UNSW Mathematics Society Presents
MATH2011/2111 Seminar

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Overview I

1. Curves and Surfaces
2. Point Set Topology
3. Differentiable Functions
4. Integration
5. Fourier Series
6. Vector Fields
7. Line Integrals
8. Surface Integrals

1. Curves and Surfaces

Curves

Definition

The parameterisation of a curve in $\mathbb{R}^n$ is a vector valued function

$$\mathbf{c} : I \to \mathbb{R}^n$$

where I is an interval on $\mathbb{R}$.

- A multiple point is a point through which the curve passes more than once.
- A curve is closed if $\mathbf{c}(a) = \mathbf{c}(b)$.

Limits and Calculus for Curves

Definition

Let $\mathbf{c} : I \to \mathbb{R}^n$ be a curve with components $c_i, i = 1, 2, \ldots, n$ and $a \in I$.

- If $\lim_{t \to a} c_i(t)$ exists for all i, then $\lim_{t \to a} \mathbf{c}(t)$ exists and

$$\lim_{t \to a} \mathbf{c}(t) = \left(\lim_{t \to a} c_1(t), \ldots, \lim_{t \to a} c_n(t) \right)$$

- If $c_i'(t)$ exists for all i, then

$$\mathbf{c}'(t) = (c_1'(t), \ldots, c_n'(t))$$

$\mathbf{c}'(t)$ can be interpreted as the tangent vector or the velocity at t and $\mathbf{c}''(t)$ is the acceleration.

Surfaces in $\mathbb{R}^3$

- **Graph**: $z = f(x, y)$.
- **Implicitly**: $F(x, y, z) = 0$.
- **Parametrically**: $\mathbf{x}(u, v)$

Past exam question

2019 Final Q1(i)

Suppose S is the surface given by $x^2 + 3y^2 - z^2 = 1$. Sketch the intersection of S with the xz and xy planes. Label your axes carefully. Identify intercepts and asymptotes if any. Identify known curves by name.

The intersection of S with the xz plane will occur when $y = 0$, and is a hyperbola

$$x^2 - z^2 = 1$$

The intersection of S with the xy plane occurs when $z = 0$, and is an ellipse

$$x^2 + 3y^2 = 1$$

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Past exam question

2. Point Set Topology

Ball

Definition

A ball around $\mathbf{a} \in \mathbb{R}^n$ of radius $\epsilon > 0$ is the set

$$B(\mathbf{a}, \epsilon) = \{\mathbf{x} \in \mathbb{R}^n : ||\mathbf{x} - \mathbf{a}|| < \epsilon\}$$

The punctured ball around $\mathbf{a}$ is the set

$$B^o(\mathbf{a}, \epsilon) = \{\mathbf{x} \in \mathbb{R}^n : 0 < ||\mathbf{x} - \mathbf{a}|| < \epsilon\}$$

Open and closed sets

Definition

Let $\Omega \subseteq \mathbb{R}^n$.

- $\mathbf{x} \in \Omega$ is an interior point if there exists an $\varepsilon > 0$ such that $B(\mathbf{x}, \varepsilon) \subseteq \Omega$.
- Ω is open if every point is an interior point.
- Ω is closed if its complement is open.
- $\mathbf{x} \in \Omega$ is a boundary point if for every $\varepsilon > 0$, there exists $\mathbf{x}_1, \mathbf{x}_2 \in B(\mathbf{x}, \varepsilon)$ such that $\mathbf{x}_1 \in \Omega$ and $\mathbf{x}_2 \notin \Omega$.

A set is closed if and only if it contains all its boundary points.

Theorem

A finite union/intersection of open sets is open.
A finite union/intersection of closed sets is closed.

Example 1

2019 Q2(i) adapted

Use the definition to show that the set $[-3, 1) \subseteq \mathbb{R}$ is not open.

Note that if a set **IS** open, then:

$$\text{For any } x_0 \in \Omega,\ \exists \epsilon > 0, \text{ s.t. } B(x_0, \epsilon) \subseteq \Omega.$$

In our question, we wish to prove that the set **IS NOT** open.
Thus, we are aiming to prove that:

$$\text{There exists an } x_0 \in \Omega, \text{ s.t. } \forall \epsilon > 0,\ B(x_0, \epsilon) \not\subseteq \Omega.$$

In simple terms, this means we can pick a particular value in the set, such that no matter how small the ball around this point is, we still cannot fully fit it into the set.

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In simple terms, this means we can pick a particular value in the set, such that no matter how small the ball around this point is, we still cannot fully fit it into the set.

Example 1 Solutions

Let's choose x_0 = -3.
Then, $\forall\epsilon > 0$, consider,

$$y_0 = x_0 - \frac{\epsilon}{2} \subseteq B(x_0, \epsilon).$$

Note that $y_0 \notin [-3, 1)$.

Thus, the existence of the point y_0 implies that we cannot fully fit any ball around -3 into [-3, 1). Therefore, the set, [-3, 1) is not open.

Limit of sequences

Definition

For a sequence $\{\mathbf{x}_i\}_{i=1}^{\infty}$ of points in $\mathbb{R}^n$, $\mathbf{x}$ is the limit of the sequence if and only if

$$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \text{ such that } \forall k \geq N : \mathbf{x}_k \in B(\mathbf{x}, \varepsilon)$$

- A sequence $\{\mathbf{x}_k\}$ converges if and only if the components of $\mathbf{x}_k$ converge to the components of $\mathbf{x}$.
- $\mathbf{x}$ is called a limit point and loosely speaking, it is a point such that it can be described by other points in the set arbitrarily close to it. Think of 0 on the real number line.
- To show a point is not a limit point, show

$$\exists \varepsilon > 0 \text{ such that } \forall N \in \mathbb{N} \; \exists k \geq N \text{ such that } \mathbf{x_k} \notin B(\mathbf{x}, \varepsilon)$$

Interior and boundary

Definition

Suppose $\Omega \subseteq \mathbb{R}^n$.

- The interior of Ω is the set of all interior points of Ω.
- The boundary of Ω is the set of all boundary points of Ω, denoted $\partial\Omega$.
- The closure of Ω is $\overline{\Omega} = \Omega \cup \partial\Omega$.

Lemma

Let $\mathbf{x} \in \mathbb{R}^n$ and $\Omega \subseteq \mathbb{R}^n$.

- $\mathbf{x}$ is an interior point of $\Omega \implies \mathbf{x}$ is a limit point of Ω.
- $\mathbf{x}$ is a limit point of $\Omega \implies \mathbf{x}$ is a boundary point or an interior point of Ω.
- $\mathbf{x}$ is in $\overline{\Omega} \implies$ there is a sequence in Ω with limit $\mathbf{x}$.

Limit of a function at a point

Definition

Let $\mathbf{b} \in \mathbb{R}^m, \Omega \subseteq \mathbb{R}^n, \mathbf{a} \in \overline{\Omega}$ and let $\mathbf{f} : \Omega \to \mathbb{R}^m$ be a function. We say that $\mathbf{f}(\mathbf{x})$ converges to $\mathbf{b}$ as $\mathbf{x} \to \mathbf{a}$ if

$$\forall \varepsilon > 0 \ \exists \delta > 0 \text{ s.t. } \forall \mathbf{x} \in \Omega : x \in B^o(\mathbf{a}, \delta) \cap \Omega \implies \mathbf{f}(\mathbf{x}) \in B(\mathbf{b}, \varepsilon).$$

If such $\mathbf{b}$ exists, then $\mathbf{b}$ is unique and we write

$$\lim_{\mathbf{x} \to \mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{b}.$$

Useful Limit Theorems

Useful Theorems and Algebra of Limits

Let $\mathbf{b} \in \mathbb{R}^m, \Omega \subseteq \mathbb{R}^n, \mathbf{a} \in \overline{\Omega}$ and let $\mathbf{f} : \Omega \to \mathbb{R}^m$ be a function. Then

$$\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{b} \iff \lim_{\mathbf{x}\to\mathbf{a}} f_i(\mathbf{x}) = b_i \text{ for all } i = 1, \ldots, m$$
$$\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{b} \iff \lim_{k\to\infty} \mathbf{f}(\mathbf{x}_k) = \mathbf{b}$$

for every sequence $\{\mathbf{x}_k\}_{k=1}^{\infty} \subseteq \Omega$ with $\lim_{k\to\infty} \mathbf{x}_k = \mathbf{a}$.

Given that, $\lim_{x\to x_0} f(x) = a$ and $\lim_{x\to x_0} g(x) = b$, then,

$$\lim_{x\to x_0} (f+g)(x) = a + b$$
$$\lim_{x\to x_0} (fg)(x) = ab$$
$$\lim_{x\to x_0} (f/g)(x) = a/b, \text{ given } b \neq 0.$$

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$$\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{b} \iff \lim_{\mathbf{x}\to\mathbf{a}} f_i(\mathbf{x}) = b_i \text{ for all } i = 1, \ldots, m$$
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Useful Limit Theorems

Pinching Theorem

Let $\Omega \subseteq \mathbb{R}^n$, let $\mathbf{a}$ be a limit point of Ω and let $f, g, h : \Omega \to \mathbb{R}$ be functions such that there exists $\delta > 0$ such that $\forall \mathbf{x} \in B(\mathbf{a}, \delta) \cap \Omega$,

$$g(\mathbf{x}) \leq f(\mathbf{x}) \leq h(\mathbf{x}).$$

Then

$$\lim_{\mathbf{x} \to \mathbf{a}} g(\mathbf{x}) = \mathbf{b} = \lim_{\mathbf{x} \to \mathbf{a}} h(\mathbf{x}) \implies \lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = \mathbf{b}$$

Example 2

2019 Q2(ii)(c)

Show that the following limit does not exist

$$\lim_{(x,y)\to(0,0)} \frac{x+y}{|x|+|y|}$$

Example 2 Solutions

To prove that the limit does not exist, we will make use of the second useful theorem shown above.

Our strategy would be to try find two particular sequences (x_k, y_k) such that they both tend towards (0,0), but where their limits, $\lim_{(x_k,y_k)\to(0,0)} \frac{x_k+y_k}{|x_k|+|y_k|}$ are not equal.

Let's first choose the sequence $x_k = \frac{1}{k}$ and $y_k = 0$ for example. Clearly, $(\frac{1}{k}, 0) \to (0,0)$ as $k \to \infty$.
Now let's compute the limit if this sequence is taken.

$$\begin{aligned}\lim_{(x_k,y_k)\to(0,0)} \frac{x_k+y_k}{|x_k|+|y_k|} &= \lim_{k\to\infty} \frac{1/k+0}{|1/k|+|0|} \\ &= 1.\end{aligned}$$

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Example 2 Solutions

We now seek a new sequence that still tends to (0,0) but then returns a limit other than 1.

This time, let's select $x_k = \frac{-1}{k}$ and $y_k = 0$. Again, clearly $(\frac{-1}{k}, 0) \to (0,0)$ as $k \to \infty$.

However, this time the limit will be,

$$\lim_{(x_k, y_k) \to (0,0)} \frac{x_k + y_k}{|x_k| + |y_k|} = \lim_{k \to \infty} \frac{-1/k + 0}{|-1/k| + |0|} = -1.$$

As the limits between the two chosen sequences are not equal, we can conclude that the limit does not exist at (0,0).

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$$= -1.$$

As the limits between the two chosen sequences are not equal, we can conclude that the limit does not exist at (0,0).

Example 3

2016 Q1(iv)(c)

Given that

$$\left|\frac{xy}{x^2 - xy + y^2}\right| \leq 1,\ \forall (x, y) \neq (0, 0)$$

show that

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^2 - xy + y^2} = 0$$

Example 3 Solutions

Note that in this question, we would want to use the Pinching Theorem. The intuition would be that, we are given information regarding the absolute value of a significant portion of our limit. If we could just show that the absolute value of our limit tends towards 0, then we would be done.

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2-xy+y^2} = \lim_{(x,y)\to(0,0)} x \cdot \frac{xy}{x^2-xy+y^2}.$$

If we then take its absolute value, we obtain,

$$\begin{aligned}\lim_{(x,y)\to(0,0)} |x| \cdot |\frac{xy}{x^2-xy+y^2}| &\leq \lim_{(x,y)\to(0,0)} |x| \\ &= 0.\end{aligned}$$

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Continuity

Definition

Let $\mathbf{a} \in \Omega \subseteq \mathbb{R}^n$. A function $\mathbf{f} : \Omega \to \mathbb{R}^m$ is continuous at $\mathbf{a}$ if

$$\lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = f(\mathbf{a})$$

$\mathbf{f}$ is said to be continuous on Ω if it is continuous at $\mathbf{a}$ for every $\mathbf{a} \in \Omega$.

Continuity is like an extension to limits. It firstly requires that the limit exists and that the limit equals to the actual value at that point.

Epsilon-Delta Interpretation

$$\forall \epsilon > 0, \exists \delta > 0, \text{ s.t. } \mathbf{x} \in B(\mathbf{a}, \delta) \cap \Omega \implies f(\mathbf{x}) \in B(f(\mathbf{a}), \epsilon).$$

Preimage

Definition

Suppose that $\Omega \subseteq \mathbb{R}^n$ and $\mathbf{f} : \Omega \to \mathbb{R}^m$ is a function. The preimage of a set $U \subseteq \mathbb{R}^m$ is defined by

$$\mathbf{f}^{-1}(U) = \{\mathbf{y} \in \mathbb{R}^n : f(\mathbf{y}) \in U\}.$$

Theorem

$\mathbf{f}$ is continuous on Ω if and only if

$$U \text{ is open in } \mathbb{R}^m \implies \mathbf{f}^{-1}(U) \text{ is open in } \mathbb{R}^n.$$

Theorem

Suppose Ω is open. If $\mathbf{f}(\mathbf{a})$ is an interior point of $\mathbf{f}(\Omega)$ and $B(\mathbf{f}(\mathbf{a}), \varepsilon) \subseteq \mathbf{f}(\Omega)$ then $\mathbf{a}$ is an interior point of $\mathbf{f}^{-1}(B(\mathbf{f}(\mathbf{a}), \varepsilon))$.

Example 4

Non-exam Question

If S is open and if f continuous, what is known about the preimage $f^{-1}(S)$? Prove it.

This is essentially asking us to prove the theorem seen above. This will require some more in-depth application of theory involving balls that we learnt.

Example 4 Solutions

Firstly, take an arbitrary $\mathbf{x_0} \in f^{-1}(S)$. Then, we have that $f(\mathbf{x_0}) \in S$.

Now, note that we have two pieces of information to work off. We have,

1. S is open. This implies that we have some ball, $B(f(\mathbf{x_0}), r) \subseteq S$.
2. f is continuous.

By the Epsilon-Delta interpretation of continuity we have that,

$$\forall \epsilon > 0, \exists \delta > 0, \text{ s.t. } \mathbf{x} \in B(\mathbf{x_0}, \delta) \cap S \implies f(\mathbf{x}) \in B(f(\mathbf{x_0}), \epsilon).$$

Since this applies to any ϵ arbitrarily small, let's just select $\epsilon < r$.

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Example 4 Solutions

Combining these two results together, we have that since $\epsilon < r$,

$$f(\mathbf{x}) \in B(f(\mathbf{x_0}), \epsilon) \subseteq B(f(\mathbf{x_0}), r) \subseteq S.$$

And so, we have that $\mathbf{x} \in f^{-1}(S)$.

Returning to the Epsilon-Delta interpretation, we then have,

$$\mathbf{x} \in B(\mathbf{x_0}, \delta) \implies \mathbf{x} \in f^{-1}(S).$$

This implies that,

$$B(\mathbf{x_0}, \delta) \subseteq f^{-1}(S).$$

And thus, $f^{-1}(S)$ is also open, as we can always find a delta-ball around any arbitrary $\mathbf{x_0}$ such that its fully contained in $f^{-1}(S)$.

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Compact sets

Definition

A set $\Omega \subseteq \mathbb{R}^n$ is bounded if there exists an $M \in \mathbb{R}$ such that $d(\mathbf{x}, \mathbf{0}) \leq M$ for all $\mathbf{x} \in \Omega$.

Definition

A set $\Omega \subseteq \mathbb{R}^n$ is compact if it is closed and bounded.

Theorem

Let $\Omega \subseteq \mathbb{R}^n$ and let $f : \Omega \to \mathbb{R}$ be continuous. Then

$$K \subseteq \Omega \text{ and } K \text{ compact} \implies f(K) \text{ compact}.$$

Path connected sets

Definition

A set $\Omega \subseteq \mathbb{R}^n$ is said to be path connected if for any $\mathbf{x}, \mathbf{y} \in \Omega$, there is a continuous function φ such that $\varphi(t) \in \Omega$ for all $t \in [0,1]$ and $\varphi(0) = \mathbf{x}$ and $\varphi(1) = \mathbf{y}$.

Theorem

Let $\Omega \subseteq \mathbb{R}^n$ and $\mathbf{f} : \Omega \to \mathbb{R}^m$ be continuous. Then

$$B \subseteq \Omega \text{ and } B \text{ path connected} \implies \mathbf{f}(B) \text{ path connected}$$

Suppose $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Then the function $\varphi : [0,1] \to \mathbb{R}^n$

$$\varphi(t) = (1-t)\mathbf{x} + t\mathbf{y}$$

is the line segment from $\mathbf{x}$ to $\mathbf{y}$.

Example 5

2019 Q2(iii)(iv) merged

a) Use the definition to prove that the unit square $S = \{(r, \theta) : 0 \le r \le 1,\ 0 \le \theta \le 1\}$ is bounded.

b) Use the definition to prove that the unit square S is path-connected.

c) Find the image of $\mathbf{f}(S)$, where $\mathbf{f}(r, \theta) = (r\cos(\frac{\pi\theta}{2}), r\sin(\frac{\pi\theta}{2}))$.

d) Is $\mathbf{f}(S)$ path-connected? Explain.

Example 5 Solutions

a)
To prove by definition, we simply need to choose a $M \in \mathbb{R}$, such that for any $\mathbf{s} \in S$, $d(\mathbf{s}, \mathbf{0}) \leq M$.

We can just choose $M = 2$, as the $\max(d(\mathbf{s}, \mathbf{0})) = \sqrt{1^2 + 1^2} = \sqrt{2}$.

b)
To prove by definition, we need to actually write down a plausible $\varphi(t)$ function.

Consider that you have any two arbitrary points (x_1, y_1) and (x_2, y_2) in S. Then we can define,

$$\varphi(t) = \begin{cases} (1-2t)x_1 + 2tx_2 & 0 \leq t < \frac{1}{2} \\ (1-2t)y_1 + 2ty_2 & \frac{1}{2} \leq t \leq 1 \end{cases}.$$

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Example 5 Solutions

c)
We have $\mathbf{f}(r,\theta) = (r\cos(\frac{\pi\theta}{2}), r\sin(\frac{\pi\theta}{2}))$. Intuitively, most of you should recognise this to be similar to polar coordinates (i.e. becomes a circle).

Let $x = r\cos(\frac{\pi\theta}{2})$ and $y = r\sin(\frac{\pi\theta}{2})$.
Then, $x^2 + y^2 = r^2$.

Furthermore, note that as $0 \leq r \leq 1$ and $0 \leq \theta \leq 1$, we have that the radius ranges anywhere from a magnitude of 0 to 1 and the angle ranges anywhere from 0 to $\frac{\pi}{2}$.
Thus, our image is actually the first quadrant of the circle.

d)
Refer back to our theorem on path-connected sets.
We have that $S \subseteq \mathbb{R}^2$ and S is path-connected. Thus $\mathbf{f}(S)$ is also path-connected.

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d)
Refer back to our theorem on path-connected sets.
We have that $S \subseteq \mathbb{R}^2$ and S is path-connected. Thus $\mathbf{f}(S)$ is also path-connected.

Example 5 Solutions

c)
We have $\mathbf{f}(r, \theta) = (r\cos(\frac{\pi\theta}{2}), r\sin(\frac{\pi\theta}{2}))$. Intuitively, most of you should recognise this to be similar to polar coordinates (i.e. becomes a circle).

Let $x = r\cos(\frac{\pi\theta}{2})$ and $y = r\sin(\frac{\pi\theta}{2})$.
Then, $x^2 + y^2 = r^2$.

Furthermore, note that as $0 \leq r \leq 1$ and $0 \leq \theta \leq 1$, we have that the radius ranges anywhere from a magnitude of 0 to 1 and the angle ranges anywhere from 0 to $\frac{\pi}{2}$.
Thus, our image is actually the first quadrant of the circle.

d)
Refer back to our theorem on path-connected sets.
We have that $S \subseteq \mathbb{R}^2$ and S is path-connected. Thus $\mathbf{f}(S)$ is also path-connected.

3. Differentiable Functions

Partial derivatives

Definition

Let $\mathbf{a} \in \Omega \subseteq \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$ be a function with coordinates x_i and standard basis vectors $\mathbf{e}_i, i \in \{1, \ldots, n\}$. The partial derivative of f in direction i is defined as

$$\frac{\partial f}{\partial x_i} = \lim_{h \to 0} \frac{f(\mathbf{a} + h\mathbf{e}_i) - f(\mathbf{a})}{h}$$

assuming the limit exists.

Practically, this can be computed by taking derivative with respect to a coordinate assuming the others are constant.

Jacobian matrix

Definition

If all partial derivatives of $\mathbf{f} : \Omega \to \mathbb{R}^m$ exist at $\mathbf{a} \in \Omega \subseteq \mathbb{R}^n$, then the Jacobian matrix of $\mathbf{f}$ at $\mathbf{a}$ is

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{a}) & \frac{\partial f_1}{\partial x_2}(\mathbf{a}) & \dots & \frac{\partial f_1}{\partial x_n}(\mathbf{a}) \\ \frac{\partial f_2}{\partial x_1}(\mathbf{a}) & \frac{\partial f_2}{\partial x_2}(\mathbf{a}) & \dots & \frac{\partial f_2}{\partial x_n}(\mathbf{a}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\mathbf{a}) & \frac{\partial f_m}{\partial x_2}(\mathbf{a}) & \dots & \frac{\partial f_m}{\partial x_n}(\mathbf{a}) \end{pmatrix}$$

Theorem

Let $\Omega \subseteq \mathbb{R}^n, \mathbf{a} \in \Omega$ be an interior point and $\mathbf{f} : \Omega \to \mathbb{R}^m$ be a function. If $\mathbf{f}$ is differentiable at $\mathbf{a}$ then all partial derivatives $\frac{\partial f_j}{\partial x_i}$ exist at $\mathbf{a}$ and

$$D\mathbf{f}(\mathbf{a}) = J\mathbf{f}(\mathbf{a})$$

Affine maps

Definition

A function $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^m$ is called affine if there exists a $\mathbf{y}_0 \in \mathbb{R}^m$ and a linear map $\mathbf{L} : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$T(\mathbf{x}) = \mathbf{y_0} + \mathbf{L}(\mathbf{x})$$

Definition

A function $\mathbf{f} : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ has an affine approximation at a point $\mathbf{a} \in \Omega$ if there is a matrix $A \in \mathbb{R}^{m \times n}$ and a vector $\mathbf{b} \in \mathbb{R}^m$ such that

$$\lim_{\mathbf{x} \to \mathbf{a}} \frac{\|\mathbf{f}(\mathbf{x}) - A\mathbf{x} - \mathbf{b}\|}{\|\mathbf{x} - \mathbf{a}\|} = 0$$

The affine approximation is

$$\mathbf{g}(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$$

Differentiability

Definition

A function $\mathbf{f}$ is said to be differentiable at $\mathbf{a}$ if $\mathbf{f}$ has an affine approximation at $\mathbf{a}$.

Theorem

A function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\mathbf{a} \in \mathbb{R}^n$ if and only if

$$\lim_{\mathbf{x}\to\mathbf{a}} \frac{\|\mathbf{f}(\mathbf{x}) - J\mathbf{f}(\mathbf{a})(\mathbf{x}-\mathbf{a}) - \mathbf{f}(\mathbf{a})\|}{\|\mathbf{x}-\mathbf{a}\|} = 0.$$

Past exam question

2012 Final Q1(b)

Find the best affine approximation to the function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$

$$\mathbf{f}(x, y) = (x^2 + y^2, \sinh(x^2 - y^2))$$

at the point $(1, 1)$.

The best affine approximation at $\mathbf{a} = (1, 1)$ is given by

$$\mathbf{g}(\mathbf{x}) = \mathbf{f}(\mathbf{a}) + J\mathbf{f}(\mathbf{a})(\mathbf{x} - \mathbf{a}).$$

Past exam question

2012 Final Q1(b)

Find the best affine approximation to the function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$

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$$\mathbf{g}(\mathbf{x}) = \mathbf{f}(\mathbf{a}) + J\mathbf{f}(\mathbf{a})(\mathbf{x} - \mathbf{a}).$$

2012 Final Q1(b) Solution

$$\mathbf{f}(x,y) = (x^2 + y^2, \sinh(x^2 - y^2))$$

The Jacobian matrix is given by

$$\begin{pmatrix} 2x & 2y \\ 2x\cosh(x^2 - y^2) & -2y\cosh(x^2 - y^2) \end{pmatrix}$$

$$\begin{aligned}
\mathbf{g}(\mathbf{x}) &= (2,0) + \begin{pmatrix} 2 & 2 \\ 2 & -2 \end{pmatrix} (\mathbf{x} - (1,1)) \\
&= (2,0) + (2(x-1) + 2(y-1), 2(x-1) - 2(y-1)) \\
&= (2x + 2y - 2, 2x - 2y)
\end{aligned}$$

Chain rule

Theorem

Let $\Omega \subseteq \mathbb{R}^n, \Omega' \subseteq \mathbb{R}^m$ and let $\mathbf{a} \in \Omega$. Suppose $\mathbf{f} : \Omega \to \mathbb{R}^m$ and $\mathbf{g} : \Omega' \to \mathbb{R}^k$ are functions such that $\mathbf{f}(\Omega) \subseteq \Omega'$. If $\mathbf{f}$ is differentiable at $\mathbf{a}$ and $\mathbf{g}$ is differentiable at $\mathbf{f}(\mathbf{a})$, then $\mathbf{g} \circ \mathbf{f} : \Omega \to \mathbb{R}^k$ is differentiable at $\mathbf{a}$ and

$$D(\mathbf{g} \circ \mathbf{f})(\mathbf{a}) = D\mathbf{g}(\mathbf{f}(\mathbf{a}))D\mathbf{f}(\mathbf{a}).$$

Chain rule (MATH2011)

Theorem

Suppose that

a) f is a differentiable function of two variables x and y,

b) x and y are differentiable functions of two variables s and t.

Then if

$$F(s,t) = f(x(s,t), y(s,t)),$$

we have

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

and

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

Past exam question

2019 Final Q1(ii)

Use the chain rule to find $\frac{\partial z}{\partial r}$, $\frac{\partial z}{\partial s}$ and $\frac{\partial^2 z}{\partial s \partial r}$ in terms of $\frac{\partial f}{\partial u}$, $\frac{\partial f}{\partial v}$, $\frac{\partial^2 f}{\partial u^2}$, $\frac{\partial^2 f}{\partial v^2}$ and $\frac{\partial^2 f}{\partial u \partial v}$, if $z = f(u, v)$ and if $u = 2r - s$ and $v = r + s^2$.

$$\frac{\partial z}{\partial r} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial r} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial r} = 2\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}$$

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial s} = -\frac{\partial f}{\partial u} + 2s\frac{\partial f}{\partial v}$$

Past exam question

2019 Final Q1(ii)

Use the chain rule to find $\frac{\partial z}{\partial r}$, $\frac{\partial z}{\partial s}$ and $\frac{\partial^2 z}{\partial s \partial r}$ in terms of $\frac{\partial f}{\partial u}$, $\frac{\partial f}{\partial v}$, $\frac{\partial^2 f}{\partial u^2}$, $\frac{\partial^2 f}{\partial v^2}$ and $\frac{\partial^2 f}{\partial u \partial v}$, if $z = f(u, v)$ and if $u = 2r - s$ and $v = r + s^2$.

$$\frac{\partial z}{\partial r} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial r} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial r} = 2\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}$$

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial s} = -\frac{\partial f}{\partial u} + 2s\frac{\partial f}{\partial v}$$

2019 Final Q1(ii) Solution

$$
\begin{aligned}
\frac{\partial^2 z}{\partial s \partial r} &= \frac{\partial}{\partial s}\frac{\partial z}{\partial r} \\
&= \frac{\partial}{\partial s}\left(2\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}\right) \\
&= 2\frac{\partial}{\partial s}\frac{\partial f}{\partial u} + \frac{\partial}{\partial s}\frac{\partial f}{\partial v} \\
&= \left(2\frac{\partial^2 f}{\partial u^2}\frac{\partial u}{\partial s} + 2\frac{\partial^2 f}{\partial v \partial u}\frac{\partial v}{\partial s}\right) + \left(\frac{\partial^2 f}{\partial u \partial v}\frac{\partial u}{\partial s} + \frac{\partial^2 f}{\partial v^2}\frac{\partial v}{\partial s}\right) \\
&= -2\frac{\partial^2 f}{\partial u^2} + (4s-1)\frac{\partial^2 f}{\partial u \partial v} + 2s\frac{\partial^2 f}{\partial v^2}
\end{aligned}
$$

Gradient

Definition

For $\Omega \subseteq \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$, the Jacobian is referred to as the gradient of f.

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right).$$

Definition (MATH2011)

Suppose f is a function of two variables and f_x, f_y are continous at (a, b). Then, the gradient of f is

$$\nabla f = (f_x(a, b), f_y(a, b))$$

$\nabla f(\mathbf{a})$ is the direction of steepest change at $\mathbf{a} \in \Omega$ and is orthogonal to the tangent plane at that point. The tangent plane can be given by the equation

$$\nabla f \cdot (\mathbf{x} - \mathbf{a}) = 0.$$

Past exam question

2019 Final Q1(d)

Does the surface

$$2x^2 - xy - xz - y^2 + 4z^2 + 4x - 4z = 30$$

and the sphere

$$x^2 + y^2 + z^2 + x - y - 3z = 12$$

touch tangentially at the point $(1, 1, -2)$? Prove your answer

Let the surface be φ_1 and the sphere be φ_2. If the surface and the sphere touch tangentially at a point, their tangent planes will be the equivalent. However, it is much simpler to prove that their normals are parallel.

Past exam question

2019 Final Q1(d)

Does the surface

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Let the surface be φ_1 and the sphere be φ_2. If the surface and the sphere touch tangentially at a point, their tangent planes will be the equivalent. However, it is much simpler to prove that their normals are parallel.

2019 Final Q1(d)

$$\nabla\varphi_1 = (4x - y - z + 4, -x - 2y, -x + 8z - 4)$$
$$\nabla\varphi_2 = (2x + 1, 2y - 1, 2z - 3)$$

$$\nabla\varphi_1(1, 1, -2) = (9, 3, -21)$$
$$\nabla\varphi_2(1, 1, -2) = (3, 1, -7)$$

Since the normals of the surface and the sphere are parallel at $(1, 1, -2)$, the surface and the sphere do touch tangentially.

2019 Final Q1(d)

$$\begin{aligned}
\nabla\varphi_1 &= (4x - y - z + 4, -x - 2y, -x + 8z - 4)\\
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\end{aligned}$$

Since the normals of the surface and the sphere are parallel at $(1, 1, -2)$, the surface and the sphere do touch tangentially.

2019 Final Q1(d)

$$\begin{aligned}\nabla\varphi_1 &= (4x - y - z + 4, -x - 2y, -x + 8z - 4)\\ \nabla\varphi_2 &= (2x + 1, 2y - 1, 2z - 3)\\ \nabla\varphi_1(1,1,-2) &= (9, 3, -21)\\ \nabla\varphi_2(1,1,-2) &= (3, 1, -7)\end{aligned}$$

Since the normals of the surface and the sphere are parallel at $(1, 1, -2)$, the surface and the sphere do touch tangentially.

Directional derivative

Definition

Let $\Omega \subseteq \mathbb{R}^n$ and let $f : \Omega \to \mathbb{R}$ be a function. Let $\mathbf{a}$ be an interior point of Ω and $\mathbf{u}$ be a unit vector in $\mathbb{R}^n$. The directional derivative of f in the direction $\mathbf{u}$ at $\mathbf{a}$ is defined by

$$D_{\mathbf{u}} f(\mathbf{a}) = \frac{\partial f}{\partial \mathbf{u}} = \lim_{h \to 0} \frac{f(\mathbf{a} + h\mathbf{u}) - f(\mathbf{a})}{h}$$

if the limit exists.

Theorem

The directional derivative of f in the direction $\mathbf{u}$ at $\mathbf{a}$ is given by

$$\frac{\partial f}{\partial \mathbf{u}} = \nabla f(\mathbf{a}) \cdot \mathbf{u}.$$

Past exam question

2019 Final Q1(c)

The luminosity, L, at a point (x, y, z) in a galaxy is given by

$$L(x, y, z) = \frac{1}{1 + x^2 + y^2 + z^4}.$$

i) Find the rate of change of luminosity at $(1, -1, 1)$ in the direction of the vector $\mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

ii) In which direction does L change most rapidly at the point $(1, -1, 1)$, and what is the maximum rate of change of L at $(1, -1, 1)$?

2019 Final Q1(c)(i)

Recall that the directional derivative can be described geometrically as the rate of change of the function in a specified direction $\mathbf{u}$. We have the gradient of L as

$$\nabla L(x,y,z) = \frac{1}{(1+x^2+y^2+z^2)^2}(-2x\mathbf{i} - 2y\mathbf{j} - 4z^3\mathbf{k})$$

$$\nabla L(1,-1,1) = (-\frac{1}{8}\mathbf{i} + \frac{1}{8}\mathbf{j} - \frac{1}{4}\mathbf{k})$$

When finding the directional derivative, we use the unit vector in the chosen direction. Here, $\hat{\mathbf{v}} = \frac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k})$.
Hence, the rate of change at $(1,-1,1)$ is

$$\nabla L \cdot \hat{\mathbf{v}} = (-\frac{1}{8}\mathbf{i} + \frac{1}{8}\mathbf{j} - \frac{1}{4}\mathbf{k}) \cdot \frac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k}) = -\frac{1}{4\sqrt{3}}$$

2019 Final Q1(c)(i)

Recall that the directional derivative can be described geometrically as the rate of change of the function in a specified direction **u**. We have the gradient of L as

$$\nabla L(x,y,z) = \frac{1}{(1+x^2+y^2+z^2)^2}(-2x\mathbf{i} - 2y\mathbf{j} - 4z^3\mathbf{k})$$

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When finding the directional derivative, we use the unit vector in the chosen direction. Here, $\hat{\mathbf{v}} = \frac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k})$.
Hence, the rate of change at $(1,-1,1)$ is

$$\nabla L \cdot \hat{\mathbf{v}} = \left(-\frac{1}{8}\mathbf{i} + \frac{1}{8}\mathbf{j} - \frac{1}{4}\mathbf{k}\right) \cdot \frac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k}) = -\frac{1}{4\sqrt{3}}$$

2019 Final Q1(c)(i)

Recall that the directional derivative can be described geometrically as the rate of change of the function in a specified direction $\mathbf{u}$. We have the gradient of L as

$$\nabla L(x, y, z) = \frac{1}{(1 + x^2 + y^2 + z^2)^2}(-2x\mathbf{i} - 2y\mathbf{j} - 4z^3\mathbf{k})$$

$$\nabla L(1, -1, 1) = (-\frac{1}{8}\mathbf{i} + \frac{1}{8}\mathbf{j} - \frac{1}{4}\mathbf{k})$$

When finding the directional derivative, we use the unit vector in the chosen direction. Here, $\hat{\mathbf{v}} = \frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k})$.

Hence, the rate of change at $(1, -1, 1)$ is

$$\nabla L \cdot \hat{\mathbf{v}} = (-\frac{1}{8}\mathbf{i} + \frac{1}{8}\mathbf{j} - \frac{1}{4}\mathbf{k}) \cdot \frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k}) = -\frac{1}{4\sqrt{3}}$$

2019 Final Q1(c)(i)

Recall that the directional derivative can be described geometrically as the rate of change of the function in a specified direction $\mathbf{u}$. We have the gradient of L as

$$\nabla L(x,y,z) = \frac{1}{(1+x^2+y^2+z^2)^2}(-2x\mathbf{i} - 2y\mathbf{j} - 4z^3\mathbf{k})$$

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When finding the directional derivative, we use the unit vector in the chosen direction. Here, $\hat{\mathbf{v}} = \frac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k})$.
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2019 Final Q1(c)(ii)

2019 Final Q1(c)

In which direction does L change most rapidly at the point $(1, -1, 1)$, and what is the maximum rate of change of L at $(1, -1, 1)$?

The maximum rate of change of L at the point $(1, -1, 1)$ occurs in the direction given by $\nabla L(1, -1, 1)$, which is

$$-\frac{1}{8}\mathbf{i} + \frac{1}{8}\mathbf{j} - \frac{1}{4}\mathbf{k}.$$

The maximum rate of change of L is

$$\sqrt{\left(-\frac{1}{8}\right)^2 + \left(\frac{1}{8}\right)^2 + \left(-\frac{1}{4}\right)^2} \approx 0.306.$$

2019 Final Q1(c)(ii)

2019 Final Q1(c)

In which direction does L change most rapidly at the point $(1,-1,1)$, and what is the maximum rate of change of L at $(1,-1,1)$?

The maximum rate of change of L at the point $(1,-1,1)$ occurs in the direction given by $\nabla L(1,-1,1)$, which is

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The maximum rate of change of L is

$$\sqrt{\left(-\frac{1}{8}\right)^2+\left(\frac{1}{8}\right)^2+\left(-\frac{1}{4}\right)^2}\approx 0.306.$$

Hessian matrix

Definition

For $\Omega \subseteq \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$, the Hessian of f at $\mathbf{a}$ is the $n \times n$ matrix

$$Hf(\mathbf{a}) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(\mathbf{a}) & \frac{\partial^2 f}{\partial x_2 \partial x_1}(\mathbf{a}) & \dots & \frac{\partial^2 f}{\partial x_n \partial x_1}(\mathbf{a}) \\ \frac{\partial^2 f}{\partial x_1 \partial x_2}(\mathbf{a}) & \frac{\partial^2 f}{\partial x_2^2}(\mathbf{a}) & \dots & \frac{\partial^2 f}{\partial x_n \partial x_2}(\mathbf{a}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n}(\mathbf{a}) & \frac{\partial^2 f}{\partial x_2 \partial x_n}(\mathbf{a}) & \dots & \frac{\partial^2 f}{\partial x_n^2}(\mathbf{a}) \end{pmatrix}$$

Taylor polynomials

Definition

Let $\Omega \subseteq \mathbb{R}^n$ be open and $f : \Omega \to \mathbb{R}$ be a function such that all partial derivatives of at most 2 exist and are continuous. Let $\mathbf{a}, \mathbf{x} \in \Omega$ be such that the line segment joining $\mathbf{a}, \mathbf{a}$ lies in Ω. The polynomial

$$P_{1,\mathbf{a}}(\mathbf{x}) = f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a})$$

is the Taylor polynomial of order 1 about $\mathbf{a}$, and

$$P_{2,\mathbf{a}}(\mathbf{x}) = f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) + \frac{1}{2}((\mathbf{x} - \mathbf{a}) \cdot Hf(\mathbf{a})(\mathbf{x} - \mathbf{a})$$

is the Taylor polynomial of order 2 about $\mathbf{a}$.

Taylor's theorem for 1st order

Theorem

There exists some point $\mathbf{z}$ on the line segment joining $\mathbf{x}$ and $\mathbf{a}$ such that

$$f(\mathbf{x}) = f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) + R_{1,\mathbf{a}}(\mathbf{x}),$$

where

$$R_{1,\mathbf{a}}(\mathbf{x}) = \frac{1}{2}((\mathbf{x} - \mathbf{a}) \cdot Hf(\mathbf{z})(\mathbf{x} - \mathbf{a})).$$

Maxima, minima and saddle points

Definition

Let $\mathbf{a} \in \Omega \subseteq \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$ be a function. Then $\mathbf{a}$ is a

- stationary point if $\nabla f(\mathbf{a}) = \mathbf{0}$.
- global maximum if $f(\mathbf{a}) \geq f(\mathbf{x})$ for all $\mathbf{x} \in \Omega$.
- global minimum if $f(\mathbf{a}) \leq f(\mathbf{x})$ for all $\mathbf{x} \in \Omega$.
- local maximum if there is an open set around $\mathbf{a}$ such that $f(\mathbf{a}) \geq f(\mathbf{x})$ for all $\mathbf{x} \in \Omega \cap A$.
- local minimum if there is an open set around $\mathbf{a}$ such that $f(\mathbf{a}) \leq f(\mathbf{x})$ for all $\mathbf{x} \in \Omega \cap A$.
- saddle point if it is a stationary point but neither a maximum nor a minimum.

Critical points

Definition

A point $\mathbf{a} \in \Omega \subseteq \mathbb{R}^n$ is a critical point of a function $f : \Omega \to \mathbb{R}$ if $\mathbf{a}$ satisfies one of the following:

- $\mathbf{a}$ is a stationary point.
- $\mathbf{a}$ lies on the boundary of Ω.
- f is not differentiable at $\mathbf{a}$.

Local minima and maxima occur at critical points.

Positive and negative definite

Definition

An $n \times n$ matrix is

- positive definite if all eigenvalues are positive
- positive semidefinite if all eigenvalues are nonnegative
- negative definite if all eigenvalues are negative
- negative semidefinite if all eigenvalues are nonpositive.

Classification of stationary points

Theorem

Let $\Omega \subseteq \mathbb{R}^n$ be open, $\mathbf{a} \in \Omega$ and let $f : \Omega \to \mathbb{R}$ be a function such that all partial derivatives of order at most 2 exist on Ω and $\nabla f(\mathbf{a}) = \mathbf{0}$. Then

- If $Hf(\mathbf{a})$ is positive definite, f has a local minimum at $\mathbf{a}$.
- If $Hf(\mathbf{a})$ is negative definite, f has a local maximum at $\mathbf{a}$.
- If f has a local minimum at $\mathbf{a}$, then $Hf(\mathbf{a})$ is positive semidefinite
- If f has a local maximum at $\mathbf{a}$, then $Hf(\mathbf{a})$ is negative semidefinite.

Sylvester's criterion

Theorem

If Δ_k is the determinant of the upper left $k \times k$ submatrix of H, then H is

- positive definite if and only if $\Delta_k > 0$ for all k.
- positive semidefinite if and only if $\Delta_k \geq 0$ for all k.
- negative definite if and only if $\Delta_k < 0$ for all odd k and $\Delta_k > 0$ for all even k.
- negative semidefinite if and only if $\Delta_k \leq 0$ for all odd k and $\Delta_k \geq 0$ for all even k.

Taylor expansion about a point (MATH2011)

Definition

The Taylor expansion of f about the point $\mathbf{x}$ is

$$f(x+h, y+k) = \sum_{n=0}^{\infty} \frac{1}{n!}(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y})^n \cdot f$$

Example

The second order Taylor expansion is

$$f(x_0+h, y_0+k) = f(x_0, y_0) + \frac{1}{1!}\left(\frac{\partial f}{\partial x}(x-x_0) + \frac{\partial f}{\partial y}(y-y_0)\right)$$
$$+\frac{1}{2!}\left(\frac{\partial^2 f}{\partial x^2}(x-x_0)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(x-x_0)(y-y_0) + \frac{\partial^2 f}{\partial y^2}(y-y_0)^2\right)$$

Critical Points (MATH2011)

Definition

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function. A critical point of f is a point (a,b) where ∇f is either zero or does not exist.

Definition

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function. A stationary point of f is a point (a, b) where ∇f is zero.

Definition

A function $f : \mathbb{R}^2 \to \mathbb{R}$ has a local maximum/minimum at (a, b) if $f(a, b)$ is the largest/smallest function value on some small disc centered at (a, b).

Second Derivative Test (MATH2011)

Theorem

Suppose the second order partial derivatives of f are continuous on a disk with centre (a, b) and suppose that $f_x(a, b) = f_y(a, b) = 0$. (This means that (a, b) is a critical point)

Let the discriminant D be

$$D = (f_{xy}(a, b))^2 - f_{xx}(a, b)f_{yy}(a, b).$$

Then

1. If $D < 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
2. If $D < 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
3. If $D > 0$ then $f(a, b)$ is a saddle point

Past exam question

2019 Final Q1(e)

Let f be the function

$$f(x, y) = x^3 + 3x^2 + 24xy + 12y^2 + 15x - 2.$$

i) Show that $(1, -1)$ is a saddle point for f.

ii) Show that f has one other critical point and find out what type it is.

iii) Write down the Taylor series expansion of f about the critical point $(1, -1)$.

2019 Final Q1(e)

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Show that $(1,-1)$ is a saddle point for f.

$$f(x,y)=x^3+3x^2+24xy+12y^2+15x-2.$$

$$\begin{aligned}
f_x(x,y) &= 3x^2+6x+24y+15\\
f_y(x,y) &= 24x+24y\\
f_{xx}(x,y) &= 6x+6\\
f_{xy}(x,y) &= 24\\
f_{yy}(x,y) &= 24\\
f_{xxx}(x,y) &= 6
\end{aligned}$$

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2019 Final Q1(e)

Show that $(1, -1)$ is a saddle point for f.

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\begin{aligned}
f_{xx}(x, y) &= 6x + 6 \\
f_{xy}(x, y) &= 24 \\
f_{yy}(x, y) &= 24
\end{aligned}
$$

$$
\begin{aligned}
D(1, -1) &= (f_{xy}(1, -1))^2 - f_{xx}(1, -1) f_{yy}(1, -1) \\
&= 24^2 - (6 + 6)(24) = 288 > 0
\end{aligned}
$$

Since $D > 0$, $(1, -1)$ is a saddle point.

2019 Final Q1(e)

2019 Final Q1(e)

Show that f has one other critical point and find out what type it is.

The gradient of f is given by

$$\nabla f(x, y) = (3x^2 + 6x + 24y + 15, 24x + 24y)^T$$

Setting $\nabla f(x, y) = \mathbf{0}$ and solving gives the critical points $(1, -1)$ and $(5, -5)$. Thus the other critical point is $(5, -5)$.

$$\begin{aligned} D(5, -5) &= (f_{xy}(5, -5))^2 - f_{xx}(5, -5) f_{yy}(5, -5) \\ &= 24^2 - (30 + 6)(24) = -288 < 0 \end{aligned}$$

Since $f_{xx}(5, -5) = 36 > 0$, $(5, 5)$ is a local minimum.

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2019 Final Q1(e)

Write down the Taylor series expansion of f about the critical point $(1, -1)$.

$$f(1,-1) + \frac{1}{1!}\Big(f_x(x-1) + f_y(y+1)\Big)$$

$$+\frac{1}{2!}\Big(f_{xx}(x-1)^2 + 2f_{xy}(x-1)(y+1) + f_{yy}(y+1)^2\Big)$$

$$+\frac{1}{3!}\Big(f_{xxx}(x-1)^3 + \dots\Big)$$

which simplifies to

$$5 + 6(x-1)^2 + 24(x-1)(y+1) + 12(y+1)^2 + (x-1)^3$$

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Lagrange multipliers

Theorem

Suppose $f : \mathbb{R} \to \mathbb{R}$ and $\varphi : \mathbb{R}^n \to \mathbb{R}$ are differentiable and

$$S = \{\mathbf{x} \in \mathbb{R}^n : \varphi(\mathbf{x}) = c\}$$

defines a smooth surface in $\mathbb{R}^n$. If f attains a local maximum or minimum at a point $\mathbf{a} \in S$, then $\nabla f(\mathbf{a})$ and $\nabla \varphi(\mathbf{a})$ are parallel. If $\nabla \varphi(\mathbf{a}) \neq \mathbf{0}$, there exist a Lagrange multiplier $\lambda \in \mathbb{R}$ such that

$$\nabla f(\mathbf{a}) = \lambda \nabla \varphi(\mathbf{a}).$$

Past exam question

2017 Final Q2(a)

A rectangular box has three of its faces on the coordinate planes and one vertex in the first octant on the paraboloid $z = 4 - x^2 - y^2$. Determine the maximum volume of the box.

First we define the volume of the box $f(x, y, z) = xyz$ and the constraint $g(x, y, z) = 4 - x^2 - y^2 - z$. Note that since the vertex is in the first octant, $x, y, z > 0$. The gradients are

$$\nabla f(x, y, z) = (yz, xz, xy)^T$$

$$\nabla g(x, y, z) = (-2x, -2y, -1)^T$$

Now check that $\nabla g(\mathbf{x}) \neq 0$ for all $\mathbf{x}$ on the constraint. Clearly $\nabla g(\mathbf{x})$ can never be $\mathbf{0}$.

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2017 Final Q2(a) Solution

The maximum on the paraboloid must satisfy

$$\nabla f(\mathbf{x}) = \lambda \nabla g(\mathbf{x}) \quad \text{and} \quad g(\mathbf{x}) = 0$$

This gives the following system of equations:

$$yz = -2\lambda x \quad xz = -2\lambda y \quad xy = -\lambda \quad x^2 + y^2 + z = 4$$

We can divide the first two equations by the third to get

$$x^2 = \frac{z}{2} \qquad y^2 = \frac{z}{2}$$

Substituting into the constraint gives

$$\frac{z}{2} + \frac{z}{2} + z = 4$$

Solving gives $z = 2$, so the point is $(1, 1, 2)$, giving a max volume of 2.

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Inverse function theorem

Theorem

Let $\Omega \subseteq \mathbb{R}^n$ be open, $\mathbf{a} \in \Omega$ and $\mathbf{f} : \Omega \to \mathbb{R}^n$ be continuously differentiable.
If $D\mathbf{f}(\mathbf{a})$ is an invertible matrix, then $\mathbf{f}$ is invertible on an open set containing $\mathbf{a}$.

$$\mathbf{f}^{-1} : \mathbf{f}(U) \to U$$

is continuously differentiable and for $\mathbf{x} \in U$,

$$D\mathbf{f}^{-1}(\mathbf{f}(\mathbf{x})) = (D\mathbf{f}(\mathbf{x}))^{-1}.$$

Inverse function theorem (MATH2011)

Theorem

Let $\mathbf{f} : \mathbb{R}^p \to \mathbb{R}^p$ be differentiable at $\mathbf{a}$.
Then if $J_{\mathbf{a}}\mathbf{f}$ is an invertible matrix, there is an inverse function $\mathbf{f}^{-1} : \mathbb{R}^p \to \mathbb{R}^p$ defined in some neighbourhood of $\mathbf{b} = \mathbf{f}(\mathbf{a})$ and

$$(J_{\mathbf{b}}\mathbf{f}^{-1}) = (J_{\mathbf{b}}\mathbf{f})^{-1}$$

Past exam question

2018 Final Q2(a)

Let $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 + x + 2y \\ x^2 + 3y \end{pmatrix}$.

i) Find the Jacobian matrix $J_{\mathbf{x}} f$.

ii) What does $J_{(0,0)} f$ and the inverse function theorem tell you about f around $(0,0)$?

The Jacobian matrix is

$$J_{\mathbf{x}} f = \begin{pmatrix} 2x+1 & 2 \\ 2x & 3 \end{pmatrix}$$

$$J_{(\mathbf{0},\mathbf{0})} f = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}.$$

Since the determinant of $J_{(\mathbf{0},\mathbf{0})} f$ is 3, and non-zero, by the Inverse Function Theorem, f has a differentiable inverse around $(0,0)$.

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4. Integration

Upper and lower sums

Definition

Consider $f : R \to \mathbb{R}$ where $R = [a, b] \times [c, d]$ is a rectangle in $\mathbb{R}^2$. Let $P_1 = \{a = x_0, x_1, \ldots, x_n = b\}$ be a partition of $[a, b]$ and $P_2 = \{c = y_0, y_1, \ldots, y_m = d\}$ be a partition of $[c, d]$. Define the upper and lower sum by

$$\underline{\mathcal{S}}_{\mathcal{P}_1,\mathcal{P}_2}(f) = \sum_{j,k} \underline{f}_{jk} \Delta x_j \Delta y_k.$$

$$\overline{\mathcal{S}}_{\mathcal{P}_1,\mathcal{P}_2}(f) = \sum_{j,k} \overline{f}_{jk} \Delta x_j \Delta y_k.$$

Riemann integration

Definition

If there exists a unique number $I \in \mathbb{R}$ such that

$$\underline{\mathcal{S}}_{\mathcal{P}_1,\mathcal{P}_2}(f) \leq I \leq \overline{\mathcal{S}}_{\mathcal{P}_1,\mathcal{P}_2}(f)$$

for every pair of partitions $\mathcal{P}_1, \mathcal{P}_2$, then f is Riemann integrable on R and

$$\iint_R f = \iint_R f(x,y)\, dA = I$$

and I is the Riemann integral of f over R.

Regions

Definition

A region $D \subseteq \mathbb{R}^2$ is y-simple if there exist continuous functions $\varphi_1, \varphi_2 : [a, b] \to \mathbb{R}$ such that $\varphi_1(x) \leq \varphi_2(x)$ for all $\mathbf{x} \in [a, b]$ and

$$D = \{(x, y) : x \in [a, b], \varphi_1(x) \leq y \leq \varphi_2(x)\}.$$

x-simple is similarly defined.
A region D is elementary if it is x-simple or y-simple.

Properties

Properties

If f and g are integrable on D, then

$$\iint_D \alpha f + \beta g = \alpha \iint_D f + \beta \iint_D g$$

$$\left| \iint_D f \right| \leq \iint_D |f|.$$

If $f(\mathbf{x}) \leq g(\mathbf{x})$ for all $\mathbf{x} \in D$, then

$$\iint_D f \leq \iint_D g.$$

If $D = D_1 \cup D_2$ and (interior D)$\cap$(interior D_2) $= \phi$, then

$$\iint_D f = \iint_{D_1} f + \iint_{D_2} f.$$

Fubini's Theorem

Theorem

Let $f : R \to \mathbb{R}$ be continuous on a rectangular domain $R = [a, b] \times [c, d]$. Then

$$\iint_R f = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy$$

Loosely speaking, this means that as long as the above conditions are satisfied, we are allowed to rearrange and manipulate our integral however we want.

Example 1

2015 Q2(ii)(a)

Consider the Euler-Poisson Integral,

$$I = \int_0^\infty e^{-x^2} dx.$$

Using Fubini's theorem, prove that,

$$I^2 = \frac{1}{4} \int_{\mathbb{R}^2} e^{-x^2-y^2} dxdy$$

Example 1 Solutions

Note that $e^{-x^2-y^2}$ is a continuous function across the entirety of $\mathbb{R}^2$. Thus we can apply Fubini's theorem.

Firstly, we have that,

$$\frac{1}{4}\int_{\mathbb{R}^2} e^{-x^2-y^2}dxdy = \frac{1}{4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}dxdy$$

Then note that $e^{-x^2-y^2}$ is an even function, and thus we get,

$$\frac{1}{4}\int_{\mathbb{R}^2} e^{-x^2-y^2}dxdy = \int_{0}^{\infty}\int_{0}^{\infty} e^{-x^2-y^2}dxdy$$

Then by Fubini's theorem,

$$\frac{1}{4}\int_{\mathbb{R}^2} e^{-x^2-y^2}dxdy = \int_{0}^{\infty} e^{-x^2}\left(\int_{0}^{\infty} e^{-y^2}dy\right)dx$$

Example 1 Solutions

We then notice that the integral inside the brackets is simply I. Thus we have,

$$\frac{1}{4}\int_{\mathbb{R}^2} e^{-x^2-y^2}\,dxdy = I\int_0^\infty e^{-x^2}\,dx$$

Then we again notice that the remaining integral is also just I, and so we finally get,

$$\frac{1}{4}\int_{\mathbb{R}^2} e^{-x^2-y^2}\,dxdy = I^2.$$

Iterated integrals for elementary regions

Theorem

Suppose D is a y-simple region bounded by $x = a, x = b, y = \varphi_1(x)$ and $y = \varphi_2(x)$ and $f : D \to \mathbb{R}$ is continuous. Then

$$\iint_D f = \int_a^b \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\, dx\, dy.$$

A similar result holds for integrals over x-simple regions.

Leibniz' rule (Differentiation under the Integral Sign)

Theorem

Let $a, b \in \mathbb{R}$ be constants, say $a \leq b$ and $\varphi_1, \varphi_2 : [a, b] \to \mathbb{R}$ be continuous functions such that $\varphi_1(x) \leq \varphi_2(x)$ for all $x \in [a, b]$. If f and $\frac{\partial f}{\partial x}$ are continuous on the region

$$D = \{(x, y) : x \in [a, b], \varphi_1(x) \leq y \leq \varphi_2(x)\}$$

then the function

$$g(x) = \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\, dx\, dy$$

has derivative

$$g^{'}(x) = \int_{\varphi_1(x)}^{\varphi_2(x)} \frac{\partial f}{\partial x}(x, y)\, dy + f(x, \varphi_2(x))\varphi_2^{'}(x) - f(x, \varphi_1(x))\varphi_1^{'}(x).$$

Example 2

2014 Q2(i)(b) adapted

Calculate

$$\frac{\mathrm{d}}{dx}\int_x^{x^2} \frac{\cos(xy)}{y}\,dy, \text{ for all } x > 1.$$

This question is essentially a direct application of Leibniz's rule.

Example 2 Solutions

First note that the integrand is continuous across the entire boundary for $x > 1$, and furthermore that $x \leq y \leq x^2$ for $x > 1$.

Additionally,

$$\begin{aligned}\frac{\partial \cos(xy)/y}{\partial x} &= \frac{y(-\sin(xy))}{y} \\ &= -\sin(xy).\end{aligned}$$

Thus, the partial derivative of the integrand is also continuous everywhere. Therefore, we can apply Leibniz's rule.

Example 2 Solutions

Applying Leibniz's rule, we get,

$$\begin{aligned}
\frac{d}{dx}\int_x^{x^2} \frac{\cos(xy)}{y}dy &= \int_x^{x^2} \frac{\partial}{\partial x}\frac{\cos(xy)}{y}dy + \frac{\cos(x(x^2))}{x^2}(2x) - \frac{\cos(x(x))}{x}(1) \\
&= \int_x^{x^2} -\sin(xy)dy + \frac{2\cos(x^3))}{x} - \frac{\cos(x^2)}{x} \\
&= \frac{1}{x}\cos(xy)\Big|_{y=x}^{y=x^2} + \frac{2\cos(x^3))}{x} - \frac{\cos(x^2)}{x} \\
&= \frac{1}{x}(\cos(x^3) - \cos(x^2)) + \frac{2\cos(x^3))}{x} - \frac{\cos(x^2)}{x} \\
&= \frac{3}{x}\cos(x^3) - \frac{2}{x}\cos(x^2).
\end{aligned}$$

Example 3

2018 Q1(iv)

Given that

$$\int_{-\infty}^{\infty} \frac{\cos(3x)}{(x^2+a^2)} dx = \frac{\pi}{a} e^{-3a}, \; a > 0$$

use differentiation under the integral sign to find

$$\int_{-\infty}^{\infty} \frac{\cos(3x)}{(x^2+4)^2} dx$$

The idea here is to construct a new 'parameter' or 'variable' that we can easily perform Leibniz's rule on, such that, the new integrand ends up looking something like our desired integrand.

Example 3 Solutions

In this case, we wish to turn:

$$\int_{-\infty}^{\infty} \frac{\cos(3x)}{(x^2+a^2)} \to \int_{-\infty}^{\infty} \frac{\cos(3x)}{(x^2+4)^2}.$$

To do so, consider setting the 'a' in the given integral as a new variable. We then have that by Leibniz's rule,

$$\begin{aligned}
\frac{d}{da}\int_{-\infty}^{\infty} \frac{\cos(3x)}{(x^2+a^2)}dx &= \frac{d}{da}\frac{\pi}{a}e^{-3a} \\
\int_{-\infty}^{\infty} \frac{\partial}{\partial a}\frac{\cos(3x)}{(x^2+a^2)}dx &= \pi(e^{-3a})(-a^{-2}-3a^{-1}) \\
\int_{-\infty}^{\infty} -2a\frac{\cos(3x)}{(x^2+a^2)^2}dx &= \pi(e^{-3a})(-a^{-2}-3a^{-1}).
\end{aligned}$$

Example 3 Solutions

Note that we end up with this result,

$$\int_{-\infty}^{\infty} \frac{\cos(3x)}{(x^2+a^2)^2} dx = \frac{\pi}{2a}(e^{-3a})(a^{-2}+3a^{-1}).$$

The integrand on the left looks extremely similar to our desired integral. If we just simply substitute in $a = 2$, we would have exactly our desired integral.

Thus, we get that,

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\cos(3x)}{(x^2+4)^2} dx &= \frac{\pi}{2(2)}(e^{-3(2)})(2^{-2}+3(2)^{-1}) \\ &= \frac{7}{16}\pi e^{-6}.\end{aligned}$$

Example 3 Solutions

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Change of variable

Theorem

Let $\Omega \subseteq \mathbb{R}^n$ and $F : \Omega \to \mathbb{R}^n$ be an injective and continuously differentiable function such that $\det(JF(\mathbf{x})) \neq 0$ for all $\mathbf{x} \in \Omega$. If f is any function that is integrable on $\Omega^{'} = F(\Omega)$ then

$$\iint_{\Omega'} (f \circ F)|\det JF|$$

Polar substitution refers to subbing $x = r\cos\theta$ and $y = r\sin\theta$ where $r > 0, \theta \in [0, 2\pi]$. The Jacobian determinant is r.

Example 4

2016 Q2(iv)

Evaluate

$$\iint_{\Omega'} x^2 \, dx \, dy$$

where Ω' is the bounded portion of the first quadrant lying between the hyperbolas

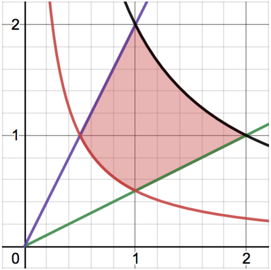
$$xy = \frac{1}{2} \text{ and } xy = 2$$

and the two straight lines

$$y = \frac{x}{2} \text{ and } y = 2x.$$

Example 4 Solutions

First we plot $\Omega' = \{(x, y) \in \mathbb{R}^2 : \frac{1}{2} \le xy \le 2, \frac{x}{2} \le y \le 2x\}$.



Now define $u = xy$ and $v = y$. Expressing x and y in terms of u and v gives

$$x = \frac{u}{v} \text{ and } y = v$$

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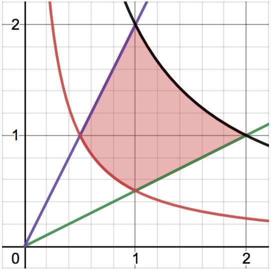

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Example 4 Solutions

The Jacobian is given by

$$J = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} \frac{1}{v} & -\frac{1}{v^2} \\ 0 & 1 \end{pmatrix}$$

and the determinant is $\det(J) = \frac{1}{v}$.

The region in the $u - v$ plane can be described by

$$\frac{1}{2} \leq u \leq 2 \quad \text{and} \quad \frac{u}{2v} \leq v \leq \frac{2u}{v}$$

which simplifies to

$$\frac{1}{2} \leq u \leq 2 \quad \text{and} \quad \sqrt{\frac{u}{2}} \leq v \leq \sqrt{2u}$$

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Example 4 Solutions

The integral is now

$$\begin{aligned}
\int_{\frac{1}{2}}^{2} \int_{\sqrt{\frac{u}{2}}}^{\sqrt{2u}} \left(\frac{u}{v}\right)^2 \cdot \frac{1}{v}\, dv\, du &= \int_{\frac{1}{2}}^{2} -\frac{u^2}{v^2}\Big|_{v=\sqrt{\frac{u}{2}}}^{v=\sqrt{2u}}\, du \\
&= \int_{\frac{1}{2}}^{2} -\frac{u^2}{2(2u)} + \frac{u^2}{2(u/2)}\, du \\
&= \int_{\frac{1}{2}}^{2} \frac{3}{4}u\, du \\
&= \frac{45}{32}
\end{aligned}$$

5. Fourier Series

Inner Products

Definition of an Inner Product

An **inner product** is a binary operation that acts on two vectors in a vector space. It can be anything that has the following properties:

1. $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$;
2. $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ iff $\mathbf{u} = \mathbf{0}$;
3. $\langle \lambda \mathbf{u} + \mu \mathbf{v}, \mathbf{w} \rangle = \lambda \langle \mathbf{u}, \mathbf{w} \rangle + \mu \langle \mathbf{v}, \mathbf{w} \rangle$;
4. $\langle \mathbf{v}, \mathbf{u} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle$,

for any vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ and for any scalars λ, μ.

Inner Products

Examples of Inner Products

1. In the vector space $\mathbb{R}^n$, the inner product can be defined as the dot product. Furthermore, recall that vectors are **orthogonal** if their dot product equal to 0.
2. For the space $C[a, b]$ containing all continuous functions in the interval $[a, b]$, the inner product can be defined as,

$$\langle f, g \rangle = \int_a^b f(x)g(x)dx.$$

Furthermore, note that f and g are **orthogonal** if $\langle f, g \rangle = 0$.

Example 1

Show that the dot product for the vector space $\mathbb{R}^n$ is a valid inner product.

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Example 1

Show that the dot product for the vector space $\mathbb{R}^n$ is a valid inner product.

Example 1 Solutions

First note that the dot product is defined as,

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y} = \sum_{\text{all i}} x_i y_i.$$

In order for the dot product to be a valid inner product, it must satisfy the above 4 conditions. Let's check through each one.

1. $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0.$

$$\begin{aligned} \langle \mathbf{x}, \mathbf{x} \rangle &= \mathbf{x} \cdot \mathbf{x} \\ &= \sum_{\text{all i}} (x_i)^2 \\ &\geq 0. \end{aligned}$$

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Example 1 Solutions

2. $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ iff $\mathbf{x} = \mathbf{0}$.

$$\langle \mathbf{x}, \mathbf{x} \rangle = \sum_{\text{all i}} (x_i)^2$$

Thus, the only way this sum is equal to $\mathbf{0}$ is if every component x_i is equal to 0. Thus the vector mx itself must be the zero vector, $\mathbf{0}$.

3. $\langle \lambda\mathbf{x} + \mu\mathbf{y}, \mathbf{z} \rangle = \lambda\langle \mathbf{x}, \mathbf{z} \rangle + \mu\langle \mathbf{y}, \mathbf{z} \rangle$

$$\langle \lambda\mathbf{x} + \mu\mathbf{y}, \mathbf{z} \rangle = (\lambda\mathbf{x} + \mu\mathbf{y}) \cdot \mathbf{z}$$

Then by the linearity of the dot product, we have,

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Example 1 Solutions

4. $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$

$$\begin{aligned}
\langle \mathbf{x}, \mathbf{y} \rangle &= \sum_{\text{all i}} x_i y_i \\
&= \sum_{\text{all i}} y_i x_i \\
&= \langle \mathbf{y}, \mathbf{x} \rangle.
\end{aligned}$$

Thus, as we have verified all 4 conditions of what constitutes an inner product, the dot product in $\mathbb{R}^n$ is a valid inner product.

Norms

Definition of a Norm

A **norm** is a map from a vector space to $\mathbb{R}$. It can be anything that has the following properties:

1. $\|\mathbf{u}\| \geq 0$;
2. $\|\mathbf{u}\| = 0$ iff $\mathbf{u} = \mathbf{0}$;
3. $\|\lambda\mathbf{u}\| = |\lambda|\|\mathbf{u}\|$;
4. $\|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$,

for any vectors $\mathbf{u}$ and $\mathbf{v}$ and any scalar λ.

- $\|\mathbf{u}\| = \sqrt{\langle \mathbf{u},\mathbf{u}\rangle}$ is a norm for any inner product $\langle\cdot,\cdot\rangle$.
- (Cauchy-Schwarz inequality) For any vectors $\mathbf{u}$ and $\mathbf{v}$ in a vector space V,

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Norms

Examples of Norms

In the vector space, $\mathbb{R}^n$, a common norm is the Euclidean norm,

$$\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + x_3^2 + \cdots + x_n^2}.$$

In the space $C[a, b]$ consisting of all the continuous functions on $[a, b]$, there exists some common norms such as,

1. $\|f\|_2 = \sqrt{\int_a^b f^2(x)dx}$. ($L^2$-norm)
2. $\|f\|_\infty = \max_{a \leq x \leq b}\{|f(x)|\}$. (Max-norm)

Example 2

Verify that $\|\mathbf{x}\|$ is indeed a valid norm.

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Example 2 Solutions

Again, this is very similar to our example on inner products. I will leave most of it to you as an exercise. I will only briefly go through the intuition for each condition.

The only condition that has a more difficult/interesting working out is the triangle inequality one which uses Cauchy-Schwarz inequality. For L^2 norm, the proof of triangle inequality requires Minkowski inequality.

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Fourier Series

Definition of Fourier Series

Fourier Series are series used to express periodic functions. They are in the form:

$$S_f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right).$$

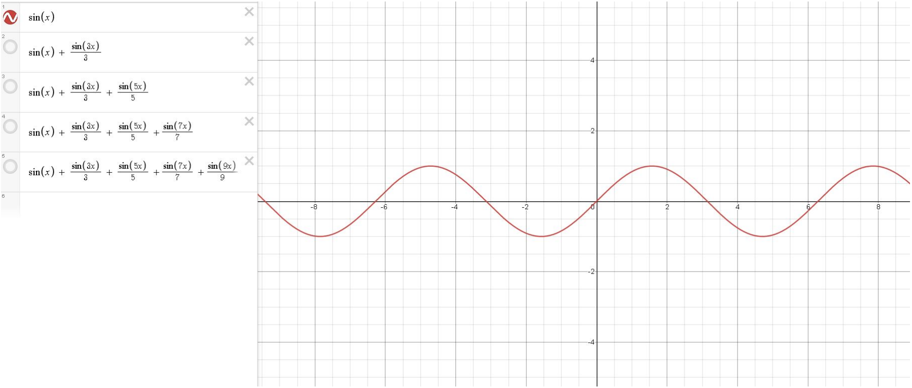
- Our task is to find a_0, a_n and b_n as well as determine where the series will converge towards the function.
- Fourier series can replicate functions that are not continuous.

Fourier Series

In a Fourier Series, we are essentially making use of the periodicity of the trigonometric functions such that we can create new functions/waves that are also periodic.

A common example is how we can build a square-looking wave out of just sine functions.
We shall look at this example in the following slides.

Fourier Series



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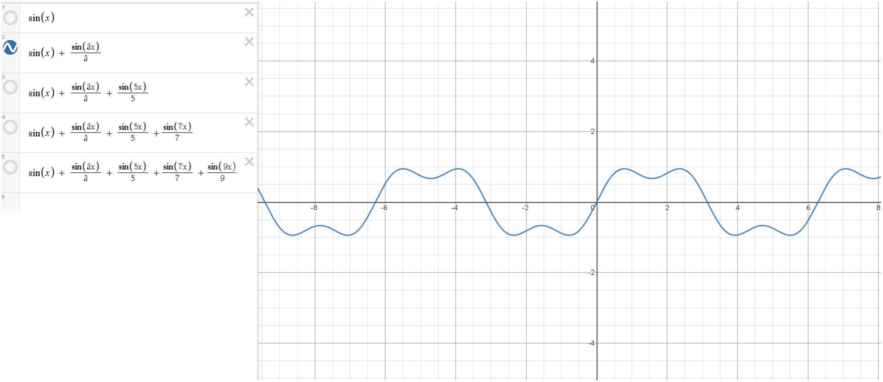

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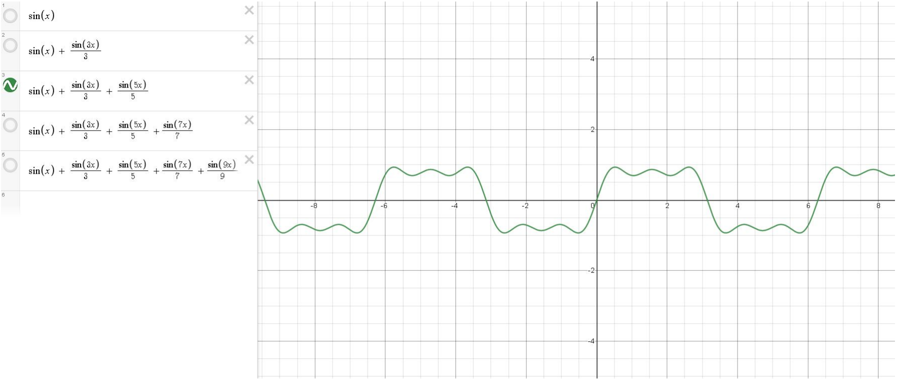

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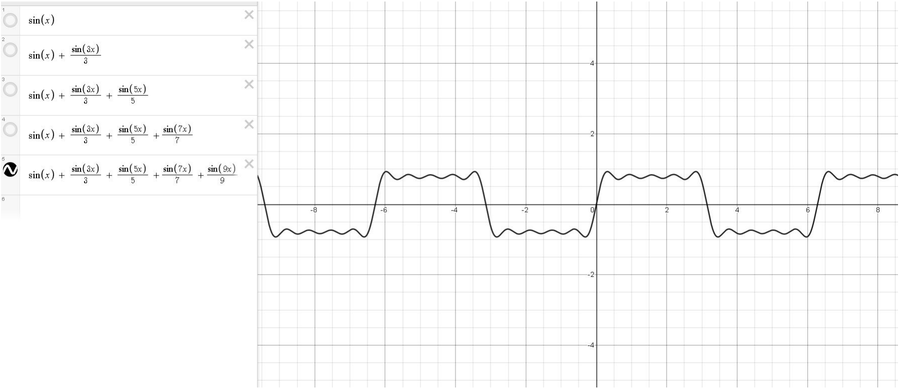

Getting Fourier Series' Coefficients

Extension from orthogonality in $\mathbb{R}^n$ to $C[-L, L]$

Firstly, recall from Linear Algebra, that if we have the standard orthonormal basis $\{\mathbf{e}_i\}$, we can decompose any vector, $\mathbf{v}$ into:

$$\mathbf{v} = \sum_{\text{all i}} \alpha_i \mathbf{e}_i, \text{ where } \alpha_i = \langle \mathbf{v}, \mathbf{e}_i \rangle = \mathbf{v} \cdot \mathbf{e}_i.$$

Next, if we have an arbitrary orthogonal basis instead, $\{\mathbf{u}_i\}$, we can extend what we did with the the standard basis and still decompose $\mathbf{v}$.

If we consider what we did with the standard basis, all we really did was **project $\mathbf{v}$** onto each component in the standard basis. We can do the same with our new $\{\mathbf{u}_i\}$ basis, leading to the following result:

$$\mathbf{v} = \sum_{\text{all i}} \frac{\mathbf{v} \cdot \mathbf{u}_i}{\|\mathbf{u}_i\|^2} \mathbf{u}_i = \sum_{\text{all i}} \frac{\langle \mathbf{v}, \mathbf{u}_i \rangle}{\|\mathbf{u}_i\|^2} \mathbf{u}_i.$$

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Getting Fourier Series' Coefficients

Extension from orthogonality in $\mathbb{R}^n$ to $C[-L, L]$

Now, let's finally apply this logic to functions, $f \in C[-L, L]$.
First note that this set, $\{1, \cos(kx), \ldots, \sin(kx), \ldots\}$ is an orthogonal set. And thus, we can derive coefficients a_n and b_n in our Fourier Series, by projecting f onto each of these trigonometric functions.

We ultimately end up with the following,

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \qquad n = 0, 1, 2, \ldots,$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \qquad n = 1, 2, \ldots.$$

Notice that often for a_0 we have to calculate it separately, as sometimes n ends up on the denominator and this causes a problem.

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Example 3

Non-exam Question

Show that the L^2-norm of $\|\cos(\frac{n\pi x}{L})\| = \sqrt{L}$.

By the definition of the L^2-norm, we have,

$$\begin{aligned}
\|\cos(\frac{n\pi x}{L})\| &= \sqrt{\int_{-L}^{L}\cos^2(\frac{n\pi x}{L})dx} \\
&= \sqrt{2\int_0^L \frac{\cos(\frac{2n\pi x}{L})+1}{2}dx} \\
&= \sqrt{\int_0^L \cos(\frac{2n\pi x}{L})dx + L} \\
&= \sqrt{\frac{L}{2n\pi}\cos(\frac{2n\pi x}{L})\Big|_{x=0}^{x=L} + L} \\
&= \sqrt{(1-1)+L}.
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Odd and Even Functions

Fourier Series

$$S_f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right).$$

Odd and Even Functions

- A function f is **odd** if $f(-x) = -f(x)$.
- For any odd function, the Fourier coefficients $\{a_n\}_{n=0,1,\ldots}$ will all be 0 and the Fourier series will be a sum of just sine functions.
- A function f is **even** if $f(-x) = f(x)$.
- For any even function, the Fourier coefficients $\{b_n\}_{n=1,2,\ldots}$ will all be 0 and the Fourier series will be a sum of just cosine functions.

Example 4

2018 Q4(iii)(a,b,c)

Let f be the function defined by

$$f(x) = \begin{cases} -x+\pi, & \text{if } 0 \le x < \pi, \\ x+\pi, & \text{if } -\pi \le x < 0, \end{cases}$$

$$f(x) = f(x+2\pi) \text{ for all } x \in \mathbb{R}.$$

a) Sketch the graph $y = f(x)$ for $-3\pi \le x \le 3\pi$.

b) Find the coefficients a_0, a_k and b_k $(k \ge 1)$ in the Fourier series of f,

$$F(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} [a_k \cos(kx) + b_k \sin(kx)].$$

Example 4

2018 Q4(iii)(a,b,c)

c) Hence or otherwise, find the coefficients a_0', a_k' and b_k' $(k \geq 1)$ in the Fourier series of g,

$$g(x) = \begin{cases} -3x, & \text{if } 0 \leq x < \pi, \\ 3x, & \text{if } -\pi \leq x < 0, \end{cases}$$

$$g(x) = g(x + 2\pi) \text{ for all } x \in \mathbb{R}.$$

Example 4 Solutions

Graphs of a) & c)

Example 4 Solutions

b)

Recall that we have the following formulae for calculating coefficients.

$$a_k = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{k\pi x}{L}\right)dx, \qquad k = 0, 1, 2, \dots,$$
$$b_k = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{k\pi x}{L}\right)dx, \qquad k = 1, 2, \dots.$$

Let's first compute a_0,

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos\left(\frac{0\pi x}{L}\right)dx \\ &= \frac{1}{\pi}\left[\int_0^{\pi}(-x+\pi)dx + \int_{-\pi}^{0}(x+\pi)dx\right] \end{aligned}$$

Example 4 Solutions

b)

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Example 4 Solutions

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a_0 &= \frac{1}{\pi}\left[\left(\frac{-x^2}{2} + \pi x\right)\Big|_{x=0}^{x=\pi} + \left(\frac{x^2}{2} + \pi x\right)\Big|_{x=-\pi}^{x=0} \right] \\
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Next, let's do it for a_k for $k > 0$,

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&\vdots \quad \text{(apply integration by parts)} \\
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Example 4 Solutions

$$a_0 = \frac{1}{\pi}\left[(\frac{-x^2}{2} + \pi x)\Big|_{x=0}^{x=\pi} + (\frac{x^2}{2} + \pi x)\Big|_{x=-\pi}^{x=0}\right]$$
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Example 4 Solutions

This ultimately simplifies to,

$$a_k = \frac{4}{\pi k^2} \quad \text{for } k = 1, 3, 5, \ldots$$

As for b_k, notice we have,

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx)\, dx.$$

Our function f is an even function, whereas sin is odd. Thus, we get a symmetric integral from $-\pi$ to π of an odd function, resulting in 0. So we have, $b_k = 0$ for all k.

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Example 4 Solutions

c)
Let's graph how $g(x)$ is meant to look like and simply perform a transformation on the Fourier Series for $f(x)$.

We end up with the following Fourier Series for $g(x)$,

$$\frac{-3\pi}{2} + \sum_{\text{all odd } k>0} \frac{12}{\pi k^2} \cos(kx).$$

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Convergence of Fourier Series

Piecewise Continuity and Differentiability

- A function f is **piecewise continuous** on $[a, b]$ if
  1. For each $x \in [a, b)$, $f(x^+)$ exists;
  2. For each $x \in (a, b]$, $f(x^-)$ exists;
  3. f is continuous on (a, b) except at (at most) a finite number of points.
- A function is piecewise continuous on $\mathbb{R}$ if it is piecewise continuous on any finite interval $[a, b] \subseteq \mathbb{R}$.
- Any continuous function is piecewise continuous.
- Essentially, a piecewise continuous function can be partitioned into a finite set of continuous "pieces".

Convergence of Fourier Series

Piecewise Continuity and Differentiability

- If f is piecewise continuous on a closed and bounded interval $[a, b]$, then $\int_a^b f(x)\,dx$ exists.
- If f is piecewise continuous, then we can compute its corresponding Fourier coefficients. However, piecewise continuity does not mean the Fourier series will converge to f.
- **Piecewise differentiability** is defined similarly to piecewise continuity. Any function f is piecewise differentiable on $[a, b]$ if
  1. For each $x \in [a, b)$, $D^+(x)$ exists;
  2. For each $x \in (a, b]$, $D^-(x)$ exists;
  3. f is differentiable on (a, b) except at (at most) a finite number of points.
- $D^+ f(x)$ is not necessarily the same as $f'(c^+)$.

Convergence of Fourier Series

Pointwise Convergence of Fourier Series

Let $c \in \mathbb{R}$ and suppose that a function $f : \mathbb{R} \to \mathbb{R}$ has the following properties:

1. f is $2L$-periodic (i.e. $f(x + 2L) = f(x)$ for all x);
2. f is piecewise continuous on $[-L, L]$;
3. $D^+ f(c)$ and $D^- f(c)$ exist.

Then if f is continuous at c,

$$S_f(c) = f(c).$$

Otherwise

$$S_f(c) = \frac{f(c^+) + f(c^-)}{2}.$$

Convergence of Fourier Series

Uniform Convergence

Let $f_k : \mathbb{R} \to \mathbb{R}$. Then f_k converges uniformly to f on $[a, b]$ if for every $\epsilon > 0$, there exists a K, such that,

$$\sup_{x \in [a,b]} |f_k(x) - f(x)| \leq \epsilon \text{ for all } k \geq K.$$

Some theorems related to Uniform Convergence

If f_k is continuous for all k and if f_k converges to f uniformly, then f is continuous on $[a, b]$.

A corollary stemming from the previous theorem is that:
If f_k is continuous for all k and f has at least one discontinuity on $[a, b]$, then f_k **cannot** converge to f uniformly.

Convergence of Fourier Series

Weierstrass Test

Let $f_k : \mathbb{R} \to \mathbb{R}$ be a sequence of functions defined on $[a, b]$. Suppose that there exists a sequence of numbers c_k such that

$$|f_k(x)| \leq c_k \text{ for all } x \in [a, b]$$

and $\displaystyle\sum_{k=1}^{\infty} c_k$ converges. Then $\displaystyle\sum_{k=1}^{\infty} f_k(x)$ converges uniformly to a function f on $[a, b]$.

Example 5

2018 Q4(iii)(d)

Let f be the function defined by

$$f(x) = \begin{cases} -x+\pi, & \text{if } 0 \le x < \pi, \\ x+\pi, & \text{if } -\pi \le x < 0, \end{cases}$$

$$f(x) = f(x+2\pi) \text{ for all } x \in \mathbb{R}.$$

d) Does the Fourier series of f converge uniformly to f on $\mathbb{R}$? Give reasons.

Example 5 Solutions

d)

To show that f_k converges to f uniformly, we can use Weierstrass Test. We need to find a c_k sequence such that $\sum_{k=1}^{\infty} c_k$ converges. Recall that the Fourier Series for f is,

$$S_f(x) = \frac{\pi}{2} + \sum_{\text{all odd } k>0} \frac{4}{\pi k^2} \cos(kx).$$

Notice that the only component we are concerned with is the summation. We need to check that $\left|\frac{4}{\pi k^2} \cos(kx)\right|$ is bounded by a sequence that converges.

Consider $\left|\frac{4}{\pi k^2} \cos(kx)\right| \leq \frac{4}{k^2}$. Note that, $\sum_{k=1}^{\infty} \frac{4}{k^2}$ converges by the p-series test.

Thus, by Weierstrass Test, we have shown that f_k converges uniformly to f.

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Thus, by Weierstrass Test, we have shown that f_k converges uniformly to f.

Example 6

2015 Q4(iv)(d)

Consider the function g, defined by letting,

$$g(x) = x^2 \text{ for all } 0 \leq x < 1,$$

and requiring that,

$$g(x+2) = g(x) \text{ and } g(-x) = -g(x) \text{ for all } x \in \mathbb{R}.$$

Show that its Fourier Series, $S_g(x)$ cannot converge to g uniformly.

Example 6 Solutions

To do this question, we simply make use of one of the theorems regarding uniform convergence shown above.
As long as g has a discontinuity anywhere, then $S_g(x)$ cannot converge to g uniformly.

Let's first sketch this graph.

Clearly, it has discontinuities at every odd integer. Thus $S_g(x)$ cannot converge to g uniformly.

Example 6 Solutions

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Convergence of Fourier Series

Mean Square Convergence

f_k converges to f within $[a, b]$ in the mean square sense if the following condition holds,

$$\lim_{k\to\infty} \int_a^b [f_k(x) - f(x)]^2 dx = 0.$$

Parseval's Theorem and Identity

If a function f is $2L$-periodic and bounded and $\int_{-L}^{L} f(x)^2 \, dx < \infty$, then the Fourier series of f converges to f in the mean squared sense.
If the above holds, then

$$\int_{-L}^{L} f(x)^2 \, dx = \|f\|_2^2 = \frac{L}{2} a_0^2 + L \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

where a_0, a_n and b_n are the corresponding Fourier coefficients.

Example 7

2019 Q2(ii)

Let f be such that $f(x+2\pi) = f(x)$ and

$$f(x) = \begin{cases} 1, & \text{if } 0 \leq |x| \leq d, \\ 0, & \text{if } d < |x| \leq \pi, \end{cases}$$

where d is some constant such that $0 < d < \pi$.

b) Determine the Fourier series F for the function f.

c) Using Parseval's identity and the Fourier series for f, find

$$\sum_{k=1}^{\infty} \frac{\sin^2(kd)}{k^2}.$$

Example 7 Solutions

b) Since f is even, $b_k = 0$ for all k. Then we need to find a_k:

$$\begin{aligned}
a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx \\
&= \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos(kx)\,dx \\
&= \frac{2}{\pi}\left(\int_{0}^{d} f(x)\cos(kx)\,dx + \int_{d}^{\pi} f(x)\cos(kx)\,dx\right) \\
&= \frac{2}{\pi}\int_{0}^{d} \cos(kx)\,dx \\
&= \frac{2}{\pi}\left[\frac{1}{k}\sin(kx)\right]_0^d \quad \text{for } k > 0 \\
&= \frac{2}{\pi k}\sin(kd).
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a_0 &= \frac{2}{\pi}\int_0^d \cos(0)\,dx \\
&= \frac{2d}{\pi} \\
F(x) &= \frac{d}{\pi} + \sum_{k=1}^{\infty} \frac{2}{\pi k}\sin(kd)\cos(kx)
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$$

c) It is clear to see that the conditions for Parseval's theorem hold, so we can apply Parseval's identity.

$$
\int_{-\pi}^{\pi} f(x)^2\,dx = \frac{\pi}{2}\left(\frac{2d}{\pi}\right)^2 + \pi\sum_{k=1}^{\infty}\left(\frac{2}{\pi k}\sin(kd)\right)^2
$$

$$
\int_{-d}^{d} 1^2\,dx = \frac{\pi}{2}\cdot\frac{4d^2}{\pi^2} + \pi\cdot\frac{4}{\pi^2}\sum_{k=1}^{\infty}\frac{\sin^2(kd)}{k^2}
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$$a_0 = \frac{2}{\pi}\int_0^d \cos(0)\,dx$$
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Example 7 Solutions

$$2d = \frac{2d^2}{\pi} + \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\sin^2(kd)}{k^2}$$

$$\sum_{k=1}^{\infty}\frac{\sin^2(kd)}{k^2} = \frac{\pi}{4}\left(2d - \frac{2d^2}{\pi}\right)$$

$$= \frac{\pi d}{2} - \frac{d^2}{2}$$

6. Vector Fields

Introduction to Vector Fields

Vector Fields

- A **vector field** is just a function that takes in vectors and outputs vectors. In 3D space, we can think of a vector field as three real-valued functions in different components:

$$\mathbf{F}(\mathbf{x}) = (F_1(\mathbf{x}), F_2(\mathbf{x}), F_3(\mathbf{x})).$$

- If you want to sketch a vector field (hopefully one in 2D), focus on individual points on the 2D plane. On each point $\mathbf{x}$, draw an arrow representing the vector $\mathbf{F}(\mathbf{x})$. The arrow direction represents the direction of the vector, and the arrow length represents the magnitude.

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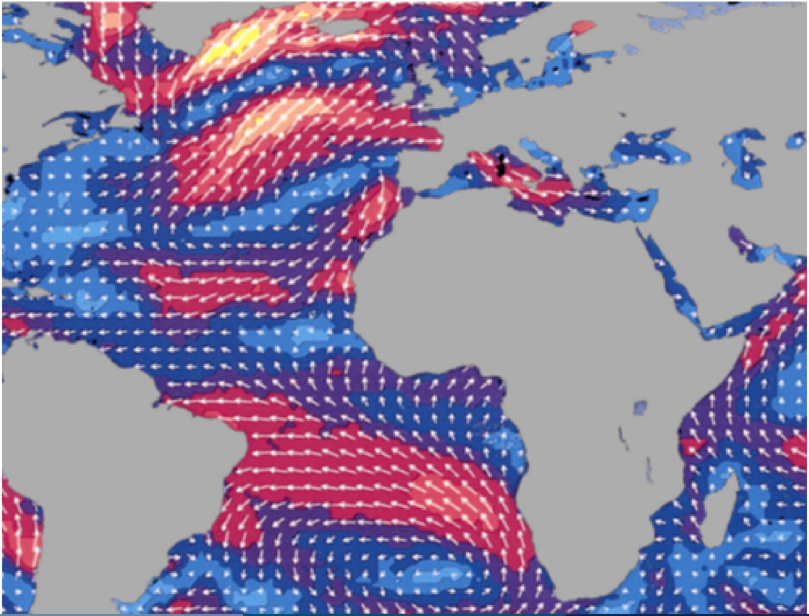
Real-life Vector Field Example



Flow Lines

Definition

A path $\mathbf{c}(t)$ is a **flow line** for a vector field $\mathbf{F}$ if

$$\mathbf{c}'(t) = \mathbf{F}(\mathbf{c}(t)).$$

- Roughly speaking, a flow line is a path whose "velocity" (speed and direction) is determined by the vector field itself.
- In relation to the previous example of ocean currents, you can think of a flow line as a path that a particle would take if it was dropped at some point in the ocean.

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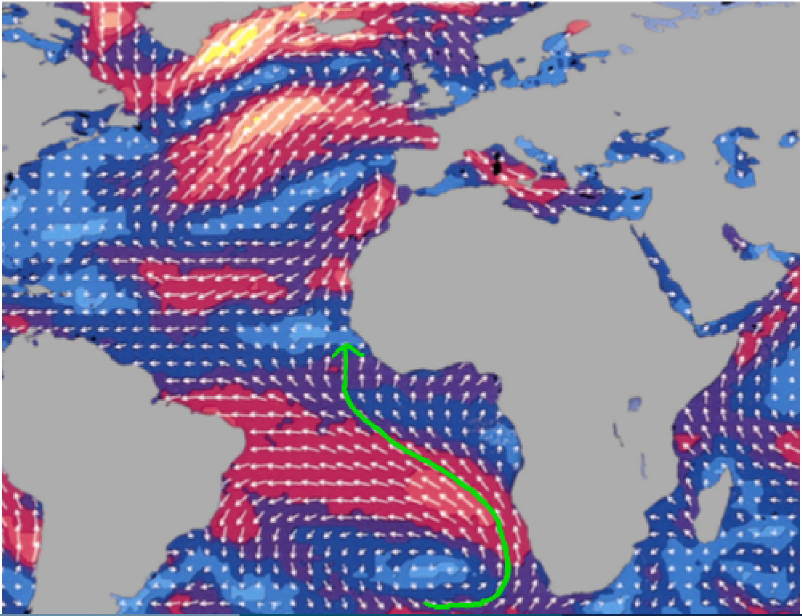

∇ and Gradient

Gradient

The gradient of a scalar function $f(x_1, x_2, \ldots, x_n)$ is just a vector of all of its partial derivatives:

$$\operatorname{grad} f = \nabla f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix}.$$

Here, the ∇ is just the operator that takes partial derivatives with respect to each input.

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Divergence

Divergence

- The divergence at a point P can be interpreted as the total outward flux at P.
- If $\mathbf{F} = (F_1, F_2, F_3)$, the **divergence** of $\mathbf{F}$ is the scalar function

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

- In the ocean currents example, the divergence at some point would be like the net current leaving that point.
- A negative divergence indicates a net inflow while a zero divergence indicates zero net flow (and the flow is said to be **incompressible**).

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Vector Fields Question

(MATH2111) S1, 2016 – Q3(i) (adapted)

Find div$\mathbf{F}$ if $\mathbf{F} = (\sin(xyz) + xy, -\log(x + y\tan^{-1}(z)), e^{xyz})$.

$$\begin{aligned}\nabla \cdot \mathbf{F} &= \frac{\partial}{\partial x}(\sin(xyz) + xy) + \frac{\partial}{\partial y}(-\log(x + y\tan^{-1}(z))) + \frac{\partial}{\partial z}e^{xyz} \\ &= yz\cos(xyz) + y - \frac{\tan^{-1}(z)}{x + y\tan^{-1}(z)} + xye^{xyz}.\end{aligned}$$

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Curl

Curl

- The curl of a vector field at a point is a measure of how the vector field swirls around that point.
- If $\mathbf{F} = (F_1, F_2, F_3)$, the **curl** of $\mathbf{F}$ is the vector field

$$\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \begin{pmatrix} \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \\ \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \end{pmatrix}$$

- If $\operatorname{curl}\mathbf{F} = \mathbf{0}$, then we say $\mathbf{F}$ is **irrotational**.
- If $\mathbf{F}$ is a 2D vector field, then we calculate $\operatorname{curl}\mathbf{F}$ by letting the third component be 0. (i.e. $F_3 = 0$.)

Curl

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- If $\mathbf{F} = (F_1, F_2, F_3)$, the **curl** of $\mathbf{F}$ is the vector field

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Vector Fields Question

(MATH2111) T1, 2021 – Q4(i)(a)

Find curl $\mathbf{F}$, given that

$$\mathbf{F}(x,y,z) = (\sin y - e^z \sin x, x\cos y + \cos z, -y\sin z + e^z \cos x.)$$

$$\begin{aligned}
\text{curl } \mathbf{F} &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \\
&= (-\sin z + \sin z,\ -e^z \sin x + e^z \sin x,\ \cos y - \cos y) \\
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7. Line Integrals

Path Integrals

Path Integrals

- A path $\mathbf{c} : [a, b] \to \mathbb{R}$ parametrises a curve $\mathcal{C}$ if $\mathbf{c}(t) = (x(t), y(t), z(t))$ traces out $\mathcal{C}$ for $a \leq t \leq b$.
- A **path integral** (or scalar line integral) is really just a regular integral, but taken over a curve instead of x, y or z.
- It is defined by

$$\int_{\mathcal{C}} f(x, y, z)\, ds = \int_a^b f(\mathbf{c}(t)) \|\mathbf{c}'(t)\|\, dt,$$

where $\mathbf{c}'$ and f are continuous, and f is the scalar field which we are integrating on.

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Path Integrals

Notes

- The value of a path integral is not affected by how we parameterise the curve.
- If $f(x, y, z) = 1$, then we get the length of $\mathcal{C}$:

$$\text{Length of } \mathcal{C} = \int_a^b \|\mathbf{c}'(t)\| \, dt.$$

- We can take out constant factors and split up additions, and even split up curves:

$$\int_{\mathcal{C}} [\lambda f_1 + f_2] \, ds = \lambda \int_{\mathcal{C}} f_1 \, ds + \int_{\mathcal{C}} f_2 \, ds$$

$$\int_{\mathcal{C}_1 + \mathcal{C}_2} f \, ds = \int_{\mathcal{C}_1} f \, ds + \int_{\mathcal{C}_2} f \, ds$$

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- A **line integral** is similar to a path integral, but we integrate over a curve in a vector field:

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt.$$

  Note that $\mathbf{c}(t)$ needs to be a continuously differentiable parametrisation of the oriented curve $\mathcal{C}$ - that is, our parametrisation must be consistent with the "direction" of $\mathcal{C}$.

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Line Integrals

Line Integrals and Work

- Line integrals are often written in the form

$$\int_{\mathcal{C}} M\,dx + N\,dy + P\,dz,$$

where M, N, P are the component functions of the vector field in question.

Given a parametrisation $\mathbf{c}(t) = (x(t), y(t), z(t))$ for $\mathcal{C}$, the above is equivalent to

$$\int_a^b [M(\mathbf{c}(t))x'(t) + N(\mathbf{c}(t))y'(t) + P(\mathbf{c}(t))z'(t)]\,dt.$$

- The **work** done by a vector field $\mathbf{F}$ along a curve $\mathcal{C}$ is just given by the line integral of $\mathbf{F}$ over $\mathcal{C}$.

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Line Integral Questions

Question 1

Let $\mathcal{C}$ be the oriented curve in $\mathbb{R}^2$ which is formed by tracing a quarter of the unit circle centered at the origin, anticlockwise from $(1, 0)$ to $(0, 1)$. Let $\mathbf{F}$ be a vector field defined by $\mathbf{F}(x, y) = (2x, 3y)$. Evaluate

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s}.$$

First, we need to parameterise the curve $\mathcal{C}$. This is easy since we are only dealing with a circle - we can use

$$\mathbf{c}(t) = (\cos t, \sin t)$$

from $t = 0$ to $\frac{\pi}{2}$.

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Next, we just apply the formula for the line integral:

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\int_{\mathcal{C}} \mathbf{F}\, d\mathbf{s} &= \int_0^{\frac{\pi}{2}} \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt \\
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&= \int_0^{\frac{\pi}{2}} \sin t \cos t\, dt \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin 2t\, dt \\
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Line Integrals

Properties of Line Integrals

Some basic properties

- Line integrals preserve linearity

$$\int_{\mathcal{C}} (\lambda \mathbf{F} + \mathbf{G}) \cdot \, d\mathbf{s} = \lambda \int_{\mathcal{C}} \mathbf{F} \, d\mathbf{s} + \int_{\mathcal{C}} \mathbf{G} \, d\mathbf{s}$$

- Reversing the orientation reverses the sign

$$\int_{-\mathcal{C}} \mathbf{F} \cdot \, d\mathbf{s} = - \int_{\mathcal{C}} \mathbf{F} \cdot \, d\mathbf{s}$$

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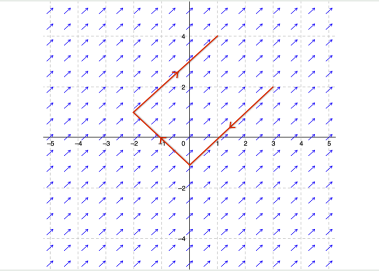
Line Integral Questions

Question

Consider the vector field $\mathbf{G}(x, y) = (1, 1)$ and the curve $\mathcal{C}$ shown in red below (it is a rectangle minus one side). Is the line integral $\int_{\mathcal{C}} \mathbf{G} \cdot d\mathbf{s}$ positive, negative or zero? Why?



Line Integral Questions

The two longer sides are traced in opposite directions - one in the direction of the vector field, and the other against it. Since the vector field is constant, these cancel out. For the shorter side, the tangential component of $\mathbf{G}$ is always $\mathbf{0}$. So overall, the line integral is zero.

Line Integrals

Fundamental Theorem of Calculus for Line Integrals

- If a vector field $\mathbf{F}$ is the gradient of some scalar field φ, then we say mF is **conservative** (and call it a **gradient field**). φ is called the **potential function** for $\mathbf{F}$.
- (Fundamental Theorem of Calculus for Line Integrals) If $\mathbf{F} = \nabla\varphi$ on a domain $\mathcal{D}$, then for every oriented curve $\mathcal{C} \in \mathcal{D}$,

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = \varphi(\mathbf{q}) - \varphi(\mathbf{p})$$

  where $\mathcal{C}$ goes from $\mathbf{p}$ to $\mathbf{q}$.
- Basically, this theorem is saying that line integrals on a gradient field only depend on the endpoints of the curve.

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- Basically, this theorem is saying that line integrals on a gradient field only depend on the endpoints of the curve.

Line Integral Questions

Question

Let $\mathbf{F}(x, y, z) = (y, x, 2z)$.

a) Evaluate the line integral

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s},$$

along the straight line segment connecting the point $A = (0, 0, 0)$ to the point $B = (1, 2, 3)$.

b) Show that the value of this integral is the same regardless of the path taken between A and B.

Line Integral Questions

a) First, we need a parametrisation of this line segment. Since $\mathcal{C}$ starts from the origin, we can just use

$$\mathbf{c}(t) = (t, 2t, 3t)$$

for $t = 0$ to 1.

Then $\mathbf{c}'(t) = (1, 2, 3)$, and

$$\begin{aligned}\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} &= \int_0^1 \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt \\ &= \int_0^1 \mathbf{F}(t, 2t, 3t) \cdot (1, 2, 3)\, dt \\ &= \int_0^1 (2t, t, 6t) \cdot (1, 2, 3)\, dt \\ &= \int_0^1 22t\, dt = \left[11t^2\right]_0^1 = 11.\end{aligned}$$

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Line Integral Questions

b) To show that the value of the integral is the same regardless of the path taken between A and B, we need to show that $\mathbf{F}$ is the gradient of some scalar field φ.

Line Integral Questions

Let us just suppose that $\mathbf{F} = \nabla\varphi$ for some real-valued function φ. Then

$$\frac{\partial\varphi}{\partial x} = y, \quad \frac{\partial\varphi}{\partial y} = x, \quad \frac{\partial\varphi}{\partial z} = 2z.$$

The idea is to reconstruct φ using its partial derivatives.

From $\dfrac{\partial\varphi}{\partial x} = y$, we see that

$$\varphi(x,y,z) = xy + f(y,z), \tag{1}$$

where f is a function in only y and z. We need f here because taking partial derivatives with respect to x can "hide" terms not involving x.

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Next, we use the fact that $\dfrac{\partial \varphi}{\partial y} = x$, by differentiating (1) with respect to y:

$$\frac{\partial \varphi}{\partial y} = x + \frac{\partial f}{\partial y},$$

and so we can deduce that

$$\frac{\partial f}{\partial y} = 0,$$

implying that

$$f(y, z) = g(z)$$

for some function g only in z. Thus, substituting this back to (1) gives

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Finally, we use the fact that $\dfrac{\partial \varphi}{\partial z} = 2z$, by differentiating (2) with respect to z:

$$\frac{\partial \varphi}{\partial z} = g'(z).$$

So

$$g'(z) = 2z \implies g(z) = z^2 + C,$$

where C is a constant. Note that we just took an ordinary derivative for g since it is a function in only one variable. Putting it all together, we have

$$\varphi(x, y, z) = xy + z^2 + C.$$

Thus, we conclude that $\mathbf{F}$ is the gradient of a real-valued function, φ, and so $\mathbf{F}$ is conservative. Therefore, to answer the question, the path integral will be the same regardless of the path taken from A to B.

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Green's Theorem

Green's theorem is very useful for finding line integrals on vector fields with complicated components, which have much simpler partial derivatives.

Suppose

- $\mathcal{D}$ is a bounded, simple region in $\mathbb{R}^2$ and $\mathcal{C}$ is its boundary (oriented in the positive (anticlockwise) direction) and
- $M(x, y)$ and $N(x, y)$ are continuously differentiable on $\mathcal{D}$.

Then

$$\oint_{\mathcal{C}} (M\,dx + N\,dy) = \iint_{\mathcal{D}} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy.$$

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Let $\mathcal{C}$ be the circle $x^2 + y^2 = 4$ oriented in a counterclockwise direction. Use Green's theorem to evaluate

$$\int_{\mathcal{C}} (y + e^{x^2 \sin 3x})\, dx + (2x + \sin(y^2 + \cos y))\, dy.$$

This is where Green's theorem is so good - the component functions $M(x, y) = y + e^{x^2 \sin 3x}$ and $N(x, y) = 2x + \sin(y^2 + \cos y)$ are very complicated. But the partial derivatives are so much simpler:

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2 - 1 = 1.$$

Of course, we should check that the conditions of Green's theorem are satisfied. Clearly $\mathcal{C}$ is the anticlockwise-oriented boundary of a bounded, simple region in $\mathbb{R}^2$, and the vector field components are continuously differentiable on this region (in fact, on all of $\mathbb{R}^2$).

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So by Green's theorem,

$$\int_{\mathcal{C}} M\, dx + N\, dy = \iint_D 1\, dx\, dy = \pi(2)^2 = 4\pi.$$

8. Surface Integrals

Surface Integrals

Surface Integrals

- Similar to how a path integral is an integral of a scalar/vector field over a curve, a **surface integral** is an integral of a scalar/vector field over a surface.
- Computing surface integrals requires parametrising a surface $\mathcal{S}$ with a function $\boldsymbol{\phi} : \mathbb{R}^2 \to \mathbb{R}^3$, where $\boldsymbol{\phi}$ "maps" out $\mathcal{S}$ on some domain $D \subseteq \mathbb{R}^2$.

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- If $\boldsymbol{\phi}(u, v)$ is a parametrisation that maps out a smooth surface $\mathcal{S}$ over a domain D, then surface integral of a scalar function f over $\mathcal{S}$ is

$$\iint_{\mathcal{S}} f(x, y, z)\, dS = \iint_{D} f(\boldsymbol{\phi}(u, v)) \|\mathbf{T_u} \times \mathbf{T_v}\| \, du \, dv,$$

where

$$\mathbf{T_u} = \frac{\partial \boldsymbol{\phi}}{\partial u} \quad \text{and} \quad \mathbf{T_v} = \frac{\partial \boldsymbol{\phi}}{\partial v}.$$

- Since $\mathbf{T_u} \times \mathbf{T_v}$ is a vector which is normal to the surface at a given point (u, v), for convenience we sometimes use the notation

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Surface Integrals

Notes on Surface Integrals

- The surface area of $\mathcal{S}$ is the surface integral of $f(x, y, z) = 1$ over $\mathcal{S}$:

$$\text{Area}(\mathcal{S}) = \iint_D \|\mathbf{T_u} \times \mathbf{T_v}\| \, du \, dv.$$

- A surface $\mathcal{S}$ is **smooth** if we can parameterise it using $\boldsymbol{\phi}(u, v)$ for $(u, v) \in D$, where
  - D is an elementary region in $\mathbb{R}^2$,
  - $\boldsymbol{\phi}$ is continuously differentiable and one-to-one except possibly on ∂D and
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Surface Integral Questions

Question

Consider a surface S given by

$$z = x^2 + y^2 \quad \text{for} \quad x^2 + y^2 \leq 1.$$

Find the area of the surface S.

We can parameterise $\mathcal{S}$ with $\boldsymbol{\phi}(x, y) = (x, y, x^2 + y^2)$ for $(x, y) \in D$, where $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$.
Then

$$\mathbf{T_x} \times \mathbf{T_y} = \begin{pmatrix} 1 \\ 0 \\ 2x \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 2y \end{pmatrix} = \begin{pmatrix} -2x \\ -2y \\ 1 \end{pmatrix}.$$

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We can parameterise $\mathcal{S}$ with $\boldsymbol{\phi}(x, y) = (x, y, x^2 + y^2)$ for $(x, y) \in D$, where $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$.

Then

$$\mathbf{T_x} \times \mathbf{T_y} = \begin{pmatrix} 1 \\ 0 \\ 2x \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 2y \end{pmatrix} = \begin{pmatrix} -2x \\ -2y \\ 1 \end{pmatrix}.$$

Surface Integral Questions

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Consider a surface S given by

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Surface Integral Questions

Thus,

$$\|\mathbf{T_x} \times \mathbf{T_y}\| = \sqrt{4x^2 + 4y^2 + 1},$$

and so

$$\text{Area}(\mathcal{S}) = \iint_D \sqrt{4x^2 + 4y^2 + 1}\, dx\, dy.$$

To compute this, we need to use polar coordinates (i.e. $x = r\cos\theta, y = r\sin\theta$). The bounds are $0 \le r \le 1$ and $-\pi < \theta \le \pi$, and the determinant of the Jacobian is r. So

$$\begin{aligned}\text{Area}(\mathcal{S}) &= \int_{-\pi}^{\pi}\int_0^1 r\sqrt{4(r\cos\theta)^2 + 4(r\sin\theta)^2 + 1}\, dr\, d\theta \\ &= 2\pi\int_0^1 r\sqrt{4r^2+1}\, dr = \frac{\pi}{6}(5\sqrt{5} - 1).\end{aligned}$$

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Surface Integrals

Surface Integrals on Vector Fields

We can also take surface integrals on vector fields, not just scalar fields.

If $\boldsymbol{\phi}(u, v)$ is a parametrisation of an oriented, smooth surface $\mathcal{S}$ on domain $\mathcal{D}$, the surface integral of a vector field F over $\mathcal{S}$ is

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \iint_{D} \mathbf{F}(\boldsymbol{\phi}(u, v)) \cdot \mathbf{n}(u, v)\, du\, dv.$$

That is, the surface integral is just the integral of the component of $\mathbf{F}$ that is normal to the surface.

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Stokes' Theorem

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Stokes' theorem basically allows us to compute line integrals over closed curves by using surface integrals. If

- $\mathcal{S}$ is a a smooth oriented surface,
- $\partial\mathcal{S}$ is its boundary oriented in the anticlockwise direction when looking down on $\mathcal{S}$ from the positive direction and
- $\mathbf{F}$ is a continuously differentiable vector field on $\mathcal{S}$,

then

$$\iint_{\mathcal{S}} (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_{\partial\mathcal{S}} \mathbf{F} \cdot d\mathbf{s}.$$

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Surface Integral Questions

Question

Let $\mathcal{S}$ be the surface $x + y + z = 0$ for $0 \le x \le 1$ and $0 \le y \le 1$, oriented so that the z-component of the unit normal is **positive**, and let $\mathbf{F}(x, y, z) = (xyz, x^2 + yz, xy + z^2)$. Use Stoke's theorem to determine the line integral

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By Stoke's theorem,

$$\oint_{\partial \mathcal{S}} \mathbf{F} \cdot d\mathbf{s} = \iint_{\mathcal{S}} \nabla \times \mathbf{F} \cdot d\mathbf{S}.$$

First we find the curl of $\mathbf{F}$:

$$\nabla \times \mathbf{F} = \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \times \begin{pmatrix} xyz \\ x^2 + yz \\ xy + z^2 \end{pmatrix} = \begin{pmatrix} x - y \\ xy - y \\ 2x - xz \end{pmatrix}.$$

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Surface Integral Questions

Next, we need to parametrise $\mathcal{S}$. We can use $\boldsymbol{\phi}(x, y) = (x, y, -(x+y))$ for $0 \le x \le 1$ and $0 \le y \le 1$. Then,

$$\mathbf{n}(x, y) = \mathbf{T_x} \times \mathbf{T_y} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Thus,

$$\begin{aligned} \iint_{\mathcal{S}} \nabla \times \mathbf{F} \cdot d\mathbf{S} &= \iint_D (x - y, xy - y, 2x + x(x+y)) \cdot (1, 1, 1)\, dx\, dy \\ &= \iint_D (3x - 2y + 2xy + x^2)\, dx\, dy, \end{aligned}$$

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Surface Integral Questions

Continuing,

$$\iint_D (3x - 2y + 2xy + x^2)\, dx\, dy = \int_0^1 \int_0^1 (3x - 2y + 2xy + x^2)\, dx\, dy$$

$$= \int_0^1 \left[\frac{3}{2}x^2 - 2xy + x^2 y + \frac{1}{3}x^3 \right]_{x=0}^{x=1} dy$$

$$= \int_0^1 \left(\frac{3}{2} - y + \frac{1}{3} \right) dy$$

$$= \left[\frac{11}{6}y - \frac{1}{2}y^2 \right]_{y=0}^{y=1}$$

$$= \frac{11}{6} - \frac{1}{2}$$

$$= \frac{4}{3}.$$

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Divergence Theorem

Gauss' Divergence Theorem

Gauss' Divergence Theorem basically allows you to compute surface integrals over "closed" surfaces using a triple integral over the 3D solid bounded by this surface. More formally, if

- the region $W \subseteq \mathbb{R}^3$ is a bounded, solid and simple region,
- $\mathcal{S}$ is its piece-wise smooth boundary, oriented such that the normal vector points outwards, and
- $\mathbf{F}$ is a C^1 vector field on W,

then

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \iiint_W \nabla \cdot \mathbf{F} \, dV.$$

This is really useful when dealing with complicated vector fields which have a much simpler expression for its divergence.

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Surface Integral Questions

Question

Let W be the unit disk in $\mathbb{R}^3$ (i.e. the unit sphere plus its entire interior), and let $\mathcal{S}$ be its surface, oriented so that the normal vector points outwards. Let $\mathbf{F}$ be a vector field defined by

$$\mathbf{F}(x, y, z) = (y \cos z^2 + 2x, y - xz^2 \sin(z), y^3 e^{x^2} + z).$$

Use Gauss' Divergence Theorem to calculate

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}.$$

Surface Integral Questions

First check the conditions of Gauss' divergence theorem. Indeed, W is bounded, solid and simple, $\mathcal{S}$ is piece-wise smooth and oriented so that the normal vector points outwards, and $\mathbf{F}$ is C^1 on W (in fact, on all of $\mathbb{R}^3$).

So, we can use Gauss' divergence theorem. To do this, we calculate the divergence first:

$$\begin{aligned}\nabla \cdot \mathbf{F} &= \frac{\partial}{\partial x}(y\cos(z^2) + 2x) + \frac{\partial}{\partial y}(y - xz^2\sin(z)) + \frac{\partial}{\partial z}(y^3 e^{x^2} + z)\\ &= 2 + 1 + 1\\ &= 4.\end{aligned}$$

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Surface Integral Questions

Thus, by Gauss' divergence theorem,

$$\begin{aligned}\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} &= \iiint_W \nabla \cdot \mathbf{F}\, dV \\ &= 4 \iiint_W dV \\ &= 4 \cdot \frac{4\pi}{3} \\ &= \frac{16\pi}{3}.\end{aligned}$$

Note that $\iiint_W dV$ is just the volume of W.

Surface Integral Questions

Thus, by Gauss' divergence theorem,

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Note that $\iiint_W \, dV$ is just the volume of W.

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Surface Integral Questions

Thus, by Gauss' divergence theorem,

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&= 4 \iiint_W \, dV \\
&= 4 \cdot \frac{4\pi}{3} \\
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\end{aligned}$$

Note that $\iiint_W \, dV$ is just the volume of W.