MATH1231/41 Algebra Part 1 Revision Session 2019 T2 Solutions

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We cannot guarantee that our answers are correct - please notify us of any errors or typos at unswmathsoc@gmail.com, or on our Facebook page. There are sometimes multiple methods of solving the same question. Remember that in the real class test, you will be expected to explain your steps and working out.

Example 1: 1231 2015 Q1.v

Prove that

$$S = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : x_1 - 2x_2 + 4x_3 = 0 \right\}$$

is a subspace of $\mathbb{R}^3$.

Clearly, $S \subseteq \mathbb{R}^3$ where $\mathbb{R}^3$ is a known vector space. Since $0 - 2(0) + 4(0) = 0$, $\mathbf{0} \in S$. So S contains a zero element.

Now suppose that $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in S$ and $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \in S$. Then

$$x_1 - 2x_2 + 4x_3 = 0, \tag{1}$$
$$y_1 - 2y_2 + 4y_3 = 0. \tag{2}$$

$(1)+(2)$ gives us $(x_1+y_1)-2(x_2+y_2)+4(x_3+y_3) = 0$, i.e. $\begin{pmatrix} x_1+y_1 \\ x_2+y_2 \\ x_3+y_3 \end{pmatrix} \in S$. Hence $(\mathbf{x}+\mathbf{y}) \in S$ and so S is closed under vector addition.

If $\lambda \in \mathbb{R}$ and $\mathbf{x} \in S$ then $\lambda \times (1)$ gives us $\lambda x_1 - 2\lambda x_2 + 4\lambda x_3 = 0$, i.e. $\begin{pmatrix} \lambda x_1 \\ \lambda x_2 \\ \lambda x_3 \end{pmatrix} \in S$. Hence $\lambda\mathbf{x} \in S$ so S is closed under scalar multiplication.

Since S is a subset of $\mathbb{R}^3$ and contains a zero element, is closed under vector addition, and is closed under scalar multiplication, then by the Subspace Theorem S is a subspace of $\mathbb{R}^3$.

Example 2: 1231 2013 Q1.i

Let

$$S = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : x_1^3 + x_2^3 + x_3^3 = 0 \right\}.$$

a) Prove that S is closed under scalar multiplication.

*b) Show that S is **not** a subspace of $\mathbb{R}^3$.*

Suppose that $\mathbf{x} \in S$ and $\lambda \in \mathbb{R}$. Then we have

$$x_1^3 + x_2^3 + x_3^3 = 0.$$

Multiplying by λ^3,

$$(\lambda x_1)^3 + (\lambda x_2)^3 + (\lambda x_3)^3 = 0.$$

Hence $\lambda\mathbf{x} \in S$, i.e. S is closed under scalar multiplication.

Note that $\mathbf{0}$ is an element of S, so to prove that S is not a subspace we will show that S is not closed under vector addition. Take $\mathbf{x} = (1, -1, 0)^T$ and $\mathbf{y} = (-2, 0, 2)^T$. Clearly $\mathbf{x}, \mathbf{y} \in S$, but

$$\mathbf{x} + \mathbf{y} = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} \notin S.$$

Hence S is not closed under vector addition. By the Subspace Theorem, S is not a subspace of $\mathbb{R}^3$.

Example 3: 1231 2015 Q1.vi

Consider the vectors in $\mathbb{R}^3$*,*

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \boldsymbol{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}, \boldsymbol{v}_3 = \begin{pmatrix} 2 \\ -3 \\ 3 \end{pmatrix}, \boldsymbol{b} = \begin{pmatrix} -1 \\ 6 \\ 3 \end{pmatrix}.$$

Prove that $\boldsymbol{b} \in \mathit{span}(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3)$*.*

We want examine the nature of solutions $(x_1, x_2, x_3)^T$ to

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{b}.$$

$\mathbf{b} \in \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ if there is at least one solution. Notice that our equation can be written in the form $A\mathbf{x} = \mathbf{b}$:

$$\begin{pmatrix} 1 & 1 & 2 \\ -1 & 2 & -3 \\ 2 & 5 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \\ 3 \end{pmatrix}.$$

Applying Gaussian elimination, we have

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ -1 & 2 & -3 & 6 \\ 2 & 5 & 3 & 3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & 3 & -1 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Clearly there are infinitely many solutions, so $\mathbf{b} \in \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

Example 4

Let $\mathbb{P}_2$ *be the vector space of all real polynomials of degree at most 2. Find three polynomials* f_1, f_2, f_3 *in* $\mathbb{P}_2$ *such that* $f_i(0) = 1$ *for* $i = 1, 2, 3$ *and* $\{f_1, f_2, f_3\}$ *is linearly independent.*

We want three polynomials such that each has a different span. The easiest way to do this is to consider the set of functions

$$
\begin{aligned}
f_1 &= a_1, \\
f_2 &= a_2 + b_2 x, \\
f_3 &= a_3 + b_3 x + c_3 x^2.
\end{aligned}
$$

Since $f_i(0) = 1$ then $a_i = 1$. The other coefficients are arbitrary constants, so set all other constants to 1:

$$
\begin{aligned}
f_1 &= 1, \\
f_2 &= 1 + x, \\
f_3 &= 1 + x + x^2.
\end{aligned}
$$

Example 5: 1241 2016 Q3.iii

The field $\mathbb{F} = GF(4)$ has elements $\{0, 1, \alpha, \beta\}$ with addition and multiplication defined by the following tables. For the vectors

$+$	0	1	α	β
0	0	1	α	β
1	1	0	β	α
α	α	β	0	1
β	β	α	1	0

$\times$	0	1	α	β
0	0	0	0	0
1	0	1	α	β
α	0	α	β	1
β	0	β	1	α

$$
\boldsymbol{b}_1 = \begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix}, \boldsymbol{b}_2 = \begin{pmatrix} \beta \\ 1 \\ 1 \end{pmatrix}, \boldsymbol{b}_3 = \begin{pmatrix} 1 \\ 0 \\ \alpha \end{pmatrix},
$$

a) show that $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3\}$ is a basis for $\mathbb{F}^3$;
b) explain why $\{\boldsymbol{b}_1, \boldsymbol{b}_1 + \boldsymbol{b}_2, \boldsymbol{b}_2 + \boldsymbol{b}_3, \boldsymbol{b}_3\}$ is a spanning set but not a basis for $\mathbb{F}^3$.

First we prove two important results. Suppose $\mathbf{x} \in \mathbb{F}^3$. Since $a + a = 0\ \forall a \in \mathbb{F}$, then

$$
\mathbf{x} + \mathbf{x} = \mathbf{0} \quad \forall \mathbf{x} \in \mathbb{F}^3. \tag{*}
$$

Also, since $a + 0 = a\ \forall a \in \mathbb{F}$, then

$$
\mathbf{x} + \mathbf{0} = \mathbf{x} \quad \forall \mathbf{x} \in \mathbb{F}^3. \tag{**}
$$

Now, note that $\dim \mathbb{F}^3 = 3$. If we can show that $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is a linearly independent set then we can apply the Dimension Theorem. For $x, y, z \in \mathbb{F}$, if

$$x\mathbf{b}_1 + y\mathbf{b}_2 + z\mathbf{b}_3 = \mathbf{0}$$

then we can represent this in the form $A\mathbf{x} = \mathbf{b}$:

$$\begin{pmatrix} 1 & \beta & 1 \\ \alpha & 1 & 0 \\ \beta & 1 & \alpha \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Carefully applying Gaussian elimination:

$$\begin{aligned}
\left(\begin{array}{ccc|c} 1 & \beta & 1 & 0 \\ \alpha & 1 & 0 & 0 \\ \beta & 1 & \alpha & 0 \end{array}\right) &\xrightarrow{R1=\alpha R1} \left(\begin{array}{ccc|c} \alpha & 1 & \alpha & 0 \\ \alpha & 1 & 0 & 0 \\ \beta & 1 & \alpha & 0 \end{array}\right) \xrightarrow{R2=R2+R1} \left(\begin{array}{ccc|c} \alpha & 1 & \alpha & 0 \\ 0 & 0 & \alpha & 0 \\ \beta & 1 & \alpha & 0 \end{array}\right) \\
&\xrightarrow{R3=\beta R3} \left(\begin{array}{ccc|c} \alpha & 1 & \alpha & 0 \\ 0 & 0 & \alpha & 0 \\ \alpha & \beta & 1 & 0 \end{array}\right) \xrightarrow{R2\leftrightarrow R3} \left(\begin{array}{ccc|c} \alpha & 1 & \alpha & 0 \\ \alpha & \beta & 1 & 0 \\ 0 & 0 & \alpha & 0 \end{array}\right) \\
&\xrightarrow{R2=R2+R1} \left(\begin{array}{ccc|c} \alpha & 1 & \alpha & 0 \\ 0 & \alpha & \beta & 0 \\ 0 & 0 & \alpha & 0 \end{array}\right).
\end{aligned}$$

Clearly, our solution $(x, y, z)^T = (0, 0, 0)^T$. So the only solution is the trivial solution, hence $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is a linearly independent set. Therefore $\dim(\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}) = \dim(\mathbb{F}^3) = 3$, and so by the Dimension Theorem $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is a basis for $\mathbb{F}^3$.

Now consider the second set of vectors in $\mathbb{F}^3$.

$$\begin{aligned}
(\mathbf{b}_1 + \mathbf{b}_2) + (\mathbf{b}_2 + \mathbf{b}_3) + (\mathbf{b}_3) &= \mathbf{b}_1 + (\mathbf{b}_2 + \mathbf{b}_2) + (\mathbf{b}_3 + \mathbf{b}_3) && \text{(associative law)} \\
&= \mathbf{b}_1 + \mathbf{0} + \mathbf{0} && \text{(Using (*))} \\
&= \mathbf{b}_1. && \text{(Using (**))}
\end{aligned}$$

Since we have written $\mathbf{b}_1$ as a linear combination of $\mathbf{b}_1 + \mathbf{b}_2$, $\mathbf{b}_2 + \mathbf{b}_3$ and $\mathbf{b}_3$, then

$$\{\mathbf{b}_1, \mathbf{b}_1 + \mathbf{b}_2, \mathbf{b}_2 + \mathbf{b}_3, \mathbf{b}_3\}$$

is a linearly dependent set. Hence the set cannot be a basis for $\mathbb{F}^3$. However since

$$\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3 \in \mathrm{span}(\{\mathbf{b}_1, \mathbf{b}_1 + \mathbf{b}_2, \mathbf{b}_2 + \mathbf{b}_3, \mathbf{b}_3\})$$

and $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is a spanning set for $\mathbb{F}^3$, then $\{\mathbf{b}_1, \mathbf{b}_1 + \mathbf{b}_2, \mathbf{b}_2 + \mathbf{b}_3, \mathbf{b}_3\}$ is a spanning set for $\mathbb{F}^3$.

Example 6: 1241 2016 Q3.iii

Consider the field $\mathbb{F} = GF(4)$, as defined in the previous example. Let $\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3$ be the vectors from the previous example. Set

$$\boldsymbol{v} = \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix}.$$

c) Find the coordinate vector of $\boldsymbol{v}$ with respect to the ordered basis $B = \{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3\}$.

The coordinate vector $\mathbf{x} = (x, y, z)^T$ of $\mathbf{v}$ will satisfy the equation

$$x\mathbf{b}_1 + y\mathbf{b}_2 + z\mathbf{b}_3 = \mathbf{z}.$$

Writing this in the form $A\mathbf{x} = \mathbf{b}$, we have

$$\begin{pmatrix} 1 & \beta & 1 \\ \alpha & 1 & 0 \\ \beta & 1 & \alpha \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix}.$$

Carefully applying Gaussian elimination:

$$\left(\begin{array}{ccc|c} 1 & \beta & 1 & \alpha \\ \alpha & 1 & 0 & 0 \\ \beta & 1 & \alpha & 0 \end{array}\right) \xrightarrow{R1=\alpha R1} \left(\begin{array}{ccc|c} \alpha & 1 & \alpha & \beta \\ \alpha & 1 & 0 & 0 \\ \beta & 1 & \alpha & 0 \end{array}\right) \xrightarrow{R2=R2+R1} \left(\begin{array}{ccc|c} \alpha & 1 & \alpha & \beta \\ 0 & 0 & \alpha & \beta \\ \beta & 1 & \alpha & 0 \end{array}\right)$$

$$\xrightarrow{R3=\beta R3} \left(\begin{array}{ccc|c} \alpha & 1 & \alpha & \beta \\ 0 & 0 & \alpha & \beta \\ \alpha & \beta & 1 & 0 \end{array}\right) \xrightarrow{R2\leftrightarrow R3} \left(\begin{array}{ccc|c} \alpha & 1 & \alpha & \beta \\ \alpha & \beta & 1 & 0 \\ 0 & 0 & \alpha & \beta \end{array}\right)$$

$$\xrightarrow{R2=R2+R1} \left(\begin{array}{ccc|c} \alpha & 1 & \alpha & \beta \\ 0 & \alpha & \beta & \beta \\ 0 & 0 & \alpha & \beta \end{array}\right).$$

From $R3$ we have $z = \alpha$ since $\alpha \times \alpha = \beta$. In $R2$ we have

$$\begin{aligned} \alpha y + \beta \times \alpha &= \beta \\ \alpha y + 1 &= \beta && (\beta \times \alpha = 1) \\ \alpha y + 1 + 1 &= \beta + 1 && (\text{Adding 1 to both sides}) \\ \alpha y &= \alpha && (1 + 1 = 0 \text{ and } \beta + 1 = \alpha) \\ y &= 1. && (\text{Since } \alpha \times 1 = \alpha) \end{aligned}$$

In $R1$ we have

$$
\begin{aligned}
\alpha x + 1 \times 1 + \alpha \times \alpha &= \beta \\
\alpha x + 1 + \beta &= \beta && (1 \times 1 = 1 \text{ and } \alpha \times \alpha = \beta) \\
\alpha x + (1+\beta) + (1+\beta) &= \beta + 1 + \beta && (\text{Adding } (1+\beta) \text{ to both sides}) \\
\alpha x &= \alpha + \beta && ((1+\beta)+(1+\beta) = 0 \text{ and } \beta + 1 = \alpha) \\
\alpha x &= 1 && (\alpha + \beta = 1) \\
x &= \beta && (\text{Since } \alpha \times \beta = 1)
\end{aligned}
$$

Hence the coordinate vector of $\boldsymbol{v}$ with respect to the basis B is

$$\mathbf{x} = \begin{pmatrix} \beta \\ 1 \\ \alpha \end{pmatrix}.$$

Example 7: 1241 2014 S2 Q3.i

Prove that the function $T : \mathbb{P}(\mathbb{R}) \to \mathbb{R}^2$ defined by

$$T(p) = \begin{pmatrix} p(0) \\ p(1) \end{pmatrix}, \textit{ for all polynomials } p \in \mathbb{P}(\mathbb{R}),$$

is a linear transformation.

For the map T to be linear, we need to show that T preserves addition and scalar multiplication. First consider addition. For any $p, q \in \mathbb{P}(\mathbb{R})$,

$$
\begin{aligned}
T(p+q) &= \begin{pmatrix} (p+q)(0) \\ (p+q)(1) \end{pmatrix} = \begin{pmatrix} p(0)+q(0) \\ p(1)+q(1) \end{pmatrix} = \begin{pmatrix} p(0) \\ p(1) \end{pmatrix} + \begin{pmatrix} q(0) \\ q(1) \end{pmatrix} \\
&= T(p) + T(q).
\end{aligned}
$$

So T preserves addition. Now consider scalar multiplication. For any $p \in \mathbb{P}(\mathbb{R})$ and $\lambda \in \mathbb{R}$,

$$
\begin{aligned}
T(\lambda p) &= \begin{pmatrix} (\lambda p)(0) \\ (\lambda p)(1) \end{pmatrix} = \begin{pmatrix} \lambda p(0) \\ \lambda p(1) \end{pmatrix} = \lambda \begin{pmatrix} p(0) \\ p(1) \end{pmatrix} \\
&= \lambda T(p).
\end{aligned}
$$

Hence T preserves scalar multiplication. Therefore since T preserves addition and scalar multiplication, then T is linear.

Example 8: 1241 2016 Q3.ii

Let V and W be vector spaces, let $T : V \to W$ be a linear transformation, and let $\boldsymbol{v}_1, \boldsymbol{v}_2 ..., \boldsymbol{v}_m$ be vectors in V.
a) Prove that if $T(\boldsymbol{v}_1), T(\boldsymbol{v}_2), ..., T(\boldsymbol{v}_m)$ are linearly independent, then $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_m$ are linearly independent.
b) Suppose that $\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_m$ are linearly independent. Is $T(\boldsymbol{v}_1), T(\boldsymbol{v}_2), ..., T(\boldsymbol{v}_m)$ linearly independent?

Suppose that $T(\mathbf{v}_1), T(\mathbf{v}_2), ..., T(\mathbf{v}_m)$ are linearly independent, and assume $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m$ are linearly dependent. Then for some $i \in \{1, 2, ..., m\}$ and constants λ_j, we have

$$\mathbf{v}_i = \sum_{j \neq i} \lambda_j \mathbf{v}_j.$$

So then

$$\begin{aligned} T(\mathbf{v}_i) &= T\left(\sum_{j\neq i} \lambda_j \mathbf{v}_j\right) \\ &= \sum_{j\neq i} T(\lambda j \mathbf{v}_j) && \text{(Since } T \text{ preserves addition)} \\ &= \sum_{j\neq i} \lambda_j T(\mathbf{v}_j). && \text{(Since } T \text{ preserves scalar multiplication)} \end{aligned}$$

Hence we have a contradiction, since $T(\mathbf{v}_1), T(\mathbf{v}_2), ..., T(\mathbf{v}_m)$ are linearly independent. Hence our assumption is incorrect, i.e. $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m$ are not linearly dependent. So we have proven, by contradiction, that $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m$ are linearly independent.

However, linear independence of $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m$ does not imply linear independence of $T(\mathbf{v}_1), T(\mathbf{v}_2), ..., T(\mathbf{v}_m)$. Consider, for example, the linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T(\mathbf{x}) = \mathbf{0} \quad \forall \mathbf{x} \in \mathbb{R}^2.$$

The standard basis vectors $\mathbf{e}_1, \mathbf{e}_2$ are linearly independent, however $xT(\mathbf{e}_1) + yT(\mathbf{e}_2) = 0$ for any choice of $x, y \in \mathbb{R}$. So $T(\mathbf{e}_1), T(\mathbf{e}_2)$ are not linearly independent. Interestingly enough, part b would be true if T were injective.

Example 9: 1231 2013 Q2,iv

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map which rotates a vector $\boldsymbol{x}$ about the origin through $\frac{\pi}{6}$ anti-clockwise and doubles its length.

a) Show that $T(\boldsymbol{e}_1) = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$*, where* $\boldsymbol{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

b) Find the matrix A such that $T(\boldsymbol{x}) = A\boldsymbol{x}$*. for all* $\boldsymbol{x} \in \mathbb{R}^2$*.*

Since T rotates a vector $(x, y)^T$ anticlockwise by $\frac{\pi}{6}$, we know that for $x > 0$ and $y \geq 0$,

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} L\cos\left(\frac{\pi}{6} + \tan^{-1}\left(\frac{y}{x}\right)\right) \\ \\ L\sin\left(\frac{\pi}{6} + \tan^{-1}\left(\frac{y}{x}\right)\right) \end{pmatrix}.$$

Since T also doubles the length of a vector $(x, y)^T$, then $L = 2\sqrt{x^2 + y^2}$. Hence

$$T\begin{pmatrix} x \\ y \end{pmatrix} = 2\sqrt{x^2 + y^2} \begin{pmatrix} \cos\left(\frac{\pi}{6} + \tan^{-1}\left(\frac{y}{x}\right)\right) \\ \\ \sin\left(\frac{\pi}{6} + \tan^{-1}\left(\frac{y}{x}\right)\right) \end{pmatrix}.$$

So

$$\begin{aligned} T\begin{pmatrix} 1 \\ 0 \end{pmatrix} &= 2\sqrt{1+0} \begin{pmatrix} \cos\left(\frac{\pi}{6} + \tan^{-1} 0\right) \\ \\ \sin\left(\frac{\pi}{6} + \tan^{-1} 0\right) \end{pmatrix} = 2\begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}. \end{aligned}$$

Also,

$$\begin{aligned} T\begin{pmatrix} 0 \\ 1 \end{pmatrix} &= 2\sqrt{0+1} \begin{pmatrix} \cos\left(\frac{\pi}{6} + \frac{\pi}{2}\right) \\ \\ \sin\left(\frac{\pi}{6} + \frac{\pi}{2}\right) \end{pmatrix} = 2\begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix}. \end{aligned}$$

Using the Matrix Representation Theorem, $T(\mathbf{x}) = A\mathbf{x}$ where

$$A = \begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix}.$$

Example 10: 1231 2018 Q1.iv

Consider the matrix $M = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$.

a) Find a basis for $\ker(M)$.

b) Find a basis for $im(M^T)$.

c) Give a geometric description of $\ker(M)$ *and* $im(M)$ *as subspaces of* $\mathbb{R}^2$.

If $\mathbf{x} \in \ker(M)$ then $M\mathbf{x} = \mathbf{0}$. Hence

$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} x + y \\ 2x + 2y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

So $x + y = 0$, i.e. $y = -x$. So

$$\mathbf{x} = x \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Therefore

$$\ker(M) = \text{span}\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\},$$

so a basis for $\ker(M)$ is

$$\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}.$$

Now, consider $\mathbf{y} \in \text{im}(M^T)$. Then

$$\mathbf{y} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2y \\ x + 2y \end{pmatrix} = (x + 2y) \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Therefore

$$\text{im}(M^T) = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\},$$

so a basis for $\text{im}(M^T)$ is

$$\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

The kernel of M is the line in $\mathbb{R}^2$, in the direction $(1, -1)^T$. The image of M is the line in $\mathbb{R}^2$, in the direction $(1, 2)^T$.

Example 11: 1241 2015 Q3.ii

Consider the mapping $T : \mathbb{P}_2 \to \mathbb{P}_3$ defined by

$$T(p)(x) = (x^2 + 1)p'(x) - 2xp(x).$$

Assuming T is linear, find the rank and nullity of T.

Let $p(x) = ax^2 + bx + c$. Then $p'(x) = 2ax + b$, and so

$$\begin{aligned} T(p)(x) &= (x^2 + 1)(2ax + b) - 2x(ax^2 + bx + c) \\ &= -bx^2 + 2(a - c)x + b. \end{aligned}$$

Hence

$$\begin{aligned} T(p) = \begin{pmatrix} -b \\ 2a - 2c \\ b \end{pmatrix} &= a \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + b \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} \\ &= 2(a - c) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}. \end{aligned}$$

So if $q \in \operatorname{im}(T)$ then

$$q = \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

So $\operatorname{rank}(T) = 2$. Since $\dim(\mathbb{P}_2) = 3$ (standard basis is $\{1, x, x^2\}$), then by the Rank-Nullity Theorem, $\operatorname{nullity}(T) = 1$.

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Eigenvalues and Eigenvectors

QUESTION 1. A linear transformation $P : V \to V$ is said to be idempotent if $P(P(\mathbf{v})) = P(\mathbf{v})$ for all $\mathbf{v} \in V$ (in other words $P^2 = P$).

(a) Show that the only possible eigenvalues for an idempotent linear transformation are 0 and 1.

(b) Show that if P is idempotent and P is neither the zero nor the identity transformation on V, then both 0 and 1 are eigenvalues.

Solution:
(a) Suppose λ is an eigenvalue of P, then $P(\mathbf{v}) = \lambda\mathbf{v}$ where $\mathbf{v}$ is nonzero.
By the definition of idempotent,

$$
\begin{aligned}
P(P(\mathbf{v})) &= P(\mathbf{v}) \\
P(\lambda\mathbf{v}) &= \lambda\mathbf{v} && \text{(since } P\mathbf{v} = \lambda\mathbf{v}\text{)} \\
\lambda P(\mathbf{v}) &= \lambda\mathbf{v} && \text{(P is linear)} \\
\lambda(\lambda\mathbf{v}) &= \lambda\mathbf{v} && \text{(Since } P\mathbf{v} = \lambda\mathbf{v}\text{)} \\
\lambda^2\mathbf{v} &= \lambda\mathbf{v} \\
(\lambda^2 - \lambda)\mathbf{v} &= \mathbf{0}.
\end{aligned}
$$

Since $\mathbf{v}$ is nonzero, $\lambda^2 - \lambda = 0$, so $\lambda = 0, 1$.
(b) Since P is not the zero map, there exists a $\mathbf{v} \in V$ with $P\mathbf{v}$ nonzero. So we can write $P(\mathbf{v}) - \mathbf{w}$ where $\mathbf{w} \in V$ and $\mathbf{w}$ nonzero.
Applying P to both sides,

$$
\begin{aligned}
P^2\mathbf{v} &= P(\mathbf{w}) \\
P(\mathbf{v}) &= P(\mathbf{w}) && \text{(from definition of idempotent)} \\
\mathbf{w} &= P(\mathbf{w}) && \text{(Since } P\mathbf{v} = \mathbf{w}\text{)}
\end{aligned}
$$

Thus $P\mathbf{w} = \mathbf{w} = 1 \times \mathbf{w}$. So 1 is an eigenvalue.
Now we prove that 0 is an eigenvalue.
Since P is not the identity map, there exists $\mathbf{u} \in V$ with $P\mathbf{u}$ not equal to $\mathbf{u}$. So we can write $P\mathbf{u} = \mathbf{u} + \boldsymbol{\epsilon}$ where $\boldsymbol{\epsilon} \in V$ and $\boldsymbol{\epsilon}$ nonzero.
Applying P to both sides,

$$
\begin{aligned}
P^2\mathbf{u} &= P(\mathbf{u} + \boldsymbol{\epsilon}) \\
P\mathbf{u} &= P\mathbf{u} + P\boldsymbol{\epsilon} \\
P\boldsymbol{\epsilon} &= \mathbf{0} \\
&= 0\boldsymbol{\epsilon}
\end{aligned}
$$

So 0 is an eigenvalue.

<u>QUESTION 2.</u> Find the eigenvalues and eigenvectors of $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$.

Solution:
$\det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 + 1$. So $\lambda = \dfrac{2 + -\sqrt{-4}}{2} = 1 + i,\, 1 - i$.
To find the eigenvectors for $1 + i$,

$\ker\begin{pmatrix}-i & 1\\ -1 & -i\end{pmatrix} = \ker\begin{pmatrix}1 & i\\ -1 & -i\end{pmatrix} = \ker\begin{pmatrix}1 & i\\ 0 & 0\end{pmatrix}$. So the eigenvector is $t\begin{pmatrix}-i\\ 1\end{pmatrix}$ for t nonzero.
To find the eigenvectors for $1-i$,
$\ker\begin{pmatrix}i & 1\\ -1 & i\end{pmatrix} = \ker\begin{pmatrix}-1 & i\\ -1 & i\end{pmatrix} = \ker\begin{pmatrix}-1 & i\\ 0 & 0\end{pmatrix}$. So the eigenvector is $t\begin{pmatrix}i\\ 1\end{pmatrix}$ for t nonzero.

QUESTION 3.

(a) Find all eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix}-4 & 5\\ 1 & 0\end{pmatrix}$

(b) Is A diagonalisable? Give reasons.

Solution:
(a)
$\det(A-\lambda I) = \det\begin{pmatrix}-4-\lambda & 5\\ 1 & -\lambda\end{pmatrix} = (-4-\lambda)(-\lambda)-(5)(1) = 0.$
Therefore $\lambda = -5, 1$ are the eigenvalues.
To find the eigenvectors for $\lambda = -5$,
$\ker\begin{pmatrix}1 & 5\\ 1 & 5\end{pmatrix} = \ker\begin{pmatrix}1 & 5\\ 0 & 0\end{pmatrix}$. So the corresponding eigenvector is $t\begin{pmatrix}-5\\ 1\end{pmatrix}$.
To find the eigenvectors for $\lambda = 1$,
$\ker\begin{pmatrix}-5 & 5\\ 1 & -1\end{pmatrix} = \ker\begin{pmatrix}-5 & 5\\ 0 & 0\end{pmatrix}$. So the corresponding eigenvector is $t\begin{pmatrix}1\\ 1\end{pmatrix}$.

(b) Yes as there are 2 distinct eigenvalues.

QUESTION 4. Evaluate A^8 if $A = \begin{pmatrix}5 & -8\\ 1 & -1\end{pmatrix}$
Solution:
We know $A^8 = MD^8M^{-1}$, where the columns of M are the eigenvectors of A and the diagonal elements of D are the eigenvalues of A.
Using the same method as in the previous questions, we can find the eigenvalues and corresponding eigenvectors to be $\lambda = 3, 1$, and $t\begin{pmatrix}4\\ 1\end{pmatrix}$, $t\begin{pmatrix}2\\ 1\end{pmatrix}$.
Therefore $M = \begin{pmatrix}4 & 2\\ 1 & 1\end{pmatrix}$ and $D = \begin{pmatrix}3 & 0\\ 0 & 1\end{pmatrix}$.
So, $A^8 = \begin{pmatrix}4 & 2\\ 1 & 1\end{pmatrix}\begin{pmatrix}3 & 0\\ 0 & 1\end{pmatrix} \times \frac{1}{2}\begin{pmatrix}1 & -2\\ -1 & 4\end{pmatrix}$
which we can then simplify.

QUESTION 5. Solve the following system of differential equations:

$$\frac{dx}{dt} = x + 2y$$
$$\frac{dy}{dt} = 3x + 2y.$$

Solution:

We put the coefficients in a matrix, $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$,

then find the eigenvalues and corresponding eigenvectors to be 4, -1 and $t\begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $t\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

The general solution to $\mathbf{y} = A\mathbf{y}$ is

$\mathbf{y}(t) = \alpha_1 e^{4t}\begin{pmatrix} 2 \\ 3 \end{pmatrix} + \alpha_2 e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ where α_1 and $\alpha_2 \in \mathbb{R}$.

Therefore $x(t) = 2\alpha_1 e^{4t} + \alpha_2 e^{-t}$ and $y(t) = 3\alpha_1 e^{4t} - \alpha_2 e^{-t}$.

QUESTION 6. Solve the following 2nd order ODE: $y'' + 4y' - 5y = 0$.

Solution:

We can turn a 2nd order linear ODE with constant coefficients into a system of 1st order equations with constant coefficients, then solve it using the method of the previous question.

Let $y_1 = y$ and $y_2 = y_1' = y'$.

Then $y_2' = y'' = 5y - 4y' = 5y_1 - 4y_2$.

Therefore we get the following system of equations,

$$y_1' = y_2$$
$$y_2' = 5y_1 - 4y_2.$$

So $\dfrac{d\mathbf{y}}{dt} = A\mathbf{y}$ where $A = \begin{pmatrix} 0 & 1 \\ 5 & -4 \end{pmatrix}$. We then find the eigenvalues and corresponding eigenvectors of A to be -5, 1 and $t\begin{pmatrix} -1 \\ 5 \end{pmatrix}$ and $t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ with t nonzero.

Therefore $\mathbf{y}(t) = \alpha_1 e^{-5t}\begin{pmatrix} -1 \\ 5 \end{pmatrix} + \alpha_2 e^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Therefore $y_1(t) = -\alpha_1 e^{-5t} + \alpha_2 e^{t}$.

Probability and Statistics

QUESTION 7. Show that the sequence defined by $p_k = \frac{7}{10}(\frac{3}{10})^k$ for $k = 0, 1, 2, ...$ is a probability distribution.

Solution:
Since $\sum_{k=0}^{\infty} \frac{7}{10}(\frac{3}{10})^k$ is a geometric series with ration between -1 and 1, we can use the limiting sum formula to simplify it.
Therefore $S_\infty = \frac{a}{1-r} = \frac{0.7}{1-0.3} = 1$.
Also, since $p_k \geq 0$, it is a probability distribution.

QUESTION 8. A certain diagnostic test for a disease is 99% sure of correctly indicating that a person has the disease when they actually do and 98% sure of correctly indicating that a person does not have a disease when they actually do not. Supoose 2% of the population actually have this disease.

(a) What is the probability that a person doesn't have the disease when they test positive (false positive)?

(b) What is the probability that a person has the disease when they test negative (false negative)?

Solution:
(a) Using Bayes' Rule,

$$\begin{aligned}\mathbb{P}(D^c \mid T) &= \frac{\mathbb{P}(T \mid D^c)\mathbb{P}(D^c)}{\mathbb{P}(T \mid D^c)\mathbb{P}(D^c) + \mathbb{P}(T \mid D)\mathbb{P}(D)} \\ &= \frac{0.02 \times 0.98}{0.02 \times 0.98 + 0.99 \times 0.02} \\ &= 0.4974.\end{aligned}$$

(b) Again, using Bayes' Rule,

$$\begin{aligned}\mathbb{P}(D \mid T^c) &= \frac{\mathbb{P}(T^c \mid D)\mathbb{P}(D)}{\mathbb{P}(T^c \mid D)\mathbb{P}(D) + \mathbb{P}(T^c \mid D^c)\mathbb{P}(D^c)} \\ &= \frac{0.01 \times 0.02}{0.01 \times 0.02 + 0.98 \times 0.98} \\ &= 0.00021.\end{aligned}$$

QUESTION 10.

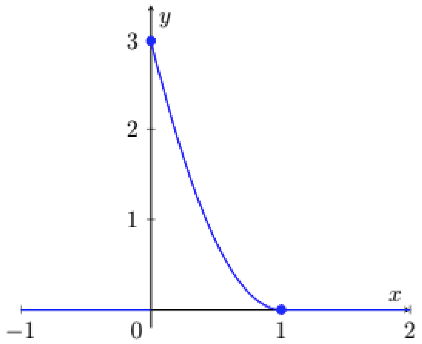
(a) Sketch the graph of $y = f(x)$.

(b) Find $\mathbb{E}(X)$ and $\mathrm{Var}(X)$.

(c) Find $\mathbb{P}(\frac{1}{2} < \sin(\pi X) < \frac{1}{\sqrt{2}})$.

(d) The median of a distribution is defined to be the real number m such that $\mathbb{P}(X \le m) = \frac{1}{2}$. Find the median of the above distribution.

Solution:
(a) The graph is a parabola between 0 and 1, with vertex $(1, 0)$ and y intercept 3, and it is 0 otherwise.



(b) To find the expectation, we integrate between 0 and 1 (since the PDF is 0 otherwise),

$$\begin{aligned}
\mathbb{E}(X) &= \int_0^1 x(3(1-x)^2)\,dx \\
&= 3\int_0^1 x - 2x^2 + x^3\,dx \\
&= 3\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) \\
&= \frac{1}{4}
\end{aligned}$$

To find the variance, we need to also compute $\mathbb{E}(X^2)$,

$$\begin{aligned}
\mathbb{E}(X^2) &= \int_0^1 x^2(3(1-x)^2)\,dx \\
&= 3\int_0^1 x^2 - 2x^3 + x^4\,dx \\
&= 3\left(\frac{1}{3} - \frac{2}{4} + \frac{1}{5}\right) \\
&= \frac{1}{10}.
\end{aligned}$$

Therefore $\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \frac{1}{10} - \frac{1}{4}^2 = \frac{3}{80}$.
(c) Note that $t \in [0, 1]$ since the PDF is 0 otherwise.
$\therefore \frac{1}{2} < \sin(\pi t) < \frac{1}{\sqrt{2}}$, so $\frac{1}{6} < t < \frac{1}{4}$ or $\frac{3}{4} < t < \frac{5}{6}$, which we get by drawing the sin graph and using a calculator.

Let F be the CDF of X. Then,

$$\begin{aligned}\mathbb{P}\left(\frac{1}{2} < \sin(\pi X) < \frac{1}{\sqrt{2}}\right) &= \mathbb{P}\left(\frac{1}{6} < X < \frac{1}{4}\right) + \mathbb{P}\left(\frac{3}{4} < X < \frac{5}{6}\right) \\ &= F\left(\frac{1}{4}\right) - F\left(\frac{1}{6}\right) + F\left(\frac{5}{6}\right) - F\left(\frac{3}{4}\right).\end{aligned}$$

To calculate F,

$$\begin{aligned}F &= \int_0^x f(t)\,dt \\ &= \int_0^x 3(1-t)^2\,dt \\ &= -(1-x)^3 - (-(1-0)^3) \\ &= 1-(1-x)^3.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{P}\left(\frac{1}{2} < \sin(\pi X) < \frac{1}{\sqrt{2}}\right) &= \left(1-\frac{1}{6}\right)^3 - \left(1-\frac{1}{4}\right)^3 + \left(1-\frac{3}{4}\right)^3 - \left(1-\frac{5}{6}\right)^3 \\ &= \frac{145}{864}.\end{aligned}$$

(d) We already have F from part c, and we need to solve $F(m) = \frac{1}{2}$.
$\therefore 1-(1-m)^3 = \frac{1}{2}$, so we can solve for m to get $m = 1 - \frac{1}{3\sqrt{2}}$.

QUESTION 11. A 6-sided die, with faces numbered 1 to 6, is suspected of being unfair so that the number 6 will occur more frequently than should happen by chance. During 300 test rolls of the die, the number 6 occurred 68 times.

(a) Write down an expression for a tail probability that measures the chance of rolling a 6 at least 68 times.

(b) Use the normal approximation to the binomial to estimate this probability.

(c) Is this evidence that the die is unfair?

Solution:
(a) $X \sim B(300, \frac{1}{6})$. Therefore,

$$\begin{aligned}\mathbb{P}(X \geq 68) &= \sum_{k=68}^{300} \mathbb{P}(X = k) \\ &= \sum_{k=68}^{300} \binom{300}{x} \frac{1}{6}^k \frac{5}{6}^{300-k}\end{aligned}$$

(b) $\mathbb{E}(X) = np = 300 \times \frac{1}{6} = 50$.

$\text{Var}(X) = np(1-p) = 50 \times \frac{5}{6} = \frac{125}{3}$. Therefore $\text{SD}(X) = \sqrt{\text{Var}(X)} = \sqrt{\frac{125}{3}}$.

$$P(X \geq 68) = P(Y \geq 67.5) = P\left(Z \geq \frac{67.5 - 50}{\sqrt{\frac{125}{3}}}\right) = \mathbb{P}(z \geq 2.71) = 1 - \mathbb{P}(z \leq 2.71) = 0.0034.$$

(c) Since $0.34\% < 5\%$, the answer is yes.