# MATH2018/2019 – Seminar solutions

## [Seminar II / II]

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The following document are full worked solutions to the questions that was discussed in the revision seminar on April 24, 2020. The solutions were written by Oscar Kamensky and Janzen Choi and transcribed by Gerald Huang, and can be used as supplement resources while preparing for final exams. Please use this resource ethically. The following document is **NOT** endorsed by the School of Mathematics and Statistics and may be prone to errors; if you spot an error, please message us here. Happy studying!

## Contents

# Part V: Ordinary Differential Equations

> Solve $\sec^2(x)\tan(y) + \dfrac{dy}{dx}\sec^2(y)\tan(x) = 0$.

SOLUTION.
This is a separable differential equation. Separating the equation into two separate ODEs, we get

$$\begin{aligned}\sec^2(x)\tan(y) + \frac{dy}{dx}\sec^2(y)\tan(x) = 0 &\implies \sec^2(x)\tan(y) = -\frac{dy}{dx}\sec^2(y)\tan(x)\\ &\implies \frac{\sec^2(x)}{\tan(x)}\,dx = -\frac{\sec^2(y)}{\tan(y)}\,dy.\end{aligned}$$

Note that $\dfrac{d}{dx}(\tan(x)) = \sec^2(x)$. So naturally, let $u = \tan(x)$. Then we have $du = \sec^2(x)\,dx$. So the left and right side integrates to

$$\ln(\tan(x)) = -\ln(\tan(y)) + C_1.$$

This can be written as

$$\ln(\tan(x)\tan(y)) = C_1 \implies \tan(x)\tan(y) = e^{C_1} = C,$$

where $C = e^{C_1}$ is still a constant.

> Solve
> $$\frac{dy}{dx} = (x+y)^2$$
> using the substitution $v = y + x$.

SOLUTION.
Let $v = y + x$. To begin, we find an expression of $\dfrac{dy}{dx}$ in terms of $v$ or $\dfrac{dv}{dx}$. Differentiating both sides with respect to $x$, we have

$$\frac{dv}{dx} = \frac{dy}{dx} + 1 \implies \frac{dy}{dx} = \frac{dv}{dx} - 1.$$

Substituting these expressions into the original expression, we have

$$\frac{dv}{dx} - 1 = (x+y)^2 = v^2 \implies \frac{dv}{dx} = (v^2 + 1).$$

This becomes a **separable ODE** so dividing both sides by $(v^2+1)$ and multiplying both sides by $dx$, we have

$$\begin{aligned}
\frac{dv}{v^2+1} &= dx \\
\int \frac{dv}{v^2+1} &= \int dx \\
\tan^{-1}(v) &= x + C \\
v &= \tan(x+C) \\
(y+x) &= \tan(x+C) \\
y &= \tan(x+C) - x.
\end{aligned}$$

> Solve
> $$y'' + 3y' + 2y = e^{-2t} + 4t^2 + 2$$
> by method of undetermined coefficients and describe the long-term steady state solution.

SOLUTION.
Begin by solving the homogeneous case and finding $y_H(t)$. This means we solve

$$y'' + 3y' + 2y = 0.$$

We calculate the **characteristic polynomial** which is given by

$$\lambda^2 + 3\lambda + 2 = 0 \implies \lambda = -2, \quad \lambda = -1.$$

Hence, for the homogeneous case, we have

$$y_H(t) = c_1 e^{-2t} + c_2 e^{-t}.$$

For our particular solution, we look at the right hand side for the form for which we want to guess. The first term contains an exponential $e^{-2t}$. So we guess something of that form, but note that $e^{-2t}$ appears in our homogeneous case, so we guess a $te^{-2t}$ form. For the $4t^2+2$ terms, we guess a polynomial of degree 2 including all terms with a degree less than 2 as well. So the guess of our particular solution is

$$\begin{aligned}
e^{-2t} &\to Ate^{-2t} \\
4t^2 + 2 &\to Bt^2 + Ct + D \\
y_p(t) &= Ate^{-2t} + \left(Bt^2 + Ct + D\right).
\end{aligned}$$

Once we have the form of our solution, we differentiate it twice and substitute into our ODE. Differentiating twice, we have

$$\begin{aligned}
y_p'(t) &= A\left(e^{-2t} - 2te^{-2t}\right) + 2Bt + C. \\
y_p''(t) &= A\left(-2e^{-2t} - 2\left(e^{-2t} - 2te^{-2t}\right)\right) + 2B \\
&= A\left(-2e^{-2t} - 2e^{-2t} + 4te^{-2t}\right) + 2B \\
&= A\left(4te^{-2t} - 4e^{-2t}\right) + 2B.
\end{aligned}$$

Substituting the two derivatives into the ODE, we have

$$\begin{aligned}
y_p''(t) + 3y_p'(t) + 2y_p(t) &= \left(A(4te^{-2t} - 4e^{-2t}) + 2B\right) \\
&\quad + 3\left(A\left(e^{-2t} - 2te^{-2t}\right) + 2Bt + C\right) \\
&\quad + 2\left(Ate^{-2t} + Bt^2 + Ct + D\right) \\
&= e^{-2t}\left[-4A + 3A\right] + te^{-2t}\left[4A - 6A + 2A\right] \\
&\quad + 2Bt^2 + t(6B + 2C) + (2B + 3C + 2D)\,.
\end{aligned}$$

Comparing this to the right hand side, we have the following system of equations

| | |
|---|---|
| Coefficient of $e^{-2t}$ | $-A = 1$ |
| Coefficient of $t^2$ | $2B = 4$ |
| Coefficient of $t$ | $6B + 2C = 0$ |
| Coefficient of $t^0$ | $2B + 3C + 2D = 2$ |

Solving for the coefficients, we have

$$A = -1, \quad B = 2, \quad C = -6, \quad D = 8.$$

Hence, we have the particular solution

$$y_p(t) = -te^{-2t} + 2t^2 - 6t + 8.$$

So our final solution is

$$y(t) = y_H(t) + y_p(t) = c_1e^{-2t} + c_2e^{-t} - te^{-2t} + 2t^2 - 6t + 8.$$

Use the method of variation of parameters to find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = \frac{e^x}{x^3}.$$

SOLUTION.
Begin by solving the homogeneous case and finding $y_H(x)$. This means we solve

$$y'' - 2y' + y = 0.$$

We calculate the **characteristic polynomial** which is given by

$$\lambda^2 - 2\lambda + 1 = 0 \implies \lambda = 1.$$

This gives us the homogeneous solution

$$y_H(t) = c_1 e^x + c_2 x e^x.$$

For the variation of parameters method, we take the two homogeneous solutions to be $y_1(x)$ and $y_2(x)$ respectively. This gives us

$$y_1(x) = e^x, \quad y_2(x) = xe^x.$$

In general, our particular solution is given by the following

$$y_p(x) = -y_1(x) \int \frac{y_2(x) f(x)}{W(y_1, y_2)}\, dx + y_2(x) \int \frac{y_1(x) f(x)}{W(y_1, y_2)}\, dx,$$

where $W(y_1, y_2)$ is the **Wronskian** determinant given by

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'.$$

This gives us

$$W\left(e^x, xe^x\right) = e^{2x}(1 + x) - xe^{2x} = e^{2x}.$$

Substituting everything into the formula for the particular solution, we have

$$\begin{aligned} y_p(x) &= -e^x \int \frac{xe^x \cdot \frac{e^x}{x^3}}{e^{2x}}\, dx + xe^x \int \frac{e^x \cdot \frac{e^x}{x^3}}{e^{2x}}\, dx \\ &= -e^x \int \frac{1}{x^2}\, dx + xe^x \int \frac{1}{x^3}\, dx \\ &= \frac{e^x}{x} - xe^x \left( \frac{1}{2x^2} \right) \\ &= \frac{e^x}{x} - \frac{e^x}{2x} \\ &= \frac{e^x}{2x}. \end{aligned}$$

So the final solution is

$$y(x) = y_H(x) + y_p(x) = c_1 e^x + c_2 x e^x + \frac{e^x}{2x}.$$

## Part VI: Laplace transforms

Find $\mathcal{L}^{-1}\left(\dfrac{s+1}{s^2+4s+5}\right)$.

Solution.
Although it may not be clear at first, this is an example of the **first shifting theorem**! Note that the denominator, when completed the square, can be expressed as

$$(s+2)^2+1^2.$$

In other words, every term in the expression can be written as some form of $(s+2)$. We can break the function up as the following

$$\frac{s+1}{s^2+4s+5}=\frac{(s+2)-1}{(s+2)^2+1^2}$$

Apply the first shifting theorem on $(s-2)$ to give us

$$\frac{(s+2)-1}{(s+2)^2+1}\to\frac{s-1}{s^2+1^2}$$

and taking the inverse Laplace transformation on this fraction allows us to split the inverse Laplace transformation into two separate transformations that are in the formula sheet provided to you; that is

$$\mathcal{L}^{-1}\left(\frac{s-1}{s^2+1}\right)=\underbrace{\mathcal{L}^{-1}\left(\frac{s}{s^2+1^2}\right)}_{g(t)}-\underbrace{\mathcal{L}^{-1}\left(\frac{1}{s^2+1^2}\right)}_{h(t)}.$$

Defining $g(t)$ and $h(t)$ to be their respective inverse Laplace transformations, we see that $g(t)=\cos(t)$ and $h(t)=\sin(t)$; this gives us

$$\mathcal{L}^{-1}\left(\frac{s-1}{s^2+1}\right)=\cos(t)-\sin(t).$$

Finally, shifting the transformation back ends up multiplying the resultant by $e^{-cs}$ where $c=2$. This gives us the final inverse Laplace transformation to be

$$\mathcal{L}^{-1}\left(\frac{s+1}{s^2+4s+5}\right)=e^{-2t}(\cos(t)-\sin(t)).$$

Find $\mathcal{L}^{-1}\left(\dfrac{e^{-2s}}{3s^4}\right)$.

Solution.

Define $f(t) = \mathcal{L}^{-1}\left(\dfrac{e^{-2s}}{3s^4}\right)$.

This is an example of the **second shifting theorem** since the inside function can be expressed as $e^{-2s}F(s)$, where $F(s) = \dfrac{1}{3s^4}$. By the second shifting theorem, we consider the inverse Laplace transformation on $\dfrac{1}{3s^4}$.

Define $g(t) = \mathcal{L}^{-1}\left(\dfrac{1}{3s^4}\right)$. This resembles the form $\dfrac{m!}{s^{m+1}}$. So transforming $\dfrac{1}{3s^4}$ into such a form with $m = 3$, we have

$$\frac{1}{3s^4} = \frac{1}{3} \times \frac{1}{6}\left(\frac{6}{s^4}\right) = \frac{1}{18}\left(\frac{3!}{s^{3+1}}\right).$$

So we have

$$g(t) = \frac{1}{18}\mathcal{L}^{-1}\left(\frac{3!}{s^{3+1}}\right) = \frac{1}{18}t^3.$$

Multiplying the expression with $u(t-2)$ and shifting every $t$ by $2$ units, we have

$$f(t) = \mathcal{L}^{-1}\left(\frac{e^{-2s}}{3s^4}\right) = \frac{1}{18}(t-2)^3u(t-2).$$

a) Find
   i) $\mathcal{L}(te^{-t}\sin 3t)$.
   ii) $\mathcal{L}^{-1}\left\{\dfrac{s+1}{s^2+4s+5} + \dfrac{e^{-2s}}{3s^4}\right\}$.

b) The function $f(t)$ is defined for $t \geq 0$ by

$$f(t) = \begin{cases} 1 & 0 \leq t \leq 1 \\ t-2 & 1 < t \leq 2 \\ 0 & t > 2. \end{cases}$$

   Express $f(t)$ in terms of the Heaviside function and hence or otherwise find $\mathcal{L}(f(t))$, the Laplace transform of $f(t)$.

c) Use the Laplace transform method to solve the differential equation

$$y'' - 4y' + 4y = e^{2t}, \quad t > 0$$

   subject to the initial condition $y(0) = 1,\ y'(0) = 0.$

Solution.

a) i) Note that the function is of the form $e^{-t} \cdot f(t)$ where $f(t) = t\sin(3t)$. This tells us to use the **first shifting theorem**! From the Laplace table provided, we see that

$$\mathcal{L}\left(t \cdot f(t)\right) = -F'(s).$$

So define

$$\begin{aligned} G(s) &= -\frac{d}{ds}\mathcal{L}\left(\sin(3t)\right) \\ &= -\frac{d}{ds}\left(\frac{3}{s^2+9}\right) \\ &= \frac{6s}{(s^2+9)^2}. \end{aligned}$$

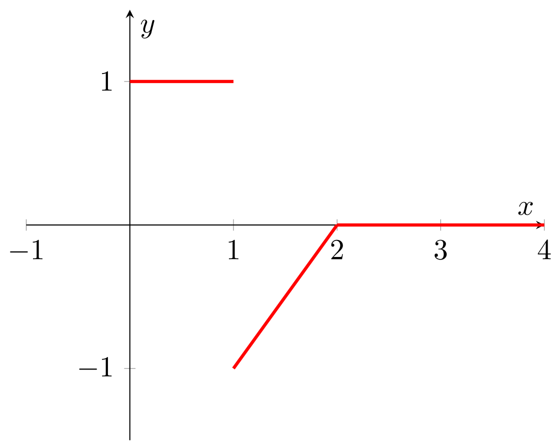
Finally, applying the first shifting theorem by shifting $F(s) = G(s-c)$, where $c = -1$, we arrive at

$$\mathcal{L}\left(te^{-t}\sin(3t)\right) = \frac{6(s+1)}{((s+1)^2+9)^2}.$$

ii) See the previous questions for the derivation. The answer should be

$$f(t) = e^{-2t}(\cos(t) - \sin(t)) + \frac{1}{18}(t-2)^3 u(t-2).$$

b) Consider the construction of $f(t)$ as below.




The idea is to construct each segment of the graph at a time and then deactivate it later.

$$\begin{aligned} f(t) = &\underbrace{u(t) - u(t-1)}_{\text{activate then deactivate}} + \underbrace{(t-2)\cdot u(t-1)}_{\text{activate}} - \underbrace{(t-2)\cdot u(t-2)}_{\text{deactivate}} \\ &= 1 - 2u(t-1) + \underbrace{(t-1)\cdot u(t-1)}_{\text{second shift theorem}} - \underbrace{(t-2)\cdot u(t-2)}_{\text{second shift theorem}}. \end{aligned}$$

Taking the Laplace transform, we have

$$\begin{aligned}\mathcal{L}\left[f(t)\right] &= \frac{1}{s} - \frac{2e^{-s}}{s} + e^{-s}\mathcal{L}(t) - e^{-2s}\mathcal{L}(t) \\ &= \frac{1}{s} - \frac{2e^{-s}}{s} + \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2}.\end{aligned}$$

d) Recall the Laplace transformations of the derivatives.

$$\begin{aligned}\mathcal{L}\left(f'(t)\right) &= sF(s) - f(0). \\ \mathcal{L}\left(f''(t)\right) &= s^2F(s) - sf(0) - f'(0).\end{aligned}$$

$$\begin{aligned}\mathcal{L}(LHS) &= \mathcal{L}(RHS) \\ \mathcal{L}(y'' - 4y' + 4y) &= \mathcal{L}(e^{2t}).\end{aligned}$$

Let $\mathcal{L}(y(t)) = Y$. Thus,

$$\left(s^2Y - s\cdot 1 - 0\right) - 4\left(sY - 1\right) + 4Y = \frac{1}{s-2}.$$

Grouping up the $Y$ terms, we have

$$\begin{aligned}Y(s^2 - 4s + 4) - s + 4 &= \frac{1}{s-2} \\ \implies Y &= \frac{s^2 - 6s + 9}{(s-2)^3}.\end{aligned}$$

To deal with $Y$, we decompose $Y$ into partial fractions. To do this, we set

$$Y = \frac{s^2 - 6s + 9}{(s-2)^3} \equiv \frac{A}{(s-2)} + \frac{B}{(s-2)^2} + \frac{C}{(s-2)^3}.$$

This gives us the following equality

$$s^2 - 6s + 9 = A(s-2)^2 + B(s-2) + C.$$

Expanding and comparing coefficients, we arrive at the results for $A$, $B$ and $C$

$$A = 1, \quad B = -2, \quad C = 1.$$

So we decompose $Y$ into

$$Y = \frac{1}{s-2} - \frac{2}{(s-2)^2} + \frac{1}{(s-2)^3}.$$

Applying the **first shifting theorem**, we have

$$G(s) = Y(s-2)$$

and we also have

$$\begin{aligned} G(s) &= \frac{1}{s} - 2\cdot\frac{1}{s^2} + \frac{1}{2}\cdot\frac{2}{s^3} \\ \implies g(t) &= 1 - 2t + \frac{1}{2}t^2. \end{aligned}$$

Finally, applying the result of the shifting theorem, we have

$$y(t) = e^{2t}\cdot g(t) = e^{2t}\left(1 - 2t + \frac{1}{2}t^2\right).$$

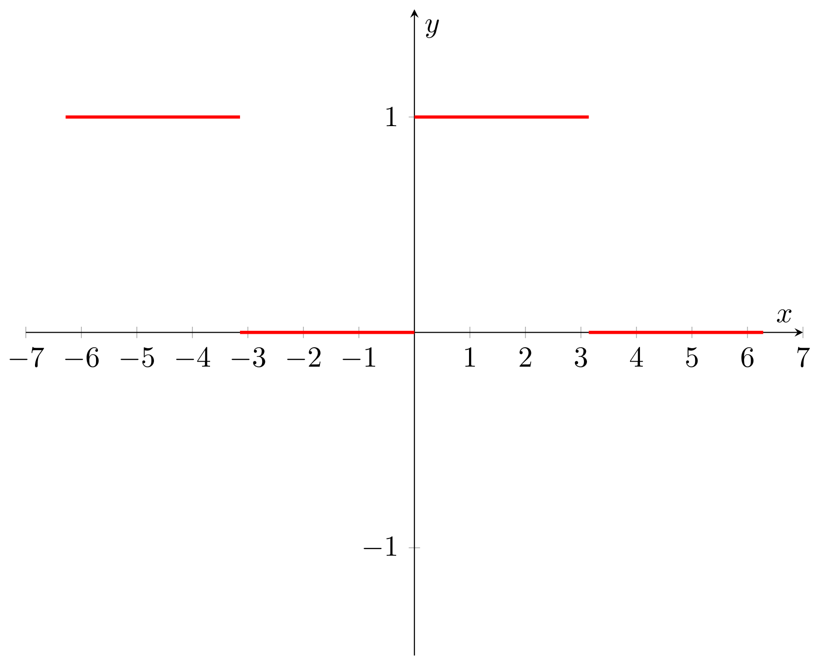
## Part VII: Fourier series

Let

$$f(x) = \begin{cases} 1 & 0 < x < \pi \\ 0 & \pi \le x \le 2\pi \\ f(x+2\pi) & \text{otherwise} \end{cases}$$

Sketch the graph of $y = f(x)$ over the interval $-2\pi \le x \le 2\pi$.

SOLUTION.



For the function $g$ given by

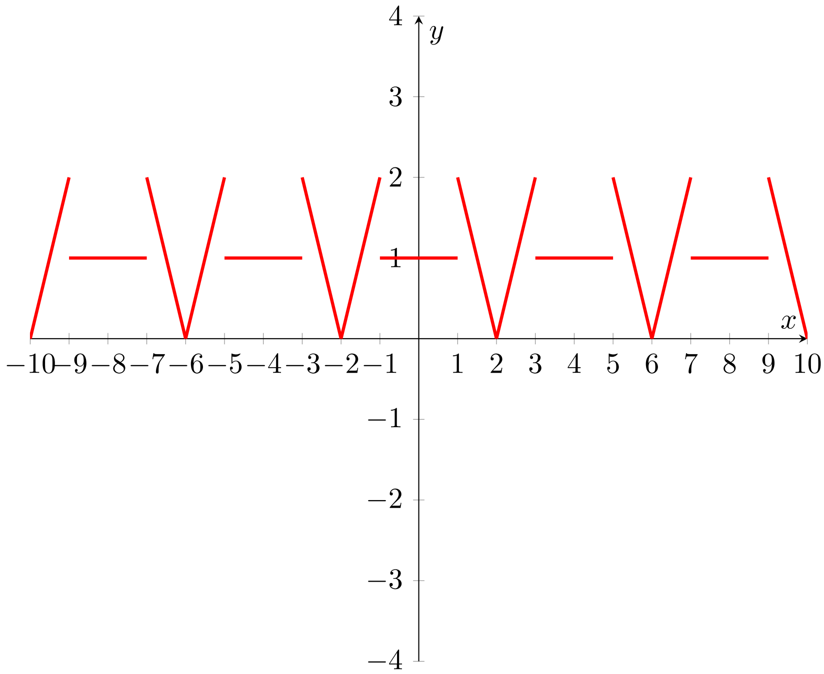
$$g(x) = \begin{cases} 1, & 0 < x < 1 \\ 4 - 2x, & 1 \leq x \leq 2, \end{cases}$$

i) Sketch over $(-10, 10)$ the graph of the function represented by the half-range Fourier **cosine** series.

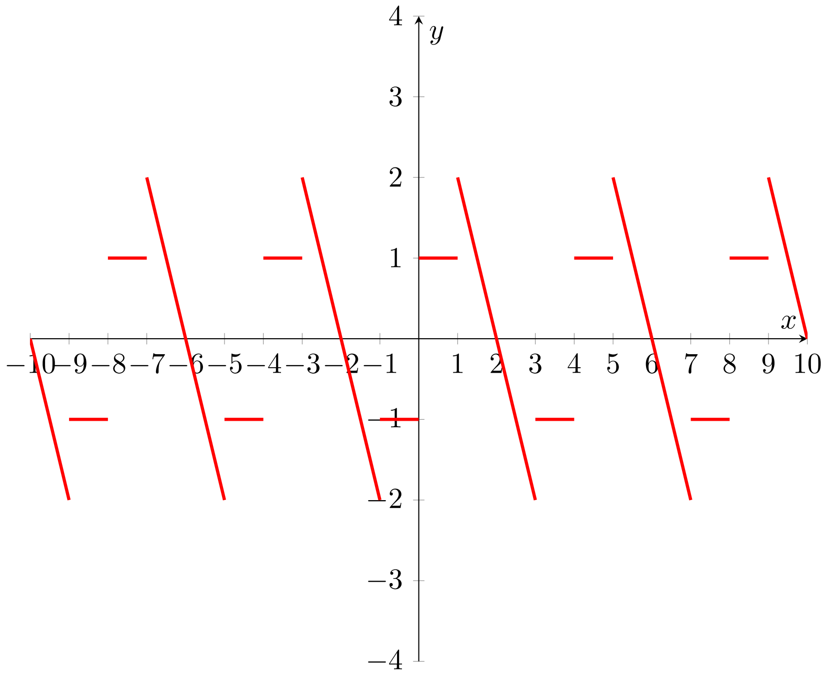
ii) Sketch over $(-10, 10)$ the graph of the function represented by the half-range Fourier **sine** series.

Solution.

i)




ii)




Describe the piecewise continuous function $f$ by

$$f(x) = \begin{cases} 1, & 0 \le x \le \frac{\pi}{2} \\ 0, & \frac{\pi}{2} \le x < \pi \end{cases}$$

i) Show that the Fourier cosine series of $f$ is given by

$$f(x) = \frac{1}{2} + \sum_{k=0}^{\infty} \frac{2(-1)^k}{\pi(2k+1)} \cos\left[(2k+1)\pi\right].$$

ii) To what value will the Fourier series converge at $x = \dfrac{\pi}{2}$?

Solution.

i) A **Fourier series** can be described as

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$$

where $L$ is the half period. In this case, the period of $f(x)$ is $2\pi$. To see why this is the case, sketch the Fourier cosine series. This gives us $L = \dfrac{2\pi}{2} = \pi$. Furthermore, the Fourier cosine series gives us the coefficients

$$\begin{aligned}
a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\ dx \\
&= \frac{2}{2\pi}\int_{0}^{\pi} f(x)\ dx && (f(x) \text{ is even}) \\
&= \frac{\pi}{2}\left(\int_{0}^{\pi/2} 1\ dx + \int_{\pi/2}^{\pi} 0\ dx\right) \\
&= \frac{1}{\pi} \times \frac{\pi}{2} \\
&= \frac{1}{2}.
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cdot\cos\left(\frac{n\pi x}{L}\right)\ dx \\
&= \frac{2}{\pi}\int_{0}^{\pi} f(x)\cdot\cos(nx)\ dx \\
&= \frac{2}{\pi}\left(\int_{0}^{\pi/2}\cos(nx)\ dx + \int_{\pi/2}^{\pi} 0\cdot\cos(nx)\ dx\right) \\
&= \frac{2}{\pi}\sin\left(\frac{n\pi}{2}\right).
\end{aligned}$$

$$\begin{aligned}
b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cdot\sin\left(\frac{n\pi x}{L}\right)\ dx \\
&= 0. && (\text{even} \times \text{odd} = \text{odd})
\end{aligned}$$

Now, consider all possible values of $n$.

When $n$ is even, we have

$$a_{2k} = \frac{2}{2k\pi}\sin\left(\frac{2k\pi}{2}\right) = \frac{1}{k\pi}\sin(k\pi) = 0.$$

Note that for any integer $k$, $\sin(k\pi) = 0$.

When $n$ is odd, we have

$$\begin{aligned}
a_{2k+1} &= \frac{2}{(2k+1)\pi}\left[\sin\left(\frac{\pi}{2}(2k+1)\right)\right] \\
&= \frac{2}{(2k+1)\pi}\left[\sin\left(k\pi + \frac{\pi}{2}\right)\right] \\
&= \frac{2}{(2k+1)\pi}\left[\cos(k\pi)\right] \\
&= \frac{2(-1)^k}{(2k+1)\pi}.
\end{aligned}$$

This conversion of $\cos(k\pi) = (-1)^k$ is a common one that often comes up, so it's worth remembering!

Hence, we have

$$f(x) = \frac{1}{2} + \sum_{k=0}^{\infty} \frac{2(-1)^k}{\pi(2k+1)} \cos[(2k+1)\pi].$$

ii) The value of convergence can be found by taking

$$\begin{aligned} \text{value} &= \frac{f\left(x^{+}\right) + f\left(x^{-}\right)}{2} \\ &= \frac{1+0}{2} \\ &= \frac{1}{2}. \end{aligned}$$

To find the value of $f(x^{+})$ and $f(x^{-})$, use the graph of the Fourier cosine series or by the definition listed in the question.

The function $f$ is given by

$$f(x) = \begin{cases} -x & \text{for } -\pi \le x \le 0 \\ x & \text{for } 0 \le x \le \pi \end{cases}$$

with $f(x + 2\pi) = f(x)$ for all $x$.

i) Make a sketch of this function for $-4\pi \le x \le 4\pi$.

ii) Is $f(x)$ odd, even or neither?

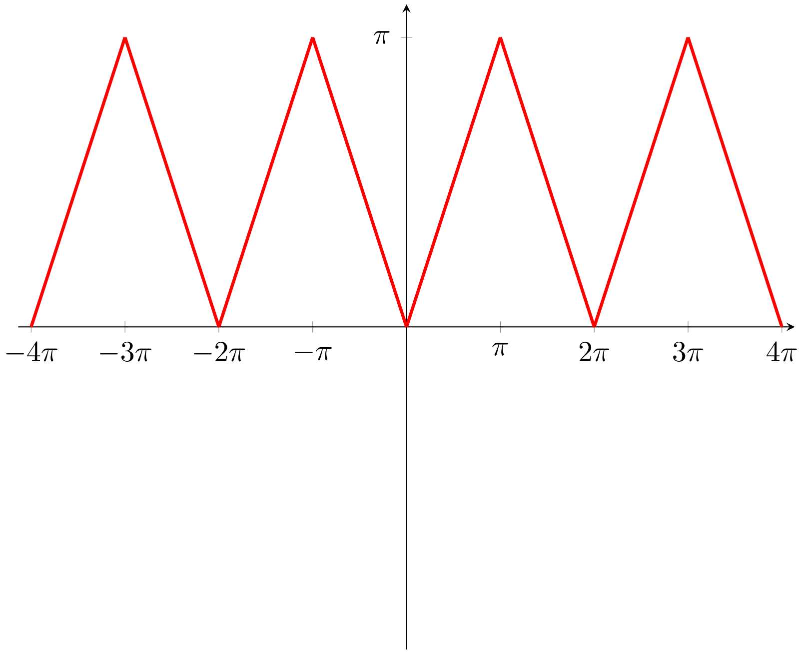
iii) Find the Fourier series of $f(x)$.

iv) By considering the value at $x = \pi$ in your answer for the Fourier series in iii), find the sum of the series

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$

Solution.

i)




ii) From the sketch, we see that $f(x)$ is **even**.

iii) A **Fourier series** for $f(x)$ can be described as

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$$

where $L$ is the half period. Here, the period of $f(x)$ is $2\pi$. This gives us $L = \dfrac{2\pi}{2} = \pi$. So we have

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ &= \frac{1}{\pi}\int_{0}^{\pi} f(x)\,dx && (f(x) \text{ is even}) \\ &= \frac{1}{\pi}\int_{0}^{\pi} x\,dx \\ &= \frac{\pi}{2}. \end{aligned}$$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\cos\left(\frac{n\pi x}{\pi}\right)\,dx \\
&= \frac{2}{\pi}\int_0^{\pi} x\cdot\cos(nx)\,dx \\
&= \frac{2}{\pi}\left[\frac{x}{n}\sin(nx)\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin(nx)\,dx\right] && \text{(integration by parts)} \\
&= \frac{2}{n^2\pi}\left[\cos(nx)-1\right] \\
&= \frac{2}{n^2\pi}\left[(-1)^n-1\right].
\end{aligned}$$

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\sin\left(\frac{n\pi x}{\pi}\right)\,dx \\
&= 0. && \text{(even} \times \text{odd = odd)}
\end{aligned}$$

Hence, the Fourier series of $f(x)$ is

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2((-1)^n-1)}{n^2\pi}\cos(nx).$$

Note that all of the even values of $n$ will disappear since we have $(-1)^{2k}-1=0$ for integer values of $k$. So we'll be left with

$$f(x) = \frac{\pi}{2} + \sum_{n=0}^{\infty}\frac{2(-1-1)(-1)}{(2n+1)^2\pi} = \frac{\pi}{2} + \sum_{n=0}^{\infty}\frac{4}{(2n+1)^2\pi}.$$

iv) Consider the value at $x=\pi$; on the one hand (by the graph), we have that $f(\pi)=\pi$. On the other hand, we have

$$f(\pi) = \frac{\pi}{2} + \sum_{n=0}^{\infty}\frac{4}{(2n+1)^2\pi}.$$

So we have

$$\begin{aligned}
\frac{\pi}{2} + \sum_{n=0}^{\infty}\frac{4}{(2n+1)^2\pi} &= \pi \\
\sum_{n=0}^{\infty}\frac{4}{(2n+1)^2\pi} &= \frac{\pi}{2} \\
\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2\pi} &= \frac{\pi}{8} \\
\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} &= \frac{\pi^2}{8}.
\end{aligned}$$

In other words,

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots = \frac{\pi^2}{8}.$$

## Part VIII: Partial Differential Equations

Consider the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} - 4\frac{\partial^2 u}{\partial y^2} = 0.$$

i) Use D'Alembert's method to find a solution in terms of arbitrary functions.
ii) Determine the particular solution satisfying $u(x,0) = 0$ and $u_y(x,0) = 8\sin(2x)$.

SOLUTION.

i) Consider the PDE

$$\frac{\partial^2 u}{\partial x^2} = c^2\frac{\partial^2 u}{\partial y^2}.$$

Using D'Alembert's method, we have

$$u(x,t) = \phi(x+ct) + \psi(x-ct).$$

D'Alembert's solution can be expressed as

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\, d\xi.$$

By setting $c = 2$, we have

$$\frac{\partial^2 u}{\partial x^2} = 4\cdot\frac{\partial^2 u}{\partial y^2} \implies \frac{\partial^2 u}{\partial x^2} - 4\cdot\frac{\partial^2 u}{\partial y^2} = 0.$$

So the solution can be expressed as

$$u(x,y) = \frac{1}{2}\left[f(x+2y) + f(x-2y)\right] + \frac{1}{4}\int_{x-2y}^{x+2y} g(\xi)\, d\xi$$

for some arbitrary functions $f$ and $g$.

ii) From part a, we have the solution expressed as

$$u(x,y) = \frac{1}{2}\left[f(x+2y) + f(x-2y)\right] + \frac{1}{4}\int_{x-2y}^{x+2y} g(\xi)\, d\xi.$$

In general, the initial displacement $u(x,0) = 0$ refers to the function $f$ while the initial velocity $u_y(x,0)$ refers to the function $g$. Hence, we have

$$\begin{aligned}
u(x,y) &= \frac{1}{2}\left[f(x+2y)+f(x-2y)\right] + \frac{1}{4}\int_{x-2y}^{x+2y} g(\xi)\, d\xi \\
&= \frac{1}{2}\left[0+0\right] + \frac{1}{4}\int_{x-2y}^{x+2y} 8\sin(2\xi)\, d\xi \\
&= 2\left[-\frac{1}{2}\cos(2\xi)\Big|_{x-2y}^{x+2y}\right] \\
&= \cos(2x-4y) - \cos(2x+4y).
\end{aligned}$$

The steady-state distribution of heat in a slab of width $L$ is given by

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0, \quad \text{for } 0 < x < L,\ 0 < y < \infty$$

$$\begin{aligned}
&U(0,y) = U(L,y) = 0, && \text{for } 0 < y < \infty \\
&U \text{ bounded} && \text{as } y \to +\infty \\
&U(x,0) = f(x), && \text{for } 0 \le x \le L.
\end{aligned}$$

Use the method of separation of variables to find the general solution $U(x,y)$, where any unknown constants are related to $U(x,0) = f(x)$. You must explicitly consider all possibilities for the separation constant in your working.

Solution.
Assume that the solution exists and is of the form

$$U(x,y) = F(x)\cdot G(y).$$

Then the PDE can be written as

$$F_{xx}G + FG_{yy} = 0 \implies \frac{F_{xx}}{F} = -\frac{G_{yy}}{G} = k$$

for some constant $k$. This gives us two separate ordinary differential equations

$$F'' = kF, \quad G'' = -kG.$$

By considering the ordinary differential equation for $F$, we have $F'' - kF = 0$ which is a second-order homogeneous differential equation. By computing the roots of the characteristic polynomial, we arrive at

$$\begin{aligned}
\lambda^2 - k\lambda = 0 \\
\lambda = \pm\sqrt{k}.
\end{aligned}$$

Now, we consider different cases for $k$.

**Case 1.** Set $k = 0$. Then our solution is of the form

$$F(x) = A_0 x + B_0$$

for some constants $A_0$ and $B_0$. By considering the boundary conditions, we have

$$F(0) = 0 \implies B_0 = 0, \qquad F(L) = 0 \implies A_0 L = 0 \implies A_0 = 0.$$

Hence, we have the trivial solution.

**Case 2.** Set $k = p^2$. Then our solution is of the form

$$F(x) = A_1 e^{px} + B_1 e^{-px}.$$

By considering the boundary conditions, we have

$$F(0) = A_1 + B_1 = 0 \implies A_1 = -B_1.$$

Furthermore,

$$F(L) = A_1 e^{pL} - A_1 e^{-pL} = 0 \implies pL = -pL \implies p = 0.$$

Thus, we have

$$A_1 - A_1 = 0 \implies A_1 = B_1 = 0.$$

Hence, we have the trivial solution.

**Case 3.** Set $k = -p^2$. Then our solution is of the form

$$A_2 \cos(px) + B_2 \sin(px).$$

By considering the boundary conditions, we have

$$F(0) = 0 \implies A_2 = 0.$$

Furthermore,

$$\begin{aligned} F(L) &= 0 + B_2 \sin(pL) = 0 \\ &\implies \sin(pL) = 0 \\ &\implies pL = \pi n, \qquad n = 0, 1, \ldots \\ &\implies p = \frac{\pi n}{L}. \end{aligned}$$

So we have

$$F_n(x) = B_n \sin\left(\frac{\pi n x}{L}\right).$$

Now consider the ordinary differential equation for $G$. We have

$$G''(y) + kG(y) = 0, \qquad \lambda^2 = p^2 \implies \lambda = \pm p.$$

Hence, a general solution for $G(y)$ is

$$G(y) = Ce^{py} + De^{-py}.$$

But we know that $u(x, y)$ is bounded so as $x \to \infty$, $u(x, y) \not\to \infty$. This means $C = 0$; else $u$ will be unbounded. Hence, our general solution for $G$ in terms of $n$ is

$$G_n(y) = D_n e^{-py} = D_n e^{-\frac{n\pi}{L}y}.$$

Finally, our solution is

$$\begin{aligned} U(x, y) &= F(x) \cdot G(y) \\ &= \sum_{n=1}^{\infty} \left( B_n \sin\left(\frac{\pi n x}{L}\right) D_n e^{-\frac{n\pi}{L}y} \right) \\ &= \sum_{n=1}^{\infty} B_n D_n \sin\left(\frac{\pi n x}{L}\right) e^{-\frac{n\pi}{L}y}. \end{aligned}$$

Using our last boundary condition, we have

$$U(x, 0) = f(x) \implies \sum_{n=1}^{\infty} B_n D_n \sin\left(\frac{\pi n x}{L}\right) = f(x).$$

This is the **sine Fourier series** which was taught in the previous topic! Using the sine Fourier series, we have

$$U(x, y) = \sum_{n=1}^{\infty} \underbrace{\left[ \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\ dx\right) \right]}_{B_n D_n} \cdot \sin\left(\frac{\pi n x}{L}\right) e^{-\frac{n\pi}{L}y}$$

A stretched wire satisfies the wave equation

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2},$$

where $u(x,t)$ is the displacement of the wire. The ends of the wire are held fixed so that

$$u(0,t) = u(\pi,t) = 0, \quad \text{for all } t.$$

i) Assuming a solution of the form $u(x,t) = F(x)G(t)$ show that

$$\frac{G''(t)}{4G(t)} = \frac{F''(x)}{F(x)} = k$$

for some constant $k$.

ii) Apply the boundary conditions to show that possible solutions for $F(x)$ are

$$F_n(x) = B_n \sin(nx)$$

where $B_n$ are constants and $n = 1, 2, 3\cdots$. You must consider all possible values of $k$.

iii) Find all possible solutions $G_n(t)$ for $G(t)$.

iv) If the initial displacement and velocity of the wire are

$$u(x,0) = 3\sin(x) + 4\sin(3x), \quad \text{and} \quad u_t(x,0) = 0,$$

find the general solution $u(x,t)$.

i) Let $u(x,t) = F(x)\cdot G(t)$. Then

$$FG'' = 4F''G \implies \frac{G''(t)}{4G(t)} = \frac{F''(x)}{F(x)} = k.$$

ii) Consider the ODE on $F(x)$. Then

$$F'' = kF = 0 \implies \lambda^2 = k.$$

Consider the cases on $k$.

**Case 1.** Set $k = 0$. Then our solution is of the form

$$F(x) = A_0 x + B_0$$

for some constants $A_0$ and $B_0$. By considering the boundary conditions, we have

$$F(0) = 0 \implies B_0 = 0 \qquad F(\pi) = 0 \implies A_0 = 0.$$

Hence, we have the trivial solution.

**Case 2.** Set $k = p^2$. Then our solution is of the form

$$F(x) = A_1 e^{px} + B_1 e^{-px}.$$

By considering the boundary conditions, we have

$$F(0) = 0 \implies A_1 = -B_1.$$

Furthermore,

$$F(\pi) = A_1 e^{px} - A_1 e^{-px} = 0 \implies pL = -pL \implies p = 0.$$

Thus, we have

$$A_1 - A_1 = 0 \implies A_1 = B_1 = 0.$$

Hence, we have the trivial solution.

**Case 3.** Set $k = -p^2$. Then our solution is of the form

$$F(x) = A_2 \cos(px) + B_2 \sin(px).$$

By considering the boundary conditions, we have

$$F(0) = 0 \implies A_2 = 0.$$

Furthermore,

$$\begin{aligned} F(\pi) &= 0 + B_2 \sin(p\pi) = 0 \\ &\implies \sin(p\pi) = 0 \\ &\implies p\pi = \pi n, \qquad n = 0, 1, \ldots \\ &\implies p = n. \end{aligned}$$

So we have

$$F_n(x) = B_n \sin(nx).$$

iii) Now consider the ordinary differential equation for $G$. We have

$$G'' - 4kG = 0, \qquad \lambda^2 + 4\pi^2 = 0 \implies \lambda = \pm 2ip.$$

Hence, a general solution for $G(y)$ is

$$G(y) = C\cos(2pt) + D\sin(2pt).$$

So our general solution for $G_n(t)$ is

$$G_n(t) = C_n \cos(2nt) + D_n \sin(2nt).$$

iv) The general solution for $u(x,t)$ is therefore

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin(nx) \left[ C_n \cos(2nt) + D_n \sin(2nt) \right].$$

Using our initial conditions, we have

$$\begin{aligned} u(x,0) = \sum_{n=1}^{\infty} B_n C_n \sin(nx) &= 3\sin(x) + 4\sin(3x) \\ \implies B_1 C_1 &= 3 \\ \implies B_3 C_3 &= 4 \\ \implies B_n C_n &= 0 \quad \text{for all } n \neq 1, 3. \end{aligned}$$

$$\begin{aligned} u_t(x,t) &= \sum_{n=1}^{\infty} B_n \sin(nx) \left[ 2C_n \sin(2nt) + 2D_n \cos(2nt) \right] \\ u_t(x,0) &= \sum_{n=1}^{\infty} 2B_n D_n \sin(nx) = 0 \\ &\implies B_n D_n = 0 \quad \text{for all } n. \end{aligned}$$

Hence, we have

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} B_n \sin(nx) \left[ C_n \cos(2nt) + D_n \sin(2nt) \right] \\ &= 3\sin(x) \cdot \cos(2t) + 4\sin(3x)\cos(6t). \end{aligned}$$