MathSoc Second Year Calculus Revision Session 2020 T1 Solutions

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We cannot guarantee that our answers are correct - please notify us of any errors or typos at unswmathsoc@gmail.com, or on our Facebook page. There are sometimes multiple methods of solving the same question. Remember that in the real class test, you will be expected to explain your steps and working out.

Seminar 2 First Half (Vector Calculus)

1. (a)

$$\begin{aligned}\nabla \cdot \mathbf{F} &= (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \cdot (x, y, z) \\ &= 1 + 1 + 1 \\ &= 3\end{aligned}$$

(b)

$$\begin{aligned}
\nabla \times \mathbf{F} &= (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) \times (x, y, z) \\
&= \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{bmatrix} \\
&= \hat{i} \det \begin{bmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z \end{bmatrix} - \hat{j} \det \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ x & z \end{bmatrix} + \hat{k} \det \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ x & y \end{bmatrix} \\
&= 0\hat{i} - 0\hat{j} + 0\hat{k}
\end{aligned}$$

(c) First we compute $\mathbf{a} \times \mathbf{F}$

$$\begin{aligned}
\mathbf{a} \times \mathbf{F} &= (a_1, a_2, a_3) \times (x, y, z) \\
&= \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{bmatrix} \\
&= \hat{i} \det \begin{bmatrix} a_2 & a_3 \\ y & z \end{bmatrix} - \hat{j} \det \begin{bmatrix} a_1 & a_3 \\ x & z \end{bmatrix} + \hat{k} \det \begin{bmatrix} a_1 & a_2 \\ x & y \end{bmatrix} \\
&= (a_2 z - a_3 y)\hat{i} - (a_1 z - a_3 x)\hat{j} + (a_1 y - a_2 x)\hat{k}
\end{aligned}$$

To compute the curl, we now compute:

$$\begin{aligned}
\det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_2 z - a_3 y & a_3 x - a_1 z & a_1 y - a_2 x \end{bmatrix} &= \hat{i} \det \begin{bmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z \end{bmatrix} - \hat{j} \det \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ x & z \end{bmatrix} + \hat{k} \det \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ x & y \end{bmatrix} \\
&= 2a_1\hat{i} + 2a_2\hat{j} + 2a_3\hat{k} \\
&= 2\mathbf{a}
\end{aligned}$$

(d) Based on the norm, we get $\|F\| = \sqrt{x^2 + y^2 + z^2}^{\frac{1}{2}}$. Therefore we need to compute:

$$
\begin{aligned}
\nabla \frac{1}{\|\mathbf{F}\|^3} &= \nabla (x^2+y^2+z^2)^{-\frac{3}{2}} \\
&= \frac{-3}{2}\left(\frac{2x}{(x^2+y^2+z^2)^{\frac{-5}{2}}}, \frac{2y}{(x^2+y^2+z^2)^{\frac{-5}{2}}}, \frac{2z}{(x^2+y^2+z^2)^{\frac{-5}{2}}}\right) \\
&= \left(\frac{-3x}{(x^2+y^2+z^2)^{-\frac{5}{2}}}, \frac{-3y}{(x^2+y^2+z^2)^{-\frac{5}{2}}}, \frac{-3z}{(x^2+y^2+z^2)^{-\frac{5}{2}}}\right)
\end{aligned}
$$

2. (a) The curve is given by $\vec{r}(t) = (\cos(t), \sin(t), t)$. Then $\vec{r}'(t) = (-\sin(t), \cos(t), 1)$

$$
\begin{aligned}
\int_C (x+y+z)ds &= \int_{t=0}^{t=2\pi} (\cos(t)+\sin(t)+t)\left\|\vec{r'(t)}\right\| dt \\
&= \int_0^{2\pi} (\cos(t)+\sin(t)+t)\sqrt{\cos^2(t)+\sin^2(t)+1dt} \\
&= \int_0^{2\pi} \sqrt{2}(\cos(t)+\sin(t)+t)dt \\
&= 2\pi\sqrt{2}
\end{aligned}
$$

(b) We let $\vec{r}(t) = (1, 2, t^2)$. Then $\vec{r}'(t) = (0, 0, 2t)$.

$$
\begin{aligned}
\int_C e^{\sqrt{z}} ds &= \int_{t=0}^{t=1} e^{\sqrt{t^2}} \sqrt{0^2+0^2+(2t)^2}dt \\
&= \int_0^1 e^{|t|}\,|2t|\,dt \\
&= \int_0^1 2te^t dt \\
&= 2\Big[(t-1)e^t\Big]_0^1 \\
&= 2
\end{aligned}
$$

3. (a) To parameterise the surface, we let the 2 parameters be x, z, therefore $y = \frac{1}{3}(4 - 2x - z)$. Therefore, $\vec{s}(x, z) = (x, \frac{4}{3} - \frac{2}{3}x - \frac{1}{3}z$. The associated domain is given by $x \in [1, 2], z \in [2, 4]$

(b) We parameterise the surface using $u = x+y+z$ and $v = x-y$. Then we have $u+v = 2x + z = 4 - 3y$ upon re-arranging the given function. Therefore, $y = \frac{1}{3}(4 - u - v)$.

Similarly, we have $u - v = 2y + z$. Back substituting, we obtain $z = u - v - \frac{2}{3}(4 - u - v) \implies z = -\frac{8}{3} + \frac{5}{3}u - \frac{1}{3}v$. Substituting into $u + v = 2x + z$, we get $x = \frac{1}{2}(u + v - (-\frac{8}{3} + \frac{5}{3}u - \frac{1}{3}v)) = \frac{4}{3} - \frac{1}{3}u + \frac{2}{3}v$. Hence, the surface can be described as:

$$\vec{s}(u, v) = \Big(\frac{4}{3} - \frac{1}{3}u + \frac{2}{3}v, \frac{1}{3}(4 - u - v), -\frac{8}{3} + \frac{5}{3}u - \frac{1}{3}v\Big)$$

This has the domain $u \in [0, 7], v \in [2, 4]$

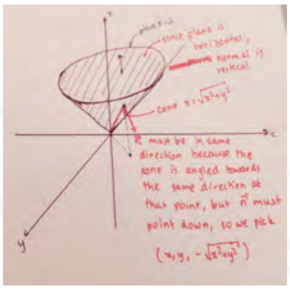
(c) We use a cylindrical co-ordinate representation, with $x = r\cos\theta, y = r\sin\theta, z = 4 - 2r\cos\theta - 3r\sin\theta$ and thus the parametric representation is given by $(r\cos\theta, r\sin\theta, 4 - 2r\cos\theta - 3r\sin\theta)$. This has the domain $r \in [0, 2], \theta \in \Big[-\frac{\pi}{2}, \frac{\pi}{2}\Big]$ since we need the right half of the xy plane and the region is a circle of radius 2.

4. (a) The boundary is given by a conical surface which has the equation $z = \sqrt{x^2 + y^2}$. Consider a cylindrical co-ordinate parameterisation. Then $x = u\cos(v), y = u\sin(v)$ and substituting, we get $z = u$ since $u \geq 0$. The parameteric representation of the top circle is given as $(0, 0, 2)$. Therefore, the overall surface is given by:

$$\mathbf{x}(u, v) = \begin{cases} (u\cos(v), u\sin(v), u) & 0 \leq z < 2, u \in [0, 2), v \in [0, 2\pi] \\ (u\cos(v), u\sin(v), 2) & z = 2, u = 2, v \in [0, 2\pi] \end{cases}$$

(b) As seen from the diagram, the unit normal clearly has to be directed upwards at $z = 2$, so the vector will be given by $\hat{\mathbf{n}} = (0, 0, 1)$.

For the curved surface of the cone, the unit normal will have to point in the same direction as the corresponding vector in the xy plane, but in the vertically opposite direction. So if we move to the position $(u\cos(v), u\sin(v), u)$, the normal vector is given by $(u\cos(v), u\sin(v), -u)$. To unitise this, we must divide by the length of this vector, which is $\sqrt{u^2\cos^2(v) + u^2\sin^2(v) + u^2} = u\sqrt{2}$. Hence, the unit normal is given by $\left(\frac{1}{\sqrt{2}}\cos(v), \frac{1}{\sqrt{2}}\sin(v), \frac{1}{\sqrt{2}}\right)$



(c) For the top of the cone, the surface vector is given by $g_1(u, v) = (u\cos(v), u\sin(v), 2)$. Hence, we have $\frac{\partial g_1}{\partial u} = (\cos(v), \sin(v), 0)$, $\frac{\partial g_1}{\partial v} = (-u\sin(v), u\cos(v), 0)$. Computing the cross product, we have:

$$\det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos(v) & \sin(v) & 0 \\ -u\sin(v) & u\cos(v) & 0 \end{bmatrix} = 0\hat{i} - 0\hat{j} + u\hat{k}$$

This vector has length u, so the unit normal would be $\frac{1}{u}(0, 0, u) = (0, 0, 1)$.

Similarly, for the lateral boundary surface, we have the vector $g_2(u, v) = (u\cos(v), u\sin(v), u) \implies \frac{\partial g_2}{\partial u} = (\cos(v), \sin(v), 1)$, $\frac{\partial g_2}{\partial v} = (-u\sin(v), u\cos(v), 0)$. Thus, we have:

$$\det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos(v) & \sin(v) & 1 \\ -u\sin(v) & u\cos(v) & 0 \end{bmatrix} = u\cos(v)\hat{i} - (-u)\sin(v)\hat{j} + u\hat{k} = (u\cos(v), u\sin(v), u)$$

This normalizes to be $\left(\frac{1}{\sqrt{2}}\cos(v), \frac{1}{\sqrt{2}}\sin(v), \frac{1}{\sqrt{2}}\right)$.

(d) To compute the surface area of the cone, we evaluate:

$$
\begin{aligned}
\iint_{\partial\Omega} dS &= \iint_{\Omega'} \|\mathbf{n}\|\, dA \\
&\qquad (\Omega \text{ is the solid, } \Omega' \text{ is the region describing restrictions on } u, v) \\
&= \int_{v=0}^{v=2\pi} \int_{u=0}^{u=2} \sqrt{(\cos(v))^2 + (\sin(v))^2 + 1^2} du\, dv \\
&+ \int_{v=0}^{v=2\pi} \int_{u=0}^{u=2} \sqrt{(u\cos(v))^2 + (-u\sin(v))^2 + 0^2} du\, dv \\
&\qquad\qquad (\text{There are 2 distinct surfaces}) \\
&= \int_{v=0}^{v=2\pi} \int_{u=0}^{u=2} \sqrt{2} du\, dv + \int_{v=0}^{v=2\pi} \int_{u=0}^{u=2} \sqrt{1} du\, dv \\
&= 4\pi\sqrt{2} + 4\pi \\
&= 4\pi(\sqrt{2} + 1)
\end{aligned}
$$

5. (a) First parameterising the line between $(0,0,-1),(0,0,1)$, we get:

$$\mathbf{r}(t) = (1-t)(0,0,-1) + t(0,0,1) = (0,0,2t-1), t \in [0,1]$$

This further implies that:

$$\mathbf{r}'(t) = (0,0,2)$$

Therefore:

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_{t=0}^{t=1} (0^2 + 0^2 + (2t-1)^2, -(2t-1), (0+1)) \cdot (0,0,2)dt \\ &= \int_0^1 2dt \\ &= 2\end{aligned}$$

(b) We now parameterise the curve $y^2 + z^2 = 1$ in the plane of $x = 0$, and thus we have $\mathbf{r}(t) = (0, \sin(t), \cos(t)), t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Therefore, we evaluate:

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (0^2 + \cos^2(t) + \sin^2(t), -\sin(t), (\cos(t)+1)) \cdot (0, -\sin(t), \cos(t))dt \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (0 + \sin^2(t) + \cos^2(t) + \cos(t))dt \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + \cos(t))dt \\ &= \pi + 2\end{aligned}$$

So the vector field is not conservative because the integrals across 2 difference paths are not equal to each other. [NOTE: This is generally what you do for disproving statements - provide counterexamples.]

6.

$$
\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{S} &= \iint_R (\sinh(x)\hat{\imath} + \cosh(y)\hat{k}) \cdot (1\hat{\imath} + 2y\hat{\jmath} - 1\hat{k}) dA \\
&\qquad\qquad (\nabla f = \nabla(x + y^2 - z) \text{ is normal to the surface}) \\
&= \int_{x=0}^{x=1} \int_{y=0}^{y=x} \sinh(x) - \cosh(y) dy\, dx \\
&= \int_{x=0}^{x=1} x\sinh(x) - \sinh(x) + \sinh(0) dx \\
&= \Big[x\cosh(x) - \sinh(x) - \cosh(x)\Big]_{x=0}^{x=1} \\
&\qquad\qquad (\text{Integrating by parts with } u = x, v' = \sinh(x)) \\
&= \cosh(1) - \sinh(1) - \cosh(1) - (0 - \sinh(0) - \cosh(0)) \\
&= 1 - \sinh(1)
\end{aligned}
$$

7.

$$
\begin{aligned}
\iint_S \mathbf{F} \cdot \hat{n} dS &= \iint_R \mathbf{F} \cdot \mathbf{n} dA \\
&= \iint_R (y, -x, z) \cdot (2x, 2y, 1) dA \\
&= \iint_R z dA \\
&= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} (4 - r^2) r dr\, d\theta \\
&\qquad (\text{Using polar co-ordinates, because the region of integration is circular}) \\
&= 2\pi\Big[2r^2 - \frac{1}{4}r^4\Big]_{r=0}^{r=2} \\
&= 2\pi \times 4 \\
&= 8\pi
\end{aligned}
$$

8. Since the vector field $\mathbf{F} = (x^2 - 2xy, x^2 + 3)$ is clearly differentiable over $\mathbb{R}^2$, we can use Green's Theorem:

$$
\begin{aligned}
\oint_C (x^2 - 2xy)dx + (x^2+3)dy &= \iint_R \frac{\partial}{\partial x}(x^2+3) - \frac{\partial}{\partial y}(x^2 - 2xy)dA \\
&= \int_{x=0}^{x=2} \int_{y=-\sqrt{8x}}^{y=\sqrt{8x}} 2x - (-2x)dy\,dx \\
&= \int_{x=0}^{x=2} 4x \cdot 2\sqrt{8x}dx \\
&= 16\sqrt{2} \int_0^2 x^{\frac{3}{2}}\,dx \\
&= 16\sqrt{2}\frac{2}{5}\left[x^{\frac{5}{2}}\right]_0^2 \\
&= \frac{32\sqrt{2}}{5} \cdot 4\sqrt{2} \\
&= \frac{256}{5}
\end{aligned}
$$

9.

$$
\begin{aligned}
\frac{1}{2}\oint_{\partial\Omega}(xdy - ydx) &= \frac{1}{2}\iint_R \frac{\partial}{\partial x}x - \frac{\partial}{\partial y}(-y)dA \\
&\quad \text{(Be careful of what the integral was, } xdy - ydx\text{, not } xdx - ydy\text{)} \\
&= \frac{1}{2}\iint_R 2dA \\
&= \iint_R dA \\
&= \text{area}(\Omega) && \text{(By definition)}
\end{aligned}
$$

10. Since the vector field is differentiable (because it consists only of polynomial functions) over W and W is a closed and bounded box, we have the following by Divergence Theorem,

$$\begin{aligned}
\iint_{\partial W} \mathbf{F} \cdot d\mathbf{S} &= \iiint_W \nabla \cdot F dV \\
&= \int_{x=0}^{x=1} \int_{y=0}^{y=2} \int_{z=0}^{z=4} \frac{\partial}{\partial x} xy + \frac{\partial}{\partial y} z^3 + \frac{\partial}{\partial z} y^2 dV \\
&= \int_{x=0}^{x=1} \int_{y=0}^{y=2} \int_{z=0}^{z=4} y dV \\
&= 1 \cdot 4 \cdot \int_0^2 y dy \\
&= 4 \cdot 2 \\
&= 8
\end{aligned}$$

11. Since the vector field is differentiable on all of $\mathbb{R}^3$, and Ω is a closed and bounded cylinder given by the equations $z = 0, z = 1, x^2 + y^2 = 1$, we can apply Divergence Theorem.

$$\begin{aligned}
\iint_S \mathbf{F} \cdot \hat{\mathbf{n}} ds &= \iint_S \mathbf{F} \cdot d\mathbf{s} \\
&= \iiint_\Omega \nabla \cdot \mathbf{F} \cdot dV \\
&= \int_{z=0}^{z=1} \int_{x=-1}^{x=1} \int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} \nabla \cdot (x, -y, z^2 - 1) dV \\
&= \int_{z=0}^{z=1} \int_{x=-1}^{x=1} \int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} (1 - 1 + 2z) dy\, dx\, dz \\
&= \int_{z=0}^{z=1} \int_{x=-1}^{x=1} \int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} 2z dy\, dx\, dz \\
&= \int_{z=0}^{z=1} \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} 2zr dr\, d\theta\, dz \\
&= \int_{z=0}^{z=1} \int_{\theta=0}^{\theta=2\pi} z \left[r^2 \right]_{r=0}^{r=1} d\theta\, dz \\
&= 2\pi \cdot \int_0^1 z dz \\
&= \pi
\end{aligned}$$

12. Method 1: Brute Force

First, we compute $\nabla \times \mathbf{F}$

$$\det \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ 2x-y & -yx^2 & -y^2z \end{bmatrix} = \hat{\imath} \det \begin{bmatrix} \partial_y & \partial_z \\ -yx^2 & -y^2z \end{bmatrix} - \hat{\jmath} \det \begin{bmatrix} \partial_x & \partial_z \\ 2x-y & -y^2z \end{bmatrix}$$
$$+ \hat{k} \det \begin{bmatrix} \partial_x & \partial_y \\ 2x-y & -yx^2 \end{bmatrix}$$
$$= \hat{\imath}(-2yz - 0) - \hat{\jmath}(0) + \hat{k}(-2xy - (-1))$$
$$= (-2yz, 0, 1 - 2xy)$$

We shall use Spherical co-ordinates to evaluate this integral. Let $x = \cos(\theta)\sin(\varphi), y = \sin(\theta)\sin(\varphi), z = \cos(\varphi)$ since $\rho = 1$. Then $\mathbf{r}(\theta, \varphi) = (\cos(\theta)\sin(\varphi), \sin(\theta)\sin(\varphi), \cos(\varphi))$.

$$\mathbf{r}_\theta = (-\sin(\theta)\sin(\varphi), \cos(\theta)\sin(\varphi), 0)$$

$$\mathbf{r}_\varphi = (\cos(\theta)\cos(\varphi), \sin(\theta)\cos(\varphi), -\sin(\varphi))$$

Thus, we set

$$\mathbf{n} = \mathbf{r}_\theta \times \mathbf{r}_\varphi = (-\cos(\theta)\sin^2(\varphi), -\sin(\theta)\sin^2(\varphi), -\sin(\varphi)\cos(\varphi))$$

But this is an inward pointing normal vector (*test by plugging in a value,* $\varphi = \frac{\pi}{2} \implies \mathbf{n} = (-\cos(\theta), -\sin(\theta), 0)$ *which points towards the origin*), so we reset

$$\mathbf{n} = (\cos(\theta)\sin^2(\varphi), \sin(\theta)\sin^2(\varphi), \sin(\varphi)\cos(\varphi))$$

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{s} = \iint_{S'} (-2yz, 0, 1-2xy) \cdot \mathbf{n} dA$$

(Where S' is the area projected by S onto the xy plane)

$$= \iint_{S'} (-2yz, 0, 1-2xy) \cdot \mathbf{n} dA$$

$$= \iint_{S'} \begin{bmatrix} -2\sin(\theta)\sin(\varphi)\cos(\varphi) \\ 0 \\ 1-2\sin(\theta)\cos(\theta)\sin^2(\varphi) \end{bmatrix} \cdot \begin{bmatrix} \cos(\theta)\sin^2(\varphi) \\ \sin(\theta)\sin^2(\varphi) \\ \sin(\varphi)\cos(\varphi) \end{bmatrix} dA$$

$$= \int_{\varphi=0}^{\varphi=\frac{\pi}{2}} \int_{\theta=0}^{\theta=2\pi} -4\sin(\theta)\cos(\theta)\cos(\varphi)\sin^3(\varphi) + \sin(\varphi)\cos(\varphi) d\theta\, d\varphi$$

$$= \int_{\varphi=0}^{\varphi=\frac{\pi}{2}} 0 + \frac{1}{2}\sin(2\varphi) \cdot 2\pi d\varphi$$

($\sin(\theta)\cos(\theta) = \frac{1}{2}\sin(2\theta)$ has odd symmetry about π)

$$= \pi \int_0^{\frac{\pi}{2}} \sin(2\varphi) d\varphi$$

$$= \pi$$

NOTE: You don't need the Jacobian because the change of variables is done originally in the parameterisation, and the normal vector already takes this into account. So don't multiply again by $\rho^2 \sin(\varphi)$. S' already describes the new region in terms of the parameters θ, φ.

Method 2: Stokes' Theorem

Since the surface is closed and bounded, and the vector field is clearly continuous and differentiable over $\mathbb{R}^3$, we can apply Stokes' Theorem.

The boundary curve of S is given by $\mathbf{r}(t) = (\cos(t), \sin(t), 0)$ since it is a circle of radius 1 in the plane $z = 0$.

Therefore, $\mathbf{r}'(t) = (-\sin(t), \cos(t), 0)$

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{s} = \int_C (2x - y, -yx^2, -yz^2) \cdot d\mathbf{r}$$

(Where $\mathbf{r}$ parameterises C, the curve bounding S)

$$\begin{aligned}
&= \int_{t=0}^{t=2\pi} (2\cos(t) - \sin(t), -\sin(t)\cos^2(t), 0) \cdot (-\sin(t), \cos(t), 0)dt \\
&= \int_0^{2\pi} (-\sin(t)(2\cos(t) - \sin(t)) - \sin(t)\cos^3(t))dt \\
&= \int_0^{2\pi} -\sin(2t) + \sin^2(t) - \sin(t)\cos^3(t)dt \\
&= \int_0^{2\pi} \sin^2(t)dt \quad \text{(The other 2 functions have odd symmetry about } \pi) \\
&= \pi
\end{aligned}$$

13. Let S denote the surface of the plane $z = y + 1$ such that $x^2 + y^2 \leq 1$ Since the surface in consideration is closed and bounded, and the vector field is continuous and differentiable (as it consists of continuous elementary functions), we may apply Stokes' Theorem.

Parameterise $\mathbf{s}(u, v) = (u\cos(v), u\sin(v), u\sin(v) + 1), u \in [0, 1], v \in [0, 2\pi]$.

Then we have:

$$\begin{aligned}
\mathbf{n} &= \mathbf{s}_u \times \mathbf{s}_v \\
&= (\cos(v), \sin(v), \sin(v)) \times (-u\sin(v), u\cos(v), u\cos(v)) \\
&= \begin{bmatrix} \sin(v)u\cos(v) - \sin(v)u\cos(v) \\ -(u\cos(v)\cos(v) - -u\sin(v)\sin(v)) \\ u\cos(v)\cos(v) - (-u\sin(v)\sin(v)) \end{bmatrix} \\
&= (0, -u, u)
\end{aligned}$$

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{s} &= \int_{\partial S} \nabla \times (4z + x^2, -2x + 3y^5, 2x^2 + 5\sin(z)) \cdot d\mathbf{s} \\
&= \iint_{\S'} \begin{bmatrix} \partial_y(2x^2 + 5\sin(z)) - \partial_z(-2x + 3y^5) \\ -\partial_x(2x^2 + 5\sin(z)) + \partial_z(4z + x^2) \\ \partial_x(-2x + 3y^5) - \partial_y(4z + x^2 \end{bmatrix} \cdot \mathbf{n} dA \\
&= \iint_{S'} (0, -4x + 4, -2) \cdot \mathbf{n} dA \\
&= \int_{v=0}^{v=2\pi} \int_{u=0}^{u=1} (0, -4u\cos(v) + 4, -2) \cdot (0, -u, u) du\, dv \\
&= \int_{v=0}^{v=2\pi} \int_{u=0}^{u=1} (4u\cos(v) - 4u - 2u) du\, dv \\
&= 2\pi \int_0^1 -6u du \\
&= -6\pi
\end{aligned}$$

14. (a) To verify that it forms a right handed system, we must prove the basis vectors in each direction correspond to the next quantity in line.

$$\mathbf{x} = ((a + w\cos(\varphi))\cos(\theta), (a + w\cos(\psi))\sin(\theta), w\sin(\varphi))$$

Then, we must have:

$$\mathbf{b}_w = \frac{\partial}{\partial w}\mathbf{x} = (\cos(\psi)\cos(\theta), \cos(\psi)\sin(\theta), \sin(\psi))$$

$$\mathbf{b}_\theta = \frac{\partial}{\partial \theta}\mathbf{x} = (-(a + w\cos(\psi))\sin(\theta), (a + w\cos(\psi))\cos(\theta), 0)$$

$$\mathbf{b}_\psi = \frac{\partial}{\partial \psi}\mathbf{x} = (-w\sin(\psi)\cos(\theta), -w\sin(\psi)\sin(\theta), w\cos(\psi))$$

The scaling factor in each case is the norm of each vector. Hence:

$$h_w = \|\mathbf{b}_w\| = 1, h_\theta = a + w\cos(\psi), h_\psi = w$$

So the standard basis vectors in the toroidal co-ordinate system is given by:

$$\mathbf{e}_w = \begin{bmatrix} \cos(\psi)\cos(\theta) \\ \cos(\psi)\sin(\theta) \\ \sin(\psi) \end{bmatrix}, \mathbf{e}_\theta = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \\ 0 \end{bmatrix}, \mathbf{e}_\psi = \begin{bmatrix} -\sin(\psi)\cos(\theta) \\ -\sin(\psi)\sin(\theta) \\ \cos(\psi) \end{bmatrix}$$

To verify it is right handed, we must prove that

$$\mathbf{e}_w \times \mathbf{e}_\theta = \mathbf{e}_\psi, \mathbf{e}_\theta \times \mathbf{e}_\psi = \mathbf{e}_w, \mathbf{e}_\psi \times \mathbf{e}_w = \mathbf{e}_\theta]$$

(b) When $\theta = \frac{\pi}{2}, w = b$, we obtain a circle given by $y = a + b\cos(\psi), z = b\sin(\psi)$ in the plane $x = 0$.

(c) When $\theta = \psi, w = b$, we obtain $x = (a + b\cos(\theta))\cos(\theta), y = (a + b\cos(\theta))\sin(\theta), z = b\sin(\theta)$.

To get a good grasp of what the shape looks like, we plug in some values. $\theta = 0, \theta = 2\pi \implies x = a+b, y = 0, z = 0$ so it starts and ends at the same point. If we fix θ, we obtain a circle, so means by varying ψ, we obtain some kind of circle perpendicular to the circle when θ is fixed. So if both vary at the same time, it traces out a spiral.

(d) Computing the arc lengths involves the same formulas:

Curve 1: $(w, \theta, \psi) = (b, \frac{\pi}{2}, t)$ where $t \in [0, 2\pi]$

$$\begin{aligned}
\int_C ds &= \int_{t=0}^{t=2\pi} \sqrt{1^2\Big(\frac{dw}{dt}\Big)^2 + (1 + \frac{1}{2}\cos(t))^2\Big(\frac{d\theta}{dt}\Big)^2 + w^2\Big(\frac{d\psi}{dt}\Big)^2} dt \\
&= \int_0^{2\pi} \sqrt{0 + 0 + b^2} dt \\
&= 2\pi \cdot b \\
&= \pi
\end{aligned}$$

Curve 2: $(w, \theta, \psi) = (b, t, t)$ where $t \in [0, 2\pi]$

$$
\begin{aligned}
\int_C ds &= \int_{t=0}^{t=2\pi} \sqrt{1^2\Big(\frac{dw}{dt}\Big)^2 + (1+\frac{1}{2}\cos(t))^2\Big(\frac{d\theta}{dt}\Big)^2 + w^2\Big(\frac{d\psi}{dt}\Big)^2}dt \\
&= \int_0^{2\pi} \sqrt{0 + (a + b\cos(t))^2 + b^2}dt \\
&= \int_0^{2\pi} \sqrt{\frac{1}{4} + (1 + \frac{1}{2}\cos(t))^2}dt \\
&= \int_0^{2\pi} \frac{1}{2}\sqrt{1 + (2+\cos(t))^2}dt \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}\Big[\sqrt{1+(2+\cos(t))^2} + \sqrt{1+(2-\cos(t))^2}\Big]dt \\
& \qquad \text{(Due to symmetry of cos, with } \cos(\pi - x) = -\cos(x)) \\
&= \int_0^{\frac{\pi}{2}} \Big[\sqrt{1+(2+\cos(t))^2} + \sqrt{1+(2-\cos(t))^2}\Big]dt \qquad (\cos(-x) = \cos(x)) \\
&\approx 7.10 \qquad \text{(WolframAlpha said so)}
\end{aligned}
$$

Seminar 2 Second Half (Fourier Series and Analysis)

Example 1.

From the definition of f, we know it is 2-periodic, so $L = 1$. Further, it is odd, so we can ignore the coefficients of the even functions. That is, $a_0[f] = a_k[f] = 0$. Then, we calculate the coefficients using the standard formula.

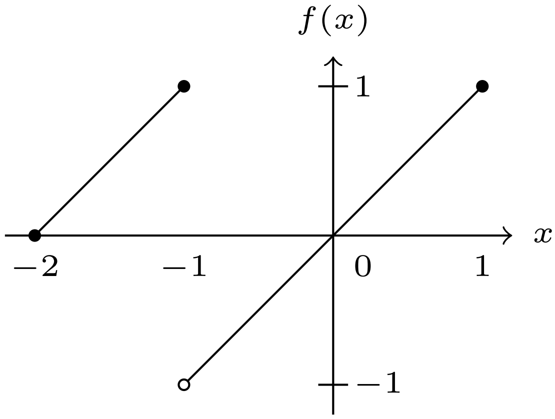
$$\begin{aligned}
b_k[f] &= \frac{2}{L}\int_0^L f(x)\sin\frac{k\pi x}{L}\mathrm{d}x \\
&= 2\int_0^1 x\sin k\pi x\mathrm{d}x \\
&= 2\left(\left[x\frac{-\cos k\pi x}{k\pi}\right]_0^1 - \int_0^1 \frac{-\cos k\pi x}{k\pi}\mathrm{d}x\right) \\
&= -2\frac{\cos k\pi}{k\pi} + \frac{2}{k\pi}\left[\frac{\sin k\pi x}{k\pi}\right]_0^1 \\
&= 2\frac{(-1)^{k+1}}{k\pi},
\end{aligned}$$

as $\sin k\pi = 0$ and $\cos k\pi = (-1)^k$ for all $k \in \mathbb{Z}$. Then, writing the series, we have

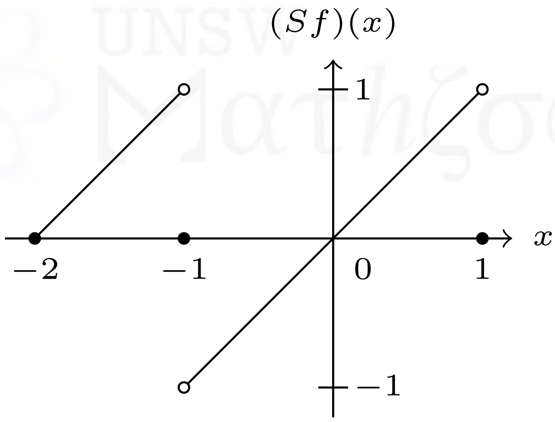
$$(Sf)(x) = \sum_{k=1}^{\infty} 2\frac{(-1)^{k+1}}{k\pi}\sin k\pi x = \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\sin k\pi x.$$

Example 2.

To draw the graph of Sf for $-2 \leq x \leq 1$ we must first consider the graph of f over the same domain:



So the Fourier series graph will be identical everywhere except the jump discontinuities at $x = -1$ and $x = 1$ (extend the graph a bit more to see the discontinuity at $x = 1$). At these discontinuities, the Fourier series will approach the average of the function coming from both sides. So, in this case, it will approach $\frac{1+-1}{2} = 0$. Putting this all together in a graph, we get:



Take special note of the point at $x = 1$. Although we don't draw anything after $x = 1$ we need to make sure to draw the single point at $x = 1$.

Example 3.

(a) To evaluate the Fourier series, we go directly to the formulas:

$$a_0 = \frac{1}{1}\int_{-1}^{1} |x| dx = 1$$

Using area underneath $|x|$ over $-1 \leq x \leq 1$.

$$a_k = \frac{1}{1}\int_{-1}^{1} |x| \cos\left(\frac{k\pi x}{1}\right) dx = 2\int_{0}^{1} x\cos(k\pi x) = \frac{2(\cos(k\pi) - 1)}{k^2\pi^2}$$

with the 2nd equality obtained using the fact that $|x|\cos(k\pi x)$ is an even function, and the last equality evaluated using Integration by Parts.

$$b_k = \frac{1}{1}\int_{-1}^{1} |x| \sin(k\pi x) dx = 0$$

since the function inside the integral is odd. This yields the Fourier series:

$$Sf(x) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{2(\cos(k\pi) - 1)}{k^2\pi^2}\cos(k\pi x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{-4}{\pi^2(2n-1)^2}\cos((2n-1)\pi x)$$

after recognising that $\cos(k\pi) = (-1)^k$, so whenever k is even, the term evaluates to 0, and when odd (that is, $k = 2n - 1$ for some $n = 1, 2, ...$, the term evaluates to $-\frac{4}{(2n-1)^2\pi^2}\cos((2n-1)\pi x)$.

(b)

$$f(0) = 0 \implies \sum_{n=1}^{\infty} \frac{4}{(2n-1)^2\pi^2}\cos(0) = \frac{1}{2} \implies \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

(c) By Parseval's identity:

$$\frac{1}{1}\int_{-1}^{1} |x|^2 dx = \frac{1^2}{2} + \sum_{n=1}^{\infty} \frac{16}{\pi^4(2n-1)^4} \implies \frac{1}{6} = \sum_{k \text{ odd}} \frac{16}{\pi^4 k^4}$$

and cross multiply obtains the result:

$$\sum_{k \text{ odd}} \frac{1}{k^4} = \frac{\pi^4}{96}.$$

Example 4.

Proving pointwise convergence is simple. Simply evaluating the limit we find

$$\lim_{\substack{n\to\infty \\ x\in[0,1)}} x^n = 0.$$

Thus, the function sequence converges to 0 on $[0,1)$.

To disprove uniform convergence, we can start by assuming the condition of uniform convergence is satisfied, and derive a contradiction. In this case, we can take $\epsilon = \frac{1}{2}$, and state

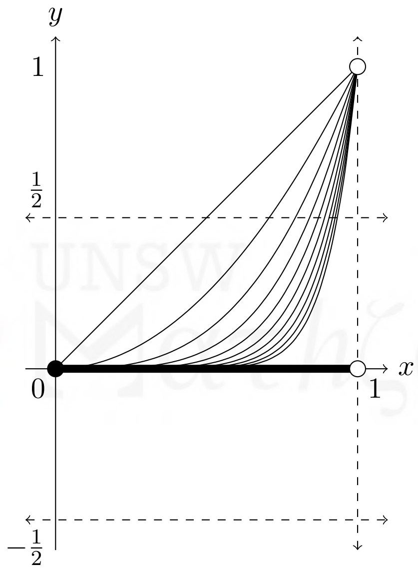
$$|f_n(x) - f(x)| = x^n < \frac{1}{2}$$

for sufficiently large $n \geq N$ and every $x \in [0,1)$. But then we must have

$$x < \frac{1}{\sqrt[n]{2}} \leq \frac{1}{\sqrt[N]{2}}.$$

So the condition is only satisfied for $x < \frac{1}{\sqrt[N]{2}} < 1$. But we assumed the condition was true for all $x \in [0,1)$, so we derive a contradiction, and thus the function sequence does not converge uniformly on $[0,1)$.

To explain why we chose $\epsilon = \frac{1}{2}$, refer to the following graph:



The solid black line along the x-axis is the function f we're taking as the sequence limit. In this case, it's just the zero function. The horizontal dashed lines are $f + \frac{1}{2}$ and $f - \frac{1}{2}$ (if f curved, they would too). Now each of the other graphs are $f_n(x) = x^n$ for varying $n \geq 1$. As we can see, no matter how large n is, there is always some point on f_n that is outside of the box, we just have to take some value closer and closer to $x = 1$. We could take $\epsilon \in (0,1)$ for the contradiction, $\epsilon = \frac{1}{2}$ is just a nice number.

Geometrically, we see that a sequence of functions converges uniformly if we can take $f + \epsilon$ and $f - \epsilon$ and "squish" the sequence between them, as $\epsilon \to 0^+$.

Example 5.

We can use Weierstrass M-test to prove series converge uniformly. Applying the test, we have

$$\left|\frac{\cos nx}{n^2 + x^2}\right| \leq \frac{1}{n^2 + x^2} \leq \frac{1}{n^2}.$$

Then, by p-test, we know the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges. Thus, by Weierstass M-test, the function

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n^2 + x^2}$$

converges uniformly for all $x \in \mathbb{R}$.

Example 6.

Again, applying Weierstrass M-test, we find

$$\sum_{k \text{ odd}} \left| \frac{-4}{k^2 \pi^2} \right| + |0| \leq \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2},$$

which converges by p-test, so the corresponding Fourier series converges uniformly to f (as it's the Fourier series of f) on $\mathbb{R}$.

Now, if a sequence of continuous functions converges uniformly to a function f, then f must be continuous. Since all the partial Fourier series $S_n f$ are continuous, this means f must be continuous also. This is true for any uniformly convergent Fourier series.