MATH1131/1141 MathSoc Calculus Revision Session 2017 S1 Solutions

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We cannot guarantee that our working is correct, or that it would obtain full marks in the exam – please notify us of any errors or typos at unswmathsoc@gmail.com, or on our Facebook page. There are sometimes multiple methods of solving the same question. Remember that in the real final exam, you will be expected to explain your steps and working out.

Problem 1

Show from the definition of the limit to infinity that

$$\lim_{x\to\infty}\left(\frac{1}{x}+3\right)=3.$$

Solution 1

Let $\varepsilon>0$. Take $M=\frac{1}{\varepsilon}$. Suppose $x>M$, i.e. $x>\frac{1}{\varepsilon}$ (note this implies $x>0$). Then $\frac{1}{x}<\varepsilon$ and so

$$\begin{aligned}\left|\left(\frac{1}{x}+3\right)-3\right| &= \left|\frac{1}{x}\right| \\ &= \frac{1}{x} \quad \left(\text{dropping absolute value signs as } x>0\Rightarrow\frac{1}{x}>0\right) \\ &< \varepsilon \quad (\text{as mentioned earlier}).\end{aligned}$$

By the definition of limits to infinity, this proves the claim. ■

Remark. The above is generally sufficient working out for an exam. However, you may be wondering how we knew what to pick for M. This is how to figure it out. Recall that we say we say that $\lim_{x\to\infty} f(x)=L$ if for every positive real number ε there exists a real number M such that if $x>M$, then $|f(x)-L|<\varepsilon$. In this problem, we have $f(x)=\frac{1}{x}+3$ and $L=3$. We want for any given $\varepsilon > 0$ to find an M (in terms of ε) such that if $x > M$ then $|f(x) - L| < \varepsilon$. The way to do this is to simplify the inequality $|f(x) - L| < \varepsilon$:

$$\begin{aligned}
&|f(x) - L| < \varepsilon \\
\iff &\left|\left(\frac{1}{x} + 3\right) - 3\right| < \varepsilon \quad \text{(using our definitions of } f(x) \text{ and } L) \\
\iff &\left|\frac{1}{x}\right| < \varepsilon \quad \text{(simplifying)} \\
\iff &\frac{1}{x} < \varepsilon \quad (\text{as } x > 0) \\
\iff &x > \frac{1}{\varepsilon}.
\end{aligned}$$

This shows that if $x > \frac{1}{\varepsilon}$, then $|f(x) - L| < \varepsilon$, as we want. Hence we will be taking M to be $\frac{1}{\varepsilon}$.

Problem 2

Show from the definition of limit to infinity that

$$\lim_{x\to\infty} e^{-2x} \sin x = 0.$$

Solution 2

Let $\varepsilon > 0$. Take $M = -\frac{\ln \varepsilon}{2}$. Suppose $x > M$, i.e. $x > -\frac{\ln \varepsilon}{2}$. Then we have

$$\begin{aligned}
\left|\left(e^{-2x} \sin x\right) - 0\right| &= \left|e^{-2x} \sin x\right| \\
&= \left|e^{-2x}\right| |\sin x| \\
&\le \left|e^{-2x}\right| \quad (\text{as } |\sin x| \le 1) \\
&= e^{-2x} \quad (\text{as } e^{-2x} > 0) \\
&< e^{-2M} \quad (\text{as } x > M \text{and } x \mapsto e^{-2x} \text{ is a strictly decreasing function}) \\
&= e^{-2\times\left(-\frac{\ln \varepsilon}{2}\right)} \quad \text{(by our definition of } M) \\
&= e^{\ln \varepsilon} \\
&= \varepsilon,
\end{aligned}$$

which proves the claim by the definition of limits to infinity. ■

Remark. As before, the above is generally sufficient working out for an exam. However, you may be wondering how we knew that it would suffice to pick $M = -\frac{\ln \varepsilon}{2}$. To figure this out, we do a similar thing to before: we simplify the inequality $|f(x) - L| < \varepsilon$ (in this case, $f(x) = e^{-2x} \sin x$ and $L = 0$). We have

$$\begin{aligned}
&|f(x) - L| < \varepsilon \\
\iff &\left|\left(e^{-2x} \sin x\right) - 0\right| < \varepsilon \\
\iff &\left|e^{-2x}\right| |\sin x| < \varepsilon \\
\iff &e^{-2x} |\sin x| < \varepsilon \quad (\text{as } e^{-2x} > 0).
\end{aligned}$$

At this stage, we know that the condition $|f(x) - L| < \varepsilon$ is equivalent to $e^{-2x}|\sin x| < \varepsilon$, so we want to find an M such that if $x > M$, then $e^{-2x}|\sin x| < \varepsilon$. Because $|\sin x| \leq 1$, we know that $e^{-2x}|\sin x| \leq e^{-2x}$. Thus if we can find an M such that if $x > M$, we have $e^{-2x} < \varepsilon$, such x will also satisfy $e^{-2x}|\sin x| < \varepsilon$, which is what we want. So our goal now is just to find an M such that if $x > M$ then $e^{-2x} < \varepsilon$.

To do this, we just solve the inequation $e^{-2x} < \varepsilon$ for x using high-school algebra:

$$\begin{aligned} & e^{-2x} < \varepsilon \\ \iff & -2x < \ln\varepsilon \quad \text{(taking logs)} \\ \iff & x > \frac{\ln\varepsilon}{-2} = \boxed{-\frac{\ln\varepsilon}{2}}. \end{aligned}$$

Note in the last step, we divided through by -2 (a negative number), which meant we had to flip the inequality sign. Thus the M we will pick is $-\frac{\ln\varepsilon}{2}$, since for this M, if $x > M$, then $e^{-2x} < \varepsilon$, which as explained, will imply $e^{-2x}|\sin x| < \varepsilon \iff |f(x) - L| < \varepsilon$, as desired.

Problem 3

Use L'Hôpital's rule to find the following limit:

$$\lim_{x\to\infty} \frac{\ln x}{x}.$$

Solution 3

As the numerator ($\ln x$) and denominator (x) both tend to ∞ as $x \to \infty$, the limit is of the indeterminate form $\frac{\infty}{\infty}$, so the limit is amenable to L'Hôpital's rule. Differentiating the numerator and denominator with respect to x, we must consider the limit

$$\lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x}.$$

This limit is clearly equal to 0. Thus by L'Hôpital's rule, we have

$$\lim_{x\to\infty} \frac{\ln x}{x} = 0.$$

Remark. Intuitively this makes sense, as we know that $\ln x$ grows much more slowly than x as $x \to \infty$ (draw a graph).

Problem 4

Find the following limit or explain why it doesn't exist:

$$\lim_{x\to 0^+} \frac{(1-\cos x)^{\frac{3}{2}}}{x - \sin x}.$$

Solution 4

As both $(1-\cos x)^{\frac{3}{2}}$ and $x - \sin x$ approaches zero as $x \to 0$, the above limit is of the indeterminate form $\frac{0}{0}$, so is amenable to L'Hôpital's rule. Differentiating the numerator and denominator, we thus obtain from L'Hôpital's rule

$$\begin{aligned}\lim_{x\to 0^+} \frac{(1-\cos x)^{\frac{3}{2}}}{x-\sin x} &\overset{\text{L'Hôpital's rule}}{=} \lim_{x\to 0^+} \frac{\frac{3}{2}(1-\cos x)^{\frac{1}{2}}\sin x}{1-\cos x} \\ &= \lim_{x\to 0^+} \frac{\frac{3}{2}\sin x}{(1-\cos x)^{\frac{1}{2}}},\end{aligned}$$

provided that the new limit exists (which, as we shall see, it does).

Now, the above limit is again of the indeterminate form $\frac{0}{0}$. However, it is clear to see that if we continue to apply L'Hôpital's rule, we would merely be going around in circles as the resulting fractions would be of similar form to the first application of L'Hôpital's rule and not be any easier to handle. In this situation, we aim to simplify the fraction.

Remembering that $2\sin^2\frac{x}{2} = 1-\cos x$ and $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$ (from double angle formulas), we can rewrite the derived limit as

$$\begin{aligned}\lim_{x\to 0^+} \frac{\frac{3}{2}\sin x}{(1-\cos x)^{\frac{1}{2}}} &= \lim_{x\to 0^+} \frac{\frac{3}{2}\times 2\sin\frac{x}{2}\cos\frac{x}{2}}{\left(2\sin^2\frac{x}{2}\right)^{\frac{1}{2}}} \\ &= \lim_{x\to 0^+} \frac{3\sin\frac{x}{2}\cos\frac{x}{2}}{\sqrt{2}\sqrt{\sin^2\frac{x}{2}}} \\ &= \lim_{x\to 0^+} \frac{3\sin\frac{x}{2}\cos\frac{x}{2}}{\sqrt{2}\sin\frac{x}{2}} \qquad (*) \\ &= \lim_{x\to 0^+} \frac{3\cos\frac{x}{2}}{\sqrt{2}} \\ &= \frac{3}{\sqrt{2}}.\end{aligned}$$

So the original limit exists and equals $\frac{3}{\sqrt{2}}$ by L'Hôpital's rule. Thus, when performing L'Hopital's, we have to critically evaluate and simplify when possible. Note in $(*)$, we used the fact that $\sqrt{\sin^2\frac{x}{2}} = \sin\frac{x}{2}$, $\underline{\textbf{since } \sin\frac{x}{2} > 0 \textbf{ (as } x > 0\textbf{)}}$, as we are taking the limit at $x \to 0^+$ (so x is positive).

Remark. In the line $(*)$ above, if we were taking the limit as $x \to 0^-$, we would have $\sin\frac{x}{2} < 0$, so $\sqrt{\sin^2\frac{x}{2}} = -\sin\frac{x}{2}$, and we would find that $\lim_{x\to 0^-} \frac{(1-\cos x)^{\frac{3}{2}}}{x-\sin x} = -\frac{3}{\sqrt{2}}$. Thus in fact while the one-sided limits exist, the limit

$$\lim_{x\to 0} \frac{(1-\cos x)^{\frac{3}{2}}}{x-\sin x}$$

does **not** exist, since the one-sided limits do not coincide.

Problem 5

Show that the function $f(x) = |x - 2|$ is continuous on $\mathbb{R}$.

Solution 5

We note that $f(x)$ can be written as

$$f(x) = \begin{cases} x - 2 & \text{if } x \geq 2 \\ 2 - x & \text{if } x < 2 \end{cases}.$$

Observe that $f(2) = |2 - 2| = 0$. Away from $x = 2$, f is defined as a polynomial function ($x - 2$ or $2 - x$), and polynomial functions are continuous everywhere, so f is continuous at all points other than 2. It is also continuous at 2, because

$$\begin{aligned} \lim_{x \to 2^+} f(x) &= \lim_{x \to 2^+} (x - 2) \quad (\text{as } f(x) = x - 2 \text{ if } x > 2) \\ &= 2 - 2 \\ &= 0 = f(2), \end{aligned}$$

and

$$\begin{aligned} \lim_{x \to 2^-} f(x) &= \lim_{x \to 2^+} (2 - x) \quad (\text{as } f(x) = 2 - x \text{ if } x < 2) \\ &= 2 - 2 \\ &= 0 = f(2). \end{aligned}$$

As the two-sided limits exist, we have that $\lim_{x \to 2} f(x)$ exists, and being equal to $0 = f(2)$. That is, $\lim_{x \to 2} f(x) = f(2)$, so f is continuous at 2. So f is continuous at every point in $\mathbb{R}$, so by definition is continuous on $\mathbb{R}$. ■

Problem 6

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with $\lim_{x \to \pm\infty} f(x) = 0$.
Show that if there is a number $\eta \in \mathbb{R}$ with $f(\eta) > 0$, then f attains a maximum value on $\mathbb{R}$.
(Note that the Max-Min Theorem, aka Extreme Value Theorem, applies to *finite* closed intervals $[a, b]$ only.)

Solution 6

Let $f(\eta) = C > 0$. It follows from the given limits and the definition of limits to $\pm\infty$ that there exist $M > 0$ and $N < 0$ such that $|f(x) - 0| < C \Rightarrow f(x) < C$ (as C is positive) whenever $x > M$ or $x < N$. Note that since $f(\eta) = C$, we must have $M > \eta$ and $N < \eta$. Now, consider the finite closed interval $[N, M]$. Since f is continuous on $\mathbb{R}$, it is also continuous on this closed interval $[N, M]$, so by the Extreme Value Theorem, f attains a maximum value $K \in \mathbb{R}$ on this interval. Since η is in this interval and $f(\eta) = C$, this maximum K is at least C, i.e. $K \geq C$. Now, by construction of N and M, we have $f(x) < C \leq K$ if $x < N$, and $f(x) < C \leq K$ if $x > M$. So f attains a maximum value of K on $\mathbb{R}$, since $f(x) \leq K$ for all real x and f attains the value K somewhere in $[N, M]$. ■

Problem 7

Show that the function $f(x) = x^3 - x^2 + 1$ has a zero between $x = -1$ and $x = 0$.

Solution 7

To solve this problem, we make use of the corollary to the Intermediate Value Theorem that states that if f is continuous on the closed interval $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs (one is strictly positive and the other strictly negative), then f has a zero in (a, b).

As $f(x)$ is a polynomial, it is continuous on the closed interval $[-1, 0]$. We have $f(-1) = -1 - 1 + 1 = -1$ and $f(0) = 1$. So $f(-1) < 0$ and $f(0) > 0$ (they have opposite signs), so f has a zero in $(-1, 0)$, i.e. between -1 and 0.

Problem 8

Show that the function $f(x) = x^2$ is differentiable everywhere (on $\mathbb{R}$).

Solution 8

Since for all $x \in \mathbb{R}$, the limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists, by definition, f is differentiable on $\mathbb{R}$. Note that the above limit is

$$\begin{aligned}
\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h \to 0} \frac{x^2 + 2hx + h^2 - x^2}{h} \\
&= \lim_{h \to 0} \frac{2hx + h^2}{h} \\
&= \lim_{h \to 0} (2x + h) \\
&= 2x,
\end{aligned}$$

so the limit does exist for any $x \in \mathbb{R}$ (and $f'(x) = 2x$ for all $x \in \mathbb{R}$). ■

Problem 9

For the curve $y^4 + x^3 - x^2 e^{3y} = 4$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Solution 9

The solution is in the slides (page 17 of the slides document). Here is a reproduction.

Write y as $y(x)$ and differentiate both sides with respect to x:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(y(x)^4 + x^3 - x^2 e^{3y(x)}\right) &= \frac{\mathrm{d}}{\mathrm{d}x}(4) \\
\Rightarrow 4y'(x)y(x)^3 + 3x^2 - 2xe^{3y(x)} - 3x^2y'(x)e^{3y(x)} &= 0 \quad \text{(product rule and chain rule)} \\
\Rightarrow 4y'y^3 + 3x^2 - 2xe^{3y} - 3x^2y'e^{3y} &= 0 \\
\Rightarrow \left(4y^3 - 3x^2e^{3y}\right)y' &= 2xe^{3y} - 3x^2 \\
\Rightarrow y' \equiv \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2xe^{3y} - 3x^2}{4y^3 - 3x^2e^{3y}}.
\end{aligned}$$

Problem 10

Use the Mean Value Theorem to show that

$$\frac{x-a}{x} < \ln\frac{x}{a} < \frac{x-a}{a}$$

if $0 < a < x$.

Solution 10

Let $0 < a < x$ be fixed. Let $f : [a, x] \to \mathbb{R}$ be the function given by $f(t) = \ln\frac{t}{a}$ for $t \in [a, x]$. Note $f'(t) = \frac{1}{t}$ and $f(a) = \ln\frac{a}{a} = \ln 1 = 0$. Since f is continuous on $[a, x]$ and differentiable on (a, x), we have from the Mean Value Theorem that for some real number ξ with $a < \xi < x$,

$$\begin{aligned}
\frac{f(x) - f(a)}{x-a} &= f'(\xi) \\
\iff \frac{\ln\frac{x}{a} - 0}{x-a} &= \frac{1}{\xi} \\
\iff \ln\frac{x}{a} &= \frac{x-a}{\xi}. \qquad (1)
\end{aligned}$$

Firstly, since $\xi > a > 0$, we have $\frac{1}{\xi} < \frac{1}{a}$, so then Equation (1) implies that $\ln\frac{x}{a} < \frac{x-a}{a}$. Secondly, we also have $\xi < x$, so $\frac{1}{\xi} > \frac{1}{x}$. Thus Equation (1) implies that $\ln\frac{x}{a} > \frac{x-a}{x}$. ■

Problem 11

Evaluate

(a) $\sin^{-1}\left(\sin\frac{5\pi}{4}\right)$

(b) $\sin\left(\cos^{-1}\frac{3}{5}\right)$.

Solution 11

Refer to the hints in the slides (page 22 of the slides document) for some further help.

(a) We have $\left(\sin \frac{5\pi}{4}\right) = \sin\left(-\frac{\pi}{4}\right)$, using the identity $\sin\theta = \sin(\pi - \theta)$. Thus

$$\begin{aligned}\sin^{-1}\left(\sin \frac{5\pi}{4}\right) &= \sin^{-1}\left(\sin\left(-\frac{\pi}{4}\right)\right) \\ &= -\frac{\pi}{4},\end{aligned}$$

since $\sin^{-1}(\sin x) = x$ for all $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (remember that the output of an inverse sine is between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$).

(b) By considering a $3, 4, 5$ right-angled triangle, the answer can be obtained. Here is an algebraic approach. Let $\theta = \cos^{-1}\frac{3}{5}$, so $\cos\theta = \frac{3}{5}$ and θ is acute (since the inverse cosine of a number in $(0, 1)$ is between 0 and $\frac{\pi}{2}$). We are asked to find $\sin\theta$. Since θ is acute, we have $\sin\theta > 0$, so $\sin\theta = +\sqrt{1 - \cos^2\theta}$. Thus the answer is

$$\begin{aligned}\sin\theta = \sqrt{1-\cos^2\theta} &= \sqrt{1 - (\cos\theta)^2} \\ &= \sqrt{1 - \left(\frac{3}{5}\right)^2} \\ &= \sqrt{\frac{16}{25}} \\ &= \frac{4}{5}.\end{aligned}$$

Problem 12

Find the slope of the inverse function of $f(x) = x^3 + x - 1$ at $x = 1$.

Solution 12

Since f is a polynomial function, it is continuously differentiable everywhere in its domain. First note that $f(1) = 1$. By the inverse function theorem, at the point $x = 1$ the slope of the inverse function g is given by:

$$\begin{aligned}g'(f(1)) &= \frac{1}{f'(1)} \\ \Rightarrow g'(1) &= \frac{1}{4} \quad \text{(since } f'(1) = \left(3x^2+1\right)\big|_{x=1} = 4\text{)}.\end{aligned}$$

Problem 13

Show that the function $f(x) = x + 2 + \frac{1}{x}$ has $y = x + 2$ as an oblique asymptote.

Solution 13

We have

$$\begin{aligned}\lim_{x\to\infty}\left(f(x)-(x+2)\right) &= \lim_{x\to\infty}\left(x+2+\frac{1}{x}-(x+2)\right)\\ &= \lim_{x\to\infty}\frac{1}{x}\\ &= 0.\end{aligned}$$

So by definition, the line $y = x + 2$ is an oblique asymptote.

Problem 14

Sketch the curve defined parametrically by

$$x = \frac{t}{2}, y = 2 + t^3, \quad t \in \mathbb{R}.$$

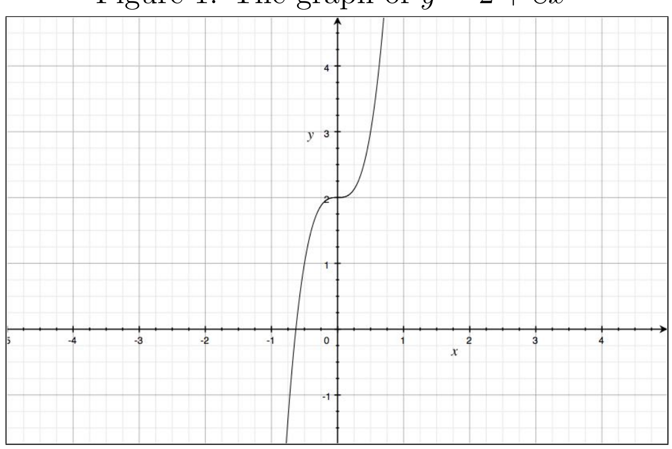
Solution 14

On the curve, we have

$$\begin{aligned}t &= 2x\\ \Rightarrow y &= 2 + (2x)^3 = 2 + 8x^3.\end{aligned}$$

Since t ranges in $\mathbb{R}$, $x = \frac{t}{2}$ takes all values in $\mathbb{R}$. So sketch $y = 2 + 8x^3$ for $x \in \mathbb{R}$. The graph is shown below.

Figure 1: The graph of $y = 2 + 8x^3$



Problem 15

Sketch the graph of the polar curve

$$r = 1 + \cos\theta.$$

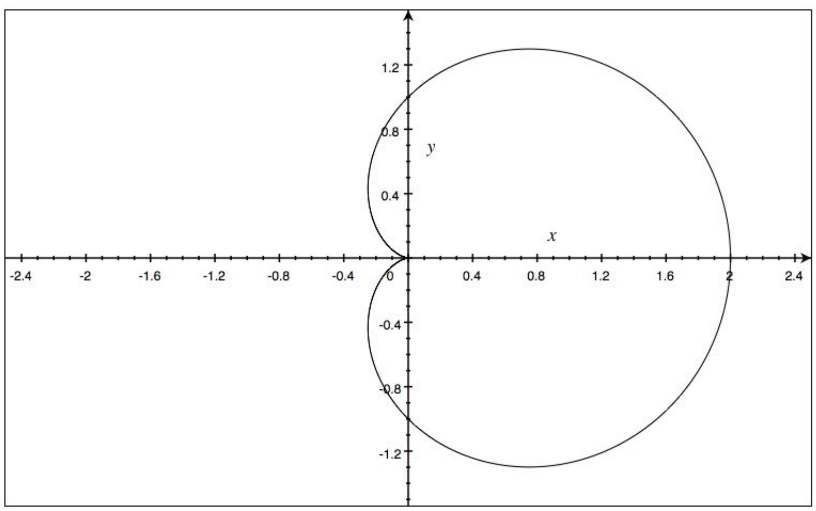
Solution 15

First sketch $y = 1 + \cos x$ to determine visually how the radius changes with respect to the angle, then use this to sketch the polar graph $r = 1 + \cos\theta$. For example, doing this, we know we only need to sketch for θ in an interval of length 2π (e.g. $[0, 2\pi]$), because the function is periodic with period 2π. We also note (even without a sketch) that we will always have $0 \leq r \leq 2$, because $-1 \leq \cos\theta \leq 1$. Furthermore, since $f(\theta) := 1 + \cos\theta$ satisfies $f(\theta) = f(-\theta)$ for all θ (even function), the graph of the polar curve will be symmetric about $\theta = 0$ (the x-axis). So we only need to sketch the curve for $0 \leq \theta \leq \pi$ and then can reflect this about the x-axis to complete the curve. We tabulate some values for $0 \leq \theta \leq \pi$ (for which we know the exact values of $\cos\theta$) below to help us sketch the curve.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	2	$1+\frac{\sqrt{3}}{2} \approx 1.87$	$1+\frac{1}{\sqrt{2}} \approx 1.71$	$\frac{3}{2}$	1	$\frac{1}{2}$	$1-\frac{1}{\sqrt{2}} \approx 0.29$	$1-\frac{\sqrt{3}}{2} \approx 0.13$	0

Using this, we should be able to sketch the curve (remember to reflect the sketch about the x-axis once sketched for $0 \leq \theta \leq \pi$). The curve is shown below. (It is an example of a famous curve known as the *cardioid*.)

Figure 2: The polar curve $r = 1 + \cos\theta$



Problem 16

Suppose $a \in (0, 1)$.

a) Show that

$$\int_0^a \ln(1-x)\,\mathrm{d}x = (a-1)\ln(1-a) - a.$$

b) By using a) and an appropriate Riemann sum, show that

$$\lim_{n\to\infty} \left\{ \left(1 - \frac{a}{n}\right)\left(1 - \frac{2a}{n}\right)\cdots\left(1 - \frac{na}{n}\right) \right\}^{\frac{a}{n}} = e^{-a}(1-a)^{a-1}.$$

c) Hence find
$$\lim_{a\to 1^-}\left\{\lim_{n\to\infty}\left\{\left(1-\frac{a}{n}\right)\left(1-\frac{2a}{n}\right)\cdots\left(1-\frac{na}{n}\right)\right\}^{\frac{a}{n}}\right\}.$$
Give reasons for your answer.

Solution 16

a) Note that using Integration by Parts, we can show that $\int \ln u \, \mathrm{d}u = u\ln u - u + C$. Hence

$$\begin{aligned}\int_0^a \ln(1-x)\,\mathrm{d}x &= -\int_0^a -\ln(1-x)\,\mathrm{d}x \\ &= -\int_1^{1-a} \ln u\ \mathrm{d}u \quad \text{(letting } u = 1-x, \text{ so } \mathrm{d}u = -\mathrm{d}x, \text{ and adjusting the bounds)} \\ &= \int_{1-a}^1 \ln u\ \mathrm{d}u \\ &= \Big[u\ln u - u\Big]_{1-a}^1 \\ &= 0 - 1 - ((1-a)\ln(1-a) - (1-a)) \\ &= (a-1)\ln(1-a) - a \quad \text{(simplifying)}.\end{aligned}$$

b) Recall that for a Riemann sum, we partition the interval in question into n subintervals as $\mathcal{P}_n = \{x_0, x_1, \ldots, x_n\}$ (so the subintervals are $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$), and then the Riemann sum is

$$S_n = \sum_{i=1}^n f(x_i^*)\,\Delta x_i,$$

where x_i^* is some chosen point in the ith subinterval $[x_{i-1}, x_i]$ and $\Delta x_i = x_i - x_{i-1}$ is the width of the ith subinterval. The integral exists if and only if $\lim\limits_{n\to\infty} S_n$ exists and equals the same value for any choice of partition and any choice of x_i^*.
For our problem, we will choose a partition of the interval of integration $[0, a]$ as $\mathcal{P}_n = \left\{0, \frac{a}{n}, \frac{2a}{n} \ldots, \frac{na}{n} = a\right\}$ (note this is a **uniform partition**). So the ith subinterval is $\left[\frac{(i-1)a}{n}, \frac{ia}{n}\right]$, for $i = 1, 2, \ldots, n$. We will choose the right endpoint as our x_i^*, i.e. $x_i^* = \frac{ia}{n}$ for each i. Also, Δx_i is just $\frac{a}{n}$ for each i, since we have used a uniform partition. Since our function f is $f(x) = \ln(1-x)$, our Riemann sum is

$$S_n = \sum_{i=1}^n \ln(1-x_i^*)\cdot\frac{a}{n} = \sum_{i=1}^n \ln\left(1-\frac{ia}{n}\right)\cdot\frac{a}{n}.$$

This sum must converge to the integral as $n\to\infty$, because we know the integral exists, as the integrand is continuous over the domain of integration. Since we know the integral's value is $(a-1)\ln(1-a) - a$, we have

$$\lim_{n\to\infty} S_n \equiv \lim_{n\to\infty}\sum_{i=1}^n \ln\left(1-\frac{ia}{n}\right)\cdot\frac{a}{n} = (a-1)\ln(1-a) - a. \tag{2}$$

Now notice that $e^{((a-1)\ln(1-a)-a)} = e^{((a-1)\ln(1-a))}e^{-a} = (1-a)^{a-1}\,e^{-a}$. So taking exponentials of both sides of Equation (3), we have

$$\begin{aligned}(1-a)^{a-1}\,e^{-a} &= \exp\left(\lim_{n\to\infty}\sum_{i=1}^{n}\ln\left(1-\frac{ia}{n}\right)\cdot\frac{a}{n}\right)\\ &= \lim_{n\to\infty}\left(\exp\left(\sum_{i=1}^{n}\ln\left(1-\frac{ia}{n}\right)\cdot\frac{a}{n}\right)\right)\\ &= \lim_{n\to\infty}\left(\prod_{i=1}^{n}\exp\left(\ln\left(1-\frac{ia}{n}\right)\cdot\frac{a}{n}\right)\right)\\ &= \lim_{n\to\infty}\left(\prod_{i=1}^{n}\left(1-\frac{ia}{n}\right)^{\frac{a}{n}}\right),\end{aligned}$$

and this is exactly what we had to prove, as we can see by writing out the last product in long notation as the question did.
In the above working, firstly, we could switch the exponential and limit because the exponential function is continuous. Secondly, we used the fact that exponential of a sum is the product of the exponentials, i.e. $\exp\left(\sum_{i=1}^{n}\beta_i\right) = \prod_{i=1}^{n}\exp\beta_i$. Finally, we used the fact that $\exp(\ln(\alpha)\,\beta) = \alpha^{\beta}$, which follows from index laws: $\exp(\ln(\alpha)\,\beta) \equiv e^{\ln(\alpha)\beta} = \left(e^{\ln\alpha}\right)^{\beta} = \alpha^{\beta}$.

c) For convenience, write $L(a) \equiv \lim_{n\to\infty}\left(\prod_{i=1}^{n}\left(1-\frac{ia}{n}\right)^{\frac{a}{n}}\right)$, so we are asked to find $\lim_{a\to 1^-} L(a)$. From (b), we have that $L(a) = (1-a)^{a-1}\,e^{-a}$. Since $(1-a)^{a-1} = (1-a)^{-(1-a)} \to 1$ as $a \to 1^-$ (proved below), we have $L(a) \to 1 \times e^{-1} = e^{-1}$ as $a \to 1^-$. Hence the answer is e^{-1}.
Note. To see that $(1-a)^{-(1-a)} \to 1$ as $a \to 1^-$, note that this is equivalent to the statement that $x^{-x} \to 1$ as $x \to 0^+$. To see why this is true, note that $x^{-x} = e^{-x\ln x} \to 1$ as $x \to 0^+$, because $-x\ln x \to 0$ as $x \to 0^+$, exponentials are continuous, and $e^0 = 1$. To prove that $-x\ln x \to 0$ as $x \to 0^+$, we can apply L'Hôpital's rule to the limit $\lim_{x\to 0^+} -\frac{\ln x}{\frac{1}{x}}$ (in fact, using this method, we can show that $x^{\alpha}\ln x \to 0$ as $x \to 0^+$ for any constant $\alpha > 0$).

Problem 17

Find $\int_1^b x^{-\frac{1}{2}}\,\mathrm{d}x$ by Riemann sums, using a uniform partition and:

$$x_i^* = \left(\frac{\sqrt{x_{i-1}}+\sqrt{x_i}}{2}\right)^2.$$

Solution 17

The width of each rectangle is $\Delta x = x_i - x_{i-1} = \frac{b-1}{n}$. Since the integrand is continuous on the interval of integration, the limit of a Riemann sum will be equal to the integral. Therefore, we have

$$\int_1^b x^{-\frac{1}{2}}\,\mathrm{d}x = \lim_{n\to\infty}\sum_{x=1}^{n}(x_i^*)^{-\frac{1}{2}}\,\Delta x$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \sum_{x=1}^{n} \left[\left(\frac{\sqrt{x_i} + \sqrt{x_{i-1}}}{2} \right)^2 \right]^{-\frac{1}{2}} \Delta x \\
&= \lim_{n\to\infty} \sum_{x=1}^{n} \frac{2}{\sqrt{x_i} + \sqrt{x_{i-1}}} \Delta x \quad \text{(algebraic simplification)} \\
&= \lim_{n\to\infty} \sum_{x=1}^{n} \frac{2(\sqrt{x_i} - \sqrt{x_{i-1}})}{(\sqrt{x_i} + \sqrt{x_{i-1}})(\sqrt{x_i} - \sqrt{x_{i-1}})} \Delta x \quad \text{(to ``rationalise the denominator'')} \\
&= \lim_{n\to\infty} \sum_{x=1}^{n} \frac{2(\sqrt{x_i} - \sqrt{x_{i-1}})}{x_i - x_{i-1}} \Delta x \\
&= 2 \lim_{n\to\infty} \sum_{x=1}^{n} \frac{\sqrt{x_i} - \sqrt{x_{i-1}}}{\Delta x} \Delta x \\
&= 2 \lim_{n\to\infty} \sum_{x=1}^{n} \left(\sqrt{x_i} - \sqrt{x_{i-1}} \right) \\
&= 2 \lim_{n\to\infty} (\sqrt{x_n} - \sqrt{x_0}) \quad \text{(telescoping sum)} \\
&= 2 \lim_{n\to\infty} \left(\sqrt{b} - \sqrt{1} \right) \quad (\text{as } x_n = b, x_0 = 1) \\
&= 2 \left(\sqrt{b} - 1 \right) \quad \text{(limit of a constant is itself)} \\
&= 2\sqrt{b} - 2.
\end{aligned}
$$

Problem 18

Does the integral

$$\int_1^\infty \frac{\mathrm{d}x}{\sqrt{x + x^3}}$$

converge? Prove your answer.

Solution 18

For $x \geq 1$, we have $\sqrt{x + x^3} > \sqrt{x^3} = x^{\frac{3}{2}}$, so $\frac{1}{\sqrt{x+x^3}} < \frac{1}{x^{\frac{3}{2}}}$. By the p-test, as $\frac{3}{2} > 1$, the improper integral

$$\int_1^\infty \frac{\mathrm{d}x}{x^{\frac{3}{2}}}$$

converges. Since $0 \leq \frac{1}{\sqrt{x+x^3}} \leq \frac{1}{x^{\frac{3}{2}}}$ for all $x \geq 1$, the comparison test implies that the integral in question converges.

Problem 19

Does the integral

$$\int_{69}^\infty \frac{\mathrm{d}x}{x \ln x}$$

converge? Prove your answer.

Solution 19

An antiderivative of $\frac{1}{x \ln x}$ is $\ln(\ln x)$ (use a substitution of $u = \ln x$ in the original integral if you can't spot this by inspection), so for $b > 69$,

$$\begin{aligned}\int_{69}^{b} \frac{\mathrm{d}x}{x \ln x} &= \Big[\ln(\ln x)\Big]_{69}^{b} \\ &= \ln(\ln b) - \ln(\ln 69) \\ &\to \infty \quad \text{as } b \to \infty.\end{aligned}$$

Hence by definition, the integral in question diverges to ∞ (so <u>does not converge</u>).

Problem 20

Does the integral

$$\int_{2}^{\infty} \frac{x^2 \ln x}{x^5 - 3} \, \mathrm{d}x$$

converge? Give reasons for your answer.

Solution 20

The answer is yes.

Let $f(x) = \frac{x^2 \ln x}{x^5 - 3}$. We note that for $x > 2$, $f(x)$ is well-defined and $f(x) > 0$. Since for all $x > 2$, $\ln x < x$, we have $f(x) < \frac{x^3}{x^5-3}$ for all $x > 2$. But $\int_2^\infty \frac{x^3}{x^5-3} \, \mathrm{d}x$ is convergent (explained below). Thus $0 < f(x) < \frac{x^3}{x^5-3}$ for all $x > 2$ (and these functions are continuous on $(2, \infty)$), so the comparison test implies that $\int_2^\infty f(x) \, \mathrm{d}x$ is convergent.

Note that $\int_2^\infty \frac{x^3}{x^5-3} \, \mathrm{d}x$ is convergent because the integrand is positive, $\frac{1}{x^2}$ is positive, and $\lim\limits_{x \to \infty} \frac{\frac{x^3}{x^5-3}}{\frac{1}{x^2}} = 1 \in (0, \infty)$. Therefore, as $\int_2^\infty \frac{1}{x^2} \, \mathrm{d}x$ is convergent (p-test), the limit form of the comparison test implies that $\int_2^\infty \frac{x^3}{x^5-3} \, \mathrm{d}x$ is convergent.

Problem 21

Does the integral

$$\int_{1}^{\infty} \frac{1}{\sqrt{1 + x + x^2}} \, \mathrm{d}x$$

converge? Prove your answer.

Solution 21

The answer is no.

The integrand is similar to $\frac{1}{\sqrt{x^2}} = \frac{1}{x}$ as $x \to \infty$, and the improper integral of $\frac{1}{x}$ diverges, so we expect the given integral to diverge. To be precise, note that $\frac{1}{\sqrt{1+x+x^2}}$ and $\frac{1}{x}$ are continuous and positive on $(1, \infty)$ and

$$\begin{aligned}\lim_{x\to\infty} \frac{\frac{1}{\sqrt{1+x+x^2}}}{\frac{1}{x}} &= \lim_{x\to\infty} \frac{x}{\sqrt{1+x+x^2}} \\ &= \lim_{x\to\infty} \frac{1}{\sqrt{\frac{1}{x^2}+\frac{1}{x}+1}} \quad \text{(dividing top and bottom by } x\text{)} \\ &= \frac{1}{\sqrt{0+0+1}} = 1 \in (0, \infty).\end{aligned}$$

Thus by the limit form of the comparison test, $\int_1^\infty \frac{1}{\sqrt{1+x+x^2}}\,\mathrm{d}x$ diverges, since $\int_1^\infty \frac{1}{x}\,\mathrm{d}x$ diverges (p-test).

Problem 22

Prove that $\cosh^2 x - \sinh^2 x = 1$.

Solution 22

We have

$$\begin{aligned}\cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 \\ &= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} \\ &= \frac{4}{4} \\ &= 1.\end{aligned}$$

An alternative method:

$$\begin{aligned}\cosh^2 x - \sinh^2 x = (\cosh x + \sinh x)(\cosh x - \sinh x) &= \left(\frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2}\right) \\ &= e^x \cdot e^{-x} \\ &= 1.\end{aligned}$$

Problem 23

a) State the domain and range of tanh and $\tanh^{-1}$ and sketch their graphs.

b) How many real numbers x satisfy the equation

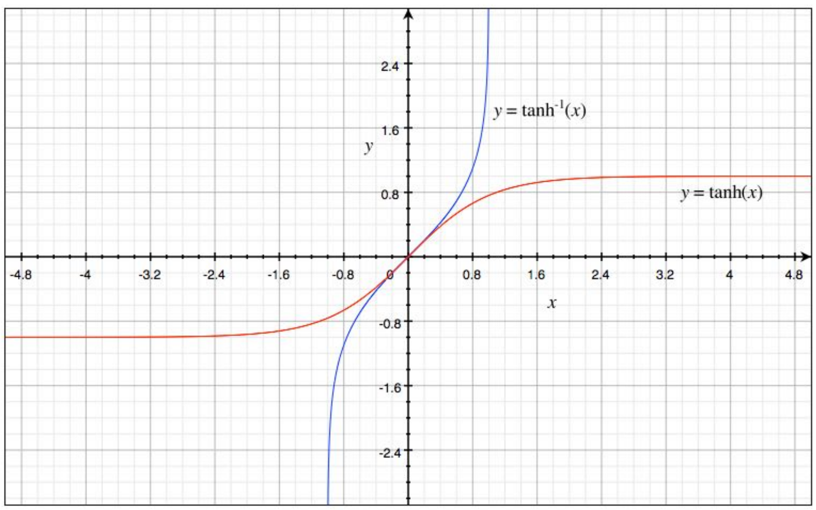
$$\tanh^{-1} x = x + 1 \ ?$$

Prove your answer explaining which theorems you have used.

Solution 23

a) The domain of tanh is $\mathbb{R}$ and the range of tanh is $(-1, 1)$. The domain of $\tanh^{-1}$ is $(-1, 1)$ and the range of $\tanh^{-1}$ is $\mathbb{R}$. (Remember the domain and range of inverse functions are reverse of each other.) You should be familiar with the graphs of the standard hyperbolic functions and their inverses. The graphs are shown below. Note that the graph of $y = \tanh x$ has horizontal asymptotes of $y = \pm 1$ and the graph of $y = \tanh^{-1} x$ has vertical asymptotes of $x = \pm 1$.

Figure 3: The graphs $\tanh x$ and $\tanh^{-1} x$



b) If we sketch the graph of $y = x + 1$ on top of our sketch in the previous part, it is clear that the answer should be that there is one (and only one) real number x that satisfies $\tanh^{-1} x = x + 1$. So we prove this below.

Let $f : (-1, 1) \to \mathbb{R}$ be the function defined by $f(x) = \tanh^{-1} x - x - 1$. Differentiating, we have

$$\begin{aligned} f'(x) &= \frac{1}{1 - x^2} - 1 \\ &> 1 - 1 \text{ for } x \in (-1, 1) \\ &= 0. \end{aligned}$$

So f is strictly increasing on its domain. Also note that $f(0) = -1$ and as $x \to 1^-$ we have $f(x) \to \infty$, so there exists some point $a \in (0, 1)$ such that $f(a) > 0$. Since f is continuous on $[0, a]$, by the Intermediate Value Theorem there exists some point $c \in (0, a)$ such that $f(c) = 0 \iff \tanh^{-1} c = c + 1$. As f is strictly monotone (increasing), this point is unique. Hence exactly one real number x satisfies the given equation.

Problem 24

Prove using the ε-δ definition of limits that $\lim_{x\to 2} x^3 = 8$.

Solution 24

Let $\varepsilon > 0$ be given. Take $\delta = \min\{\frac{\varepsilon}{19}, 1\}$. Let $0 < |x-2| < \delta$. Then $|x-2| < \frac{\varepsilon}{19}$ and also $1 \le x \le 3$, so $|x^2+2x+4| \le 3^2 + 2\times 3 + 4 = 19$, since x^2+2x+4 is an increasing and positive function for $x \ge 1$. Thus

$$\begin{aligned} 0 < |x-2| < \delta &\Rightarrow |x-2| \cdot \left|x^2+2x+4\right| < \frac{\varepsilon}{19}\cdot 19 \\ &\Rightarrow \left|x^3-8\right| < \varepsilon. \end{aligned}$$

This implies by definition of limits that $\lim_{x\to 2} x^3 = 8$.

Problem 25

Evaluate the following limit or explain why it doesn't exist:

$$\lim_{x\to 2^-} \frac{|x^2-4|}{x-2}.$$

Solution 25

We have that $|x^2-4| = 4-x^2$ for $x<2$, since $x^2-4<0$ if x is smaller than 2 and sufficiently close to 2 (you can sketch the graph to see this). Hence

$$\begin{aligned} \lim_{x\to 2^-} \frac{|x^2-4|}{x-2} &= \lim_{x\to 2^-} \frac{4-x^2}{x-2} \\ &= \lim_{x\to 2^-} \frac{(2-x)(2+x)}{x-2} \\ &= \lim_{x\to 2^-} (-(2+x)) \\ &= -4. \end{aligned}$$

Problem 26

Calculate the limit

$$L = \lim_{x\to\infty} \frac{1}{x}\int_x^{4x} \cos\left(\frac{1}{u}\right) \mathrm{d}u.$$

Solution 26

The aim is to use L'Hôpital's Rule on the quotient $\frac{\int_x^{4x}\cos\left(\frac{1}{u}\right)\mathrm{d}u}{x}$. It is clear that the denominator goes to ∞ as $x\to\infty$. To use L'Hôpital's Rule, we need to show that the numerator also does this.

To show this, note that for any $u > \frac{4}{\pi}$, we have $0 < \frac{1}{u} < \frac{\pi}{4} \Rightarrow \frac{1}{\sqrt{2}} < \cos\left(\frac{1}{u}\right) < 1$. So for $x > \frac{4}{\pi}$, we can integrate this inequality from x to $4x$ to obtain

$$\int_x^{4x} \frac{1}{\sqrt{2}}\, \mathrm{d}u < \int_x^{4x} \cos\left(\frac{1}{u}\right) \mathrm{d}u < \int_x^{4x} 1\, \mathrm{d}u, \quad x > \frac{4}{\pi},$$

which becomes

$$\frac{3x}{\sqrt{2}} < \int_x^{4x} \cos\left(\frac{1}{u}\right) \mathrm{d}u < 3x, \quad x > \frac{4}{\pi}.$$

Now taking limits as $x \to \infty$, it follows from the Pinching Theorem that $\int_x^{4x} \cos\left(\frac{1}{u}\right) \mathrm{d}u \to \infty$ as $x \to \infty$. Thus we can indeed use L'Hôpital's Rule here.

So using L'Hôpital's Rule, we have

$$L = \lim_{x\to\infty} \frac{\frac{\mathrm{d}}{\mathrm{d}x}\int_x^{4x} \cos\left(\frac{1}{u}\right) \mathrm{d}u}{1}. \tag{3}$$

Now, note that

$$\begin{aligned}\int_x^{4x} \cos\left(\frac{1}{u}\right) \mathrm{d}u &= \int_x^{1} \cos\left(\frac{1}{u}\right) \mathrm{d}u + \int_1^{4x} \cos\left(\frac{1}{u}\right) \mathrm{d}u \\ &= -\int_1^{x} \cos\left(\frac{1}{u}\right) \mathrm{d}u + \int_1^{4x} \cos\left(\frac{1}{u}\right) \mathrm{d}u,\end{aligned}$$

so by the Fundamental Theorem of Calculus, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\int_x^{4x} \cos\left(\frac{1}{u}\right) \mathrm{d}u &= \frac{\mathrm{d}}{\mathrm{d}x}\left(-\int_1^{x} \cos\left(\frac{1}{u}\right) \mathrm{d}u + \int_1^{4x} \cos\left(\frac{1}{u}\right) \mathrm{d}u\right) \\ &= -\cos\left(\frac{1}{x}\right) + 4\cos\left(\frac{1}{4x}\right).\end{aligned}$$

Note that we used the chain rule to get the derivative of $\int_1^{4x} \cos\left(\frac{1}{u}\right) \mathrm{d}u$ as being $4\cos\left(\frac{1}{4x}\right)$. Now, substituting this into Equation (1), we have

$$\begin{aligned}L &= \lim_{x\to\infty} \frac{-\cos\left(\frac{1}{x}\right) + 4\cos\left(\frac{1}{4x}\right)}{1} \\ &= \frac{-1+4}{1} \\ &= 3,\end{aligned}$$

since $\cos\left(\frac{1}{x}\right) \to 1$ as $x \to \infty$, since $\frac{1}{x} \to 0$ and $\cos 0 = 1$ (and the cosine function is continuous). Hence the answer is $L = 3$.

Problem 27

The area $A(t)$ of an arbitrary convex quadrilateral $\mathcal{Q}$ with given side lengths a, b, c, d depends on the sum $t = \alpha + \beta$ of either pair of opposite angles, and is given by *Bretschneider's formula*:

$$A(t) = \sqrt{(s-a)(s-b)(s-c)(s-d) - \frac{1}{2}abcd\,(1+\cos t)},$$

where $s = \frac{a+b+c+d}{2}$ is the *semi-perimeter* of $\mathcal{Q}$.

a) Explain why the area A of a convex quadrilateral with fixed sides a, b, c, d is maximal if the sum of either pair of opposite angles is π.

b) Show that the area function $A : [0, \pi] \to \mathbb{R}$ as defined above is invertible, and that the inverse function B is differentiable on $(A(0), A(\pi))$.

c) Show that
$$B'(A_0) = \frac{4A_0}{abcd},$$
where
$$A_0 = \sqrt{(s-a)(s-b)(s-c)(s-d) - \frac{1}{2}abcd}.$$

Solution 27

a) Note that $A(t)$ is maximised if and only if the expression under the square root is maximised. Since $(s-a)(s-b)(s-c)(s-d)$ is fixed, the only way to affect the expression for $A(t)$ is by affecting the $-\frac{1}{2}abcd\,(1+\cos t)$ term (since this is the only part that depends on t). Since $1+\cos t \geq 0$ and $abcd > 0$, and we are *subtracting* off $\frac{1}{2}abcd\,(1+\cos t)$, we will maximise A precisely when this subtracted amount is 0 (otherwise, we are subtracting off a positive amount, making A smaller). Hence to maximise A, we want $1 + \cos t = 0 \iff t = \pi$, i.e. sum of opposite angles is π.

b) Note that $1 + \cos t$ is a monotone function on $[0, \pi]$, and the function
$$\phi(u) := \sqrt{(s-a)(s-b)(s-c)(s-d) - \frac{1}{2}abcdu}$$
is also a monotone function. Since $A(t) = \phi(1+\cos t)$, A is the *composition* of two monotone functions. Since the composition of monotone functions is monotone, A is monotone. This implies that A is invertible.

The reason that the inverse function $B \equiv A^{-1}$ is differentiable on $(A(0), A(\pi))$ is that A is differentiable on $(0, \pi)$, with
$$\begin{aligned}
A'(t) &= \frac{\frac{1}{2}abcd\sin t}{2\sqrt{(s-a)(s-b)(s-c)(s-d) - \frac{1}{2}abcd\,(1+\cos t)}} \\
&= \frac{abcd\sin t}{4\sqrt{(s-a)(s-b)(s-c)(s-d) - \frac{1}{2}abcd\,(1+\cos t)}}.
\end{aligned}$$

Since this derivative is never 0 for $t \in (0, \pi)$ (and is always well-defined here since the denominator is never 0, as it is strictly positive when $t \in (0, \pi)$, as it is then the area of a non-degenerate quadrilateral), it follows from the *Inverse Function Theorem* that the inverse B is differentiable on $(A(0), A(\pi))$.

c) We use the Inverse Function Theorem. Observe that $A_0 = A\left(\frac{\pi}{2}\right)$. Then the Inverse Function Theorem implies that

$$B'(A_0) = \frac{1}{A'\left(\frac{\pi}{2}\right)}$$

$$\begin{aligned}
&= \left.\frac{1}{\frac{abcd\sin t}{4\sqrt{(s-a)(s-b)(s-c)(s-d)-\frac{1}{2}abcd(1+\cos t)}}}\right|_{t=\frac{\pi}{2}} \qquad \text{(using our expression for } A'(t) \text{ from before)}\\
&= \frac{1}{\frac{abcd}{4\sqrt{(s-a)(s-b)(s-c)(s-d)-\frac{1}{2}abcd(1+0)}}}\\
&= \frac{4\sqrt{(s-a)(s-b)(s-c)(s-d)-\frac{1}{2}abcd(1+0)}}{abcd}\\
&= \frac{4A_0}{abcd}. \quad \blacksquare
\end{aligned}$$

Problem 28

a) State carefully the Mean Value Theorem.

b) Use the Mean Value Theorem to prove that if $a < b$, then

$$0 < \tan^{-1} b - \tan^{-1} a \leq b - a.$$

c) Using (b) or otherwise, prove that the improper integral

$$I = \int_1^\infty \left[\tan^{-1}\left(t + \frac{1}{t^2}\right) - \tan^{-1} t\right] \mathrm{d}t$$

converges.

Solution 28

a) *Mean Value Theorem.*
Let $a < b$ be real numbers. If the (real-valued) function f is continuous on $[a, b]$ and differentiable on (a, b), then there exists a $c \in (a, b)$ with $f'(c) = \frac{f(b)-f(a)}{b-a}$.

b) Let $a < b$ be real numbers. Let $f(t) := \tan^{-1} t$. We obtain the first part of the inequality, i.e. that $\tan^{-1} b - \tan^{-1} a > 0$, from the fact that the inverse tan function is strictly increasing, which is true because $f'(t) = \frac{1}{1+t^2} > 0$ for all $t \in \mathbb{R}$. Now, since f is continuous on $[a, b]$ and differentiable on (a, b), the Mean Value Theorem tells us that for some c with $a < c < b$, we have

$$\begin{aligned}
\frac{f(b) - f(a)}{b-a} &= f'(c)\\
\iff \frac{\tan^{-1} b - \tan^{-1} a}{b-a} &= \frac{1}{1+c^2}.
\end{aligned}$$

Since $\frac{1}{1+c^2} \leq 1$, we have $\frac{\tan^{-1} b - \tan^{-1} a}{b-a} \leq 1 \Rightarrow \tan^{-1} b - \tan^{-1} a \leq b - a$, and thus we have the other part of the inequality too.

c) Note that by part (b), since $t + \frac{1}{t^2} > t$ for all $t \geq 1$, we have

$$0 < \tan^{-1}\left(t + \frac{1}{t^2}\right) - \tan^{-1} t < \underbrace{\frac{1}{t^2}}_{=\left(t+\frac{1}{t^2}\right)-t}$$

for all $t \geq 1$. Since $\int_1^\infty \frac{1}{t^2}\,\mathrm{d}t$ converges (using the p-test), it follows from the comparison test that the integral in question converges.

Problem 29

1. Determine whether the following improper integrals converge.

   (a) $\int_0^\infty e^{-\sqrt{x}}\,\mathrm{d}x$

   (b) $\int_3^\infty \frac{x}{\sqrt{x^6-1}}\,\mathrm{d}x$

2. [Absolute integrability implies integrability] Suppose that f is a continuous real-valued function on $[a, \infty)$, where $a \in \mathbb{R}$ is a constant. Show that if

$$\int_a^\infty |f(x)|\,\mathrm{d}x \quad \text{converges,}$$

then

$$\int_a^\infty f(x)\,\mathrm{d}x \quad \text{converges.}$$

(Bonus exercise: Show by finding a counterexample that the converse is not true.)

Solution 29

1. (a) This is convergent. Using the substitution $u = \sqrt{x}$, the integral is equivalent to

$$\int_0^\infty 2ue^{-u}\,\mathrm{d}u.$$

This integral converges, as we can show using Integration by Parts:

$$\begin{aligned}
\int_0^\infty 2ue^{-u}\,\mathrm{d}u &= \Big[-e^{-u}\cdot 2u\Big]_0^\infty + \int_0^\infty 2e^{-u}\,\mathrm{d}u \\
&= 0 - 0 - 2\Big[e^{-u}\Big]_0^\infty \quad (\text{note } ue^{-u} \to 0 \text{as } u \to \infty) \\
&= -2\,(0-1) \\
&= 2,
\end{aligned}$$

i.e. it is convergent. Thus the original integral is convergent.

(b) This is convergent. Here we provide a **sketch** of the solution. Remember, in the exam, you must show full working.
Let $f(x) = \frac{x}{\sqrt{x^6-1}}$. Then it is easy to show that $\lim_{x\to\infty} \frac{f(x)}{\frac{1}{x^2}} = 1$. (The motivation for choosing $\frac{1}{x^2}$ is that for large x, $f(x)$ behaves like $\frac{x}{\sqrt{x^6}} = \frac{x}{x^3} = \frac{1}{x^2}$.) Now we can use the limit form of the comparison test and the p-test to conclude that the original integral converges (since the improper integral of $\frac{1}{x^2}$ converges).

2. First we show that

$$\int_a^\infty (|f(x)| + f(x))\,\mathrm{d}x$$

is convergent.

Observe that for all $x > a$, we have

$$0 \le |f(x)| + f(x) \le 2\,|f(x)|\,.$$

As $f(x)$ is continuous, so are $|f(x)|$ and $|f(x)| + f(x)$. Therefore, the above and the inequality form of the comparison test imply that $\int_a^\infty (|f(x)| + f(x))\,\mathrm{d}x$ is convergent, since $\int_a^\infty 2\,|f(x)|\,\mathrm{d}x$ is convergent, due to the problem's assumptions.

Now, we have

$$\begin{aligned}\int_a^\infty f(x)\,\mathrm{d}x &= \int_a^\infty ((|f(x)| + f(x)) - |f(x)|)\,\mathrm{d}x \\ &= \int_a^\infty (|f(x)| + f(x))\,\mathrm{d}x - \int_a^\infty |f(x)|\,\mathrm{d}x, \quad \text{(splitting up the integral)} \quad (*)\end{aligned}$$

which is a finite number, being the difference of two convergent improper integrals. Thus $\int_a^\infty f(x)\,\mathrm{d}x$ is convergent.

Remark. The step $(*)$ is valid because we know the two integrals we got from splitting up the previous integral were each convergent integrals (due to what we proved at the start and the problem's assumptions). In general, it is **not** valid and does not make sense to do this splitting up if one of the resulting improper integrals is divergent.

For the bonus exercise in this problem, an example of a function that is integrable but not absolutely integrable on $(0, \infty)$ is $\frac{\sin x}{x}$. That is, $\int_0^\infty \left|\frac{\sin x}{x}\right|\,\mathrm{d}x$ is divergent but $\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x$ is convergent (in fact, the latter equals $\frac{\pi}{2}$). Proof of these facts is left as an exercise for the interested reader.