Vector Fields

Topic and goal

Topic: vector fields and the basic operations on them (gradient, flux, circulation), with emphasis on how they naturally appear in line and surface integrals.

Goal: build intuition first, then give clean definitions and practical computation tools.

Motivation and big picture

Vector fields appear whenever *every point* of a region has both:

- a **direction**, and
- a **magnitude** (strength / speed / intensity).

Typical examples:

- **Force fields:** gravitational or electrostatic force at each point (force on a unit mass/charge).
- **Velocity fields:** fluid velocity at each point (steady flow).
- **Heat flow:** a heat-flux vector pointing in the direction of heat transfer; its magnitude measures how much heat crosses a unit area per unit time.

Why we study them:

1. They are the natural objects behind **line integrals of the second type** (work of a force along a curve, discharge across a curve).
2. They are central for **surface integrals** (flux through a surface) and **volume integrals** (divergence).
3. They unify many major integral relations that connect boundary integrals to integrals over the region inside.

Connection to what you already know:

- scalar fields like $T(x, y, z)$ (temperature), and partial derivatives,
- line integrals of scalar functions (first type), and integrals like $P\,dx + Q\,dy$ along curves (second type),
- multivariable integration (double and triple integrals).

Two quick clarifications: what is D and what is an “arrow”?

What is D? $\mathbb{R}^2$ means the entire plane (all points (x, y)). A set $D \subseteq \mathbb{R}^2$ is simply a **region of the plane** (for example: a disk, a rectangle, a ring, or even the whole plane). We define a vector field *on D* because sometimes the formula is only valid there.

What is an “arrow”? An **arrow** is just the usual picture of a vector: a directed segment.

- direction of the arrow $\Rightarrow$ direction of the vector,
- length of the arrow $\Rightarrow$ magnitude of the vector.

Scalar fields and vector fields

Definition (Scalar field)**.** *A scalar field on a region $D \subseteq \mathbb{R}^n$ is a function*

$$\varphi : D \to \mathbb{R}, \qquad \varphi(x_1, \ldots, x_n).$$

Easier to understand. A scalar field puts **one number** at each point (temperature, pressure, density, potential).

Definition (Vector field in $\mathbb{R}^2$)**.** *Let $D \subseteq \mathbb{R}^2$. A vector field on D is a function*

$$\vec{F} : D \to \mathbb{R}^2, \qquad (x, y) \mapsto \vec{F}(x, y) = \langle P(x, y),\ Q(x, y)\rangle.$$

Definition (Vector field in $\mathbb{R}^3$)**.** *Let $E \subseteq \mathbb{R}^3$. A vector field on E is a function*

$$\vec{F} : E \to \mathbb{R}^3, \qquad (x, y, z) \mapsto \vec{F}(x, y, z) = \langle P(x, y, z),\ Q(x, y, z),\ R(x, y, z)\rangle.$$

Easier to understand. At each point you attach a small arrow whose components are the numbers P, Q (in $\mathbb{R}^2$) or P, Q, R (in $\mathbb{R}^3$).

Example (Micro-example: uniform wind)**.**

$$\vec{F}(x, y) = \langle 1, 0\rangle$$

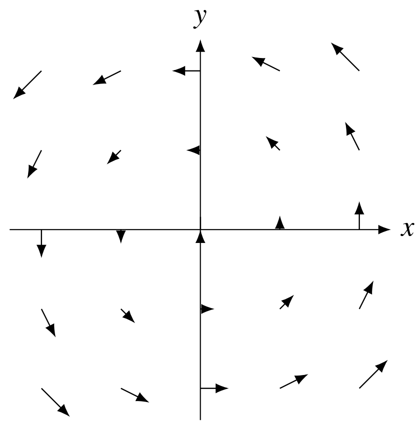
assigns the same unit arrow to every point: constant wind blowing to the right.

Picture 1: a vector field in the plane

Consider

$$\vec{F}(x, y) = \langle -y,\ x\rangle.$$

This field represents rotation around the origin.



$\vec{F}(x, y) = \langle -y, x \rangle$

Example (Why the arrows are tangent to circles)**.** *Let $\vec{r} = \langle x, y \rangle$. Then*

$$\vec{r} \cdot \vec{F}(x, y) = \langle x, y \rangle \cdot \langle -y, x \rangle = -xy + xy = 0.$$

So $\vec{F}(x, y)$ is perpendicular to the radial direction $\vec{r}$, meaning it is tangent to circles centered at the origin.

Recall: level curves, tangent lines, and gradient fields

Level curves and tangent lines

If $f(x, y)$ is a scalar function, a **level curve** is the set of points where the function is constant:

$$f(x, y) = c.$$

A **tangent line** at a point on a curve is the best straight-line approximation near that point (the instantaneous direction of the curve).

Gradient

Definition (Gradient)**.** *For a differentiable scalar field $u : \mathbb{R}^3 \to \mathbb{R}$,*

$$\vec{\nabla} u(x, y, z) = \left\langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z} \right\rangle.$$

For $u : \mathbb{R}^2 \to \mathbb{R}$,

$$\vec{\nabla} u(x, y) = \left\langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right\rangle.$$

Easier to understand:

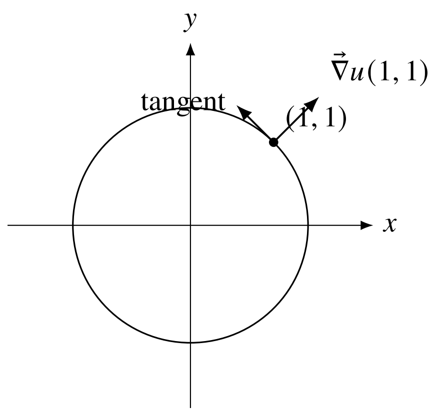
- $\vec{\nabla} u$ points in the direction where u increases **fastest**.
- $\|\vec{\nabla} u\|$ measures how fast the function increases in that best direction.
- $\vec{\nabla} u$ is perpendicular to level curves (and in $\mathbb{R}^3$, perpendicular to level surfaces).

Picture 2: a level curve and a gradient vector

Take $u(x, y) = x^2 + y^2$. Level curves are circles $x^2 + y^2 = c$ and

$$\vec{\nabla} u(x, y) = \langle 2x,\ 2y\rangle.$$

At $(1, 1)$ the gradient is $\langle 2, 2\rangle$: outward and perpendicular to the circle.



Flux and circulation: why line/surface integrals exist

A vector field can be accumulated in two main ways:

- along a **curve** (line integral) ⇒ work / circulation,
- across a **surface** (surface integral) ⇒ flux.

Line integral of a vector field (circulation / work)

Definition (Line integral along a curve)**.** *Let $\vec{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$ and let a curve γ be parametrized by*

$$\vec{r}(t) = \langle x(t), y(t)\rangle, \qquad t \in [\alpha, \beta].$$

Then

$$\int_\gamma \vec{F} \cdot d\vec{r} = \int_\alpha^\beta \Big(P(x(t), y(t))\, x'(t) + Q(x(t), y(t))\, y'(t)\Big)\, dt.$$

Equivalently, $\int_\gamma \vec{F} \cdot d\vec{r} = \int_\gamma P\, dx + Q\, dy$.

Easier to understand. You walk along the curve. At each point, you take the component of $\vec{F}$ in the direction you move, and you add it up along the path. For a force field, this integral equals **work**.

Example (Work in a constant force field)**.** *Let $\vec{F} = \langle 2, 3\rangle$ and move from $A(0, 0)$ to $B(1, 2)$. Parametrize the straight segment by $\vec{r}(t) = \langle t, 2t\rangle$, $0 \le t \le 1$. Then $\vec{r}'(t) = \langle 1, 2\rangle$ and*

$$\int_\gamma \vec{F} \cdot d\vec{r} = \int_0^1 \langle 2, 3\rangle \cdot \langle 1, 2\rangle\, dt = \int_0^1 (2 + 6)\, dt = 8.$$

Flux through a surface (surface integral)

Definition (Flux through an oriented surface)**.** *Let $\vec{F}$ be a vector field in $\mathbb{R}^3$ and let S be an oriented surface with unit normal $\vec{n}$. The flux of $\vec{F}$ through S is*

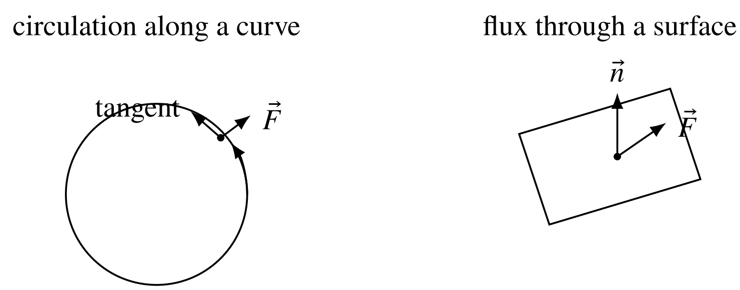
$$\iint_S \vec{F} \cdot \vec{n}\, dS.$$

Easier to understand. Only the part of $\vec{F}$ perpendicular to the surface counts. If $\vec{F}$ is a fluid velocity field, flux measures how much fluid crosses the surface per unit time.

Example (Micro-example: flux through a unit square)**.** *Let $\vec{F} = (0, 0, 1)$ and let S be a unit square in the plane $z = 0$ with upward unit normal $\vec{n} = (0, 0, 1)$. Then $\vec{F} \cdot \vec{n} = 1$ everywhere and the flux is*

$$\iint_S 1\, dS = 1.$$

Picture 3: circulation vs. flux (geometric intuition)



Conservative vector fields (potential fields)

Definition (Conservative field)**.** *A vector field $\vec{F}$ is conservative if there exists a scalar function f such that*

$$\vec{F} = \vec{\nabla} f.$$

The function f is called a potential function.

Easier to understand. If $\vec{F}$ is conservative, then the line integral between two points depends only on the endpoints (path independence).

Example (Finding a potential in the plane (when possible))**.** *Let $\vec{F} = \langle 2x, 2y \rangle$. Try $f(x, y) = x^2 + y^2$. Then $\vec{\nabla} f = \langle 2x, 2y \rangle = \vec{F}$. So $\vec{F}$ is conservative with potential f.*

Operations on vector fields: divergence and curl

Divergence

Definition (Divergence)**.** *For $\vec{F} = \langle P, Q, R \rangle$ in $\mathbb{R}^3$,*

$$\nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

This is a scalar field.

Easier to understand (core meaning). Divergence measures **net outward flow near a point**.

- $\nabla \cdot \vec{F} > 0$: source-like (more leaves than enters),
- $\nabla \cdot \vec{F} < 0$: sink-like (more enters than leaves),
- $\nabla \cdot \vec{F} = 0$: no net creation or loss locally.

Flux-per-volume interpretation. For a tiny closed surface S enclosing a tiny volume ΔV around p:

$$\operatorname{div} \vec{F}(p) = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \iint_S \vec{F} \cdot \vec{n}\, dS.$$

Example. *If $\vec{F}(x, y, z) = \langle x, y, z \rangle$, then $\nabla \cdot \vec{F} = 1 + 1 + 1 = 3 > 0$, so the field behaves like a uniform source everywhere.*

Curl (vorticity / local rotation)

Definition (Curl)**.** *For $\vec{F} = \langle P, Q, R \rangle$ in $\mathbb{R}^3$,*

$$\nabla \times \vec{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle.$$

Easier to understand. Curl measures how much the field tends to **swirl** around a point (local spinning of tiny particles in a flow).

Key integral relationships (conceptual overview)

These are the three most important ideas you will use again and again:

- A boundary line integral can be converted into an area integral (in the plane).
- Flux through a closed surface can be converted into a volume integral of divergence.
- Circulation around a closed spatial curve can be converted into a surface integral of curl.

Even if you do not memorize the full statements yet, remember the philosophy:

$$\text{boundary integral} \quad \longleftrightarrow \quad \text{integral over the region inside.}$$

How to think when solving problems

Recognizing conservative fields (and finding potentials)

Typical workflow in the plane:

1. Compute $\partial P/\partial y$ and $\partial Q/\partial x$.
2. If they do not match, the field is not conservative (in that region).
3. If they match, then try to build a potential f by integrating P with respect to x and correcting by a function of y (or vice versa).

Important: the region must be "nice enough" (no holes / singular points) for the usual sufficiency direction.

Evaluating line integrals

1. Parametrize the curve $\vec{r}(t)$.
2. Compute $\vec{r}'(t)$ and plug into $\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)$.
3. Integrate over the parameter interval.

Shortcut: if the field is conservative and you know a potential f, then

$$\int_\gamma \vec{F} \cdot d\vec{r} = f(B) - f(A).$$

Using flux ideas

Always remember: flux uses only the component in the normal direction:

$$\text{flux density} = \vec{F} \cdot \vec{n}.$$

Common mistakes (so you avoid them)

- Forgetting that $\partial P/\partial y = \partial Q/\partial x$ is not enough if the region has a hole or a singularity.
- Mixing up the two different line integrals: $\int f\, ds$ (scalar along length) vs. $\int P\, dx + Q\, dy$ (vector-field work/circulation).
- Ignoring orientation: reversing curve direction flips the sign of $\int_\gamma \vec{F} \cdot d\vec{r}$; reversing the surface normal flips the sign of flux.
- Parametrization mistakes: wrong derivatives $x'(t), y'(t)$ or wrong bounds.
- Treating flux as if tangential components matter (they do not): only $\vec{F} \cdot \vec{n}$ contributes.

What comes next

Once vector fields feel natural, line and surface integrals stop looking like random formulas and become geometry:

- line integrals $\Rightarrow$ work and circulation,
- surface integrals $\Rightarrow$ flux,
- gradient/divergence/curl connect local change to global integral relations.