UNSW Mathematics Society Presents
MATH2221/2121 Seminar Part 2

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Overview

3. Initial-Boundary Value Problems in 1D (mostly MATH2221)

4. Generalised Fourier Series

5. Initial-Boundary Value Problems in 2D

3. Initial-Boundary Value Problems in 1D (mostly MATH2221)

Inner Products and Norms (MATH2121)

Definition 1

The **inner product** of continuous functions f, g defined on $[a, b] \to \mathbb{R}$ is

$$\langle f, g \rangle = \int_a^b f(x)g(x)\, dx.$$

f and g are orthogonal if $\langle f, g \rangle = 0$

Definition 2

The **norm** of a function f defined on $[a, b]$ is

$$\|f\| = \sqrt{\langle f, f \rangle} = \sqrt{\int_a^b [f(x)]^2\, dx}.$$

Example 1

Let $f(x) = \sin x$ and $g(x) = \cos x$ be defined on $[0, \pi]$. Find $\langle f, g \rangle$ and $\|f\|$.

We have

$$\begin{aligned}
\langle f, g \rangle &= \int_0^\pi \sin x \cos x \, dx \\
&= \frac{1}{2} \int_0^\pi \sin 2x \, dx \\
&= \frac{1}{4} [-\cos 2x]_0^\pi \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
\|f\| &= \sqrt{\int_0^\pi \sin^2 x \, dx} \\
&= \sqrt{\int_0^\pi \frac{1 - \cos 2x}{2} \, dx} \\
&= \sqrt{\frac{1}{2} \left[x - \frac{1}{2} \sin 2x \right]_0^\pi} \\
&= \sqrt{\frac{\pi}{2}}.
\end{aligned}$$

Cauchy-Schwarz Inequality (MATH2121)

Theorem 1

The **Cauchy-Schwarz Inequality** states that

$$|\langle f, g\rangle| \leq \|f\|\|g\|$$

.

Corollary 1

The **Triangle Inequality** states that

$$\|f + g\| \leq \|f\| + \|g\|$$

.

Linear Two-Point Boundary Value Problem

$$Lu = f \text{ for } a < x < b, \text{ with } B_1 u = \alpha_1 \text{ and } B_2 u = \alpha_2$$

where

$$Lu = a_2 u'' + a_1 u' + a_0 u$$

is **2nd-order linear differential operator**, and

$$B_1 u = b_{11} u'(a) + b_{10} u(a),$$

$$B_2 u = b_{21} u'(b) + b_{20} u(b),$$

are **the boundary operators**.

Theorem 2

If the homogeneous problem $(f \equiv 0, \alpha_1 = 0, \alpha_2 = 0)$ has only the trivial solution $u \equiv 0$, then for every choice of $f, \alpha_1,$ and α_2, the inhomogeneous problem has a unique solution.

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Example 2

Consider the following boundary value problem:

$$Lu = p(D)u = f(x); u(a) = \alpha_1; u(b) = \alpha_2; a < b,$$

where p is a second order polynomial. For the following cases, under what circumstances does a unique solution to the boundary value problem exist:

a) $p(\lambda)$ has two distinct real roots

b) $p(\lambda)$ has a repeated real root

c) $p(\lambda)$ has roots with non-zero imaginary parts?

The BVP will have a unique solution iff the homogeneous problem has only the trivial solution. The homogeneous problem is

$$Lu = 0, u(a) = 0, u(b) = 0.$$

a) Since $p(\lambda)$ has two distinct real roots, then

$$u = Ae^{\lambda_1 x} + Be^{\lambda_2 x},$$
$$u(a) = 0 \Rightarrow Ae^{\lambda_1 a} = -Be^{\lambda_2 a},$$
$$u(b) = 0 \Rightarrow Ae^{\lambda_1 b} = -Be^{\lambda_2 b}.$$

If $A = 0$, then $B = 0$, and vice versa. Consider $A, B \neq 0$, then dividing LHS by RHS and equating the results gives

$$-\frac{A}{B} = e^{(\lambda_2 - \lambda_1)a} = e^{(\lambda_2 - \lambda_1)b} \Rightarrow a = b,$$

which is a contradiction $(a < b)$. Therefore $A = B = 0$, i.e. $u \equiv 0$ for $Lu = 0$. Therefore the solution to the BVP is unique.

b) Since $p(\lambda)$ has a repeated real root, then

$$
\begin{aligned}
u &= Ae^{\lambda x} + Bxe^{\lambda x}, \\
u(a) &= 0 \Rightarrow Ae^{\lambda a} + Bae^{\lambda a} = 0, \\
u(b) &= 0 \Rightarrow Ae^{\lambda b} + Bbe^{\lambda b} = 0, \\
&\Rightarrow 0 = Ae^{\lambda a} + Bae^{\lambda a} = Ae^{\lambda b} + Bbe^{\lambda b} \\
\Rightarrow e^{\lambda a}(A+Ba) &= e^{\lambda b}(A+Bb) \\
&\Rightarrow a = b \text{ or } A = B = 0.
\end{aligned}
$$

Since $a < b$, therefore $A = B = 0$, i.e. $u \equiv 0$ for $Lu = 0$. Therefore the solution to the BVP is unique.

c) Since $p(\lambda)$ has roots with non-zero imaginary parts, then

$$
\begin{aligned}
u &= e^{\alpha x}(A\cos\beta x + B\sin\beta x),\\
u(a) = 0 &\Rightarrow e^{\alpha a}(A\cos\beta a + B\sin\beta a) = 0,\\
u(b) = 0 &\Rightarrow e^{\alpha b}(A\cos\beta b + B\sin\beta b) = 0,\\
\text{Since } e^{\alpha a}, e^{\alpha b} > 0\\
&\Rightarrow 0 = A\cos\beta a + B\sin\beta a = A\cos\beta b + B\sin\beta b\\
&\Rightarrow \beta a = \beta b + 2\pi.
\end{aligned}
$$

Therefore non-trivial solution exists, so solution to BVP is not unique.

Adjoint Operator and Lagrange Identity

Definition 3

Let

$$Lu = a_2 u'' + a_1 u' + a_0 u$$

be a linear second-order differential operator. Then define the **formal adjoint** $L^* v$

$$L^* v = (a_2 v)'' - (a_1 v)' + a_0 v,$$

and the **bilinear concomitant** $P(u, v)$

$$P(u, v) = u'(a_2 v) - u(a_2 v)' + u(a_1 v).$$

Theorem 3

The **Lagrange Identity** is

$$\langle Lu, v \rangle = \langle u, L^* v \rangle + [P(u, v)]_a^b \,.$$

Adjoint Operator

Example 3

Let
$$Lu = xu'' - e^x u' + (x^2 + 1)u.$$
Find the formal adjoint.

The formal adjoint is

$$\begin{aligned} L^* v &= (a_2 v)'' - (a_1 v)' + a_0 v \\ &= (xv)'' + (e^x v)' + (x^2 + 1)v \\ &= (xv' + v)' + e^x v' + e^x v + (x^2 + 1)v \\ &= xv'' + 2v' + e^x v' + e^x v + (x^2 + 1)v \\ &= xv'' + (e^x + 2)v' + (e^x + x^2 + 1)v. \end{aligned}$$

Formal Self-adjointness

Definition 4

An operator L is **formally self-adjoint** if $L = L^*$.

Theorem 4

A second-order linear differential operator L is formally self-adjoint if and only if it can be written as

$$Lu = -(pu')' + qu$$

for some p, q.

Transforming to Formally Self-adjoint Form

Any ODE of the form

$$a_2 u'' + a_1 u' + a_0 u = f(x),$$

can be transformed to formally self-adjoint form using the integrating factor

$$\exp(\int \frac{a_1}{a_2} dx).$$

Example 4

Write $Lu = xu'' + xu' - 2u = f$ in formally self-adjoint form.

Firstly, find the integrating factor

$$\begin{aligned}\exp(\int \frac{a_1}{a_2} dx) &= \exp(\int \frac{x}{x} dx),\\ &= e^x.\end{aligned}$$

Transforming to Formally Self-adjoint Form

Then, make coefficient of u'' equal 1 and multiply by integrating factor

$$
\begin{aligned}
& & xu'' + xu' - 2u &= f \\
&\iff & u'' + u' - \frac{2}{x}u &= \frac{f}{x} \\
&\iff & e^x u'' + e^x u' - \frac{2e^x}{x}u &= \frac{e^x f}{x} \\
&\iff & \left(e^x u'\right)' - \frac{2e^x}{x}u &= \frac{e^x f}{x} \\
&\iff & -\left(e^x u'\right)' + \frac{2e^x}{x}u &= -\frac{e^x f}{x}.
\end{aligned}
$$

So, we have the formally self-adjoint form $Lu = -(pu')' + qu = g$, where

$$
p = e^x, \qquad q = \frac{2e^x}{x}, \qquad g = -\frac{e^x f}{x}.
$$

Lagrange Identity for Self-adjoint Operators

Theorem 5

Any formally self-adjoint operator $Lu = -(pu')' + qu$ satisfies the identity

$$\langle Lu, v\rangle - \langle u, Lv\rangle = \sum_{i=1}^{2} \left(B_i u R_i v - R_i u B_i v\right),$$

where

$$\begin{aligned} B_1 u &= b_{11} u'(a) + b_{10} u(a), \\ B_2 u &= b_{21} u'(b) + b_{20} u(b), \\ R_1 u &= \frac{p(a)u(a)}{b_{11}} = -\frac{p(a)u'(a)}{b_{10}}, \\ R_2 u &= -\frac{p(b)u(b)}{b_{21}} = \frac{p(b)u'(b)}{b_{20}}. \end{aligned}$$

Fredholm Alternative

Theorem 6 (Fredholm Alternative)

Consider the system

$$\begin{aligned} Lu &= f && \text{for } a < x < b, \\ b_{11}u' + b_{10}u &= \alpha_1 && \text{at } x = a, \\ b_{21}u' + b_{20}u &= \alpha_2 && \text{at } x = b. \end{aligned}$$

1. If the homogeneous system (that is, $f \equiv 0$ and $\alpha_1 = \alpha_2 = 0$) has only the trivial solution, then the inhomogeneous system has a unique solution for every choice of f, α_1, α_2.
2. Else, the inhomogeneous system has a solution if and only if

$$\langle f, v \rangle = \alpha_1 R_1 v + \alpha_2 R_2 v$$

for every solution v to the homogeneous system. In this case, $u + Cv$ is also a solution for any constant C.

Fredholm Alternative

Example 5

Under what conditions does

$$Lu = u'' + u = f$$

where $u(0) = \alpha_1$, $u(\pi) = \alpha_2$ have a solution?

$v = A\sin x$ is a non-trivial solution to the homogeneous system $u'' = -u$. Transforming to formally self-adjoint form

$$Lu = -(-u')' + u,$$

gives $p(x) = -1$, which means that

$$R_1 v = -\frac{p(a)v'(a)}{b_{10}} = -\frac{p(0)Acos(0)}{1} = -\frac{(-1)A}{1} = A,$$

$$R_2 v = \frac{p(b)v'(b)}{b_{20}} = \frac{p(\pi)Acos(\pi)}{1} = \frac{(-1)A(-1)}{1} = A.$$

Fredholm Alternative

So, we apply the Fredholm alternative to deduce that there is a solution if and only if

$$\langle f, v\rangle = \alpha_1 R_1 v + \alpha_2 R_2 v$$

$$\langle f, v\rangle = A\int_0^{\pi} f(x)\sin x\,dx = A\alpha_1 + A\alpha_2.$$

That is,

$$\int_0^{\pi} f(x)\sin x\,dx = \alpha_1 + \alpha_2.$$

4. Generalised Fourier Series

Fourier Series

You may have already seen trigonometric Fourier series. If not:

Definition 5

Suppose $f : \mathbb{R} \to \mathbb{R}$ is bounded, piecewise continuous, and $2L$-periodic. Then the n^{th} **real Fourier polynomial** of f is

$$(S_n f)(x) = \frac{a_0[f]}{2} + \sum_{k=1}^{n} \left(a_k[f] \cos \frac{k\pi x}{L} + b_k[f] \sin \frac{k\pi x}{L} \right),$$

where

$$a_0[f] = \frac{1}{L} \int_{-L}^{L} f(x)\, dx,$$

and for $1 \le k \le n$,

$$a_k[f] = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{k\pi x}{L}\, dx, \quad b_k[f] = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{k\pi x}{L}\, dx.$$

Fourier Series

A couple of results help us when dealing with even or odd functions.

Theorem 7

If f is an even function, then

$$a_0[f] = \frac{2}{L}\int_0^L f(x)\,dx, \quad a_k[f] = \frac{2}{L}\int_0^L f(x)\cos\frac{k\pi x}{L}\,dx,$$

and $b_k[f] = 0$.
If f is an odd function, then

$$b_k[f] = \frac{2}{L}\int_0^L f(x)\sin\frac{k\pi x}{L}\,dx,$$

and $a_0[f] = a_k[f] = 0$.

Fourier Series

A **Fourier series** is the limit of the sequence $\{S_n f\}_{n=1}^{\infty}$ denoted Sf.

Example 6

Find the real Fourier series of f where $f(x) = x$ for $-1 < x \leq 1$, and requiring that $f(x+2) = f(x)$ for all $x \in \mathbb{R}$.

So f is 2-periodic, and odd. Thus $a_0[f] = a_k[f] = 0$. Then

$$b_k[f] = 2\int_0^1 x \sin k\pi x \, dx = 2\frac{\sin \pi k - \pi k \cos \pi k}{\pi^2 k^2}.$$

Now, $\sin \pi k = 0$ and $\cos \pi k = (-1)^k$ as k is an integer, so

$$(Sf)(x) = \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \sin k\pi x.$$

Fourier Series

At jump discontinuities, the Fourier series approaches the average of the function value either side of the discontinuity.

Theorem 8

Suppose f has a jump discontinuity at a. That is, the limits

$$f(a^+) = \lim_{x \to a^+} f(x), \quad f(a^-) = \lim_{x \to a^-} f(x)$$

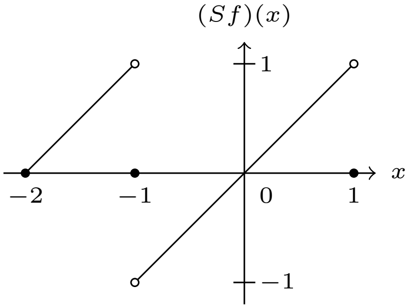
both exist, but $f(a^+) \neq f(a^-)$. Then

$$\lim_{n \to \infty} (S_n f)(a) = \frac{f(a^+) + f(a^-)}{2}.$$

This requires the one-sided derivatives existing at a. For the functions you deal with, this will almost always be the case.

Example 7

For the previous example, draw the graph of $(Sf)(x)$ for $-2 \le x \le 1$



Wherever the function is continuous, we can simply draw f. At each jump discontinuity, the Fourier series approaches the average of the function, in this case 0.

Half-Range Expansions

We can extend functions that are defined on an interval $[0, L]$ to an interval $[-L, L]$ to find their Fourier series as follows.

Definition 6

Suppose f is defined on the interval $[0, L]$. Then the

1. **Odd extension** of f is defined to be

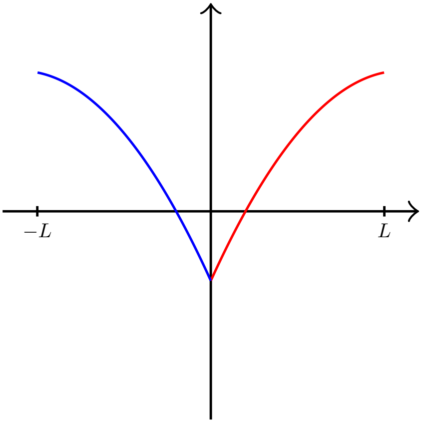
$$g(x) = \begin{cases} f(x), & 0 < x \leq L; \\ 0, & x = 0; \\ -f(-x), & -L \leq x < 0. \end{cases}$$

2. **Even extension** of f is defined to be

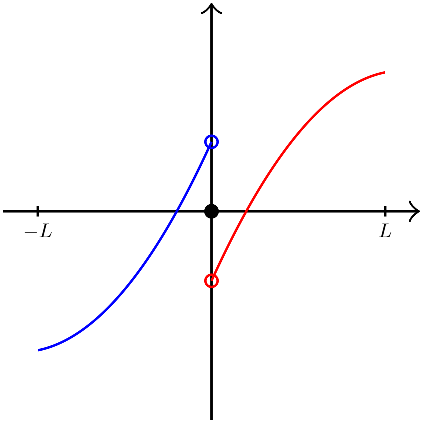
$$g(x) = \begin{cases} f(x), & 0 \leq x \leq L; \\ f(-x), & -L \leq x < 0. \end{cases}$$

Half-Range Expansions

Even Extension



Odd Extension



Half-Range Expansions

Example 8

Find the Fourier cosine series of f, where $f(x) = x$ for $0 < x \leq 1$.

To begin, we find the even expansion of f:

$$g(x) = \begin{cases} x, & 0 \leq x \leq 1; \\ -x, & -1 \leq x < 0 \end{cases} = |x|.$$

$$\begin{aligned} a_k[g] &= 2\int_0^1 x \cos k\pi x dx \\ &= 2\left[x\frac{\sin k\pi x}{k\pi}\right]_0^1 - 2\int_0^1 \frac{\sin k\pi x}{k\pi} dx \\ &= \frac{2}{(k\pi)^2}\left[\cos k\pi x\right]_0^1 \\ &= \begin{cases} \frac{-4}{k^2\pi^2}, & k \text{ odd}; \\ 0, & k \text{ even}. \end{cases} \end{aligned}$$

Half-Range Expansions

$$\begin{aligned} a_0[g] &= 2\int_0^1 x dx \\ &= 2\frac{[x^2]_0^1}{2} \\ &= 1 \end{aligned}$$

Also $b_k[g] = 0$, thus,

$$(Sg)(x) = \frac{1}{2} - \frac{4}{\pi^2} \sum_{k \text{ odd}} \frac{\cos k\pi x}{k^2}.$$

Orthogonality of Functions

Definition 7

The **inner product** of two functions f and g, on the interval (a, b), and with respect to the weight function w, is

$$\langle f, g \rangle_w = \int_a^b w(x) f(x) g(x)\, dx,$$

where $w(x) > 0$ on (a, b).

Definition 8

Two functions f and g are **orthogonal** over (a, b) with respect to the weight w if

$$\langle f, g \rangle_w = 0.$$

Orthogonality of Sets of Functions

We can extend this notion to sets of functions.

Definition 9

A set of functions $\{f_1, f_2, \cdots\}$ is **orthogonal** on the interval (a, b) and with respect to the weight function w if for all $n \neq m$ we have

$$\langle f_n, f_m \rangle_w = 0.$$

That is, every function is orthogonal (with respect to w) to every other function.

Orthogonality of Sets of Functions

Example 9 (2018 MATH2221 Problem Set 8, 3i)

Show that
$$y_n(x) = \sin\left(\frac{n\pi x}{a}\right), \qquad n = 1, 2, 3, \cdots$$
is an orthogonal set on $[0, a]$ and with respect to $w(x) = 1$.

If $n \neq m$, we have

$$\begin{aligned}
\langle y_n, y_m \rangle_1 &= \int_0^a \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi x}{a}\right) dx \\
&= \frac{1}{2} \int_0^a \left\{ \cos\left[\frac{(n-m)\pi x}{a}\right] - \cos\left[\frac{(n+m)\pi x}{a}\right] \right\} dx \\
&= 0.
\end{aligned}$$

Thus, the set is orthogonal.

Norms

Finally, for notational convenience, we define the norm.

Definition 10

The **norm** of a function f over the interval (a, b) with respect to the weight function w is

$$\|f\|_w = \sqrt{\langle f, f \rangle_w} = \sqrt{\int_a^b w(x)\,[f(x)]^2\,dx}.$$

Norms

Example 10

Using the orthogonal set from the previous example, find $\|y_n\|_1$.

$$\begin{aligned}
\|y_n\|_1 &= \sqrt{\int_0^a \sin^2\left(\frac{n\pi x}{a}\right)\, dx} \\
&= \sqrt{\frac{1}{2}\int_0^a \left[1 - \cos\left(\frac{2n\pi x}{a}\right)\right] dx} \\
&= \sqrt{\frac{1}{2}\left[x - \frac{a}{2n\pi}\sin\left(\frac{2n\pi x}{a}\right)\right]_0^a} \\
&= \sqrt{\frac{a}{2}}.
\end{aligned}$$

Generalised Fourier Series

With these definitions, we can finally define the generalised Fourier series of a function.

Definition 11

Let $\{\phi_1, \phi_2, \cdots\}$ be an orthogonal set of functions over (a, b) with respect to w. The **generalised Fourier series** of a function f is

$$F(x) = \sum_{n=1}^{\infty} c_n \phi_n(x),$$

where

$$c_n = \frac{\langle f, \phi_n \rangle_w}{\|\phi_n\|_w^2}.$$

c_n is the nth Fourier coefficient of f with respect to given orthogonal set of functions.

Completeness (MATH2221 only)

Completeness of an orthogonal set is important when using generalised Fourier series. A Fourier series using an incomplete set may not accurately represent the function.

Definition 12

An orthogonal set S is called **complete** if no non-trivial function $f \in L_2(a, b, w)$ (functions on (a, b) for which $\|f\|_w$ exists and is finite) is orthogonal to every function in S. That is, if, for every $\phi \in S$,

$$\langle f, \phi \rangle_w = 0,$$

then

$$\|f\|_w = 0.$$

This can be thought of as a set that forms a basis for $L_2(a, b, w)$.

Parseval's Identity

Theorem 9 (Parseval's Identity)

Let $\{\phi_1, \phi_2, \cdots\}$ be a complete orthogonal set. If A_k denotes the k^{th} Fourier coefficient of f, then

$$\|f\|_w^2 = \sum_{k=1}^{\infty} A_k^2 \|\phi_k\|_w^2.$$

For those in MATH2221, the converse holds. If Parseval's identity is true for every function $f \in L_2(a, b, w)$, then the orthogonal set is complete.

Example 11

Find the Fourier coefficients for $f(x) = x^2$ over $(-\pi, \pi)$, and hence show

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

First, we find the coefficients. Since f is even, we have $b_k[f] = 0$, and

$$a_0[f] = \frac{2}{\pi} \int_0^{\pi} x^2 \, dx = \frac{2\pi^2}{3},$$

$$a_k[f] = \frac{2}{\pi} \int_0^{\pi} x^2 \cos kx \, dx = \frac{4(-1)^k}{k^2}.$$

Parseval's Identity (cont.)

To apply Parseval's Identity, we find

$$\|1\|_1^2 = 2\int_0^\pi 1^2\,dx = 2\pi,$$
$$\|\cos kx\|_1^2 = 2\int_0^\pi \cos^2 kx\,dx = \pi.$$

Then we also calculate

$$\|f\|_1^2 = 2\int_0^1 x^4\,dx = \frac{2\pi^5}{5}.$$

Finally, using Parseval's Identity, we find

$$\frac{2\pi^5}{5} = \left(\frac{2\pi^2}{2\cdot 3}\right)^2 \cdot 2\pi + \sum_{k=1}^{\infty}\left(\frac{4(-1)^k}{k^2}\right)^2 \cdot \pi,$$

Parseval's Identity (cont.)

Rearranging gives the result.

$$\frac{2\pi^5}{5} = \frac{2\pi^5}{9} + 16\pi \sum_{k=1}^{\infty} \frac{1}{k^4},$$

$$\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\frac{8\pi^5}{45}}{16\pi} = \frac{\pi^4}{90},$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

Example 12

Q2. iii) Consider the function $f : [-1, 1] \to \mathbb{R}$ defined by

$$f(x) = 2x^2 - 1 \text{ for } -1 \leq x \leq 1.$$

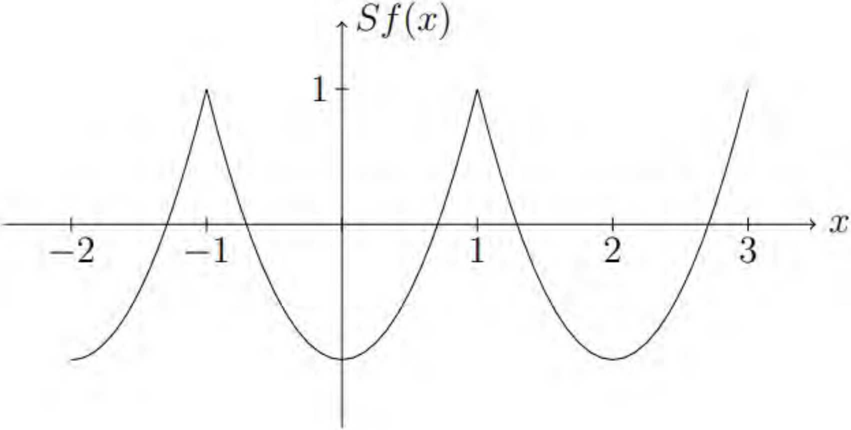
a) Sketch the graph of $Sf(x)$, the Fourier series of $f(x)$ (with period 2), for $-2 \leq x \leq 3$.

b) Find $Sf(x)$.

c) Use Parseval's identity to find the whole number M such that

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + ... = \frac{\pi^4}{M}.$$

a)



b) Since f is an even function,

$$\begin{aligned} A_0 &= \frac{2}{1}\int_0^1 (2x^2-1)dx \\ &= 2[\frac{2}{3}x^3 - x]_0^1 \\ &= -\frac{2}{3}. \end{aligned}$$

For $n \geq 1$

$$\begin{aligned} A_n &= \frac{2}{1}\int_0^1 (2x^2-1)\cos n\pi x dx \\ &= \frac{2}{n\pi}\left([(2x^2-1)\sin n\pi x]_0^1 - \int_0^1 4x\sin n\pi x dx\right) \\ &= \frac{8}{(n\pi)^2}\left([x\cos n\pi x]_0^1 - \int_0^1 \cos n\pi x dx\right) \end{aligned}$$

b) cont.

$$A_n = \frac{8}{\pi^2}\frac{(-1)^n}{n^2}$$
$$Sf(x) = -\frac{1}{3} + \frac{8}{\pi^2}\Sigma_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi x.$$

c) From Parseval's identity,

$$\int_0^1 f(x)^2 dx = \left(\frac{A_0}{2}\right)^2 \int_0^1 1^2 dx + \sum_{n=1}^{\infty} A_n^2 \int_0^1 \cos^2 n\pi x dx$$

$$\begin{aligned}
\int_0^1 f(x)^2 dx &= \int_0^1 (4x^4 - 4x^2 + 1) dx \\
&= [\frac{4}{5}x^5 - \frac{4}{3}x^3 + x]_0^1 \\
&= \frac{4}{5} - \frac{4}{3} + 1 \\
&= \frac{7}{15} \\
\int_0^1 \cos^2 n\pi x dx &= \frac{1}{2}\int_0^1 (1 + \cos 2n\pi x) dx \\
&= \frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
\frac{7}{15} &= \frac{A_0^2}{4} + \sum_{n=1}^{\infty} \frac{A_n^2}{2} \\
&= \frac{(-\frac{2}{3})^2}{4} + \sum_{n=1}^{\infty} \frac{1}{2} \left(\frac{8}{\pi^2} \frac{(-1)^n}{n^2} \right)^2 \\
\frac{32}{\pi^4} \sum_{n=1}^{\infty} \frac{1}{n^4} &= \frac{7}{15} - \frac{1}{9} \\
\sum_{n=1}^{\infty} \frac{1}{n^4} &= \frac{\pi^4}{90} \\
M &= 90.
\end{aligned}$$

Sturm-Liouville Equations

Definition 13

An ODE in the form

$$(p(x)u')' + (\lambda r(x) - q(x))\,u = 0, \quad a < x < b$$

where p, q, r are all real-valued functions, with

$$p(x) > 0, r(x) > 0$$

over the interval (a, b) is called a **Sturm-Liouville equation**.

Throughout this subsection, we will assume p, q, r satisfy the conditions in the definition above.

Eigenstuffs

If we let $Lu = -\left(p(x)u'\right)' + q(x)u$ be a self-adjoint operator, then the general Sturm-Liouville equation can be written as $Lu = \lambda ru$. Using this form, we can define a few terms.

Definition 14

Write $Lu = \lambda ru$ over (a, b). Then a non-trivial solution ϕ to this equation is said to be an **eigenfunction** of L with **eigenvalue** λ. Then (ϕ, λ) is referred to as an **eigenpair**.

Sturm-Liouville Equations

Example 13

Write Legendre's equation

$$(1-x^2)u'' - 2xu' + \nu(\nu+1)u = 0$$

as a Sturm-Liouville equation and identify the eigenvalue. Over what domain is this equation Sturm-Liouville?

We can write the equation as

$$\Big((1-x^2)u'\Big)' + (\nu(\nu+1) - 0)u = 0,$$

which, using the previous notation, has

$$p(x) = 1 - x^2, \quad q(x) = 0, \quad r(x) = 1,$$

and eigenvalue $\lambda = \nu(\nu+1)$.
We require $p(x) > 0$, which is true when $-1 < x < 1$.

Regular Sturm-Liouville Eigenproblems

Definition 15

A **regular Sturm-Liouville eigenproblem** is of the form

$$\begin{aligned} Lu &= \lambda r u && \text{for } a < x < b, \\ B_1 u = b_{11} u' + b_{10} u &= 0 && \text{at } x = a, \\ B_2 u = b_{21} u' + b_{20} u &= 0 && \text{at } x = b, \end{aligned}$$

where $a, b, b_{10}, b_{11}, b_{20}, b_{21}$ are all (finite) reals, with

$$p(a) \neq 0 \quad \text{and} \quad p(b) \neq 0,$$

$$|b_{10}| + |b_{11}| \neq 0 \quad \text{and} \quad |b_{20}| + |b_{21}| \neq 0.$$

Regular Sturm-Liouville Eigenproblems

Example 14

Consider the Sturm-Liouville problem $u'' + \lambda u = 0$ where $u(0) = u(1) = 0$. Find the eigenpairs.

This Sturm-Liouville eigenproblem is regular, defined on $(0, 1)$. We first try to find possible values of λ. Let $k > 0$, then

1. If $\lambda = 0$, we find $u = Ax + B$, which yields $u \equiv 0$ after solving for the initial conditions.
2. If $\lambda = k^2 > 0$, we find $u = A\cos kx + B\sin kx$, and we finally get a non-trivial solution, $u = B\sin n\pi x$ for $n \in \mathbb{Z}^+$. This means $\lambda = n^2\pi^2$.
3. If $\lambda = -k^2 < 0$, we find $u = A\cosh kx + B\sinh kx$, which again yields $u \equiv 0$.

So, our eigenpairs are $(\sin n\pi x, n^2\pi^2)$ for $n = 1, 2, \ldots$.

Regular Sturm-Liouville Eigenproblems

There are a few properties of these kinds of problems that are important.

Theorem 10

The eigenvalues from a regular Sturm-Liouville eigenproblem are real.

Theorem 11

A regular Sturm-Liouville eigenproblem has an infinite sequence of eigenfunctions $\phi_1, \phi_2, \phi_3, \cdots$ which form a complete orthogonal system over the interval (a, b) with respect to the weight function $r(x)$. Further, their corresponding eigenvalues satisfy

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots ,$$

with $\lambda_k \to \infty$ as $k \to \infty$.

Example 15

Show that the eigenfunctions from the previous example are orthogonal.

We can reason that the eigenvalues are all distinct, and they come from a regular Sturm-Liouville eigenproblem, but calculating the inner products, we find for $n \neq m$,

$$\begin{aligned}\langle \phi_n, \phi_m \rangle &= \int_0^1 \sin n\pi x \sin m\pi x \, dx \\ &= \frac{1}{2} \int_0^1 \left(\cos (n-m)\pi x - \cos (n+m)\pi x \right) \, dx \\ &= \frac{1}{2} \left[\frac{\sin (n-m)\pi x}{(n-m)\pi} - \frac{\sin (n+m)\pi x}{(n+m)\pi x} \right]_0^1 \\ &= 0.\end{aligned}$$

Regular Sturm-Liouville Eigenproblems

Example 16

Consider the following ODE

$$\sqrt{1-x^2}u'' - \frac{x}{\sqrt{1-x^2}}u' + \frac{\nu^2}{\sqrt{1-x^2}}u = 0.$$

a) Transform this into Sturm-Liouville form.

b) If the corresponding eigenfunctions were to form an orthogonal set on the interval $[a,b]$, what equality must any two of these functions obey?

a) First, we identify the integrating factor to be $p(x) = \exp(\int -\frac{x}{\sqrt{1-x^2}}/\sqrt{1-x^2}dx) = \sqrt{1-x^2}$.

Regular Sturm-Liouville Eigenproblems

Then the equation can be rearranged to

$$\left(\sqrt{1-x^2}\,u'\right)' + \frac{\nu^2}{\sqrt{1-x^2}}u = 0,$$

which has $q(x) = 0, \quad r(x) = \frac{\nu^2}{\sqrt{1-x^2}}$ and eigenvalues $\lambda = \nu^2$.

b) The eigenfunctions will form an orthogonal set over $[a, b]$ if they come from a regular Sturm-Liouville eigenproblem. For this, we need them to satisfy two boundary conditions:

$$\begin{aligned} B_1\phi = b_{11}\phi' + b_{10}\phi \quad &= 0 \quad \text{at } x = a, \\ B_2\phi = b_{21}\phi' + b_{20}\phi \quad &= 0 \quad \text{at } x = b, \end{aligned}$$

where $b_{10}, b_{11}, b_{20}, b_{21}$ are all real constants, with

$$|b_{10}| + |b_{11}| \neq 0 \quad \text{and} \quad |b_{20}| + |b_{21}| \neq 0.$$

Singular Sturm-Liouville Eigenproblems

Theorem 12

Consider the singular Sturm-Liouville eigenproblem

$$x^2 u'' + xu' + \left(\lambda x^2 - \nu^2\right) u = 0 \quad \text{for } 0 < x < l,$$
$$u(x) \text{ bounded with } xu'(x) \to 0 \text{ as } x \to 0^+,$$
$$c_1 u' + c_0 u = 0 \text{ at } x = l,$$

where $c_0 c_1 \geq 0$. The solution eigenpairs are

$$\left(J_\nu(k_j x), k_j^2\right) \text{ for } j \geq 1,$$

where k_j is the j^{th} positive solution to $c_0 J_\nu(kl) + c_1 k J_\nu'(kl) = 0$.
If $c_0 = \nu = 0$, then $(1, 0)$ is an additional eigenpair.

Singular Sturm-Liouville Eigenproblems

Example 17

Find all eigenpairs of

$$x^2u'' + xu' + (\lambda x^2 - 1)u = 0 \text{ for } 0 < x < 1,$$
$$u'(1) = 0,$$

where u is bounded at 0.

Let k_j be the j^{th} positive solution to

$$kJ_1'(k) = 0.$$

Then the eigenpairs are

$$(J_1(k_jx), k_j^2).$$

5. Initial-Boundary Value Problems in 2D

Separation of Variables

To solve IBPs in two (or more) variables, it is usually helpful to assume the solution is made up of univariate parts, and solve each one independently. This generally results in more than one solution, which can be superimposed to find the general solution.

Separation of Variables

Example 18

Use the separation of variables $u(x,t) = X(x)T(t)$ to find two ODEs for X and T for

$$\nabla^2 u = 0.$$

Let $u(x,t) = X(x)T(t)$, so that

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial t^2} = X''T + XT'' = 0.$$

This can be rearranged so that, for some constant λ,

$$\frac{X''}{X} = -\frac{T''}{T} = -\lambda.$$

That is,

$$X'' + \lambda X = 0, \qquad T'' - \lambda T = 0.$$

Solving IBPs in 2D

There is a somewhat general method of solving IBPs you will find in this course.

1. Use the separation of variables $u = X(x)T(t)$ (or similar);
2. Determine the conditions on X and T implied by the boundary conditions, which usually result in a separation constant, λ;
3. Either
    3.1 Solve the resulting ODEs for different cases of the separation constant, ($\lambda < 0$, $\lambda = 0$, $\lambda > 0$ usually), or
    3.2 Solve the resulting ODEs using methods previously taught;
4. Superimpose the resulting solutions to find the general solution.

Solving IBPs in 2D

Example 19

Consider the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \text{ for } 0 \leq x \leq 1, t \geq 0,$$
$$u(0,t) = u(1,t) = 0,$$
$$u(x,0) = f(x),$$

where f is a piecewise continuous function. Find u.

Let $u(x,t) = X(x)T(t)$, so that

$$XT' = X''T \text{ for } 0 \leq x \leq 1, t \geq 0,$$
$$X(0) = X(1) = 0.$$

We can't yet use the last boundary condition.

Solving IBPs in 2D (cont.)

Now we obtain:

$$\frac{X''}{X} = \frac{T'}{T}$$

and we set this equal to a separation constant $-\lambda$ that will help us find a basis for the solutions.

$$\frac{X''}{X} = \frac{T'}{T} = -\lambda \implies X'' = -\lambda X, \quad T' = -\lambda T$$

Rearranging, we have:

$$X'' + \lambda X = 0 \qquad\qquad T' + \lambda T = 0.$$

Solving IBPs in 2D (cont.)

Letting $k > 0$, we take three cases:

1. If $\lambda = 0$, then $X'' = 0$, and we find $X \equiv 0$ using the boundary conditions: $A \cdot 0 + B = 0, A + B = 0 \implies A = 0, B = 0$.
2. If $\lambda = -k^2 < 0$, then $X'' - k^2 X = 0$, $X(x) = Ae^{kx} + Be^{-kx} \equiv 0$ which means $A = 0, B = 0$.
3. If $\lambda = k^2 > 0$, then $X'' + k^2 X = 0$, and we find that $X = B \sin n\pi x$, where $n \in \mathbb{Z}^+$ (when $\lambda = n^2\pi^2$) since
   $X(x) = A\cos(kx) + B\sin(kx), X(0) = 0 \implies A = 0$.

So, $\lambda = n^2\pi^2$, and $X(x) = B \sin n\pi x$ for $n = 1, 2, \ldots$.

Solving IBPs in 2D (cont.)

Now we tackle T. Since we know λ now, we have

$$T' + n^2\pi^2 T = 0$$

for some $n \in \mathbb{Z}^+$. This can be easily solved to give us

$$T(t) = Ce^{-n^2\pi^2 t}.$$

Combining this with our solution for X (and letting $A_n = BC$, we get

$$u_n(x,t) = A_n e^{-n^2\pi^2 t} \sin n\pi x.$$

That is,

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-n^2\pi^2 t} \sin n\pi x,$$

for some choice of A_n's.

Solving IBPs in 2D (cont.)

Finally, we can use the initial condition, yielding

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin n\pi x = f(x).$$

This is the half-range Fourier sine series of f (with the range $(0,1)$, so

$$A_n = 2\int_0^1 f(x)\sin n\pi x\,dx.$$

If we were given an explicit f, we could evaluate this to get the final solution for u.

Laplacian in Polar Coordinates

Sometimes we approach problems which are easier to express in polar coordinates. For example, problems defined on the unit disk, or which are radially symmetric.

Theorem 13

Suppose u is defined in polar coordinates (r, θ). Then

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

Laplacian in Polar Coordinates

Example 20

The temperature $u(r, \theta)$ of a circular surface obeys

$$\nabla^2 u = 0 \text{ for } 0 \leq r < 1,$$
$$u(1, \theta) = f(\theta),$$
$$u \text{ continuous.}$$

Since u is continuous, we require $u(r, \theta) = u(r, \theta + 2\pi)$. Now use the separation of variables $u(r, \theta) = R(r)H(\theta)$. Then,

$$H(\theta) = H(\theta + 2\pi),$$
$$r^2 R'' H + rR' H + RH'' = 0.$$

Laplacian in Polar Coordinates (cont.)

This is because:

$$\frac{1}{r}\partial_r(rR'H) + \frac{1}{r^2}\partial_{\theta\theta}(RH) = 0$$

obtains:

$$\frac{H}{r}(R' + rR'') + \frac{1}{r^2}RH'' = 0$$

since H is independent of r and R is independent of H. Multiplying through by r^2 gets the result obtained above. From here, one may execute the following step:

$$H(r^2R'' + rR') + RH'' = 0 \implies \frac{r^2R'' + rR'}{R} = -\frac{H''}{H} = \lambda$$

For a suitable separation constant λ,

$$r^2R'' + rR' - \lambda R = 0, \qquad\qquad H'' + \lambda H = 0.$$

Laplacian in Polar Coordinates (cont.)

Since the ODE for R is Cauchy-Euler, we find H first (so we know λ). Considering all three cases, the only possible periodic case is if H is constant ($\lambda = 0$), or when $\lambda > 0$, say $\lambda = k^2$ where $k > 0$. If $\lambda = -k^2$, then $H(\theta) = Ae^{k\theta} + Be^{-k\theta}$ which is clearly not periodic. This gives us

$$H = A\cos k\theta + B\sin k\theta.$$

Here, we have merged the 2 cases - if $k = 0$, we end up with $H(\theta) = B$ which is a constant. Using our periodic condition, we find

$$B = A\cos 2k\pi + B\sin 2k\pi,$$

from which we deduce $k \in \mathbb{Z}^+$.
Thus, $H(\theta) = A\cos k\theta + B\sin k\theta$ and $\lambda = k^2$ for $k \in \mathbb{N}$.

Laplacian in Polar Coordinates (cont.)

Now we have

$$r^2 R'' + rR' - k^2 R = 0.$$

This is Cauchy-Euler, with indicial equation

$$n(n-1) + n - k^2 = 0,$$

which has solutions $n = \pm k$. If $k \neq 0$, we have

$$R(r) = Cr^k + Dr^{-k}.$$

If $k = 0$, we have

$$R(r) = Cr^0 + Dr^0 \ln r = C + D \ln r.$$

Note, however, that we require R to be bounded at 0, so $D = 0$ in both cases, and $R(r) = Cr^k$ for $k \in \mathbb{N}$.

Laplacian in Polar Coordinates (cont.)

Finally, we find the general solution for u,

$$u(r,\theta) = A_0 + \sum_{k=1}^{\infty} r^k (A_k \cos k\theta + B_k \sin k\theta),$$

for some choice of A_k and B_k (with C having been merged in). Since $u(1,\theta) = f(\theta)$, we have

$$u(1,\theta) = A_0 + \sum_{k=1}^{\infty} (A_k \cos k\theta + B_k \sin k\theta) = f(\theta).$$

So, the A_k and B_k are chosen to be the trigonometric Fourier coefficients of f.

Elliptic Operators

We can write a general partial differential operator (in $\mathbb{R}^d$) in the form

$$Lu = -\sum_{j=1}^{d}\sum_{k=1}^{d} a_{jk}(\mathbf{x})\partial_j\partial_k u + \sum_{k=1}^{d} b_k(\mathbf{x})\partial_k u + c(\mathbf{x})u.$$

If we let A be the matrix formed from second order coefficients (the a_{ij}s), then we can define what it means to be elliptic.

Definition 16

A second-order linear partial differential operator is called **elliptic** in Ω if there exists a $c > 0$ such that, for all $\mathbf{x} \in \Omega$ and $\mathbf{y} \in \mathbb{R}^d$,

$$\mathbf{y}^T A(\mathbf{x})\mathbf{y} \geq c\|\mathbf{y}\|^2.$$

Elliptic Operators

It is generally difficult to work with the previous definition of an elliptic operator, so we can apply the following theorem instead.

Theorem 14

Let the symmetric part of A be A^{sy}. Then let λ_k be the k^{th} eigenvalue of A^{sy} (i.e. $\frac{A+A^T}{2}$). The operator is then elliptic on Ω if and only if there exists a $c > 0$ such that, for all $\mathbf{x} \in \Omega$ and $1 \leq k \leq d$,

$$\lambda_k(\mathbf{x}) \geq c.$$

A Note on the Eigenvalue Condition

The condition $\lambda_k(\mathbf{x}) > 0$ is not sufficient, as it is possible for the eigenvalues to approach 0 on the set. We need them to be bounded away from 0.

Elliptic Operators

Example 21

Define the operator

$$L = -(2x+4)\partial_1^2 + 3\partial_1\partial_2 + \partial_2\partial_1 - (2x+1)\partial_2^2 + (x+2)\partial_1 - 3$$

with respect to x. Is this operator elliptic on $\Omega = (0,1)$?

The relevant matrix and its symmetric part are

$$A = \begin{pmatrix} 2x+4 & -3 \\ -1 & 2x+1 \end{pmatrix}, \qquad A^{\text{sy}} = \begin{pmatrix} 2x+4 & -2 \\ -2 & 2x+1 \end{pmatrix}.$$

A^{sy} has eigenvalues $2x$ and $2x+5$, but there is no $c > 0$ such that $2x \geq c$ for all $x \in \Omega$, so the operator isn't elliptic on Ω. It is, however, elliptic on every $[a, 1)$ for $0 < a < 1$.

Elliptic Eigenproblems

The theory behind elliptic eigenproblems is complex, and generally, you don't need to know it in too much detail for this course. Instead, we will go through solving an elliptic eigenproblem. The method is very similar to that of other IBPs in 2D, however, we now have an eigenvalue to worry about, which sometimes complicates things.

Elliptic Eigenproblems

Example 22

Let Ω be the unit square $0 < x < 1$ and $0 < y < 1$. Let

$$\begin{gathered}\nabla^2 u + \lambda u = 0 \text{ on } \Omega,\\ u(x,0) = u(x,1) = 0,\\ \frac{\partial u}{\partial x}(0,y) = \frac{\partial u}{\partial x}(1,y) = 0.\end{gathered}$$

Find the eigenpairs of this system.

Letting $u(x,y) = X(x)Y(y)$ and p be the separation constant, we can write this as

$$\begin{gathered}X'' + pX = 0 = Y'' + (\lambda - p)Y,\\ Y(0) = Y(1) = 0,\\ X'(0) = X'(1) = 0.\end{gathered}$$

Elliptic Eigenproblems (cont.)

Since X seems easier, we solve it for three cases of p. Let $k > 0$.

1. If $p = 0$, then $X'' = 0$, and we find $X = A$ for some constant.
2. If $p = k^2 > 0$, then $X'' + k^2 X = 0$, so $X = A\cos kx + B\sin kx$ for some A, B. Using the initial conditions, we find $X = A\cos m\pi x$ where $m \in \mathbb{Z}^+$ (so $p = m^2\pi^2$).
3. If $p = -k^2 < 0$, then $X'' - k^2 X = 0$, so $X = A\cosh kx + B\sinh kx$ for some A, B. The initial conditions result in $X \equiv 0$, so we ignore this case.

So we have $X = A$ (whence $p = 0$), and $X = A\cos m\pi x$ (whence $p = m^2\pi^2$). These can be combined to yield

$$X = A\cos m\pi x$$

for $m = 0, 1, 2, \cdots$, and $p = m^2\pi^2$.

Elliptic Eigenproblems (cont.)

Now, we can attack Y. If we let $\mu = \lambda - p$, then we can write

$$Y'' + \mu Y = 0.$$

As with X, we consider three cases.

1. If $\mu = 0$, then $Y'' = 0$ and using the initial conditions, we get $Y \equiv 0$, so we ignore this case.
2. If $\mu = k^2$, then $Y'' + k^2 Y = 0$, so $Y = A \sin ky + B \cos ky$ for some A, B. Using the initial conditions, we find $Y = A \sin n\pi y$ where $n \in \mathbb{Z}^+$ (so $\mu = n^2\pi^2$).
3. If $\mu = -k^2 < 0$, then $Y'' - k^2 Y = 0$, so $Y = A \cosh ky + B \sinh ky$ for some A, B. The initial conditions result in $Y \equiv 0$, so we ignore this case.

So, we get $Y = A \sin n\pi y$ for $n = 1, 2, \cdots$, and $\mu = n^2\pi^2$.

Elliptic Eigenproblems (cont.)

Finally, we can return to our initial separation and put things together. The eigenvalues are $\lambda_{m,n} = \mu + p = (m^2 + n^2)\pi^2$, with corresponding eigenfunctions

$$\phi_{m,n}(x, y) = \cos m\pi x \sin n\pi y.$$

Superimposing these functions will give us the general solution.