UNSW Mathematics Society Presents

MATH2018/2019 Workshop

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Overview II

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1. Functions of Several Variables

Partial Derivatives

Example: 19T1, Q1(i)(b)

Consider the function

$$z = f(x, y) = x^2y^3 + e^{4x}\sin(y).$$

Is f an increasing or decreasing function in the x direction at the point $(0, \frac{\pi}{2})$?

When we consider how a function is changing in a certain direction, we are thinking about partial derivatives!

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$$\frac{\partial f}{\partial x}(x, y) = 2xy^3 + 4e^{4x}\sin(y).$$

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When we consider how a function is changing in a certain direction, we are thinking about partial derivatives! We have that

$$\frac{\partial f}{\partial x}(x, y) = 2xy^3 + 4e^{4x}\sin(y).$$

So

$$\frac{\partial f}{\partial x}(0, \frac{\pi}{2}) = 4\sin(\frac{\pi}{2}) = 4.$$

So the function is increasing in the x direction!

Multi-variable Chain Rule

Original example

Suppose that $u = x^2 + 4y$ where $x = 3st$ and $y = 4s \ln t$. Find $(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial s})$.

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Here, u is given as a function of x and y. We're then given each of those variables as a function of s and t. So to find the rates of change with respect to s and t, we need to use a chain rule.

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Multi-variable Chain Rule

If $z = f(x, y)$, $x = x(u, v)$, $y = y(u, v)$, then:

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}; \qquad (*)$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}. \qquad (**)$$

Multi-variable Chain Rule

Original example

Suppose that $u = x^2 + 4y$ where $x = 3st$ and $y = 4s \ln t$. Find $(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial s})$.

First, find how quickly u is changing with respect to x and y. That is,

$$\frac{\partial u}{\partial x} = 2x, \text{ and}$$
$$\frac{\partial u}{\partial y} = 4.$$

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$$\begin{aligned}\frac{\partial u}{\partial t} &= 2x \times 3s + 4 \times \frac{4s}{t} \\ &= 6xs + \frac{16s}{t}.\end{aligned}$$

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Suppose that $u = x^2 + 4y$ where $x = 3st$ and $y = 4s \ln t$. Find $(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial s})$.

But, looking at this formula

$$\frac{\partial u}{\partial t} = 6xs + \frac{16s}{t},$$

we're not actually done! Our answer needs to be in terms of t and s since those are the variables we're considering to be the inputs.

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we're not actually done! Our answer needs to be in terms of t and s since those are the variables we're considering to be the inputs. So simply replace x with $x = 3st$ to obtain

$$\begin{aligned}\frac{\partial u}{\partial t} &= 6 \times 3st \times s + \frac{16s}{t} \\ &= 18s^2t + \frac{16s}{t}.\end{aligned}$$

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Suppose that $u = x^2 + 4y$ where $x = 3st$ and $y = 4s \ln t$. Find $(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial s})$.

Do the same thing with $\frac{\partial u}{\partial s}$ to obtain

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Suppose that $u = x^2 + 4y$ where $x = 3st$ and $y = 4s\ln t$. Find $(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial s})$.

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\frac{\partial u}{\partial s} &= 2x \times 3t + 4 \times 4\ln t \\
&= 6xt + 16\ln t \\
&= 6 \times 3st \times t + 16\ln t \\
&= 18st^2 + 16\ln t.
\end{aligned}$$

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Suppose that $u = x^2 + 4y$ where $x = 3st$ and $y = 4s\ln t$. Find $(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial s})$.

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$$\begin{aligned}\frac{\partial u}{\partial s} &= 2x \times 3t + 4 \times 4\ln t \\ &= 6xt + 16\ln t \\ &= 6 \times 3st \times t + 16\ln t \\ &= 18st^2 + 16\ln t.\end{aligned}$$

So

$$\left(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial s}\right) = \left(18s^2t + \frac{16s}{t}, 18st^2 + 16\ln t\right).$$

Taylor Series

Recall from first year maths that a function $f(x)$ at some point $(a, f(a))$ can be approximated as an infinite-degree polynomial.

Single-Variable Taylor Series

That is,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ...$$

It turns out that there's a very similar formula for functions

$$f(x, y)$$

of two variables!

Taylor Series

Multi-Variable Taylor Series

The Taylor Series of the function $f(x, y)$ at the point (a, b) is given by

$$f(x,y) = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$$
$$+ \frac{1}{2!}\left[\frac{\partial^2 f}{\partial x^2}(a,b)(x-a)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(a,b)(x-a)(y-b)\right.$$
$$\left.+ \frac{\partial^2 f}{\partial y^2}(a,b)(y-b)^2\right] + \cdots .$$

All you will really need is these first and second-degree terms!

Taylor Series

Original example

Consider the scalar field $f(x,y) = \sin(x+y^2)$. Calculate the Taylor series expansion of the function about the point $(4,4)$ up to and including the quadratic terms.

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First, we have

$$f(4, 4) = \sin(4 + 4^2) = \sin 20.$$

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First, we have

$$f(4, 4) = \sin(4 + 4^2) = \sin 20.$$

Then, go through the **first-order** derivatives, and substitute in $(1, 0)$. We have that

$$\frac{\partial f}{\partial x}(x, y) = \cos(x + y^2) \implies \frac{\partial f}{\partial x}(4, 4) = \cos 20, \text{ and}$$
$$\frac{\partial f}{\partial y}(x, y) = 2y\cos(x + y^2) \implies \frac{\partial f}{\partial y}(4, 4) = 8\cos 20.$$

Taylor Series

Original example

Consider the scalar field $f(x, y) = \sin(x + y^2)$. Calculate the Taylor series expansion of the function about the point $(4, 4)$ up to and including the quadratic terms.

Then, move onto the second derivatives! We have that

$$\frac{\partial^2 f}{\partial x^2}(x, y) = -\sin(x + y^2) \implies \frac{\partial^2 f}{\partial x^2}(4, 4) = -\sin 20,$$

$$\frac{\partial^2 f}{\partial x \partial y}(x, y) = -2y\sin(x + y^2) \implies \frac{\partial^2 f}{\partial x \partial y}(4, 4) = -8\sin 20, \text{ and}$$

$$\frac{\partial^2 f}{\partial y^2}(x, y) = 2\cos(x + y^2) - 4y^2\sin(x + y^2)$$

$$\implies \frac{\partial^2 f}{\partial y^2}(4, 4) = 2\cos 20 - 64\sin 20.$$

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Consider the scalar field $f(x, y) = \sin(x + y^2)$. Calculate the Taylor series expansion of the function about the point $(4, 4)$ up to and including the quadratic terms.

We have $f(4,4) = \sin 20$, $\dfrac{\partial f}{\partial x}(4,4) = \cos 20$, $\dfrac{\partial f}{\partial y}(4,4) = 8\cos 20$,

$$\frac{\partial^2 f}{\partial x^2}(4,4) = -\sin 20, \frac{\partial^2 f}{\partial x \partial y}(4,4) = -8\sin 20, \text{ and}$$

$$\frac{\partial^2 f}{\partial y^2}(4,4) = 2\cos 20 - 64\sin 20.$$

Taylor Series

Original example

Consider the scalar field $f(x,y)=\sin\left(x+y^2\right)$. Calculate the Taylor series expansion of the function about the point $(4,4)$ up to and including the quadratic terms.

We have $f(4,4)=\sin 20$, $\dfrac{\partial f}{\partial x}(4,4)=\cos 20$, $\dfrac{\partial f}{\partial y}(4,4)=8\cos 20$,

$$\frac{\partial^2 f}{\partial x^2}(4,4)=-\sin 20,\ \frac{\partial^2 f}{\partial x \partial y}(4,4)=-8\sin 20,\ \text{and}$$

$$\frac{\partial^2 f}{\partial y^2}(4,4)=2\cos 20-64\sin 20.$$

$$\text{So } f(x,y)\approx \sin 20+\cos 20(x-4)+8\cos 20(y-4)-$$
$$\tfrac{1}{2}(\sin 20(x-4)^2-16\sin 20(x-4)(y-4)+(2\cos 20-64\sin 20)(y-4)^2).$$

Application to Error Approximations

Example

The volume V of a cone with radius r and perpendicular height h is given by $V = \frac{1}{3}\pi r^2 h$. Determine the maximum absolute error and the maximum percentage error in calculating V given that $r = 5$cm and $h = 3$cm to the nearest millimetre.

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Error Approximation

Since

$$f(x,y) - f(x_0,y_0) \approx \frac{\partial f}{\partial y}(y - y_0) + \frac{\partial f}{\partial x}(x - x_0),$$

the maximum error of f in terms of the errors in x and y is given by

$$|\Delta f| \leq \left|\frac{\partial f}{\partial x}\right| |\Delta x| + \left|\frac{\partial f}{\partial y}\right| |\Delta y|.$$

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So in our example, we'd have the equation

$$|\Delta V| \leq \left|\frac{\partial V}{\partial r}\right| |\Delta r| + \left|\frac{\partial V}{\partial h}\right| |\Delta h|.$$

Since V is a function of r and h, we easily have that

$$\frac{\partial V}{\partial r} = \frac{2\pi r h}{3} \qquad \text{and} \qquad \frac{\partial V}{\partial h} = \frac{\pi r^2}{3}.$$

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$$|\Delta r| \text{ and } |\Delta h|?$$

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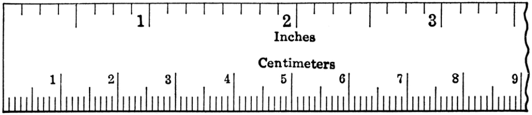
But what are the values of

$$|\Delta r| \text{ and } |\Delta h|?$$

An important detail given in the question is that we're finding r and h to the nearest *millimetre*. This means that r and h each have a maximum error of 0.5 mm (or 0.05 cm).



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So the maximum absolute error is given by

$$\begin{aligned}
|\Delta V| &\leq \left|\frac{2\pi r h}{3}\right|_{(r,h)=(5,3)} \times 0.05 + \left|\frac{\pi r^2}{3}\right|_{(r,h)=(5,3)} \times 0.05 \\
&= 10\pi \times 0.05 + \frac{25\pi}{3} \times 0.05 \\
&\approx 2.87.
\end{aligned}$$

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Finally, the maximum percentage error is given by

$$\left|\frac{\Delta V}{V}\right| \leq \frac{2.87}{75\pi/3} \approx 3.65\%.$$

Leibniz' Rule

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$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{u(x)}^{v(x)} f(x,t)\,\mathrm{d}t = \int_{u(x)}^{v(x)} \frac{\partial f}{\partial x}(x,t)\,\mathrm{d}t + f(x,v(x))v'(x) - f(x,u(x))u'(x).$$

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Effectively, this tells us how to differentiate an integral when the bounds and/or the integrand are given in terms of another variable.

To explain this formula intuitively, the **sources of change** in the integral when x changes are that

1. Increases in the function itself $f(x,t)$ will cause the integral to increase,
2. Increases in the upper limit will cause the integral to increase, and
3. Increases in the lower limit will cause the integral to decrease.

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Leibniz' Rule with Constant Bounds (A special case)

For u and v constant functions,

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{u(x)}^{v(x)} f(x,t)\,\mathrm{d}t = \int_{u(x)}^{v(x)} \frac{\partial f}{\partial x}(x,t)\,\mathrm{d}t + \cancel{f(x,v(x))v'(x)} - \cancel{f(x,u(x))u'(x)},$$

that is,

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{a}^{b} f(x,t)\,\mathrm{d}t = \int_{a}^{b} \frac{\partial f}{\partial x}(x,t)\,\mathrm{d}t.$$

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Original example

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$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{-x^2}^{2x}\frac{1}{y}\sin\left(xy\right)\mathrm{d}y.$$

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$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{-x^2}^{2x}\frac{1}{y}\sin\left(xy\right)\mathrm{d}y=\int_{-x^2}^{2x}\frac{1}{y}\times y\cos\left(xy\right)\mathrm{d}y$$

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$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\int_{-x^2}^{2x}\frac{1}{y}\sin\left(xy\right)\mathrm{d}y &= \int_{-x^2}^{2x}\frac{1}{y}\times y\cos\left(xy\right)\mathrm{d}y\\ &\quad + 2\times\frac{1}{2x}\times\sin\left(x\times 2x\right)\end{aligned}$$

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$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\int_{-x^2}^{2x}\frac{1}{y}\sin(xy)\,\mathrm{d}y &= \int_{-x^2}^{2x}\frac{1}{y}\times y\cos(xy)\,\mathrm{d}y \\ &\quad + 2\times\frac{1}{2x}\times\sin(x\times 2x) \\ &\quad - (-2x)\frac{1}{x^2}\times\sin\left(x\times -x^2\right)\end{aligned}$$

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$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\int_{-x^2}^{2x}\frac{1}{y}\sin(xy)\,\mathrm{d}y &= \int_{-x^2}^{2x}\frac{1}{y}\times y\cos(xy)\,\mathrm{d}y \\ &\quad + 2\times\frac{1}{2x}\times\sin(x\times 2x) \\ &\quad - (-2x)\frac{1}{x^2}\times\sin(x\times -x^2) \\ &= \frac{2}{x}\sin(2x^2)-\frac{1}{x}\sin(x^3).\end{aligned}$$

Leibniz' Rule

17S2, Q1(e)

You are given the following integral

$$\int_0^a \frac{1}{(x^2+a^2)^{1/2}}\,\mathrm{d}x = \sinh^{-1}(1).$$

Use Leibniz' rule to evaluate

$$\int_0^a \frac{1}{(x^2+a^2)^{3/2}}\,\mathrm{d}x.$$

Somehow we need to transform our integral in the following fashion:

$$\int_0^a \frac{1}{(x^2+a^2)^{1/2}}\,\mathrm{d}x \mapsto \int_0^a \frac{1}{(x^2+a^2)^{3/2}}\,\mathrm{d}x.$$

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What operation will we need to use here?

We are effectively decreasing the power to which $(x^2 + a^2)$ is raised. Hopefully we can see that a good choice here would be differentiation of the integral, since then, by using Leibniz' rule, we can move the derivative inside the integral.

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Let's differentiate both sides with respect to a. We'll get

$$\frac{\mathrm{d}}{\mathrm{d}a} \int_0^a \frac{1}{(x^2 + a^2)^{1/2}} \, \mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}a} \sinh^{-1}(1) = 0.$$

Now we can apply Leibniz' rule to the left-hand side of the equation.

Leibniz' Rule

We have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}a}\int_0^a \frac{1}{(x^2+a^2)^{1/2}}\,\mathrm{d}x &= \int_0^a \frac{\partial}{\partial a}\left(\frac{1}{(x^2+a^2)^{1/2}}\right)\,\mathrm{d}x \\ &\quad + \frac{1}{(a^2+a^2)^{1/2}} \times \frac{\mathrm{d}a}{\mathrm{d}a} - 0.\end{aligned}$$

Leibniz' Rule

We have

$$\frac{\mathrm{d}}{\mathrm{d}a}\int_0^a \frac{1}{(x^2+a^2)^{1/2}}\,\mathrm{d}x = \int_0^a \frac{\partial}{\partial a}\left(\frac{1}{(x^2+a^2)^{1/2}}\right)\mathrm{d}x \\ + \frac{1}{(a^2+a^2)^{1/2}} \times \frac{\mathrm{d}a}{\mathrm{d}a} - 0.$$

Simplifying and differentiating gives us

$$\frac{\mathrm{d}}{\mathrm{d}a}\int_0^a \frac{1}{(x^2+a^2)^{1/2}}\,\mathrm{d}x = -a\int_0^a \frac{1}{(x^2+a^2)^{3/2}}\,\mathrm{d}x + \frac{1}{\sqrt{2}a}.$$

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However, we know that

$$\frac{\mathrm{d}}{\mathrm{d}a}\int_0^a \frac{1}{(x^2+a^2)^{1/2}}\,\mathrm{d}x = 0,$$

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which means that

$$-a\int_0^a \frac{1}{(x^2+a^2)^{3/2}}\,\mathrm{d}x + \frac{1}{\sqrt{2}a} = 0.$$

Hence, by a little bit of rearranging we obtain

$$\int_0^a \frac{1}{(x^2+a^2)^{3/2}}\,\mathrm{d}x = \frac{1}{\sqrt{2}a^2},$$

and we are done.

2. Extreme Values

Critical Points

For univariate functions, we find the critical points of a function by finding the points where the derivative is equal to zero.

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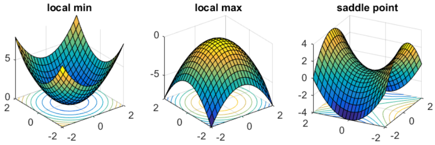
For univariate functions, we find the critical points of a function by finding the points where the derivative is equal to zero.

Likewise, a critical point of a multi-variate function is simply a point where all the **first-order** partial derivatives are 0.

And just like in the case of univariate functions, we can have multiple types of critical points, that is,

- A local minimum
- A local maximum
- A saddle point



Critical Points

Classification of Critical Points

We define the expression $\mathcal{D} := \dfrac{\partial^2 f}{\partial x^2} \cdot \dfrac{\partial^2 f}{\partial y^2} - \left(\dfrac{\partial^2 f}{\partial x \partial y}\right)^2$.

Critical Points

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We define the expression $\mathcal{D} := \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y} \right)^2$.

- If $\mathcal{D} > 0$ and $\frac{\partial^2 f}{\partial x^2} < 0$ at crit $= (a, b)$, then the critical point is a local maximum.
- If $\mathcal{D} > 0$ and $\frac{\partial^2 f}{\partial x^2} > 0$ at crit $= (a, b)$, then the critical point is a local minimum.
- If $\mathcal{D} < 0$ at crit $= (a, b)$, then the critical point is a saddle point.
- If $\mathcal{D} = 0$ at crit $= (a, b)$, then the test is inconclusive. You will need to use other methods (e.g. general reasoning about the shape of the function) to make a proper conclusion.

Critical Points

15S2 Q1(d)

Find and classify the critical points of

$$h(x, y) = 2x^3 + 3x^2y + y^2 - y.$$

Provide the function value at these critical points.

We want

$$\frac{\partial h}{\partial x} = 6x^2 + 6xy = 0 \qquad (*)$$

$$\frac{\partial h}{\partial y} = 3x^2 + 2y - 1 = 0. \qquad (**)$$

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$$\frac{\partial h}{\partial y} = 3x^2 + 2y - 1 = 0. \qquad (**)$$

The first equation $(*)$ looks fairly easy to factorise, so we have

$$x(x + y) = 0,$$

which leads to two cases.

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First of all, x could be equal to 0. Then we substitute that into equation $(**)$. In that case $2y - 1 = 0$, which means that $y = \frac{1}{2}$.

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First of all, x could be equal to 0. Then we substitute that into equation $(**)$. In that case $2y - 1 = 0$, which means that $y = \frac{1}{2}$.

Alternatively, x could be equal to $-y$. Then we substitute that into our formula to obtain

$$3y^2 + 2y - 1 = 0,$$

which implies that $y = -1$ or $y = \frac{1}{3}$.

Critical Points

So now we only need to classify our critical points! That's actually quite formulaic. We calculate our **second-order derivatives**

$$\frac{\partial^2 h}{\partial x^2}(x,y) = 12x + 6y, \quad \frac{\partial^2 h}{\partial y^2}(x,y) = 2, \text{ and } \frac{\partial^2 h}{\partial x \partial y}(x,y) = 6x.$$

It's best then to get a generic formula for $\mathcal{D}$, and we have

$$\mathcal{D} = 2(12x + 6y) - (6x)^2 = 24x + 12y - 36x^2.$$

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- At $(0, \frac{1}{2})$ we have $\mathcal{D}(0, \frac{1}{2}) = 6 > 0$ and $\frac{\partial^2 f}{\partial x^2}(0, \frac{1}{2}) = 3 > 0$. So $(0, \frac{1}{2})$ is a local minimum.

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- At $(1, -1)$ $\mathcal{D}(1, -1) = -24 < 0$. So $(1, -1)$ is a saddle point.

Critical Points

All we need to do now is find the function values at our critical points.

Recall that $h(x, y) = 2x^3 + 3x^2y + y^2 - y$.

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Recall that $h(x, y) = 2x^3 + 3x^2y + y^2 - y$.

- At $(0, \frac{1}{2})$ we have $h(0, \frac{1}{2}) = -\frac{1}{4}$.
- At $(-\frac{1}{3}, \frac{1}{3})$ we have $h(-\frac{1}{3}, \frac{1}{3}) = -\frac{5}{27}$.
- At $(1, -1)$ we have $h(1, -1) = 1$.

Lagrange Multipliers

Original example

Suppose that the surface area of a cylinder is 50cm^2. What is its maximum possible volume?

Lagrange Multipliers

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Whenever we're maximising a quantity subject to a constraint, we can't simply use an unrestricted optimisation problem as we just did. In these cases, we will want to use the method of Lagrange multipliers.

Gradient of a Multi-variate Function

The gradient of a function f of several variables $x_1, x_2, ..., x_n$ is given by

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix}.$$

Intuitively, ∇f is the collection of all first order partial derivatives of a function composed as a vector!

Lagrange Multipliers

Lagrange Multipliers

The critical values (local max or min) of a function $z = f(x_1, x_2, ..., x_n)$, subject to the constraint $g(x_1, x_{2,}..., x_n) = 0$, satisfy the equation

$$\nabla f(x_1, x_2, ..., x_n) = \lambda \nabla g(x_1, x_2, ..., x_n),$$

as well as the equation

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When we use the method of Lagrange multipliers, we will often find ourselves solving a set of simultaneous equations of $n + 1$ variables.

Note that in the case of Lagrange multipliers, we don't ever need to check the 'nature' of our critical points. We can just find all of them. The maximum will be the option that has the highest function value, and the minimum will be the option that has the lowest function value.

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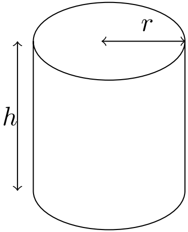

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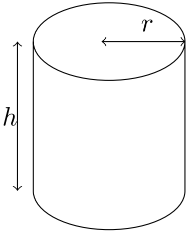

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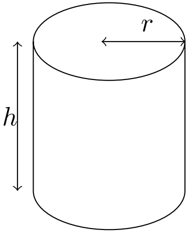

Let the radius of the cylinder be r and the height of the cylinder be h. Then our surface area is $\pi r^2 + 2\pi rh$, and our volume is $\pi r^2 h$.

Lagrange Multipliers

Original example

Suppose that the surface area of a cylinder with a base but no lid is 16cm^2. What is its maximum possible volume?

Let the radius of the cylinder be r and the height of the cylinder be h. Then our surface area is $\pi r^2 + 2\pi rh$, and our volume is $\pi r^2 h$. So we are trying to maximise the function

$$f(r, h) = \pi r^2 h,$$

subject to the constraint

$$g(r, h) = \pi r^2 + 2\pi rh - 16\pi = 0.$$

Lagrange Multipliers

Now we need to calculate the gradient of each function. We have

$$\nabla f(r,h) = \begin{pmatrix} 2\pi r h \\ \pi r^2 \end{pmatrix},$$

and

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We have, then, that

$$\begin{pmatrix} 2\pi rh \\ \pi r^2 \end{pmatrix} = \lambda \begin{pmatrix} 2\pi r + 2\pi h \\ 2\pi r \end{pmatrix}.$$

Remember, after we 'solve' this equation, we'll need to substitute it back into our constraint. So we need to find a relationship between r and h that we can use in our constraint.

Lagrange Multipliers

Start with

$$2\pi rh = \lambda(2\pi r + 2\pi h) \tag{1}$$

$$\pi r^2 = \lambda(2\pi r). \tag{2}$$

Let's try to factorise both equations:

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So one case is that $r = 0$, in which case $h = 0$. However, usually cases like these are an edge case. When you substitute your solution into the constraint, you will find it is impossible. The other option is

$$rh - \lambda(r+h) = 0 \tag{5}$$

$$r - 2\lambda = 0. \tag{6}$$

Lagrange Multipliers

Usually the aim is to eliminate λ from your equations by making it the subject of both and equating the two expressions for λ.

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$$\lambda = \frac{rh}{r+h},$$

and

$$\lambda = \frac{r}{2},$$

which means that

$$\frac{rh}{r+h} = \frac{r}{2}.$$

So

$$r = h.$$

Lagrange Multipliers

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$$3h^2 = 16.$$

So clearly there is only one non-negative solution for h, that is,

$$h = \frac{4}{\sqrt{3}}.$$

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$$h = \frac{4}{\sqrt{3}}.$$

So we only have one critical point to choose from, and clearly our maximum volume is

$$\begin{aligned} V &= 2\pi \left(\frac{4}{\sqrt{3}}\right)^3 \\ &= \frac{128\pi}{3\sqrt{3}}. \end{aligned}$$

3. Vector Calculus

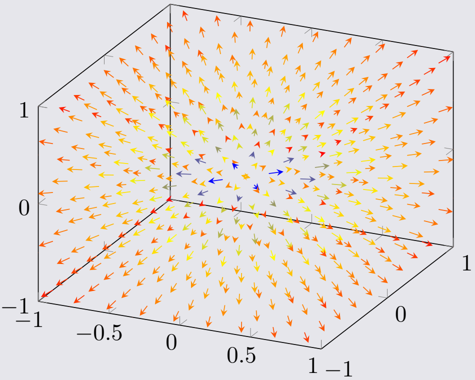
Curves

A curve is an equation like $\mathbf{c}(t) = (x(t), y(t), z(t))$ that takes a parameter t and traces out coordinates based on how the parameter varies. As t is allowed to vary within an interval, say $[0, 1]$, these coordinates trace out a path in three dimensions.

Line Integrals

Vector Fields

A vector field is an equation like $\mathbf{F}(x, y, z) = (x^2, y^2, x \sin z)$ that maps a point in three dimensions to a three-dimensional vector $\mathbf{F}(x, y, z) = (F_1(x, y, z), F_2(x, y, z), F_3(x, y, z))$.



Line Integrals

Line Integrals

If $\mathcal{C}$ is a curve that is parametrised by $\mathbf{c}(t)$ over the interval $[a,b]$, the **line integral** of the vector field $\mathbf{F}$ over the curve $\mathcal{C}$ is defined as

$$\begin{aligned}\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\boldsymbol{r} &= \int_{\mathcal{C}} F_1(\mathbf{c}(t))\,\mathrm{d}x + F_2(\mathbf{c}(t))\,\mathrm{d}y + F_3(\mathbf{c}(t))\,\mathrm{d}z \\ &= \int_a^b F_1(\mathbf{c}(t))x'(t)dt + F_2(\mathbf{c}(t))y'(t)dt + F_3(\mathbf{c}(t))z'(t)dt \\ &= \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)dt.\end{aligned}$$

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Remember, the line integral tells us how much work the vector field $\mathbf{F}$ does over the curve $\mathcal{C}$.

Line Integrals

Example: 19T1 Q3c(v), *adapted*

A force field $\mathbf{F}$ is given by

$$\mathbf{F} = 2y\mathbf{i} + xyz\mathbf{j} + 3e^{x+y}\mathbf{k},$$

and the path $\mathcal{C}$ is the straight line from $(4,3,1)$ to $(6,3,5)$ in $\mathbb{R}^3$. Find a parametric representation for the path $\mathcal{C}$, and evaluate the work integral $\int_{\mathcal{C}} \mathbf{F}\,\mathrm{d}\mathbf{r}$.

Alternative Form

Note: They could have also written this as

$$\int_{\mathcal{C}} 2y\,\mathrm{d}x + xyz\,\mathrm{d}y + 3e^{x+y}\,\mathrm{d}z.$$

Line Integrals

We begin by first parameterising $\mathcal{C}$ into a straight path.

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Straight Line Parametrisation

A straight line path L from A to B can always be parametrised by the formula

$$\mathbf{c}(t) = (1-t)A + tB,$$

for $t \in [0,1]$. Another way of looking at it is the formula

$$\mathbf{c}(t) = A + t\overrightarrow{AB}$$

for $t \in [0,1]$.

Line Integrals

Hence, we can parametrise the path $\mathcal{C}$ by $\mathbf{c}(t) = (4,3,1)^T + t(2,0,4)^T$ for $t \in [0,1]$. So

- $x(t) = 4 + 2t$, and $\mathrm{d}x = 2\,\mathrm{d}t$
- $y(t) = 3$, and $\mathrm{d}y = 0\,\mathrm{d}t$.
- $z(t) = 1 + 4t$, and $\mathrm{d}z = 4\,\mathrm{d}t$.

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Hence,

$$\begin{aligned}\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\boldsymbol{r} &= \int_0^1 2 \times 3 \times 2\,\mathrm{d}t + 0 + 3e^{4+2t+3} 4\,\mathrm{d}t \\ &= \int_0^1 12 + 12e^{2t+7}\,\mathrm{d}t \\ &= 12 + 6(e^9 - e^7).\end{aligned}$$

Divergence and Curl

The divergence and curl of a vector field $\mathbf{F} = (F_1, F_2, F_3)^T$ are given by the following equations.

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Divergence

$$\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_2}{\partial z}.$$

Curl

$$\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

It turns out that curl can be pretty important when calculating the line integral of a vector field.

Theorem

If a vector field $\mathbf{G}$ has **zero curl** (i.e. it is irrotational), then it can be expressed as the gradient of some other function ϕ. That is,

$$\mathbf{G} = \nabla \phi$$

for some function ϕ. $\mathbf{G}$ is also called a conservative field.

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But why do we care about this?

Theorem: Line Integrals on Conservative Fields

For a conservative/irrotational field $\mathbf{F}$, recall that you can find a function ϕ such that $\mathbf{F} = \nabla\phi$. Then suppose $\mathcal{C}$ is a curve that starts at the point A and ends at the point B. We have

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}r = \phi(B) - \phi(A).$$

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$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}r = \phi(B) - \phi(A).$$

In other words, we can use this function ϕ much like we would the antiderivative of a single-variable function.

Divergence and Curl

18S1 Q1(a)

Consider the scalar field

$$\phi(x, y, z) = xe^{z-1} + \cos y$$

and let $\mathbf{F} = \nabla\phi$.

- What is $\nabla \times \mathbf{F}$?
- Hence, or otherwise, calculate the line integral $\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}r$ along the straight line path $\mathcal{C}$ from $(1, 0, 1)$ to $(5, \pi, 1)$.

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- Hence, or otherwise, calculate the line integral $\displaystyle\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}r$ along the straight line path $\mathcal{C}$ from $(1, 0, 1)$ to $(5, \pi, 1)$.

So first of all, what's $\mathbf{F}$? Breaking down our question, we can see that $\mathbf{F} = \nabla\phi$, which means that we get the curl for free. So

$$\nabla \times \mathbf{F} = 0$$

since gradients are irrotational.

Line Integrals Continued

Now we're on our way to calculating the integral! Using the fundamental theorem of line integrals, we can very easily show that

Line Integrals Continued

Now we're on our way to calculating the integral! Using the fundamental theorem of line integrals, we can very easily show that

$$\begin{aligned}\int_{\mathcal{C}} \mathbf{F} \cdot \,\mathrm{d}r &= \phi(5, \pi, 1) - \phi(1, 0, 1) \\ &= (5 \times e^{1-1} + \cos \pi) - (1 \times e^{1-1} + \cos 0) \\ &= 2.\end{aligned}$$

Divergence and Curl

18S2 Q4(i) - *modified*

Consider the vector field

$$\mathbf{F} = yz^2\mathbf{i} + (xz^2 + y)\mathbf{j} + (2xyz + 3)\mathbf{k}.$$

- Show that $\mathbf{F}$ is a conservative vector field by evaluating $\operatorname{curl}(\mathbf{F})$.
- The path $\mathcal{C}$ in $\mathbb{R}^3$ starts at the point $(3, 4, 7)$ and subsequently travels anti-clockwise four complete revolutions around the circle $x^2 + y^2 = 25$ within the plane $z = 7$, returning to the point $(3, 4, 7)$. It then moves diagonally in a straight line to the point $(4, 1, 0)$. Using the first part or otherwise, evaluate the work integral $\displaystyle\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\boldsymbol{r}$.

Divergence and Curl

Recall the formula defined on slide 60 for the curl, that is,

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$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}.$$

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It can be a bit tedious to calculate curl, but we'll be all good as long as we keep our formula in mind!

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It can be a bit tedious to calculate curl, but we'll be all good as long as we keep our formula in mind! We have

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz^2 & xz^2 & 2xyz+3 \end{vmatrix}.$$

Divergence and Curl

$\operatorname{curl}\mathbf{F} =$

Divergence and Curl

$$\begin{aligned}
\operatorname{curl}\mathbf{F} &= \mathbf{i}\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^2+y & 2xyz+3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ yz^2 & 2xyz+3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ yz^2 & xz^2+y \end{vmatrix} \\
&= \mathbf{i}\left(\frac{\partial}{\partial y}(2xyz+3) - \frac{\partial}{\partial z}(xz^2+y)\right) - \mathbf{j}\left(\frac{\partial}{\partial x}(2xyz+3) - \frac{\partial}{\partial z}(yz^2)\right) \\
&\quad + \mathbf{k}\left(\frac{\partial}{\partial x}(xz^2+y) - \frac{\partial}{\partial y}(yz^2)\right) \\
&= (2xz-2xz)\mathbf{i} - (2yz-2yz)\mathbf{j} + (z^2-z^2)\mathbf{k} = \mathbf{0}.
\end{aligned}$$

Hence, $\mathbf{F}$ is a conservative vector field.

Looking back at our question ...

18S2 Q4(i) - *modified*

The path $\mathcal{C}$ in $\mathbb{R}^3$ starts at the point $(3, 4, 7)$ and subsequently travels anti-clockwise four complete revolutions around the circle $x^2 + y^2 = 25$ within the plane $z = 7$, returning to the point $(3, 4, 7)$. It then moves diagonally in a straight line to the point $(4, 1, 0)$. Using the first part or otherwise, evaluate the work integral $\displaystyle\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\boldsymbol{r}$.

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When we have a conservative vector field, we don't need to go to the trouble of parametrising our curve. We can really just find ϕ such that $\mathbf{F} = \nabla\phi$, and then calculate $\phi(B) - \phi(A)$.

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When we have a conservative vector field, we don't need to go to the trouble of parametrising our curve. We can really just find ϕ such that $\mathbf{F} = \nabla \phi$, and then calculate $\phi(B) - \phi(A)$.

Here, $B = (4, 1, 0)$, and $A = (3, 4, 7)$.

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$$\begin{aligned}\frac{\partial \phi}{\partial x} &= yz^2 \\ \frac{\partial \phi}{\partial y} &= xz^2 + y \\ \frac{\partial \phi}{\partial z} &= 2xyz + 3.\end{aligned}$$

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To solve this, we can start with one of these equations. Integrating,

$$\frac{\partial \phi}{\partial x} = yz^2 \implies \phi(x, y, z) = xyz^2 + C(y, z).$$

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So we're close, but now we need to try to solve for $C(y, z)$. We've basically squeezed everything we can out of this first equation, so we need to start using the next two equations.

Divergence and Curl

$$\frac{\partial \phi}{\partial y} = xz^2 + y \qquad \frac{\partial \phi}{\partial z} = 2xyz + 3$$

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Since we now know that $\phi(x, y, z) = xyz^2 + C(y, z)$,

$$\frac{\partial \phi}{\partial y} = xz^2 + y \implies xz^2 + y = xz^2 + \frac{\partial C}{\partial y}(y, z) \implies \frac{\partial C}{\partial y}(y, z) = y.$$

Now we have

$$C(y, z) = \frac{y^2}{2} + C_1(z).$$

Substituting this into our original equation, we obtain

$$\phi(x, y, z) = xyz^2 + \frac{y^2}{2} + C_1(z).$$

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Substituting this into our original equation, we obtain

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And you guessed it: it's time to use our final equation to finish this off.

Divergence and Curl

$$\frac{\partial \phi}{\partial z} = 2xyz + 3$$

Divergence and Curl

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Finally,

$$\phi(x,y,z) = xyz^2 + \frac{y^2}{2} + C_1(z) \implies \frac{\partial \phi}{\partial z} = 2xyz + C_1'(z) \implies C'(z) = 3.$$

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So $C(z) = 3z + C$, and now we've effectively identified the function! That is,

$$\phi(x, y, z) = xyz^2 + \frac{y^2}{2} + 3z + C.$$

Back to the question, all we have to do now is calculate that

$$\begin{aligned}\int_{\mathcal{C}} \mathbf{F} \cdot \, \mathrm{d}r &= \phi(4,1,0) - \phi(3,4,7) \\ &= (4 \times 0^2 + \frac{1^2}{2} + 3 \times 0) - (3 \times 7^2 + \frac{4^2}{2} + 3 \times 7) \\ &= -\frac{351}{2}.\end{aligned}$$

Double Integrals

Example

Evaluate

$$\iint_{\Omega} xy^2 \, \mathrm{d}A$$

where Ω is the region in the first quadrant bounded by the parabola $y = 6 - x^2$, the line $y = x$, and the y-axis.

Double Integrals

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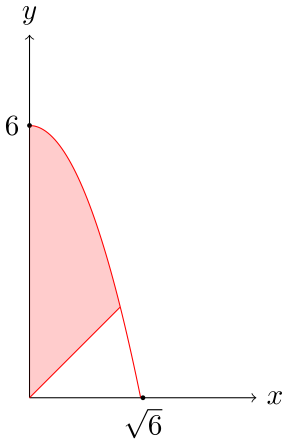
When doing a double integral over an obscure non-linear region, the best thing to do is to sketch the domain.

Double Integrals

Our region looks like this:



Double Integrals

Double Integrals

So our two options are to

1. Calculate $\int_0^2 \int_x^{6-x^2} xy^2 \,\mathrm{d}y \,\mathrm{d}x$, or
2. Calculate $\int_0^2 \int_0^y xy^2 \,\mathrm{d}x \,\mathrm{d}y + \int_2^6 \int_0^{\sqrt{6-y}} xy^2 \,\mathrm{d}x \,\mathrm{d}y$.

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We'll use the first formula because it's simpler. In general, we want to take the integral whose inner bounds vary smoothly as we vary the other variable.

Double Integrals

So we have

$$\iint_\Omega xy^2 \, \mathrm{d}A = \int_0^2 \int_x^{6-x^2} xy^2 \, \mathrm{d}y \, \mathrm{d}x$$

Double Integrals

So we have

$$\iint_{\Omega} xy^2 \, \mathrm{d}A = \int_0^2 \int_x^{6-x^2} xy^2 \, \mathrm{d}y \, \mathrm{d}x$$
$$= \int_0^2 \left[\frac{1}{3}xy^3\right]_{y=x}^{y=6-x^2} \, \mathrm{d}x$$

Double Integrals

So we have

$$\begin{aligned}\iint_\Omega xy^2\,\mathrm{d}A &= \int_0^2 \int_x^{6-x^2} xy^2\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_0^2 \left[\frac{1}{3}xy^3\right]_{y=x}^{y=6-x^2} \mathrm{d}x \\ &= \int_0^2 \frac{1}{3}\left(x(6-x^2)^3 - x^4\right)\,\mathrm{d}x\end{aligned}$$

Double Integrals

So we have

$$
\begin{aligned}
\iint_\Omega xy^2 \,\mathrm{d}A &= \int_0^2 \int_x^{6-x^2} xy^2 \,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^2 \left[\frac{1}{3}xy^3\right]_{y=x}^{y=6-x^2} \mathrm{d}x \\
&= \int_0^2 \frac{1}{3}\left(x(6-x^2)^3 - x^4\right)\,\mathrm{d}x \\
&= \left[-\frac{1}{24}(6-x^2)^4 - \frac{1}{15}x^5\right]_{x=0}^{x=2} \\
&= \frac{256}{5}.
\end{aligned}
$$

Double Integrals: Polar Coordinates

Now that we understand how to parse double integrals, we can also look into integrating over a different coordinate system.

Double Integrals: Polar Coordinates

Consider an integral over a region Ω that can be expressed as

$$\iint_{\Omega} f(x, y)\,\mathrm{d}x\,\mathrm{d}y.$$

If we can express the same region in terms of polar coordinates, that is,

$$(x, y) \in \Omega \Leftrightarrow (r, \theta) \in \Psi,$$

then

$$\iint_{\Psi} f(r\cos\theta, r\sin\theta){\color{red} r\,\mathrm{d}r\,\mathrm{d}\theta}.$$

Double Integrals: Polar Coordinates

Example

Use polar coordinates to evaluate the integral

$$\iint_{\Omega} e^{-(x^2+y^2)} dA,$$

where Ω is the quadrant of the circle $x^2 + y^2 \leq R^2$ that lies in the first quadrant. Hence, find $\int_0^\infty e^{-x^2}\,\mathrm{d}x$.

First of all, let's convert the region Ω into polar coordinates. If $x^2 + y^2 \leq R^2$, then what does that tell us about (r, θ)?

Double Integrals: Polar Coordinates

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First of all, let's convert the region Ω into polar coordinates. If $x^2 + y^2 \leq R^2$, then what does that tell us about (r, θ)? We have that

$$0 \leq r \leq R, \text{ and } 0 \leq \theta \leq \frac{\pi}{2}.$$

So instead, we have

$$\int_0^{\frac{\pi}{2}} \int_0^R e^{-r^2} r\,\mathrm{d}r\,\mathrm{d}\theta.$$

Double Integrals: Polar Coordinates

So now we just need to calculate this simple double integral. We have

$$\iint_{\Omega} e^{-(x^2+y^2)} dA = \int_0^{\frac{\pi}{2}} \int_0^R e^{-r^2} r \, \mathrm{d}r \, \mathrm{d}\theta$$

Double Integrals: Polar Coordinates

So now we just need to calculate this simple double integral. We have

$$\begin{aligned}\iint_{\Omega} e^{-(x^2+y^2)} dA &= \int_0^{\frac{\pi}{2}} \int_0^R e^{-r^2} r \, \mathrm{d}r \, \mathrm{d}\theta \\ &= \int_0^{\frac{\pi}{2}} \left[-\frac{1}{2} e^{-r^2} \right]_{r=0}^{R} \mathrm{d}\theta \\ &= \int_0^{\frac{\pi}{2}} \frac{1}{2}(1 - e^{-R^2}) \, \mathrm{d}\theta \\ &= \frac{\pi}{4}(1 - e^{-R^2}).\end{aligned}$$

Double Integrals: Polar Coordinates

Back to the second part ...

Second part of question

Hence, find $\int_0^\infty e^{-x^2}\,\mathrm{d}x$.

We somehow need to leverage this identity, that

$$\iint_\Omega e^{-(x^2+y^2)}\,\mathrm{d}A = \frac{\pi}{4}(1 - e^{-R^2}),$$

into calculating $\int_0^\infty e^{-x^2}\,\mathrm{d}x$.

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into calculating $\int_0^\infty e^{-x^2}\,\mathrm{d}x$. Well, first of all, we should take the limit as R goes to infinity. That gives us

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,\mathrm{d}x\,\mathrm{d}y = \frac{\pi}{4}.$$

Double Integrals: Polar Coordinates

Since our function is separable in x and y, we have that

$$\int_0^\infty e^{-x^2}\,\mathrm{d}x \int_0^\infty e^{-y^2}\,\mathrm{d}y = \frac{\pi}{4}.$$

But the two integrals on the left-hand side are the same, so we have

$$\left(\int_0^\infty e^{-x^2}\,\mathrm{d}x\right)^2 = \frac{\pi}{4}.$$

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$$\left(\int_0^\infty e^{-x^2}\,\mathrm{d}x\right)^2 = \frac{\pi}{4}.$$

So ultimately, square-rooting both sides (since both sides must be positive), we have that

$$\int_0^\infty e^{-x^2}\,\mathrm{d}x = \frac{\sqrt{\pi}}{2}.$$

4. Matrices

Diagonalisation

Example

The matrix $A = \begin{pmatrix} -5 & 6 & 0 \\ -3 & 4 & 0 \\ -3 & 3 & 1 \end{pmatrix}$ is diagonalisable with eigenvalues -2 and 1.

- Verify that $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector of A.
- Diagonalise the matrix A.

Diagonalisation

Eigenvalues & Eigenvectors

An eigenvector of a matrix A is a vector $\mathbf{v}$ such that

$$A\mathbf{v} = \lambda\mathbf{v}$$

for some scalar λ.

So we just need to multiply A by the vector, obtaining

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So we just need to multiply A by the vector, obtaining

$$\begin{aligned} A\mathbf{v} &= \begin{pmatrix} -5 & 6 & 0 \\ -3 & 4 & 0 \\ -3 & 3 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \\ &= -2 \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}. \end{aligned}$$

So this vector is an eigenvector of eigenvalue -2.

However, for the second part of the question, we are using a process called diagonalisation, which requires us to find the eigenvectors of A.

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You might remember from first year a process called finding the characteristic polynomial, but we won't go through that here. Instead, this question actually gives us a bit more information that we can use to solve the problem a little more efficiently.

Eigenvalues & Eigenvectors

Given information

The matrix $A = \begin{pmatrix} -5 & 6 & 0 \\ -3 & 4 & 0 \\ -3 & 3 & 1 \end{pmatrix}$ is diagonalisable with eigenvalues -2 and 1.

First of all, we're not given a complete set of eigenvalues.

Given information

The matrix $A = \begin{pmatrix} -5 & 6 & 0 \\ -3 & 4 & 0 \\ -3 & 3 & 1 \end{pmatrix}$ is diagonalisable with eigenvalues -2 and 1.

First of all, we're not given a complete set of eigenvalues. However, we can easily rectify that! Recall that the trace of a matrix (the sum of the diagonal elements) is equal to the sum of the eigenvalues. So we have

$$-5 + 4 + 1 = (-2) + 1 + \lambda_3,$$

which implies that $\lambda_3 = 1$. So our three eigenvalues are -2, 1, and 1.

Eigenvalues & Eigenvectors

We already know the eigenvector with eigenvalue -2, so we can move on to finding the eigenvectors with eigenvalue 1. Now,

$$A\mathbf{v} = \mathbf{v} \implies (A - I)\mathbf{v} = \mathbf{0},$$

so we need to find the kernel of $A - I$.

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We row-reduce $A - I$, giving us

$$A - I = \begin{pmatrix} -6 & 6 & 0 \\ -3 & 3 & 0 \\ -3 & 3 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

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Thus, we can find two simple eigenvectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Diagonalisation

Diagonal Matrix

A **diagonal matrix** is one where the only nonzero elements are on the diagonals.

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Process of Diagonalisation

If a $n \times n$ matrix A is diagonalisable, then there exists a matrix Q such that

$$\boxed{A = QDQ^{-1}} \iff \boxed{D = Q^{-1}AQ}$$

where D is an $n \times n$ matrix with eigenvalue entries along the diagonal. Q is a matrix where the eigenvectors are arranged in an order corresponding to the eigenvalues in D.

Diagonalisation

First, find Q.

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

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Diagonalisation

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Then, find D.

$$\begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

And then it's as simple as just writing it out:

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \right)^{-1}.$$

Conics and Quadrics

A great example of a conic/quadric is $x^2 + y^2 + 3z^2 = 5$! And perhaps you already know what this shape will look like.

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A great example of a conic/quadric is $x^2 + y^2 + 3z^2 = 5$! And perhaps you already know what this shape will look like. In general, it is not too hard to classify shapes that only have x^2, y^2, and z^2 terms, like this one. There is a list of simple rules you can use to identify the type of conic (2D) / quadric (3D) surface.

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Unfortunately, when we have terms like xy, yz, and xz, things get a bit more complicated. But in these cases, our shapes are just a rotation of a conic/quadric with only x^2, y^2 and z^2 terms. To identify this rotation, as expected, we use **eigenvectors**.

Conics and Quadrics

Example

For the quadric surface

$$5x^2 + 3y^2 + 4z^2 + 4xz + 4yz = 1,$$

find the principal axes, type of surface, and the points, if any, on the curve which are closest to the origin.

For questions about conic and quadric surfaces:

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- We first need to express our curve in the form $\mathbf{x}^T A\mathbf{x} = c$, where A is **symmetric**.
- We then find the eigenvalues and eigenvectors of A.
- Then we diagonalise A into the form $A = QDQ^T$, where Q is **orthogonal**. Our new axis will be in the form $X = Q^T\mathbf{x}$.

Conics and Quadrics

We can write the curve $5x^2 + 3y^2 + 4z^2 + 4xz + 4yz = 1$ as

$$\boldsymbol{x}^T A \boldsymbol{x} = 1,$$

with

$$\boldsymbol{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

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We can write the curve $5x^2 + 3y^2 + 4z^2 + 4xz + 4yz = 1$ as

$$\boldsymbol{x}^T A \boldsymbol{x} = 1,$$

with

$$\boxed{\boldsymbol{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 5 & 0 & 2 \\ 0 & 3 & 2 \\ 2 & 2 & 4 \end{pmatrix}.}$$

Now let's calculate the eigenvalues and eigenvectors of our matrix A.

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This is a first-year problem, so we won't dedicate time in this workshop to doing this calculation. For now, let's take for granted that our eigenvectors are

$$\begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix} \text{ with eigenvalue } 1,$$

$$\begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \text{ with eigenvalue } 4, \text{ and}$$

$$\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \text{ with eigenvalue } 7.$$

Conics and Quadrics

Now we can diagonalise! But remember our steps from before ...

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Orthogonal Matrices

An orthogonal matrix Q is a matrix whose columns are **orthonormal vectors**. That is, the columns all have magnitude 1 and are mutually orthogonal.

So we could very well just say that

$$A = QDQ^{-1}$$

for

$$Q = \begin{pmatrix} -1 & -2 & 2 \\ -2 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}$$

and

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{pmatrix},$$

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but unfortunately then Q woudn't be an orthogonal matrix.

The great news is that the vectors are already orthogonal (and in fact for a symmetric matrix they will be!). Now we just need to scale the vectors to make their magnitude equal to 1.

So now, instead, we have

$$Q = \begin{pmatrix} -\frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{pmatrix}.$$

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So now, we have

$$A = \begin{pmatrix} -\frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ -\frac{4}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{pmatrix} \begin{pmatrix} -\frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ -\frac{4}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{pmatrix}^T,$$

where the transpose comes from the fact that the inverse of an orthogonal matrix is simply its transpose.

Conics and Quadrics

We're in the home stretch now! Let's just remember that

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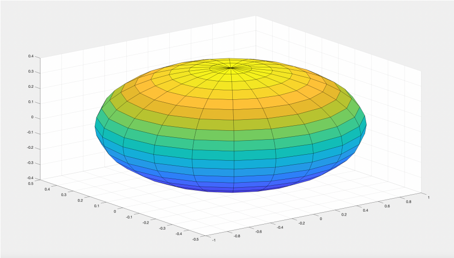
$$\mathbf{X}^T D \mathbf{X} = 1,$$

which means that $X^2 + 4Y^2 + 7Z^2 = 1$.

Conics and Quadrics

So we can identify this as an ellipsoid (because all the signs are positive). Your lecture notes will have an authoritative guide on how to identify the different basic quadrics.



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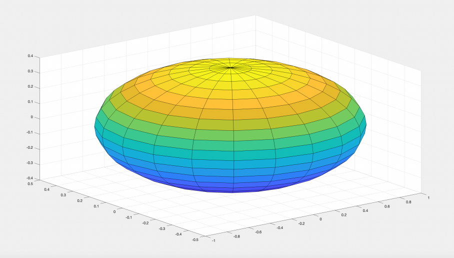

So now that we've found the basic shape in the $\mathbf{X}$-domain, let's try to find the original shape. We can think of this as a mapping between two surfaces:

$$\mathbf{x} \xrightarrow{Q^T\mathbf{x}=\mathbf{X}} \mathbf{X}, \text{ and } \mathbf{X} \xrightarrow{Q\mathbf{X}=\mathbf{x}} \mathbf{x}.$$

Conics and Quadrics

So we have a natural bijection $Q : \mathbf{X} \mapsto \mathbf{x}$. That is, to get any point of significance on our actual quadric, we can take the corresponding point on the quadric on the $X - Y - Z$ axis and multiply it by Q.

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So the $(1,0,0)^T$ axis gets mapped to the $Q(1,0,0)^T$ axis, the $(0,1,0)^T$ axis gets mapped to the $Q(0,1,0)^T$ axis, and the $(0,0,1)^T$ axis gets mapped to the $Q(0,0,1)^T$ axis.

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Example

For the quadric surface

$$5x^2 + 3y^2 + 4z^2 + 4xz + 4yz = 1,$$

find the principal axes, type of surface, and the points, if any, on the curve which are farthest from, and closest to, the origin.

Conics and Quadrics

This is an ellipsoid with principal axes

$$\begin{pmatrix} -\frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{pmatrix}, \begin{pmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}, \text{ and } \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}.$$

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$$\begin{pmatrix} -\frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{pmatrix}, \begin{pmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}, \text{ and } \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}.$$

The point closest to the origin on the $X - Y - Z$ plane is

$$\left(0, 0, \frac{1}{\sqrt{7}}\right),$$

so the point closest to the origin on our original quadric is

$$Q \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\sqrt{7}} \end{pmatrix} = \left(\frac{2}{3\sqrt{7}}, \frac{1}{3\sqrt{7}}, \frac{2}{3\sqrt{7}}\right).$$

Example

Find the general solution to the system of differential equations

$$\begin{aligned} y_1' &= -3y_1 + 4y_2, \\ y_2' &= 8y_1 + y_2, \end{aligned}$$

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Before we can do anything, let's try to convert this to matrix form, that is, $\mathbf{y}' = A\mathbf{y}$. We have

$$\mathbf{y}' = \begin{pmatrix} -3 & 4 \\ 8 & 1 \end{pmatrix} \mathbf{y}.$$

General Solution to $\mathbf{y} = A\mathbf{y}$

If A has n linearly independent eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_n$, with eigenvalues λ_1, λ_2, ... λ_n, then our general solution is

$$\mathbf{y} = \sum_{k=1}^{n} c_k \mathbf{v}_k e^{\lambda_k t}.$$

The eigenvalues are the roots of the equation

$$\begin{aligned}\det(A-\lambda I) &= \begin{vmatrix} -3-\lambda & 4 \\ 8 & 1-\lambda \end{vmatrix} \\ &= (\lambda-1)(\lambda+3)-32 \\ &= (\lambda+7)(\lambda-5).\end{aligned}$$

So our eigenvalues are -7 and 5.

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$$\begin{aligned} A+7I &= \begin{pmatrix} 4 & 4 \\ 8 & 8 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \implies \mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ A-5I &= \begin{pmatrix} -8 & 4 \\ 8 & -4 \end{pmatrix} \sim \begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix} \implies \mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}. \end{aligned}$$

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Wrapping it all up

So our general solution is given by the formula

$$\mathbf{y} = \sum_{k=1}^{n} c_k \mathbf{v}_k e^{\lambda k t} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-7t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{5t}$$
$$= \begin{pmatrix} c_1 e^{-7t} + c_2 e^{5t} \\ -c_1 e^{-7t} + 2c_2 e^{5t} \end{pmatrix}.$$

5. Ordinary Differential Equations

What is an ordinary differential equation (ODE)?

- A differential equation is an equation that relates an unknown function with its derivatives.
- Ordinary differential equations only involve ordinary derivatives, rather than partial derivatives.

First-Order Differential Equations

Example

Solve

$$(x^2 - y)\frac{dy}{dx} = \sin x + 2x(x^2 - y)$$

First-Order Differential Equations

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$$(x^2 - y)\frac{dy}{dx} = \sin x + 2x(x^2 - y) \qquad \text{with} \quad y(0) = 2$$

First, we should try to **classify** this ODE. Clearly, this is a first-order ODE, and is non-linear because of the $y\dfrac{dy}{dx}$ term. This tells us that we need to find a way to separate the variables.

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First-Order Differential Equations

Example

Solve

$$(x^2 - y)\frac{dy}{dx} = \sin x + 2x(x^2 - y) \qquad \text{with} \quad y(0) = 2$$

Let $v = x^2 - y$. Then,

$$\begin{aligned}(x^2 - y)\frac{dy}{dx} &= \sin x + 2x(x^2 - y), \\ v\frac{dy}{dx} &= \sin x + 2xv.\end{aligned}$$

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$$\begin{aligned}(x^2 - y)\frac{dy}{dx} &= \sin x + 2x(x^2 - y),\\ v\frac{dy}{dx} &= \sin x + 2xv.\end{aligned}$$

Looks a bit better, but now we would like the $\dfrac{dy}{dx}$ term to be something like $\dfrac{dv}{dx}$, so that we end up with an ODE in terms of v and x only.

First-Order Differential Equations

Example

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$$(x^2 - y)\frac{dy}{dx} = \sin x + 2x(x^2 - y) \qquad \text{with} \quad y(0) = 2$$

But we can get $\dfrac{dy}{dx}$ in terms of $\dfrac{dv}{dx}$. Since $v = x^2 - y$, we have $y = x^2 - v$, and so
$$\frac{dy}{dx} = 2x - \frac{dv}{dx}.$$

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Putting this back in our ODE, we get

$$v(2x - \frac{dv}{dx}) = \sin x + 2xv,$$
$$v\frac{dv}{dx} = -\sin x.$$

First-Order Differential Equations

Example

Solve
$$(x^2 - y)\frac{dy}{dx} = \sin x + 2x(x^2 - y) \qquad \text{with} \quad y(0) = 2$$

From here, we can just integrate both sides with respect to x, and substitute back $v = x^2 - y$.

$$\begin{aligned} \int v\frac{dv}{dx}\,dx &= \int (-\sin x)\,dx, \\ \int v\,dv &= \int (-\sin x)\,dx, \\ \frac{1}{2}v^2 &= \cos x + C, \\ (x^2 - y)^2 &= 2\cos x + C'. \end{aligned}$$

First-Order Differential Equations

Example

Solve

$$(x^2 - y)\frac{dy}{dx} = \sin x + 2x(x^2 - y) \qquad \text{with} \quad y(0) = 2$$

Finally, using the initial value $y(0) = 2$, we find C' and therefore the particular solution.

$$\begin{aligned}(0^2 - 2)^2 &= 2\cos(0) + C', \\ 4 &= 2 + C',\end{aligned}$$

so $C' = 2$.

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$$\begin{aligned}(0^2 - 2)^2 &= 2\cos(0) + C', \\ 4 &= 2 + C',\end{aligned}$$

so $C' = 2$.
Hence, the particular solution is given by

$$(x^2 - y)^2 = 2\cos x + 2.$$

Second-order Differential Equations

Example

Use the method of variation of parameters to solve the second-order differential equation

$$y'' + 2y' + y = e^{-x} \sec^2 x.$$

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The first step, as with any non-homogeneous ODE, is to find the homogeneous solution. This requires us to determine the auxiliary equation (aka characteristic polynomial).

Second-order Differential Equations

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Use the method of variation of parameters to solve the second-order differential equation

$$y'' + 2y' + y = e^{-x}\sec^2 x.$$

The first step, as with any non-homogeneous ODE, is to find the homogeneous solution. This requires us to determine the auxiliary equation (aka characteristic polynomial).

Here, the auxiliary equation is $\lambda^2 + 2\lambda + 1 = 0$, so we have a double root $\lambda = -1$. Thus, the homogeneous solution is

$$y_h = Ae^{-x} + Bxe^{-x}.$$

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$$y_h = Ae^{-x} + Bxe^{-x}.$$

Now it's time to find the particular solution, using the method of variation of parameters.

Second-order Differential Equations

Example

Use the method of variation of parameters to solve the second-order differential equation

$$y'' + 2y' + y = e^{-x} \sec^2 x.$$

Variation of Parameters

Suppose that the second order ODE $y'' + p(x)y' + q(x) = f(x)$ has homogeneous solution $y_h = Ay_1(x) + By_2(x)$. Then a particular solution is given by

$$y_P = -y_1(x) \int \frac{y_2(x) f(x)}{W(x)}\, dx + y_2(x) \int \frac{y_1(x) f(x)}{W(x)}\, dx,$$

where $W(x)$ is the Wronskian of y_1 and y_2.

Second-order Differential Equations

Example

Use the method of variation of parameters to solve the second-order differential equation

$$y'' + 2y' + y = e^{-x} \sec^2 x.$$

In our case, $y_1 = e^{-x}$ and $y_2 = xe^{-x}$. We should first find their Wronskian:

Wronskian

The Wronskian of $y_1(x)$ and $y_2(x)$ is defined as

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

Second-order Differential Equations

Example

Use the method of variation of parameters to solve the second-order differential equation

$$y'' + 2y' + y = e^{-x} \sec^2 x.$$

$$\begin{aligned} W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} &= \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & -xe^{-x} + e^{-x} \end{vmatrix} \\ &= -xe^{-2x} + e^{-2x} + xe^{-2x} \\ &= e^{-2x}. \end{aligned}$$

Example

Use the method of variation of parameters to solve the second-order differential equation

$$y'' + 2y' + y = e^{-x}\sec^2 x.$$

Plugging this into the variation of parameters formula, a particular solution is

$$\begin{aligned}
y_P &= -y_1(x)\int \frac{y_2(x)f(x)}{W(x)}\,dx + y_2(x)\int \frac{y_1(x)f(x)}{W(x)}\,dx \\
&= -e^{-x}\int \frac{xe^{-x}e^{-x}\sec^2 x}{e^{-2x}}\,dx + xe^{-x}\int \frac{e^{-x}e^{-x}\sec^2 x}{e^{-2x}}\,dx \\
&= -e^{-x}\int x\sec^2 x\,dx + xe^{-x}\int \sec^2 x\,dx.
\end{aligned}$$

Example

Use the method of variation of parameters to solve the second-order differential equation

$$y'' + 2y' + y = e^{-x} \sec^2 x.$$

With the help of integration by parts, we have

$$\begin{aligned} y_P &= -e^{-x} \int x \sec^2 x \, dx + xe^{-x} \int \sec^2 x \, dx \\ &= -e^{-x} \left(x \tan x - \int \tan x \, dx \right) + xe^{-x} \tan x \\ &= -e^{-x} (x \tan x + \ln|\cos x|) + xe^{-x} \tan x \\ &= -e^{-x} \ln|\cos x|. \end{aligned}$$

Note that we omit the constant of integration because we only want one particular solution.

Example

Use the method of variation of parameters to solve the second-order differential equation

$$y'' + 2y' + y = e^{-x}\sec^2 x.$$

Finally, putting it all together, the solution to the ODE is

$$y = Ae^{-x} + Bxe^{-x} - e^{-x}\ln|\cos x|.$$

Forced Oscillations and Resonance

- Every object in the world around us has an infinite set of natural frequencies that it can freely oscillate (vibrate) at.
- Forced oscillations occur when a periodic force is applied to a given system.
- When the frequency of the periodic force matches a natural frequency of an object, the object starts to resonate, which is a critical issue when dealing with real-life systems.
- Damping is the process of removing energy from a system and consequentially reduces the amplitude of oscillation.

Forced Oscillations and Resonance

ODE for Modelling Forced Oscillations

When dealing with a system subject to simple forced oscillations, we often need to solve equations of the form

$$my'' + cy' + ky = F_0 \sin \omega t,$$

where

- $m > 0$ is mass,
- $c \geq 0$ is the dampening coefficient,
- $k > 0$ is the spring constant (how "stiff" the system is),
- F_0 is the amplitude of the periodic force, and
- ω is the frequency of the periodic force.

Forced Oscillations and Resonance

Example

A forced vibrating system is represented by

$$y'' \; + \; 2cy' \; + \; y = 2\sin t,$$

Describe the y displacement of the system over time for $c = 0, 1, 2$, as well as the long term steady state solution.

Forced Oscillations and Resonance

Example

A forced vibrating system is represented by

$$y'' + 2cy' + y = 2\sin t,$$

Describe the y displacement of the system over time for $c = 0, 1, 2$, as well as the long term steady state solution.

First, we need the homogeneous solution. To save time, we should try to find the homogeneous solution in terms of c, and then substitute $c = 0, 1, 2$.

Forced Oscillations and Resonance

Example

A forced vibrating system is represented by

$$y'' + 2cy' + y = 2\sin t,$$

Describe the y displacement of the system over time for $c = 0, 1, 2$, as well as the long term steady state solution.

First, we need the homogeneous solution. To save time, we should try to find the homogeneous solution in terms of c, and then substitute $c = 0, 1, 2$.
The characteristic equation is $\lambda^2 + 2c\lambda + 1 = 0$, which has roots $\lambda = -c \pm \sqrt{c^2 - 1}$.

Forced Oscillations and Resonance

Example

A forced vibrating system is represented by

$$y'' + 2cy' + y = 2\sin t,$$

Describe the y displacement of the system over time for $c = 0, 1, 2$, as well as the long term steady state solution.

Case 1: $c = 1$.

For $c = 1$, there is only one (double) root, $\lambda = -1$. Thus, the homogeneous solution is

$$y_h = Ae^{-t} + Bte^{-t}.$$

Forced Oscillations and Resonance

Example

A forced vibrating system is represented by

$$y'' + 2cy' + y = 2\sin t,$$

Describe the y displacement of the system over time for $c = 0, 1, 2$, as well as the long term steady state solution.

Case 1: $c = 1$.

For the particular solution, we should guess $y_P = a\cos t + b\sin t$. Then

$$\begin{aligned} y_P' &= -a\sin t + b\cos t, \\ y_P'' &= -a\cos t - b\sin t = -y_P. \end{aligned}$$

Forced Oscillations and Resonance

Example

A forced vibrating system is represented by

$$y'' + 2cy' + y = 2\sin t,$$

Describe the y displacement of the system over time for $c = 0, 1, 2$, as well as the long term steady state solution.

Case 1: $c = 1$.
Substituting back in, we eventually have

$$-2a\sin t + 2b\cos t = 2\sin t,$$

so $a = -1$ and $b = 0$, and thus $y_P = -\cos t$. The solution is then

$$y = Ae^{-t} + Bte^{-t} - \cos t.$$

The long term steady state solution is $y = -\cos t$.

Forced Oscillations and Resonance

Example

A forced vibrating system is represented by

$$y'' \ + \ 2cy' \ + \ y = 2\sin t,$$

Describe the y displacement of the system over time for $c = 0, 1, 2$, as well as the long term steady state solution.

Case 2: $c = 2$.

In this case, we have two roots, $\lambda = -2 \pm \sqrt{3}$. Thus, the homogeneous solution is

$$y_h = Ae^{t(-2+\sqrt{3})} + Be^{t(-2-\sqrt{3})}.$$

Forced Oscillations and Resonance

Example

A forced vibrating system is represented by

$$y'' \; + \; 2cy' \; + \; y = 2\sin t,$$

Describe the y displacement of the system over time for $c = 0, 1, 2$, as well as the long term steady state solution.

Case 2: $c = 2$.

For the particular solution, we again guess $y_P = a\cos t + b\sin t$. Using the expressions for y_P' and y_P'' found earlier, substituting y_P in the ODE results in

$$-4a\sin t + 4b\cos t = 2\sin t,$$

and so $a = -1/2$ and $b = 0$. Thus the particular solution is $y_P = -\frac{1}{2}\cos t$.

Forced Oscillations and Resonance

Example

A forced vibrating system is represented by

$$y'' \;+\; 2cy' \;+\; y = 2\sin t,$$

Describe the y displacement of the system over time for $c = 0, 1, 2$, as well as the long term steady state solution.

Case 2: $c = 2$.

Putting it all together, the solution in this case is

$$y = Ae^{t(-2+\sqrt{3})} + Be^{t(-2-\sqrt{3})} - \frac{1}{2}\cos t.$$

The long term steady state solution is $y = -\frac{1}{2}\cos t$, because the exponents $-2 \pm \sqrt{3}$ are both less than 0.

Example

A forced vibrating system is represented by

$$y'' + 2cy' + y = 2\sin t,$$

Describe the y displacement of the system over time for $c = 0, 1, 2$, as well as the long term steady state solution.

Case 3: $c = 0$.

In this case, we also have two roots, $\lambda = \pm i$. Thus, the homogeneous solution is

$$y_h = A\cos t + B\sin t.$$

How about the particular solution?

Example

A forced vibrating system is represented by

$$y'' \ + \ 2cy' \ + \ y = 2\sin t,$$

Describe the y displacement of the system over time for $c = 0, 1, 2$, as well as the long term steady state solution.

Case 3: $c = 0$. This time, we **can't** just guess the particular solution to be in the form $y_P = a\cos t + b\sin t$, because this involves terms in the homogeneous solution.
So, for the particular solution, we should guess t times the guess used for the previous cases,

$$y_P = t(a\cos t + b\sin t).$$

Forced Oscillations and Resonance

Example

A forced vibrating system is represented by

$$y'' + 2cy' + y = 2\sin t,$$

Describe the y displacement of the system over time for $c = 0, 1, 2$, as well as the long term steady state solution.

Case 3: $c = 0$.
Then,

$$\begin{aligned} y_P &= t(a\cos t + b\sin t), \\ y_P' &= t(-a\sin t + b\cos t) + a\cos t + b\sin t, \\ y_P'' &= t(-a\cos t - b\sin t) - 2a\sin t + 2b\cos t. \end{aligned}$$

Forced Oscillations and Resonance

Example

A forced vibrating system is represented by

$$y'' + 2cy' + y = 2\sin t,$$

Describe the y displacement of the system over time for $c = 0, 1, 2$, as well as the long term steady state solution.

Case 3b: $c = 0$.
Substituting back in, we eventually have

$$-2a\sin t + 2b\cos t = 2\sin t,$$

so $a = -1$ and $b = 0$. The particular solution is then

$$y_P = -t\cos t.$$

Forced Oscillations and Resonance

Example

A forced vibrating system is represented by

$$y'' + 2cy' + y = 2\sin t,$$

Describe the y displacement of the system over time for $c = 0, 1, 2$, as well as the long term steady state solution.

Case 3b: $c = 0$.
Finally, putting it all together, the solution for this case is

$$y = A\cos t + B\sin t - t\cos t.$$

Notice that as $t \to \infty$, the magnitude of y also goes to infinity. This is because in this case, the forcing frequency matches the natural frequency given by the homogeneous solution.

6. Laplace Transforms

Laplace Transforms

- Laplace transforms provide a mathematical tool to solve linear ODEs and systems of linear ODEs.
- Roughly, the idea is to transform the given ODE into an algebraic equation, and solve it using algebraic operations.
- Then, we transform the algebraic equation back to yield the solution to the ODE.

The Heaviside Function

Example

Use the integral definition to prove that

$$\mathcal{L}(u(t-a)) = \frac{e^{-as}}{s}$$

where $u(t-a)$ is the Heaviside step function.

The Heaviside Function

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where $u(t-a)$ is the Heaviside step function.

Integral Definition

The Laplace transform of a function $f(t)$ is

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)dt = F(s)$$

The Heaviside Function

Heaviside Function Definition

The Heaviside function is defined as

$$u(t-a) = \begin{cases} 0 & t < a \\ 1 & t > a. \end{cases}$$

The Heaviside Function

Heaviside Function Definition

The Heaviside function is defined as

$$u(t-a) = \begin{cases} 0 & t < a \\ 1 & t > a. \end{cases}$$

Because the Heaviside function is "split" at a, we should try to split the integral into two at a as well:

$$\begin{aligned} \mathcal{L}(u(t-a)) &= \int_0^\infty e^{-st} u(t-a)\,dt \\ &= \int_0^a e^{-st} u(t-a)\,dt + \int_a^\infty e^{-st} u(t-a)\,dt \\ &= \int_0^a e^{-st}(0)\,dt + \int_a^\infty e^{-st}(1)\,dt = \boxed{\int_a^\infty e^{-st}\,dt.} \end{aligned}$$

The Heaviside Function

Example

Use the integral definition to prove that

$$\mathcal{L}(u(t-a)) = \frac{e^{-as}}{s}$$

where $u(t-a)$ is the Heaviside step function.

Continuing, we have

$$\begin{aligned}\mathcal{L}(u(t-a)) = \int_a^\infty e^{-st}\,dt &= \lim_{b\to\infty}\int_a^b e^{-st}\,dt \\ &= \lim_{b\to\infty}\left(-\frac{e^{-sb}}{s} + \frac{e^{-sa}}{s}\right) \\ &= \boxed{\frac{e^{-as}}{s}.}\end{aligned}$$

Shifting Theorems

Example

Find $\mathcal{L}(\sin(t)u(t-\pi))$.

Here, there is a cut at $t=\pi$, but no shift. We need to introduce a shift in order to use one of the shifting theorems. Notice that

$$\sin t = \sin(\pi - t) = -\sin(t-\pi).$$

Shifting Theorems

Example

Find $\mathcal{L}(\sin(t)u(t-\pi))$.

Here, there is a cut at $t = \pi$, but no shift. We need to introduce a shift in order to use one of the shifting theorems. Notice that

$$\sin t = \sin(\pi - t) = -\sin(t - \pi).$$

Since the Laplace transform is linear, we have

$$\mathcal{L}(\sin(t)u(t-\pi)) = -\mathcal{L}(\sin(t-\pi)u(t-\pi)).$$

Shifting Theorems

Example

Find $\mathcal{L}(\sin(t)u(t-\pi))$.

Second Shifting Theorem

$$\mathcal{L}(f(t-a)u(t-a)) = e^{-as}F(s)$$

Laplace Transform For sin

$$\mathcal{L}(\sin bt) = \frac{b}{s^2+b^2}$$

Since $\mathcal{L}$ is linear, we apply the above to get

$$\mathcal{L}(\sin(t)u(t-\pi)) = -\mathcal{L}(\sin(t-\pi)u(t-\pi)) = \boxed{-\frac{e^{-\pi s}}{s^2+1}}.$$

Inverse Laplace Transform

- We also want to be able to go backwards, i.e. find the inverse Laplace transform of a function.
- To do this, we need to rely on the table of Laplace transforms, and often need to do some manipulation to make a function "look" like the tabulated transforms.
- Partial fractions might be useful for manipulation.
- Also, we might have to apply some shifting theorems "backwards".

Inverse Laplace Transform

Example

Find $\mathcal{L}^{-1}\left(\dfrac{1}{s^2 - 10}\right)$.

Inverse Laplace Transform

Example

Find $\mathcal{L}^{-1}\left(\dfrac{1}{s^2-10}\right)$.

This looks suspiciously similar to this:

Laplace Transform of sinh

$$\mathcal{L}(\sinh bt) = \frac{b}{s^2-b^2}.$$

Inverse Laplace Transform

Example

Find $\mathcal{L}^{-1}\left(\frac{1}{s^2-10}\right)$.

We can use a simple manipulation, and exploit the fact that the inverse Laplace transform is linear:

$$\begin{aligned}\mathcal{L}\left(\frac{1}{s^2-10}\right) &= \mathcal{L}\left(\frac{1}{s^2-\sqrt{10}^2}\right)\\ &= \frac{1}{\sqrt{10}}\mathcal{L}\left(\frac{\sqrt{10}}{s^2-\sqrt{10}^2}\right)\\ &= \frac{1}{\sqrt{10}}\sinh t\sqrt{10}.\end{aligned}$$

Inverse Laplace Transform

Example

Find $\mathcal{L}^{-1}\left(\dfrac{3s+1}{(s+1)(s-1)}\right)$.

Inverse Laplace Transform

Example

Find $\mathcal{L}^{-1}\left(\dfrac{3s+1}{(s+1)(s-1)}\right)$.

Partial fractions essentially lets us "split" the denominator of functions like the above. In this case, our aim is to find a and b such that

$$\frac{3s+1}{(s+1)(s-1)} = \frac{a}{s+1} + \frac{b}{s-1}.$$

Inverse Laplace Transform

Example

Find $\mathcal{L}^{-1}\left(\dfrac{3s+1}{(s+1)(s-1)}\right)$.

Partial fractions essentially lets us "split" the denominator of functions like the above. In this case, our aim is to find a and b such that

$$\frac{3s+1}{(s+1)(s-1)} = \frac{a}{s+1} + \frac{b}{s-1}.$$

So we want

$$3s+1 = a(s-1) + b(s+1),$$

which, after comparing coefficients of s, results in $a = 1$ and $b = 2$.

Inverse Laplace Transform

Example

Find $\mathcal{L}^{-1}\left(\dfrac{3s+1}{(s+1)(s-1)}\right)$.

Laplace Transform

$$\mathcal{L}\left(e^{-at}\right) = \frac{1}{s+a}$$

Using this, and the relevant Laplace transform, we have

$$\begin{aligned}
\mathcal{L}^{-1}\left(\frac{3s+1}{(s+1)(s-1)}\right) &= \mathcal{L}^{-1}\left(\frac{1}{s+1} + \frac{2}{s-1}\right) \\
&= \mathcal{L}^{-1}\left(\frac{1}{s+1}\right) + \mathcal{L}^{-1}\left(\frac{2}{s-1}\right) \\
&= e^{-t} + 2e^{t}.
\end{aligned}$$

Inverse Laplace Transform

Example

Find $\mathcal{L}^{-1}\left(\dfrac{2}{(s+3)^2+4}\right)$.

Inverse Laplace Transform

Example

Find $\mathcal{L}^{-1}\left(\dfrac{2}{(s+3)^2+4}\right)$.

This time, we should not try to use partial fractions (technically we can, but involves complex numbers and gets messy). We should note that this is very similar to this...

Laplace Transform of sin

$$\mathcal{L}(\sin bt) = \frac{b}{s^2+b^2}$$

It's just that the s is instead $s+3$.

Inverse Laplace Transform

Example

Find $\mathcal{L}^{-1}\left(\dfrac{2}{(s+3)^2+4}\right)$.

First Shifting Theorem

$$\mathcal{L}\left(e^{-at}f(t)\right) = F(s+a)$$

However, we can use the First Shifting Theorem backwards to immediately get

$$\mathcal{L}^{-1}\left(\frac{2}{(s+3)^2+4}\right) = e^{-3t}\sin(2t).$$

Inverse Laplace Transform

Example

Find $\mathcal{L}^{-1}\left(e^{-4t}\frac{2}{(s+3)^2+4}\right)$.

Inverse Laplace Transform

Example

Find $\mathcal{L}^{-1}\left(e^{-4t}\frac{2}{(s+3)^2+4}\right)$.

Because of the e^{-4t} term, we should try to use the Second Shifting Theorem backwards. Recall:

Second Shifting Theorem

$$\mathcal{L}(f(t-a)u(t-a)) = e^{-as}F(s)$$

But from the previous question, we already know that

$$\mathcal{L}^{-1}\left(\frac{2}{(s+3)^2+4}\right) = e^{-3t}\sin(2t).$$

Inverse Laplace Transform

Example

Find $\mathcal{L}^{-1}\left(e^{-4t}\dfrac{2}{(s+3)^2+4}\right)$.

Because of the e^{-4t} term, we should try to use the Second Shifting Theorem backwards. Recall:

Second Shifting Theorem

$$\mathcal{L}(f(t-a)u(t-a)) = e^{-as}F(s)$$

Thus,

$$\mathcal{L}^{-1}\left(e^{-4t}\frac{2}{(s+3)^2+4}\right) = e^{-3(t-4)}\sin(2(t-4))u(t-4).$$

Laplace Transform and Systems of ODEs

Example

Solve the system of differential equations

$$\frac{dx}{dt} + 2x + y = 0, \quad \frac{dy}{dt} + x + 2y = 0$$

where $x(0) = 0$ and $y(0) = 1$.

We start by taking the Laplace transform of each side of the equations.

$$\begin{aligned}
\mathcal{L}\left(\frac{dx}{dt} + 2x + y\right) &= \mathcal{L}(0) \\
\mathcal{L}\left(\frac{dx}{dt}\right) + 2\mathcal{L}(x) + \mathcal{L}(y) &= \mathcal{L}(0) \\
\mathcal{L}(x') + 2\mathcal{L}(x) + \mathcal{L}(y) &= \mathcal{L}(0)
\end{aligned}$$

Laplace Transform and Systems of ODEs

Example

Solve the system of differential equations

$$\frac{dx}{dt} + 2x + y = 0, \quad \frac{dy}{dt} + x + 2y = 0$$

where $x(0) = 0$ and $y(0) = 1$.

Laplace transform of First Derivative

$$\mathcal{L}(f'(t)) = sF(s) - f(0)$$

$$\begin{aligned} &\mathcal{L}(x') + 2\mathcal{L}(x) + \mathcal{L}(y) = \mathcal{L}(0), \\ &sX - x(0) + 2X + Y = 0, \end{aligned}$$

$$sX + 2X + Y = 0 \implies \boxed{X = -\frac{Y}{s+2}}.$$

Laplace Transform and Systems of ODEs

Example

Solve the system of differential equations

$$\frac{dx}{dt} + 2x + y = 0, \quad \frac{dy}{dt} + x + 2y = 0$$

where $x(0) = 0$ and $y(0) = 1$.

Then, we Laplace transform the second equation:

$$\begin{aligned}
\mathcal{L}\left(\frac{dy}{dt} + x + 2y\right) &= \mathcal{L}(0), \\
sY - y(0) + X + 2Y &= 0, \\
(s+2)Y + X &= 1.
\end{aligned}$$

Laplace Transform and Systems of ODEs

Example

Solve the system of differential equations

$$\frac{dx}{dt} + 2x + y = 0, \quad \frac{dy}{dt} + x + 2y = 0$$

where $x(0) = 0$ and $y(0) = 1$.

Next, substitute our previous result $X = -\dfrac{Y}{s+2}$ to get

$$\begin{aligned}
(s+2)Y - \frac{Y}{s+2} &= 1, \\
(s+2)^2 Y - Y &= s+2, \\
((s+2)^2 - 1)Y &= s+2,
\end{aligned}$$

$$Y = \frac{s+2}{s^2+4s+3} = \frac{s+2}{(s+3)(s+1)}.$$

Example

Solve the system of differential equations

$$\frac{dx}{dt} + 2x + y = 0, \quad \frac{dy}{dt} + x + 2y = 0$$

where $x(0) = 0$ and $y(0) = 1$.

Substituting Y back into $X = -\dfrac{Y}{s+2}$, we also get

$$X = -\frac{1}{(s+3)(s+1)}.$$

Laplace Transform and Systems of ODEs

Example

Solve the system of differential equations

$$\frac{dx}{dt} + 2x + y = 0, \quad \frac{dy}{dt} + x + 2y = 0$$

where $x(0) = 0$ and $y(0) = 1$.

Finally, we now need to find the inverse Laplace transforms of X and Y to obtain the solutions to the system of ODEs. We need to use partial fractions again:

$$X = -\frac{1}{(s+3)(s+1)} = \frac{a}{s+3} + \frac{b}{s+1},$$

$$-1 = a(s+1) + b(s+3),$$

$$a + b = 0, \quad a + 3b = -1,$$

$$a = 1/2, \quad b = -1/2.$$

Laplace Transform and Systems of ODEs

Example

Solve the system of differential equations

$$\frac{dx}{dt} + 2x + y = 0, \quad \frac{dy}{dt} + x + 2y = 0$$

where $x(0) = 0$ and $y(0) = 1$.

Thus,

$$X = \frac{1}{2}\left(\frac{1}{s+3} - \frac{1}{s+1}\right),$$

and so applying the inverse Laplace transform on X gives

$$\mathcal{L}^{-1}(X) = x = \frac{1}{2}\left(e^{-3t} - e^{-t}\right).$$

Example

Solve the system of differential equations

$$\frac{dx}{dt} + 2x + y = 0, \quad \frac{dy}{dt} + x + 2y = 0$$

where $x(0) = 0$ and $y(0) = 1$.

We take the same partial fractions approach with Y:

$$Y = \frac{s+2}{(s+3)(s+1)} = \frac{a}{s+3} + \frac{b}{s+1},$$

$$s + 2 = a(s+1) + b(s+3),$$

$$a + b = 1, \quad a + 3b = 2,$$

$$a = 1/2, \quad b = 1/2.$$

Laplace Transform and Systems of ODEs

Example

Solve the system of differential equations

$$\frac{dx}{dt} + 2x + y = 0, \quad \frac{dy}{dt} + x + 2y = 0$$

where $x(0) = 0$ and $y(0) = 1$.

Thus,

$$Y = \frac{1}{2}\left(\frac{1}{s+3} + \frac{1}{s+1}\right).$$

Applying the inverse Laplace transform, we have

$$\mathcal{L}^{-1}(Y) = y = \frac{1}{2}\left(e^{-3t} + e^{-t}\right).$$

Laplace Transform and Systems of ODEs

Example

Solve the system of differential equations

$$\frac{dx}{dt} + 2x + y = 0, \quad \frac{dy}{dt} + x + 2y = 0$$

where $x(0) = 0$ and $y(0) = 1$.

Hence, the solution to this system of ODEs is

$$x = \frac{1}{2}\left(e^{-3t} - e^{-t}\right), \quad y = \frac{1}{2}\left(e^{-3t} + e^{-t}\right).$$

We can (should) verify this solution by plugging it back into the system. First, the initial conditions are clearly satisfied. Next, we find

$$\frac{dx}{dt} = \frac{1}{2}\left(-3e^{-3t} + e^{-t}\right), \quad \frac{dy}{dt} = \frac{1}{2}\left(-3e^{-3t} - e^{-t}\right).$$

Example

Solve the system of differential equations

$$\frac{dx}{dt} + 2x + y = 0, \quad \frac{dy}{dt} + x + 2y = 0$$

where $x(0) = 0$ and $y(0) = 1$.

Plugging these back in, we confirm that

$$\begin{aligned}\frac{dx}{dt} + 2x + y &= \frac{1}{2}\left(-3e^{-3t} + e^{-t}\right) + e^{-3t} - e^{-t} + \frac{1}{2}\left(e^{-3t} + e^{-t}\right) \\ &= \left(-\frac{3}{2} + 1 + \frac{1}{2}\right)e^{-3t} + \left(\frac{1}{2} - 1 + \frac{1}{2}\right)e^{-t} \\ &= 0,\end{aligned}$$

Laplace Transform and Systems of ODEs

Example

Solve the system of differential equations

$$\frac{dx}{dt} + 2x + y = 0, \quad \frac{dy}{dt} + x + 2y = 0$$

where $x(0) = 0$ and $y(0) = 1$.

and also that

$$\begin{aligned}\frac{dy}{dt} + x + 2y &= \frac{1}{2}\left(-3e^{-3t} - e^{-t}\right) + \frac{1}{2}\left(e^{-3t} - e^{-t}\right) + e^{-3t} + e^{-t} \\ &= \left(-\frac{3}{2} + \frac{1}{2} + 1\right)e^{-3t} + \left(-\frac{1}{2} - \frac{1}{2} + 1\right)e^{-t} \\ &= 0.\end{aligned}$$

So our solution is correct. Yay!

7. Fourier Series

Fourier Series

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Fourier Series

- Many applications in engineering involve phenomena that have a cyclic or periodic behaviour.
- Fourier series provide us with a tool to analyse periodic phenomena by transforming periodic signals into a sum (finite or infinite) of sines and cosines.
- As will be seen in the next topic, knowing how to calculate Fourier series becomes crucial to being able to solve certain partial differential equations.

Fourier series with arbitrary period

Fourier series of a $2L$-periodic function

A $2L$-periodic function f has Fourier series representation

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L},$$

where the Fourier coefficients $a_0, a_1, \ldots$ satisfy

$$\begin{aligned} a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\,dx \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \text{ for } n = 1, 2, \ldots \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \text{ for } n = 1, 2, \ldots. \end{aligned}$$

Periodic Extensions

Not all functions are defined on all of $\mathbb{R}$. We can, however, extend them to one in a few different ways.

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2. *Even periodic extension.* Reflect f over the y-axis, and then repeat f by $2L$. The result is an even, $2L$-periodic function.

Periodic Extensions

Not all functions are defined on all of $\mathbb{R}$. We can, however, extend them to one in a few different ways.

Three types of periodic extensions

1. *Periodic extension.* Repeat f by L to the left and right.
2. *Even periodic extension.* Reflect f over the y-axis, and then repeat f by $2L$. The result is an even, $2L$-periodic function.
3. *Odd periodic extension.* Reflect f in the line $y = x$, and then repeat f by $2L$. The result is an even, $2L$-periodic function.

Periodic Extensions

- The Fourier series of the even/odd periodic extension of a function is known as its half-range expansion.

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- The Fourier series of the even/odd periodic extension of a function is known as its half-range expansion.
- The half-range expansion of an even periodic extension is a Fourier cosine series (i.e. $b_n = 0$), and the half-range expansion of an odd periodic extension is a Fourier sine series (i.e. $a_0, a_n = 0$).

Periodic Extensions

- The Fourier series of the even/odd periodic extension of a function is known as its half-range expansion.
- The half-range expansion of an even periodic extension is a Fourier cosine series (i.e. $b_n = 0$), and the half-range expansion of an odd periodic extension is a Fourier sine series (i.e. $a_0, a_n = 0$).
- Upshot: we can avoid calculating unnecessary coefficients when finding half-range expansions!

Half-range Expansion Example

Semester 1, 2017 Exam, Q4(a)(i) – (ii)

Define the function f on $[0, \pi)$ as $f(x) = x/2$.

(i) Sketch the odd periodic extension of f for $-4\pi \leq x \leq 4\pi$.

(ii) Show that the Fourier sine series of f is

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx).$$

Half-range Expansion Example – Part (i)

Solution, (i). We need to reflect f in the line $y = x$, and then repeat it up to the bounds $x = \pm 4\pi$. (To do on iPad: sketch.)

Half-range Expansion Example – Part (ii)

Solution, (ii). The Fourier series of an odd function will just be a sine series, so it suffices to prove that

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \stackrel{?}{=} \frac{(-1)^{n+1}}{n}.$$

Half-range Expansion Example – Part (ii)

Solution, (ii). The Fourier series of an odd function will just be a sine series, so it suffices to prove that

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \stackrel{?}{=} \frac{(-1)^{n+1}}{n}.$$

Since each function in the integrand is odd, their product is an even function, so we can write

$$b_n = \frac{2}{\pi}\int_{0}^{\pi} {\color{red}f(x)}\sin(nx)\,dx$$

Half-range Expansion Example – Part (ii)

Solution, (ii). The Fourier series of an odd function will just be a sine series, so it suffices to prove that

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As is often the case, we will need to use integration by parts to evaluate this integral. With the choice $u = x$ and $v' = \sin(nx)$, we have

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$$b_n = \frac{-(-1)^n}{n} = \frac{(-1)^{n+1}}{n}.$$

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A plot of the extension can aid you in finding this value, if available.

Convergence Example

Semester 1, 2017 Exam, Q4(a)(iii)

(iii) To what does the Fourier sine series of f converge to at $x = \pi$?

Solution. The extension is discontinuous at $x = \pi$, so we can either calculate the value

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or we could use our plot from (i) to visually see this. Either way, the Fourier series converges to 0 at $x = \pi$.

Harder Example: Parseval's Identity

Semester 1, 2014 Exam, Q4(iii) – modified

Prove Parseval's identity in the case of a 2-periodic function, i.e. that the Fourier coefficients of f satisfy

$$\int_{-1}^{1} f(x)^2 dx = 2a_0^2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2.$$

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Proof. Working from the LHS, we have

$$\begin{aligned}\int_{-1}^{1} f(x)^2\,dx &= \int_{-1}^{1} f(x) f(x)\,dx \\ &= \int_{-1}^{1} f(x)\left(a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi x + b_n \sin n\pi x\right)dx.\end{aligned}$$

Harder Example: Parseval's Identity

Distributing, we have

$$\int_{-1}^{1} f(x)^2\,dx = a_0 \int_{-1}^{1} f(x)\,dx + \int_{-1}^{1} \sum_{n=1}^{\infty} a_n f(x) \cos n\pi x\,dx$$
$$+ \int_{-1}^{1} \sum_{n=1}^{\infty} b_n f(x) \sin n\pi x\,dx.$$

We can now attack each of these three integrals separately.

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To evaluate the first integral, notice that

$$a_0 \int_{-1}^{1} f(x)\,dx = a_0 \cdot 2 \cdot \frac{1}{2}\int_{-1}^{1} f(x)\,dx$$

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Harder Example: Parseval's Identity

To evaluate the second integral, we can interchange the sum and the integral so that

$$\int_{-1}^{1} \sum_{n=1}^{\infty} a_n f(x) \cos n\pi x \, dx = \sum_{n=1}^{\infty} \int_{-1}^{1} a_n f(x) \cos(n\pi x) \, dx$$

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&= \sum_{n=1}^{\infty} a_n \int_{-1}^{1} f(x) \cos(n\pi x) \, dx \\
&= \sum_{n=1}^{\infty} a_n a_n \\
&= \sum_{n=1}^{\infty} a_n^2.
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Harder Example: Parseval's Identity

The third integral may be computed in a similar way so that

$$\int_{-1}^{1} \sum_{n=1}^{\infty} b_n f(x) \sin n\pi x \, dx = \sum_{n=1}^{\infty} b_n^2.$$

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$$\int_{-1}^{1} \sum_{n=1}^{\infty} b_n f(x) \sin n\pi x \, dx = \sum_{n=1}^{\infty} b_n^2.$$

Hence,

$$\int_{-1}^{1} f(x) \, dx = 2a_0^2 + \sum_{n=1}^{\infty} a_n^2 + \sum_{n=1}^{\infty} b_n^2 = 2a_0^2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2$$

as claimed. $\square$

8. Partial Differential Equations

Partial Differential Equations

- Most problems in fluid and solid mechanics, quantum mechanics, electromagnetism, and other areas of physics can be modelled as partial differential equations (PDEs), i.e. differential equations involving partial derivatives.

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- There are many PDEs that cannot be solved exactly, but can be solved approximately by numerical methods. We will focus on those we can solve.

The 1D Heat Equation

Heat Equation

For some predetermined constant c, the 1D heat equation is the PDE

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{or} \quad u_t = c^2 u_{xx}.$$

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to keep the ends of the bar at a constant temperature of $0°$, and the initial condition

$$u(x,0) = f(x) \quad \text{for all } x$$

to specify the initial temperature distribution across the bar.

The 1D Heat Equation with Insulation

Insulated Heat Equation

If we take the heat equation as before, we can instead impose the boundary conditions

$$u_x(0,t) = u_x(L,t) = 0 \quad \text{for all } t$$

to keep the ends of the bar insulated, i.e. to stop heat flow at the ends. The heat in the bar will therefore evenly distribute itself over time.

Separation of Variables

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Given a PDE in variables x and t,

1. Assume the solution is of the form $F(x)G(t)$ for some functions F and G in just one variable.
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3. Solve each ODE for the given boundary conditions.
4. Combine the solutions such that they satisfy the initial conditions.

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- There are almost always infinitely many solutions to the 2 ODEs when applying this method. The superposition (i.e. the sum) of all of the **nontrivial** solutions (i.e. not the zero function) gives the overall solution to the PDE.

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- Adapting this solution to fit the initial conditions will likely also necessitate the use of Fourier series.
- Exam questions on this will always be split up into parts to guide you through the process of solving such an equation.

Solving the Insulated Heat Equation

Semester 2, 2013 Exam, Q4(b)

The temperature $u(x,t)$, in Celsius, of a bar of length π satisfies the heat equation $u_t = 4u_{xx}$. The ends of the bar are insulated so that

$$u_x(0,t) = u_x(\pi,t) = 0 \quad \text{for all } t.$$

Solving the Insulated Heat Equation – Part (i)

Semester 2, 2013 Exam, Q4(b)(i)

(i) Assuming a solution of the form $u(x,t) = F(x)G(t)$, show that

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for some constant λ.

Solution, (i). Substituting this form of u into the given PDE, we have

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Solving the Insulated Heat Equation – Part (ii)

Semester 2, 2013 Exam, Q4(b)(ii)

(ii) You may assume that $\lambda = -k^2$ for some $k \geq 0$ yields nontrivial solutions to the equations in (i). Apply the boundary conditions to show that possible solutions of F are

$$F_n(x) = B_n \cos nx, \quad k = n = 0, 1, 2, \ldots$$

for some constants B_n.

Solution, (ii). We try to find solutions to

$$\frac{F''}{F} = -k^2 \quad \iff \quad F'' = -k^2 F.$$

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so $k\pi = n\pi$, $n = 0, 1, 2, \ldots$, i.e. $k = n$.

Solving the Insulated Heat Equation – Part (ii)

We conclude that, for each $n = 0, 1, 2, \ldots$ we have a solution

$$F(x) = F_n(x) = B_n \cos(nx),$$

where the constants B_n depend upon n.

Solving the Insulated Heat Equation – Part (iii)

Semester 2, 2013 Exam, Q4(b)(iii)

(iii) Find all possible solutions $G(t) = G_n(t)$.

Solution, (iii). The previous part forces $k = n$, so we need to find solutions to

$$G' = -4n^2 G.$$

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Solving the Insulated Heat Equation – Part (iii)

Semester 2, 2013 Exam, Q4(b)(iii)

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$$G(t) = G_n(t) = e^{-4n^2 t}$$

for each $n = 0, 1, 2, \ldots$.

Solving the Insulated Heat Equation – Part (iv)

Semester 2, 2013 Exam, Q4(b)(iv)

In a previous question of this exam, you are asked to find the Fourier series of the even periodic extension of the function

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x < \frac{\pi}{2} \\ 0 & \text{if } \frac{\pi}{2} \le x < \pi. \end{cases}$$

We will assume that we found the correct answer, i.e. that

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right) \cos(nx).$$

(iv) If the initial temperature distribution of the bar is $u(x,0) = f(x)$, express the general solution $u(x,t)$ as a Fourier cosine series.

Solving the Insulated Heat Equation – Part (iv)

Solution, (iv). By superposition, a general solution is

$$u(x,t) = \sum_{n=0}^{\infty} F_n(x)G_n(t) = \sum_{n=0}^{\infty} B_n \cos(nx)e^{-4n^2t}.$$

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$$u(x,0) = \sum_{n=0}^{\infty} B_n \cos(nx) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right) \cos(nx).$$

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$$B_0 = \frac{1}{2} \quad \text{and} \quad B_n = \frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right).$$

Solving the Insulated Heat Equation – Part (v)

Semester 2, 2013 Exam, Q4(b)(v)

(v) What is the equilibrium temperature as $t \to \infty$?

Solution, (v). We can write

$$u(x,t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right) \cos(nx) e^{-4n^2 t},$$

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so it follows that $u(x,t) \to \frac{1}{2}$ in equilibrium as $t \to \infty$.

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- An earlier Fourier series often ties into these questions in exams, since they are the most intensive part of the process.

Wave Equation

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For some predetermined constant c, the 1D wave equation is the PDE

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{or} \quad u_{tt} = c^2 u_{xx}.$$

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to keep the endpoints of the string fixed, and the initial conditions

$$u(x,0) = f(x) \quad \text{and} \quad u_t(x,0) = \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x) \quad \text{for all } x$$

to specify the initial displacement and velocity.

D'Alembert's Solution to the Wave Equation

D'Alembert's Solution

It can be shown that, for arbitrary functions ϕ and ψ of one variable, the general solution to the 1D wave equation is

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If we know the initial displacement f and velocity g, then it is possible to derive the more precise solution

$$u(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds.$$

Solving the Wave Equation

- We can solve the wave equation using D'Alembert's solution where appropriate, or we can also use the more general method of separation of variables detailed for the heat equation. The procedure is analogous.
- The major difference in practice when using separation of variables is that you end up solving two second-order ODEs, which slightly changes your *eigenfunctions* (i.e. the solutions F_n and G_n).

Harder Example: Proving D'Alembert's Solution

Semester 1, 2017 Exam, Q4(c)

Consider the one-dimensional wave equation

$$u_{tt} = 9u_{xx}.$$

D'Alembert's solution to this wave equation is

$$u(x,t) = \phi(x+3t) + \psi(x-3t)$$

for arbitrary functions ϕ and ψ. If the initial displacement of the wave is $u(x,0) = g(x)$ and the initial velocity is $u_t(x,0) = 0$, prove that

$$u(x,t) = \frac{g(x+3t) + g(x-3t)}{2}.$$

Proof. This is mostly an exercise in multivariable differentiation from last seminar! Let $v(x,t) = x + 3t$ and $w(x,t) = x - 3t$ so that

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and so

$$\phi(x) = \frac{g(x) - C}{2} + C = \frac{g(x) + C}{2}.$$

Harder Example: Proving D'Alembert's Solution

Hence,

$$u(x,t) = \frac{g(v)+C}{2} + \frac{g(w)-C}{2} = \frac{g(x+3t)+g(x-3t)}{2}$$

as claimed. $\square$

The End

Thanks for watching/attending!
Best of luck with all of your exams!