MATH1131/1141 MathSoc Algebra Revision Session 2017 S1 Solutions

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Problem 1

Consider the triangle ABC with $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{AC} = \mathbf{b}$. Let E be the midpoint of AC and D be the midpoint of BC. The lines AD and BE meet at the point F. Assume these points are in $\mathbb{R}^3$ so that the cross product is defined.

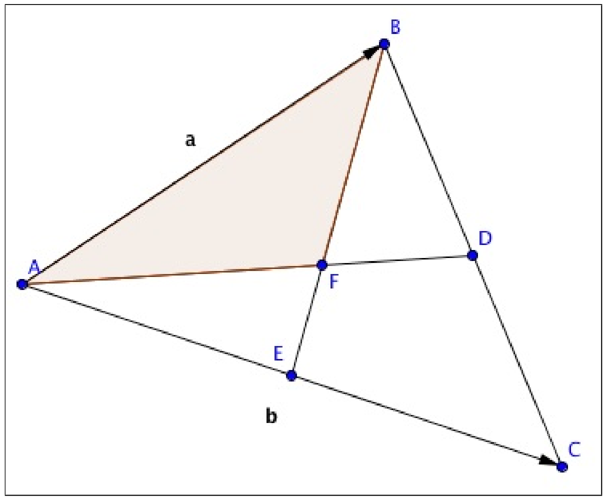
(a) Write down the area of the triangle ABC in terms of quantities involving $\mathbf{a}$ and $\mathbf{b}$.

(b) Show that $\overrightarrow{BE} = \frac{1}{2}\mathbf{b} - \mathbf{a}$ and find the vector $\overrightarrow{AD}$.

(c) Given that $\overrightarrow{BF} = \frac{2}{3}\overrightarrow{BE}$ and $\overrightarrow{AF} = \frac{2}{3}\overrightarrow{AD}$, find in terms of quantities involving $\mathbf{a}$ and $\mathbf{b}$ the area of $\triangle ABF$.

Solution 1

The triangle is drawn below. Make sure to draw a diagram for such problems – it helps immensely! The triangle ABF is shaded.



(a) The answer is $\text{Area}(\triangle ABC) = \frac{1}{2}\|\mathbf{a}\times\mathbf{b}\|$ (standard formula for area of a triangle spanned by two vectors in $\mathbb{R}^3$).

(b) From the geometric definition of addition of vectors, we know that $\overrightarrow{AB}+\overrightarrow{BE}=\overrightarrow{AE}$. Thus

$$\begin{aligned}\overrightarrow{BE} &= \overrightarrow{AE}-\overrightarrow{AB}\\ &= \frac{1}{2}\mathbf{b}-\mathbf{a},\end{aligned}$$

since $\overrightarrow{AB}=\mathbf{a}$ (given) and $\overrightarrow{AE}=\frac{1}{2}\mathbf{b}$, since E is the midpoint of A and C and $\overrightarrow{AC}=\mathbf{b}$ (see diagram above).

(c) We have

$$\begin{aligned}\text{Area}(\triangle ABF) &= \frac{1}{2}\left\|\overrightarrow{AB}\times\overrightarrow{AF}\right\| \quad \text{(area of a triangle spanned by } \overrightarrow{AB} \text{ and } \overrightarrow{AF}\text{)}\\
&= \frac{1}{2}\left\|\mathbf{a}\times\left(\frac{2}{3}\overrightarrow{AD}\right)\right\| \quad \left(\text{as } \overrightarrow{AB}=\mathbf{a} \text{ and } \overrightarrow{AF}=\frac{2}{3}\overrightarrow{AD}\right)\\
&= \frac{1}{2}\cdot\frac{2}{3}\left\|\mathbf{a}\times\overrightarrow{AD}\right\| \quad (*)\\
&= \frac{1}{3}\left\|\mathbf{a}\times\left(\frac{1}{2}(\mathbf{a}+\mathbf{b})\right)\right\| \quad (**)\\
&= \frac{1}{3}\cdot\frac{1}{2}\|\mathbf{a}\times(\mathbf{a}+\mathbf{b})\| \quad (*)\\
&= \frac{1}{6}\|\mathbf{a}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}\| \quad \text{(distributive law of cross products)}\\
&= \frac{1}{6}\|\mathbf{a}\times\mathbf{b}\|.\end{aligned}$$

Note in steps with $(*)$ above, we used the fact that $\|\mathbf{u}\times(\alpha\mathbf{v})\| = |\alpha|\,\|\mathbf{u}\times\mathbf{v}\|$ for all scalars $\alpha\in\mathbb{R}$ and vectors $\mathbf{u},\mathbf{v}\in\mathbb{R}^3$. This is true due to the fact that $\mathbf{u}\times(\alpha\mathbf{v})=\alpha(\mathbf{u}\times\mathbf{v})$ and $\|\alpha\mathbf{v}\|=|\alpha|\,\|\mathbf{v}\|$. The line marked $(**)$ used the fact that $\overrightarrow{AD}=\frac{1}{2}(\mathbf{a}+\mathbf{b})$, which follows from the fact that D is the midpoint of B and C, and the definition of $\mathbf{a}$ and $\mathbf{b}$. Also, note that the final line above follows from $\mathbf{a}\times\mathbf{a}=\mathbf{0}$. This is true for any $\mathbf{a}\in\mathbb{R}^3$, and can be shown by using the geometric definition of the cross product* and noting that the angle θ

* Namely $\mathbf{u}\times\mathbf{v}=\|\mathbf{u}\|\,\|\mathbf{v}\|\sin\theta\,\widehat{\mathbf{n}}$, where θ is the angle between $\mathbf{u},\mathbf{v}\in\mathbb{R}^3$ and $\widehat{\mathbf{n}}\in\mathbb{R}^3$ is the vector orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ and with orientation given by the *right-hand rule*.

between **a** and **a** is 0.

Problem 2

Let $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 3 \\ 6 \\ 2 \end{pmatrix}$. Find:

(a) $\mathbf{a} \cdot \mathbf{b}$

(b) $\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c})$

(c) the angle θ between **b** and **c**.

Solution 2

(a) We have

$$\begin{aligned}
\mathbf{a} \cdot \mathbf{b} &= \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \\
&= 1 \times 1 + 2 \times 0 + (-1) \times 1 \\
&= 1 + 0 - 1 \\
&= 0.
\end{aligned}$$

(b) We have

$$\begin{aligned}
\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c}) &= \begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad \text{(scalar triple product formula)} \\
&= \begin{vmatrix} 1 & 0 & 1 \\ 1 & 2 & -1 \\ 3 & 6 & 2 \end{vmatrix} \\
&= \begin{vmatrix} 1 & 0 & 1 \\ 0 & 2 & -2 \\ 0 & 6 & -1 \end{vmatrix} \quad (\text{using } R_2 \rightsquigarrow R_2 - R_1, R_3 \rightsquigarrow R_3 - 3R_1) \\
&= \begin{vmatrix} 1 & 0 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{vmatrix} \quad (\text{using } R_3 \rightsquigarrow R_3 - 3R_2) \\
&= 1 \times 2 \times 5 \quad \text{(determinant of triangular matrix is product of diagonal elements)} \\
&= 10.
\end{aligned}$$

Another valid method would be to first calculate $\mathbf{a} \times \mathbf{c}$ and then dot this with **b**.

(c) We have

$$\begin{aligned}\cos\theta &= \frac{\mathbf{b}\cdot\mathbf{c}}{\|\mathbf{b}\|\,\|\mathbf{c}\|}\\ &= \frac{3+0+2}{\sqrt{2}\sqrt{9+36+4}}\\ &= \frac{5}{\sqrt{2}\sqrt{49}}\\ &= \frac{5}{7\sqrt{2}}.\end{aligned}$$

Thus $\theta = \cos^{-1}\left(\frac{5}{7\sqrt{2}}\right)$.

Problem 3

1. Show that the line $\ell : \mathbf{x} = t\begin{pmatrix}1\\3\\1\end{pmatrix}$ lies on the plane $\Pi : x - y + 2z = 0$.
2. Consider the line $\ell : \frac{x-6}{5} = \frac{y-3}{2} = z+1$ and the plane $\Pi : 2x + y + z = 1$. Find where these intersect.
3. Convert $\mathbf{x} = \begin{pmatrix}1\\1\\1\end{pmatrix} + \lambda_1\begin{pmatrix}1\\1\\1\end{pmatrix} + \lambda_2\begin{pmatrix}1\\2\\2\end{pmatrix}$ to Cartesian form.

Solution 3

1. A point on ℓ has $x = t$, $y = 3t$, $z = t$ for some $t \in \mathbb{R}$. Substituting these into the plane's equation, we have

$$\begin{aligned}x - y + 2z &= t - 3t + 2t\\ &= 0,\end{aligned}$$

so the point $t\begin{pmatrix}1\\3\\1\end{pmatrix}$ satisfies Π's equation for any $t \in \mathbb{R}$, that is, ℓ lies on the plane Π.

■

2. A point $\begin{pmatrix}x\\y\\z\end{pmatrix}$ on ℓ has $x = 6 + 5\lambda$, $y = 3 + 2\lambda$, $z = -1 + \lambda$ for some $\lambda \in \mathbb{R}$ (this can be shown by converting ℓ to parametric vector form first). Substitute these into Π's equation to find the value of λ at the intersection point:

$$\begin{aligned}2x + y + z &= 1\\ \Rightarrow 2(6+5\lambda) + (3+2\lambda) + (-1+\lambda) &= 1\\ \Rightarrow 13\lambda + 14 &= 1\end{aligned}$$

$$\Rightarrow 13\lambda = -13$$
$$\Rightarrow \lambda = -1.$$

This is the value of λ on the line at the point of intersection. To find the actual point of intersection, we just substitute $\lambda = -1$ into the line parametrisation $x = 6 + 5\lambda$, $y = 3 + 2\lambda$, $z = -1 + \lambda$. This gives $x = 6 - 5 = 1, y = 3 - 2 = 1, z = -1 - 1 = -2$. Thus the point of intersection is $\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$.

Remark. You should check your answer by making sure this point actually lies on the plane, i.e. substitute it into Π's equation and see that it holds. If the answer you got did not satisfy Π's equation, you would know you made a mistake somewhere. (A point of intersection has to satisfy Π's equation, as it must lie on the plane if such a point of intersection exists.)

3. We give two methods.
Method 1 – Inspection
Note that $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ represents a plane since the two direction vectors are non-parallel. We observe that any point $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ on the plane has $y = z$ (since both are $1 + \lambda_1 + 2\lambda_2$). There is only one plane that has every point $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ on it satisfy $y = z$, and that is the plane $y = z$. Hence the plane has Cartesian equation $y = z$ (i.e. $y - z = 0$, $x \in \mathbb{R}$ free).
Method 2 – Usual process
A normal to the plane is

$$\begin{aligned}
\mathbf{n} &= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{vmatrix} \\
&= \mathbf{i} \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \\
&= 0\mathbf{i} - (2 - 1)\,\mathbf{j} + (2 - 1)\,\mathbf{k} \\
&= \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.
\end{aligned}$$

Thus using the point-normal form with $\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ being a point on the plane, the plane has Cartesian equation

$$\begin{aligned} \mathbf{n} \cdot \mathbf{x} &= \mathbf{n} \cdot \mathbf{a} \\ \iff 0x - y + z &= 0 - 1 + 1 \\ \iff -y + z &= 0, \end{aligned}$$

which agrees with the answer in Method 1.

Problem 4

Consider two points $\mathbf{u}$, $\mathbf{v}$ in $\mathbb{R}^3$ with position vectors $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$ and respectively.

(a) Find the cross product $\mathbf{u} \times \mathbf{v}$.

(b) Hence find the Cartesian equation of the plane parallel to $\mathbf{u}$ and $\mathbf{v}$ and passing through the point $\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$.

(c) Find the distance of $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ to the plane in (b).

Solution 4

(a) The answer is

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{vmatrix} \\ &= \mathbf{i} \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} \\ &= 1\mathbf{i} - 2\mathbf{j} + (2-3)\,\mathbf{k} \\ &= \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}. \end{aligned}$$

(b) We have $\mathbf{n} := \mathbf{u} \times \mathbf{v} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$ being a normal to the plane. A point on the plane is $\mathbf{a} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$. Hence the plane has Cartesian equation

$$\begin{aligned} \mathbf{n} \cdot \mathbf{x} &= \mathbf{n} \cdot \mathbf{a} \\ \iff x - 2y - z &= -9 \quad (\text{as } \mathbf{n} \cdot \mathbf{a} = -9). \end{aligned}$$

(c) Let X be the point $(1,1,1)$ and A the point $(1,4,2)$ on the plane. Note $\overrightarrow{AX} = (1,1,1)^T - (1,4,2)^T = (0,-3,-1)^T$ Then the desired distance d is

$$\begin{aligned} d &= \left\|\text{proj}_{\mathbf{n}}\overrightarrow{AX}\right\| \\ &= \frac{\left|\overrightarrow{AX}\cdot\mathbf{n}\right|}{\|\mathbf{n}\|} \quad (*) \\ &= \frac{\left|(0,-3,-1)^T\cdot(1,-2,-1)^T\right|}{\left\|(1,-2,-1)^T\right\|} \\ &= \frac{|0+6+1|}{\sqrt{1+4+1}} \\ &= \frac{7}{\sqrt{6}}. \end{aligned}$$

Note in $(*)$, we made use of the general fact that $\|\text{proj}_{\mathbf{b}}\mathbf{a}\| = \frac{|\mathbf{a}\cdot\mathbf{b}|}{\|\mathbf{b}\|}$ (this is true for any $\mathbf{a},\mathbf{b}\in\mathbb{R}^n$, for any n, with $\mathbf{b}\neq\mathbf{0}$).

Problem 5

The non-zero point Q has position vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$. The vector $\overrightarrow{OQ}$ makes angles α,β and γ respectively with the x,y and z axes.

(a) By considering the vector $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, show that

$$a = \sqrt{a^2+b^2+c^2}\cos\alpha.$$

(b) Deduce that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

(c) If the angles α and β are complementary, what can be said about the vector $\overrightarrow{OQ}$?

Solution 5

(a) Consider the right-angled triangle with vertices (a,b,c), $(0,0,0)$, $(a,0,0)$. We observe it has hypotenuse $\sqrt{a^2+b^2+c^2}$, and side a (with angle α between them). This gives the desired equation through high school right-angle trigonometry. The result can also be proved by noting that $\mathbf{q}\cdot\mathbf{e}_1 = a$ and $\underbrace{\mathbf{q}}_{=\sqrt{a^2+b^2+c^2}} \cdot \underbrace{\mathbf{e}_1}_{=1} = \|\mathbf{q}\|\,\|\mathbf{e}_1\|\cos\alpha$, where $\mathbf{q} = (a,b,c)^T$.

(b) Rearranging the previous result gives $\cos^2\alpha = \frac{a^2}{a^2+b^2+c^2}$. Repeating similarly for b and c gives the three equations

$$\begin{aligned}\cos^2\alpha &= \frac{a^2}{a^2+b^2+c^2}\\ \cos^2\beta &= \frac{b^2}{a^2+b^2+c^2}\\ \cos^2\gamma &= \frac{c^2}{a^2+b^2+c^2}.\end{aligned}$$

Summing these three equations gives

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{a^2+b^2+c^2}{a^2+b^2+c^2} = 1.$$

(c) If $\alpha + \beta = \frac{\pi}{2}$, then $\sin\alpha = \cos\beta$, using the identity $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$. The Pythagorean identity thus implies that $1 = \cos^2\alpha + \sin^2\alpha = \cos^2\alpha + \cos^2\beta$. Hence from the previous part's result, we must have $\cos^2\gamma = 0$, which implies that $\gamma = \frac{\pi}{2}$ (note $\gamma \in [0, \pi]$, as it is an angle between two vectors). This means that the angle Q makes to the z-axis is 90°, i.e. $\overrightarrow{OQ}$ lies in the x-y plane (i.e. $c = 0$).

Problem 6

Let $z = 5 + 5i$ and $w = 2 + i$. Find:

(a) $2z + 3\overline{w}$

(b) $z(w-1)$

(c) $\dfrac{z}{w}$.

Solution 6

(a) We have

$$\begin{aligned}2z + 3\overline{w} &= 2(5+5i) + 3(2-i)\\ &= 16 + 7i.\end{aligned}$$

(b) We have

$$\begin{aligned}z(w-1) &= z((2+i)-1)\\ &= (5+5i)(1+i)\\ &= (5-5) + i(5+5)\\ &= 10i.\end{aligned}$$

(c) We have

$$\begin{aligned}\frac{z}{w} &= \frac{z\overline{w}}{w\overline{w}} \\ &= \frac{(5+5i)(2-i)}{|w|^2} \\ &= \frac{(10+5)+i\,(-5+10)}{4+1} \\ &= \frac{15+5i}{5} \\ &= 3+i.\end{aligned}$$

Problem 7

Let the set $\mathcal{S}$ in the complex plane defined by

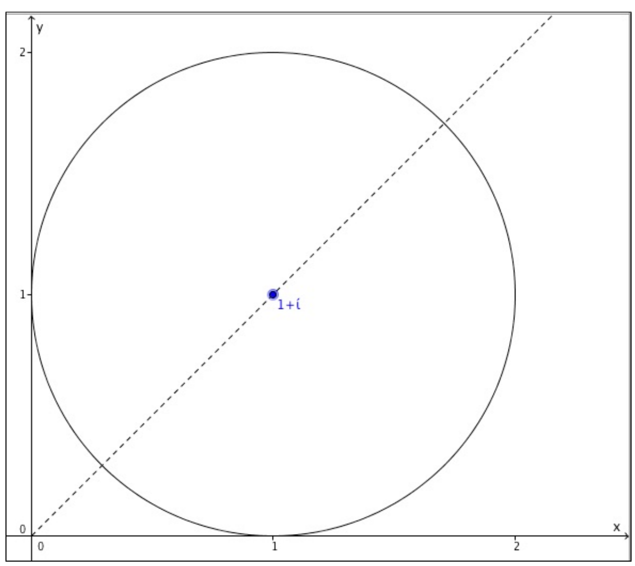
$$\mathcal{S} = \{z \in \mathbb{C} : |z-(1+i)| = 1\}\,.$$

(a) Sketch the set $\mathcal{S}$ on a labelled Argand diagram.

(b) By considering your sketch, or otherwise, find the maximum value of $|z|$ for $z \in \mathcal{S}$.

Solution 7

(a) The set $\mathcal{S}$ is a circle in the complex plane centred at $1+i$ and with radius 1. This is sketched below. We have also drawn a ray connecting the origin to the circle centre for use in the next part.



(b) By inspection of the diagram, the minimum value of $|z|$ on $\mathcal{S}$ is $\sqrt{2}-1$ and the maximum value is $\sqrt{2}+1$. (This is because the circle centre is distance $\sqrt{2}$ from the origin and the circle has radius 1.) These occur at the points of intersection of the dashed line with the circle in our diagram.

Problem 8

Let $z = \cos\theta + i\sin\theta$, where $\theta \in \mathbb{R}$.

(a) Prove that

$$z^n + \frac{1}{z^n} = 2\cos n\theta \quad \text{and} \quad z^n - \frac{1}{z^n} = 2i\sin n\theta$$

for any positive integer n.

(b) Deduce that

$$(2\cos\theta)^4(2i\sin\theta)^4 = \left(z^2 - \frac{1}{z^2}\right)^4.$$

(c) Hence show that $\cos^4\theta\sin^4\theta = \dfrac{1}{128}(\cos 8\theta - 4\cos 4\theta + 3)$.

Solution 8

(a) Note by De Moivre's Theorem, we have for all integers n that $z^n = \cos n\theta + i\sin n\theta$. Thus $\operatorname{Re}(z^n) = \cos n\theta$ and $\operatorname{Im}(z^n) = \sin n\theta$. Now, replacing n with $-n$ (which is also an integer for n integer), we have $z^{-n} = \cos(-n\theta) + i\sin(-n\theta) = \cos n\theta - i\sin n\theta = \overline{z^n}$. Thus for all positive integers n, we have

$$\begin{aligned} z^n + \frac{1}{z^n} &= z^n + z^{-n} \\ &= z^n + \overline{z^n} \\ &= 2\operatorname{Re}(z^n) \\ &= 2\cos n\theta, \end{aligned}$$

and

$$\begin{aligned} z^n - \frac{1}{z^n} &= z^n - z^{-n} \\ &= z^n - \overline{z^n} \\ &= 2i\operatorname{Im}(z^n) \\ &= 2i\sin n\theta. \end{aligned}$$

(b) We have using the previous result

$$\begin{aligned} (2\cos\theta)^4(2i\sin\theta)^4 &= \left(z + \frac{1}{z}\right)^4\left(z - \frac{1}{z}\right)^4 \\ &= \left[\left(z + \frac{1}{z}\right)\left(z - \frac{1}{z}\right)\right]^4 \\ &= \left(z^2 - \frac{1}{z^2}\right)^4 \quad \text{(difference of two squares).} \end{aligned}$$

(c) Note that

$$\begin{aligned}
\left(z^2 - \frac{1}{z^2}\right)^4 &= z^8 - 4z^4 + 6 - 4z^{-4} + z^{-8} \quad \text{(binomial expansion)} \\
&= \left(z^8 + \frac{1}{z^8}\right) - 4\left(z^4 + \frac{1}{z^4}\right) + 6 \\
&= 2\cos 8\theta - 8\cos 4\theta + 6 \quad \text{(using part (a)'s result)} \\
&= 2\left(\cos 8\theta - 4\cos 4\theta + 3\right).
\end{aligned}$$

Therefore, from (b), we have

$$\begin{aligned}
(2\cos\theta)^4 (2i\sin\theta)^4 &= 2\left(\cos 8\theta - 4\cos 4\theta + 3\right) \\
\Rightarrow 2^4 \times 2^4 \cos^4\theta \sin^4\theta &= 2\left(\cos 8\theta - 4\cos 4\theta + 3\right) \quad \text{(note } i^4 = 1\text{)} \\
\Rightarrow \cos^4\theta \sin^4\theta &= \frac{1}{2^7}\left(\cos 8\theta - 4\cos 4\theta + 3\right),
\end{aligned}$$

which yields the desired result since $2^7 = 128$.

Problem 9

Let

$$S = e^{i\theta} + \frac{e^{3i\theta}}{3} + \frac{e^{5i\theta}}{3^2} + \frac{e^{7i\theta}}{3^3} + \cdots.$$

(a) Prove that

$$S = \frac{3\left(3e^{i\theta} - e^{-i\theta}\right)}{10 - 6\cos 2\theta}.$$

(b) Hence, or otherwise, find the sum

$$T := \sin\theta + \frac{\sin 3\theta}{3} + \frac{\sin 5\theta}{3^2} + \frac{\sin 7\theta}{3^3} + \cdots.$$

Section 9

(a) Note that S is a geometric series with common ratio $r = \frac{e^{2i\theta}}{3}$. Since $|r| = \frac{1}{3} < 1$, the sum converges, so using the geometric series formula, we have

$$\begin{aligned}
S &= \frac{e^{i\theta}}{1 - r} \\
&= \frac{e^{i\theta}}{1 - \frac{e^{2i\theta}}{3}} \\
&= \frac{3e^{i\theta}}{3 - e^{2i\theta}} \\
\Rightarrow S &= \frac{3e^{i\theta}\left(3 - e^{-2i\theta}\right)}{\left|3 - e^{2i\theta}\right|^2},
\end{aligned}$$

by multiplying top and bottom by the conjugate of $3-e^{2i\theta}$, which is $3-e^{-2i\theta}$ (conjugate of a difference is the difference of the conjugates). Now, we note that

$$\begin{aligned}\left|3-e^{2i\theta}\right|^2 &= \left|3-\cos 2\theta - i\sin 2\theta\right|^2\\ &= (3-\cos 2\theta)^2+\sin^2 2\theta\\ &= 9-6\cos 2\theta+1 \quad (\text{note } \cos^2 2\theta+\sin^2 2\theta=1)\\ &= 10-6\cos 2\theta.\end{aligned}$$

Putting this into the result established above, we have

$$\begin{aligned}S &= \frac{3e^{i\theta}\left(3-e^{-2i\theta}\right)}{\left|3-e^{2i\theta}\right|^2}\\ &= \frac{3e^{i\theta}\left(3-e^{-2i\theta}\right)}{10-6\cos 2\theta}\\ &= \frac{3\left(3e^{i\theta}-e^{-i\theta}\right)}{10-6\cos 2\theta}.\end{aligned}$$

This proves the claim. ■

(b) Note that T is just the imaginary part of S (since each term in the sum T is the imaginary part of the corresponding term in S and the sum of imaginary parts is equal to the imaginary part of the sum). Thus

$$\begin{aligned}T &= \operatorname{Im}(S)\\ &= \operatorname{Im}\left(\frac{3\left(3e^{i\theta}-e^{-i\theta}\right)}{10-6\cos 2\theta}\right)\\ &= \frac{3}{10-6\cos 2\theta}\times \operatorname{Im}\left(3e^{i\theta}-e^{-i\theta}\right) \quad (\text{factoring out real expressions})\\ &= \frac{3}{10-6\cos 2\theta}\times\left(\operatorname{Im}\left(3e^{i\theta}\right)-\operatorname{Im}\left(e^{-i\theta}\right)\right)\\ &= \frac{3}{10-6\cos 2\theta}\times(3\sin\theta-(-\sin\theta))\\ &= \frac{3\times 4\sin\theta}{10-6\cos 2\theta}\\ &= \frac{6\sin\theta}{5-3\cos 2\theta}.\end{aligned}$$

Problem 10

A system of three equations in three unknowns x, y and z has been reduced to the following form

$$\left[\begin{array}{ccc|c} 2 & 0 & -4 & b_1\\ 3 & 1 & -2 & b_2\\ -2 & -1 & 0 & b_3\end{array}\right].$$

(a) Reduce the matrix to row-echelon form.

(b) For which value(s) of b_1, b_2, b_3 will the system have no solution?

(c) For which value(s) of b_1, b_2, b_3 will the system have infinitely many solutions?

(d) For the value(s) of b_1, b_2, b_3 determined in part (c), find the general solution (in terms of b_1, b_2 and b_3).

Solution 10

(a) We write the augmented part as having one column each for b_1, b_2, b_3, as this means we don't have to keep track of algebra when doing row reduction. Our augmented matrix is thus

$$\left[\begin{array}{ccc|ccc} 2 & 0 & -4 & 1 & 0 & 0 \\ 3 & 1 & -2 & 0 & 1 & 0 \\ -2 & -1 & 0 & 0 & 0 & 1 \end{array}\right].$$

The first column of the augmented part represents the coefficient of b_1 in each row, and similarly for the second and third columns (i.e. they represent the coefficient of b_2 and b_3 (respectively) in each row). First perform $R_1 \rightsquigarrow \frac{1}{2}R_1$ to obtain

$$\left[\begin{array}{ccc|ccc} \boxed{1} & 0 & -2 & \frac{1}{2} & 0 & 0 \\ 3 & 1 & -2 & 0 & 1 & 0 \\ -2 & -1 & 0 & 0 & 0 & 1 \end{array}\right].$$

(It is helpful to box or circle your pivot entries as we have done above, to help you see them clearly.) Now perform $R_2 \rightsquigarrow R_2 - 3R_1$, $R_3 \rightsquigarrow R_3 + 2R_1$:

$$\left[\begin{array}{ccc|ccc} \boxed{1} & 0 & -2 & \frac{1}{2} & 0 & 0 \\ 0 & \boxed{1} & 4 & -\frac{3}{2} & 1 & 0 \\ 0 & -1 & -4 & 1 & 0 & 1 \end{array}\right].$$

Finally, perform $R_3 \rightsquigarrow R_3 + R_2$ to obtain

$$\left[\begin{array}{ccc|ccc} \boxed{1} & 0 & -2 & \frac{1}{2} & 0 & 0 \\ 0 & \boxed{1} & 4 & -\frac{3}{2} & 1 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 1 & 1 \end{array}\right].$$

Thus a row-echelon form is

$$\left[\begin{array}{ccc|c} \boxed{1} & 0 & -2 & \frac{1}{2}b_1 \\ 0 & \boxed{1} & -4 & -\frac{3}{2}b_1 + b_2 \\ 0 & 0 & 0 & -\frac{1}{2}b_1 + b_2 + b_3 \end{array}\right].$$

(b) The system has no solution if and only if the right-hand column in the row-echelon form is leading. From the above row-echelon form, we thus see the answer is when b_1, b_2, b_3 satisfy $-\frac{1}{2}b_1 + b_2 + b_3 \neq 0$.

(c) The system has infinitely many solutions if and only if it has a solution and has a non-leading column in the row-echelon form of the coefficient matrix. We can see from above that the row-echelon form of the coefficient matrix indeed has a non-leading column (column 3). Also, there exists a solution to the system if and only if $-\frac{1}{2}b_1 + b_2 + b_3 = 0$ (so that the right-hand column is non-leading). So the answer is when b_1, b_2, b_3 are such that $-\frac{1}{2}b_1 + b_2 + b_3 = 0$.

(d) Assume b_1, b_2, b_3 are such that $-\frac{1}{2}b_1 + b_2 + b_3 = 0$. Then our row-echelon matrix is

$$\left[\begin{array}{ccc|c} \boxed{1} & 0 & -2 & \frac{1}{2}b_1 \\ 0 & \boxed{1} & -4 & -\frac{3}{2}b_1 + b_2 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

To get the general solution, we see column 3 is non-leading, so set $x_3 = \alpha$ (a free parameter). Then using back-substitution, row 2 implies that $x_2 = -\frac{3}{2}b_1 + b_2 + 4x_3 = -\frac{3}{2}b_1 + b_2 + 4\alpha$. Also, row 1 implies that $x_1 = \frac{1}{2}b_1 + 2x_3 = \frac{1}{2}b_1 + 2\alpha$. The general solution is thus

$$\begin{aligned} \mathbf{x} &= \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}b_1 + 2\alpha \\ -\frac{3}{2}b_1 + b_2 + 4\alpha \\ \alpha \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}b_1 \\ -\frac{3}{2}b_1 + b_2 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}. \end{aligned}$$

Problem 11

Consider the following system of linear equations.

$$x + y - z = 2$$
$$2x + 3y + z = 6$$

(a) Using Gaussian Elimination, find the general solution to the system of equations.

(b) Hence or otherwise, find the solution to the system with the property that the sum of the x, y and z coordinates is 0.

Solution 11

(a) Writing the system in augmented matrix form, we have

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 2 \\ 2 & 3 & 1 & 6 \end{array}\right].$$

Perform the elementary row operation $R_2 \rightsquigarrow R_2 - 2R_1$ to obtain

$$\left[\begin{array}{ccc|c} \boxed{1} & 1 & -1 & 2 \\ 0 & \boxed{1} & 3 & 2 \end{array}\right].$$

The third column is non-leading, so set $z = \alpha$ (free parameter). Then using back-substitution, we find

$$\begin{aligned} y &= 2 - 3z \quad \text{(row 2)} \\ \Rightarrow y &= 2 - 3\alpha, \end{aligned}$$

and

$$\begin{aligned} x &= 2 - y + z \quad \text{(row 1)} \\ &= 2 - (2 - 3\alpha) + \alpha \\ \Rightarrow x &= 4\alpha. \end{aligned}$$

Thus the general solution to the system is

$$\begin{aligned} \mathbf{x} &= \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ &= \begin{pmatrix} 4\alpha \\ 2 - 3\alpha \\ \alpha \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}. \end{aligned}$$

(b) Using (a), a solution to this system with sum of coordinates equal to 0 will be

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4\alpha \\ 2 - 3\alpha \\ \alpha \end{pmatrix}$$

where $x + y + z = 0$, i.e.

$$\begin{aligned} 4\alpha + (2 - 3\alpha) + \alpha = 0 \\ \Rightarrow 2\alpha = -2 \\ \Rightarrow \alpha = -1. \end{aligned}$$

So putting $\alpha = -1$, the desired solution (which is unique) is

$$\begin{aligned} \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} 4 \times (-1) \\ 2 + 3 \\ -1 \end{pmatrix} \\ &= \begin{pmatrix} -4 \\ 5 \\ -1 \end{pmatrix}. \end{aligned}$$

Problem 12

Consider the matrix $M = \begin{pmatrix} 2 & i \\ 1+i & \alpha \end{pmatrix}$.

(a) Find the conditions on α so that the matrix is invertible.

(b) Hence find M^{-1} for $\alpha = 1$.

Solution 12

(a) The matrix is invertible iff its determinant is non-zero. The determinant is

$$\begin{aligned}\det M &= 2\alpha - i(1+i)\\ &= 2\alpha - i + 1.\end{aligned}$$

So the matrix is invertible iff $2\alpha - i + 1 \neq 0$.

(b) When $\alpha = 1$, the matrix M is $M = \begin{pmatrix} 2 & i \\ 1+i & 1 \end{pmatrix}$ and its determinant is $2 - i + 1 = 3 - i$ (putting $\alpha = 1$ in the determinant formula obtained in the previous part). So using the formula for the inverse of a 2×2 matrix, we have

$$\begin{aligned}M^{-1} &= \frac{1}{3-i}\begin{pmatrix} 1 & -i \\ -1-i & 2 \end{pmatrix}\\ &= \frac{3+i}{10}\begin{pmatrix} 1 & -i \\ -1-i & 2 \end{pmatrix}\\ &= \frac{1}{10}\begin{pmatrix} 3+i & -i(3+i) \\ (-1-i)(3+i) & 2(3+i) \end{pmatrix}\\ &= \frac{1}{10}\begin{pmatrix} 3+i & 1-3i \\ -2-4i & 6+2i \end{pmatrix}.\end{aligned}$$

Remark. Remember, if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\boxed{A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$ (i.e. divide by determinant, swap main diagonal elements, and negate the others), assuming $\det A \neq 0$.

Problem 13

Consider the matrix $A = \begin{pmatrix} 3 & 3 & 0 \\ 3 & 3 & 0 \\ 0 & 0 & -2 \end{pmatrix}$

(a) Write down the matrix $A - \lambda I$, where λ is a scalar parameter and I is the identity matrix.

(b) Find the determinant D of $A - \lambda I$ as a function of λ.

(c) Find all solutions to $D = 0$.

(d) For $\lambda = 6$, find the general solution to the linear system $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

(e) Give a geometric interpretation of your solution to the previous part.

Solution 13

(a) The matrix $A - \lambda I$ is just A with λ subtracted from its diagonal elements, i.e.

$$A - \lambda I = \begin{pmatrix} 3-\lambda & 3 & 0 \\ 3 & 3-\lambda & 0 \\ 0 & 0 & -2-\lambda \end{pmatrix}.$$

(b) Expand the determinant along the third column, so we get

$$\begin{aligned} D &= (-2-\lambda)\begin{vmatrix} 3-\lambda & 3 \\ 3 & 3-\lambda \end{vmatrix} \\ &= (-2-\lambda)\left((3-\lambda)^2 - 9\right) \\ &= (-2-\lambda)\left((3-\lambda+3)(3-\lambda-3)\right) \quad \text{(difference of two squares)} \\ &= (-2-\lambda)(6-\lambda)(-\lambda) \\ &= -\lambda(\lambda+2)(\lambda-6). \end{aligned}$$

(c) From the above answer, we can see that the solutions to $D = 0$ are $\lambda = 0, -2$ or 6.

(d) Put $\lambda = 6$ in (a), so we are solving $B\mathbf{x} = \mathbf{0}$ where

$$B = A - 6I = \begin{pmatrix} -3 & 3 & 0 \\ 3 & -3 & 0 \\ 0 & 0 & -8 \end{pmatrix}.$$

To solve this, row-reduce B as usual (note that since the right-hand sides are all 0, we don't need to worry about putting an augmented column of 0's, since they will stay as 0's no matter what elementary row operations are used. Just keep in your mind an imaginary augmented column of 0's.). Use the operation $R_2 \rightsquigarrow R_2 + R_1$ to obtain the row-echelon form

$$\begin{pmatrix} \boxed{-3} & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \boxed{-8} \end{pmatrix}.$$

Column 2 is non-leading, so set $y = t$ (free parameter). Then row 3 tells us $-8z = 0 \Rightarrow z = 0$. Also, row 1 gives us $-3x + 3y = 0 \Rightarrow x = y = t$. So the general solution is

$$\begin{aligned} \mathbf{x} &= \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ &= \begin{pmatrix} t \\ t \\ 0 \end{pmatrix} \\ &= t\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}. \end{aligned}$$

(e) The solutions to the system $(A - 6I)\,\mathbf{x} = \mathbf{0}$ lie along the line through the origin spanned by $\begin{pmatrix}1\\1\\0\end{pmatrix}$.

Aside. A more advanced geometric interpretation is as follows: we have $(A - 6I)\,\mathbf{x} = \mathbf{0} \iff A\mathbf{x} - 6I\mathbf{x} = \mathbf{0} \iff A\mathbf{x} = 6\mathbf{x}$ (as $I\mathbf{x} = \mathbf{x}$). Thus the solutions to the system in (d) are the solutions to the equation $A\mathbf{x} = 6\mathbf{x}$. This tells us that the vectors along the line through the origin spanned by $\begin{pmatrix}1\\1\\0\end{pmatrix}$ are all simply scaled by a factor of 6 (and have the same direction) when acted upon by A. (Furthermore, these are the only vectors with this property, as they formed the general solution to the system in (d).) These ideas lead in to the study of *eigenvalues and eigenvectors*, which is beyond the scope of MATH1131/1141, but will be taught in MATH1231/1241.