MATH2621 Revision Seminar Solutions

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Example 1

Set	Open/Closed	Bounded	Compact	Connected	Simply C.	Region	Domain
S_1	Open	Yes	No	Yes	Yes	Yes	Yes
S_2	Closed	Yes	Yes	Yes	Yes	Yes	No
S_3	Closed	Yes	Yes	Yes	Yes	No	No
S_4	Open	Yes	No	Yes	No	Yes	Yes
S_5	Open	No	No	Yes	No	Yes	Yes

Note: Simply C. = Simply Connected. Important observations:

1. To justify S_1 is open, you can generally just quote the fact that a disc that doesn't contain it's boundary is automatically open. The more formal proof entails the following:

   Let $\epsilon = 1 - |z|$, where $z = x + iy \in S_1$. Let $y \in B(z, \epsilon)$. Then $|y - z| < \epsilon = 1 - |z|$. Hence, $|y - z| + |z| < 1$. By triangle inequality, $|y - z| + |z| \geq |y - z + z| = |y|$ and thus $|y| < 1$. Since the choice of $y \in B(z, \epsilon)$ was arbitrary, and for every $z \in S_1$, $B(z, \epsilon) \subseteq S_1$, S_1 must therefore be open.

2. S_3 is connected. We can use the path $c = \{z \in S_3 : z = tp + (1-t)p, t \in [0, 1]\} = \{p\}$, and since such a set is always contained in S_3, S_3 must be connected. Additionally, the point p is an boundary point of S_3 because every epsilon ball about p contains p but also contains elements outside p, and so any other point that is not p is an exterior point, and S_3 has no interior. This is why S_3 also is NOT a region.

3. There are sets that are neither open nor closed, and sets that are both open and closed. An example of a set that is neither open nor closed is:

$$S = \{z \in \mathbb{C} : \mathrm{Re}(z) \in (0, \infty), \mathrm{Im}(z) \in [0, \infty)$$

   Since it contains some of it's boundary points (on the positive imaginary axis) but not all (for example, those on the positive real axis). The only sets in $\mathbb{C}$ that are open and closed are $\varnothing, \mathbb{C}$. $\varnothing$ is open because it doesn't contain any of its boundary points, and thus $\mathbb{C}$ is closed for that reason. And $\mathbb{C}$ is open because it can be written as a union of open balls in $\mathbb{C}$, and for this reason, $\varnothing$ is closed.

Example 2

1. Note that it suffices to work out what happens to the boundaries, the region will be that which is inside the boundary. Consider the boundary line $z = c + iy$ (where $c \in [0, 4]$), then the image of that will be $(1 + i) \cdot (c + iy) + 2 = i(y + c) + 2 - y + c$. Letting $u = 2 - y + c, v = y + c$, we obtain the new equation to be $u + v = 2c + 2$, and since we restrict $y \in [-0.5, 5]$, we obtain the image:

2. The line passing through $z = 1, z = 3 + 4i$ is given by $z = t(1) + (1 - t)(3 + 4i) = (1-2t)+(4-4t)i$. Thus, the image will be $(1+i)z+2 = (1+i)((1-2t)+i(4-4t))+2 = (1-2t-(4-4t))+i(1-2t+4-4t) = (-3+2t)+i(5-6t)$. Letting $u = -3+2t, v = 5-6t$, we get $3u + v = -4$ as the image, with no restriction because t is unrestricted.

Example 3

By Extended Triangle inequality, we have $|z^4 - 1| \geq ||z|^4 - 1| = |R^4 - 1| = R^4 - 1 \geq 15$ since $R \geq 2$ and thus $R^4 - 1 > 0$. Since both sides of the inequality are greater than 0, we may reciprocate both sides to yield:

$$\left|\frac{1}{z^4-1}\right| \leq \frac{1}{15}$$

as required.

Example 4

By definition of a limit, we seek a δ such that for every $\epsilon > 0$, we have $0 < |z-(1+i)| < \delta \implies |z^2-2i| < \epsilon$.

$$\begin{aligned} |z^2-2i| &= |z-(1+i)||z+(1+i)| \\ &= |z-(1+i)||z-(1+i)+(2+2i)| \\ &\leq \delta(\delta+2\sqrt{2}) && \text{(By Triangle Inequality)} \\ &< \epsilon \end{aligned}$$

Where we select δ such that $\delta <$ the positive solution of $\delta^2+2\sqrt{2}\delta-\epsilon = 0 \implies \delta = \frac{-2\sqrt{2}+\sqrt{8+4\epsilon}}{2}$.

Example 5

Consider the path $z = iy$, then the limit becomes:

$$\lim_{y\to 0}\frac{0}{0+iy} = 0$$

Consider the path $z = x+0i$, then the limit becomes:

$$\lim_{x\to 0}\frac{x}{x+0i} = 1$$

Since the limits along the 2 different paths are different, the limit expression does not exist.

Example 6

1. $f_1(z) = (x+iy)(x^2+y^2) = (x^3+xy^2)+i(yx^2+y^3)$. Then by the Cauchy-Riemann Equations:

$$\frac{\partial u}{\partial x} = 3x^2+y^2, \quad \frac{\partial u}{\partial y} = 2xy$$

$$\frac{\partial v}{\partial x} = 2xy, \quad \frac{\partial v}{\partial y} = 3y^2 + x^2$$

We have $2xy = -2xy \implies xy = 0$. We also have $3x^2 + y^2 = -3y^2 - x^2 \implies 4x^2 = -4y^2 \implies x^2 = -y^2 \implies x = y = 0$. So the function is differentiable only at $x = y = 0$.

Note that this function is not holomorphic at $z = 0$.

2. $f_2(z) = x^2 + iy^2 \implies 2x = 2y, 0 = -0 \implies x = y$. Hence the function is differentiable $z = x + ix$.

3. Similar to the previous part, we require when $\frac{x}{|x|} = \frac{|y|}{y} \implies xy = |xy|$ (upon noting that the derivative of $|x| = \frac{x}{|x|}$). The above is only true when $xy > 0$. This also means that the function is holomorphic in this region, since it's differentiable on the open set.

Example 7

$u(x, y) = \cos x \cosh y \implies \partial_x^2 u = -\cos x \cosh y, \partial_y^2 u = \cos x \cosh y \implies \partial_x^2 u + \partial_y^2 u = 0$. Hence u is harmonic. The harmonic conjugate is given by solving the CRE's. Let v be the harmonic conjugate so that $\partial_x v = -\cos x \sinh y \implies v(x, y) = -\sin x \sinh y + f(y)$. $\partial_y v = -\sin x \cosh y \implies f'(y) = 0 \implies f(y) = C$. Hence the harmonic conjugate is $v(x, y) = -\sin x \sinh y + C$.

Example 8

Using the same idea as above, you should obtain $v(x, y) = \frac{-y}{x^2+y^2}$.

Example 9

1. $e^z = 2i \implies x = \log 2i = \ln|2i| + i\arg(2i) = \ln 2 + i\left(\frac{\pi}{2} + 2k\pi\right)$.

2. $\cos z = 3 \implies \frac{e^{iz}+e^{-iz}}{2} = 3$. Thus upon rearrangement, we seek to solve:

$$e^{2iz} - 6e^{iz} + 1 = 0 \implies e^{iz} = \frac{6 \pm \sqrt{36-4}}{2} = 3 \pm 2\sqrt{2}$$

Thus $iz = \log(3 \pm 2\sqrt{2}) = \ln|3 \pm 2\sqrt{2}| + i(\arg(3 \pm 2\sqrt{2}) \implies z = -i\ln 3 \pm 2\sqrt{2} + \arg(3 \pm 2\sqrt{2})$.

3. $\cosh z = -4 \implies e^z + e^{-z} = -8 \implies e^{2z} + 8e^z + 1 = 0$. Hence by Quadratic formula, we obtain:

$$e^z = \frac{-8 \pm \sqrt{64-4}}{2} = -4 \pm \sqrt{15} \implies z = \ln|-4 \pm \sqrt{15}| + i\text{arg}(-4 \pm \sqrt{15})$$

Example 10

We have $\sin z = i\cos z \implies \sin^2 z = -\cos^2 z \implies \sin^2 z + \cos^2 z = 0$ but this is not valid since we know $\sin^2 z + \cos^2 z = 1$.

Example 11

1. Using the principal argument of $\frac{1+\sqrt{3}i}{2} = e^{\frac{\pi}{3}i}$, we can simplify the inside expression to $e^{i\pi}$.

$$\text{pv}(e^{i\pi})^{1-i} = \exp((1-i)\text{Log}(e^{i\pi}) = \exp((1-i)(0+i\pi)) = \exp(i\pi + \pi) = -e^{\pi}$$

2. $i^i = \exp(i\log i) = \exp(i(0 + i(\frac{\pi}{2} + 2k\pi))) = \exp(-\frac{pi}{2} - 2k\pi)$, where $k \in \mathbb{Z}$.

3. $$\lim_{z\to 0} \exp\left(\frac{1}{z^2}\text{Log}(\cos z)\right) = \exp\left(\lim_{z\to 0}\frac{\text{Log}(\cos z)}{z^2}\right) = e^{\frac{-1}{2}}$$

   Upon using L'Hopital's rule twice.

Example 12

1. Note that Log is analytic everywhere except along the negative real axis. So the function $\text{Log}(iz)$ is analytic everywhere except along the positive imaginary axis. (You can get this geometrically).

2. Since the limit as $z \to 0$ doesn't exist, we can discount that. $\text{Log}(z+1)$ is not analytic for $z + 1 = x \in \mathbb{R}^-$ so we have $z \in (-\infty, -1]$. Hence g is analytic everywhere except $(-\infty, -1] \cup \{0\} \subseteq \mathbb{R}$.

Example 13

1. Note that: $f(z) = \exp(\frac{1}{2}\text{Log}(z+1))$, so the function is analytic everywhere except $z \in (-\infty, -1]$.

2. $f(z) = \exp(\frac{1}{2}\text{Log}(z^2 - 1))$ is analytic except on $z^2 - 1 = x$ for $x \leq 0$, and hence $z = \pm i\sqrt{-x+1}, x \leq 0$.

Example 14

Solving for z, we obtain:

$$z = \frac{2w}{1-w}$$

Substituting into the region that we had originally obtained, we simplify the requirement down to:

$$|3w - 1| \leq |w - 1|$$

Which is a circle. One may brute force the rest by substituting $w = x + iy$ and squaring both sides of the inequality to obtain an equation.

Example 15

Let the complex number on the line be given by $z = (2-2y) + iy$. Then 3 points on the line are $z = 2, i, 4 - i$, which maps to the points $w = \frac{2-i}{5}, -\frac{i}{2}, \frac{1}{4}$. By constructing perpendicular bisectors to 2 of the lines and finding their intersection point, we arrive at the centre $\frac{1}{8} - \frac{1}{4}i$, and radius of $\frac{\sqrt{5}}{8}$.