Vectors (2D/3D)

Big idea: what a vector is

A **vector** is a quantity with **magnitude** (length) and **direction**.It can be drawn as an **arrow** and written algebraically by its **components**.

Scalars vs vectors. A **scalar** is described by just a number (with units): mass, time, temperature. A **vector** needs both a size and a direction: displacement, velocity, force.

1. Geometric picture: directed segments

A vector can be represented by a directed segment $\overrightarrow{AB}$.

- A is the **initial point** (tail), B is the **terminal point** (head).
- The **length** of the arrow is the vector's magnitude.
- Two vectors are **equal** if they have the same length and direction (you may slide an arrow parallel to itself).

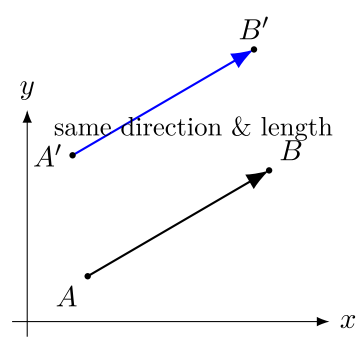

Figure 1: A vector is an arrow. You can translate it without changing it.

2. Standard position and components

2.1 Standard position

Because only direction and length matter, we often move a vector so its tail is at the origin O. Then the **terminal point coordinates** become the vector's components.

2.2 Component form in 2D and 3D

$$\text{2D: } \mathbf{v} = \langle v_1, v_2 \rangle \in \mathbb{R}^2, \qquad \text{3D: } \mathbf{v} = \langle v_1, v_2, v_3 \rangle \in \mathbb{R}^3.$$

The numbers v_1, v_2, v_3 are called the vector's **components**.

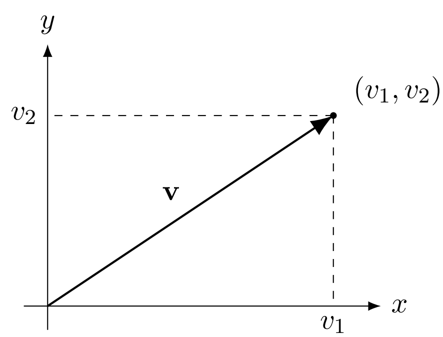

Figure 2: In standard position, components are the "moves" in each coordinate direction.

3. Components from two points

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$. The vector from P to Q is the coordinate change:

$$\overrightarrow{PQ} = \langle x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1 \rangle.$$

Meaning: to go from P to Q you add the vector:

$$(x_1, y_1, z_1) + \langle x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1 \rangle = (x_2, y_2, z_2).$$

Example 1 (components)

Find $\overrightarrow{PQ}$ for $P(1, -2, 3)$ and $Q(4, 2, -1)$:

$$\overrightarrow{PQ} = \langle 4 - 1,\ 2 - (-2),\ -1 - 3 \rangle = \langle 3, 4, -4 \rangle.$$

4. Length (magnitude) of a vector

4.1 Length from components

For $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$,

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}.$$

Why? In 3D, we reduce to the Pythagorean theorem twice: first in the xy-plane (projection), then with the z-component.

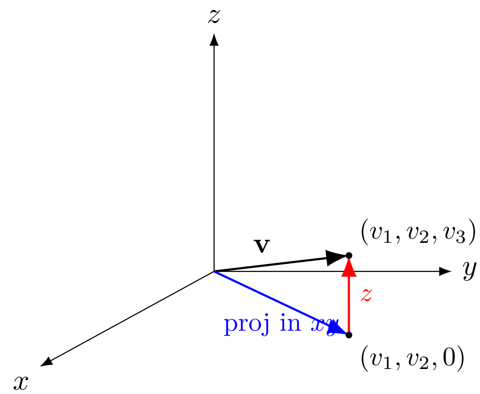

Figure 3: Length in 3D: use projection to the xy-plane, then apply Pythagoras.

Example 2 (length)

For $\mathbf{v} = \langle 3, 2, 1 \rangle$,

$$\|\mathbf{v}\| = \sqrt{3^2 + 2^2 + 1^2} = \sqrt{14}.$$

4.2 The zero vector

$$\mathbf{0} = \langle 0, 0 \rangle \text{ or } \langle 0, 0, 0 \rangle$$

has length 0 and no direction.

5. Vector algebra (componentwise)

Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, $k \in \mathbb{R}$.

5.1 Addition and subtraction

$$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1,\ u_2 + v_2,\ u_3 + v_3 \rangle, \qquad \mathbf{u} - \mathbf{v} = \langle u_1 - v_1,\ u_2 - v_2,\ u_3 - v_3 \rangle.$$

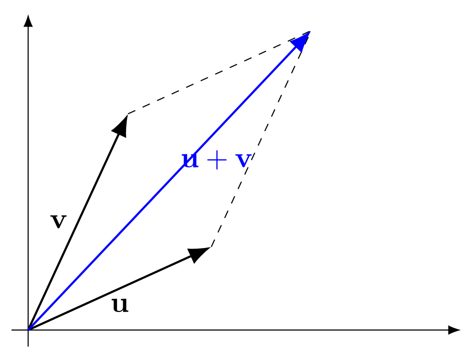

Figure 4: Addition: parallelogram law (or head-to-tail).

5.2 Scalar multiplication

$$k\mathbf{u} = \langle ku_1,\ ku_2,\ ku_3 \rangle, \qquad \|k\mathbf{u}\| = |k|\,\|\mathbf{u}\|\,.$$

If $k > 0$ the direction stays the same; if $k < 0$ it reverses.

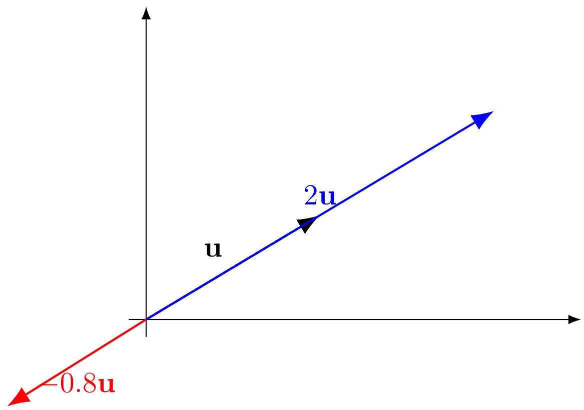

Figure 5: Scaling changes length by $|k|$; negative k reverses direction.

6. Unit vectors and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ notation

6.1 Standard unit vectors

$$\mathbf{i} = \langle 1, 0, 0 \rangle, \qquad \mathbf{j} = \langle 0, 1, 0 \rangle, \qquad \mathbf{k} = \langle 0, 0, 1 \rangle.$$

Any vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ can be written as

$$\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}.$$

6.2 Direction (unit vector)

If $\mathbf{v} \neq \mathbf{0}$, then the **unit vector in the direction of $\mathbf{v}$** is

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|}, \qquad \|\hat{\mathbf{v}}\| = 1.$$

So every nonzero vector decomposes as

$$\mathbf{v} = \|\mathbf{v}\|\ \hat{\mathbf{v}} \quad (\text{length}) \times (\text{direction}).$$

Example 3 (unit vector)

For $\mathbf{v} = \langle 3, 4, 0 \rangle$ we have $\|\mathbf{v}\| = 5$, hence

$$\hat{\mathbf{v}} = \frac{1}{5}\langle 3, 4, 0 \rangle.$$

7. Dot product: algebra and geometry

7.1 Definition (component formula)

For $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$,

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \quad \text{(a scalar)}.$$

7.2 Geometric meaning

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \; \|\mathbf{b}\| \cos\theta,$$

where θ is the angle between the vectors (for $0 \le \theta \le \pi$).

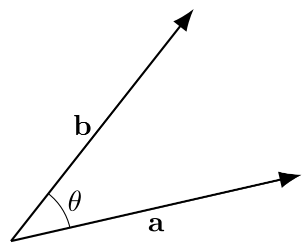

Figure 6: Dot product encodes the angle: $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos\theta$.

7.3 Signs and orthogonality

- If $\mathbf{a} \cdot \mathbf{b} > 0$, then $\theta < 90^\circ$ (acute angle).
- If $\mathbf{a} \cdot \mathbf{b} < 0$, then $\theta > 90^\circ$ (obtuse angle).
- If $\mathbf{a} \cdot \mathbf{b} = 0$ and neither vector is zero, then $\theta = 90^\circ$, i.e. $\mathbf{a} \perp \mathbf{b}$.

Example 4 (angle in space)

Let $P = (1, 0, 0)$, $Q = (0, 1, 0)$, $R = (0, 0, 2)$. Find the angle at P in triangle PQR.

Compute

$$\overrightarrow{PQ} = Q - P = \langle -1, 1, 0 \rangle, \qquad \overrightarrow{PR} = R - P = \langle -1, 0, 2 \rangle.$$

Then

$$\overrightarrow{PQ} \cdot \overrightarrow{PR} = (-1)(-1) + 1 \cdot 0 + 0 \cdot 2 = 1.$$

Lengths:

$$\left\|\overrightarrow{PQ}\right\| = \sqrt{(-1)^2 + 1^2 + 0^2} = \sqrt{2}, \qquad \left\|\overrightarrow{PR}\right\| = \sqrt{(-1)^2 + 0^2 + 2^2} = \sqrt{5}.$$

So

$$\cos\theta = \frac{\overrightarrow{PQ} \cdot \overrightarrow{PR}}{\left\|\overrightarrow{PQ}\right\| \; \left\|\overrightarrow{PR}\right\|} = \frac{1}{\sqrt{2}\sqrt{5}} = \frac{1}{\sqrt{10}}, \qquad \theta = \arccos\!\left(\frac{1}{\sqrt{10}}\right).$$

8. Components via dot product (projection idea)

If $\hat{\mathbf{u}}$ is a unit vector, then

$$\mathbf{a} \cdot \hat{\mathbf{u}} = \|\mathbf{a}\| \cos\theta$$

is exactly the **scalar component** of $\mathbf{a}$ in the direction $\hat{\mathbf{u}}$.

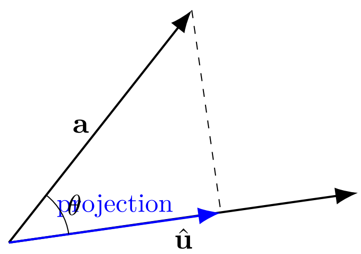

Figure 7: Projection viewpoint: $\mathbf{a} \cdot \hat{\mathbf{u}}$ equals the signed length of the projection of $\mathbf{a}$ onto $\hat{\mathbf{u}}$.

Example 5 (component along a direction)

Let $\mathbf{a} = \langle 2, 3, 6\rangle$ and $\hat{\mathbf{u}} = \frac{1}{3}\langle 2, 1, 2\rangle$ (check it is unit: $\sqrt{4+1+4} = 3$). Then the component of $\mathbf{a}$ along $\hat{\mathbf{u}}$ is

$$\mathbf{a} \cdot \hat{\mathbf{u}} = \langle 2, 3, 6\rangle \cdot \frac{1}{3}\langle 2, 1, 2\rangle = \frac{1}{3}(4+3+12) = \frac{19}{3}.$$

9. A quick geometry application: planes and normal vectors

Consider the set of points (x, y, z) satisfying

$$x + 2y + 3z = 0.$$

Let $\mathbf{n} = \langle 1, 2, 3\rangle$ and $\mathbf{r} = \langle x, y, z\rangle = \overrightarrow{OP}$. Then

$$\mathbf{n} \cdot \mathbf{r} = x + 2y + 3z = 0$$

means $\mathbf{r} \perp \mathbf{n}$ (a 90° angle). So the set is a **plane through the origin** with **normal vector** $\langle 1, 2, 3\rangle$.

10. Summary

- Vectors live in $\mathbb{R}^2$ or $\mathbb{R}^3$ and are written by components: $\langle v_1, v_2\rangle$, $\langle v_1, v_2, v_3\rangle$.
- From points: $\overrightarrow{PQ} = \langle x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1\rangle$.
- Length: $\|\langle v_1, v_2, v_3\rangle\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$.
- Addition/scaling are componentwise; scaling changes length by $|k|$.

- Dot product: $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = \|\mathbf{a}\| \, \|\mathbf{b}\| \cos\theta$.
- $\mathbf{a} \cdot \mathbf{b} = 0 \iff$ perpendicular (for nonzero vectors).
- $\mathbf{a} \cdot \hat{\mathbf{u}}$ gives the component of $\mathbf{a}$ along direction $\hat{\mathbf{u}}$.

Practice (20 problems, no solutions)

Try to solve without looking back, then check definitions.

A. Components and vectors between points (5)

1. Compute $\overrightarrow{PQ}$ for $P(-2,5,1)$ and $Q(4,-1,7)$.
2. Find Q if $P(1,2,3)$ and $\overrightarrow{PQ} = \langle -4,6,1\rangle$.
3. Are $A(0,0,0)$, $B(2,4,6)$, $C(3,6,9)$ collinear? Use vectors.
4. Find a point R such that $\overrightarrow{PR} = 2\,\overrightarrow{PQ}$ where $P(1,0,-1)$ and $Q(3,2,4)$.
5. Let $\mathbf{v} = \langle a,b,c\rangle$. Write the vector from (x,y,z) to $(x+a, y+b, z+c)$.

B. Length and unit vectors (5)

6. Compute $\|\langle -1,2,2\rangle\|$.
7. Find a unit vector in the direction of $\langle 6,-2,3\rangle$.
8. Find all scalars k such that $\|k\langle 3,4,0\rangle\| = 10$.
9. Compute the distance between $P(1,-1,2)$ and $Q(5,2,-4)$ using vector length.
10. For what values of t does $\langle t, 2t, 1-t\rangle$ have length 3?

C. Addition, subtraction, scaling (5)

11. If $\mathbf{u} = \langle 2,-1,4\rangle$ and $\mathbf{v} = \langle -3,5,0\rangle$, compute $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$.
12. Compute $3\mathbf{u} - 2\mathbf{v}$ for the vectors in (11).
13. Find $\mathbf{w}$ such that $\mathbf{u}+\mathbf{w} = \mathbf{v}$.
14. If $\mathbf{a} = \langle 1,1,1\rangle$ and $\mathbf{b} = \langle 2,0,-1\rangle$, compute $\mathbf{a}+\mathbf{b}$ and $-2\mathbf{a}$.
15. Show (by components) that $\mathbf{u} + (\mathbf{v}+\mathbf{w}) = (\mathbf{u}+\mathbf{v}) + \mathbf{w}$.

D. Dot product, angles, orthogonality, components (5)

16. Compute $\langle 1,2,3\rangle \cdot \langle -2,0,5\rangle$.
17. Decide if $\langle 2,-1,0\rangle$ is perpendicular to $\langle 1,2,0\rangle$.
18. Find the angle between $\langle 1,0,1\rangle$ and $\langle 0,1,1\rangle$.
19. Let $\hat{\mathbf{u}} = \dfrac{1}{\sqrt{5}}\langle 1,2,0\rangle$. Compute the component of $\mathbf{a} = \langle 3,1,4\rangle$ along $\hat{\mathbf{u}}$.

20. Describe the set of points (x, y, z) satisfying $2x - y + z = 0$ using a normal vector idea.

End of notes.