MATH1081 Revision

Set Theory, Number Theory and Graph Theory

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Table of Contents

Sets - Definitions

Definition (Set)

A set is a well-defined collection of distinct objects. An element of a set is any object in the set. If $x \in S$, then x is an element of S.

- The cardinality of a set S, $|S|$, is the number of elements in S.
- A set S is a subset of a set T iff each element of S is also an element of T. Notation: $S \subseteq T$
- A set S is a proper subset of a set T iff S is a subset of T and $S \neq T$. Notation: $S \subsetneq T$ or $S \subset T$.

Sets - Definitions (continued)

Definition (Power set)

The power set $P(S)$ of a set S is the set of all possible subsets of S.

- The number of subsets of S is $|P(S)| = 2^{|S|}$
- $\emptyset$ is the empty set. Note that $\emptyset \subseteq S$, where S is a set. Also, $\emptyset \subseteq P(S)$
- The complement of a set A is A^c, 'not A'.

Inclusion-Exclusion Principle

$|A \cup B| = |A| + |B| - |A \cap B|$

Laws of Set Algebra

- Commutative Law
  - $A \cap B = B \cap A$
  - $A \cup B = B \cup A$
- Associative Law
  - $A \cap (B \cap C) = (A \cap B) \cap C$
  - $A \cup (B \cup C) = (A \cup B) \cup C$
- Distributive Law
  - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
  - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- Absorption Law
  - $A \cap (A \cup B) = A$
  - $A \cup (A \cap B) = A$
- Identity Law
  - $A \cap U = U \cap A = A$
  - $A \cup \emptyset = \emptyset \cup A = A$
- Idempotent Law
  - $A \cap A = A$
  - $A \cup A = A$

Laws of Set Algebra (continued)

- Double Complement Law
  - $(A^c)^c = A$
- Difference Law
  - $A - B = A \cap B^c$
- Domination
  - $A \cap \emptyset = \emptyset \cap A = \emptyset$
  - $A \cup U = U \cup A = U$
- Intersection and Union with the Complement
  - $A \cap A^c = A^c \cap A = \emptyset$
  - $A \cup A^c = A^c \cup A = U$
- De Morgan's Laws
  - $(A \cup B)^c = A^c \cap B^c$
  - $(A \cap B)^c = A^c \cup B^c$

Laws of Set Algebra: Example

Example

Simplify $(A \cup B) \cup (C \cap A) \cup (A \cap B)^c$ using the laws of set algebra.

Make sure to state the Laws of Set Algebra!

Drawing Venn diagrams can help you to visualise, but they do not count as formal proofs.

Proofs

Hints for proofs

To prove that $S \subseteq T$, we assume that $x \in S$ and show that $x \in T$.
To prove that $S = T$, we show that $S \subseteq T$ and $T \subseteq S$.

Examples

1. Prove that $A = \{\cos(x) = \frac{1}{\sqrt{6}} \mid x \in \mathbb{R}\}$ is a proper subset of $B = \{\cos(2x) = \frac{-2}{3} \mid x \in \mathbb{R}\}$.
2. Prove that $A = \{x \geq 2 \mid x \in \mathbb{R}\}$ is a proper subset of $B = \{3x^2 + x \geq 14 \mid x \in \mathbb{R}\}$. (Class test 1 v1B S2 2018)

Generalised Set Operations

Generalised Set Operations

- $\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n$
- $\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n$

Example

Let $I = 1, 2, 3, \ldots$ be the index set. For each $i \in I$, let $A_i = [0, \frac{1}{i}]$ be the set of real numbers between 0 and $\frac{1}{i}$, including 0 and $\frac{1}{i}$.
Find:

1. $\bigcup_{i \in I} A_i$
2. $\bigcap_{i \in I} A_i$

Functions

Definition (Cartesian product)

- An ordered pair (a, b) is a collection of 2 objects in a specified order. $(a, b) \neq (b, a)$.
- The Cartesian product of two sets A and B, denoted by $A \times B$, is the set of all ordered pairs, the first from A and the second from B.

Definition (Function)

A function f from a set X to a set Y is a subset of $X \times Y$ such that for every $x \in X$ there is exactly one $y \in Y$ for which (x, y) belongs to f. (On a graph, each vertical line intersects the graph once.)

Functions continued

Definition (domain, codomain, range, image)

- If f is a function from X to Y then we write $f : X \to Y$, where X is the domain of f (set of inputs) and Y is the codomain of f (possible outputs).
- The range of f is the set of all values of f (actual outputs)
- The image of a set is the subset of the codomain that comes from the subset of the input given.
- The inverse image of a set is the subset of the inputs that comes from the given subset of outputs.

Functions continued

- A function $f : X \to Y$ is injective or one-to-one iff for every $y \in Y$, there is at most one $x \in X$ such that $f(x) = y$. If $f(x_1) = f(x_2)$ then $x_1 = x_2$. (On a graph, each horizontal line intersects the graph at most once.)
- A function $f : X \to Y$ is surjective or onto, by definition, if for all $y \in Y$, there exists an x such that $f(x) = y$. We can prove that a function is surjective iff the range of f is the same as the codomain of f. (On a graph, each horizontal line intersects the graph at least once.)
- A function $f : X \to Y$ is bijective iff f is both injective and surjective.

Example (Class Test 1 v1B S2 2018)

Let $X = \{0 \leq x \leq \frac{\pi}{2} \mid x \in \mathbb{R}\}$. A function $f : X \to \mathbb{R}$ is defined by $f(x) = \sin(x)$. Find the range of f. Is f injective, surjective, and/or a bijection? Give reasons.

Sequences

Properties of summation:

- $\sum_{k=m}^{n} (a_k + b_k) = \sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k$
- $\sum_{k=m}^{n} (\lambda a_k) = \lambda \sum_{k=m}^{n} a_k$

Properties of product:

- $\prod_{k=m}^{n} a_k b_k = (\prod_{k=m}^{n} a_k)(\prod_{k=m}^{n} b_k)$
- Note: $\prod_{k=m}^{n} (a_k + b_k) \neq \prod_{k=m}^{n} a_k + \prod_{k=m}^{n} b_k$

Sequences continued

Change of summation index:

- For any sequence a_k and any integer d we have:

$$\sum_{k=m}^{n} a_k = \sum_{k=m+d}^{n+d} a_{k-d}$$

Example

Prove that if k is a positive integer then

$$\frac{3}{k} + \frac{4}{k+1} - \frac{7}{k+2} = \frac{10k+6}{k(k+1)(k+2)}.$$

Hence simplify $\sum_{k=1}^{n} \frac{5k+3}{k(k+1)(k+2)}$

Table of Contents

Integers and Factorisation

Definition

If $b = am$, then $a|b$. We say that 'a divides b'.

- Remember, $a|0$, because 0 is a multiple of every integer.
- If $a|b$ and $a|c$, then $a|sb + tc$ for all integers s and t.

Useful Property: If n has no prime factor less than or equal to $\sqrt{n}$, then it is prime.

Example

Let $a, m \in \mathbb{Z}$. Prove that if $a|m$ and $a+1|m$ then $a(a+1)|m$.

Integers and Factorisation (continued)

Definition

$\gcd(a, b) =$ 'greatest common divisor of a and b'.

- If $\gcd(a, b) = 1$, then a and b are coprime.
- $\gcd(a, b) \times \text{lcm}(a, b) = ab$
- Tip: To find the $\gcd(a, b)$, multiply all the prime factors common to both.
- Tip: To find the $\text{lcm}(a, b)$, take the smallest product that includes all the factors of both a and b.

Extended Euclidean Algorithm

The Extended Euclidean Algorithm is used to find integer solutions x and y to the equation $ax + by = \gcd(a, b)$

Example

Find $x, y \in \mathbb{Z}$ if $16758x + 14175y = 63$

The Bezout Property

If an equation is of the form $ax + by = c$

- if $c = \gcd(a, b)$ then the equation has integer solutions
- if c is a multiple of $\gcd(a, b)$ then the equation has integer solutions
- if c is not a multiple of $\gcd(a, b)$ then the equation has no integer solutions

Modular Arithmetic

If $a \equiv b \pmod{m}$ then:

- a and b are congruent modulo m
- $a \bmod m = b \bmod m$
- $m | (a - b)$
- $a = b + km$, i.e. 'a and b differ by a factor of m'

Simplifying exponents with mod

Example

Simplify $6^{54321} \bmod 100$

- When simplifying, try and reach a result that is 1 or -1.
- If you can't get down to 1 or -1, look out for a repeating pattern in the numbers.
- To find the last 2 digits of a number, simplify it mod 100.

Inverses

Definition (inverse)

If $ab \equiv 1 \pmod{m}$ then a and b are inverses modulo m.

You can find the inverse of a number by using the Euclidean algorithm, after re-writing the congruence equation.

Solving congruence equations

Method to solve a congruence equation

Typical question: "Find all x such that $ax \equiv b \pmod{m}$".
This can be re-written as $ax + my = b$, where $y \in \mathbb{Z}$.
3 cases:

1. CASE 1: If $\gcd(a, m)$ is not a factor of b, then the equation has no integer solutions.
2. CASE 2: If $\gcd(a, m) = 1$: Let c be the inverse of a. Multiply both sides of the original equation by c to get $cax \equiv bc \pmod{m}$. Since $ca = 1$, this can be simplified to $x = bc \pmod{m}$. So to find x, we find c which is the inverse of a, and multiply it by b, then take $\pmod{m}$.

Solving congruence equations (continued)

Method to solve a congruence equation (continued)

1. CASE 3: If $\gcd(a, m) \neq 1$: Divide all numbers (a, b, m) by the highest digit possible until $\gcd(a, m) = 1$. We can now use method 2. This will give the solution in terms of modulo $\frac{m}{d}$. To give your answer in terms of $(\bmod\, m)$, add multiples of $\frac{m}{d}$ to your solution so that you get d solutions.

Give your answer in terms of the original modulus, unless the question asks otherwise.

Solving congruence equations: Example

Example

Solve the linear congruence $52x \equiv 8 \pmod{60}$

Table of Contents

Relations - Definitions

Definition

- A relation R from a set A to a set B is a subset of $A \times B$.
  - if $(a, b) \in R$ we say that a is related to b (by R), and we write aRb.
  - if $(a, b) \notin R$ we write that $a \not{R} b$.
- A function is a relation $R \subseteq A \times B$ with the special property that for every $a \in A$ there is exactly one $b \in B$ such that aRb.

- We will mainly consider binary relations on a set A, that is, relations of the form $R \subseteq A \times A$.

Relations - Representations

Arrow Diagram

List the elements of A and the elements of B, and then draw an arrow from a to b for each pair $(a, b) \in R$.

Matrix M_R

Arrange the elements of A and B in some order $a_1, a_2, \ldots$ and $b_1, b_2, \ldots,$ and then form a rectangular array of numbers where

$$m_{i,j} = \begin{cases} 1 & \text{if } a_i R b_j \\ 0 & \text{if } a_i \not R b_j \end{cases}$$

- The matrix M_R has $|A|$ rows and $\mid B \mid$ columns.
- The matrix changes if the elements are arranged in a different order.

Example

Example

Let $R = \{(a,a),(a,b),(b,a),(b,b),(d,a),(b,c)\}$ be a relation on the set $A = \{a,b,c,d\}$. Draw the arrow diagram of R and write down the matrix of R.

meme

EVERYONE:*USES ARROW DIAGRAMS*

Relations - Properties

- We say that a relation R on a set A is reflexive if for every $a \in A$,

$$aRa$$

(that is, every element is related to itself).

- We say that a relation R on a set A is symmetric when for every $a, b \in A$,

$$aRb \text{ implies } bRa$$

(that is, if a is related to b, then b is related to a).

Relations - Properties (Continued)

- We say that a relation R on a set A is antisymmetric when for every $a, b \in A$,

$$aRb \text{ and } bRa \text{ implies } a = b$$

(that is, if a and b are related to each other then they must be identical).

- We say that a relation R on a set A is transitive when for every $a, b, c \in A$,

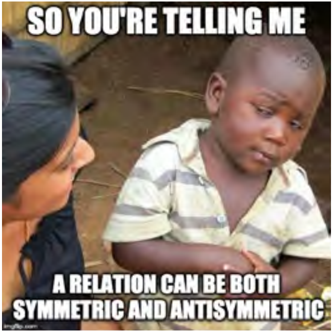
$$aRb \text{ and } bRc \text{ implies } aRc$$

(that is, if a is related to b and b is related to c then a is related to c).

meme

WATCH OUT

Note that "antisymmetric" is not the opposite of "symmetric". A relation can be both symmetric and antisymmetric.



Properties in terms of Arrow Diagrams and Matrices

	arrow diagram	matrix
reflexive	we must have at every dot	diagonal entries are all 1
symmetric	if we have then we must have	for $i \neq j$, $m_{i,j} = m_{j,i}$
antisymmetric	we cannot have	for $i \neq j$, $m_{i,j}$ and $m_{j,i}$ cannot both be 1
transitive	(i) if we have , then we must have , and (ii) if we have , then we must have	for every nonzero entry in the matrix product M^2 the corresponding entry in M must be 1

Source: MATH1081 Topic 5 Notes F. Kuo/T. Britz/D. Chan/D. Trenerry/C. Greenhill

Equivalence Relations and Equivalence Classes - Definitions

Equivalence Relations

- A relation which is reflexive, symmetric and transitive is called an equivalence relation.
- We often write $\sim$ to denote an equivalence relation: $a \sim b$ reads "a is equivalent to b" (with respect to $\sim$).

Equivalence Classes

- Let $\sim$ be an equivalence relation on a set A. For every element $a \in A$, the equivalence class of a with respect to $\sim$, denoted by $[a]$, is the set
$$[a] = \{x \in A \mid x \sim a\}.$$
- We let $A/\sim$ denote the set of equivalence classes. Thus $A/\sim \subseteq P(A)$.

Equivalence Relations and Equivalence Classes - Theorem

Theorem

Let $\sim$ be an equivalence relation on a set A. Then

- For all $a \in A$, $a \in [a]$. Hence,
  - every element of A belongs to at least one equivalence class.
  - every equivalence class contains at least one element.
- For all $a, b \in A$, $a \sim b$ if and only if $[a] = [b]$.
- For all $a, b \in A$, $a \nsim b$ if and only if $[a] \cap [b] = \emptyset$.
  - Hence, any two equivalence classes are either equal or disjoint.

meme

Example

2012 Semester 2 Q1. (iv)

Define a relation $\sim$ on $\mathbb{R}$ by $x \sim y$ if and only if there exists $k \in \mathbb{Z}$ such that $x - y = 2k\pi$

1. Show that $\sim$ is an equivalence relation.
2. Write $\mathbb{R}$ as the union of pairwise disjoint equivalence classes $[x]$ for all $[x] \in \mathbb{R}$.

Example

2015 Semester 1 Q1. (iv)

Define a relation $\sim$ on the set of complex numbers by

$$z \sim w \iff |z-1| = |w-1|$$

1. Prove that $\sim$ is an equivalence relation on S.
2. Draw a sketch which shows the equivalence class of $\{2i\}$.

Partitions

Definition

A partition of a set A is a collection of disjoint nonempty subsets of A whose union equals A. When this holds, we say that these sets partition A.

Theorem

Let A be a set.

- The equivalence classes of an equivalence relation on A partition A.
- Any partition of A can be used to form an equivalence relation on A.

Partial Orders

- A reflexive, antisymmetric and transitive relation is called a partial order.
- We often write $\preceq$ to denote a partial order: $a \preceq b$ reads "a precedes b".
- A set A together with a partial order $\preceq$ is called a partialy ordered set or a poset. We denote this by $(A, \preceq)$.
- We say that two elements $a, b \in A$ are comparable with respect to a partial order $\preceq$ if and only if at least one of $a \preceq b$ or $b \preceq a$ holds.
- A partial order in which every pair of two elements are comparable is called a total order or a linear order.

Hasse Diagrams

Construction

- Find all pairs with $a \prec b$ such that there is no c with $a \prec c \prec b$,
- Draw a line between a and b, with a positioned lower than b in the diagram.

Posets

Definitions

Let $(A, \preceq)$ be a poset. An element $x \in A$ is called

- A maximal element iff there is no element $a \in A$ with $a \prec a$;
- A minimal element iff there is no element $a \in A$ with $a \prec x$;
- The greatest element iff $a \preceq x$ for all $a \in A$;
- The least element iff $x \preceq a$ for all $a \in A$.

Greatest and Least Elements

- The greatest element in a poset is unique if it exists.
- The least element in a poset is unique if it exists.

Posets

Definitions

Let $S \subseteq A$

- an upper bound for S is an element $b \in A$ such that $s \preceq b$ for every $s \in S$.
- a lower bound for S is an element $b \in A$ such that $b \preceq s$ for every $s \in S$.
- the least upper bound for S (if it exists) is the least element for the set of upper bounds.
- the greatest lower bound for S (if it exists) is the greatest element for the set of lower bounds.

Example

2013 Semester 1 Q2. (i)

Suppose that $A = \{2, 3, 5, 6, 15, 30, 35, 70, 105, 210\}$

1. Draw the Hasse Diagram for the partially ordered set $(A, |)$, where $a|b$ means that a divides b. (You do not have to prove this is a partial order.)
2. Find, if they exist, all
    1. greatest elements
    2. least elements
    3. maximal elements
    4. minimal elements
3. Find two elements of A that do not have a greatest lower bound and explain why they do not.

Table of Contents

Graph Theory - Definitions

Definitions

- Loosely speaking, a graph is a set of dots (vertices) and dot-connecting lines (edges).
- Formally, a graph G consists of
  - A finite set V whose elements are called the vertices of G;
  - A finite set E whose elements are called the edges of G;
  - A function that assigns to each edge $e \in E$ an unordered pair of vertices, called the endpoints of e. This function is called the edge-endpoint function.

Don't get confused

Note that these graphs are not related to the graphs of functions.

Graph Theory - More Definitions...

More Definitions/Terminology

- If the edge $e \in E$ has endpoints $v, w \in V$ then we say that
  - the edge e connects the vertices v and w;
  - the edge e is incident with the vertices v and w;
  - the vertices v and w are the endpoints of the edge e;
  - the vertices v and w are adjacent;
  - the vertices v and w are neighbours.
- Two edges with the same endpoints are multiple or parallel.
- A loop is an edge that connects a vertex to itself.
- The degree of a vertex v, denoted by $deg(v)$, is the number of edges incident with v, counting any loops twice.
- An isolated vertex is one with degree 0, and a pendant vertex is one with degree 1.

The Handshaking Theorem

Theorem

The total degree of a graph is twice the number of edges

$$2|E| = \sum_{v \in V} deg(v)$$

By the Handshaking Theorem, the total degree of a graph must be even and the number of odd vertices must be even.

Types of Graphs

Simple Graph

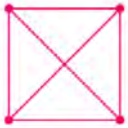
A simple graph is a graph with no loops or parallel edges.

Complete Graph

The complete graph $K_n (n \geq 1)$ is a simple graph with

- n vertices;
- exactly one edge between each pair of distinct vertices.

Hence, K_n has $C(n, 2)$ edges.



K_4

K_5

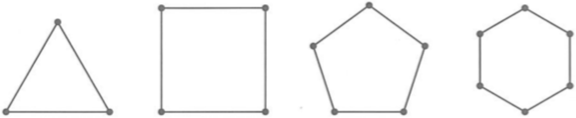
K_6

Types of Graphs

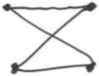
Cyclic Graph

The cyclic graph $C_n(n \geq 3$ consists of

- n vertices $v_1, v_2, ..., v_n$;
- n edges $\{v_1, v_2\}, \{v_2, v_3\}, ..., \{v_{n-1}, v_n\}, \{v_n, v_1\}$.



C_3 C_4 C_5 C_6



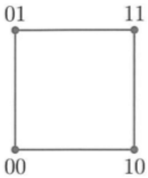
alt. C_4

Types of Graphs

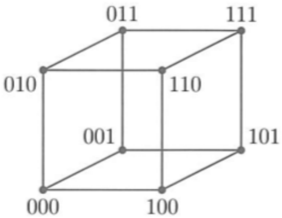
n-cube

The n-cube Q_n is the simple graph with

- vertices for each bit string $a_1 a_2 \ldots a_n$ of length n, where $a_i \in \{0, 1\}$;
- an edge between vertices $a_1 a_2 \ldots a_n$ and $b_1 b_2 \ldots b_n$ if and only if $a_j \neq b_j$ for exactly one $j \in \{1, \ldots, n\}$.
    - Two vertices are adjacent if and only if they differ by one bit.



Q_2



Q_3

Bipartite Graphs

Definition

A graph is bipartite iff its vertex set V can be partitioned into subsets V_1, V_2 so that every edge has an endpoint in V_1 and an endpoint in V_2. That is, no vertex is adjacent to any other vertex in the same subset.

Useful Result

A graph is bipartite if and only if it contains no odd cycles.

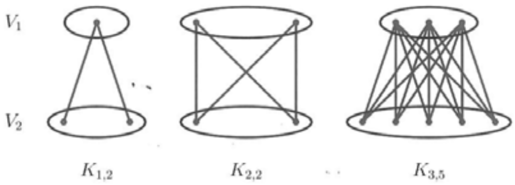
Complete Bipartite Graphs

Complete Bipartite Graph

The complete bipartite graph $K_{m,n}$ is the simple bipartite graph with vertex set $V_1 \cup V_2$, with

- V_1 containing m vertices and V_2 containing n vertices;
- edges between every vertex V_1 and every vertex in V_2.

$K_{m,n}$ has m+n vertices and mn edges.



Subgraph

Subgraph

Let G_1 and G_2 be two graphs with vertex sets V_1 and V_2 and edge sets E_1 and E_2, respectively. Then G_1 is a subgraph of G_2 , and we write $G_1 \subseteq G_2$, iff

- $V_1 \subseteq V_2$;
- $E_1 \subseteq E_2$;
- each edge in G_1 has the same endpoints as in G_2.

Pictorially, a graph is obtained by deleting some edges and/or vertices is a subgraph. If a vertex is deleted then all edges incident with it are also deleted.

Example. $G_1 \subseteq G_2$.

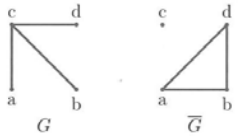
Complementary Graph

Complementary Graph

Let G be a simple graph. The complementary graph $\bar{G}$ of G is a simple graph with

- the same vertex set as G;
- an edge joining two vertices if and only if they are not adjacent in G.

Example.



Adjacency Matrix

Adjacency Matrix

- Let G be a graph with an ordered listing of vertices $v_1, v_2, .., v_n$. The adjacency matrix of G is the $n \times n$ matrix $A = [a_{ij}]$ with

$$a_{ij} = \#\text{edges connecting } v_i \text{ and } v_j$$

- The entries a_{ij} depend on the order in which the vertices have been numbered.
  - Changing the vertex order corresponds to permuting rows and columns
- The adjacency matrix A is symmetric, i.e., $A = A^T$.

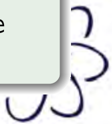
Example

2013 Semester 1 Q2 (ii)

A graph H on the vertices A, B, C, D (in that order) has adjacency matrix:

$$M = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

1. Draw the graph H.
2. M^3 has the number 12 in position 1,2 (that is, in the first row and second column). What does this mean in terms of the graph?



Walks

Walks

- A walk in a graph G is an alternating sequences of vertices v_i and edges e_i in G

$$v_0 e_1 v_1 e_2 v_2 \ldots v_{n-1} e_n v_n$$

- The length of the walk is the number of edges involved (n above).
- A closed walk is one that starts and ends in the same vertex.
- In a simple graph, a walk can be specified by stating the vertices alone.

Paths and Circuits

Paths and Circuits

- A path is a walk with no repeated edges.
- A circuit is a closed walk with no repeated edges.
- A path $v_0 e_1 v_1 e_2 v_2 \ldots v_{n-1} e_n v_n$ is simple iff all the v_i are distinct, i.e. there are no repeated vertices.
- A circuit $v_0 e_1 v_1 e_2 v_2 \ldots v_{n-1} e_n v_n$ is simply iff $v_1, \ldots, v_n$ are distinct (but $v_0 = v_n$).

meme


WALK
PATH
CIRCUIT
LEAVING A QUESTION THAT YOU CAN'T DO AS AN EXERCISE FOR THE MARKER

Some Theorems

Theorem

Let a and b be vertices in a graph. If there is a walk from a to b, then there is a simple path from a to b.

Another Theorem

If A is the adjacency matrix for G with ordered vertices $v_1, \ldots, v_n$, then the number of walks of length k from v_i to v_j in G is given by the entry in the ith row and jth column of A^k.

Connectivity

Connectivity

- Vertices a, b of a graph G are connected in G iff there is a walk from a to b.
- A graph G is connected iff every pair of distinct vertices is connected in G.
- Let G be a graph with vertex set V. The relation $\sim$ on V defined by

  $$v_i \sim v_j \text{ if and only if } v_i \text{ is connected to } v_j \text{ in } G$$

  is an equivalence relation.
- The equivalence classes of this relation are the connected components of G. Two vertices are in the same connected component if and only if they are connected in G.

Connectivity

A Connection to the Adjacency Matrix

Let G be a graph with vertices $v_1, ..., v_n$ and adjacency matrix A. Let

$$C = I + A + A^2 + ... + A^{n-1}.$$

Then G is connected if and only if C has no zero entries.

Euler/Hamilton Paths/Circuits

Definitions

- Let G be a graph.
- An Euler path in G is a path that includes every edge of G exactly once.
- A Hamiltonian path in G is a simple path that includes every vertex of G exactly once.
- An Eulerian circuit in G is a circuit that includes every edge of G exactly once.
- A Hamiltonian circuit in G is a simple circuit which includes every vertex of G exactly once (counting the start-vertex once).

Theorems for Euler Circuits/Paths

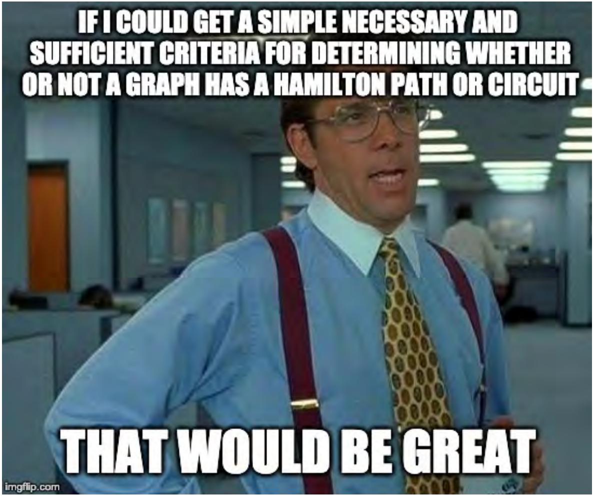
Theorem for Euler Circuit

Let G be a connected graph. An Euler circuit exists if and only if G has even vertex degrees. i.e. there are no vertices with odd degree.

Theorem for Euler Path

Let G be a connected graph. An Euler path which is not a circuit exists if and only if G has exactly two vertices of odd degree.

On Hamilton Circuits/Paths



On Hamilton Circuits/Paths

Notes

- No simple necessary and sufficient criteria are known that determine whether a graph has a Hamiltonian circuit or path.
- Note that a graph with a vertex of degree 1 cannot have a Hamiltonian circuit.
- If a graph G has a Hamilton circuit, then the circuit must include all edges incident with vertices of degree 2.
- A Hamilton path or circuit uses at most 2 edges incident with any one vertex.

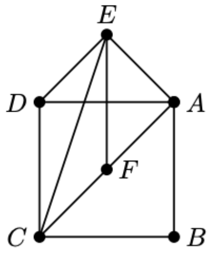
Theorem

Let G be a connected and simple graph with $n \geq 3$ vertices, such that each vertex has degree at least $n/2$. Then G has a Hamilton circuit.

Example

2014 Semester 1 Q2 (iii)

Consider the following graph G:



Giving reasons, show that

1. G is not bipartite
2. G contains an Euler path
3. G contains a Hamilton Circuit

Isomorphic Graphs

Isomorphic Graphs

- Let G and G' be graphs with vertices V and V' and edges E and E' respectively. Then G is isomorphic to G', and we write $G \simeq G'$, iff there are two bijections $f : V \to V'$ and $g : E \to E'$, such that e is incident with v in G if and only if $g(e)$ is incident with $f(v)$ in G'.
    - Roughly speaking, two graphs are isomorphic iff they are the same except for edge and vertex labelings.
    - In this case, $\deg(v) = \deg(f(v))$
- Two simple graphs G and G' are isomorphic iff there is a bijection $f : V \to V'$ such that for all $v_1, v_2 \in V$, v_1 and v_2 are adjacent in G if and only if $f(v_1)$ and $f(v_2)$ are adjacent in G'.

meme

Isomorphic Graphs

Invariants

A property of a graph G is an invariant iff G' also has this property whenever $G' \simeq G$.

Some graph invariants are

- the number of vertices;
- the number of edges;
- the total degree;
- the number of vertices of a given degree;
- bipartiteness, number of connected components, connectedness;
- having a vertex of some degree n adjacent to a vertex of some degree m;
- the number of circuits of a given length;
- the existence of an Euler/Hamilton circuit

Isomorphic Graphs

Tips

- The easiest way to show that graphs G and G' are NOT isomorphic is to find an invariant property that holds for G but not for G'.
- To prove that simple graphs G and G' ARE isomorphic, we need to find an isomorphism between them; i.e. a bijection $f : V \to V'$ satisfying the condition for isomorphism.
- If G and G' have n vertices then there are $n!$ bijections from V to V'. If n is large then it is very hard to find an isomorphism among all $n!$ bijections.

Planar Graphs

Planar Graphs

- A graph G is planar iff it can be drawn in the plane so that no edge crosses another.
- Such a drawing is called a planar map or planar representation of G.
- A planar map divides the plane into a finite number of regions. Exactly one of these regions is unbounded.
- A planar graph can have different planar representations (or maps), but the number of regions is the same for all planar representations.

Planar Graphs

Euler's Formula

If G is a connected planar graph with e edges and v vertices, and if r is the number of regions in a planar representation of G, then

$$v - e + r = 2.$$

More information!

- The degree of a region R in a planar representation is the number of edges (counting repetitions) traversed in going around the boundary of R.
- In summing all region degrees, each edge contributes twice, so

$$2|E| = \text{ the sum of region degrees.}$$

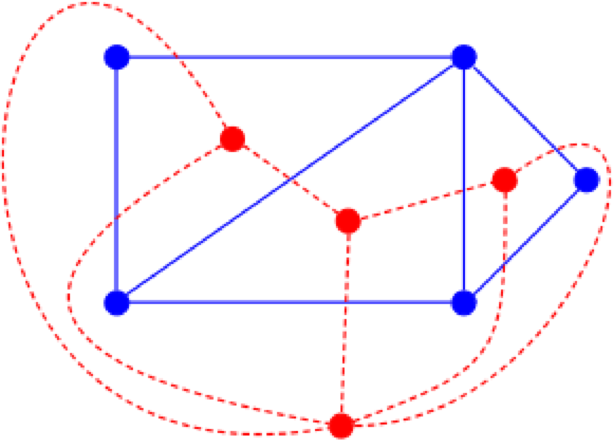
- By the Handshaking Lemma, it follows that the sum of region degrees equals the sum of vertex degrees.

Dual

Dual

- The dual of a planar map G is a graph G^* given as follows:
  - for each region R_i of G, there is an associated vertex v_i^* in G^*;
  - for each edge e in G that is surrounded by one region R_i, there is an associated loop in G^* at vertex v_i^*.
  - for each edge e of G that separates two regions R_1 and R_2, there is an edge e^* in G^* that connected vertices v_1^* and v_2^* corresponding to R_1 and R_2 respectively;
- The dual G^* is a planar graph.

Dual



(source:https://en.wikipedia.org/wiki/Dual$_g raph/media/File : Duals_g rap$

Planar Graphs

More information

If G is a simple connected planar graph with at least 3 vertices, then every region degree is at least 3.

- To have a region of degree 1, G must have a loop.
- To have a region of degree 2, G must have parallel edges.

Planar Graphs

Theorem

Theorem. If G is a connected planar simple graph with e edges and $v \geq 3$ vertices, then

- $e \leq 3v - 6$;
- $e \leq 2v - 4$ if G has no circuits of length 3.

This theorem is useful for proving that some graphs are NOT planar. It is an example of the principle that, the more edges a graph has, the harder it is for it to be planar.

Example

2014 Semester 1 Q2 (iv)

1. State Euler's Formula for a connected planar graph having v vertices, e edges and r regions.
2. Show that if G is a connected planar simple graph with $v \geq 3$ vertices, then
$$e \leq 3v - 6$$
.
3. Hence show that a connected planar simple graph with $v \geq 3$ vertices has at least one vertex of degree less than or equal to 6.

Elementary Subdivision, Homeomorphisms and Kuratowski's Theorem

Subdivision

- Suppose that G has an edge e with endpoints v and w. Let G' be the graph obtained from G by replacing e by a path $ve'v'e''w$.
- Such an operation is called an elementary subdivision.
- If G is planar then so is G'.
- Two graphs are homeomorphic iff each can be obtained from a common graph by elementary subdivisions.
- If G is planar, then so is any graph homeomorphic to G.

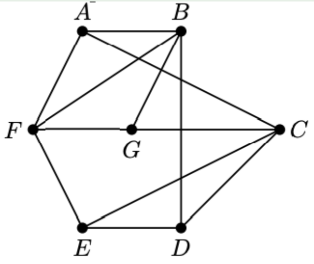
Kuratowski's Theorem

A graph is planar if and only if it does not contain a subgraph homeomorphic to $K_{3,3}$ or K_5.

Example

2012 Semester 2 Q2 (iv)

1. State Kuratowski's theorem characterising non-planar graphs.
2. Show that the following graph is not planar.



Trees

Trees

- A tree is a connected graph with no circuits.
- A tree has no loops or multiple edges so it is simple.
- Theorem. Any tree T is planar.
- A spanning tree in a graph G is a subgraph that is a tree and contains every vertex of G.
- Theorem. Every connected graph contains a spanning tree.
- Theorem. A connected graph with n vertices is a tree if and only if it has exactly $n-1$ edges.

Trees

More Tree Stuff

- A weighted graph is a graph whose edges have been given numbers called weights. The weight of an edge e is denoted by $w(e)$.
- The weight of a subgraph in a weighted graph G is the sum of the weights of the edges in the subgraph.
- A minimal spanning tree in a weighted graph G is a spanning tree whose weight is less than or equal to the weight of any other spanning tree.
- There can be more than one minimal spanning tree in a graph.

Kruskal's Algorithm

Kruskal's Algorithm

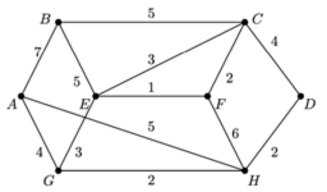
Minimal spanning trees if weighted graphs G on n vertices are found using Kruskal's Algorithm.

- Start with the tree $T := \emptyset$.
- Sort the edges of G into increasing order of weight.
- Going down the list, add an edge to T if and only if it does not form a circuit with edges already in T.
- Continue this process until T has $n - 1$ edges.
- Then T is a minimal spanning tree for G.

Example

2014 Semester 2 Question Q2 (iii)

1. Use Kruskal's algorithm to construct a minimal spanning tree T for the following weighted graph. Make a table showing the details of each step in your application of the algorithm.
2. Is T also a tree showing the shortest path from A to every other vertex? Explain.



Dijkstra's Algorithm

Dijksta's Algorithm

Given a connected weighted graph G and a particular vertex v_0, we want to find a shortest path from v_0 to v for each vertex v in G (here, a shortest path is one with minimal total weight). The union of these paths forms a minimal v_0-path spanning tree for G.

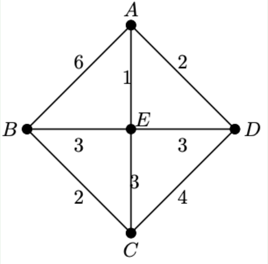
Dijkstra's Algorithm.

- Start with the subgraph T consisting of v_0 only.
- Consider all edges e with one vertex in T and the other vertex v not in T.
- Of these edges, choose an edge e giving a shortest path from v_0 to v.
- Add this edge e and vertex v to T.
- Continue this process until T contains all the vertices of G.
- Then T is a minimal v_0-path spanning tree for G.

Example

2014 Semester 1 Q2 (v)

Consider the following weighted graph:



Use Dijkstra's algorithm find a spanning tree that gives the shortest paths from A to every other vertex of the graph. Make a table showing the details of each step in your application of the algorithm.

WARNING

!!!

The minimal a-path spanning tree is typically NOT a minimal spanning tree.

Assorted Examples

2014 Semester 2 Q2 (ii)

A connected planar map has vertices of degrees 6, 5, 5, 2, 2, 1, 1.

1. How many regions does the map have?
2. Draw an example of such a planar map.
3. Does the graph have an Euler path? Give reasons.

Tips + Common Mistakes for Relations and Graph Theory

- Confuse the two graph algorithms
- Confuse necessary and sufficient conditions when solving Hamilton and other existence questions (e.g., state that the minimum degree of a graph on n vertices is less than n/2, so therefore the graph has no Hamilton circuit).
- Forget lines/elements, or wrongly add lines, in Hasse diagrams.

Tips + common mistakes

- Don't have to complete exam in order, move on if you get stuck on a question and come back to it
- Remember to write conclusions to proofs
- Remember logical explanations and flow signs

GOOD LUCK


AM I THE ONLY ONE AROUND HERE
WHO'S GOING TO HD DISCRETE
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