MATH1131/1141 MathSoc Calculus Revision Session 2020 T1 Solutions

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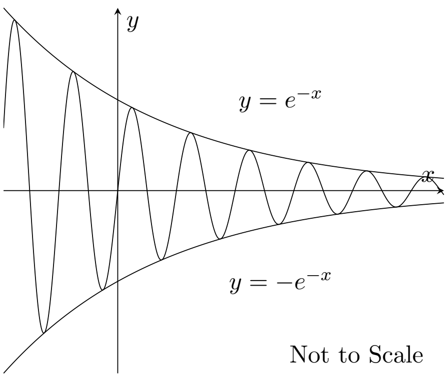
We cannot guarantee that our answers are correct - please notify us of any errors or typos at unswmathsoc@gmail.com, or on our Facebook page. There are sometimes multiple methods of solving the same question. Remember that in the real class test, you will be expected to explain your steps and working out.

Chapter 2

Example:
Use the pinching theorem to evaluate

$$\lim_{x\to\infty} e^{-x}\sin(x).$$

Solution:



Since $-1 \leq \sin x \leq 1$,

$$-e^{-x} \leq e^{-x}\sin x \leq e^{-x}$$

since $e^{-x} > 0$.
Now,

$$\lim_{x\to\infty}(-e^{-x}) = 0$$
$$\lim_{x\to\infty} e^{-x} = 0$$

Hence, $\lim\limits_{x\to\infty} e^{-x}\sin x = 0$ by pinching theorem.

Example:
Evaluate the limit:

$$\lim_{x\to\infty}\frac{10x^2+3x+\sin x}{5x^2+3x-2}.$$

Solution:

$$\begin{aligned}
\lim_{x\to\infty}\frac{10x^2+3x+\sin x}{5x^2+3x-2} &= \lim_{x\to\infty}\frac{10(\frac{x^2}{x^2})+3(\frac{x}{x^2})+(\frac{\sin x}{x^2})}{5(\frac{x^2}{x^2})+3(\frac{x}{x^2})-2(\frac{1}{x^2})}\\
&= \frac{10+0+0}{5+0+0}\\
&= \frac{10}{5}\\
&= 2
\end{aligned}$$

Example:
Evaluate the limit

$$\lim_{x\to\infty}\frac{1}{x-\sqrt{x^2-6x-4}}.$$

Solution:

$$\begin{aligned}
\lim_{x\to\infty}\frac{1}{x-\sqrt{x^2-6x-4}} &= \lim_{x\to\infty}\frac{1}{x-\sqrt{x^2-6x-4}}\times\frac{x+\sqrt{x^2-6x-4}}{x+\sqrt{x^2-6x-4}}\\
&= \lim_{x\to\infty}\frac{x+\sqrt{x^2-6x-4}}{x^2-(x^2-6x-4)} = \lim_{x\to\infty}\frac{x+\sqrt{x^2-6x-4}}{6x+4}\\
&= \lim_{x\to\infty}\frac{\frac{x}{x}+\sqrt{\frac{x^2}{x^2}-6(\frac{x}{x^2})-4\frac{1}{x^2}}}{6(\frac{x}{x})+4(\frac{1}{x})}\\
&= \frac{1+\sqrt{1}}{6}\\
&= \frac{1}{3}
\end{aligned}$$

Example: Evaluate the limit

$$\lim_{x \to 0} \frac{x^2 e^x}{1 - \cos \pi x}.$$

Solution:

$$\begin{aligned}
\lim_{x \to 0} \frac{x^2 e^x}{1 - \cos \pi x} &= \lim_{x \to 0} \frac{2xe^x + x^2 e^x}{\pi \sin(\pi x)} && \text{(by L'Hopital's rule)} \\
&= \lim_{x \to 0} \frac{2e^x + 2xe^x + 2xe^x + x^2 e^x}{\pi^2 \cos(\pi x)} && \text{(by L'Hopital's rule)} \\
&= \frac{2e^0 + 2 \times (0) \times e^0 + 2 \times (0) \times e^0 + (0)^2 \times e^0}{\pi^2 \cos(\pi \times 0)} \\
&= \frac{2}{\pi^2}
\end{aligned}$$

Example:

Use the ϵ-M definition of the limit to prove that

$$\lim_{x \to \infty} \frac{e^x}{\cosh x} = 2.$$

Solution: Let $f(x) = \dfrac{e^x}{\cosh x} = \dfrac{2e^x}{e^x + e^{-x}}$

$$\begin{aligned}
|f(x) - 2| &= \left| \frac{2e^x}{e^x + e^{-x}} - 2 \right| \\
&= \left| \frac{2e^x - 2(e^x + e^{-x})}{e^x + e^{-x}} \right| \\
&= \left| -\frac{2e^{-x}}{e^x + e^{-x}} \right| \\
&= \frac{2e^{-x}}{e^x + e^{-x}}
\end{aligned}$$

Suppose $|f(x) - 2| < \epsilon$, where ϵ is a small positive value.

$$\begin{aligned}
\frac{2e^{-x}}{e^x + e^{-x}} &< \epsilon \\
\frac{2}{e^{2x} + 1} &< \epsilon \\
\frac{2}{\epsilon} &< (e^{2x} + 1) && \text{(since } e^{2x} + 1 > 0 \text{ and } \epsilon > 0\text{)} \\
e^{2x} &> \frac{2}{\epsilon} - 1 \\
x &> \frac{1}{2} \ln\left(\frac{2}{\epsilon} - 1\right)
\end{aligned}$$

There exists a value of M for every small positive value of ϵ, given by $\frac{1}{2} \ln\left(\frac{2}{\epsilon} - 1\right)$, such that, if $x > M$, then $|f(x) - 2| < \epsilon$ holds true.

Hence, $\lim_{x\to\infty} \dfrac{e^x}{\cosh x} = 2$.

Example:

The function f is defined by

$$f(x) = \begin{cases} 3 - x & 0 \leq x < 1 \\ (x-2)^2 + 1 & 1 \leq x \leq 3. \end{cases}$$

Does $\lim_{x\to 1} f(x)$ exist? Give brief reasons for your answer.

Note: You need to use the property of continuous functions: $\lim_{x\to a} f(x) = f(a)$ to solve this problem.

Solution: We must check the left and right hand limits at 1:

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (3-x) = 3 - 1 = 2,$$
$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} ((x-2)^2 + 1) = (1-2)^2 + 1 = 2.$$

Since the left and right hand limit are equal, the two-sided limit, $\lim_{x\to 1} f(x)$, exists.

Example:

A function g is defined by

$$g(x) = \begin{cases} \frac{|x^2-16|}{x-4} & \text{if } x \neq 4 \\ \alpha & \text{if } x = 4. \end{cases}$$

By considering the left and right hand limits at $x = 4$, show that no value of α can make g continuous at the point $x = 4$.

Solution: For the function g to be continuous at $x = 4$, the limit must exist (that is, $\lim_{x\to 4^-} g(x) = \lim_{x\to 4^+} g(x)$ and be equal to $f(4)$.

$$\begin{aligned} \lim_{x\to 4^-} g(x) &= \lim_{x\to 4^-} \frac{|x^2-16|}{x-4} \\ &= \lim_{x\to 4^-} \frac{-(x^2-16)}{x-4} \qquad \text{(since } x^2 - 16 < 0 \text{ for values of } x \text{ close to but less than 4)} \\ &= \lim_{x\to 4^-} \frac{-(x+4)(x-4)}{x-4} \\ &= \lim_{x\to 4^-} [-(x+4)] \\ &= -(4+4) = -8 \end{aligned}$$

$$
\begin{aligned}
\lim_{x\to 4^+} g(x) &= \lim_{x\to 4^+} \frac{|x^2-16|}{x-4} \\
&= \lim_{x\to 4^+} \frac{(x^2-16)}{x-4} \qquad \text{(since } x^2-16>0 \text{ for values of } x \text{ close to but greater than 4)} \\
&= \lim_{x\to 4^+} \frac{(x+4)(x-4)}{x-4} \\
&= \lim_{x\to 4^+} (x+4) \\
&= (4+4) = 8
\end{aligned}
$$

Since the left and right hand limits are not equal, the limit of g at $x = 4$ does not exist. Therefore, the function cannot be continuous, regardless of the value of α.

Chapter 3

Example:
Show that the equation $e^{-7x} = -2\cos(16x)$ has a unique solution for $x \in [0, \frac{\pi}{16}]$.

Solution:
Proving that solution exists using IVT:
Consider the function f given by $f(x) = e^{-7x} + 2\cos(16x)$.

$$
\begin{aligned}
f(0) &= e^{-7(0)} + 2\cos[16(0)] \\
&= 1 + 2 \\
&= 3 > 0 \\
f\left(\frac{\pi}{16}\right) &= e^{-7\pi/16} + 2\cos\left[16\left(\frac{\pi}{16}\right)\right] \\
&= e^{-7\pi/16} - 2 \\
&\approx -1.75 < 0
\end{aligned}
$$

Hence, 0 lies between $f(0)$ and $f\left(\frac{\pi}{16}\right)$.

Furthermore, as f consists of a sum of an exponential function and a cosine function which are continuous and defined over the real numbers, f itself is continuous over the interval $x \in \left[0, \frac{\pi}{16}\right]$. Hence, by the **Intermediate Value Theorem**, there must exist at least one $c \in \left[0, \frac{\pi}{16}\right]$ such that $f(c) = 0$.

Proving that a solution is unique:
Now,

$$f'(x) = -7e^{-7x} - 32\sin(16x).$$

Note that $-7e^{-7x} < 0$ for $x \in \left[0, \frac{\pi}{16}\right]$ since $e^t > 0$ for all real t.

Furthermore, $-32\sin(16x) \leq 0$ for $x \in \left[0, \frac{\pi}{16}\right]$ since $\sin(t) \geq 0$ for $t \in [0, \pi]$.

Hence, $f'(x) < 0$ for $x \in \left[0, \frac{\pi}{16}\right]$. f is monotonically decreasing over this interval.
$\therefore$ $f(x) = e^{-7x} + 2\cos(16x)$ has exactly one zero over the interval $x \in \left[0, \frac{\pi}{16}\right]$.
$\Longleftrightarrow$ For the equation $e^{-7x} = -2\cos(16x)$ there must exist a unique solution for $x \in \left[0, \frac{\pi}{16}\right]$.

Example:
Consider the three functions:

$$\begin{aligned}
&f:\ \mathbb{R} \to \mathbb{R} && f(x) = \frac{x^2}{1+x^2} \\
&g:\ (0,3) \to \mathbb{R} && g(x) = (x-1)^2 \\
&h:\ [1,5] \to \mathbb{R} && h(x) = \sqrt{1 + \ln x + \sin x \cos x}
\end{aligned}$$

Only one of these functions has a maximum value on its given domain. Which one is it? Give reasons for your answer.

Solution: Out of the three functions, only h is defined on a closed interval.
The function $\sqrt{x}$ is continuous and defined for $x > 0$ and functions $\ln x$, $\sin x$ and $\cos x$ are all continuous over $x \in [1,5]$.
Now,

$$\begin{aligned}
1 + \ln x + \sin x \cos x &= 1 + \ln x + \frac{1}{2}\sin 2x \\
&\geq 1 + \ln 1 + \left(-\frac{1}{2}\right) && \text{(for } x \in [1,5]) \\
&= \frac{1}{2} > 0.
\end{aligned}$$

Hence, h is also continuous over $x \in [1,5]$, since h is a composition of continuous functions, and is defined at all points in the given interval.
By maximum-minimum value theorem, it is guaranteed that h has maximum value on its given domain.

Chapter 4

Example:
Consider the function f defined by

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

(a) Given that $\lim_{x\to\infty} xe^{-x} = 0$, evaluate the limit

$$\lim_{h\to 0} \frac{e^{-1/h^2}}{h}.$$

(b) Using the definition of a derivative, determine whether f is differentiable at $x = 0$.

Solution:

(a)

$$\begin{aligned}
\lim_{h\to 0} \frac{e^{-1/h^2}}{h} &= \lim_{h\to 0} \left[\left(\frac{e^{-1/h^2}}{h^2} \right) \times (h) \right] \\
&= \lim_{h\to 0} \frac{e^{-1/h^2}}{h^2} \times \lim_{h\to 0} h && \text{Let } x = 1/h^2. \\
& && \text{As } h \to 0,\ x \to \infty. \\
&= \lim_{x\to\infty} xe^{-x} \times \lim_{h\to 0} h \\
&= 0 \times 0 \\
&= 0
\end{aligned}$$

(b) We need to find the left and right hand limit of difference quotient when $x = 0$. That is

$$\begin{aligned}
\lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} &= \lim_{h\to 0^+} \frac{e^{-1/h^2} - 0}{h} \\
&= \lim_{h\to 0^+} \frac{e^{-1/h^2}}{h} \\
&= 0 && \text{(from (a))}
\end{aligned}$$

and, similarly,

$$\begin{aligned}
\lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h} &= \lim_{h\to 0^-} \frac{e^{-1/h^2} - 0}{h} \\
&= \lim_{h\to 0^-} \frac{e^{-1/h^2}}{h} \\
&= 0.
\end{aligned}$$

Since the left and right hand limits of the difference quotient exist and are equal at $x = 0$ the function must be differentiable at $x = 0$.

Example: Given that the function h, defined by

$$h(x) = \begin{cases} e^{3x}, & x \leq 0 \\ q(x), & x > 0 \end{cases}$$

is differentiable at $x = 0$, and q is a monic quadratic function, find the expression for $q(x)$.

Solution:
For the function to be differentiable at $x = 0$, it must also be continuous at $x = 0$, and the left and right hand limit of the difference quotient must be equal.
Let $q(x) = x^2 + bx + c$.

Method 1: Using the definition of the derivative

As for continuity,

$$\begin{aligned}
\lim_{x\to 0^-} h(x) &= \lim_{x\to 0^-} e^{3x} = e^{3(0)} = 1 = h(0) \\
\lim_{x\to 0^+} h(x) &= \lim_{x\to 0^+} (x^2+bx+c) = c.
\end{aligned}$$

For h to be continuous at $x=0$, $\lim_{x\to 0^+} h(x) = \lim_{x\to 0^-} h(x) = h(0)$, and so $c=1$.
As for difference quotient,

$$\begin{aligned}
\lim_{k\to 0^-} \frac{h(0+k)-h(0)}{k} &= \lim_{k\to 0^+} \frac{h(0+k)-h(0)}{k} \\
\lim_{k\to 0^-} \frac{e^{3k}-e^{3(0)}}{k} &= \lim_{k\to 0^+} \frac{k^2+bk+c-e^{3(0)}}{k} \\
\lim_{k\to 0^-} \frac{e^{3k}-1}{k} &= \lim_{k\to 0^+} \frac{k^2+bk+1-1}{k} && \text{(since } c=1 \text{ for continuity)} \\
\lim_{k\to 0^-} \frac{3e^{3k}}{1} &= \lim_{k\to 0^+} (k+b) && \text{(by L'Hopital's rule)} \\
3e^{3(0)} &= b
\end{aligned}$$

$\therefore b=3$.
Hence, if h is differentiable at $x=0$, the function q must be defined by $q(x)=x^2+3x+1$.

Method 2: Using the theory for piecewise-defined functions
As for continuity,

$$\begin{aligned}
q(0) &= e^{3(0)} \\
0^2+b(0)+c &= 1 \\
c &= 1.
\end{aligned}$$

Now, noting that $\frac{d}{dx}e^{3x} = 3e^{3x}$ and $q'(x) = 2x+b$,

$$\begin{aligned}
q'(0) &= 3e^{3(0)} \\
2(0)+b &= 3 \\
b &= 3.
\end{aligned}$$

Hence, $q(x) = x^2+3x+1$

Example:

Find the equation of the tangent at the origin to the curve implicitly defined by

$$e^x + \sin(y) = xy + 1$$

Solution:

Differentiating both sides with respect to x:

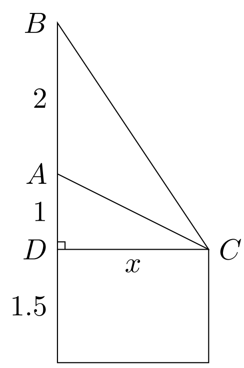
$$\begin{aligned}
\frac{d}{dx}(e^x + \sin(y)) &= \frac{d}{dx}(xy+1) \\
e^x + \cos(y)\frac{dy}{dx} &= y + x\frac{dy}{dx} \\
(\cos(y) - x)\frac{dy}{dx} &= y - e^x \\
\frac{dy}{dx} &= \frac{y - e^x}{\cos(y) - x} \\
&= \frac{0 - e^0}{\cos(0) - 0} && \text{(at the origin)} \\
&= -1.
\end{aligned}$$

Hence, the equation of the tangent is given by

$$\begin{aligned}
y - 0 &= -1 \cdot (x - 0) \\
y &= -x.
\end{aligned}$$

Example:

A statue 2 metres high stands on a pillar 2.5 metres high. A person, whose eye is 1.5m above the ground, stands at a distance x metres from the base of the pillar. The diagram shows the above information, with the person's eye being at C. Let $\angle BCA = \theta$ and $\angle ACD = \phi$.



(a) Prove that $\dfrac{d}{dt}(\cot^{-1} t) = -\dfrac{1}{1+t^2}$.

(b) Show that $\theta = \cot^{-1}\dfrac{x}{3} - \cot^{-1} x$.

(c) Hence, find the distance x that maximise the angle θ.

Solution:

(a) Let $x = \cot^{-1} t \implies t = \cot(x) = \tan(\frac{\pi}{2} - x)$.

$$\begin{aligned}
\frac{dt}{dx} &= -\sec^2(\frac{\pi}{2} - x) \\
&= -\operatorname{cosec}^2 x \\
&= -(1 + \cot^2 x) \\
&= -(1 + t^2) \\
\therefore \quad \frac{dx}{dt} &= -\frac{1}{1+t^2}
\end{aligned}$$

Now, $\dfrac{dx}{dt} = \dfrac{d}{dt}(\cot^{-1} t)$, and so $\dfrac{d}{dt}(\cot^{-1} t) = -\dfrac{1}{1+t^2}$.

(b)

$$\cot\phi = \cot\angle ACD = \frac{DC}{AD} = \frac{x}{1} = x$$

$$\phi = \cot^{-1} x \qquad \text{(since } 0 < \phi < \tfrac{\pi}{2}\text{)}$$

$$\cot(\theta+\phi) = \cot\angle BCD = \frac{DC}{BD} = \frac{x}{1+2} = \frac{x}{3}$$

$$\theta + \phi = \cot^{-1}\frac{x}{3} \qquad \text{(since } 0 < \theta + \phi < \tfrac{\pi}{2}\text{)}$$

Now,

$$\theta = (\theta+\phi) - (\phi)$$

$$\theta = \cot^{-1}\frac{x}{3} - \cot^{-1} x.$$

(c) To maximise the angle θ, we will find differentiate θ with respect to x:

$$\begin{aligned}\frac{d\theta}{dx} &= -\frac{1/3}{1+(x/3)^2} - \left(-\frac{1}{1+x^2}\right)\\ &= -\frac{3}{9+x^2} + \frac{1}{1+x^2}\\ &= \frac{-2(x^2-3)}{(1+x^2)(9+x^2)}.\end{aligned}$$

For a maximum,

$$\begin{aligned}\frac{d\theta}{dx} &= 0\\ \frac{-2(x^2-3)}{(1+x^2)(9+x^2)} &= 0\\ (x^2-3) &= 0\\ x &= \sqrt{3}. \qquad \text{(since } x \text{ is positive)}\end{aligned}$$

Now, we must test whether this point is a maximum or not.

Method 1: 'Sign' tables

x	$\sqrt{3}^-$	$\sqrt{3}$	$\sqrt{3}^+$
(x^2-3)	$-$	0	$+$
$\dfrac{-2}{(1+x^2)(9+x^2)}$	$-$	$-$	$-$
$\dfrac{d\theta}{dx}$	$+$	0	$-$
	/	–	\

Hence, θ is a maximum when $x = \sqrt{3}$.

Method 2: Second derivative

$$\begin{aligned}\frac{d^2\theta}{dx^2} &= \frac{6x}{(9+x^2)^2} - \frac{2x}{(1+x^2)^2} \\ &= -\frac{\sqrt{3}}{12} && \text{(at } x = \sqrt{3}\text{)} \\ &< 0\end{aligned}$$

This means that the curve representing the relationship between θ and x is concave down at the stationary point at $x = \sqrt{3}$, and so a maximum must be attained.

Example: Suppose $-1 < x < y < 1$. By applying the Mean Value Theorem to the function $f(t) = \sin^{-1} t$ on the interval $[x, y]$, prove that

$$\sin^{-1} y - \sin^{-1} x \geq y - x.$$

Solution: Since $\sin^{-1}$ is differentiable on (x, y) and continuous on $[x, y]$, there exists $c \in (x, y)$ such that

$$\frac{\sin^{-1} y - \sin^{-1} x}{y - x} = f'(c)$$

by Mean Value Theorem.

$$\begin{aligned}f'(c) &= \frac{1}{\sqrt{1-c^2}} \\ &\geq \frac{1}{\sqrt{1-0^2}} \\ &= 1\end{aligned}$$

Hence,

$$\begin{aligned}\frac{\sin^{-1} y - \sin^{-1} x}{y - x} &\geq 1 \\ \sin^{-1} y - \sin^{-1} x &\geq y - x.\end{aligned}$$

Some Basic Set Notation and Logic Symbols

$\cup$ - union (or)

$\cap$ - intersection (and)

$\{\}$ - set grouping symbols

$\in$ - 'is an element of'

$:$ or $|$ - 'such that'

$\setminus$ - 'excluding'

$\exists$ - 'there exists'

$\forall$ - 'for all'

$\iff$ - 'if and only if'

$\implies$ - 'implies'