -1, \mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, solutions have the form

$$\mathbf{x} = Ae^{-3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + Be^{-t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$



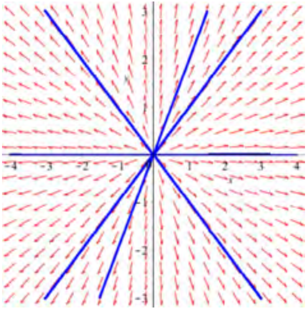
Deficient Node

Here $\lambda_1 = \lambda_2 \in \mathbb{R}$, and the eigenspace is one dimensional (out of the scope of the course).



Star Node

Here $\lambda_1 = \lambda_2 \neq 0$ and all nonzero vectors are eigenvectors. Therefore, all trajectories are either being attracted or repelled by the equilibrium points in a straight line.



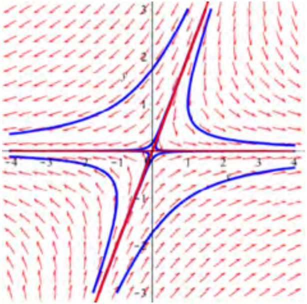
Saddle Point

Here, $\lambda_1 < 0 < \lambda_2$.

1. Draw lines representing the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ of λ_1 and λ_2 respectively. Trajectories on $\mathbf{v}_1$ and $\mathbf{v}_2$ are straight lines. Those on $\mathbf{v}_1$ are attracted towards the equilibrium point, whereas those on $\mathbf{v}_2$ are repelled.
2. Draw all other trajectories starting close to $\mathbf{v}_1$, initially approaching the equilibrium point. Such trajectories are eventually repelled from the equilibrium point, approaching $\mathbf{v}_2$.

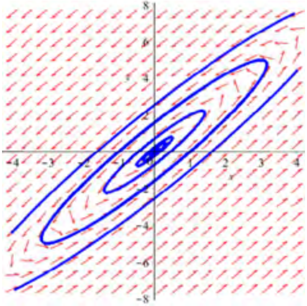
When $\lambda_1 = -1, \mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\lambda_2 = 1$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, solutions have the form

$$\mathbf{x} = Ae^{-t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + Be^{t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



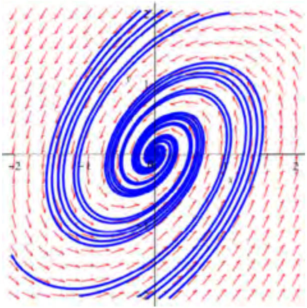
Centre

Here λ_1, λ_2 are purely imaginary. If eigenvalues are purely imaginary, the trajectory forms orbits around the equilibrium point as $e^{it} = \cos(t) + i\sin(t)$. We can determine whether the orbit is clockwise or anticlockwise by finding the value of $d\mathbf{x}/dt$ at a point.



Spiral

Here $\lambda_1 = \alpha + i\beta = \bar{\lambda_2}$. The trajectory will spiral inward when $\alpha < 0$ or outward when $\alpha > 0$. We can determine whether the spiral is is clockwise or anticlockwise by finding the value of $d\mathbf{x}/dt$ at a point.



Example 13: MATH2121 2018 T2 2.iii)

Solve for $x(t)$ and $y(t)$ and determine the type and stability of the equilibrium point of the following system of differential equations:

$$\frac{dx}{dt} = x + y;$$
$$\frac{dy}{dt} = 2x.$$

The system can be written as

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Equilibrium points

Let $A = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$. By solving $\det(A - \lambda I) = 0$, we find that the eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = 2$. The eigenvectors are given by $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
Hence, the solutions to the differential equation can be written in the form

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

By equating the components, we get $x(t) = c_1 e^{-t} + c_2 e^{2t}$ and $y(t) = -2c_1 e^{-t} + c_2 e^{2t}$.
To find the equilibrium point, we solve $x + y = 0$ and $2x = 0$ simultaneously. The only solution is the point $(0, 0)$. Since $\lambda_1 < 0 < \lambda_2$, the equilibrium point $(0, 0)$ is a saddle point. Since $\mathrm{Re}(\lambda_2) > 0$ then $(0, 0)$ is an unstable equilibrium point.
Hence $(0, 0)$ is an unstable saddle point.

First integrals (2221 only)

Definition 17

A function $G : \mathbb{R}^n \to \mathbb{R}$ is a **first integral** for a system of ODEs

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x})$$

if $G\left(\mathbf{x}(t)\right)$ is constant for every solution $\mathbf{x}(t)$.

Geometrically: G is a first integral iff

$$\nabla G(\mathbf{x}) \perp \mathbf{F}(\mathbf{x}) \text{ for all } \mathbf{x}.$$

First integrals (2221 only)

Example 14

Consider the system of ODEs

$$\begin{aligned}\frac{dx_1}{dt} &= x_1 x_2, \\ \frac{dx_2}{dt} &= -x_1^2.\end{aligned}$$

Prove that $G(\mathbf{x}) = x_1^2 + x_2^2$ is a first integral.

Set the function $\mathbf{F}(\mathbf{x})$ as the RHS of the system:

$$\mathbf{F}(\mathbf{x}) = \begin{pmatrix} x_1 x_2 \\ -x_1^2 \end{pmatrix}.$$

First integrals (2221 only)

We want to find the gradient of G:

$$\nabla G(\mathbf{x}) = \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix}.$$

So

$$\nabla G(\mathbf{x}) \cdot \mathbf{F}(\mathbf{x}) = \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix} \cdot \begin{pmatrix} x_1 x_2 \\ -x_1^2 \end{pmatrix} = 2x_1^2 x_2 - 2x_2 x_1^2 = 0.$$

This means that all solutions to the system of ODEs must be mapped to a constant under G. That is, if (x_1, x_2) is a solution then $x_1^2 + x_2^2 = C$ for *some* C. In other words, all the solutions to the system of ODEs lie on *some* circle around the origin.