UNSW Mathematics Society Presents
MATH2521/2621 Seminar

Presented by Felix, Wendy, Raymond, and the constant desire to convince people that imaginary numbers are vital to everyday life

Overview I

Overview II

4. Series
    - Power Series
    - Taylor Series
    - Laurent Series
    - Singularities
    - Residues

1. Intro to Complex Analysis

Basic Rules and Ideas

1. A complex number is a number that can be written as $z = x + iy$ (cartesian form) and can also be written as $z = re^{i\theta}$ (Euler form).

2. Normal rules of addition, multiplication and subtraction hold. With division, we multiply numerator and denominator by the conjugate of the denominator to make the denominator purely real.

3. Geometrically, multiplying complex numbers involves scaling and rotation about the origin. Addition involves shifting in the direction of the vector that you have added.

Important Ideas

1. $|z|^2 = z\bar{z}$
2. The principal argument of a complex number is the argument θ of a complex number z such that $-\pi < \theta \leq \pi$

Inequalities and Sets

Triangle Inequality

The triangle inequality states:

$$|w + z| \leq |w| + |z|.$$

Circle Inequality

The circle inequality states:

$$||w| - |z|| \leq |w - z|.$$

Topology and Sets

Types of Points

Consider a set S. Then an element $x \in S$ must be one of the following:

1. interior point: There exists an $\epsilon > 0$ such that $B(x, \epsilon) \subseteq S$, where $B(x, \epsilon)$ is the open ball about x or radius ϵ.
2. exterior point: There exists an $\epsilon > 0$ such that $B(x, \epsilon) \cap S = \varnothing$.
3. boundary point: None of the above. More formally, for every $\epsilon > 0$, $B(x, \epsilon)$ overlaps with S and S^C at elements EXCEPT for x.

Topology and sets

Types of Sets I

Consider a set S. Then it can described using the following terms:

1. Open: If every $x \in S$ is an interior point.
2. Closed: If the complement of S is open.
3. Closure: The set of all points in S plus all its boundary points.
4. Bounded: If there exists an $M > 0, x \in S$ such that $S \subset B(x, M)$.
5. Compact: If it is closed and bounded.
6. Region: If it is an open set with none, some, or all of its boundary points.

Arcs

Types of Arcs

1. Polygonal path: A polygonal path is a set of finite line segments with the end point of a line segment equal to the initial point of the next line segment.
2. Closed polygonal path: A polygonal path is where the end point of the final line segment is the start point of the first line segment.
3. Simple: If it does not cross over itself at any point in time.

the complement of a simple closed polygonal path is made up of an interior and exterior

Example

Example

Describe the following sets in terms of if they are open, closed, bounded, compact, connected, simply connected, regions or domains.

1. $S_1 = \{z \in \mathbb{C} : |z| < 4\}$
2. $S_2 = \{z \in \mathbb{C} : |z| \leq 4\}$
3. $S_3 = \{p\}$
4. $S_4 = \{z \in \mathbb{C} : 1 < |z| < 4\}$
5. $S_5 = \{z \in \mathbb{C} : |z| > 4\}$

Solution

Set	Open/Closed	Bounded	Compact	Connected
S_1	Open	Yes	No	Yes
S_2	Closed	Yes	Yes	Yes
S_3	Closed	Yes	Yes	Yes
S_4	Open	Yes	No	Yes
S_5	Open	No	No	Yes

Example

Set	Simply Connected	Region	Domain
S_1	Yes	Yes	Yes
S_2	Yes	Yes	No
S_3	Yes	No	No
S_4	No	Yes	Yes
S_5	No	Yes	Yes

Functions

Complex function

A complex function is one whose domain, or whose range, or both, is a subset of the complex plane $\mathbb{C}$ that is not a subset of the real line $\mathbb{R}$

Most of the concepts from real analysis carry over to complex functions such as domain, codomain, range, image and preimage but with extra functions like $z \to |z|$, the Arg function and $z \to \bar{z}$.

Fractional Linear Transformations

Complex function

These are the functions of the form

$$f(z) = \frac{az + b}{cz + d}$$

where a, b, c, $d \in \mathbb{C}$ and $ad - bc \neq 0$.

If f is a fractional linear transformation and z varies on a line or on a circle, then $f(z)$ varies on a line or on a circle.
What happens when $z \to -d/c$?

Example

Example

Consider the fractional linear transformation $T(z) = \frac{z}{z+2}$. Find the image of the circle $C := \{z \in \mathbb{C} : |z - 1| = 1\}$.

Solution

The image of a circle by a FLT is either a circle or a line. Since $-2 \notin C$ we have $T(C)$ is bounded and so C is a circle. Now $1 = |z-1|^2 = (z-1)(\bar{z}-1) = |z|^2 - z - \bar{z} + 1$ that is equivalent to

$$|z|^2 = z + \bar{z}.$$

Observe that

$$T_M(z) = \frac{z(\bar{z}+2)}{|z+2|^2} = \frac{|z|^2 + 2z}{|z+2|^2}$$

and thus

$$u = \frac{|z|^2 + z + \bar{z}}{|z+2|^2} = 2\frac{|z|^2}{|z+2|^2} = 2|T_M(z)|^2 = 2(u^2 + v^2)$$

We obtain that $(u - 1/4)^2 + v^2 = 1/16$ and thus $T_M(C)$ is the circle centre $1/4$ with radius $1/4$.

Estimating Sizes of functions

Bounding functions by a size

This just basically involves using Extended Triangle inequality to bound function sizes given some size of z.

Example

Suppose that $f(z) = \frac{1}{z^4-1}$ for all $z \in \mathbb{C} - \{\pm 1, \pm i\}$. Show that $|f(z)| \leq \frac{1}{15}$ for $|z| > 2$.

Solution

By Extended Triangle inequality, we have
$|z^4 - 1| \geq ||z|^4 - 1| = |R^4 - 1| = R^4 - 1 \geq 15$ since $R \geq 2$ and thus $R^4 - 1 > 0$. Since both sides of the inequality are greater than 0, we may reciprocate both sides to yield:

$$\left|\frac{1}{z^4 - 1}\right| \leq \frac{1}{15}$$

2. Limits and Differentiability

Existence of limits

1. A limit is said to not exist if the function attains different values along different paths.
2. A limit is said to exist, that is, there is a unique $l \in \mathbb{C}$ with $\lim_{z \to z_0} f(z) = l$ if the following statement holds true: For every $\epsilon > 0$, there is a $\delta > 0$ such that $0 < |z - z_0| < \delta \implies |f(z) - l| < \epsilon$.
   Simply put: $f(z)$ gets close to l whenever z gets close to z_0.

Example

Example 6

Prove that $\lim_{z\to 1+i} z^2 = 2i$ using the definition of limits.

Example 7

Prove that the following limit does not exist:

$$\lim_{z\to 0} \frac{\mathrm{Re}(z)}{z}$$

Example 6 Solution

By definition of a limit, we seek a δ such that for every $\epsilon > 0$, we have $0 < |z - (1+i)| < \delta \implies |z^2 - 2i| < \epsilon$.

$$\begin{aligned} |z^2 - 2i| &= |z - (1+i)||z + (1+i)| \\ &= |z - (1+i)||z - (1+i) + (2+2i)| \\ &\leq \delta(\delta + 2\sqrt{2}) && \text{(By Triangle Inequality)} \\ &< \epsilon \end{aligned}$$

Where we select δ such that $\delta <$ the positive solution of $\delta^2 + 2\sqrt{2}\delta - \epsilon = 0 \implies \delta = \frac{-2\sqrt{2}+\sqrt{8+4\epsilon}}{2}$.

Example 7 Solution

Consider the path $z = iy$, then the limit becomes:

$$\lim_{y \to 0} \frac{0}{0 + iy} = 0$$

Consider the path $z = x + 0i$, then the limit becomes:

$$\lim_{x \to 0} \frac{x}{x + 0i} = 1$$

Since the limits along the 2 different paths are different, the limit expression does not exist.

Limit Properties

Properties and relationships

1. $\lim_{z \to z_0} f(z) \pm g(z) = L_1 \pm L_2$
2. $\lim_{z \to z_0} f(z)g(z) = L_1 L_2$
3. $\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{L_1}{L_2}$ provided that $L_2 \neq 0$.

Note that for polynomial function $f(z) = \sum_{k=0}^{n} a_k z^k$, for positive integer n, and $a_k \in \mathbb{C}$ for each k, we have the more specific limit:

$$\lim_{z \to a} f(z) = f(a)$$

Continuity

Definition: Continuity

A function is said to be continuous if:

$$\lim_{z \to a} f(z) = f(a)$$

That is, it's function value is equal to the limit of the function as that point.

As a result, we can say that the sum and product of continuous functions are always continuous. The quotient of 2 continuous functions, provided that the denominator does not evaluate to 0, is also continuous. If a function is continuous for each value of z on it's domain S, then we say that f is continuous on S.

Differentiability

Definition: Differentiability

1. The function values get close to each other quicker than the inputs get closer to each other.
2. A function is <u>differentiable</u> if the following limit exists:

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0)$$

Note: Differentiability implies continuity.

The differentiation rules from real numbers apply as usual.

Cauchy-Riemann Equations

Cauchy-Riemann Equations

The Cauchy-Riemann Equations state that a function $f(x+iy) = u(x,y) + iv(x,y)$ is differentiable at an interior point $z = a \in \mathrm{dom}(f)$ if and only if the partial derivatives of u, v all exist and are continuous and:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

at $x + iy = a$. The derivative, provided it exists, of f is given by $f'(z) = u_x(x,y) + iv_x(x,y)$.

To find out where a function is differentiable, you solve the 2 equations simultaneously and solve for all possible pairs of values of x, y.

Examples

Example 8

Where are the following functions differentiable?

1. $f_1(z) = z|z|^2$
2. $f_2(x+iy) = x^2 + iy^2$
3. $f_3(x+iy) = |x| + i|y|$

Example 8 Solution

1. $f_1(z) = (x+iy)(x^2+y^2) = (x^3+xy^2) + i(yx^2+y^3)$. Then by the Cauchy-Riemann Equations:

$$\frac{\partial u}{\partial x} = 3x^2 + y^2, \quad \frac{\partial u}{\partial y} = 2xy, \quad \frac{\partial v}{\partial x} = 2xy, \quad \frac{\partial v}{\partial y} = 3y^2 + x^2$$

   We have $2xy = -2xy \implies xy = 0$. We also have $3x^2 + y^2 = 3y^2 + x^2 \implies 2x^2 = 2y^2$. Therefore, $x = \pm y$. Thus $x = y = 0$ is the only solution.
2. $f_2(z) = x^2 + iy^2 \implies 2x = 2y, 0 = -0 \implies x = y$. Hence the function is differentiable $z = x + ix$.
3. Similar to the previous part, we require when $\frac{x}{|x|} = \frac{|y|}{y} \implies xy = |xy|$ (upon noting that the derivative of $|x| = \frac{x}{|x|}$). The above is only true when $xy > 0$.

Holomorphic

Definitions

A function is said to be holomorphic at a if the function is differentiable in some neighbourhood of a (an open disk with centre a). A function that is holomorphic everywhere is called entire.

Thus if the function is differentiable on an open set, it is holomorphic in that set.

Holomorphic-ness

A function can only ever be holomorphic on an open set. So if a function is differentiable on a closed set, it will NOT be holomorphic.

Harmonic Functions

Definition

Let D be a domain in $\mathbb{R}^3$. A function of 2 variables u is harmonic if it satisfies Laplace's equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

and al first and 2nd partials are continuous in D. The harmonic conjugate is a function of 2 variables v so that the Cauchy-Riemann equations are satisfied.

Harmonic Conjugates

It has absolutely nothing to do with the actual conjugate of a complex number.

Properties of Harmonic Conjugates

Corollaries

1. $-u$ is the harmonic conjugate of v.
2. If u is a harmonic on a simply connected domain, then u has a harmonic conjugate on D.
3. Harmonic conjugates of u only ever differ by a constant.
4. Let f be a function holomorphic at $z = a$. Then $f(z)$ admits a power series expansion about a (not needed for now, but is a master-key for later).
5. Let f, g be 2 holomorphic functions on D and C be a smooth curve in D. If $f(z) = g(z)$ for each $z \in C$, then $f(z) = g(z)$ for each $z \in D$.

Examples

Example 9

Show that $\cos x \cosh y$ is harmonic and find its harmonic conjugate.

Example 10

Show that $\frac{x}{x^2+y^2}$ is harmonic and find its harmonic conjugate.

Example 9 Solution

$u(x,y) = \cos x \cosh y \implies \partial_x^2 u = -\cos x \cosh y, \partial_y^2 u = \cos x \cosh y \implies \partial_x^2 u + \partial_y^2 u = 0$. Hence u is harmonic.
The harmonic conjugate is given by solving the CRE's. Let v be the harmonic conjugate so that
$\partial_x v = -\cos x \sinh y \implies v(x,y) = -\sin x \sinh y + f(y)$.
$\partial_y v = -\sin x \cosh y \implies f'(y) = 0 \implies f(y) = C$. Hence the harmonic conjugate is $v(x,y) = -\sin x \sinh y + C$.

Example 10 Solution

Using the same idea as above, you should obtain $v(x, y) = \frac{-y}{x^2+y^2}$.

Exponential Function

Definition

The exponential series is defined by

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \ \ \forall z \in \mathbb{C}.$$

Properties

- $\exp(x+iy) = e^x(\cos y + i \sin y) \ \ \forall x, y \in \mathbb{R}$
- Periodicity: $\exp(z) = \exp(z + 2\pi i k) \ \ \forall k \in \mathbb{Z}$.

Even More Properties of Exponential

Properties of Exponential Function

For all $z, w \in \mathbb{C}$

- $\exp(0) = 1$
- $\exp(z + w) = \exp(z)\exp(w)$
- $\exp(-z) = \exp(z)^{-1}$
- $\exp(z) \neq 0$
- $\exp'(z) = \exp(z)$
- If a function $f : \mathbb{C} \to \mathbb{C}$ satisfies $f(0) = 1$ and $f'(z) = f(z)$ for all $z \in \mathbb{C}$, then $f(z) = \exp(z)$ for all $z \in \mathbb{C}$.

Hyperbolic Functions

Definition

The hyperbolic cosine and sine functions are defined by

$$\cosh(z) = \frac{e^z + e^{-z}}{2} = \sum_{n \in \mathbb{N}} \frac{z^{2n}}{(2n)!},$$

$$\sinh(z) = \frac{e^z - e^{-z}}{2} = \sum_{n \in \mathbb{N}} \frac{z^{2n+1}}{(2n+1)!}.$$

Properties

For all $w, z \in \mathbb{C}$,

- $\cosh(z + w) = \cosh(z)\cosh(w) + \sinh(z)\sinh(w)$
- $\sinh(z + w) = \sinh(z)\cosh(w) + \cosh(z)\sinh(w)$
- $\cosh'(z) = \sinh(z), \sinh'(z) = \cosh(z)$.

The Properties Start Coming and They Don't Stop Coming

Properties of Hyperbolic Sine and Cosine Functions

For all $z \in \mathbb{C}$ and $k \in \mathbb{Z}$,

- $\cosh(-z) = \cosh(z)$
- $\sinh(-z) = -\sinh(z)$
- $\cosh(z + 2\pi i k) = \cosh(z)$
- $\sinh(z + 2\pi i k) = \sinh(z)$
- $\cosh^2(z) - \sinh^2(z) = 1.$

Trigonometric Functions

Definition

The cosine and sine functions are defined by

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} = \sum_{n\in\mathbb{N}} \frac{(-1)^n z^{2n}}{(2n)!},$$

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} = \sum_{n\in\mathbb{N}} \frac{(-1)^n z^{2n+1}}{(2n+1)!}.$$

Properties

- $\sinh(iz) = i\sin(z), \cosh(iz) = \cos(z)$
- $\sin(iz) = i\sinh(z), \cos(iz) = \cosh(z)$

What Do You Mean You've Seen This? It's Brand New

Properties of Complex Sine and Cosine Functions

For all $w, z \in \mathbb{C}$,

- $\cos(-z) = \cos(z)$
- $\sin(-z) = -\sin(z)$
- $\cos'(z) = -\sin(z)$
- $\sin'(z) = \cos(z)$
- $\cos(z + 2\pi k) = \cos(z)$
- $\sin(z + 2\pi k) = \sin(z)$
- $\cos(z + w) = \cos(z)\cos(w) - \sin(z)\sin(w)$
- $\sin(z + w) = \sin(z)\cos(w) + \cos(z) sin(w)$
- $\cos^2(z) + \sin^2(z) = 1.$

Logarithm

Definition

The principal branch of the logarithm is

$$\mathrm{Log}(z) = \ln|z| + i\mathrm{Arg}(z),$$

where $\mathrm{Arg}(z)$ takes values in range $(-\pi, \pi]$, and $\mathrm{Log}(z)$ is differentiable on $\mathbb{C}\backslash(-\infty, 0]$.

The multi-valued logarithm is

$$\log(z) = \ln|z| + i\arg z = \ln|z| + i(\mathrm{Arg} z + 2k\pi), k \in \mathbb{Z}$$

Branches of Logarithms

Definition

For any real number θ, a branch of the logarithm is defined by

$$\log_\theta(z) = \ln|z| + i\arg z,$$

where $\theta < \arg z \leq \theta + 2\pi$, and $\{z : z = 0 \text{ or } \arg z = \theta\}$ is the branch cut.

Note: $\log_\theta(z)$ is differentiable everywhere, with derivative $\frac{d}{dz}\log_\theta z = \frac{1}{z}$, except on branch cut.

Choosing Branch of Logarithm

Example

Define a branch $\log_\theta$ of the multi-valued function $\log(z)$ that is continuous at $z = -1$ and has value $5\pi i$ there. What is $\log_\theta(-i)$ for this branch?

The principal branch is not continuous at -1, so must choose different branch cut, for example, take branch cut along positive real axis. This means that

$$2k\pi < \arg z \leq 2k\pi + 2\pi, k \in \mathbb{Z}.$$

Since $\log_\theta(-1) = i \arg z = 5\pi i$, then choose $4\pi < \arg z \leq 6\pi$.
This gives $\theta = 4\pi$, and the branch is defined by

$$\log_{4\pi}(z) = \ln|z| + i \arg z, 4\pi < \arg z \leq 6\pi.$$

Hence, $\log_{4\pi}(-i) = \ln|-i| + i[-\frac{\pi}{2} + 3(2\pi)] = \frac{11\pi i}{2}$

Example: Differentiability of Logarithm

Example

Determine where $f(z) = \operatorname{Log}(2iz - 1)$ is differentiable.

The principal branch of the logarithm is differentiable everywhere except on branch cut, which means the values of z where f is not differentiable are

$$
\begin{aligned}
2iz - 1 &= \lambda, \lambda \in (-\infty, 0] \\
z &= \frac{1}{2i}(1 + \lambda) \\
z &= \frac{-i}{2}(1 + \lambda) \\
z &\in [-\frac{i}{2}, \infty)
\end{aligned}
$$

So z is differentiable everywhere except for the segment of imaginary axis from $-\frac{i}{2}$ upwards.

MATH2621 2014 Final Q1(ii)

Example

Find all complex numbers z such that $\sin(z) = 2i$.

Firstly, use the definition of $\sin(z)$ to write

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} = 2i.$$

Let $w = e^{iz}$, and multiply by w to get the quadratic equation

$$w^2 + 4w - 1 = 0.$$

Use the quadratic equation to solve for w, giving

$$w = \frac{-4 \pm \sqrt{20}}{2} = -2 \pm \sqrt{5} = e^{iz}.$$

MATH2621 2014 Final Q1(ii) Continued

Example

Find all complex numbers z such that $\sin(z) = 2i$.

Solve for iz using logarithms.

When $w = -2 + \sqrt{5}$, then

$$\begin{aligned} iz &= \log(-2+\sqrt{5}) \\ &= \ln|-2+\sqrt{5}| + \arg(-2+\sqrt{5}) \\ &= \ln(-2+\sqrt{5}) + i(0+2k\pi) \\ z &= -i\ln(\sqrt{5}-2) + 2k\pi, k \in \mathbb{Z}. \end{aligned}$$

MATH2621 2014 Final Q1(ii) Continued

Example

Find all complex numbers z such that $\sin(z) = 2i$.

Similarly, when $w = -2 - \sqrt{5}$, then

$$\begin{aligned} iz &= \log(-2 - \sqrt{5}) \\ &= \ln|-2 - \sqrt{5}| + \arg(-2 - \sqrt{5}) \\ &= \ln(2 + \sqrt{5}) + i(\pi + 2k\pi) \\ z &= -i\ln(2 + \sqrt{5}) + (2k+1)\pi, k \in \mathbb{Z}. \end{aligned}$$

Complex Powers

Definition

The principal branch of z^α is defined by

$$\mathrm{PV}z^\alpha = \exp(\alpha \mathrm{Log} z).$$

The multi-valued function z^α is defined by

$$z^\alpha = \exp(\alpha \log z) = \exp(\alpha \mathrm{Log} z + 2\pi i k \alpha), k \in \mathbb{Z}.$$

Evaluating Complex Power

Example

Find $(-i)^{2i}$ and PV $(-i)^{2i}$.

$$
\begin{aligned}
(-i)^{2i} &= \exp[2i\log(-i)] \\
&= \exp\{2i[\ln|-i| + i(-\frac{\pi}{2} + 2k\pi)]\} \\
&= \exp[-2(-\frac{\pi}{2} + 2k\pi)] \\
&= \exp(\pi - 4k\pi), k \in \mathbb{Z} \\
\mathrm{PV}(-i)^{2i} &= \exp(\pi).
\end{aligned}
$$

Complex Power Again but Slightly Spicier

Example

Evaluate $\lim_{z\to 0} \mathrm{PV}(\cos z)^{\frac{1}{z^2}}$.

$$\begin{aligned}\lim_{z\to 0} \mathrm{PV}(\cos z)^{\frac{1}{z^2}} &= \lim_{z\to 0} \exp\left(\frac{1}{z^2}\mathrm{Log}(\cos z)\right)\\ &= \exp\left(\lim_{z\to 0}\frac{\mathrm{Log}(\cos z)}{z^2}\right).\end{aligned}$$

Use L'Hopital's rule to evaluate this.

$$\begin{aligned}\lim_{z\to 0} \mathrm{PV}(\cos z)^{\frac{1}{z^2}} &= \exp\left(\lim_{z\to 0}\frac{\left(\frac{-\sin z}{\cos z}\right)}{2z}\right)\\ &= \exp\left(\lim_{z\to 0}\frac{-\sec^2 z}{2}\right) = \exp(-\frac{1}{2}).\end{aligned}$$

Differentiability of Complex Powers

Example

Where is the principal branch of $f(z) = \sqrt{z+1}$ analytic?

$$f(z) = \exp\left(\frac{1}{2}\mathrm{Log}(z+1)\right)$$

So function is analytic everywhere except $z \in (-\infty, -1]$.

Differentiability Again

Example

Where is the principal branch of $f(z) = \sqrt{z^2 - 1}$ analytic?

$$f(z) = \exp\left(\frac{1}{2}\mathrm{Log}(z^2 - 1)\right)$$

So function is analytic everywhere except $z^2 \in (-\infty, 1]$, which means function is analytic everywhere except $\{\lambda i : \lambda \in \mathbb{R}\} \cup [-1, 1]$.

3. Contour Integration

Curves

Definition

A **curve** in $\mathbb{C}$ is a continuous function $\gamma : [a, b] \to \mathbb{C}$.

The **initial point** of the curve is $\gamma(a)$ and the **final point** is $\gamma(b)$.

The **range** of a curve is the set $\{\gamma(t) : t \in [a, b]\}$.

A curve is **closed** if $\gamma(a) = \gamma(b)$, and **simple** if $\gamma(s) \neq \gamma(t)$ when $s < t$, except for possibly $s = a, t = b$, in which case it is a **simple closed** curve/**Jordan** curve.

What's That Curve

Example

Classify the following curves as simple and/or closed.

Simple curves do not cross over themselves, except for possibly the initial and final points. Closed curves have the same initial and final point.

The first curve is simple but not closed. The second curve is closed but not simple. The third curve is simple and closed (Jordan curve).

Curves

We can combine and flip curves, as you might expect.

Definition

Let $\alpha : [a,b] \to \mathbb{C}$ and $\beta : [c,d] \to \mathbb{C}$ be curves, with $\alpha(b) = \beta(c)$. Then the **join of α and β** is

$$(\alpha \sqcup \beta)(t) = (\alpha + \beta)(t) = \begin{cases} \alpha(t), & a \leq t \leq b; \\ \beta(t), & c \leq t \leq d. \end{cases}$$

Definition

Let $\gamma : [a,b] \to \mathbb{C}$ be a curve. Then the **reverse curve** $\gamma^* : [-b,-a] \to \mathbb{C}$ is

$$\gamma^*(t) = \gamma(-t).$$

Parameterisation Of Lines and Circles

Example

a) Parameterise the straight line between distinct points z_0 and z_1.
b) Parameterise the circle with centre z_0 and radius r.

a)

$$\gamma(t) = z_0 + t(z_1 - z_0) \text{ for } t \in [0, 1]$$

Notice that this is the parametric vector form of lines.

b)

$$\gamma(t) = z_0 + re^{it} \text{ for } t \in [0, 2\pi]$$

Derivatives of Curves

Derivatives are used a lot in contour integration, so we define it for curves in the complex plane.

Definition

Suppose $\gamma : [a, b] \to \mathbb{C}$ is a curve, with $\gamma(t) = \gamma_1(t) + \gamma_2(t)i$ and γ_1, γ_2 are real-valued (real and imaginary components). Then we define the **derivative**

$$\gamma'(t) = \gamma_1'(t) + \gamma_2'(t)i.$$

We say that γ is **continuously differentiable** if the derivative exists and is continuous on $[a, b]$.
We say that γ is **smooth** if it is continuously differentiable, and $\gamma'(t) \neq 0$ for all $t \in [a, b]$.
We say that γ is **piecewise smooth** if it is a finite join $\gamma = \alpha_1 + \alpha_2 + \cdots + \alpha_n$, and all α_i are smooth.

Smooth Criminal

Example

Is the curve $\gamma : [-1, 1] \to \mathbb{C}$ given by

$$\gamma(t) = |t| + it$$

smooth? Piecewise smooth?

The derivative of $|t|$ doesn't exist at $t = 0$, so the curve is not smooth. However, this curve can be split into

$$\gamma_1(t) = t + it, t \in [0, 1],$$
$$\gamma_2(t) = -t + it, t \in [-1, 0].$$

Then both γ_1 and γ_2 are smooth, and $\gamma = \gamma_1 + \gamma_2$. Thus, γ is piecewise smooth.

Curve Length

Almost every curve you'll deal with will be piecewise smooth, but keep in mind the piecewise smooth condition if you're asked to define the terms.

Definition

The **length** of a piecewise smooth curve $\gamma : [a, b] \to \mathbb{C}$ is

$$\text{Length}(\gamma) = \int_a^b |\gamma'(t)| dt.$$

The Lengths You Go to Pursue Your Passion of Maths

Example

Find the length of the curve

$$\gamma(t) = Re^{it},$$

for $t \in [0, 2n\pi]$, $n \in \mathbb{N}$, and $R > 0$.

$\gamma'(t) = Rie^{it}$, so the length of our curve is:

$$\begin{aligned}
\text{Length}(\gamma) &= \int_0^{2n\pi} |Rie^{it}| dt \\
&= \int_0^{2n\pi} R dt \\
&= 2n\pi R.
\end{aligned}$$

Contours

Definition

A **contour** is the oriented range of a piecewise smooth curve γ. In other words, it is the range $\mathrm{Range}(\gamma)$ with some orientation describing how this set should be traversed.

Generally these are described as a set in the complex plane, traversed in some manner. If the contour is simple (doesn't cut itself), then we traverse it **anticlockwise** or **clockwise**.

Complex Integration

Definition

We define the **integral** of a complex-valued function $f : [a, b] \to \mathbb{C}$ where $f(t) = u(t) + v(t)i$ and u, v are both real-valued as

$$\int_a^b f(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt.$$

Effectively, we treat the imaginary unit i as just another constant.

Complex Integration

Just as with real integrals, we have some familiar identities.

Theorem

Let $f : [a, b] \to \mathbb{C}$ and $g : [a, b] \to \mathbb{C}$. Further, let $h : [c, d] \to [a, b]$ be a differentiable with $h(c) = a$, $h(d) = b$, and $\lambda, \mu \in \mathbb{C}$. Then

- $\displaystyle\int_a^b \lambda f(t) + \mu g(t)dt = \lambda \int_a^b f(t)dt + \mu \int_a^b g(t)dt,$
- $\displaystyle\int_c^d f(h(t))h'(t)dt = \int_a^b f(t)dt,$
- $\displaystyle\int_a^b f'(t)g(t)dt = [f(t)g(t)]_a^b - \int_a^b f(t)g'(t)dt,$
- $\displaystyle\left|\int_a^b f(t)dt\right| \leq \int_a^b |f(t)|dt.$

Complex Integration

Example

Integrate $f(t) = te^{it}$ over $[0, 2\pi]$.

$$\begin{aligned}\int_0^{2\pi} te^{it} dt &= \left[t\frac{e^{it}}{i}\right]_0^{2\pi} - \int_0^{2\pi} \frac{e^{it}}{i} dt \\ &= -2\pi i - \left[\frac{e^{it}}{i^2}\right]_0^{2\pi} \\ &= -2\pi i.\end{aligned}$$

Complex Integration but This is Your Last Moment of Peace

Example

Using the previous example, deduce that

$$\int_0^{2\pi} t\cos t\,dt = 0.$$

We have

$$\begin{aligned}\int_0^{2\pi} t\cos t\,dt &= \int_0^{2\pi} \operatorname{Re}\left(te^{it}\right) dt\\ &= \operatorname{Re}\left(\int_0^{2\pi} te^{it} dt\right)\\ &= \operatorname{Re}(-2\pi i)\\ &= 0.\end{aligned}$$

Line Integration

Definition

Given a piecewise smooth curve $\gamma : [a, b] \to \mathbb{C}$, we define the **complex line integral**

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

You can think of this as using the substitution $z = \gamma(t)$.

Line Integration

Theorem

Let $\lambda, \mu \in \mathbb{C}$, α, β be piecewise smooth curves, and f, g be complex functions defined on $\text{Range}(\gamma)$. Further, let $\gamma = \alpha + \beta$. Then

- $\displaystyle\int_\alpha \lambda f(z) + \mu g(z)dz = \lambda \int_\alpha f(z)dz + \mu \int_\alpha g(z)dz,$
- $\displaystyle\int_{\alpha^*} f(z)dz = -\int_\alpha f(z)dz,$
- $\displaystyle\int_\gamma f(z)dz = \int_\alpha f(z)dz + \int_\beta f(z)dz.$

Line Integration

Example

Find $\int_\gamma \bar{z} dz$ where γ is the straight line from 0 to $1+i$.

The parameterisation of γ is

$$\begin{aligned} \gamma(t) &= 0 + t(1 + i - 0) = (1+i)t, t \in [0,1]. \\ \gamma'(t) &= 1 + i. \end{aligned}$$

Now, evaluate the integral

$$\begin{aligned} \int_\gamma \bar{z} dz &= \int_0^1 (1-i)t(1+i) dt \\ &= \int_0^1 2t dt \\ &= 1. \end{aligned}$$

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Examples

Evaluate the integral $\int_\gamma \bar{z}dz$, where $\gamma(t) = t + it^2$ for $t \in [0, 1]$.

$$\begin{aligned}
\int_\gamma \bar{z}dz &= \int_0^1 f(\gamma(t))\gamma'(t)dt \\
&= \int_0^1 \overline{(t + it^2)}(1 + 2it)dt \\
&= \int_0^1 (t - it^2)(1 + 2it)dt \\
&= \int_0^1 (2t^3 + it^2 + t)dt \\
&= 1 + \frac{i}{3}
\end{aligned}$$

Reparameterisation

We could write a curve $\gamma(t) = t$ on $[0, 1]$, or $\delta(t) = t + 1$ on $[-1, 0]$. These describe the same curve in different ways, so we formalise this.

Definition

Suppose that $\gamma : [a, b] \to \mathbb{C}$ is a curve, and $h : [c, d] \to [a, b]$ is a continuous bijection such that $h(c) = a$ and $h(d) = b$. Then we call $\gamma \circ h : [c, d] \to \mathbb{C}$ a **reparameterisation of** γ.

Definition

Let γ, δ be piecewise smooth curves, where δ is a reparameterisation of γ, and f be complex-valued defined on $\text{Range}(\gamma)$. Then

$$\int_\gamma f(z)dz = \int_\delta f(z)dz.$$

ML Lemma/Estimation Lemma

Lemma

Let γ be a piecewise smooth curve and f be a complex-valued function defined on $\mathrm{Range}(\gamma)$. Then

$$\left|\int_\gamma f(z)dz\right| \leq ML,$$

where L is the length of γ, and M is a maximiser of $|f|$ on $\mathrm{Range}(\gamma)$.

The estimation; of your enthusiasm; for integration (whoops there's an upper bound)

Example

Let γ be the circle centred at the origin with radius 3, and $f(z) = \dfrac{z^2 - 2i}{(z-4i)(z+1)}$. Show that $\left|\int_\gamma f\right| \leq 33\pi$.

Since $|z| = 3$, then by triangle inequality,

$$|z^2 - 2i| \leq |z|^2 + 2 = 3^2 + 2 = 11.$$

Using circle inequality gives

$$|z - 4i||z+1| \geq \Big||z| - 4\Big| \cdot \Big||z| - 1\Big| = |3-4||3-1| = 2.$$

So $|f(z)| \leq \frac{11}{2}$. Since γ is a circle, $L = 2\pi \times 3 = 6\pi$. Using the estimation lemma, $\left|\int_\gamma f\right| \leq \dfrac{11}{2} \times 6\pi = 33\pi$

Contour Integration

Definition

Given a contour Γ, we define the **contour integral**

$$\int_{\Gamma} f(z)dz = \int_{\gamma} f(z)dz,$$

where γ is any parameterisation of Γ. This is well-defined, as the complex line integral is independent of parameterisation.

Cauchy-Goursat Theorem

Cauchy-Goursat

Suppose that Ω is a simply connected domain, that $f \in H(\Omega)$, and that Γ is a closed contour in Ω. Then

$$\int_{\Gamma} f(z)dz = 0.$$

Suppose that Ω is a bounded domain whose boundary consists of finitely many contours $\Gamma_1, \Gamma_2, \cdots, \Gamma_n$. Further, suppose f is holomorphic on an open set containing $\overline{\Omega}$. Then

$$\int_{\partial\Omega} f(z)dz = \sum_{k=1}^{n} \int_{\Gamma_k} f(z)dz = 0.$$

Cauchy-Goursat but Nothing is Scary

Example

Find $\displaystyle\int_{\gamma_1} \frac{1}{z-2i} dz$, where γ_1 is the circle centred at -1 of radius $\sqrt{2}$ traversed anti-clockwise.

Let $f(z) = \dfrac{1}{z-2i}$. Then f is analytic on $\mathbb{C}\backslash\{2i\}$, so is analytic on and inside simple closed contour γ_1. By Cauchy-Goursat Theorem,

$$\int_{\gamma_1} \frac{1}{z-2i} = 0.$$

Cauchy-Goursat but It's Sudden Death

Example

Find

$$\int_{-\infty}^{\infty} \frac{x^2 - 1}{(x^2 + 1)^2} dx$$

by considering an integral of

$$f(z) = \frac{1}{(z + i)^2}.$$

Let Γ_R be the upper semicircular arc of radius $R > 0$ around 0, and $\Gamma_x = [-R, R]$. $f(z)$ is holomorphic on the set $\{z \in \mathbb{C} : \mathrm{Im}(z) > -1\}$, and the join of Γ_R and Γ_x (say Γ) lies inside this domain. Thus, by Cauchy-Goursat,

$$\int_{\Gamma} f(z) dz = 0.$$

Cauchy-Goursat Continued

Now, we can evaluate each part of the contour integral separately. Note that on Γ_R, by the circle inequality we have

$$|f(z)| = \left|\frac{1}{(z+i)^2}\right| \leq \frac{1}{(R-1)^2},$$

when $R > 1$. Since $\text{Length}(\Gamma_R) = \pi R$, by ML lemma,

$$\lim_{R\to\infty}\left|\int_{\Gamma_R} f(z)dz\right| \leq \lim_{R\to\infty}\frac{1}{(R-1)^2}\cdot \pi R = 0.$$

So, we deduce that the integral is zero as $R \to \infty$.

Cauchy-Goursat but Scroll Back Up To Recall the Question

Along $\mathit{\Gamma}_x$, we have

$$\int_{\mathit{\Gamma}_x} f(z)dz = \int_{-R}^{R} \frac{1}{(x+i)^2}\,dx = \int_{-R}^{R} \frac{(x-i)^2}{(x^2+1)^2}\,dx.$$

Combining this with the integral along $\mathit{\Gamma}_R$, we have

$$\begin{aligned}\lim_{R\to\infty} \operatorname{Re}\left(\int_{\mathit{\Gamma}} f(z)dz\right) &= \operatorname{Re}\left(\int_{-\infty}^{\infty} \frac{(x-i)^2}{(x^2+1)^2}\,dx\right)\\ &= \int_{-\infty}^{\infty} \frac{x^2-1}{(x^2+1)^2}\,dx\\ &= 0.\end{aligned}$$

Consequences of Cauchy-Goursat

Independence of Contour

Suppose Ω is a simply connected domain, f is holomorphic on Ω, and Γ, Δ are two contours with the same initial and final points. Then

$$\int_{\Gamma} f(z)dz = \int_{\Delta} f(z)dz.$$

Existence of Primitives

Suppose that Ω is a simply connected domain in $\mathbb{C}$, and that $f \in H(\Omega)$. Then there exists a function F on Ω such that

$$\int_{\Gamma} f(z)dz = F(q) - F(p)$$

for all contours Γ in Ω from p to q. Further, F is differentiable, and $F' = f$. F is the **primitive** or **anti-derivative** of f.

Primitive Question: Level Primitive

Example

Let γ be the straight line from 0 to $1+2i$ followed by the parabola from $1+2i$ to $1-i$. Find $\int_\gamma \sin 2z dz$.

$f(z) = \sin 2z$ has primitive $F(z) = -\frac{1}{2}\cos 2z$, which is analytic on $\mathbb{C}$.

$$\begin{aligned}\int_\gamma \sin 2z dz &= F(1-i) - F(0) \\ &= -\frac{1}{2}[\cos(2(1-i)) - 1] \\ &= \frac{1}{2}[1 - \cos(2(1-i))].\end{aligned}$$

Primitive Question: Level Logarithm Crossover

Example

Let γ be any contour from i to $-i$ which lies entirely in the set where $\text{Re}(z) \le 0$. Calculate $\int_\gamma (z-1)^{-1} dz$.

$\text{Log}(z-1)$ is not analytic on $z \in (-\infty, 1]$, and since γ must pass through this branch cut, then need to use a different branch of the log as the primitive.
Choose branch cut to be on the positive real axis instead, so the branch is

$$\log_0(z-1) = \ln|z-1| + i\arg(z-1), 0 < \arg(z-1) \le 2\pi,$$

which is analytic everywhere except $z \in [1, \infty)$. Hence,
$F(z) = \log_0(z-1)$ is analytic on domain γ is in.

Primitive Question Continued: Level Return to Primitive

Example

Let γ be any contour from i to $-i$ which lies entirely in the set where $\mathrm{Re}(z) \leq 0$. Calculate $\int_\gamma (z-1)^{-1} dz$.

$$\begin{aligned}
\int_\gamma \frac{1}{z-1} dz &= F(-i) - F(i) \\
&= \log_0(-i-1) - \log_0(i-1) \\
&= \ln|-1-i| + \arg(-1-i) - \ln|-1+i| - \arg(-1+i) \\
&= \ln\sqrt{2} + i(-\frac{3\pi}{4} + 2\pi) - \ln\sqrt{2} + i(-\frac{3\pi}{4}) \\
&= \frac{1}{2}\pi i.
\end{aligned}$$

UNSW Mathematics Society Presents

MATH2521/2621 Seminar

Presented by Felix, Wendy, Raymond, and the constant desire to convince people that imaginary numbers are vital to everyday life

Cauchy's Integral Formula

Cauchy's Integral Formula

Suppose that Ω is a simply connected domain, f is holomorphic on Ω, Γ is a simple closed contour in Ω, and $w \in \operatorname{Int}(\Gamma)$. Then

$$f(w) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{z - w} dz.$$

This allows us to handle integration of functions that aren't holomorphic at a point (to some extent).

Cauchy's Integral Formula Example

Examples

Find

$$\int_{|z|=2} \frac{e^z dz}{z-1}$$

Cauchy's Integral Formula Example

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We apply Cauchy's Integral formula with $f(z) = e^z, w = 1,$

Cauchy's Integral Formula Example

Examples

Find

$$\int_{|z|=2} \frac{e^z dz}{z-1}$$

We apply Cauchy's Integral formula with $f(z) = e^z, w = 1$,

$$\int_{|z|=2} \frac{e^z dz}{z-1} = 2\pi i f(1) = 2\pi i e$$

Power Series

Using Cauchy's Integral Formula, we can prove that any holomorphic function can be written as a power series.

Theorem

Suppose that f is holomorphic on the ball $B(z_0, R)$, and Γ is a simple closed contour in $B(z_0, R)$ with $z_0 \in \mathrm{Int}(\Gamma)$. Then, for all $w \in B(z_0, R)$,

$$f(w) = \sum_{n=0}^{\infty} c_n (w - z_0)^n,$$

where

$$c_n = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{(z - z_0)^{n+1}} dz.$$

This shows that holomorphic functions are actually infinitely differentiable.

Cauchy's Generalised Integral Formula

Cauchy's Generalised Integral Formula

Suppose that Ω is a simply connected domain, f is holomorphic on Ω, Γ is a simple closed contour in Ω, and $w \in \mathrm{Int}(\Gamma)$. Then

$$f^{(n)}(w) = \frac{n!}{2\pi i} \int_{\Gamma} \frac{f(z)}{(z-w)^{n+1}} dz.$$

This is one of the most useful theorems of the course.

Cauchy's Generalised Integral Formula

Examples

Evaluate

$$\int_{\Gamma} \frac{1}{z^n} dz,$$

where Γ is the unit circle, and $n \in \mathbb{Z}$.

Cauchy's Generalised Integral Formula

Examples

Evaluate

$$\int_\Gamma \frac{1}{z^n}\,dz,$$

where Γ is the unit circle, and $n \in \mathbb{Z}$.

Let $f(z) = 1$ and $w = 0$. Then if $n > 0$, we can use Cauchy's generalised integral formula to evaluate

$$\int_\Gamma \frac{1}{z^n}\,dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(0) = \begin{cases} 2\pi i, & n = 1; \\ 0, & n > 1. \end{cases}$$

Cauchy's Generalised Integral Formula

Examples

Evaluate
$$\int_{\Gamma} \frac{1}{z^n} dz,$$
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If $n \leq 0$, then $\frac{1}{z^n}$ is entire, so by Cauchy-Goursat,

$$\int_{\Gamma} \frac{1}{z^n} dz = 0.$$

MATH2621 2017 Final Q3(ii)

Examples

(a) Show that if f is an entire function and $|f(z)| \leq |z| + 1$, then $f^{(k)}(0) = 0$ for $k \geq 2$.
(b) Deduce that there are complex constants a, b such that $f(z) = az + b$ and moreover $|a| \leq 1$ and $|b| \leq 1$.

Examples

(a) Show that if f is an entire function and $|f(z)| \leq |z| + 1$, then $f^{(k)}(0) = 0$ for $k \geq 2$.
(b) Deduce that there are complex constants a, b such that $f(z) = az + b$ and moreover $|a| \leq 1$ and $|b| \leq 1$.

Using Cauchy's Generalised Integral formula, we have

$$f^{(k)}(0) = \frac{k!}{2\pi i} \int_{\Gamma} \frac{f(z)}{z^{k+1}} dz.$$

MATH2621 2017 Final Q3(ii)

Let the contour Γ be the circle centred 0 with radius R oriented in the anti-clockwise direction. Note that

$$\left|\frac{f(z)}{z^{k+1}}\right| \le \frac{|z|+1}{|z|^{k+1}}$$

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Using the ML-Lemma,

$$0 \leq |f^{(k)}(0)| \leq \frac{k!}{2\pi} 2\pi R \frac{R+1}{R^{k+1}} = k! \frac{R+1}{R^k}$$

MATH2621 2017 Final Q3(ii)

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Using the ML-Lemma,

$$0 \leq |f^{(k)}(0)| \leq \frac{k!}{2\pi} 2\pi R \frac{R+1}{R^{k+1}} = k! \frac{R+1}{R^k}$$

For $k \geq 2$,

$$\lim_{R \to \infty} \frac{R+1}{R^k} = 0$$

so by the pinching theorem, $|f^{(k)}(0)| = 0$.

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Since f is entire, we can write

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

where

$$c_n = \frac{f^{(n)}(0)}{n!}$$

MATH2621 2017 Final Q3(ii)

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where

$$c_n = \frac{f^{(n)}(0)}{n!}$$

From the above part, we know $c_n = 0$ for $n \geq 2$. Hence $f(z) = c_0 + c_1 z$.

$$|c_0| = |f(0)| \leq |0| + 1 = 1$$

$$|c_1| = |f'(0)| \leq \frac{R+1}{R}.$$

MATH2621 2017 Final Q3(ii)

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$$|c_0| = |f(0)| \leq |0| + 1 = 1$$

$$|c_1| = |f'(0)| \leq \frac{R+1}{R}.$$

Since $\frac{R+1}{R} \to 1$ as $R \to \infty$, $|c_1| \leq 1$.

Liouville's and Morera's Theorems

Liouville

Suppose f is bounded and entire. Then f is constant.

Morera

Suppose that Ω is a domain, f is continuous on Ω, and

$$\int_\Gamma f(z)dz = 0$$

for every closed contour $\Gamma \subseteq \Omega$. Then f is holomorphic on Ω.

This is, to some extent, the converse of Cauchy-Goursat.

4. Series

Power Series

Definition

A **(complex) power series** is an expression of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where z, z_0, and z are complex. The largest $R > 0$ such that the power series converges in $B(z_0, R)$ is called the **radius of convergence**. If the series converges only at z_0, we say $R = 0$. If it converges everywhere, then we say $R = \infty$.

Theorem

A power series can be integrated and differentiated term-by-term inside its radius of convergence.

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Examples

Suppose that

$$h(z) = \sum_{n=1}^{\infty} n(z+3)^n$$

for all z for which the sum converges.

(a) Find the centre and radius of convergence of the pwoer series.

(b) Find an expression for h as an elementary function.

(c) Where is h analytic?

2016 Final Exam Q2v

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for all z for which the sum converges.

(a) Find the centre and radius of convergence of the pwoer series.

(b) Find an expression for h as an elementary function.

(c) Where is h analytic?

(a) By inspection the centre of convergence is -3. Define $a_n = n(z+3)^n$. By the ratio test, the sum converges when

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{n+1}{n}(z+3) \right| = |z+3| < 1$$

Hence the radius of convergence is 1.

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(b)

$$\begin{aligned}
h(z) &= (z+3)\sum_{n=1}^{\infty} n(z+3)^{n-1} \\
&= (z+3)\frac{d}{dz}\sum_{n=1}^{\infty}(z+3)^n \\
&= (z+3)\frac{d}{dz}\frac{1}{1-(z+3)} \\
&= (z+3)\frac{d}{dz}\frac{-1}{z+2} \\
&= \frac{z+3}{(z+2)^2}
\end{aligned}$$

(b)

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&= (z+3)\frac{d}{dz}\frac{-1}{z+2} \\
&= \frac{z+3}{(z+2)^2}
\end{aligned}$$

(c) h is analytic on $B(-3, 1)$

2015 Final Exam Q2iv

Examples

Suppose that

$$f(z) = \sum_{n=1}^{\infty} \frac{1}{n}(z+1)^n$$

in a particular region.

(a) Find the centre and radius of convergence of the power series.

(b) Write down the power series that represents $f'(z)$.

(c) Find an expression for f as an elementary function.

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(a) By inspection the centre of convergence is -1. Define $a_n = \frac{1}{n}(z+1)^n$. By the ratio test, the sum converges when

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{n}{n+1}(z+1)\right| = |z+1| < 1$$

Hence the radius of convergence is 1.

2015 Final Exam Q2iv

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Hence the radius of convergence is 1.

(b) We can simply differentiate the power series term by term.

$$f'(z) = \sum_{n=1}^{\infty}(z+1)^{n-1} = \sum_{n=0}^{\infty}(z+1)^n.$$

2015 Final Exam Q2iv

(c)

$$f'(z) = \sum_{n=0}^{\infty}(z+1)^n = \frac{1}{1-(z+1)} = -\frac{1}{z}$$

2015 Final Exam Q2iv

(c)

$$f'(z) = \sum_{n=0}^{\infty} (z+1)^n = \frac{1}{1-(z+1)} = -\frac{1}{z}$$

By integrating we obtain

$$f(z) = -\operatorname{Log}_0(z) + C$$

where Log_0 denotes the complex logarithm with branch cut along the positive real axis.

(c)

$$f'(z) = \sum_{n=0}^{\infty} (z+1)^n = \frac{1}{1-(z+1)} = -\frac{1}{z}$$

By integrating we obtain

$$f(z) = -\operatorname{Log}_0(z) + C$$

where Log_0 denotes the complex logarithm with branch cut along the positive real axis.

From the power series representation, $f(-1) = 0$, so we have

$$-\operatorname{Log}_0(-1) + C = -i\pi + C = 0$$

Hence

$$f(z) = -\operatorname{Log}_0(z) + i\pi$$

Taylor Series

Taylor series can be defined for complex functions exactly like real functions.

Definition

The **Taylor series** of a holomorphic function f "around" or "with centre" z_0 is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n.$$

A Taylor series around 0 ($z_0 = 0$) is called a **Maclaurin series**.

You are expected to know common Maclaurin series, like e^x and $\sin x$ from first year.
Their complex analogues are identical.

Taylor Series Example

Examples

Determine the Taylor series of

$$f(z) = (z+2)e^{3z}$$

about the point $z = 0$.

Taylor Series Example

Using the Taylor series for the exponential function

$$
\begin{aligned}
f(z) &= (z+2)e^{3z} \\
&= (z+2)\sum_{k=0}^{\infty}\frac{(3z)^k}{k!} \\
&= \sum_{k=0}^{\infty}\frac{3^k z^{k+1}}{k!} + 2\sum_{k=0}^{\infty}\frac{(3z)^k}{k!} \\
&= \sum_{k=1}^{\infty}\frac{3^{k-1} z^{k}}{(k-1)!} + 2\sum_{k=0}^{\infty}\frac{(3z)^k}{k!} \\
&= 2 + \sum_{k=1}^{\infty}\left[\frac{3^{k-1}}{(k-1)!} + 2\frac{3^k}{k!}\right] z^k
\end{aligned}
$$

Laurent Series

If a function isn't holomorphic at a point, then to get a power series near that point, you'd need to find several around it. In this case, it can be useful to discuss series defined on annuli.

Laurent's Theorem

Let A be the annulus $A = B(z_0, R_2) \setminus \overline{B(z_0, R_1)}$, and $R_1 < r < R_2$. If f is holomorphic on A, then, for every $w \in A$,

$$f(w) = \sum_{n=-\infty}^{\infty} c_n (w - z_0)^n,$$

where

$$c_n = \frac{1}{2\pi i} \int_{\partial B(z_0, r)} \frac{f(z)}{(z - z_0)^{n+1}} dz.$$

This is called the **Laurent series** of f on the annulus A.

Example

Examples

Find the Laurent series of

$$f(z) = \frac{1}{z(z-1)(z-2)}$$

in the "annulus" $\{z \in \mathbb{C} : |z-1| > 1\}$.

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Find the Laurent series of

$$f(z) = \frac{1}{z(z-1)(z-2)}$$

in the "annulus" $\{z \in \mathbb{C} : |z-1| > 1\}$.

First, we expand into partial fractions:

$$f(z) = \frac{1}{z(z-1)(z-2)} = \frac{1}{2}\frac{1}{z} - \frac{1}{z-1} + \frac{1}{2}\frac{1}{z-2}.$$

Example

Now, we expand it out into convergent geometric series, only in terms of $(z-1)$:

$$\begin{aligned} f(z) &= \frac{1}{2}\frac{1}{z-1}\frac{1}{1+\frac{1}{z-1}} - \frac{1}{z-1} + \frac{1}{2}\frac{1}{z-1}\frac{1}{1-\frac{1}{z-1}} \\ &= \frac{1}{2}\frac{1}{z-1}\sum_{n=0}^{\infty}\frac{(-1)^n}{(z-1)^n} - \frac{1}{z-1} + \frac{1}{2}\frac{1}{z-1}\sum_{n=0}^{\infty}\frac{1}{(z-1)^n} \\ &= \sum_{n=-\infty}^{\infty} c_n (z-1)^n, \end{aligned}$$

where

$$c_n = \begin{cases} 1, & n = -3, -5, -7, \cdots ; \\ 0, & \text{otherwise.} \end{cases}$$

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Examples

Suppose that

$$f(z) = \frac{2}{z+2i} - \frac{5}{z-4}.$$

Find the Laurent series for f in powers of $(z-2)$ that converges when $z = i$.

MATH2621 2015 Final Q2(i)

Examples

Suppose that

$$f(z) = \frac{2}{z+2i} - \frac{5}{z-4}.$$

Find the Laurent series for f in powers of $(z-2)$ that converges when $z = i$.

The singularities are at $-2i$ and 4, which are respectively at a distance of $\sqrt{8}$ and 2 from 2. Since the distance of i from 2 is $\sqrt{5}$, we aim to find the Laurent series in the annulus $2 < |z-2| < \sqrt{8}$.

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$$\begin{aligned}
f(z) &= \frac{2}{z+2i} - \frac{5}{z-4} \\
&= \frac{2}{(z-2)+(2+2i)} - \frac{5}{(z-2)-2} \\
&= \frac{2}{2+2i}\frac{1}{1+\frac{z-2}{2+2i}} - \frac{5}{z-2}\frac{1}{1-\frac{2}{z-2}} \\
&= \frac{2}{2+2i}\sum_{k=0}^{\infty}(-1)^k\left(\frac{z-2}{2+2i}\right)^k - \frac{5}{z-2}\sum_{k=0}^{\infty}\left(\frac{2}{z-2}\right)^k \\
&= \frac{2}{2+2i}\sum_{k=0}^{\infty}(-1)^k\left(\frac{z-2}{2+2i}\right)^k - 5\sum_{k=0}^{\infty}\frac{2^k}{(z-2)^{k+1}}.
\end{aligned}$$

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&= \frac{2}{2+2i}\sum_{k=0}^{\infty}(-1)^k\left(\frac{z-2}{2+2i}\right)^k - \frac{5}{z-2}\sum_{k=0}^{\infty}\left(\frac{2}{z-2}\right)^k \\
&= \frac{2}{2+2i}\sum_{k=0}^{\infty}(-1)^k\left(\frac{z-2}{2+2i}\right)^k - 5\sum_{k=0}^{\infty}\frac{2^k}{(z-2)^{k+1}}.
\end{aligned}
$$

Hence $f(z) = \sum_{k=-\infty}^{\infty} c_k (z-2)^k$ where $c_k = \frac{2(-1)^k}{(2+2i)^{k+1}}$ for $k \geq 0$ and $c_k = -\frac{5}{2^{k+1}}$ for $k < 0$.

Singularities

Definition

An **isolated singularity** of f is a point z_0 for which f is holomorphic on $B^\circ(z_0, r)$ for some $r > 0$, but is not differentiable at z_0.

Definition

Suppose f has an isolated singularity at z_0, and has Laurent coefficients c_n. Assume $f \not\equiv 0$ so that there is at least one non-zero c_n. Then we have three exclusive and exhaustive possibilities:

1. No $n < 0$ have $c_n \neq 0$. We say that f has a **removable singularity** at z_0.
2. Some non-zero, finite number of $n < 0$ have $c_n \neq 0$. We say that f has a **pole** at z_0.
3. Infinitely many $n < 0$ have $c_n \neq 0$. We say that f has an **essential singularity** at z_0.

Rather than using Laurent series, it can be easier to evaluate a limit, in some cases.

Theorem

Suppose f has an isolated singularity at z_0, and $f \not\equiv 0$. Then

1. If $\lim_{z \to z_0} f(z)$ exists, we have a removable singularity.
2. If $\lim_{z \to z_0} (z - z_0)^k f(z)$ exists for $k = n$, but not for $k = 0, \cdots, n-1$, we have a pole (of order n).
3. If $\lim_{z \to z_0} (z - z_0)^k f(z)$ doesn't exist for any k, we have an essential singularity.

Poles and Zeroes

Definition

Suppose f has a pole at z_0. Then there is an $M < 0$ such that $c_M \neq 0$ and $c_n = 0$ for all $n < M$. We say that f has a **pole of order** $-M$ at z_0, or that the pole has order $-M$. A **simple pole** is a pole of order 1.

Definition

Suppose a non-constant function f has a removable singularity at z_0. If there is an $M > 0$ such that $c_M \neq 0$ and $c_n = 0$ for all $n < M$, then we say that f has a **zero of order** M at z_0. A **simple zero** is a zero of order 1.

Example

Examples

Classify the singularities of

$$f(z) = \frac{1}{z^2 \sin z}$$

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$$\lim_{z\to 0} z^3 f(z) = \lim_{z\to 0} z^3 \frac{1}{z^2 \sin z} = \lim_{z\to 0} \frac{z}{\sin z} = \lim_{z\to 0} \frac{1}{\cos z} = 1,$$

so f has a pole of order 3 at $z = 0$.

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so f has a pole of order 3 at $z = 0$.

$$\lim_{z\to k\pi} (z - k\pi) f(z) = \lim_{z\to k\pi} \frac{1}{z^2} \frac{z - k\pi}{\sin z} = \frac{1}{k^2\pi^2} \lim_{z\to k\pi} \frac{1}{\cos z} = \frac{(-1)^k}{k^2\pi^2}$$

so f has simple poles at $z = k\pi$ for $k \neq 0$.

Example

Examples

Classify the singularities of the following functions

$$g(z) = \frac{z(z-\pi)^2}{\sin^2 z}$$

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$$g(z) = \frac{z(z-\pi)^2}{\sin^2 z}$$

g has singularities at $z = k\pi, k \in \mathbb{Z}$.
$z = 0$ is a simple pole, $z = \pi$ is a removable singularity, and the other singularities are poles of order 2.

Examples

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Classify the singularities of

$$f(z) = \exp\left(z + \frac{1}{z}\right).$$

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$$f(z) = \exp\left(z + \frac{1}{z}\right).$$

By definition of the exponential function we have

$$\exp\left(z + \frac{1}{z}\right) = \sum_{i=0}^{\infty} \frac{(z + z^{-1})^i}{i!} = \sum_{i=0}^{\infty} \frac{1}{i!} \sum_{j=0}^{i} \binom{i}{j} z^{i-2j}$$

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Let c_n denote the coefficient of z^n. For any $n < 0$, the summation will give a z^n term when $i = j = -n$ since $i - 2j = n$. Since all the coefficients in the above expansion are positive, this means that $c_n \neq 0$ for any $n < 0$. Since infinitely many $n < 0$ have $c_n \neq 0$, f has an essential singularity at $z = 0$.

Residues

Residues allow us to extend our methods beyond holomorphic functions.

Definition

Suppose f is holomorphic on $B^\circ(z_0, r)$ for some $r > 0$, with Laurent coefficients c_n in $B^\circ(z_0, r)$. The **residue** of f at z_0 is

$$\operatorname{Res}(f, z_0) = \operatorname{Res}(f(z);\, z = z_0) = c_{-1}.$$

Definition

If f has a pole of order N at z_0, then

$$\operatorname{Res}(f, z_0) = \frac{1}{(N-1)!} \lim_{z \to z_0} \frac{d^{N-1}}{dz^{N-1}} (z - z_0)^N f(z).$$

Examples

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From the previous example, $g(z) = \frac{z(z-\pi)^2}{\sin^2 z}$ has singularities at $z = k\pi$. Calculate the residues at these singularities.

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Since $z = \pi$ is a removable singularity,

$$\operatorname{Res}(g, \pi) = 0.$$

Since $z = 0$ is a simple pole,

$$\operatorname{Res}(g, 0) = \lim_{z \to 0} z g(z) = \lim_{z \to 0} \left(\frac{z}{\sin z} \right)^2 (z - \pi)^2 = \pi^2$$

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From the previous example, $g(z) = \frac{z(z-\pi)^2}{\sin^2 z}$ has singularities at $z = k\pi$. Calculate the residues at these singularities.

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Since $z = k\pi, k \neq 0, 1$ is a pole of order 2,

$$\mathrm{Res}(g, k\pi) = \lim_{z \to k\pi} \frac{d}{dz}(z - k\pi)^2 g(z)$$

Unfortuantely this is horrible. We will instead try to find the coefficient of $z - k\pi$.

Examples

First we find the taylor series for $\sin^2 z$ around $z = k\pi$.

$$\sin^2 z = \sin^2(z - k\pi) = \left(\sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)!} (z - k\pi)^{2k+1} \right)^2$$

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$$\sin^2 z = \sin^2(z - k\pi) = \left(\sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)!}(z-k\pi)^{2k+1}\right)^2$$

Then

$$\begin{aligned}
\frac{1}{\sin^2 z} &= \left(\frac{1}{(z-k\pi) - \frac{(z-k\pi)^3}{3!} + \frac{(z-k\pi)^5}{5!} - \dots}\right)^2 \\
&= \frac{1}{(z-k\pi)^2}\left(\frac{1}{1 - \frac{(z-k\pi)^2}{3!} + \frac{(z-k\pi)^4}{5!} - \dots}\right)^2 \\
&= \frac{1}{(z-k\pi)^2}\sum_{i=0}^{\infty}\left(\frac{(z-k\pi)^2}{3!} - \frac{(z-k\pi)^4}{5!} + \dots\right)^i
\end{aligned}$$

Examples

We only need to care about the $\frac{1}{(z-k\pi)^2}$ term. We simply need to rewrite the term

$$\frac{z(z-\pi)^2}{(z-k\pi)^2}$$

in powers of $z-k\pi$ and the residue is the coefficient of $(z-k\pi)^{-1}$.

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$$\frac{z(z-\pi)^2}{(z-k\pi)^2} = \frac{[(z-k\pi)+k\pi][(z-k\pi)+\pi(k-1)]^2}{(z-k\pi)^2}$$

Examples

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By selectively expanding, the required coefficient is

$$[\pi(k-1)]^2 + 2k\pi^2(k-1) = \pi^2(3k^2-4k+1)$$

As a sanity check, if we substitute in $k = 0$ or 1, we obtain the same residues as determined above.

Cauchy's Residue Theorem

Cauchy's Residue Theorem

Suppose Ω is a domain, and that Γ is a simple closed contour with standard (anticlockwise) orientation in Ω. Further, let f be holomorphic on Ω, and $\mathrm{Int}(\Gamma) \cap \Omega = \mathrm{Int}(\Gamma) \setminus \{z_1, z_2, \cdots, z_K\}$. Then

$$\int_\Gamma f(z)dz = 2\pi i \sum_{k=1}^{K} \mathrm{Res}(f, z_k).$$

This theorem is used mostly in evaluating integrals around singularities, and expands the kinds of integrals we can now evaluate using complex analysis methods.

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Examples

Suppose that

$$f(z) = \frac{e^{\frac{1}{z}}}{(z-2)(z+2)}$$

Let Γ_R denote the circle centre 0 with radius R, traversed in the anti-clockwise direction.

(a) Find all the singular points of f, classify them, and find the residue of f at each pole.

(b) Explain why

$$\lim_{R\to\infty} \int_{\Gamma_R} f(z)dz = 0.$$

(c) Hence calculate the integral

$$\int_{\Gamma_1} f(z)dz.$$

(a) $z = 2$ and $z = -2$ are simple poles.

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$$\begin{aligned}
\operatorname{Res}(f, 2) &= \lim_{z \to 2} (z-2) f(z) \\
&= \lim_{z \to 2} \frac{e^{\frac{1}{z}}}{z+2} \\
&= \frac{e^{\frac{1}{2}}}{4} \\
\operatorname{Res}(f, -2) &= \lim_{z \to -2} (z+2) f(z) \\
&= \lim_{z \to -2} \frac{e^{\frac{1}{z}}}{z-2} \\
&= -\frac{e^{-\frac{1}{2}}}{4}
\end{aligned}$$

$z = 0$ is an essential singularity.

(b) Note that the length of the contour is $2\pi R$. By the ML-lemma,

$$\left|\int_{\Gamma_R} f(z)dz\right| \leq 2\pi R\frac{e^{\frac{1}{R}}}{R^2-4}$$

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$$\left|\int_{\Gamma_R} f(z)dz\right| \leq 2\pi R\frac{e^{\frac{1}{R}}}{R^2-4}$$

Since

$$\lim_{R\to\infty} 2\pi R\frac{e^{\frac{1}{R}}}{R^2-4} = 0,$$

by the pinching theorem,

$$\lim_{R\to\infty}\left|\int_{\Gamma_R} f(z)dz\right| = 0,$$

and hence

$$\lim_{R\to\infty}\int_{\Gamma_R} f(z)dz = 0,$$

MATH2621 2015 Final Q2(iii)

Using Cauchy's Residue Theorem, for $R > 2$,

$$\int_{\Gamma_R} f(z)dz = 2\pi i[\mathrm{Res}(f, 2) + \mathrm{Res}(f, -2) + \mathrm{Res}(f, 0)]$$

MATH2621 2015 Final Q2(iii)

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Using (ii),

$$2\pi i[\mathrm{Res}(f,2) + \mathrm{Res}(f,-2) + \mathrm{Res}(f,0)] = 0$$

Solving gives

$$\mathrm{Res}(f,0) = \frac{1}{4}\left(e^{-\frac{1}{2}} - e^{\frac{1}{2}}\right)$$

MATH2621 2015 Final Q2(iii)

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Using Cauchy's Residue Theorem again

$$\int_{\Gamma_1} f(z)dz = 2\pi i \,\text{Res}(f,0) = \frac{1}{2}\pi i\left(e^{-\frac{1}{2}} - e^{\frac{1}{2}}\right)$$

Examples

Consider the function f defined by

$$f(z) = \frac{z}{2z^4 - 5z^2 + 2}$$

(a) Find all the singularities of f and classify these.
(b) Find the residues of f at the singularities that lie inside the unit circle in $\mathbb{C}$.
(c) Hence find $\displaystyle\int_0^{2\pi} \frac{d\theta}{8\cos^2\theta + 1}$

MATH2621 2017 Q3(iii)

(a) Factorising the denominator gives

$$2z^4 - 5z^2 + 2 = (2z^2 - 1)(z^2 - 2) = (\sqrt{2}z + 1)(\sqrt{2}z - 1)(z + \sqrt{2})(z - \sqrt{2})$$

So there are simple poles as $z = \pm\sqrt{2}, \pm\frac{1}{\sqrt{2}}$

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So there are simple poles as $z = \pm\sqrt{2}, \pm\frac{1}{\sqrt{2}}$

(b)

$$\operatorname{Res}\left(f, \frac{1}{\sqrt{2}}\right) = \lim_{z \to \frac{1}{\sqrt{2}}} \left(z - \frac{1}{\sqrt{2}}\right) f(z) = \frac{1}{6}$$

$$\operatorname{Res}\left(f, -\frac{1}{\sqrt{2}}\right) = \lim_{z \to -\frac{1}{\sqrt{2}}} \left(z + \frac{1}{\sqrt{2}}\right) f(z) = \frac{1}{6}$$

(c) Using Cauchy's Residue Theorem,

$$\int_\gamma \frac{z\,dz}{2z^4+5z^2+2} = 2\pi i \left[\operatorname{Res}\left(f, \frac{1}{\sqrt{2}}\right) + \operatorname{Res}\left(f, -\frac{1}{\sqrt{2}}\right)\right] = \frac{2\pi i}{3}$$

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Using the parameterisation $z = e^{i\theta}, \theta \in [0, 2\pi]$

$$\begin{aligned}\int_\gamma \frac{z\,dz}{2z^4+5z^2+2} &= \int_\gamma \frac{dz}{z[2(z+z^{-1})^2+1]}\\ &= \int_0^{2\pi} \frac{ie^{i\theta}\,d\theta}{e^{i\theta}(8\cos^2\theta+1)}\\ &= i\int_0^{2\pi} \frac{d\theta}{8\cos^2\theta+1}.\end{aligned}$$

MATH2621 2017 Q3(iii)

(c) Using Cauchy's Residue Theorem,

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Hence we have

$$\int_0^{2\pi} \frac{d\theta}{8\cos^2\theta+1} = \frac{2\pi}{3}.$$

Solving integrals

Examples

Use complex analysis methods to show that

$$\int_{-\pi}^{\pi} \frac{\cos 2\theta}{5 + 4\cos\theta} d\theta = \frac{\pi}{6}$$

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Let $z = e^{i\theta}$ and γ represent the unit circle. $dz = ie^{i\theta}$.

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Let $z = e^{i\theta}$ and γ represent the unit circle. $dz = ie^{i\theta}$.

$$\begin{aligned}
\int_{-\pi}^{\pi} \frac{\cos 2\theta}{5 + 4\cos\theta} d\theta &= \int_\gamma \frac{\frac{z^2+z^{-2}}{2}}{5 + 4\left(\frac{z+z^{-1}}{2}\right)} \frac{1}{iz} dz \\
&= \frac{1}{2i} \int_\gamma \frac{z^4 + 1}{z^2(2z^2 + 5z + 2)} dz \\
&= \frac{1}{2i} \int_\gamma \frac{z^4 + 1}{z^2(2z + 1)(z + 2)} dz
\end{aligned}$$

Solving integrals

We now simply need to evaluate the residues at $z = 0$ and $z = -\frac{1}{2}$, with

$$f(z) = \frac{z^4 + 1}{z^2(2z+1)(z+2)}$$

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$$\operatorname{Res}\left(f, -\frac{1}{2}\right) = \lim_{z \to -\frac{1}{2}} \left(z + \frac{1}{2}\right) f(z) = \lim_{z \to -\frac{1}{2}} \frac{z^4 + 1}{2z^2(2z+1)(z+2)} = \frac{17}{12}$$

$$\operatorname{Res}(f, 0) = \lim_{z \to 0} \frac{d}{dz}(z^2 f(z)) = \lim_{z \to 0} \frac{d}{dz} \frac{z^4 + 1}{(2z+1)(z+2)} = -\frac{5}{4}$$

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Hence, using Cauchy's Residue Theorem,

$$\int_{-\pi}^{\pi} \frac{\cos 2\theta}{5 + 4\cos\theta} d\theta = \frac{1}{2i} \cdot 2\pi i \left(\frac{17}{12} - \frac{5}{4}\right) = \frac{\pi}{6}$$

End

Good luck with your exams!