MATH1081 Revision

Proofs and Combinatorics

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Proofs

“Guessing before proving! Need I remind you that it is so that all important discoveries have been made?”

-Henri Poincaré, The Value of Science

Summary

Main umbrella:

- Direct/Indirect(Contradiction) Proofs
- Proof by Exhaustion
- Constructive/Non-Constructive Proofs
- Proof by Induction

Special cases:

- If and only If Proofs
- Existence and Uniqueness Proofs
- Limit Proofs

Direct Proofs

Definition

Direct Proofs: A direct proof of a conditional statement $p \to q$ is constructed when the **first step is the assumption that** p **is true**; subsequent steps are constructed using rules of inference, with the **final step showing that** q **must also be true.**

Intuition

- Useful for proofs involving numbers instead of abstract variables.
- Start on one side of the equation and work towards the other.
- Work backwards/Plan a solution and invert it.

Example and Structure (Direct Proofs)

Structure

- Proving q is true implies p is true: "Since p is true... Then q is true."
- Proving "if p then q": "Suppose p is true... Hence q is true."

Example

Prove that the product of two odd numbers is odd.

Solution

Let x and y be odd numbers such that $x = 2m + 1$, $y = 2n + 1$, where $n, m \in \mathbb{Z}$.

$x \times y = (2m + 1) \times (2n + 1) = 2([mn] + [m + n]) + 1.$

Therefore $xy = 2i + 1$ where $i = mn + m + n$, which is odd by definition.

Therefore the product of two odd numbers is odd as required.

Generalization

Often times, Direct Proofs can be converted to a universal statement or "all" statement through a process called generalisation.

Proofs by Contradiction

Definition

Proof by Contradiction: A form of proof that establishes the truth or validity of a proposition by first **assuming that the opposite proposition is true**, and then **shows that such an assumption leads to a contradiction**.

Intuition

- Useful when proving objects related to infinity (irrationality, primes).
- Usually used when trying to "not".
- Usually used as a last resort, as there exist and element of "guessing and checking".
- Begin by assuming the negation of the statement to be proved.
- Proof by contrapositive is very similar.

Example and Structure (Indirect Proofs)

Structure

- Proving statement A is true:

  Suppose that A is false...
  Therefore...
  But this is also false...
  Therefore the original assumption was also false. That is A is true.

- Proving if A then B is true:

  Suppose that A is true but B is false...
  Therefore...
  But this is a contradiction...
  So if A is true, then B is true.

Example (Rosen Q8, pg 91)

Prove that if n is a perfect square, $n+2$ is not a perfect square.

Solution

Suppose n is a perfect square, then $n = k^2$ for some integer $k \in \mathbb{Z}^+$.

For a contradiction, suppose that $n + 2$ is also a perfect square, where $n + 2 = l^2$ for some integer $l \in \mathbb{Z}^+$.

Then:

$$\begin{aligned} n + 2 - n &= l^2 - k^2 \\ l^2 - k^2 &= 2 \\ (l + k)(l - k) &= 2 \end{aligned}$$

Since k, l are both integers, $(k + l), (k - l)$ have to also both be integers, or that one of them equals 1 and the other 2, since $l > k$.

Solving simultaneously, we obtain $k = \frac{1}{2}$ or $k = -\frac{1}{2}$ which is impossible as we assume that $k \in \mathbb{Z}$.

Thus, this is a contradiction and we conclude that if n is a perfect square, $n + 2$ cannot be a perfect square. This is similar for the $n, l \in \mathbb{Z}^-$ scenario except opposite.

Proofs by Exhaustion

Definition

Proof by Exhaustion: A proof where the statement to be proved is **split into a finite number of cases** or sets of equivalent cases and each type of case is checked to see if the proposition in question holds.

Intuition

- Useful when there is a "separation" division of the problem into a small number of cases; for example, questions involving absolute values, congruence or divisibility.
- First state all cases and check that all possibilities are covered before proving each one.

Example and Structure (Proofs by Exhaustion)

Structure

The object to be proven can only take on n possible forms which are: Case A, Case B,..., Case N.

Case A:...
Case B:...
...
Case N:...

Example (2015 T2 3(iii))

Prove that if x is a real number, then:

$$|x| + 2|x - 3| \leq x^2 - 5x + 10$$

Solution

Intuition

- This should hold true for all $x \in \mathbb{R}$.
- Therefore, there should be 3 cases. Why?

Proof.

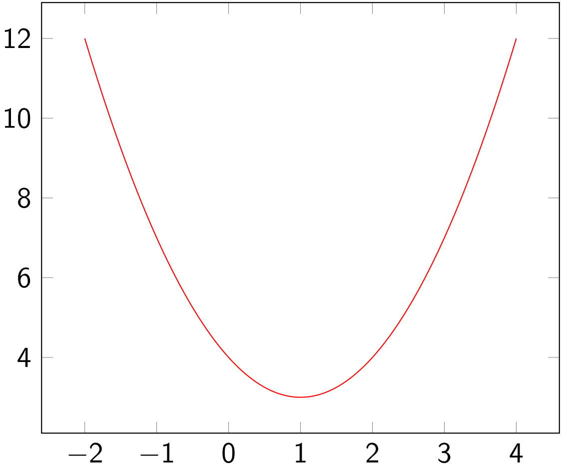
Let $f(x) = x^2 - 5x + 10 - |x| - 2|x - 3|$. We aspire to prove $f(x) = 0$.

Case 1: $x < 0$. Expanding the absolute values, we obtain:

$$\begin{aligned} f(x) &= x^2 - 5x + 10 + x - 2(3 - x) \\ &= x^2 - 2x + 4 \end{aligned}$$

The RHS equation has no real roots and lies above the x axis. Therefore $f(x) \geq 0 \quad \forall x < 0$.

Sample Plots



Solution cont.

Case 2: $0 \leq x < 3$. Expanding the absolute values, we obtain:

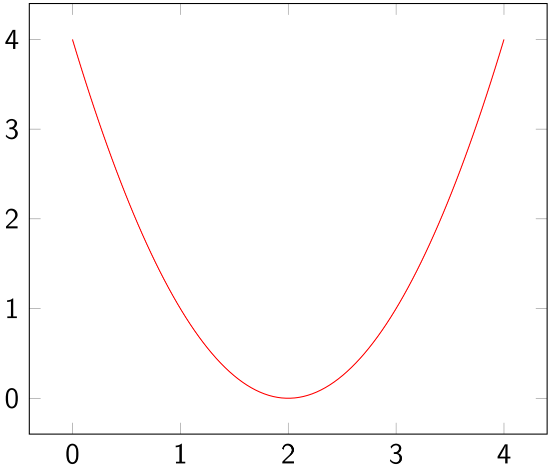
$$f(x) = x^2 - 5x + 10 - x - 2(3 - x)$$

Rearranging, we obtain the equation:

$$f(x) = x^2 - 4x + 4$$

The RHS is a concave up parabola with its only zero at $x = 2$. Therefore, $f(x) \geq 0 \quad \forall x \in [0, 3)$.

Sample Plots



Solution cont.

Case 3: $x \geq 3$. Expanding the absolute values, we obtain:

$$f(x) = x^2 - 5x + 10 - x - 2(x-3)$$

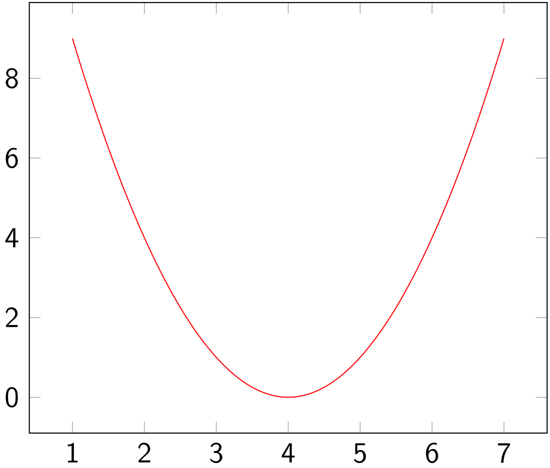
Rearranging, we obtain the equation:

$$f(x) = x^2 - 8x + 16$$

The RHS is a concave up parabola with its only zero at $x = 4$. Therefore, the $f(x) \geq 0 \quad \forall x \geq 3$.

Clearly, in every case, $f(x) \geq 0$ which implies $|x| + 2|x-3| \leq x^2 - 5x + 10$ given x is a real number.

Sample Plots



Constructive Proofs

Definition

Constructive Proofs: A method of proof that demonstrates the existence of a mathematical object by **creating or providing a method** for creating the object.

Intuition

- Use for "there exist" questions.
- Useful for questions involving matrices, integrals, functions etc.
- Might be useful to find a specific example first, before generalising it to a object creating method.

Example and Structure (Constructive Proofs)

Example

Prove that there exist irrational x, y such that xy is rational.

Structure

Proving the existence of a structure x: Let there exist a, b, c such that... Therefore x is true.

Further Question

Example

Prove that there exist irrational x, y such that x^y is rational.

Say we have a number $\sqrt{2}^{\sqrt{2}}$. If it is rational, $x = y = \sqrt{2}$. If it is not rational, $x = \sqrt{2}^{\sqrt{2}}, y = \sqrt{2}$. This means $x^y = \sqrt{2}^2 = 2$ which is also rational as required.

“Some” Statements

Definition

An existential statement, or **“some”** statement, asserts that there exists something (one object or more) which satisfies a certain condition.

Intuition

- Easiest way to verify is by providing an example or counterexample.

Non-Constructive Proofs

Definition

Non-Constructive Proofs: A proof that proves the existence of a particular kind of object without providing an example.

Intuition

- Useful for situations where computing exact numbers are hard/tedious.
- Often requires additional and specific background knowledge.

Example and Structure (Non-Constructive Proofs)

Example

Prove that for every cubic equation with real coefficients, there is at least one real root.

Structure

State theorem/principle that satisfying the proving conditions. Therefore the theorem is true.

Solution

Intuition

Some ways to approach this:

- Complex Conjugate Root Theorem
- Intermediate Value Theorem

Since the coefficients of the cubic polynomial is real, by the complex conjugate root theorem, any complex roots must exist in conjugate pairs.

Case 1: 1 pair of complex roots, 1 real root.

Case 2: no complex roots, 3 real root.

Therefore, for every cubic equation with real coefficients, there is at least one real root as required.

Proof by Induction

Definition

Mathematical Induction: Proving technique used to prove that a property holds for every natural number n and so on.

Intuition

- Useful when you need to proof for an variable n, and there is a certain relationship between n and $n+1$.
- You guys done this in high school.

Law of Induction

Intuition

For a given statement, if $n = 0$ is true, and if $n = k$ is true implies $n = k + 1$ is true, then setting $k = 0, n = 1$ is true, $n = 2$ is true, ...

Structure

- Define predicate $P(n)$.
- **Basis Case (Step 1)**: Prove true for $P(n_0)$;
- **Inductive Step (Step 2)**: Assume true for $P(k)$, then prove true for $P(k + 1)$ given $P(k)$ is true.
- **Inductive Hypothesis Conclusion (Step 3)**: As we have proved the basis step and the inductive step, it follows by induction that the equation is true for all integers $n \geq 0$.

Strong Law of Induction

Intuition

"The difference between weak induction and strong induction only appears in induction hypothesis. In weak induction, we only assume that particular statement holds at k-th step, while in strong induction, we assume that the **particular statement holds at all the steps from the base case to k-th step**."

Structure

- Define predicate $P(n)$.
- **Basis Case (Step 1)**: Prove true for $P(n_0), P(n_1)$;
- **Inductive Step (Step 2)**: Assume true for $P(k)$, $P(k+1)$ then prove true for $P(k+2)$ given $P(k), P(k+1)$ is true.
- **Inductive Hypothesis Conclusion (Step 3)**: As we have proved the basis step and the inductive step, it follows by induction that the equation is true for all integers $n \geq 0$.

Induction Examples

2011 S2 3(iii)

Prove that if n is a positive integer, then:

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ... + \frac{1}{\sqrt{n}} < 2\sqrt{n}$$

Solution

Proof: Let P(n) be the predicate:

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ... + \frac{1}{\sqrt{n}} < 2\sqrt{n}$$

Basis Step: Solve for $P(0)$.

$$\frac{1}{1} < 2\sqrt{1}$$

$\therefore$ P(0) is true.

Inductive Step: Let $k \geq 0$ and assume true for $P(k)$, i.e.

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ... + \frac{1}{\sqrt{k}} < 2\sqrt{k}$$

And consequentially, prove true for $P(k+1)$, i.e.

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ... + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} < 2\sqrt{k+1}$$

Solution cont.

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+...+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}}<2\sqrt{k}+\frac{1}{\sqrt{k+1}}$$

Therefore, using the inductive hypothesis, it suffices to prove:

$$2\sqrt{k}+\frac{1}{\sqrt{k+1}}<2\sqrt{k+1}$$

$$\begin{aligned}
&4(k^2+k)<4k^2+4k+1\\
\Rightarrow &2\sqrt{k^2+k}<2k+1\\
\Rightarrow &2\sqrt{k^2+k}+1<2k+2\\
\Rightarrow &2\sqrt{k}\sqrt{k+1}+1<2\sqrt{k+1}\sqrt{k+1}\\
\Rightarrow &2\sqrt{k}+\frac{1}{\sqrt{k+1}}<2\sqrt{k+1}
\end{aligned}$$

This is obviously true, therefore true for $P(k+1)$. Thus if $P(k)$ is true, $P(k+1)$ is true.
Conclusion: As we have proved the basis step and the inductive step, it follows by induction that the equation is true for all integers $n \geq 0$.

If and only If Proofs

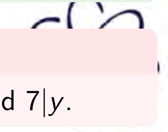
Definition

IFF: A bi-conditional logical connective between statements, i.e. $A \to B$ and $B \to A$.

Structure

Proving x is an A iff x is a B:

Firstly, let x be an A...
Therefore x is a B...
Conversely let x be a B...
Therefore x is an A.



Example (Tutorial Question)

For integers x and y, show that $7|x^2 + y^2$ if and only if $7|x$ and $7|y$.

Solution

Proof: We first prove $7|x, 7|y \rightarrow 7|x^2 + y^2$

Since $x|7$ and $y|7$, $x = 7n, y = 7m$ for some integer n, m. Therefore $x^2 + y^2 = 7 \times 7 \times (n^2 + m^2)$ which is obviously divisible by 7, since $n^2 + m^2$ integer, as required.

Solution cont.

Proof: Next we prove $7|x^2 + y^2 \to 7|x, 7|y$.

We attempt to prove this by contradiction. Suppose that there exists x, y not divisible by 7 such that $7|x^2 + y^2$.

We set $x = 7n + a, y = 7m + b$ where $n, m \in \mathbb{Z}$ and $a, b \in \{1, 2, 3, 4, 5, 6\}$.

Thus,
$x^2 + y^2 = (49n^2 + 14an + a^2) + (49m^2 + 14bm + b^2) = 7(7n^2 + 2an + 7m^2 + 2bm) + (a^2 + b^2)$.

Hence $7|x^2 + y^2 \equiv a^2 + b^2 mod 7$ and for $7|x^2 + y^2$ to be true, we require $a^2 + b^2 = 0 mod 7$.

We can prove this is never true by exhaustion of cases. Therefore the original assumption was false and $7|x^2 + y^2 \to 7|x, 7|y$.

Thus we have proved both cases and $7|x^2 + y^2 \leftrightarrow 7|x, 7|y$.

Further Examples

2014 S2 3iv

Let x be an integer. Prove that:

$$x = \left\lfloor \frac{x}{2} \right\rfloor + \left\lceil \frac{x}{2} \right\rceil$$

Solution

Suppose $x = \left\lfloor \frac{x}{2} \right\rfloor + \left\lceil \frac{x}{2} \right\rceil$, then since x is the sum of two integers, x is an integer.

Conversely suppose x is an integer. We consider the case where x is even and odd.

If x is even, then we write $x = 2k$, for some integer k. Hence $\left\lfloor \frac{x}{2} \right\rfloor + \left\lceil \frac{x}{2} \right\rceil = \left\lfloor \frac{2k}{2} \right\rfloor + \left\lceil \frac{2k}{2} \right\rceil = k + k = 2k = x$.

If x is odd, then we write $x = 2k + 1$, for some integer k. Hence, $\left\lfloor \frac{x}{2} \right\rfloor + \left\lceil \frac{x}{2} \right\rceil = \left\lfloor \frac{2k+1}{2} \right\rfloor + \left\lceil \frac{2k+1}{2} \right\rceil = k + k + 1 = 2k + 1 = x$. The result follows.

Existence and Uniqueness Proofs

Definition

Existence and Uniqueness: A statement that asserts that there is **one and only one** object having the given property.

Structure

- Show that there exists an object with the required property.
- Show that there cannot be two different objects with the property (given two objects that both satisfy the property, they must be identical):

Suppose that x and y both have the property... Therefore $x = y$.

Limit Proofs

Definition

Multiple Quantifiers: A statement with more than one quantifier like "all" or "some".

Structure

Let ϵ be given. We choose $M = ...$ such that when $x > \delta, \epsilon > M$ This proves the limit ... as required.

Converse, Negation and Contra-positive

Definition

Converse: Switching the hypothesis and conclusion of a conditional statement, i.e. “if B then A” is the converse of “if A then B”.

Definition

Negation: The negation of a statement is the assertion that the statement is false.

Definition

Contra-positive: Switching the hypothesis and conclusion of a conditional statement and negating both, i.e. “if not A then not B” is the contra-positive of “if A then B”.

Negation

Definition

Negation: The negation of a statement is the assertion that the statement is false ($\sim$ operator).

Definitions

- The negation of "and" is "or".
- The negation of $\forall$ is $\exists$.

Example (Limit Proofs)

2018 s2 3iv

Let $a_1, a_2, a_3, \ldots$ be a sequence of real numbers. The definition of the limit of the sequence, $lim_{n\to\infty} a_n = l$, is:

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, |a_n - l| < \epsilon \tag{1}$$

- Write in symbolic form the negation of (1) and simplify your answer so that the negation symbol is not used.
- By working directly from the definition, prove that $lim_{n\to\infty} \frac{n}{3n+1} = \frac{1}{3}$.

Solution

The negation is:

$$\forall N \in \mathbb{N}, \exists \epsilon > 0, \exists n \geq N, |a_n - l| < \epsilon$$

For $\epsilon > 0$ and $n \geq \frac{1}{9\epsilon}$, we have

$$\begin{aligned} \left|\frac{n}{3n+1} - \frac{1}{3}\right| &= \left|\frac{1}{3} \times \frac{1}{3n+1}\right| \\ &\leq \frac{1}{3} \times \frac{1}{3n} \\ &= \frac{1}{9n} \\ &< \epsilon \end{aligned}$$

Thus, by the definition of the limit given above, $lim_{n\to\infty} \frac{n}{3n+1} = \frac{1}{3}$ as required.

Example

2011 S1 3(iii)

Suppose that m and n are integers. Consider the following statement: *If m and n are relatively prime, then there exist integers x and y satisfying $mx + ny = 2$.*

- What is the converse of this statement?
- What is the contrapositive of this statement?
- Which of these statements are true? (Give reasons for your answers.)

Solution

- If there exist integers x and y satisfying $mx + ny = 2$, then m and n are relatively prime.
- If m and n are not relatively prime, then there does not exist integers satisfying $mx + ny = 2$.
- Which one is true?

Propositions

Proposition: "It is impossible for any number which is a power greater than the second to be written as a sum of two like powers. I have a truly marvelous demonstration of this proposition which this margin is too narrow to contain."

- Pierre de Fermat

Logical Operators

- $\sim$ symbolises "not"
- $\wedge$ symbolises "and"
- $\vee$ symbolises "or"
- $\oplus$ symbolises "exclusive or"
- $\rightarrow$ symbolises "implies"
- $\leftrightarrow$ symbolises "if and only if" (i.e. equivalence)

Compound Propositions

Compound Proposition Example

James is on hella drugs and will be happy only if he either studies or he parties tonight! (If he does both, he will die of a drug overdose) Please express this as a compound proposition given the definitions below.

Definitions

- x: James is not on cocaine!
- y: James will party tonight!
- z: James will study tonight!
- a: James will be unhappy!

Solution

$$[(\sim x) \wedge (z)] \oplus [(\sim x) \wedge (y)] \rightarrow (\sim a)$$

Example

2011 S1 Q3v

Consider the following compound statement: *If you do not study hard or if you go out, then you will find the exam difficult.*

- Using appropriate letters for the component propositions, translate the compound statement into symbolic notation.
- Write down the negation of the symbolic statement.
- Express this negation concisely in English.

Solution

Let:

- $x =$ I am not studying hard.
- $y =$ I will go out.
- $z =$ I will find the exam difficult.

The statement is: $x \vee y \to z$.

The negation of the statement is $(\sim x) \wedge (\sim y) \to (\sim z)$.

In English, this comes out as "If I study hard and not go out, I will not find the exam difficult.

Truth Tables

Definition

Truth tables help **determine the truth values** of a certain **compound proposition** by analysing the truth values of its constituent propositions.

Definitions

- **Tautology**: *A propositional form that is always true, no matter what the truth values of the propositional variables that occur in it.*
- **Contradiction**: *A propositional form that is always false.*
- **Contingency**: *A propositional form that is sometimes true and sometimes false.*

Defintions

Essential Definitions

p	q	$p \vee q$	$p \wedge q$	$p \oplus q$	$p \to q$	$p \leftrightarrow q$
T	T	T	T	F	T	T
T	F	T	F	T	F	F
F	T	T	F	T	T	F
F	F	F	F	F	T	T

Truth Table Example

2011 S2 Q3i

Use truth tables to show that the following propositional formula is a tautology:

$$(p \vee q) \rightarrow (\sim (r \rightarrow q) \rightarrow p)$$

Solution

p	q	r	$p \vee q$	$r \rightarrow q$	$\sim (r \rightarrow q)$	$\sim (r \rightarrow q) \rightarrow p$	$(p \vee q) \rightarrow (\sim (r \rightarrow q) \rightarrow p)$
T	T	T	T	T	F	T	T
T	T	F	T	T	F	T	T
T	F	T	T	F	T	T	T
T	F	F	T	T	F	T	T
F	T	T	T	T	F	T	T
F	T	F	T	T	F	T	T
F	F	T	F	F	T	F	T
F	F	F	F	T	F	T	T

Therefore, this is a tautology.

Law of Logical Equivalence

Definition

Logical Equivalence: *When two propositions(compound) have an if and only if relationship.*
Note: *Logical Equivalence can be dis-proven by conflicting truth table values or through judicious selection of variables resulting in conflicting form.*

Theorem

Two propositional forms P and Q are logically equivalent if and only if P$\leftrightarrow$Q is a tautology.

Logical Equivalence Example

2011 S1 Q3iv

Use standard logical equivalences to simplify the following logical expression:

$$(p \vee q) \vee \sim (\sim p \vee q) \vee (q \vee r)$$

2017 S2 3iii

Show, using standard logical equivalences, that $(q \vee \sim r) \rightarrow p$ is logically equivalent to $(r \vee p) \wedge (q \rightarrow p)$

Solution 1

$$
\begin{aligned}
&(p \vee q) \vee \sim (\sim p \vee q) \vee (q \vee r) \\
&= (p \vee q) \vee (p \wedge \sim q) \vee (q \vee r) \qquad \text{De Morgan's Law} \\
&= [(p \vee q) \vee p] \wedge [(p \vee q) \vee \sim q] \vee (q \vee r) \qquad \text{Distribution Law} \\
&= (p \vee q) \wedge (p \vee t) \vee (q \vee r) \qquad \text{Associative, Negation Law} \\
&= p \vee (q \wedge t) \vee (q \vee r) \qquad \text{Idempotent, Domination Law} \\
&= p \vee q \vee r \qquad \text{Idempotent, Identity Law}
\end{aligned}
$$

Solution 2

$$
\begin{aligned}
&(q \vee \sim r) \to p \\
&= \sim (q \vee \sim r) \vee p && \text{Alternative Form} \\
&= (\sim q \wedge r) \vee p && \text{De Morgan's Law} \\
&= p \vee (\sim q \wedge r) && \text{Commutative Law} \\
&= (p \vee \sim q) \wedge (p \vee r) && \text{Distributive Law} \\
&= (r \vee p) \wedge (\sim q \vee p) && \text{Commutative Law} \\
&= (r \vee p) \wedge (q \to p) && \text{Alternative Form}
\end{aligned}
$$

Logical Implication

Definition

Let P and Q be propositional forms. Suppose that in any case where P **is true,** Q **is also true**. Then we say that P **logically implies** Q, and write $P \implies Q$.

Theorem

Two propositional forms P and Q are logically equivalent if and only if $P \leftrightarrow Q$ is a tautology.

Arguments - Rules of Inferences

Definitions

- **Modus Ponens**: "P implies Q and P is asserted to be true, therefore Q must be true."
- **Converse Fallacy**: $P \rightarrow Q$ does not imply $Q \rightarrow P$.
- **Modus Tollens**: P implies Q to the negation of Q implies the negation of P is valid. (Proof by Contradiction).
- **Hypothetical Syllogism**: $P \rightarrow Q$ and $Q \rightarrow R$ implies $P \rightarrow R$.

Final Tips

Common Mistakes

- Forgetting to state Laws of Set Algebra/Logic even when asked to provide them.
- In excitement of completing a proof/calculation problem, forget to write the conclusion.
- Using Venn diagrams or essays instead of concise written proofs (not ok in this course).
- Proofs should be striaght to the point and not excessively waffled.

Table of Contents

The Outer View

Math as a transformation

Solving real world problems with math involves:

1. Analysing the problem and writing it down in maths
2. Solving the problem
3. Converting mathematical solution into a real world solution

The Outer View

Math as a transformation

Solving real world problems with math involves:

1. Analysing the problem and writing it down in maths
2. Solving the problem
3. Converting mathematical solution into a real world solution

But we can transform problems within maths!

Counting becomes easier if we transform our problems into simpler ones.

The Universal Problem

Problem statement:

If I have **a** apples and **b** baskets to put them in, how many different ways can I arrange them?

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I claim that I can turn any Perms and Combs problem in 1081 into a version of this.

The Universal Problem

Problem statement:

If I have **a** apples and **b** baskets to put them in, how many different ways can I arrange them?

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I claim that I can turn any Perms and Combs problem in 1081 into a version of this.

This is an actual method I use to decompose tough problems!

Exponentials

You should already know this

$b^a = b * b * b * \ldots * b$ (a times)

Exponentials

You should already know this

$b^a = b * b * b * ... * b$ (a times)

Set theoretic perspective

$|B_1 \times B_2 \times ... \times B_a| = |B_1| \times |B_2| \times ... \times |B_a|$

Exponentials

You should already know this

$b^a = b * b * b * ... * b$ (a times)

Set theoretic perspective

$|B_1 \times B_2 \times ... \times B_a| = |B_1| \times |B_2| \times ... \times |B_a|$

Link to Problem

- Apples are unique
- Baskets are unique
- Baskets can contain unlimited apples

Factorials

You should already know this

$n! = n \times (n-1) \times ... \times 2 \times 1$

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Expression

$b!$

Permutations

Question

There are 5 people in a race. How many ways can we award 1nd , 2rd and 3st place?

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awards = apples, people = baskets.

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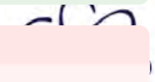

Intuition

We discount the permutations of the empty $(b - a)$ baskets, since rearranging them doesn't change the solution; divide by $(b - a)!$

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$P(b,a) = \frac{b!}{(b-a)!}$

Combinations

Question

Out of a room of 8 math students, how many ways can I pick 3 "volunteers"?

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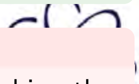

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We discount the permutations of the *a* baskets in use, since switching the apples doesn't create a new solution like it did before; divide by *a*!

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Expression

$C(b, a) = \binom{b}{a} = \frac{b!}{a!(b-a)!}$

Strategy: Removing permutations post-hoc

Technique

Count the solutions as if they are each unique and valid, and then divide by k! to get the real solution.
Only works sometimes!! Always gut check and test edge cases.

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Examples

This operation gets us from

- factorials $\rightarrow$ permutations
- permutations $\rightarrow$ combinations
- combinations $\rightarrow$ multinomials

Problem

2017 S2 Q4 iv

6 people of different heights are in line to buy some tasty ice cream:

1. Compute the number of ways the six people can be arranged so that the **first three** are ordered according to height tallest to shortest.
2. Compute the number of ways the six people can be arranged so that the **first four** are ordered according to height tallest to shortest.
3. Compute the number of ways the six people can be arranged so that the **first three** are ordered according to height tallest to shortest and the **last three** are ordered according to height tallest to shortest.

Simplified Problem

2017 S2 Q4 iv

6 different sized apples are put in a line of 1-apple-sized baskets. Find the number of arrangements such that:

1. The **first 3** apples are in increasing-size order.
2. The **first 4** apples are in increasing-size order.
3. The **first 3** apples are in increasing-size order and the **last 3** apples are in increasing-size order.

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2. The **first 4** apples are in increasing-size order.
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Answers

a) $\frac{6!}{3!}$ b) $\frac{6!}{4!}$ c) $\frac{6!}{3!3!}$

Problem

2017 S2 Q4 iv

Consider a standard 52 card pack. How many seven-card hands contain:

1. exactly 3 hearts, 2 spades and 2 clubs?
2. either no hearts or no spades or no clubs?
3. at least one heart, one spade and one club?

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1. exactly 3 hearts, 2 spades and 2 clubs?
2. either no hearts or no spades or no clubs?
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Answers

$$a)\binom{13}{3}\binom{13}{2}\binom{13}{2}$$

$$b)3\binom{39}{7} - 3\binom{26}{7} + 1\binom{13}{7}$$

$$c)\binom{52}{7} - (b)$$

Combinations 2

Question

$$x_1 + x_2 + ... + x_b = a$$

Combinations 2

Question

$$x_1 + x_2 + ... + x_b = a$$

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Numbers/Apples are distributed over the pronumerals/baskets.

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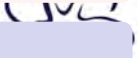

Expression

$C(a + b - 1, a) = \binom{a+b-1}{a} = \frac{(a+b-1)!}{a!(b-1)!}$

Combinations 2

2018 S1 Q4 i

How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 83$$

if $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ are non-negative integers

1. with no further restrictions?
2. with x_k being an odd number for every k?

Combinations 2

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Answers

a) $C(89, 6)$ b) $C(44, 6)$

Strategy: Transformations

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More generally, it's reframing the problem in an easier way.
In this form of usage, it is similar to u-substitution.
Shifting the problem can save a headache down the road.

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Examples

- $x_i = 2k_i + 1$ (if x_i is odd) $x_i = 2k_i$ (if x_i is even)
- $x_i = a - y_i$ (if $0 \leq x_i \leq a$ then $0 \leq y_i \leq a$ as well)
- $P(E) = 1 - P(\sim E)$

Combinations 2

2017 S2 Q4 i

Consider the equation

$$x_1 + x_2 + x_3 + x_4 = 70$$

Where x_1, x_2, x_3, x_4 are non-negative integers. How many solutions are there with all x_i satisfying $0 \le x \le 20$?

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Answer

$C(13, 3)$

Inclusion-Exclusion

2017 S2 Q4 i

Consider the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 45$$

Where x_1, x_2, x_3, x_4 are non-negative integers. How many solutions are there with all x_i satisfying $0 \leq x \leq 20$?

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Answer

$P(6,0)\binom{49}{4} - P(6,1)\binom{28}{4} + P(6,2)\binom{7}{4}$

Problem

2017 S2 Q4 i

The expression

$$(m + a + t + h)^{1081}$$

is expanded and like terms collected.

1. How many terms are there?
2. What is the coefficient of

$$m^{255}a^{265}t^{275}h^{286}?$$

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Answers

a) $C(1804, 3)$ b) $\frac{1081!}{255!265!275!286!}$

Multinomials

It's like an extension of combinations

$C(a, \{k_1, k_2, ...k_b\}) = \frac{a!}{k_1!k_2!...k_b!}$

But don't write it with that notation in the exam!

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Link to Problem

- Each apple is unique
- Each basket is unique
- Each basket can contain only a specific number of apples (k_i)

Probability

If we assume that every outcome within a set is equally likely, then we can evenly divide probability over it. Probability of an event is the number of ways it can happen over the total number of things that could happen. Or if we want to get real fancy:

$$P(E) = \frac{N(E)}{N(E) + N(\overline{E})}$$

In our course, probability problems are just extended counting problems.

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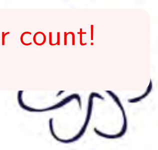
In our course, probability problems are just extended counting problems.

Make sure your final answer is a probability (≤ 1) not a number count! Classic mistake.



Extension

Question Traits

- Each apple is unique/interchangeable
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Try different combinations of these traits and see what you get.
Add another variable: Does the basket care about the order of the apples within it?

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Add another variable: Does the basket care about the order of the apples within it?

If you're really sharp, you can make yourself a formula sheet.

Definitions

Common Sense

If there are k pigeon holes and $k+1$ pigeons to fit into them, then at least one pigeon hole will have at least 2 pigeons.

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Generalised Common Sense

If there are k slots and there are n items to put in them, at least one slot will have at least $\lceil \frac{n}{k} \rceil$ items in it.

Pigeonhole Principle Strategies

- If you're asked to prove/show something in Q4, it's usually a pigeonhole principle problem.
- Look for k and $k+1$ (when possible); then you can work backwards to figure out how to construct your proof.
- If you're struggling to imagine the problem, try doing it with smaller numbers (1 dimension instead of 3, or a set of 4 instead of a set of 9) and work your way up.

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Don't forget to write a conclusion!

Problems

2016 S1 Q4 iv

Suppose that 26 integers are chosen from the set $S = 1, 2, .., 50$. By writing these numbers as $2^k m$ with m odd, prove that one of the chosen numbers is a multiple of another of the chosen numbers

Problems

2014 S1 Q4 iii

Let A be any set of 10 distinct positive integers less than 100. Define the term "element-sum" of a set to be the sum of all the elements of the set.

1. How many subsets are there of A
2. Show that the largest element-sum of any subset of A is 945
3. Use the pigeon-hole principle to deduce that there must be at least two different subsets of A having the same element sum.

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2014 S1 Q4 iii

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Extension

4. Prove that you can find a pair of *disjoint* subsets of A: S_1 and S_2 with the same element-sum. (Disjoint means $S_1 \cap S_2 = \{\}$)

Problems

2014 S1 Q4 iii

The n points $P_i(x_i, y_i, z_i), i = 1, 2..n$ in $\mathbb{R}^3$ have integer coordinates, meaning all $x_i, y_i, z_i \in \mathbb{Z}$.

1. When does the mid-point of the line segment P_1P_2 have integer coordinates?
2. Giving reasons, find the least value for n do which it must be true that at least one of the midpoints of all the possible line segments P_iP_j (for $i \neq j$) has integer coordinates.

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2014 S1 Q4 iii

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Answers

2) $2^3 + 1 = 9$

Homogenous Recurrence

First Order

$$a_n - ca_{n-1} = 0$$

$$a_n = c^n a_0$$

Homogenous Recurrence

Second Order

$$a_n - c_1 a_{n-1} - c_2 a_{n-2} = 0$$

Step 1: write out and solve the characteristic equation:

$$\lambda^2 - c_1\lambda - c_2$$

$$\lambda_1, \lambda_2 = \frac{c_1 \pm \sqrt{c_1^2 + 4c_2}}{2}$$

We put these into the formula:

$$a_n = A\lambda_1^n + B\lambda_2^n$$

And then use the values a_0 and a_1 to solve for A and B

Homogenous Recurrence

To describe a series using a recurrence relation we need

1. The relation
2. The starting values

So when we create our general equation, we need to reflect this information.

$$a_n = A\lambda_1^n + B\lambda_2^n$$

has λ_1 and λ_2 to represent the recurrence relation, and A and B to represent the starting values.

Homogenous Recurrence

Double ups

$$a_n - 2a_{n-1} + a_n - 2 = 0$$

When solving for this, we find $\lambda_1 = \lambda_2 = 1$, so our equation would look like:

$$a_n = A(1)^n + B(1)^n$$

$$a_n = A + B$$

$$a_n = C$$

Which is a constant C. Unfortunately, if we set $a_0 = 2$ and $a_1 = 3$, we find that the equation can't be a constant! This error comes from the fact that we only have 1 variable (C) to represent the possible starting conditions, where we actually need 2.

Homogenous Recurrence

$$a_n - 2a_{n-1} + a_n{-}2 = 0$$

To solve this, we need two distinguishable variables in our equation, and we get that by multiplying one of them by n:

$$a_n = nA\lambda_1^n + B\lambda_1^n$$

$$a_n = (An + B)\lambda_1^n$$

Non-Homogenous Recurrence

$$a_n - c_1 a_{n-1} + c_2 a_n - 2 = f(n)$$

Non-Homogenous Recurrence

$$a_n - c_1 a_{n-1} + c_2 a_n - 2 = f(n)$$

Explanation

We break it into homogenous and non-homogenous components.

$$a_n = h_n + p_n$$

The homogenous solution is the same as seen before, the non-homogenous takes the same form as $f(n)$.

Non-Homogenous Recurrence

Forms of P(n)

If $f(n)$ is a constant, p_n should be a constant.
If $f(n)$ is a polynomial, p_n should be a polynomial with the same degree.
If $f(n)$ is an exponential, p_n should be an exponential $\times$ a constant.
If $f(n)$ is a product of the aforementioned forms, p_n should similarly be an product of those forms.

Non-Homogenous Recurrence

Double ups

But one caveat: if we ever double up terms, then we must multiply by n to maintain its identity! Always prefer to times-by-n on p_N not h_n.

Non-Homogenous Recurrence

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Triple ups

If there are three of the same term, then multiply the last one by n again to keep it unique.

Problems

2014 S2 Q4 i

1. Solve the recurrence

$$a_n = 4a_{n-1} + 5a_{n-2}$$

subject to the initial conditions $a_0 = -1$, $a_1 = 7$.

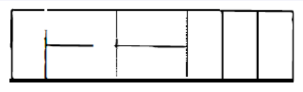
2. Find a particular solution of

$$a_n - 4a_{n-1} - 5a_{n-2} = 5^n, \quad \text{for} n \geq 2.$$

Problems

2018 S2 Q4 v

A straight path of width 2 is to be laid using the 1-unit by 2-unit paving slabs. Here is an example of a path of 8 units:



Let a_n be the number of ways to lay a path of with 2 units and length n units.

1. Find a_1, a_2, a_3.
2. Obtain a recurrence relation for a_n. Explain your answer

Problems

2018 S2 Q4 v

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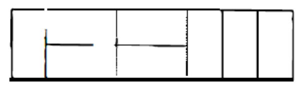

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Extension

Solve the recurrence relation

Problems

2016 S2 Q4 i

1. Find the solution to the recurrence relation

$$a_n - 4_{n-1} + 4_{n-2} = n.$$

2. What form of solution would you try if the right hand side were replaced by 2^n?

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2016 S2 Q4 i

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