UNSW Mathematics Society Presents

MATH2089 Workshop

Presented by Wendy Ji and Gerald Huang

Overview I

1. Probability Theory
2. Random Sampling and CLT
3. Confidence intervals
4. Hypothesis Testing
5. Analysis of Variance (ANOVA)
6. Curve Fitting
7. Numerical Differentiation
8. Numerical Integration
9. ODEs: Initial-Value Problems
10. ODEs: Boundary Value Problems
11. Parabolic Equations

1. Probability Theory

(2018, Semester 2) Probability Theory

Let X follow a Bernoulli distribution:

$$p(x) = \begin{cases} 1 - \pi & \text{if } x = 0 \\ \pi & \text{if } x = 1 \end{cases}$$

where $0 < \pi < 1$.

(i) Show that $\mathbb{E}(X) = \pi$.

(ii) Show that $\mathbb{V}\text{ar}(X) = \pi(1 - \pi)$.

(2018, Semester 2) Probability Theory

(i)

$$\mathbb{E}(X) = \sum_{x \in X} x \cdot p(x)$$

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(2018, Semester 2) Probability Theory

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(2018, Semester 2) Probability Theory

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So we have
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(2018, Semester 2) Probability Theory

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(Example Paper) Probability Theory

Let X be a random variable whose probability density function is

$$f(x) = \begin{cases} \frac{1}{\mu} e^{-x/\mu} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases},$$

where $\mu > 0$ is a parameter. It can be shown that $\mathbb{E}(X) = \mu$.

(i) Use integration to show that $\mathbb{E}\left(X^2\right) = 2\mu^2$.

(ii) Hence show that it is not generally true that

$$\mathbb{E}(g(X)) = g\left[\mathbb{E}(X)\right].$$

(i) Note that X is a continuous function, so we have

$$\begin{aligned}
\mathbb{E}\left(X^2\right) &= \int_{-\infty}^{\infty} x^2 f(x)\,dx \\
&= \underbrace{\int_{-\infty}^{0} x^2 f(x)\,dx}_{=0} + \int_{0}^{\infty} x^2 f(x)\,dx \\
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\end{aligned}$$

We shall integrate by parts. Here, we will differentiate x^2 and integrate $e^{-x/\mu}$.

Then we have

$$\begin{aligned} u &= x^2, & u' &= 2x \\ v &= -\mu e^{-x/\mu}, & v' &= e^{-x/\mu} \end{aligned}$$

(Example Paper) Probability Theory

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Our integral becomes

$$\begin{aligned} \int_0^\infty x^2 f(x)\,dx &= -\mu x^2 e^{-x/\mu}\Big|_0^\infty + 2\mu \int_0^\infty x e^{-x/\mu}\,dx \\ &= 0 + 2\mu \int_0^\infty x e^{-x/\mu}\,dx. \end{aligned}$$

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So we have

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So we have

$$\mathbb{E}\left(X^2\right) = \frac{1}{\mu}\int_0^\infty x^2 f(x)\,dx = 2\int_0^\infty x e^{-x/\mu}\,dx. \qquad (*)$$

Notice that the integral in (*) looks *suspiciously* similar to $E(X)$. It's simply missing a factor of $1/\mu$!

So we have

$$\begin{aligned}
\mathbb{E}\left(X^2\right) &= 2\mu \underbrace{\int_0^\infty x\left(\frac{1}{\mu}e^{-x/\mu}\right)\,dx}_{=\mathbb{E}(X)} \\
&= 2\mu\mathbb{E}(X) \\
&= 2\mu\cdot\mu && (\mathbb{E}(X)=\mu) \\
&= 2\mu^2.
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But we see that

$$g\left[\mathbb{E}(X)\right] = \left[\mathbb{E}(X)\right]^2 = \mu^2 \neq 2\mu^2.$$

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But we see that

$$g\left[\mathbb{E}(X)\right] = \left[\mathbb{E}(X)\right]^2 = \mu^2 \neq 2\mu^2.$$

Hence, it is not true in general that $\mathbb{E}\left[g(X)\right] = g\left[\mathbb{E}(X)\right]$.

(2010, Semester 1) Probability Theory

Let X and Y be independent random variables with $X \sim \text{Poisson}(2)$ and $Y \sim \text{Exp}(1)$.

(i) Calculate $\mathbb{P}(X = 1)$.

(ii) Calculate $\mathbb{P}(Y < 1)$.

(iii) Calculate $\mathbb{P}(X = 1 \text{ or } Y < 1)$

(i) Note that the Poisson distribution is **discrete**. So we find that the Poisson distribution with $\lambda = 2$ has probability mass

$$\mathbb{P}(X = x) = \frac{2^x \cdot e^{-2}}{x!}.$$

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(ii) Note that the Exponential distribution is **continuous**. So we find that the Exponential distribution with $\lambda = 1$ has probability density

$$f_X(x) = \begin{cases} e^{-x} & x \geq 0 \\ 0 & x < 0 \end{cases}.$$

Since the Exponential distribution is continuous, then we have

$$\mathbb{P}(Y < 1) = \int_0^1 f_Y(y)\,dy = \int_0^1 e^{-y}\,dy = 1 - e^{-1}.$$

(2010, Semester 1) Probability Theory

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(iii) We can apply the inclusion/exclusion principle since some may overlap.

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$$\mathbb{P}(X = 1 \text{ or } Y < 1) = \mathbb{P}(X = 1) + \mathbb{P}(Y < 1) - \mathbb{P}(X = 1 \text{ and } Y < 1).$$

(2010, Semester 1) Probability Theory

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$$\mathbb{P}(X = 1 \text{ or } Y < 1) = \mathbb{P}(X = 1) + \mathbb{P}(Y < 1) - \mathbb{P}(X = 1 \text{ and } Y < 1).$$

But since X and Y are independent, we have

$$\mathbb{P}(X = 1 \text{ and } Y < 1) = \mathbb{P}(X = 1)\mathbb{P}(Y < 1).$$

Putting everything together, we have

$$\begin{aligned}\mathbb{P}(X = 1 \text{ or } Y < 1) &= \mathbb{P}(X = 1) + \mathbb{P}(Y < 1) - \mathbb{P}(X = 1)\mathbb{P}(Y < 1) \\ &= 2e^{-2} + \left(1 - e^{-1}\right) - 2e^{-2}\left(1 - e^{-1}\right) \\ &= 2e^{-2} + \left(1 - e^{-1}\right) - 2e^{-2} + 2e^{-3} \\ &= 1 - e^{-1} + 2e^{-3} \approx 0.732\ldots.\end{aligned}$$

(2009, Semester 1) Probability Theory

Suppose that X and Y are independent **standard normal** variables:

$$X \sim N(0,1), \quad Y \sim N(0,1).$$

(i) What is the distribution of $X+Y$?

(ii) Calculate $\mathbb{P}(X+Y<1)$.

(i) Note that the distribution of a sum of normal variables is another normal variable. So $X + Y$ is another normal distribution.

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$$A + B \sim N(\mu_A + \mu_B, \sigma_A^2 + \sigma_B^2).$$

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$$A + B \sim N(\mu_A + \mu_B, \sigma_A^2 + \sigma_B^2).$$

So we see that

$$X + Y \sim N(0 + 0, 1 + 1) = N(0, 2).$$

(2009, Semester 1) Probability Theory

(ii) Note that $X + Y$ is just another random variable. For ease, let's just call it $W = X + Y$. Then this is the same as finding

$$\mathbb{P}(W < 1),$$

where $W \sim N(0, 2)$. To find this, we can standardise our random variable to find

$$\mathbb{P}\left(Z < \frac{W - \mu}{\sigma}\right),$$

where $Z \sim N(0, 1)$. This is equivalent to finding

$$\mathbb{P}\left(Z < \frac{W - 0}{\sqrt{2}} < \frac{1 - 0}{\sqrt{2}}\right) = \mathbb{P}\left(Z < \frac{1}{\sqrt{2}}\right) = 0.77935.$$

(MATH2801 – 2018, Semester 1) Probability Theory

Let A and B be two events in some sample space, with $\mathbb{P}(A) > 0$ and $\mathbb{P}(B) > 0$.

(i) Show that, if A and B are independent, they cannot be mutually exclusive.

(ii) Show that, if A and B are mutually exclusive, they cannot be independent.

(iii) Suppose now that A and B are mutually exclusive. Show that

$$\mathbb{P}(A \mid A \text{ or } B) = \frac{\mathbb{P}(A)}{\mathbb{P}(A) + \mathbb{P}(B)}.$$

(MATH2801 – 2018, Semester 1) Probability Theory

(i) Let A and B be independent events and suppose that they are mutually exclusive. This means that A and B do not share any common elements. In other words,

$$\mathbb{P}(A \text{ and } B) = 0.$$

But because A and B are independent, then

$$\mathbb{P}(A \text{ and } B) = \mathbb{P}(A)\mathbb{P}(B).$$

So we have

$$\mathbb{P}(A)\mathbb{P}(B) = \mathbb{P}(A \text{ and } B) = 0 \implies \mathbb{P}(A)\mathbb{P}(B) = 0.$$

So either $\mathbb{P}(A) = 0$ or $\mathbb{P}(B) = 0$. But we stated that $\mathbb{P}(A) > 0$ and $\mathbb{P}(B) > 0$, so we hit a contradiction. Thus, A and B cannot be mutually exclusive.

(MATH2801 – 2018, Semester 1) Probability Theory

(ii) Now let A and B be mutually exclusive events. This means that

$$\mathbb{P}(A \text{ and } B) = 0.$$

But if A and B were independent, then

$$\mathbb{P}(A)\mathbb{P}(B) = \mathbb{P}(A \text{ and } B) = 0 \implies \mathbb{P}(A)\mathbb{P}(B) = 0.$$

Using a similar argument to (i), this means that A and B cannot be independent.

(iii) Finally, using the conditional probability formula,

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \text{ and } B)}{\mathbb{P}(B)},$$

we have

$$\mathbb{P}(A \mid A \text{ or } B) = \frac{\mathbb{P}\left[A \text{ or } \ (A \text{ and } B)\right]}{\mathbb{P}(A \text{ or } B)} = \frac{\mathbb{P}(A)}{\mathbb{P}(A \text{ or } B)}.$$

But, because A and B are mutually exclusive, then

$$\mathbb{P}(A \text{ or } B) = \mathbb{P}(A) + \mathbb{P}(B).$$

That is,

$$\mathbb{P}(A \mid A \text{ or } B) = \frac{\mathbb{P}(A)}{\mathbb{P}(A) + \mathbb{P}(B)}.$$

(MATH2901 – 2018 Midsession) Probability Theory

Let X and Y be continuous random variables with joint density function

$$f_{X,Y}(x,y) = \frac{1}{8}(x+y) \quad \text{for } 0 < x < 2 \text{ and } 0 < y < 2.$$

Determine the marginal density $f_Y(y)$.

(MATH2901 – 2018 Midsession) Probability Theory

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Determine the marginal density $f_Y(y)$.

The marginal density is given by

$$\begin{aligned} f_Y(y) &= \int_{x \in X} f_{X,Y}(x,y)\, dx \\ &= \frac{1}{8}\int_0^2 (x+y)\, dx \\ &= \frac{1}{8}\left(\frac{1}{2}x^2 + xy\right)\Big|_0^2 = \frac{1}{4}\left(y+1\right). \end{aligned}$$

2. Random Sampling and CLT

(2018, Semester 2) Random Samples

Assume we have a random sample $\{X_1, \ldots, X_n\}$ of size n from X. The probability π can be estimated from this sample as

$$\hat{p} = \frac{1}{n} \sum_{i=1}^{n} X_i.$$

Show that the standard error of $\hat{p}$ is

$$\sqrt{\frac{\pi(1-\pi)}{n}}.$$

You can assume that each X follows a Bernoulli distribution as given in slide X.

(2018, Semester 2) Random Samples

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Some facts about random samples:

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(2018, Semester 2) Random Samples

Some facts about random samples:

- Each X_i is independent from every other event.
- Each X_i is identically distributed; they have the same mean and variance.

(2018, Semester 2) Random Samples

We use the additional facts that, since X_i are independent, then

$$\mathbb{V}\mathrm{ar}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathbb{V}\mathrm{ar}\left(X_i\right)$$

and

$$\mathbb{V}\mathrm{ar}\left(aX + b\right) = a^2 \mathbb{V}\mathrm{ar}(X).$$

Then

$$\mathbb{V}\mathrm{ar}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\mathbb{V}\mathrm{ar}\left(\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n} \mathbb{V}\mathrm{ar}(X_i).$$

(2018, Semester 2) Random Samples

Putting all of this together, we have

$$\begin{aligned}
\mathbb{V}\mathrm{ar}(\hat{p}) &= \mathbb{V}\mathrm{ar}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) \\
&= \frac{1}{n^2}\sum_{i=1}^{n} \mathbb{V}\mathrm{ar}(X_i) \\
&= \frac{n}{n^2} \cdot \mathbb{V}\mathrm{ar}(X) && (X_i\text{'s are identically distributed}) \\
&= \frac{\pi(1-\pi)}{n}.
\end{aligned}$$

The standard error is given by approximating the standard deviation with the standard deviation of the random sample supplied. So we see that

$$\mathrm{SE} = \sigma_{\hat{p}} = \sqrt{\mathbb{V}\mathrm{ar}(\hat{p})} = \sqrt{\frac{\pi(1-\pi)}{n}},$$

as required.

3. Confidence intervals

(2017, Semester 1) Confidence intervals

Wildlife biologists inspect 153 deers taken by hunters and find 32 of them carrying ticks that test positive for Lyme disease.
Create a two-sided 96% confidence interval for the percentage of deer that may carry such ticks.

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Step 1. Figure out whether to use the normal distribution or the t-distribution.

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Create a two-sided 96% confidence interval for the percentage of deer that may carry such ticks.

Step 1. Figure out whether to use the normal distribution or the t-distribution.
Criteria:

Is the number of samples > 30? If no, use t-distribution.

(2017, Semester 1) Confidence intervals

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Create a two-sided 96% confidence interval for the percentage of deer that may carry such ticks.

Step 2. Find the significance level. Since this test is two-sided, then 4% is shared on both extreme sides of the distribution. So the significance level is

$$\alpha = \frac{1 - 0.96}{2} = 0.02.$$

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So we consider the z-score of $1 - \alpha = 1 - 0.02 = 0.98$.

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$$z_{0.98} = \texttt{norminv}(0.98) = 2.054.$$

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Create a two-sided 96% confidence interval for the percentage of deer that may carry such ticks.

Step 3. Find the standard error. Recall that

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Here, n is the number of deers and $\hat{\pi}$ is the percentage of deers carrying ticks. So

$$n = 153,\ \hat{\pi} = \frac{32}{153} \approx 0.209.$$

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$$n = 153,\ \hat{\pi} = \frac{32}{153} \approx 0.209.$$

So,

$$\mathrm{SE} = \sqrt{\frac{0.209(1-0.209)}{153}}.$$

(2017, Semester 1) Confidence intervals

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Create a two-sided 96% confidence interval for the percentage of deer that may carry such ticks.

Step 4. The two-sided confidence interval becomes

$$\begin{aligned}\mathrm{CI} = (\hat{\pi} - z_{0.98} \times \mathrm{SE}, \hat{\pi} + z_{0.98} \times \mathrm{SE}) &= (0.209 - 0.068, 0.209 + 0.068) \\ &= (0.141, 0.277).\end{aligned}$$

(2018, Semester 1) Confidence intervals

In August this year, Roy Morgan Research published a poll on Rugby viewership of New Zealanders. The poll, of 6,422 randomly selected New Zealanders, found that 43.6% of them watch Rugby on the television.

Find a 95% confidence interval for the true proportion of New Zealanders who watch Rugby on the television.

(2018, Semester 1) Confidence intervals

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Find a 95% confidence interval for the true proportion of New Zealanders who watch Rugby on the television.

We find the significance level. Since it is a proportion, we are taking a two-sided confidence interval. So 5% is shared on both extreme ends of the distribution.

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$$\alpha = \frac{1 - 0.95}{2} = 0.025.$$

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So we are finding the z-score of $1 - \alpha = 1 - 0.025 = 0.975$.

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We find the significance level. Since it is a proportion, we are taking a two-sided confidence interval. So 5% is shared on both extreme ends of the distribution. This gives us

$$\alpha = \frac{1 - 0.95}{2} = 0.025.$$

So we are finding the z-score of $1 - \alpha = 1 - 0.025 = 0.975$. As per the MATLAB output, we have

$$z_{0.975} = \texttt{norminv}(0.975) \approx 1.9600.$$

(2018, Semester 1) Confidence intervals

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The next step is to find the standard error.

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The next step is to find the standard error. As per the formula, we have

$$\mathrm{SE} = \frac{\sigma}{\sqrt{n}} = \sqrt{\frac{p(1-p)}{n}}.$$

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$$\mathrm{SE} = \frac{\sigma}{\sqrt{n}} = \sqrt{\frac{p(1-p)}{n}}.$$

Here, our sample size is $6,422$ with the proportion of people watching Rugby being 0.436.

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Find a 95% confidence interval for the true proportion of New Zealanders who watch Rugby on the television.

The next step is to find the standard error. As per the formula, we have

$$\mathrm{SE} = \frac{\sigma}{\sqrt{n}} = \sqrt{\frac{p(1-p)}{n}}.$$

Here, our sample size is $6,422$ with the proportion of people watching Rugby being 0.436. So we have

$$p = 0.436,\ n = 6422.$$

(2018, Semester 1) Confidence intervals

In August this year, Roy Morgan Research published a poll on Rugby viewership of New Zealanders. The poll, of 6,422 randomly selected New Zealanders, found that 43.6% of them watch Rugby on the television.
Find a 95% confidence interval for the true proportion of New Zealanders who watch Rugby on the television.

This gives us

$$\mathrm{SE} = \sqrt{\frac{0.436 \times 0.564}{6422}} \approx 0.006187962.$$

(2018, Semester 1) Confidence intervals

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Find a 95% confidence interval for the true proportion of New Zealanders who watch Rugby on the television.

This gives us

$$\mathrm{SE} = \sqrt{\frac{0.436 \times 0.564}{6422}} \approx 0.006187962.$$

Our confidence interval is simply

$$\mathrm{CI} = 0.436 \pm z_{0.975} \times \mathrm{SE} = (0.4238715, 0.4481284).$$

4. Hypothesis Testing

(2017, Semester 2) Hypothesis Testing

The final step in the manufacture of graphite electrodes is graphitizing in an electric furnace, which reduces the resistance of the electrode so that it will not burn up in use. The graphitizing process is a slow one and the electrodes must remain in the electric furnace for several weeks before the graphitizing is completed. In order to increase production and meet all the customers' demands, the production manager in a large producer of these electrodes carried out a study to determine ho the resistance of an electrode varies with the length of time it spends in the graphitizing furnace. 15 specimens were tested.
Assume the predictor X is Time and the response variable Y is Resistance. The linear regression model is given by

$$Y = \beta_0 + \beta_1 X + \epsilon.$$

Perform a hypothesis test to determine whether the variable X is significant in this model, at the 5% level of significance.

(2017, Semester 2) Hypothesis Testing

Step 1. Write out the hypotheses.

(2017, Semester 2) Hypothesis Testing

Step 1. Write out the hypotheses. Here, we want to determine whether X is significant. So

$$H_0 : \beta_1 = 0, \quad H_a : \beta_1 \neq 0.$$

Step 2. Determine the test statistic. Since only 15 specimens were tested, we use the t-test. To use the t-test, we need to find the degree of freedom. Since β_1 and β_0 are required, then the degree of freedom is $n-2$ where n is the number of samples. So the degree of freedom is $15-2=13$. Next, we determine t-statistic probability region. Since it is a two-sided test, we have

$$1-\frac{\alpha}{2}=1-0.025=0.975.$$

Our t-statistic is $t_{13,0.975}=\texttt{tinv}(0.975,13)=2.160$.

Step 3. Determine the rejection region. We reject the null hypothesis if

$$\hat{\beta}_1 \notin \left[-t_{13,0.975}\frac{s}{\sqrt{s_{xx}}}, t_{13,0.975}\frac{s}{\sqrt{s_{xx}}}\right].$$

We can use the observed t-statistic to see that

$$t_{\text{observed}} = \frac{\text{Estimate}}{\text{SE}} = -14.384;$$

Note this will be given to you as a MATLAB output as below:

Estimated Coefficients:

	Estimate	SE	tStat	pValue
(Intercept)	5.5931	0.22314	25.066	2.164e-12
x	-0.16506	0.011475	-14.384	2.3125e-09

(2017, Semester 2) Hypothesis Testing

Using the observed t-statistic, this is the same as rejecting the null hypothesis if

$$t_{\text{observed}} > t_{13,0.975} \text{ or } t_{\text{observed}} < -t_{13,0.975}.$$

That is, reject the null hypothesis if the observed value falls outside of the range $[-2.160, 2.160]$.

(2017, Semester 2) Hypothesis Testing

Step 4. Write out the conclusion. We reject H_0 since there is strong evidence that there is a correlation between Time and Resistance.

(2018, Semester 2) Hypothesis Testing

In August this year, Roy Morgan Research published a poll on Rugby viewership of New Zealanders. The poll, of 6,422 randomly selected New Zealanders, found that 43.6% of them watch Rugby on the television.

Assume Rugby New Zealand (the organising body for the sport) want to be able to demonstrate that Rugby viewership is in excess of 40% of New Zealanders, using a sample of size n.

(i) What are the appropriate null and alternative hypotheses?

(ii) What is the distribution of the sample proportion $\hat{p}$, if the null hypothesis is true?

(iii) Show that, for the relevant hypothesis test at the 0.05 significance level, the rejection region for $\hat{p}$ can be expressed as

$$\left(0.4 + \frac{0.806}{\sqrt{n}}, 1\right].$$

(2018, Semester 2) Hypothesis Testing

(i)

$$H_0 : \mu = 0.4, \quad H_a : \mu > 0.4.$$

(ii) The distribution is approximately normal with

$$\mu = 0.4, \quad \sigma = \sqrt{\frac{0.4(1-0.4)}{\sqrt{n}}} = \sqrt{\frac{0.24}{n}}.$$

(iii) We can standardise our distribution so that it is *asymptotically* normal – this gives us

$$Z = \frac{X - \mu}{\sigma} = \frac{X - 0.4}{\sqrt{0.24/n}}.$$

Next, we see that our significance level is at 0.05; note that this is a one-sided test so we consider the z-score of $1 - 0.05 = 0.95$. The rejection region occurs when $Z > \texttt{norminv}(0.95) = 1.6449$. This gives us the following inequality

$$\frac{X - 0.4}{\sqrt{0.24/n}} > 1.6449.$$

(2018, Semester 2) Hypothesis Testing

We can rearrange the expression for X:

$$X - 0.4 > \frac{1.6449\sqrt{0.24}}{\sqrt{n}}$$

$$X > 0.4 + \frac{1.6449\sqrt{0.24}}{\sqrt{n}} \approx 0.4 + \frac{0.806}{\sqrt{n}}.$$

So we reject the null hypothesis when $\hat{p}$ is in the region

$$\left(0.4 + \frac{0.806}{\sqrt{n}}, 1\right].$$

(2010, Semester 1) Hypothesis Testing

Concrete is vulnerable to shock vibrations, which may cause hidden damage to the material. In a study of vibration phenomena, an experiment is carried out and data is reported: including the variables ppv – peak particle velocity (mm/sec), and Ratio – ratio of ultrasonic pulse velocity after impact to that before impact in concrete prisms. Investigators fit the simple linear linear regression model:

$$\text{Ratio} = \beta_0 + \beta_1 \times \text{ppv} + \epsilon.$$

(i) Write the null and alternative hypotheses to test whether the variable ppv is significant in predicting the variable Ratio.

(ii) Carry out the test at the 1% significance level, given $n = 30$.

(2010, Semester 1) Hypothesis Testing

(i)

$$H_0 : \beta_1 = 0, \quad H_a : \beta_1 \neq 0.$$

(i)

$$H_0 : \beta_1 = 0, \quad H_a : \beta_1 \neq 0.$$

(ii) We carry out the hypothesis test at 1% significance level. Begin by identifying the test statistic: t-test with degree of freedom $n-2$. In this case, we have the test statistic t_{28}.

(2010, Semester 1) Hypothesis Testing

Note that we are also given the regression analysis for these problems.

(2010, Semester 1) Hypothesis Testing

Note that we are also given the regression analysis for these problems. Looking at the following chart, we find the observed test statistic:

Predictor	Coef	SE Coef	T	P
Constant	1.00007	0.00131	761.91	0.000
ppv	-0.00001484	0.00000190	-7.80	0.000

(2010, Semester 1) Hypothesis Testing

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Predictor	Coef	SE Coef	T	P
Constant	1.00007	0.00131	761.91	0.000
ppv	-0.00001484	0.00000190	-7.80	0.000

This tells us that $t_{\text{observed}} = -7.80$. Finally, since the hypothesis test is two-sided, then we find the p-value to be

$$p = 2\mathbb{P}\left(T > |-7.8|\right) = 2\mathbb{P}\left(T > 7.8\right),$$

where $T \sim t_{28}$.

(2010, Semester 1) Hypothesis Testing

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This tells us that $t_{\text{observed}} = -7.80$. Finally, since the hypothesis test is two-sided, then we find the p-value to be

$$p = 2\mathbb{P}\left(T > |-7.8|\right) = 2\mathbb{P}\left(T > 7.8\right),$$

where $T \sim t_{28}$. This gives us a p-value of 0, which means that there is **strong** evidence against the null hypothesis. We reject H_0 as a result, meaning ppv has a significant effect on Ratio.

5. Analysis of Variance (ANOVA)

(2017, Semester 1) ANOVA Test

To see how much of a difference time of day makes on the speed at which the could download files, a college sophomore performed an experiment. He placed a file on a remote server and then proceeded to download it at three different time periods of the day. He downloaded the file 24 times in all, 8 times in each time period. The downloading times (in seconds) are summarised in the table below:

Early (7am)	Evening (5pm)	Late night (12am)
68	299	216
138	367	175
75	331	274
186	257	171
68	260	187
217	269	213
93	252	221
90	200	139

(2017, Semester 1) ANOVA Test

An ANOVA table was partially constructed to summarise the data:

Source	df	SS	MS	F
Treatment	(1)	105635	(4)	(5)
Error	21	(3)	2549	
Total	(2)	159164		

(2017, Semester 1) ANOVA Test

An ANOVA table was partially constructed to summarise the data:

Source	df	SS	MS	F
Treatment	(1)	105635	(4)	(5)
Error	21	(3)	2549	
Total	(2)	159164		

(1) The degree of freedom of the treatment is simply $k - 1$ where k is the number of treatments. So (1) is $3 - 1 = 2$.

(2017, Semester 1) ANOVA Test

An ANOVA table was partially constructed to summarise the data:

Source	df	SS	MS	F
Treatment	(1)	105635	(4)	(5)
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(1) The degree of freedom of the treatment is simply $k - 1$ where k is the number of treatments. So (1) is $3 - 1 = 2$.

(2) The total degree of freedom is simply $n - 1$ where n is the number of cases. So (2) is $24 - 1 = 23$.

(2017, Semester 1) ANOVA Test

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Treatment	(1)	105635	(4)	(5)
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(2) The total degree of freedom is simply $n - 1$ where n is the number of cases. So (2) is $24 - 1 = 23$.

(3) The sum of squares is simply the mean square $\times$ degree of freedom. So (3) is $2549 \times 21 = 53529$.

(2017, Semester 1) ANOVA Test

An ANOVA table was partially constructed to summarise the data:

Source	df	SS	MS	F
Treatment	2	105635	(4)	(5)
Error	21	53529	2549	
Total	23	159164		

(4) The mean square can be found by dividing the sum of squares by the degree of freedom. So (4) is simply $105635/2 = 52817.5 \approx 52818$.

(2017, Semester 1) ANOVA Test

An ANOVA table was partially constructed to summarise the data:

Source	df	SS	MS	F
Treatment	2	105635	(4)	(5)
Error	21	53529	2549	
Total	23	159164		

(4) The mean square can be found by dividing the sum of squares by the degree of freedom. So (4) is simply $105635/2 = 52817.5 \approx 52818$.

(5) The observed value of the F-statistic is given by the mean square of the treatment divided by the mean square of the error. That is,

$$F = \frac{\text{MS}_{\text{Treatment}}}{\text{MS}_{\text{Error}}} = \frac{52818}{2549} \approx 20.72.$$

(2017, Semester 1) ANOVA Test

Using a significance level of $\alpha = 0.05$, carry out the ANOVA F-test to determine whether there is a difference in average downloading time among the three time periods.

(2017, Semester 1) ANOVA Test

Step 1. Write out the null and alternative hypothesis. The Null Hypothesis is the claim that we want to find evidence against in favour of the alternative hypothesis.

$$H_0 : \mu_1 = \mu_2 = \mu_3, \quad H_a : \text{ not all means are the same.}$$

Note that saying *not all the means are the same* is **NOT** the same as saying $\mu_1 \neq \mu_2 \neq \mu_3$.

(2017, Semester 1) ANOVA Test

Step 2. Identify the test statistic. Since we're using an ANOVA test, the appropriate test statistic is to use the F-test. Recall that the F-test takes two parameters, $k-1$ and $n-k$ degrees of freedom where k is the *number of treatments* and n is the total number of cases. Here, we are comparing three different scenarios and so we have $k=3$ treatments. So we consider the test statistic $f_{2,21}$.

(2017, Semester 1) ANOVA Test

Step 3. Calculate the *observed value* of the test statistic. We calculated the observed value in the previous question to be $F = 20.72$.

(2017, Semester 1) ANOVA Test

Step 4. Determine the rejection region. The rejection criterion is the following:

$$\textit{Reject } H_0 \textit{ if } F > f_{2,21;0.95}.$$

Here, we consider the region 0.95 since the F-test is a **one-sided** test statistic and we consider the significance level $\alpha = 0.05$. This gives us

$$f_{2,21;0.95} = \texttt{finv}(0.95, 2, 21) = 3.467.$$

(2017, Semester 1) ANOVA Test

Step 5. Write out the conclusion. We reject H_0 since there is very strong evidence that the downloading time among the three times is not all the same.

(2017, Semester 2) ANOVA Test

Data were collected on 387 new vehicles for year 2004. The variables include type of drivetrain (FWD = front wheel drive, AWD=all wheel drive, RWD=rear wheel drive) and fuel efficiency for highway travel (HwayMPG, larger values imply greater efficiency). This data is summarised in the table below.

FWD	AWD	RWD
$\bar{x}_1 = 29.60$	$\bar{x}_2 = 23.01$	$\bar{x}_3 = 25.46$
$s_1 = 5.91$	$s_2 = 3.97$	$s_3 = 2.71$
$n_1 = 215$	$n_2 = 78$	$n_3 = 94$

(2017, Semester 2) ANOVA Test

An ANOVA table was partially constructed to summarise the data:

Source	df	SS	MS	F
Treatment	(1)	2885	(4)	(5)
Error	384	(3)	24.42	
Total	(2)	12261.1		

(2017, Semester 2) ANOVA Test

An ANOVA table was partially constructed to summarise the data:

Source	df	SS	MS	F
Treatment	(1)	2885	(4)	(5)
Error	384	(3)	24.42	
Total	(2)	12261.1		

(1) $k - 1$ where k is the number of treatments. So (1) is 2.

(2017, Semester 2) ANOVA Test

An ANOVA table was partially constructed to summarise the data:

Source	df	SS	MS	F
Treatment	(1)	2885	(4)	(5)
Error	384	(3)	24.42	
Total	(2)	12261.1		

(1) $k - 1$ where k is the number of treatments. So (1) is 2.

(2) $n - 1$ where n is the total number of cases. So (2) is

$$(215 + 78 + 94) - 1 = 386.$$

(2017, Semester 2) ANOVA Test

An ANOVA table was partially constructed to summarise the data:

Source	df	SS	MS	F
Treatment	(1)	2885	(4)	(5)
Error	384	(3)	24.42	
Total	(2)	12261.1		

(1) $k - 1$ where k is the number of treatments. So (1) is 2.

(2) $n - 1$ where n is the total number of cases. So (2) is

$$(215 + 78 + 94) - 1 = 386.$$

(3) We can use the fact that $SS_{\text{Total}} = SS_{\text{Treatment}} + SS_{\text{Error}}$. So (3) becomes

$$12261.1 - 2885 = 9376.1.$$

(2017, Semester 2) ANOVA Test

An ANOVA table was partially constructed to summarise the data:

Source	df	SS	MS	F
Treatment	2	2885	(4)	(5)
Error	384	9476.1	24.42	
Total	386	12261.1		

(4) The mean square can be computed by taking the sum of squares and dividing it by the degree of freedom. This gives us a value of $2885/2 = 1442.5$.

(2017, Semester 2) ANOVA Test

An ANOVA table was partially constructed to summarise the data:

Source	df	SS	MS	F
Treatment	2	2885	(4)	(5)
Error	384	9476.1	24.42	
Total	386	12261.1		

(4) The mean square can be computed by taking the sum of squares and dividing it by the degree of freedom. This gives us a value of $2885/2 = 1442.5$.

(5) The F-statistic can be computed using the formula

$$F = \frac{\text{MS}_{\text{Treatment}}}{\text{MS}_{\text{Error}}} = \frac{1442.5}{24.42} \approx 59.07.$$

(2017, Semester 2) ANOVA Test

Using a significance level of $\alpha = 0.05$, carry out the ANOVA F-test to determine whether there is a difference in highway fuel efficiency among the three types of drivetrains.

(2017, Semester 2) ANOVA Test

Using a significance level of $\alpha = 0.05$, carry out the ANOVA F-test to determine whether there is a difference in highway fuel efficiency among the three types of drivetrains.

Step 1. The hypotheses are

$$H_0 : \mu_1 = \mu_2 = \mu_3, \quad H_a : \text{ not all the means are the same.}$$

(2017, Semester 2) ANOVA Test

Using a significance level of $\alpha = 0.05$, carry out the ANOVA F-test to determine whether there is a difference in highway fuel efficiency among the three types of drivetrains.

Step 2. We are applying an ANOVA Test, so we use the F-test with $k-1, n-k$ degrees of freedom. This gives us the test statistic $f_{2,384}$.

(2017, Semester 2) ANOVA Test

Using a significance level of $\alpha = 0.05$, carry out the ANOVA F-test to determine whether there is a difference in highway fuel efficiency among the three types of drivetrains.

Step 3. The observed value is what we calculated using the ANOVA table:

$$F = 59.07.$$

(2017, Semester 2) ANOVA Test

Using a significance level of $\alpha = 0.05$, carry out the ANOVA F-test to determine whether there is a difference in highway fuel efficiency among the three types of drivetrains.

Step 4. The rejection criterion is as follows:

$$\textit{Reject } H_0 \textit{ if } F > f_{2,384;0.95},$$

since our significance level is $\alpha = 0.05$. This gives

$$f_{2,384;0.95} = \texttt{finv}(0.95, 2, 384) = 3.0192$$

(2017, Semester 2) ANOVA Test

Using a significance level of $\alpha = 0.05$, carry out the ANOVA F-test to determine whether there is a difference in highway fuel efficiency among the three types of drivetrains.

Step 5. We reject H_0 since there is very strong evidence that fuel efficiency among drivetrains are not all the same.

6. Curve Fitting

Interpolation and Least Squares Regression

Interpolation creates a curve which passes through every given data point (x_i, y_i). Least squares regression is used when there are more data points than unknown coefficients. The method is based on minimising the sum of squares of the errors between the fitted curve, and the given data (x_i, y_i)

Different-degree Polynomials

An order n polynomial $y = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... a_0$ is used to fit n+1 data points. Substituting the given data leads to the following matrix (called the Vandermonde matrix), which can be solved using your method of choice (probably Gaussian elimination).

$$\begin{bmatrix} x_0^n & x_0^{n-1} & x_0^{n-2} & \dots & x_0 & 1 \\ x_1^n & x_1^{n-1} & x_1^{n-2} & \dots & x_1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ x_n^n & x_n^{n-1} & x_n^{n-2} & \dots & x_n & 1 \end{bmatrix} \begin{bmatrix} a_n \\ a_{n-1} \\ \vdots \\ a_0 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{bmatrix}.$$

Lagrange Interpolating Polynomial

Consider fitting a quadratic function to 3 data points. The Lagrange polynomial of order 2 is

$$y = y_0 \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} + y_1 \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} + y_2 \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_1 - x_0)}$$

The general n$^{\text{th}}$ order Lagrange interpolating polynomial is

$$y = \sum_{i=0}^{n} (y_i \prod_{j=0, j \neq i}^{n} \frac{x - x_j}{x_i - x_j})$$

Newton Interpolating Polynomials

The n$^{\text{th}}$ order Newton interpolating polynomial is

$$y = c_0 + c_1(x - x_0) + c_2(x - x_0)(x - x_1) + ... + c_n(x - x_0)(x - x_1)...(x - x_{n-1})$$

,where $\mathrm{c}_n = f[x_n, x_{n-1}, ...x_0] = \frac{f[x_n, x_{n-1}, ...x_1] - f[x_{n-1}, x_{n-2}, ...x_0]}{x_n - x_0}$

For uniformly spaced data, where $h = \frac{x_n - x_0}{n}$, the Newton-Gregory interpolating polynomials can be used. The n$^{\text{th}}$ order Newton-Gregory forward interpolating polynomial is

$$y = c_0 + c_1(x - x_0) + c_2(x - x_0)(x - x_0 - h) + ... + c_n(x - x_0)(x - x_1) + ...$$

$+[\mathrm{x}\text{-}\mathrm{x}_0 - (n-1)h]$, where $c_n = \frac{\Delta^n f_0}{n! h^n}$, and Δ is the forward difference

Splines

A spline is a curve from one data point to the next data point. A series of splines is used to fit to n data points. For a natural cubic spline (so a cubic curve joins each pair of adjacent points), the equation to describe the spline from (x_i, y_i) to (x_{i+1}, y_{i+1}) is $y = a_i h_i^3 + b_i h_i^2 + c_i h_i + d_i$ where $h_i = x_{i+1} - x_i$. The following matrix is then solved

$$\begin{bmatrix} 2(h_0+h_1) & h_1 & & & \\ h_1 & 2(h_1+h_2) & h_2 & & \\ & h_2 & 2(h_2+h_3) & h_3 & \\ & & & \ddots & \\ & & & h_{n-2} & 2(h_{n-2}+h_{n-1}) \end{bmatrix} \begin{Bmatrix} S_1 \\ S_2 \\ S_3 \\ \vdots \\ S_{n-1} \end{Bmatrix} = 6 \begin{Bmatrix} f[x_2,x_1]-f[x_1,x_0] \\ f[x_3,x_2]-f[x_2,x_1] \\ f[x_4,x_3]-f[x_3,x_2] \\ \vdots \\ f[x_n,x_{n-1}]-f[x_{n-1},x_{n-2}] \end{Bmatrix}$$

Which leads to solving the coefficieints using
$a_i = \frac{S_{i+1}-S_i}{6h_i}, b_i = \frac{S_i}{2}, c_i = \frac{y_{i+1}-y_i}{h_i} - \frac{h_i S_{i+1} - 2h_i S_i}{6h}, d_i = y_i$

Linear Regression

For line of best fit $y = ax + b$, a and b can be solved with

$$\begin{bmatrix} \sum x_i & n \\ \sum x_i^2 & \sum x_i \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \sum y_i \\ \sum x_i y_i \end{pmatrix}$$

Non-Linear Regression

For $y = ax^b$, this can be changed to a linear regression question by transforming into $\ln y = \ln a + b \ln x$, which is solved with

$$\begin{bmatrix} \sum \ln x_i & n \\ \sum (\ln x_i)^2 & \sum \ln x_i \end{bmatrix} \begin{pmatrix} b \\ \ln a \end{pmatrix} = \begin{pmatrix} \sum \ln y_i \\ \sum \ln x_i \ln y_i \end{pmatrix}$$

Non-Linear Regression

For $y = ae^{bx}$, this can be changed to a linear regression question by transforming into $\ln y = \ln a + bx$, which is solved with

$$\begin{bmatrix} \sum x_i & n \\ \sum x_i^2 & \sum x_i \end{bmatrix} \begin{pmatrix} b \\ \ln a \end{pmatrix} = \begin{pmatrix} \sum \ln y_i \\ \sum x_i \ln y_i \end{pmatrix}$$

Polynomial Regression

For second-order polynomial$y = a_0 + a_1 x + a_2 x^2$, the coefficients are solved using

$$\begin{bmatrix} n & \sum x_i & \sum x_i^2 \\ \sum x_i & \sum x_i^2 & \sum x_i^3 \\ \sum x_i^2 & \sum x_i^3 & \sum x_i^4 \end{bmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \sum y_i \\ \sum x_i y_i \\ \sum x_i^2 y_i \end{pmatrix}$$

Multiple Regression

This is an extension of linear regression for 2 independent variables x_1 and x_2. The 'line' of best fit is $y = a_0 + a_1 x_1 + a_2 x_2$, and the coefficients are solved using

$$\begin{bmatrix} n & \sum x_{1i} & \sum x_{2i} \\ \sum x_{1i} & \sum x_{1i}^2 & \sum x_{1i}x_{2i} \\ \sum x_{2i} & \sum x_{1i}x_{2i} & \sum x_{2i}^2 \end{bmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \sum y_i \\ \sum x_{1i}y_i \\ \sum x_{2i}y_i \end{pmatrix}$$

(Past Exam Question) Linear Regression

The heat-transfer coefficient (h) in a forced convection heat transfer in cross-flow past a cylinder at room temperature is found to vary with the velocity of the fluid (v) flowing past the cylinder as follows:

v_i(m/s)	2	4	6	8
h_i(W/m^2K)	6,000	10,000	13,000	15,000

Use linear regression analysis and find an equation relating h and v.

Recall that for line of best fit $y = ax + b$

$$\begin{bmatrix} \sum x_i & n \\ \sum x_i^2 & \sum x_i \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \sum y_i \\ \sum x_i y_i \end{pmatrix}$$

(Past Exam Question) Linear Regression

$$
\begin{aligned}
\text{Let } h &= av + b \\
\sum v_i &= 2 + 4 + 6 + 8 = 20 \\
\sum h_i &= 1000(6 + 1 + 13 + 15) = 44,000 \\
\sum v_i^2 &= 2^2 + 4^2 + 6^2 + 8^2 = 120 \\
\sum v_i h_i &= 1000(2 \cdot 6 + 4 \cdot 10 + 6 \cdot 13 + 8 \cdot 15) = 250,000 \\
n &= 4
\end{aligned}
$$

$$
\begin{bmatrix} 20 & 4 \\ 120 & 20 \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 44,000 \\ 250,000 \end{pmatrix}
$$

(Past Exam Question) Linear Regression

$$\left[\begin{array}{cc|c} 20 & 4 & 44,000 \\ 120 & 20 & 250,000 \end{array}\right]$$

$$\downarrow R_2^{(1)} = R_2 - 6R_1$$

$$\left[\begin{array}{cc|c} 20 & 4 & 44,000 \\ 0 & -4 & -14,000 \end{array}\right]$$

$$\begin{aligned} -4b &= -14,000 \\ b &= 3,500 \\ 20a + 4(3,500) &= 44,000 \\ a &= 1,500 \\ \therefore h &= 1,500v + 3,500 \end{aligned}$$

7. Numerical Differentiation

Finite Difference Formulae

Type of approximation	Formula	Truncation error
Forward differences	$f_i' = (f_{i+1} - f_i)/(\Delta x)$ $f_i'' = (f_{i+2} - 2f_{i+1} + f_i)/(\Delta x)^2$ $f_i''' = (f_{i+3} - 3f_{i+2} + 3f_{i+1} - f_i)/(\Delta x)^3$ $f_i'''' = (f_{i+4} - 4f_{i+3} + 6f_{i+2} - 4f_{i+1} + f_i)/(\Delta x)^4$	$O(\Delta x)$
Backward differences	$f_i' = (f_i - f_{i-1})/(\Delta x)$ $f_i'' = (f_i - 2f_{i-1} + f_{i-2})/(\Delta x)^2$ $f_i''' = (f_i - 3f_{i-1} + 3f_{i-2} - f_{i-3})/(\Delta x)^3$ $f_i'''' = (f_i - 4f_{i-1} + 6f_{i-2} - 4f_{i-3} + f_{i-4})/(\Delta x)^4$	$O(\Delta x)$
Central differences	$f_i' = (f_{i+1} - f_{i-1})/(2\,\Delta x)$ $f_i'' = (f_{i+1} - 2f_i + f_{i-1})/(\Delta x)^2$ $f_i''' = (f_{i+2} - 2f_{i+1} + 2f_{i-1} - f_{i-2})/(2(\Delta x)^3)$ $f_i'''' = (f_{i+2} - 4f_{i+1} + 6f_i - 4f_{i-1} + f_{i-2})/(\Delta x)^4$	$O(\Delta x^2)$

Higher Order Finite Difference Formulae

Type of formula	Formula	Truncation error
Forward differences	$f_i' = (-f_{i+2} + 4f_{i+1} - 3f_i)/(2(\Delta x))$	$O(\Delta x)^2$
	$f_i'' = (-f_{i+3} + 4f_{i+2} - 5f_{i+1} + 2f_i)/(\Delta x)^2$	
	$f_i''' = (-3f_{i+4} + 14f_{i+3} - 24f_{i+2} + 18f_{i+1} - 5f_i)/(2(\Delta x)^3)$	
	$f_i'''' = (-2f_{i+5} + 11f_{i+4} - 24f_{i+3} + 26f_{i+2} - 14f_{i+1} + 3f_i)/(\Delta x)^4$	
Backward differences	$f_i' = (3f_i - 4f_{i-1} + f_{i-2})/(2(\Delta x))$	$O(\Delta x^2)$
	$f_i'' = (2f_i - 5f_{i-1} + 4f_{i-2} - f_{i-3})/(\Delta x)^2$	
	$f_i''' = (5f_i - 18f_{i-1} + 24f_{i-2} - 14f_{i-3} + 3f_{i-4})/(2(\Delta x)^3)$	
	$f_i'''' = (3f_i - 14f_{i-1} + 26f_{i-2} - 24f_{i-3} + 11f_{i-4} - 2f_{i-5})/(\Delta x)^4$	
Central differences	$f_i' = (-f_{i+2} + 8f_{i+1} - 8f_{i-1} + f_{i-2})/(12(\Delta x))$	$O(\Delta x^4)$
	$f_i'' = (-f_{i+2} + 16f_{i+1} - 30f_i + 16f_{i-1} - f_{i-2})/(12(\Delta x)^2)$	
	$f_i''' = (-f_{i+3} + 8f_{i+2} - 13f_{i+1} + 13f_{i-1} - 8f_{i-2} + f_{i-3})/(8(\Delta x)^3)$	
	$f_i'''' = (-f_{i+3} + 12f_{i+2} - 39f_{i+1} + 56f_i - 39f_{i-1} + 12f_{i-2} - f_{i-3})/(6(\Delta x)^4)$	

Richardson's Extrapolation for Differentiation

If D_1, D_2 are approximate derivatives of step size h_1 and h_2 respectively, then Richardson's extrapolation gives

$$D \approx D_2 + \frac{D_2 - D_1}{(\frac{h_1}{h_2})^n - 1}$$

When $h_2 = \frac{h_1}{2}$, then

$$D \approx D_2 + \frac{D_2 - D_1}{2^n - 1} + O(h_2^{n+2})$$

(Past Exam Question) Finite Difference Approximation

Compute forward difference approximation of $O(\Delta x)$ and central difference approximation of $O(\Delta x)^2$ for the first derivative of $f(x) = e^x + x$ at $x = 1$, using a value of $\Delta x = 0.25$.

Calculate the percentage relative errors for each approximation by comparing with exact solution and discuss your results.

The exact solution is $f'(1) = e^1 + 1 = 3.71828$

(Past Exam Question) Finite Difference Approximation

Forward difference approximation of $O(\Delta x)$ is

$$\begin{aligned}
f_i' &= \frac{f_{i+1} - f_i}{\Delta x} \\
f'(1) &= \frac{f(1.25) - f(1)}{0.25} \\
&= \frac{(e^{1.25} + 1.25) - (e^1 + 1)}{0.25} \\
&= 4(e^{1.25} - e + 0.25) \\
\text{Relative error } &= \frac{3.71828 - 4(e^{1.25} - e + 0.25)}{3.71828} \times 100\% \\
&= -9.95\%
\end{aligned}$$

(Past Exam Question) Finite Difference Approximation

Central difference approximation of $O(\Delta x)^2$ is

$$
\begin{aligned}
f_i' &= \frac{f_{i+1} - f_{i-1}}{2\Delta x} \\
f'(1) &= \frac{f(1.25) - f(0.75)}{2(0.25)} \\
&= \frac{(e^{1.25} + 1.25) - (e^{0.75} + 1)}{0.5} \\
&= 2(e^{1.25} - e^{0.75} + 0.5) \\
\text{Relative error } &= \frac{3.71828 - 2(e^{1.25} - e^{0.75} + 0.5)}{3.71828} \times 100\% \\
&= -0.764\%
\end{aligned}
$$

(Past Exam Question) Finite Difference Approximation

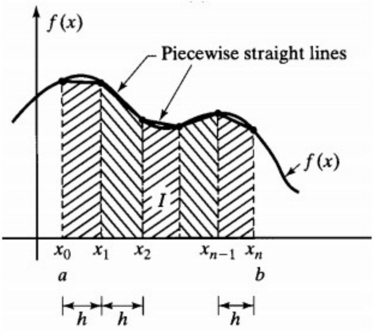
Smaller error for central difference approximation ase error size is comparable to $O(\Delta x)^2$, while error size for forward difference approximation is comparable to $O(\Delta x)$

8. Numerical Integration

Numerical Integration Methods

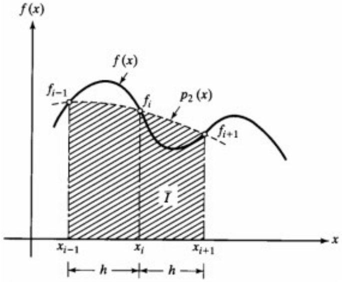
Rule	Formula	Order
Trapezoidal	$\int_a^b f(x)dx \approx \frac{h}{2}(f_0 + 2\sum_{i=1}^{n-1} f_i + f_n)$	$O(h^2)$
Simpsons $\frac{1}{3}$	$\frac{h}{3}(f_0 + 4\sum_{i=1,3,5...}^{n-1} f_i + 2\sum_{i=2,4,6...}^{n-2} f_i + f_n)$	$O(h^4)$
Simpsons $\frac{3}{8}$	$\frac{3h}{8}(f_0 + 3\sum_{i=1,4,7...}^{n-2} (f_i + f_{i+1}) + 2\sum_{i=3,6,9...}^{n-3} f_i + f_n)$	$O(h^4)$

Trapezoidal Rule



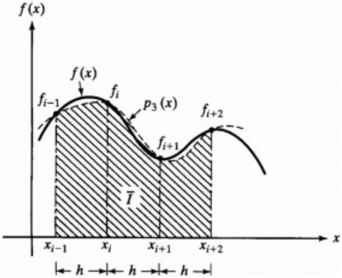
Rule	Formula	Order
Trapezoidal	$\int_a^b f(x)dx \approx \frac{h}{2}(f_0 + 2\sum_{i=1}^{n-1} f_i + f_n)$	$O(h^2)$

Simpsons $\frac{1}{3}$ Rule



Rule	Formula	Order
Simpsons $\frac{1}{3}$	$\frac{h}{3}(f_0 + 4\sum_{i=1,3,5...}^{n-1} f_i + 2\sum_{i=2,4,6...}^{n-2} f_i + f_n)$	$O(h^4)$

Simpsons $\frac{3}{8}$ Rule



Rule	Formula	Order
Simpsons $\frac{3}{8}$	$\frac{3h}{8}\left(f_0 + 3\sum_{i=1,4,7\ldots}^{n-2}(f_i + f_{i+1}) + 2\sum_{i=3,6,9\ldots}^{n-3} f_i + f_n\right)$	$O(h^4)$

Richardson's Extrapolation for Integration

If I_1, I_2 are approximate integrals of step size h_1 and h_2 respectively, then Richardson's extrapolation gives

$$I \approx I_2 + \frac{I_2 - I_1}{(\frac{h_1}{h_2})^n - 1}$$

When $h_2 = \frac{h_1}{2}$, then

$$I \approx I_2 + \frac{I_2 - I_1}{2^n - 1} + O(h_2^{n+2})$$

(Past Exam Question) Trapezoidal and Simpsons 1/3 Rules

Integrate the function $\int_{-3}^{5}(4x+5)^3dx$ using (i) Simpson's and (ii) trapezoidal rules, with n=4.

Compute the percentage relative errors for the numerical solutions obtained in (i) and (ii) with respect to exact solution and discuss your results.

The exact solution is $\int_{-3}^{5}(4x+5)^3dx = 24264$

(Past Exam Question) Trapezoidal and Simpsons 1/3 Rules

x	-3	-1	1	3	5
$(4x+5)^3$	-343	1	729	4913	15625

(i)

$$\int_{-3}^{5} (4x+5)^3 dx \approx \frac{2}{2}[-343 + 2(1 + 729 + 4913) + 15625]$$
$$= 26568$$
$$\text{Relative error } = \frac{24264 - 26568}{24264} \times 100\%$$
$$= -9.50\%$$

(Past Exam Question) Trapezoidal and Simpsons 1/3 Rules

(ii)

$$
\begin{aligned}
\int_{-3}^{5} (4x+5)^3 dx &\approx \frac{2}{3}[-343 + 4(1+4913) + 2(729) + 15625] \\
&= 24750 \\
\text{Relative error } &= \frac{24264 - 24750}{24264} \times 100\% \\
&= -2.00\%
\end{aligned}
$$

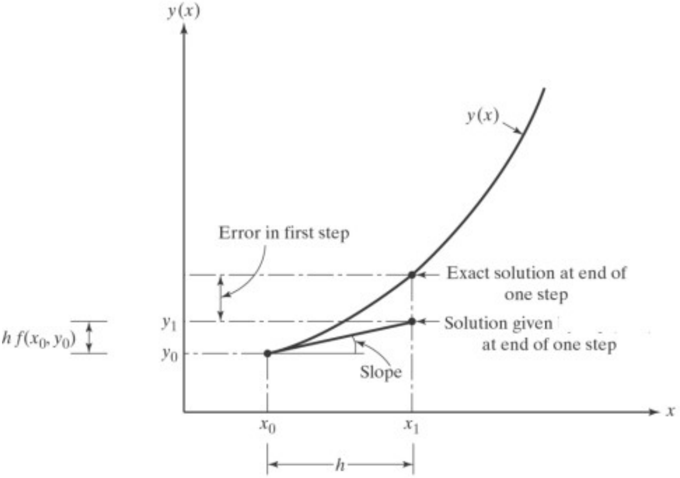
Smaller error when using Simpson's rule, as Simpson's rule approximation is $O(h^4)$, while trapezoidal rule is $O(h^2)$

9. ODEs: Initial-Value Problems

A glorious list of methods to solve IVPs

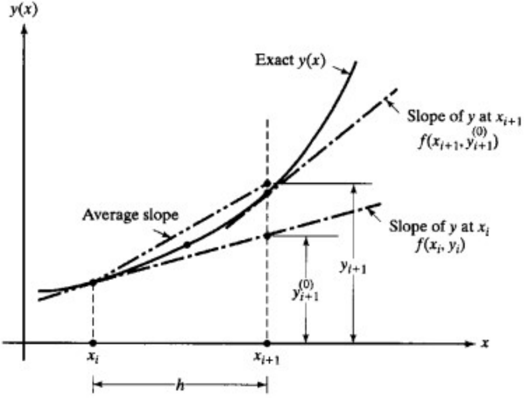
Method	Formula	Order
Euler's Method	$y_{i+1} = y_i + hf(x_i, y_i)$	$O(h)$
Higher Order Euler's Method	$y_{i+1} = y_i + hf(x_i, y_i) + \frac{h^2}{2!}f'(x_i, y_i)$	$O(h^2)$
	$y_{i+1} = y_i + hf(x_i, y_i) + \frac{h^2}{2!}f'(x_i, y_i) + \frac{h^3}{3!}f''(x_i, y_i)$	$O(h^3)$
Heun's Method	$y_{i+1}^{(0)} = y_i + hf(x_i, y_i)$ $y_{i+1} = y_i + h[\frac{f(x_i, y_i) + f(x_{i+1}, y_{i+1}^{(0)})}{2}]$	$O(h^2)$
Modified Euler's Method	$\mathrm{y}_{i+\frac{1}{2}} = y_i + \frac{h}{2}f(x_i, y_i)$ $y_{i+1} = y_i + hf(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}})$	$O(h^2)$
Runge-Kutta	$\mathrm{y}_{i+1} = y_i + h(c_1k_1 + c_2k_2 + ...c_nk_n)$	$O(h^n)$

Euler's Method



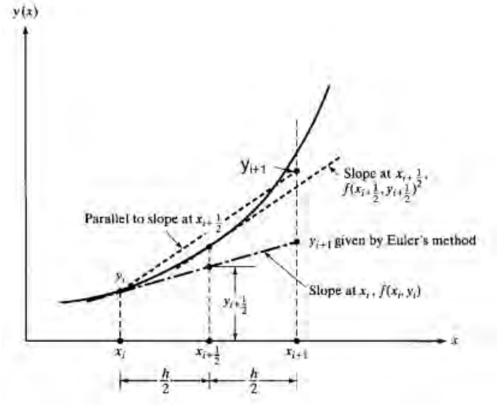
Method	Formula	Order
Euler's Method	$y_{i+1} = y_i + hf(x_i, y_i)$	$O(h)$

Heun's Method



Method	Formula	Order
Heun's Method	$y^{(0)}_{i+1} = y_i + hf(x_i, y_i)$ $y_{i+1} = y_i + h[\frac{f(x_i,y_i)+f(x_{i+1},y^{(0)}_{i+1})}{2}]$	$O(h^2)$

Modified Euler's Method



Method	Formula	Order
Modified Euler's Method	$y_{i+\frac{1}{2}} = y_i + \frac{h}{2} f(x_i, y_i)$ $y_{i+1} = y_i + hf(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}})$	$O(h^2)$

Adams Predictor-Corrector Method

Similarly to Heun's Method, first predict $y_{i+1}^{(0)}$ using Adams-Bashforth open formula, and then correct to y_{i+1} using Adams-Moulton closed formula

Adams-Bashforth Open Formulas

Order of the formula (n)	$k=1$ α_{n1}	$k=2$ α_{n2}	$k=3$ α_{n3}	$k=4$ α_{n4}	$k=5$ α_{n5}	$k=6$ α_{n6}	Local truncation error, $O(h^{n+1})$
1	1						$\frac{1}{2}h^2 f'(\xi)$
2	$\frac{3}{2}$	$-\frac{1}{2}$					$\frac{5}{12}h^3 f''(\xi)$
3	$\frac{23}{12}$	$-\frac{16}{12}$	$\frac{5}{12}$				$\frac{9}{24}h^4 f'''(\xi)$
4	$\frac{55}{24}$	$-\frac{59}{24}$	$\frac{37}{24}$	$-\frac{9}{24}$			$\frac{251}{720}h^5 f^{(4)}(\xi)$
5	$\frac{1901}{720}$	$-\frac{2774}{720}$	$\frac{2616}{720}$	$-\frac{1274}{720}$	$\frac{251}{720}$		$\frac{475}{1440}h^6 f^{(5)}(\xi)$
6	$\frac{4277}{1440}$	$-\frac{7923}{1440}$	$\frac{9982}{1440}$	$-\frac{7298}{1440}$	$\frac{2877}{1440}$	$-\frac{475}{1440}$	$\frac{19087}{60480}h^7 f^{(6)}(\xi)$

The most commonly used is the 4th-order method:
$y_{i+1}^{(0)} = y_i + \frac{h}{24}(55f_i - 59f_{i-1} + 37f_{i-2} - 9f_{i-3})$. Note that the required initial three f(x,y) values are found using a previous method such as Runge-Kutta. The global truncation error is $O(h^n)$.

Adams-Moulton Closed Formulas

Order of the formula (n)	$k=1$ α_{n1}	$k=2$ α_{n2}	$k=3$ α_{n3}	$k=4$ α_{n4}	$k=5$ α_{n5}	$k=6$ α_{n6}	Local truncation error, $O(h^{n+1})$
1	1						$-\frac{1}{2}h^2 f'(\xi)$
2	$\frac{1}{2}$	$\frac{1}{2}$					$-\frac{1}{12}h^3 f''(\xi)$
3	$\frac{5}{12}$	$\frac{8}{12}$	$-\frac{1}{12}$				$-\frac{1}{24}h^4 f'''(\xi)$
4	$\frac{9}{24}$	$\frac{19}{24}$	$-\frac{5}{24}$	$\frac{1}{24}$			$-\frac{19}{720}h^5 f^{(4)}(\xi)$
5	$\frac{251}{720}$	$\frac{646}{720}$	$-\frac{264}{720}$	$\frac{106}{720}$	$-\frac{19}{720}$		$-\frac{27}{1440}h^6 f^{(5)}(\xi)$
6	$\frac{475}{1440}$	$\frac{1427}{1440}$	$-\frac{798}{1440}$	$\frac{482}{1440}$	$-\frac{173}{1440}$	$\frac{27}{1440}$	$-\frac{863}{60480}h^7 f^{(6)}(\xi)$

The most commonly used is the 4th-order method:
$y_{i+1} = y_i + \frac{h}{24}(9f_{i+1} + 19f_i - 5f_{i-1} + f_{i-2})$, where $f_{i+1} = f(x_{i+1}, y_{i+1}^{(0)})$ was found from Adams-Bashforth open formula

(Past Exam Question) Euler's and Runge-Kutta Method

The fourth-order Runge-Kutta method can be written as

$$y_{i+1} = y_i + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where

$$\begin{aligned}
k_1 &= f(x_i, y_i) \\
k_2 &= f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}hk_1) \\
k_3 &= f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}hk_2) \\
k_4 &= f(x_i + h, y_i + hk_3)
\end{aligned}$$

(Past Exam Question) Euler's and Runge-Kutta Method

Find the solution of the initial value problem $y' = 3y + 3t$ with $y(0) = 1$ at $t = 0.2$

(i) Using Euler's method with $h = 0.2$

$$\begin{aligned} y(0.2) = y_1 &= y_0 + hf(t_0, y_0) \\ &= 1 + 0.2(3 \cdot 1 + 3 \cdot 0) \\ &= 1.6 \end{aligned}$$

(Past Exam Question) Euler's and Runge-Kutta Method

(ii) Using the fourth-order Runge-Kutta method with $h = 0.2$

$$
\begin{aligned}
k_1 &= f(t_0, y_0) \\
&= 3 \cdot 1 + 3 \cdot 0 \\
&= 3 \\
k_2 &= f(t_0 + \frac{1}{2}h, y_0 + \frac{1}{2}hk_1) \\
&= f(0 + \frac{0.2}{2}, 1 + \frac{0.2}{2}3) \\
&= f(0.1, 1.3) \\
&= 3 \cdot 1.3 + 3 \cdot 0.1 \\
&= 4.2
\end{aligned}
$$

(Past Exam Question) Euler's and Runge-Kutta Method

$$
\begin{aligned}
k_3 &= f(0 + \frac{0.2}{2}, 1 + \frac{0.2}{2}4.2) \\
&= f(0.1, 1.42) \\
&= 3 \cdot 1.42 + 3 \cdot 0.1 \\
&= 4.56 \\
k_4 &= f(0 + 0.2, 1 + 0.2 \cdot 4.56) \\
&= f(0.2, 1.912) \\
&= 3 \cdot 1.912 + 3 \cdot 0.2 \\
&= 6.336 \\
y_1 &= 1 + \frac{0.2}{6}(3 + 2 \cdot 4.2 + 2 \cdot 4.56 + 6.336) \\
&= 1.8952
\end{aligned}
$$

(Past Exam Question) Euler's and Runge-Kutta Method

(iii) Compare the results with the exact solution $y(t) = \frac{4}{3}e^{3t} - t - \frac{1}{3}$ and find the percentage errors for the results obtained in (i) and (ii)

$$\begin{aligned} y(0.2) &= \frac{4}{3}e^{3\cdot 0.2} - 0.2 - \frac{1}{3} \\ &= 1.8962 \\ \text{(i) Percentage error } &= \frac{1.8962 - 1.6}{1.8962} \times 100\% \\ &= 15.6\% \\ \text{(ii) Percentage error } &= \frac{1.8962 - 1.8952}{1.8962} \times 100\% \\ &= 0.051\% \end{aligned}$$

(Past Exam Question) Euler's and Runge-Kutta Method

(iv) Why do you have improvement in the case of the Runge-Kutta method?

Error is smaller for fourth-order Runge-Kutta method as it has global truncation error $O(h^4)$ while Euler's method has global truncation error $O(h)$

(Past Exam Question) Simultaneous Differential Equations

Consider the second order differential equation

$$2x''(t) - 5x'(t) - 3x(t) = 45e^{2t}$$

Reformulate this equation as a system of two first order differential equations

$$\begin{aligned} y(t) &= x'(t) \\ 2y'(t) - 5y(t) - 3x(t) &= 45e^{2t} \end{aligned}$$

10. ODEs: Boundary Value Problems

Boundary Conditions (not really formulae, just terminology)

Dirichlet condition:

$$y(x_0) = f(x) \text{ or constant}$$

Neumann condition:

$$\frac{dy}{dx}|_{x_0} = f(x) \text{ or constant}$$

Mixed condition:

$$\frac{dy}{dx}|_{x_0} + cy(x_0) = f(x) \text{ or constant}$$

(Past Exam Question) Boundary Value Problems

For a fin with a constant cross section, the governing equation for the heat transfer is given by $kA\frac{d^2\theta}{dx^2} - hP\theta = 0$ where $\theta = T - T_\infty$.

(a) Divide the fin into N equal parts and write the finite difference form of the equation above that relates the temperature inside of the fin at node i, T_i, to the temperatures at the neighbouring nodes using the central-differencce formula.

Let Δx be distance between nodes

Finite difference form of the equation is

$$kA\frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{\Delta x^2} - hP\theta_i = 0$$

(Past Exam Question) Boundary Value Problems

(b) Using the boundary conditions $\theta_0 = 10$ and $\theta_N = 1$ write the coefficients A_i, B_i, C_i and D_i to obtain the system of linear equations for all interior points

$$A_i\theta_{i-1} + B_i\theta_i + C_i\theta_{i+1} = D_i,\ i = 1, 2, ..., N-1$$

$$\frac{kA}{\Delta x^2}\theta_{i-1} - (\frac{2kA}{\Delta x^2} + hP)\theta_i + \frac{kA}{\Delta x^2}\theta_{i+1} = 0$$

$$i = 1 : A_1 = 0, B_1 = -\frac{2kA}{\Delta x^2} - hP, C_i = \frac{kA}{\Delta x^2}, D_i = -\frac{10kA}{\Delta x^2}$$

$$i = 2, 3, ...N-2 : A_i = \frac{kA}{\Delta x^2}, B_i = -\frac{2kA}{\Delta x^2} - hP, C_i = \frac{kA}{\Delta x^2}, D_i = 0$$

$$i = N-1 : A_{N-1} = \frac{kA}{\Delta x^2}, B_{N-1} = -\frac{2kA}{\Delta x^2} - hP, C_{N-1} = 0,$$

$$D_{N-1} = -\frac{kA}{\Delta x^2}$$

(c) Express the system of equations in matrix form and discuss which method you can use for solving this system of equations without solving it.

$$\begin{bmatrix} -\frac{2kA}{\Delta x^2} - hP & \frac{kA}{\Delta x^2} & 0 & 0 & \cdots & 0 \\ \frac{kA}{\Delta x^2} & -\frac{2kA}{\Delta x^2} - hP & \frac{kA}{\Delta x^2} & 0 & \cdots & 0 \\ 0 & \frac{kA}{\Delta x^2} & -\frac{2kA}{\Delta x^2} - hP & \frac{kA}{\Delta x^2} & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \frac{kA}{\Delta x^2} & -\frac{2kA}{\Delta x^2} - hP & \frac{kA}{\Delta x^2} \\ 0 & 0 & \cdots & 0 & \frac{kA}{\Delta x^2} & -\frac{2kA}{\Delta x^2} - hP \end{bmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \vdots \\ \theta_{N-2} \\ \theta_{N-1} \end{pmatrix} = \begin{pmatrix} -\frac{10kA}{\Delta x^2} \\ 0 \\ 0 \\ \vdots \\ 0 \\ -\frac{kA}{\Delta x^2} \end{pmatrix}$$

This is a tridiagonal matrix (TDMA), which can be solved by eliminating the bottom diagonal of the three diagonals, and then back substituting (refer to next slide for solving TDMA).

(Past Exam Question) Boundary Value Problems

Given the general tridiagonal matrix

$$
\begin{bmatrix}
B_1 & C_1 & 0 & 0 & \dots & 0 \\
A_2 & B_2 & C_2 & 0 & \dots & 0 \\
0 & A_3 & B_3 & C_3 & 0 & 0 \\
0 & 0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & 0 \\
0 & \dots & 0 & A_{N-2} & B_{N-2} & C_{N-2} \\
0 & 0 & \dots & 0 & A_{N-1} & B_{N-1}
\end{bmatrix}
\begin{pmatrix}
y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{N-2} \\ y_{N-1}
\end{pmatrix}
=
\begin{pmatrix}
D_1 - A_1 Y_0 \\ D_2 \\ D_3 \\ \vdots \\ \vdots \\ D_{N-1} - C_{N-1} Y_N
\end{pmatrix}
$$

(Past Exam Question) Boundary Value Problems

First eliminate A_i using

$B_1^{(1)} = B_1, B_i^{(1)} = B_i - \frac{A_i}{B_{i-1}^{(1)}} C_{i-1}, D_1^{(1)} = D_1, D_i^{(1)} = D_i - \frac{A_i}{B_{i-1}^{(1)}} D_{i-1}$

$$\begin{bmatrix} B_1 & C_1 & 0 & 0 & ... & 0 \\ 0 & B_2^{(1)} & C_2 & 0 & ... & 0 \\ 0 & 0 & B_3^{(1)} & C_3 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & ... & 0 & 0 & B_{N-2}^{(1)} & C_{N-2} \\ 0 & 0 & ... & 0 & 0 & B_{N-1}^{(1)} \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{N-2} \\ y_{N-1} \end{pmatrix} = \begin{pmatrix} D_1^{(1)} \\ D_2^{(1)} \\ D_3^{(1)} \\ \vdots \\ \vdots \\ D_{N-1}^{(1)} \end{pmatrix}$$

(Past Exam Question) Boundary Value Problems

Then solve for y_i by back substituting

$$
\begin{aligned}
y_{N-1} &= \frac{D_{N-1}^{(1)}}{B_{N-1}^{(1)}} \\
y_i &= \frac{D_i^{(1)} - C_i y_{i+1}}{B_i^{(1)}},\ i = 1, 2, ...N-2
\end{aligned}
$$

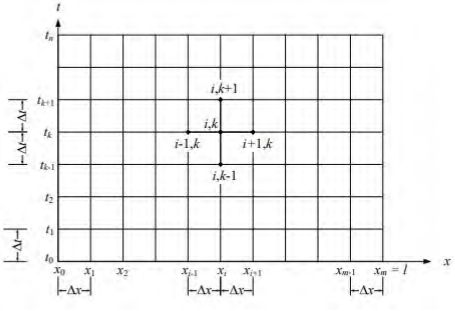
11. Parabolic Equations

Heat Conduction Equation

The common parabolic equation tested is the heat conduction equation:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

The two variables means that there is variation in the time (t), and space (x) variables. The mesh used is



So T_i^k represents the temperature at mesh point (x_i, t_k)

Explicit Method: Forward Time Centred Space (FTCS) scheme

Using the forward difference approximation for time, and the central difference approximation for space, the heat conduction equation is rewritten as

$$\frac{T_i^{k+1} - T_i^k}{\Delta t} = \alpha \frac{T_{i+1}^k - 2T_i^k + T_{i-1}^k}{(\Delta x)^2}$$

which rearranges to

$$T_i^{k+1} = sT_{i+1}^k + (1 - 2s)T_i^k + sT_{i-1}^k \text{ ,where } s = \frac{\alpha \Delta t}{(\Delta x)^2}$$

and the stability condition is

$$s = \frac{\alpha \Delta t}{(\Delta x)^2} \leq \frac{1}{2}$$

Implicit Method

The general heat equation is now

$$-\theta s T_{i+1}^{k+1} + (1+2\theta s) T_i^{k+1} - \theta s T_{i-1}^{k+1} = (1-\theta) s T_{i+1}^{k} + [1-2(1-\theta)s] T_i^{k} + (1-\theta) s T_{i-1}^{k}$$

With special cases for $\theta = 1$ (fully-implicit):

$$-sT_{i+1}^{k+1} + (1+2s)T_i^{k+1} - sT_{i-1}^{k+1} = T_i^k$$

$\theta = 0.5$ (semi-implicit/Crank-Nicolson Method):

$$-0.5sT_{i+1}^{k+1} + (1+s)T_i^{k+1} - 0.5sT_{i-1}^{k+1} = 0.5sT_{i+1}^{k} + (1-s)T_i^{k} + 0.5sT_{i-1}^{k}$$

$\theta = 0$ (fully-implicit/explicit):

$$T_i^{k+1} = sT_{i+1}^{k} + (1-2s)T_i^{k} + sT_{i-1}^{k}$$

Parabolic Equation Example but it's actually eternal damnation

The initial temperature of a metal rod of length 1 m is given by $T(x,0) = x^2, 0 < x < 1$. The temperatures of the rod at x=0 and x=1 are given by $T(0,t) = 0, T(1,t) = 1, t > 0$.
Using $\alpha = 0.75, \Delta x = 0.2$ and $\Delta t = 0.02$, determine the temperature distribution in the rod for $0 \leq t \leq 0.1$ using explicit method by solving numerically a parabolic, heat conduction equation $\frac{\partial T(x,t)}{\partial t} = \alpha \frac{\partial^2 T(x,t)}{\partial x^2}$

Check for stability:

$$s = \frac{\alpha \Delta t}{(\Delta x)^2} \leq \frac{1}{2}, s = \frac{(0.75)(0.02)}{(0.2)^2} = 0.375 \leq \frac{1}{2}$$

Which means that this is stable

Parabolic Equation Example

Find boundary conditions

$$T(0,t) = 0, T(1,t) = 1, T(x,0) = x^2$$

Which means that at all times, the beginning of the rod is at $0°$ and the end of the rod is at $1°$, while at t=0:

$$T(0,0) = 0^2 = 0$$
$$T(0.2,0) = 0.2^2 = 0.04$$
$$T(0.4,0) = 0.4^2 = 0.16$$
$$T(0.6,0) = 0.6^2 = 0.36$$
$$T(0.8,0) = 0.8^2 = 0.64$$
$$T(1,0) = 1^2 = 1$$

Parabolic Equation Example

Using

$$T_i^{k+1} = sT_{i+1}^k + (1-2s)T_i^k + sT_{i-1}^k$$

$$\begin{aligned}
&\text{At } t_1 = 0.2 : \\
&\quad T_1^1 = 0.375T_2^0 + (1-2\cdot 0.375)T_1^0 + 0.375T_0^0 \\
&\qquad = 0.375(0.16) + (1-2\cdot 0.375)(0.04) + 0.375(0) = 0.07 \\
&\quad T_2^1 = 0.375T_3^0 + (1-2\cdot 0.375)T_2^0 + 0.375T_1^0 \\
&\qquad = 0.375(0.36) + (1-2\cdot 0.375)(0.16) + 0.375(0.04) = 0.19 \\
&\quad T_3^1 = 0.375T_4^0 + (1-2\cdot 0.375)T_3^0 + 0.375T_2^0 \\
&\qquad = 0.375(0.64) + (1-2\cdot 0.375)(0.36) + 0.375(0.16) = 0.39 \\
&\quad T_4^1 = 0.375T_5^0 + (1-2\cdot 0.375)T_4^0 + 0.375T_3^0 \\
&\qquad = 0.375(1) + (1-2\cdot 0.375)(0.64) + 0.375(0.36) = 0.67
\end{aligned}$$

Parabolic Equation Example

Using

$$T_i^{k+1} = sT_{i+1}^k + (1-2s)T_i^k + sT_{i-1}^k$$

$$\begin{aligned}
&\text{At } t_2 = 0.4 :\\
&\quad T_1^2 = 0.375T_2^1 + (1-2\cdot 0.375)T_1^1 + 0.375T_0^1\\
&\qquad = 0.375(0.19) + (1-2\cdot 0.375)(0.07) + 0.375(0) = 0.0887\\
&\quad T_2^2 = 0.375T_3^1 + (1-2\cdot 0.375)T_2^1 + 0.375T_1^1\\
&\qquad = 0.375(0.39) + (1-2\cdot 0.375)(0.19) + 0.375(0.07) = 0.2200\\
&\quad T_3^2 = 0.375T_4^1 + (1-2\cdot 0.375)T_3^1 + 0.375T_2^1\\
&\qquad = 0.375(0.67) + (1-2\cdot 0.375)(0.39) + 0.375(0.19) = 0.4200\\
&\quad T_4^2 = 0.375T_5^1 + (1-2\cdot 0.375)T_4^1 + 0.375T_3^1\\
&\qquad = 0.375(1) + (1-2\cdot 0.375)(0.67) + 0.375(0.39) = 0.6887
\end{aligned}$$

Parabolic Equation Example

Using the wonders of MATLAB, I then went to the afterlife and retrieved the rest of the answers (vertical is t, horizontal is x)

	Var1	Var2	Var3	Var4	Var5	Var6
1	0	0.0400	0.1600	0.3600	0.6400	1
2	0	0.0700	0.1900	0.3900	0.6700	1
3	0	0.0887	0.2200	0.4200	0.6887	1
4	0	0.1047	0.2458	0.4458	0.7047	1
5	0	0.1183	0.2679	0.4679	0.7183	1
6	0	0.1300	0.2868	0.4868	0.7300	1

Elliptic and Hyperbolic Equations

The common elliptic equation tested is the Poisson equation:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = f(x, y)$$

Using the central difference formula for x and y gives

$$\frac{T_{i+1}^k - 2T_i^k + T_{i-1}^k}{(\Delta x)^2} + \frac{T_{i+1}^k - 2T_i^k + T_{i-1}^k}{(\Delta y)^2} = f(x, y)$$

The common hyperbolic equation tested is the wave equation:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

Using the central difference formula for x and y gives

$$\frac{u_{i+1}^k - 2u_i^k + u_{i-1}^k}{(\Delta x)^2} = \frac{1}{c^2}\frac{u_{i+1}^k - 2u_i^k + u_{i-1}^k}{(\Delta t)^2}$$