MATH1131/1141 MathSoc Calculus Revision Session 2019 T1 Solutions

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We cannot guarantee that our answers are correct - please notify us of any errors or typos at unswmathsoc@gmail.com, or on our Facebook page. There are sometimes multiple methods of solving the same question. Remember that in the real class test, you will be expected to explain your steps and working out.

Example 1: MATH1131 2012 S1 Q4.(iii)

Prove that

$$\lim_{x\to\infty} \frac{x^2-2}{x^2+3} = 1$$

Since $\left|\dfrac{x^2-2}{x^2+3} - 1\right| = \left|\dfrac{-5}{x^2+3}\right| = \dfrac{5}{x^2+3} \leq \dfrac{5}{x^2}$, *we set* $\dfrac{5}{x^2} < \epsilon$.

Here, $x > \sqrt{\dfrac{5}{\epsilon}}$. *Let* $M = \sqrt{\dfrac{5}{\epsilon}}$. *Then if* $x > M$, $\left|\dfrac{x^2-2}{x^2+3} - 1\right| < \epsilon$.

So by the ϵ-M *definition of the limit,* $\displaystyle\lim_{x\to\infty} \frac{x^2-2}{x^2+3} = 1$.

Example 2: MATH1131 2015 S2 Q1.(i).c)

Evaluate

$$\lim_{x\to\pi}\frac{x-\pi}{\sin 3x}$$

$(x-\pi)\to 0$ *and* $\sin 3x \to 0$ *as* $x\to\pi$.

$$\begin{aligned}\lim_{x\to\pi}\frac{(x-\pi)}{\sin 3x} &= \lim_{x\to\pi}\frac{\frac{d}{dx}(x-\pi)}{\frac{d}{dx}\sin 3x} && \text{(L'H)}\\ &= \lim_{x\to\pi}\frac{1}{3\cos 3x}\\ &= \frac{1}{3\cos 3\pi}\\ &= -\frac{1}{3}\end{aligned}$$

Use of L'Hopital's rule was justified as the final limit exists.

Example 3: Useful Limit Technique for 1141

Evaluate

$$\lim_{x\to 0^+} x^x$$

$$\begin{aligned}x^x &= e^{\ln x^x}\\ &= e^{x\ln x}\\ \lim_{x\to 0^+} x^x &= \lim_{x\to 0^+} e^{x\ln x}\\ &= \exp\left(\lim_{x\to 0^+}(x\ln x)\right)\\ &= \exp\left(-\lim_{x\to 0^+}\left(\frac{-\ln x}{\frac{1}{x}}\right)\right)\end{aligned}$$

Here, numerator and denominator $\to\infty$ *as* $x\to 0^+$. *So if*

$$\lim_{x\to 0^+}\frac{\frac{d}{dx}(-\ln x)}{\frac{d}{dx}(\frac{1}{x})}$$

exists then we can apply L'Hopital's Rule.

$$\lim_{x\to 0^+} \frac{\frac{d}{dx}(-\ln x)}{\frac{d}{dx}(\frac{1}{x})} = \lim_{x\to 0^+} \frac{\dfrac{-1}{x}}{\dfrac{-1}{x^2}} \tag{L'H}$$
$$= \lim_{x\to 0^+} x$$
$$= 0$$

Limit exists, so we can use L'Hopital's Rule.

So $\lim\limits_{x\to 0^+} x^x = \exp\left(-0\right) = \exp(0) = 1$

Example 4

Evaluate

$$\lim_{x\to\infty} \frac{x+\sin x}{x-\sin x}$$

$$\lim_{x\to\infty} \frac{\dfrac{d}{dx}(x+\sin x)}{\dfrac{d}{dx}(x-\sin x)} = \lim_{x\to\infty} \frac{1+\cos x}{1-\cos x}$$

This limit does not exist since $\cos x$ *oscillates as* $x\to\infty$*, so we cannot use L'Hopital's Rule. Instead, let us use a simpler approach.*

$$\lim_{x\to\infty} \frac{x+\sin x}{x-\sin x} = \lim_{x\to\infty} \frac{1+\dfrac{\sin x}{x}}{1-\dfrac{\sin x}{x}}$$
$$= \frac{1+0}{1-0}$$
$$= 1$$

Example 5: MATH1131 2014 S2 Q1.(v).b)

Show that the equation

$$e^x = x + 2$$

has a solution in the interval [0,2].

Let $f(x) = e^x - x - 2$. We want to find when $e^x = x + 2$, i.e. when $f(x) = 0$.

First note that, since f is an elementary function, then f is continuous on the interval $[0, 2]$. Also

$$f(0) = -1$$

and

$$f(2) = e^2 - 4 > 2^2 - 4 = 0$$

, so

$$f(0) \leq 0 \leq f(2)$$

. Then by the Intermediate Value Theorem, there is some $c \in [0, 2]$ such that $f(c) = 0$.

Thus $e^x = x + 2$ for some $x \in [0, 2]$

Example 6: MATH1131 2015 S2 Q2.(i)

Find a and b such that the function

$$f(x) = \begin{cases} x^2 + ax + b & x < 0 \\ \cos 2x & x \geq 0 \end{cases}$$

is differentiable.

If f is differentiable, f must be continuous. So, $\lim\limits_{x\to 0} f(x) = f(0)$.

Thus $\lim\limits_{x\to 0}(x^2 + ax + b) = 1 \Rightarrow b = 1$.

For f to be differentiable at $x = 0$, $\lim\limits_{h\to 0} \dfrac{f(0+h) - f(0)}{h}$ must exist. Then

$$\lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h}$$

$$\lim_{h\to 0^+} \frac{\cos 2h - 1}{h} = \lim_{h\to 0^-} \frac{(h^2 + ah + 1) - 1}{h}$$

By L'Hopital's Rule,

$$\lim_{h\to 0^+} \frac{\cos 2h - 1}{h} = \lim_{h\to 0^+} (-2\sin 2h) = 0$$

So

$$\lim_{h\to 0^-} (h + a) = 0 \Rightarrow a = 0$$

Hence

$$f(x) = \begin{cases} x^2 + 1 & x < 0 \\ \cos 2x & x \geq 0 \end{cases}$$

Example 7: MATH1131 2014 S1 Q4.(v).b)

Suppose that $-1 < x < y < 1$. Prove that

$$\sin^{-1} y - \sin^{-1} x \geq y - x$$

by applying the Mean Value Theorem on the function $f(t) = \sin^{-1} t$ on the interval $[x, y]$.

Let $f(x) = \sin^{-1} t$ *for* $t \in [x, y]$.
Since $|x| < 1$ *and* $|y| < 1$ *then* f *is continuous on the closed interval* $[x, y]$ *and differentiable on the open interval* (x, y). *So by the Mean Value Theorem, there exists some* $c \in [x, y]$ *such that*

$$\frac{f(y) - f(x)}{y - x} = f'(c)$$

But

$$\begin{aligned} f'(c) &= \frac{d}{dx} \sin^{-1} x \\ &= \frac{1}{\sqrt{1 - c^2}} \\ &\geq 1 \end{aligned}$$

so $\dfrac{f(y) - f(x)}{y - x} \geq 1$.

Thus $f(y) - f(x) \geq y - x$, *i.e.* $\sin^{-1} y - \sin^{-1} x \geq y - x$

Example 8: MATH1141 2013 S1 Q4.(iv)

Suppose $f : [0, 2] \to [0, 8]$ is continuous and differentiable on its domain.

(a) By considering the function $g(x) = f(x) - x^3$, prove that there is a real number $\xi \in [0, 2]$ such that $f(\xi) = \xi^3$, stating any theorems you use.

$g(0) = f(0)$ *and* $g(2) = f(2) - 8$.
$0 \leq f(0) \leq 8$ *and* $-8 \leq f(2) - 8 \leq 0$, *so*
$f(2) - 8 \leq 0 \leq f(0)$
$\Rightarrow g(2) \leq 0 \leq g(0)$.

Since f *is continuous on* $[0, 2]$ *then* g *is continuous on* $[0, 2]$. *So by the Intermediate Value Theorem, there exists some* $\xi \in [0, 2]$ *such that* $g(\xi) = 0$, *i.e.* $f(\xi) = \xi^3$.

(b) Now suppose that $f(0) = 0$ and $f(2) = 8$. Explain why $f'(\eta) = 4$ for some real $\eta \in (0, 2)$, stating any theorems you use.

f *is continuous on the closed interval* $[0, 2]$ *and differentiable on the open interval* $(0, 2)$ *so by the Mean Value Theorem, there exists some* $\eta \in (0, 2)$ *such that*

$$\frac{f(2) - f(0)}{2 - 0} = f'(\eta)$$

So $f'(\eta) = \dfrac{8 - 0}{2 - 0} = 4$

Example 9: MATH1141 2014 S1 Q3.(i)

Let $g(x) = 3x - \cos 2x - 1$ for all $x \in \mathbb{R}$. Explain why g has a differentiable inverse function $h = g^{-1}$, and calculate $h'(-2)$.

g *is an elementary function, so* g *is differentiable. Further,* $g'(x) = 3 + 2\sin 2x \geq 1$ *so* $g'(x) \neq 0$. *Then by the Inverse Function Theorem,* g *has a differentiable inverse function* $h = g^{-1}$. *Also,* $h'(x) = \dfrac{1}{g'(h(x))}$.

$g(0) = -2$, *so* $h(-2) = 0$. *Then* $h'(-2) = \dfrac{1}{g'(0)} = \dfrac{1}{3}$.

Example 10: MATH1131 2016 S1 Q2.(ii)

Sketch the polar curve $r(\theta) = 1 - \cos\theta$ for $0 \leq \theta < 2\pi$.

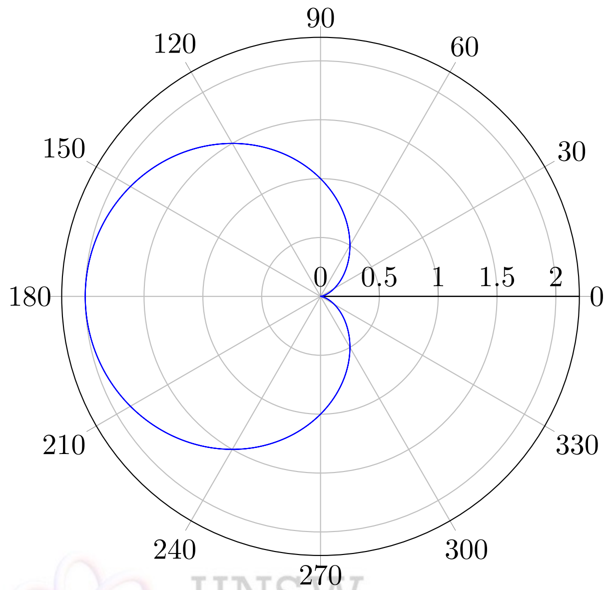
$r(0) = 0$, $r(\frac{\pi}{2}) = 1$, $r(\pi) = 2$, $r(\frac{3\pi}{2}) = 1$.

Here, $x = r\cos\theta = \cos\theta - \cos^2\theta$ *and* $y = r\sin\theta = \sin\theta - \sin\theta\cos\theta$, *so*

$x'(\theta) = \sin\theta(2\cos\theta - 1)$ and $y'(\theta) = \cos\theta - \cos 2\theta$. Thus,

$$\frac{dy}{dx} = \frac{\cos\theta - \cos 2\theta}{\sin\theta(2\cos\theta - 1)}.$$

Here, $\dfrac{dy}{dx} = 0$ when $\theta = \dfrac{2\pi}{3}, \dfrac{4\pi}{3}$ and $\dfrac{dy}{dx} \to \infty$ when $\theta = \dfrac{\pi}{3}, \pi, \dfrac{5\pi}{3}$.



Example 11: MATH1141 2012 S1 Q4.(i)

Consider the polar curve $r(\theta) = 1 + \cos 2\theta$.

(a) Prove that the curve is symmetric about the x-axis and also about the y-axis.

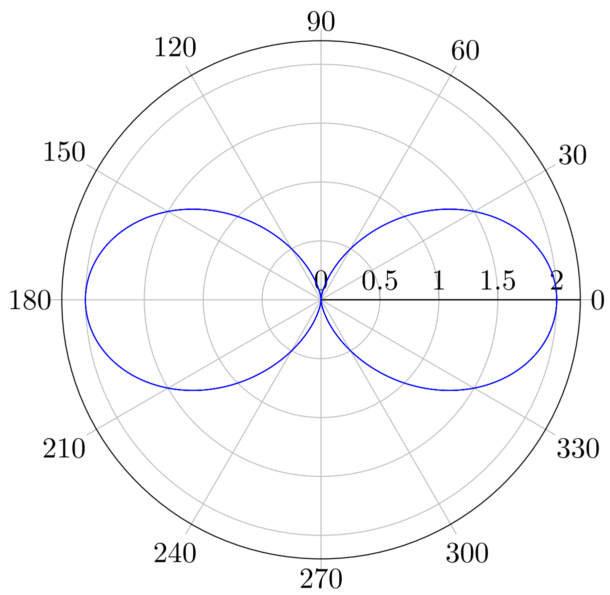
$x(\theta) = (1 + \cos 2\theta)\cos\theta$ and $y(\theta) = (1 + \cos 2\theta)\sin\theta$.

Here, $x(-\theta) = (1 + \cos(-2\theta))\cos(-\theta) = (1 + \cos 2\theta)\cos\theta = x(\theta)$, so $r(\theta)$ is symmetric about the x-axis.

$y(\pi - \theta) = (1 + \cos(2(\pi - \theta)))\sin(\pi - \theta) = (1 + cos(2\pi - 2\theta))\sin\theta = (1 + \cos 2\theta)\sin\theta = y(\theta)$, so $r(\theta)$ is also symmetric about the y-axis.

(b) Sketch the curve (Not required to find derivative).

Since we have symmetry about both axes, we only need to inspect the first quadrant. Here, $r(0) = 2$, $r(\frac{\pi}{2}) = 0$.



Example 12: MATH1131 2015 S2 Q2.(vi)

(a) Calculate the upper Riemann sum of the function $f(x) = x^2$ for the partition $P_n = \left\{0, \frac{1}{n}, \frac{2}{n}, ..., 1\right\}$ of the interval $[0, 1]$, where n is a positive integer.

$$\begin{aligned}
\overline{S}_{P_n}(f) &= \sum_{k=1}^{n} \left(\frac{k}{n} - \frac{k-1}{n}\right)\overline{f}_k \\
&= \sum_{k=1}^{n} \frac{1}{n}\left(\frac{k}{n}\right)^2 \\
&= \frac{1}{n^3}\sum_{k=1}^{n} k^2 \\
&= \frac{1}{6}\frac{(n+1)(2n+1)}{n^2}
\end{aligned}$$

(b) Find the value of the definite integral

$$\int_0^1 x^2 dx.$$

$$
\begin{aligned}
\int_0^1 x^2 dx &= \lim_{n\to\infty} \overline{S}_{P_n}(f) \\
&= \lim_{n\to\infty} \frac{1}{6}\frac{(n+1)(2n+1)}{n^2} \\
&= \frac{1}{6}(2) \\
&= \frac{1}{3}
\end{aligned}
$$

Example 13: MATH1131 2015 S2 Q2.(iv)

Use the First Fundamental Theorem of Calculus to find

$$\frac{d}{dx}\left(\int_{\cos x}^{\sin x} e^{1-t^2} dt\right)$$

$$
\begin{aligned}
\frac{d}{dx}\left(\int_{\cos x}^{\sin x} e^{1-t^2} dt\right) &= \frac{d}{dx}\left(\int_{\cos x}^{\sin x} e^{1-t^2} dt + \int_0^{\cos x} e^{1-t^2} dt - \int_0^{\cos x} e^{1-t^2} dt\right) \\
&= \frac{d}{dx}\left(\int_0^{\sin x} e^{1-t^2} dt - \int_0^{\cos x} e^{1-t^2} dt\right) \\
&= \frac{d}{dx}\left(\int_0^{\sin x} e^{1-t^2} dt\right) - \frac{d}{dx}\left(\int_0^{\cos x} e^{1-t^2} dt\right) \\
&= e^{1-\sin^2 x}(\cos x) - e^{1-\cos^2 x}(-\sin x) \\
&= e^{\cos^2 x}\cos x + e^{\sin^2 x}\sin x
\end{aligned}
$$

Example 14: MATH1131 2016 S1 Q3.(i)

Find:

(a)

$$\int x^2\sqrt{3+x^3}dx$$

Let $u = 3 + x^3$. Then $u'(x) = 3x^2$ and so,

$$\int x^2\sqrt{3+x^3}dx = \frac{1}{3}\int 3x^2\sqrt{3+x^3}dx$$

$$= \frac{1}{3}\int \sqrt{u}du$$

$$= \frac{1}{3}(\frac{2}{3}u\sqrt{u}) + C$$

$$= \frac{2u\sqrt{u}}{9} + C$$

(b)

$$\int xe^{3x}dx$$

Let $u = x$, $dv = e^{3x}dx$. Then $du = dx$, $v = \frac{1}{3}e^{3x}$. So by Integration by Parts:

$$\int xe^{3x}dx = x\frac{1}{3}e^{3x} - \int \frac{1}{3}e^{3x}dx$$

$$= \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C$$

Example 15: Important Techniques for 1131/1141

(a) Find using integration by substitution:

$$\int x\sqrt{x+1}dx$$

Let $u = \sqrt{x+1}$*, then* $x = u^2 - 1$ *and* $dx = 2udu$*. Then,*

$$\begin{aligned}
\int x\sqrt{x+1}dx &= \int (u^2-1)(u)(2u)du \\
&= 2\int (u^4 - u^2)du \\
&= 2\left(\frac{u^5}{5} - \frac{u^3}{3}\right) + C \\
&= \frac{2}{15}\left(3(\sqrt{x+1})^5 - 5(\sqrt{x+1})^3\right) + C \\
&= \frac{2}{15}(x+1)\sqrt{x+1}(3(x+1)-5) + C \\
&= \frac{2}{15}(x+1)(3x-2)\sqrt{x+1} + C
\end{aligned}$$

(b) Find using integration by substitution:

$$\int \frac{1}{(1+x^2)^{\frac{3}{2}}}dx$$

Let $x = \tan u$*, then* $dx = \sec^2(u)du$*. So,*

$$\begin{aligned}
\int \frac{1}{(1+x^2)^{\frac{3}{2}}}dx &= \int \frac{1}{(1+\tan^2 u)^{\frac{3}{2}}}\sec^2(u)du \\
&= \int \frac{\sec^2 u}{(\sec^2 u)^{\frac{3}{2}}}du \\
&= \int \frac{\sec^2 u}{\sec^3 u}du \\
&= \int \frac{1}{\sec u}du \\
&= \int \cos(u)du \\
&= \sin(u) + C \\
&= \sin(\tan^{-1} x) + C \\
&= \frac{x}{\sqrt{x^2+1}} + C
\end{aligned}$$

(c) Find using integration by parts:

$$\int e^x \cos(x) dx$$

Let

$$I = \int e^x \cos(x) dx$$

Then by applying integration by parts with $u = e^x$, $dv = \cos(x)dx$, *we have*

$$I = e^x \sin(x) - \int e^x \sin(x) dx$$

Here we need to apply integration by parts again, this time with $u = e^x$ *and* $dv = \sin(x)$. *So we have*

$$\begin{aligned} I &= e^x \sin(x) - \left[-e^x \cos(x) + \int e^x \cos(x) dx \right] \\ &= e^x \sin(x) + e^x \cos(x) - I \\ &= \frac{1}{2}\Big(e^x \sin(x) + e^x \cos(x) \Big) \end{aligned}$$

Example 16

Find

$$\int x e^{x^2} dx$$

Let $u = x^2$, *then* $du = (2x)dx$. *So*

$$\begin{aligned} \int x e^{x^2} dx &= \int \frac{1}{2} e^{x^2} (2x) dx \\ &= \frac{1}{2} \int e^u du \\ &= \frac{1}{2} e^u + C \\ &= \frac{1}{2} e^{x^2} + C \end{aligned}$$

Example 17: MATH1131 2014 S2 Q1.(ii)

Does the following improper integral converge? If so, find its value. If not, show that it diverges.

$$\int_e^\infty \frac{dx}{x+\ln x}$$

$$\begin{aligned}\ln x &\leq x, \forall x > 0\\ x+\ln x &\leq 2x\\ \frac{1}{x+\ln x} &\geq \frac{1}{2x}\end{aligned}$$

Here,

$$\int_e^\infty \frac{1}{2x}dx = \frac{1}{2}\int_e^\infty \frac{1}{x}dx$$

By the p-test, the above integral diverges. So by the comparison test,

$$\int_e^\infty \frac{1}{x+\ln x}dx$$

diverges.

Example 18: MATH1131 2015 S2 Q1.(v)

Determine whether or not the following improper integral converges. Give reasons for your answer.

$$\int_2^\infty \frac{x^2+\sqrt{x}}{x^{\frac{8}{3}}-x^2-1}dx$$

$$\begin{aligned}\lim_{x\to\infty}\frac{\left(\dfrac{x^2+\sqrt{x}}{x^{\frac{8}{3}}-x^2-1}\right)}{\left(\dfrac{1}{x^{\frac{2}{3}}}\right)} &= \lim_{x\to\infty}\frac{x^{\frac{8}{3}}+x^{\frac{7}{6}}}{x^{\frac{8}{3}}-x^2-1}\\ &= \lim_{x\to\infty}\frac{1+x^{\frac{-3}{2}}}{1-x^{\frac{-2}{3}}-x^{\frac{-8}{3}}}\\ &= \frac{1+0}{1-0+0}\\ &= 1\\ &> 0\end{aligned}$$

By the p-test,

$$\int_2^\infty \frac{1}{x^{\frac{2}{3}}}dx$$

diverges. So by the limit comparison test,

$$\int_{2}^{\infty} \frac{x^2 + \sqrt{x}}{x^{\frac{8}{3}} - x^2 - 1} dx$$

also diverges.

Example 19: MATH1131 2014 S1 Q2.(ii)

(a) Give the definitions of $\sinh x$ and $\cosh x$ in terms of the exponential function.

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

(b) Use your definitions to prove that $\sinh(2x) = 2\sinh x \cosh x$

$$\begin{aligned}
\sinh(2x) &= \frac{e^{2x} - e^{-2x}}{2} \\
&= \frac{(e^x)^2 - (e^{-x})^2}{2} \\
&= \frac{(e^x - e^{-x})(e^x + e^{-x})}{2} \\
&= 2\left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right) \\
&= 2\sinh x \cosh x
\end{aligned}$$

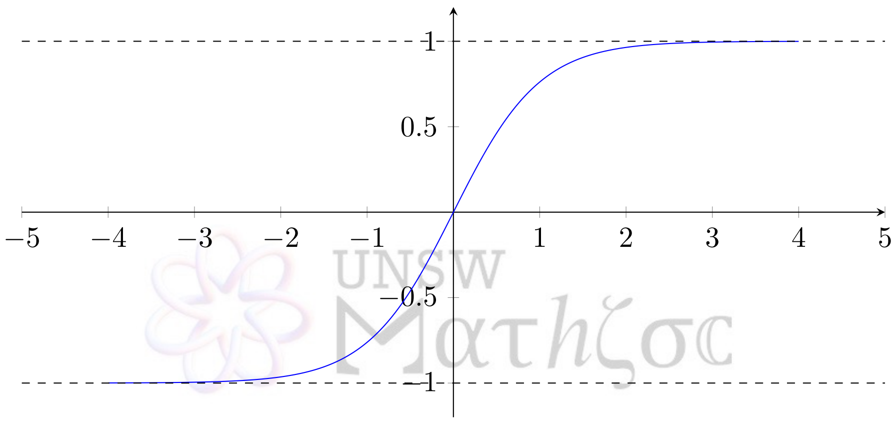
Example 20: MATH1141 2012 S1 Q4.(ii)

(a) Express $\tanh x$ in terms of exponentials.

$$
\begin{aligned}
\tanh x &= \frac{\sinh x}{\cosh x} \\
&= \frac{\frac{e^x - e^{-x}}{2}}{\frac{e^x + e^{-x}}{2}} \\
&= \frac{e^x - e^{-x}}{e^x + e^{-x}}
\end{aligned}
$$

(b) Sketch the graph $y = \tanh x$.

$\tanh 0 = 0,\ \lim_{x\to\infty} \tanh x = 1,\ \lim_{x\to-\infty} \tanh x = -1,$
$\frac{d}{dx}\tanh x = \operatorname{sech}^2 x = \frac{1}{\cosh^2 x} > 0.$



(c) Find

$$\lim_{x\to\infty} \frac{1 - \tanh x}{e^{-2x}}$$

$$
\begin{aligned}
\lim_{x\to\infty} \frac{1 - \tanh x}{e^{-2x}} &= \lim_{x\to\infty} \frac{1 - \frac{e^x - e^{-x}}{e^x + e^{-x}}}{e^{-2x}} \\
&= \lim_{x\to\infty} \frac{(e^x + e^{-x}) - (e^x - e^{-x})}{e^{-2x}(e^x + e^{-x})} \\
&= \lim_{x\to\infty} \frac{2e^{-x}}{e^{-x} + e^{-3x}} \\
&= \lim_{x\to\infty} \frac{2}{1 + e^{-2x}} \\
&= \frac{2}{1+0} \\
&= 2
\end{aligned}
$$

(d) Explain why the improper integral converges.

$$\int_0^\infty (1 - \tanh x) dx$$

First, note that

$$\int_0^\infty (1 - \tanh x) dx = \int_0^1 (1 - \tanh x) dx + \int_1^\infty (1 - \tanh x) dx$$

So,

$$\int_0^\infty (1 - \tanh x) dx$$

converges if

$$\int_1^\infty (1 - \tanh x) dx$$

converges. Now consider e^{-2x}.

$$\begin{aligned} e^{2x} &= (e^x)^2 \\ &\geq x^2 \\ e^{-2x} &\leq \frac{1}{x^2} \end{aligned}$$

By the p-test,

$$\int_1^\infty \frac{1}{x^2} dx$$

converges. So by the comparison test,

$$\int_1^\infty e^{-2x} dx$$

converges. But since

$$\lim_{x \to \infty} \frac{1 - \tanh x}{e^{-2x}} = 2 > 0$$

then by the limit comparison test,

$$\int_1^\infty (1 - \tanh x) dx$$

converges. Therefore,

$$\int_0^\infty (1 - \tanh x) dx$$

converges.

(e) Compute

$$\int_0^\infty (1-\tanh x)dx$$

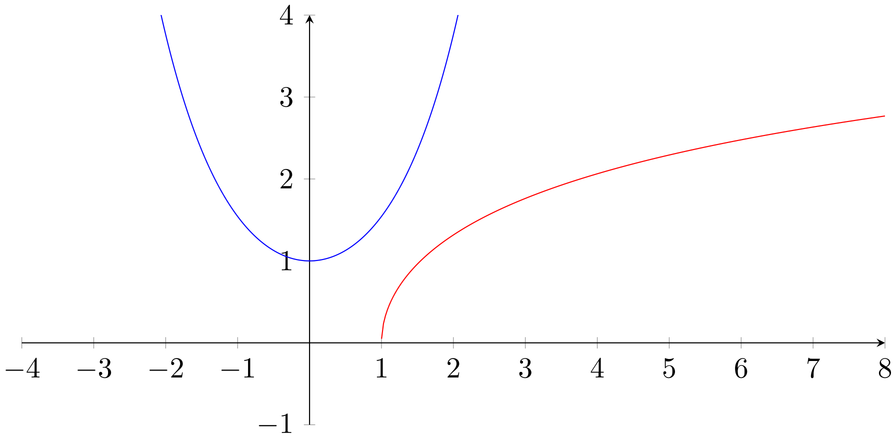
$$\begin{aligned}
\int_0^\infty (1-\tanh x)dx &= \lim_{R\to\infty}\int_0^R (1-\tanh x)dx \\
&= \lim_{R\to\infty}\int_0^R \left(1-\frac{\sinh x}{\cosh x}\right)dx \\
&= \lim_{R\to\infty}\int_0^R \left(1-\frac{\frac{d}{dx}\cosh x}{\cosh x}\right)dx \\
&= \lim_{R\to\infty}\Big[x-\ln|\cosh x|\Big]_0^R \\
&= \lim_{R\to\infty}(R-\ln|\cosh R|) \\
&= \lim_{R\to\infty}\left(R-\ln\frac{e^R+e^{-R}}{2}\right) \\
&= \lim_{R\to\infty}\left(R-\ln\left(e^R(1+e^{-2R})\right)+\ln 2\right) \\
&= \lim_{R\to\infty}\left(R-\ln\left(e^R\right)-\ln\left(1+e^{-2R}\right)+\ln 2\right) \\
&= \lim_{R\to\infty}\left(-\ln\left(1+e^{-2R}\right)+\ln 2\right) \\
&= \ln 2
\end{aligned}$$

Example 21: MATH1131 2016 S1 Q3.(i)

Sketch, on one set of axes, the graphs of $y=\cosh x$ and $y=\cosh^{-1}x$.

Let $f(x)=\cosh x$. $f(-x)=f(x)$ so the graph $y=f(x)$ has symmetry about y-axis. Then to obtain an inverse graph, we must restrict the domain of f, either to $(-\infty,0]$ or $[0,\infty)$. Conventionally, we choose the positive domain.

So let $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\cosh x$. Then $f^{-1}:[1,\infty)\to[0,\infty)$, $f^{-1}(x)=\cosh^{-1}x$.
Graphs of $y=f(x)$ and $y=f^{-1}(x)$:



Example 22: MATH1141 2016 S1 Q3.(iv)

(a) Carefully state the first fundamental theorem of calculus.

Let the function $f : [a, b] \to \mathbb{R}$ be continuous over its domain. Then the function $F : [a, b] \to \mathbb{R}$ defined by

$$F(x) = \int_a^x f(t)dt$$

is continuous on $[a, b]$, differentiable on (a, b), and has derivative $F'(x) = f(x)$.

(b) For $\alpha > 0$ and $n > 0$, determine whether the improper integral

$$\int_0^\infty ue^{\alpha u^n} du$$

converges or diverges. Give reasons for your answer.

First note that since

$$\int_0^\infty ue^{\alpha u^n} du = \int_0^1 ue^{\alpha u^n} du + \int_1^\infty ue^{\alpha u^n} du$$

then

$$\int_0^\infty ue^{\alpha u^n} du$$

diverges if

$$\int_1^\infty ue^{\alpha u^n} du$$

diverges. Now consider $ue^{\alpha u^n}$.

$$\begin{aligned} ue^{\alpha u^n} &= u(e^{u^n})^\alpha \\ &\geq u(u^n)^\alpha = u^{\alpha n+1} = \frac{1}{u^{-\alpha n-1}} \end{aligned}$$

$\alpha > 0$ *and* $n > 0$ *so* $(-\alpha n - 1) \leq 1$. *Then by the p-test,*

$$\int_1^\infty \frac{1}{u^{-\alpha n-1}} du$$

diverges. So by the comparison test,

$$\int_1^\infty ue^{\alpha u^n} du$$

diverges. Therefore

$$\int_0^\infty ue^{\alpha u^n} du$$

diverges.

(c) Using L'Hopital's Rule, find, without integration, $\lim_{x\to\infty} f(x)$, where

$$f(x) = \frac{\left(\int_0^x ue^{3u^2} du\right)^2}{\int_0^x ue^{6u^2} du}$$

Since both integrals are of the form

$$\int_0^\infty ue^{\alpha u^n} du$$

they both diverge as $x \to \infty$. *So the numerator and denominator both go to infinity. Therefore we can apply L'Hopital's Rule, provided the limit*

$$\lim_{x\to\infty} \frac{\frac{d}{dx}\left(\int_0^x ue^{3u^2} du\right)^2}{\frac{d}{dx}\left(\int_0^x ue^{6u^2} du\right)}$$

exists. By applying the first fundamental theorem of calculus,

$$\lim_{x\to\infty} \frac{\frac{d}{dx}\left(\int_0^x ue^{3u^2}du\right)^2}{\frac{d}{dx}\left(\int_0^x ue^{6u^2}du\right)} = \lim_{x\to\infty} \frac{2xe^{3x^2}\int_0^x ue^{3u^2}du}{xe^{6x^2}}$$

$$= 2\lim_{x\to\infty} \frac{\int_0^x ue^{3u^2}du}{e^{3x^2}}$$

Here, the numerator and denominator both go to infinity again. So by applying L'Hopital's Rule once again:

$$\lim_{x\to\infty} \frac{\dfrac{d}{dx}\int_0^x ue^{3u^2}du}{\dfrac{d}{dx}e^{3x^2}} = \lim_{x\to\infty} \frac{xe^{3x^2}}{6xe^{3x^2}}$$

$$= \lim_{x\to\infty} \frac{1}{6}$$

$$= \frac{1}{6}$$

Since the limit exists, then

$$\lim_{x\to\infty} \frac{\frac{d}{dx}\left(\int_0^x ue^{3u^2}du\right)^2}{\frac{d}{dx}\left(\int_0^x ue^{6u^2}du\right)} = 2\left(\frac{1}{6}\right) = \frac{1}{3}$$

And since this limit exists, then

$$\lim_{x\to\infty} f(x) = \frac{1}{3}$$

(d) Show that the function f is an even function, that is, $f(-x) = f(x)$.

Let $g(u) = ue^{\alpha u^2}$, $\alpha > 0$.
Then $g(-u) = (-u)e^{\alpha(-u)^2} = -ue^{\alpha u^2} = -g(u)$, *i.e.* g *is an odd function.*

So,

$$\int_{-x}^{x} g(u)du = \int_{-x}^{x} ue^{\alpha u^2}du = 0$$

Now,

$$\begin{aligned}
\int_0^{-x} ue^{\alpha u^2}du &= \int_0^x g(u)du + \int_x^{-x} g(u)du \\
&= \int_0^x g(u)du - \int_{-x}^x g(u)du \\
&= \int_0^x g(u)du - 0 \\
&= \int_0^x g(u)du = \int_0^x ue^{\alpha u^2}du
\end{aligned}$$

So by applying this result to $f(-x)$:

$$\begin{aligned}
f(-x) &= \frac{\left(\int_0^{-x} ue^{3u^2}du\right)^2}{\int_0^{-x} ue^{6u^2}du} \\
&= \frac{\left(\int_0^{x} ue^{3u^2}du\right)^2}{\int_0^{x} ue^{6u^2}du} \\
&= f(x)
\end{aligned}$$

Thus f is an even function.

Example 23: MATH1141 2014 S1 Q3.(iii)

Use the ϵ-M definition of the limit to prove that

$$\lim_{x\to\infty} \frac{e^x}{\cosh x} = 2$$

First consider the absolute difference between the function and the value 2.

$$\begin{aligned}
\left|\frac{e^x}{\cosh x} - 2\right| &= \left|\frac{2e^x - 2(e^x + e^{-x})}{e^x + e^{-x}}\right| \\
&= \left|\frac{-2e^{-x}}{e^x + e^{-x}}\right| \\
&= \frac{2}{e^{2x}+1} \\
&\le \frac{2}{e^{2x}}
\end{aligned}$$

Set $M = \ln\sqrt{\frac{2}{\epsilon}}$. Then if $x > M$ we have $\frac{2}{e^{2x}} < \epsilon$, i.e. $\left|\frac{e^x}{\cosh x} - 2\right| < \epsilon$

In other words, for each $\epsilon > 0$ there is $M > 0$ such that, if $x > M$ then $\left|\dfrac{e^x}{\cosh x} - 2\right| < \epsilon$.

So by ϵ-M definition, $\displaystyle\lim_{x\to\infty} \frac{e^x}{\cosh x} = 2$.

Bonus Question: During 10.min Break

Let the function f be continuous and differentiable on $\mathbb{R}$, and

$$|f'(x)| \leq \sqrt{x}$$

for all $x \geq 0$. Find the limit

$$\lim_{x\to\infty} \frac{f(x+1) - f(x)}{x}$$

Let $f = f(t)$ and fix $x \geq 0$. Consider f on the interval $[x, x+1]$. f is continuous on $[x, x+1]$ and differentiable on $(x, x+1)$. So by the Mean Value Theorem, there exists some $c \in (x, x+1)$ such that

$$\frac{f(x+1) - f(x)}{(x+1) - x} = f'(c)$$

So then

$$\frac{f(x+1) - f(x)}{1} = f'(c)$$
$$|f(x+1) - f(x)| = \left|f'(c)\right|$$
$$|f(x+1) - f(x)| \leq \sqrt{c}$$

Since $c \in (x, x+1)$ then $\sqrt{c} \leq \sqrt{x+1}$, so

$$|f(x+1) - f(x)| \leq \sqrt{x+1}$$

$$-\sqrt{x+1} \leq f(x+1) - f(x) \leq \sqrt{x+1}$$

$$-\frac{\sqrt{x+1}}{x} \leq \frac{f(x+1) - f(x)}{x} \leq \frac{\sqrt{x+1}}{x}$$

Since $\lim_{x \to \infty} -\frac{\sqrt{x+1}}{x} = 0$ *and* $\lim_{x \to \infty} \frac{\sqrt{x+1}}{x} = 0$, *then by the pinching theorem,*

$$\lim_{x \to \infty} \frac{f(x+1) - f(x)}{x} = 0$$