MATH1081 MathSoc Topics 1-2, 5 Revision Session 2019 T1 Solutions

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We cannot guarantee that our answers are correct - please notify us of any errors or typos at unswmathsoc@gmail.com, or on our Facebook page. There are sometimes multiple methods of solving the same question. Remember that in the real class test, you will be expected to explain your steps and working out.

Set Theory

1. Simplify $(A \cup B) \cup (C \cap A) \cup (A \cap B)^c$ using the laws of set algebra.

Solution:

$$
\begin{aligned}
&(A \cup B) \cup (C \cap A) \cup (A \cap B)^c \\
&= (A \cup B) \cup (C \cap A) \cup (A^c \cup B^c) && \text{(de Morgan's)} \\
&= A \cup B \cup (C \cap A) \cup A^c \cup B^c && \text{(associative)} \\
&= A \cup A^c \cup B \cup B^c \cup (C \cap A) && \text{(commutative)} \\
&= U \cup U \cup (C \cap A) && \text{(union with complement)} \\
&= U \cup (C \cap A) = U && \text{(domination)}
\end{aligned}
$$

2. Prove that $A = \{\cos(x) = \frac{1}{\sqrt{6}} \mid x \in \mathbb{R}\}$ is a proper subset of $B = \{\cos(2x) = \frac{-2}{3} \mid x \in \mathbb{R}\}$.

Solution:
Let $x \in A$.
So $\cos x = \frac{1}{\sqrt{6}}$.
We can draw a right-angled triangle and use Pythagoras' theorem to find that $\sin x = \frac{\sqrt{5}}{\sqrt{6}}$.

$$\begin{aligned}\cos 2x &= \cos^2\theta - \sin^2\theta \\ &= (\frac{1}{\sqrt{6}})^2 - (\frac{\sqrt{5}}{\sqrt{6}})^2 \\ &= \frac{1}{6} - \frac{5}{6} \\ &= \frac{-2}{3}.\end{aligned}$$

Hence, $x \in B$
Therefore, $A \subseteq B$
To prove A is a proper subset of B, we find an element in B that's not in A.

$$\begin{aligned}\cos 2x &= \frac{-2}{3} \\ 2x &= \cos^{-1}\frac{2}{3} + \pi \\ x &= \frac{1}{2}(\cos^{-1}\frac{2}{3} + \pi)\end{aligned}$$

But substituting this into A gives: $\cos(\frac{1}{2}(\cos^{-1}\frac{2}{3} + \pi)) = -0.408 \neq \frac{1}{\sqrt{6}}$
$\therefore A \subsetneq B$

3. Prove that $A = \{x \geq 2 \mid x \in \mathbb{R}\}$ is a proper subset of $B = \{3x^2 + x \geq 14 \mid x \in \mathbb{R}\}$.

Solution:
Let $x \in A$. Thus, $x \geq 2$. So,

$$\begin{aligned}3x^2 + x &= x(3x+1) \\ &\geq 14\end{aligned}$$

since $3x + 1 \geq 7$.
Hence, $x \in B$, so $A \subseteq B$.
To prove A is a proper subset of B, we find an element in B that's not in A.
Consider $x = -4 < 2$, thus $3x^2 + x = 44 \geq 14$.
Since x is not greater than or equal to 2, $A \subsetneq B$.

4. Let $I = \{1, 2, 3, ...\}$ be the index set. For each $i \in I$, Let $A_i = [0, \frac{1}{i}]$ be the set of real numbers between, and including, 0 and $\frac{1}{i}$. Find: (a) $\bigcup_{i \in I} A_i$, (b) $\bigcap_{i \in I} A_i$

Solution:

(a) $= [0,1] \cup [0,\frac{1}{2}] \cup [0,\frac{1}{3}] \cup ... = [0,1]$

(b) $= [0,1] \cap [0,\frac{1}{2}] \cap [0,\frac{1}{3}] \cap ... = \{0\}$

5. Let $X = \{0 \le x \le \frac{\pi}{2} \mid x \in \mathbb{R}\}$. A function $f : X \to \mathbb{R}$ is defined by $f(x) = \sin(x)$. Find the range of f. Is f injective, surjective, and/or a bijection? Give reasons.

Solution:

Since $\sin(0) = 0$ and $\sin(\frac{\pi}{2}) = 1$ and the graph is increasing and continuous over the domain, the range is $[0,1]$.

Since $f'(x) = \cos(x)$, which is strictly positive over the domain, $f(x)$ is monotonically increasing and is thus injective.

f is not surjective because the range does not equal the codomain.

f is not a bijection because it is injective but not surjective.

6. Prove that if k is a positive integer then

$$\frac{3}{k} + \frac{4}{k+1} - \frac{7}{k+2} = \frac{10k+6}{k(k+1)(k+2)}.$$

Hence simplify $\sum_{k=1}^{n} \frac{5k+3}{k(k+1)(k+2)}$.

Solution:

$$\begin{aligned}\frac{3}{k} + \frac{4}{k+1} - \frac{7}{k+2} &= \frac{3(k+1)(k+2) + 4k(k+2) - 7k(k+1)}{k(k+1)(k+2)} \\ &= \frac{10k+6}{k(k+1)(k+2)}.\end{aligned}$$

Now, by using the change of summation index,

$$\begin{aligned}\sum_{k=1}^{n} \frac{5k+3}{k(k+1)(k+2)} &= \frac{1}{2}[\sum_{k=1}^{n} \frac{3}{k} + \sum_{k=1}^{n} \frac{4}{k+1} - \sum_{k=1}^{n} \frac{7}{k+2}] \\ &= \frac{1}{2}[\sum_{k=1}^{n} \frac{3}{k} + \sum_{u=2}^{n+1} \frac{4}{u} - \sum_{l=3}^{n+2} \frac{7}{l}] \\ &= \frac{1}{2}[\sum_{k=3}^{n} \frac{3}{k} + 3 + \frac{3}{2} + \sum_{u=3}^{n+1} \frac{4}{u} + 2 - \sum_{l=3}^{n+2} \frac{7}{l} - \frac{7}{n+1} - \frac{7}{n+2}] \\ &= \frac{1}{2}[\frac{13}{2} - \frac{3}{n+1} - \frac{7}{n+2}].\end{aligned}$$

Number Theory

1. Let $a, m \in \mathbb{Z}$. Prove that if $a|m$ and $a+1|m$ then $a(a+1)|m$.

Solution:

$$ka = m \tag{1}$$

$$k_1(a+1) = m \tag{2}$$

Multiplying (1) by $(a+1)$ gives

$$ka(a+1) = ma + m. \tag{3}$$

Multiplying (2) by a gives

$$k_1a(a+1) = ma. \tag{4}$$

(3) - (4) gives

$$ka + ka^2 - k_1a(a+1) = m$$
$$ka(a+1) - k_1a(a+1) = m.$$

Therefore, $m = kk_1(a(a+1))$.

2. Find $x, y \in \mathbb{Z}$ if $16758x + 14175y = 63$.

Solution:

$$\begin{aligned}
\underline{16758} &= 1 \times \underline{14175} + \underline{2583}\\
\underline{14175} &= 5 \times \underline{2583} + \underline{1260}\\
\underline{2583} &= 2 \times \underline{1260} + \underline{63}\\
\underline{1260} &= 20 \times \underline{63} + 0.
\end{aligned}$$

Now, by re-writing the remainders,

$$\begin{aligned}
\underline{63} &= \underline{2583} \text{ - } 2 \times \underline{63}\\
&= \underline{2583} \text{ - } 2(\underline{14175} \text{ - } 5 \times \underline{2583})\\
&= 11 \times \underline{2583} \text{ - } 2 \times \underline{14175}\\
&= 11(\underline{16758} \text{ - } \underline{14175}) \text{ - } 2 \times \underline{14175}.\\
&= 11 \times \underline{16758} \text{ - } 13 \times \underline{14175}
\end{aligned}$$

Thus, $16758 \times 11 + 14175 \times (\text{-}13) = 63$.
So $x = 11$, $y = \text{-}13$.

3. Simplify 6^{54321} mod 100.

Solution:

$$\begin{aligned}
6^1 &\equiv 6 \pmod{100}\\
6^2 &\equiv \underline{36} \pmod{100}\\
6^3 &\equiv 36 \times 6 \equiv 216 \equiv 16 \pmod{100}\\
6^4 &\equiv 16 \times 6 \equiv 96 \equiv -4 \pmod{100}\\
6^5 &\equiv (-4) \times 6 \equiv -24 \pmod{100}\\
6^6 &\equiv (-24) \times 6 \equiv -144 \equiv -44 \pmod{100}\\
6^7 &\equiv (-44) \times 6 \equiv -264 \equiv \underline{36} \pmod{100}.
\end{aligned}$$

So the numbers repeat every 5 steps from now on.
Therefore, $6^{54321} \equiv 6^{54316} \equiv 6^{54311} \ldots \equiv 6^6 \equiv -44 \equiv 56 \pmod{100}$.

4. Solve the linear congruence $52x \equiv 8 \pmod{60}$

Solution:
We can 'divide' by gcd(52, 60) = 4. So $13x \equiv 2 \pmod{15}$.
We can use the Euclidean algorithm to find the inverse c of 13 (mod 15).

$$\begin{aligned}
\underline{15} &= 1 \times \underline{13} + \underline{2}\\
\underline{13} &= 6 \times \underline{2} + \underline{1}\\
\underline{2} &= 2 \times \underline{1} + 0.
\end{aligned}$$

By rewriting the remainders,

$$\begin{aligned}
\underline{1} &= \underline{13} - 6 \times \underline{2}\\
&= \underline{13} - 6(\underline{15} - \underline{13})\\
&= 7 \times \underline{13} - 6 \times \underline{15}.
\end{aligned}$$

So $c = 7$ is an inverse of 13 (mod 15).
Hence the solution to $13x \equiv 2 \pmod{15}$ is $x \equiv 7 \times 2 = 14 \pmod{15}$. which is the solution to the original congruence equation.
By adding multiples of 15 to 14, we can find the solution in terms of the original modulus,
$x \equiv 14, 29, 44, 59 \pmod{60}$.

Relations

1. Define a relation $\sim$ on $\mathbb{R}$ by $x \sim y$ if and only if there exists $k \in \mathbb{Z}$ such that $x-y = 2k\pi$

   (a) Show that $\sim$ is an equivalence relation.

   (b) Write $\mathbb{R}$ as the union of pairwise disjoint equivalence classes $[x]$ for all $[x] \in \mathbb{R}$.

Solution:

   (a) Let $x, y, z \in \mathbb{R}$. Then $x - x = 0 = 2 \times 2\pi$. Thus, $\sim$ is reflexive.
   Now, let x, y satisfy $x \sim y$. Then there exists $m \in \mathbb{Z}$ such that $x - y = 2m\pi$. Because $y - x = 2(-m)\pi$, we have $y \sim x$, so $\sim$ is symmetric.
   Lastly, let x, y, z satisfy $x \sim y$ and $y \sim z$. There there exists $k, j \in \mathbb{Z}$ such that $x-y = 2k\pi$ and $y - z = 2j\pi$. Because $x - z = (x - y) + (y - z) = 2(k + j)\pi$, we have $x \sim z$, so $\sim$ is transitive. Consequently, $\sim$ is an equivalence relation.

   (b) We can write $\mathbb{R} = \bigcup\{[x] : x \in (-\pi, \pi]\} = \bigcup_{x\in(-\pi,\pi]}\{x + 2k\pi : k \in \mathbb{Z}\}$ or $\mathbb{R} = \bigcup\{[x] : x \in [0, 2\pi)\} = \bigcup_{x\in(-\pi,\pi]}\{x + 2k\pi : k \in \mathbb{Z}\}$.

2. Define a relation $\sim$ on the set of complex numbers by

$$z \sim w \iff |z - 1| = |w - 1|$$

   (a) Prove that $\sim$ is an equivalence relation on S.

   (b) Draw a sketch which shows the equivalence class of $\{2i\}$.

Solution:

   (a) Let $z, w, u \in \mathbb{Z}$. Clearly, $|z - 1| = |z - 1|$ so we have $z \sim z$. That is, $\sim$ is reflexive.
   Suppose that $z \sim w$. We have $|z - 1| = |w - 1|$, which implies that $|w - 1| = |z - 1|$. That is, $w \sim z$. Hence, $\sim$ is symmetric.
   Lastly, suppose that $z \sim w$ and $w \sim u$. Then $|z - 1| = |w - 1|$ and $|w - 1| = |u - 1|$. We conclude that $|z - 1| = |u - 1|$. That is, $z \sim u$. Hence, $\sim$ is transitive. Consequently, $\sim$ is an equivalence relation.

   (b) The equivalence class of $2i$ is the set of $z \in \mathbb{C}$ such that $z \sim 2i$. That is, the solutions of $|z - 1| = |2i - 1| = \sqrt{5}$. This is a circle centred at $(1, 0)$ with radius $\sqrt{5}$.

3. Suppose that $A = \{2, 3, 5, 6, 15, 30, 35, 70, 105, 210\}$

   (a) Draw the Hasse Diagram for the partially ordered set $(A, |)$, where $a|b$ means that a divides b. (You do not have to prove this is a partial order.)

   (b) Find, if they exist, all

      (i) greatest elements

      (ii) least elements

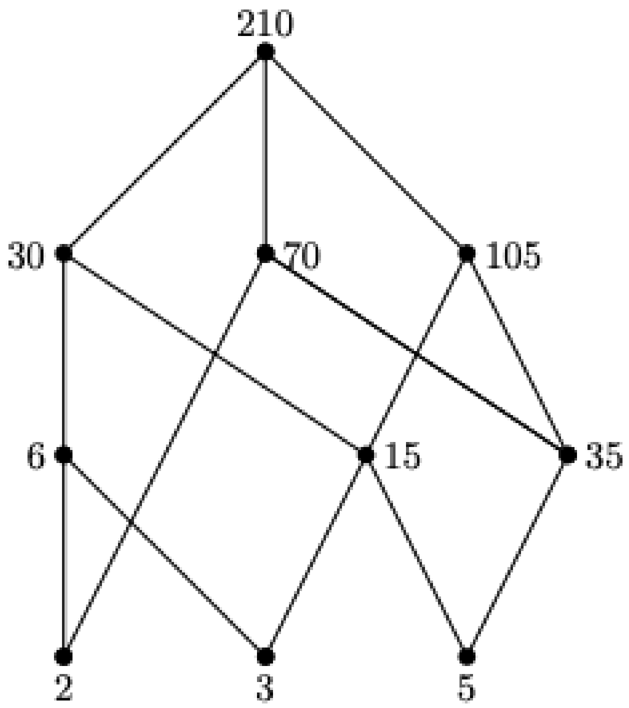
(iii) maximal elements

(iv) minimal elements

(c) Find two elements of A that do not have a greatest lower bound and explain why they do not.

Solution:

(a) The Hasse Diagram for A is given below.



(b) (i) The greatest element is 210.

(ii) There is no least element.

(iii) The maximal elements are: 210.

(iv) The minimal elements are: 2, 3, 5.

(c) 2 and 3 have no element below them so they have no greatest lower bound.

Graph Theory

1. A graph H on the vertices A, B, C, D (in that order) has adjacency matrix:

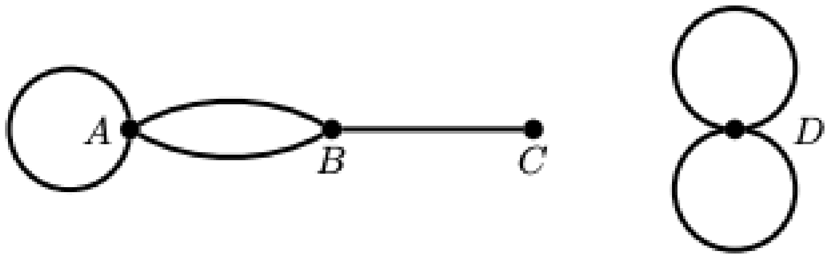
$$M = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

(a) Draw the graph H.

(b) M^3 has the number 12 in position 1,2 (that is, in the first row and second column). What does this mean in terms of the graph?

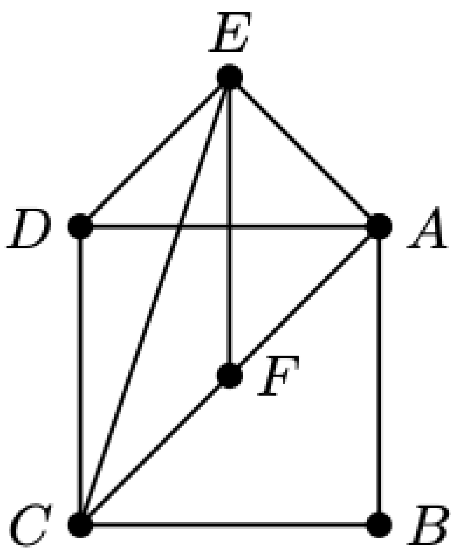
Solution:

(a) The graph of H is given below.



(b) The number 12 in position 1,2 of M^3 indicates that there are 12 walks of length 3 from A to B.

2. Consider the following graph G:



Giving reasons, show that

(a) G is not bipartite

(b) G contains an Euler path

(c) G contains a Hamilton Circuit

Solution:

(a) There are odd cycles. For example, EAD.

(b) There are exactly two vertices of odd degree (D and F). An example of an Euler path in G is $DEABCDAFCEF$.

(c) An example of a Hamilton circuit in G is $ABCDEFA$.

3. (a) State Euler's Formula for a connected planar graph having v vertices, e edges and r regions.

(b) Show that if G is a connected planar simple graph with $v \geq 3$ vertices, then $e \leq 3v - 6$.

(c) Hence, show that a connected planar simple graph with $v \geq 3$ vertices has at least one vertex of degree less than or equal to 6.

Solution:

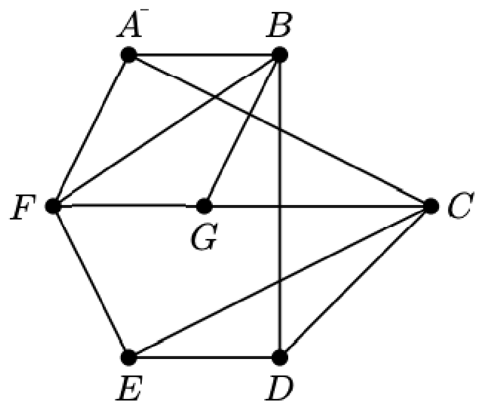
(a) $v - e + r = 2$

(b) For a connected planar simple graph G with at least 3 vertices, each region in the dual graph of G must have degrees at least 3, and as the dual and G have the same number of edges, sum of degrees of the dual $= 2e \geq 3r$. So, $\frac{2}{3}e \geq r = e - v + 2$ by Euler's formula. Rearranging, we get $e \leq 3v - 6$.

(c) If all the vertices have degree strictly greater than 6, then the sum of the degrees is greater than $6v$, so $2e > 6v$. But from (b), $2e \leq 6v - 12$, a contradiction.

4. (a) State Kuratowski's theorem characterising non-planar graphs.

(b) Show that the following graph is not planar.



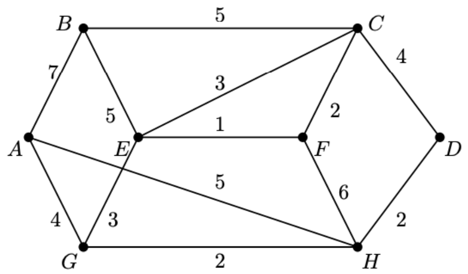
Solution:

(a) Kuratowski's Theorem states that a graph is non-planar if and only if it has a subgraph that is homeomorphic to K_5 or to $K_{3,3}$.

(b) Delete the edges BF and CD. The resulting graph is homeomorphic to $K_{3,3}$. By Kuratowski's Theorem, the graph is not planar.

5. (a) Use Kruskal's algorithm to construct a minimal spanning tree T for the following weighted graph. Make a table showing the details of each step in your application of the algorithm.

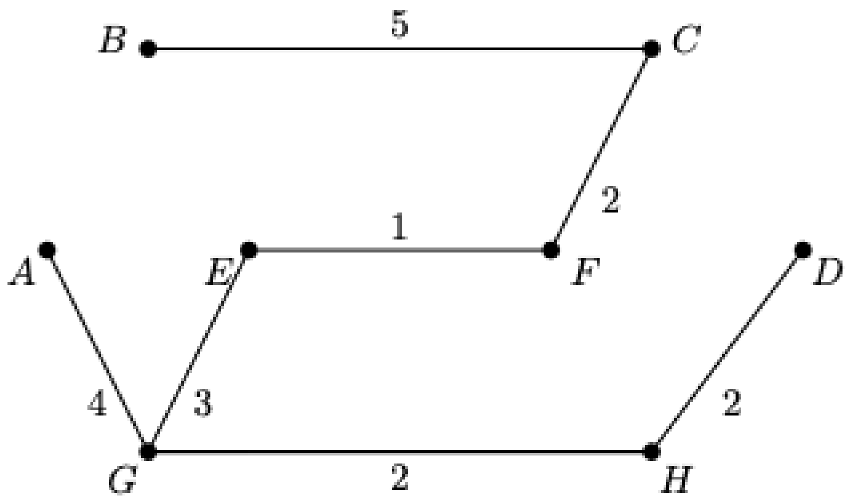
(b) Is T also a tree showing the shortest path from A to every other vertex? Explain.



Solution:

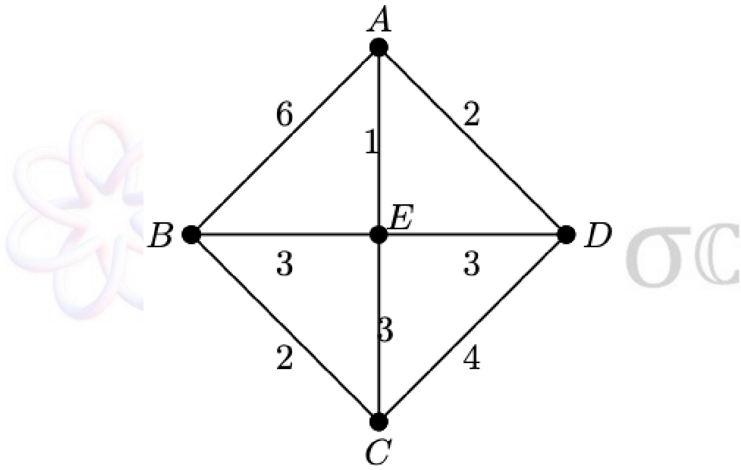
(a) Given a graph with n vertices, construct a table of edges sorted in ascending order by cost. Kruskal's algorithm is to choose the first $n - 1 = 7$ edges that do not form a cycle. The edges form a minimal spanning tree.

Edge(cost)	Chosen
EF(1)	Yes
HD(2)	Yes
FC(2)	Yes
GH(2)	Yes
EG(3)	Yes
EC(3)	No
AG(4)	Yes
CD(4)	No
CB(5)	Yes
⋮	No



(b) No, since the shortest path from A to B has cost 7 in the graph, the cost of travelling from A to B in the minimal spanning tree is 15.

6. Consider the following weighted graph:

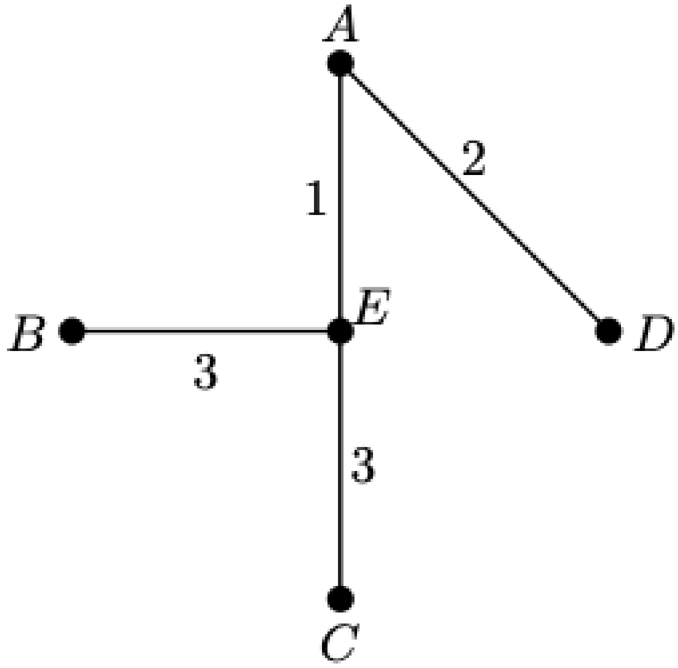


Use Dijkstra's algorithm find a spanning tree that gives the shortest paths from A to every other vertex of the graph. Make a table showing the details of each step in your application of the algorithm.

Solution:

Starting with A, we have

step	candidate edges (total distance)	new edge	new vertex v	$d(v, A)$
1	AB(6), AE(1), AD(2)	AE	E	1
2	AB(6), AD(2), EB(4), ED(4), EC(4)	AD	D	2
3	AB(6), DC(6), EB(4), EC(4)	EB	B	4
4	BC(6), DC(6), EC(4)	EC	C	4



7. A connected planar map has vertices of degrees 6, 5, 5, 2, 2, 1, 1.

(a) How many regions does the map have?

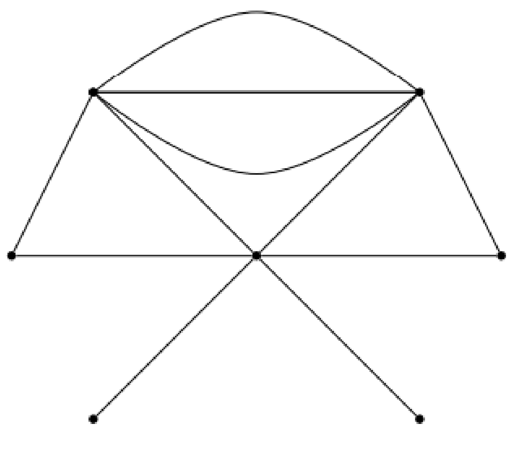
(b) Draw an example of such a planar map.

(c) Does the graph have an Euler path? Give reasons.

Solution:

(a) It is given that the number of vertices 7. By the handshaking lemma, the number of edges is 11. By Euler's formula, the number of regions is therefore $2 + 11 - 7 = 6$.

(b) An example of such a planar map is given below.



(c) There is no Euler path since there are more than two vertices of odd degree.