Cylinders & Quadric Surfaces (in $\mathbb{R}^3$)

What we mean by "cylinder" and "quadric"

Big picture. In analytic geometry we describe 3D surfaces by equations in x, y, z.

- A **cylinder** is (often) what you get when a *2D curve* is "extruded" in some direction. In coordinates, the most common easy test is: **one variable does not appear**.
- A **quadric surface** is defined by a **second-degree** equation in x, y, z (squares and possibly mixed terms). Many familiar surfaces (sphere, ellipsoid, paraboloids, cones, hyperboloids, and also some cylinders) are quadrics.

Why we care later. These surfaces show up constantly as *boundaries of regions* (triple integrals), as *surfaces for flux integrals*, and as *geometry for intersection curves* (e.g., "surface $\cap$ cylinder" gives a space curve).

1. Cylinders (in $\mathbb{R}^3$)

1.1 The recognition rule (the fast test)

If an equation in x, y, z does **not involve one variable**, then the surface repeats the same cross-section for all values of that missing variable.

- If z does not appear: you get a cylinder **parallel to the z-axis**.
- If x does not appear: cylinder parallel to the x-axis.
- If y does not appear: cylinder parallel to the y-axis.

Reason (simple but precise). For example, if the equation is $F(x, y) = 0$, then for any fixed pair (x, y) that satisfies it, *every* (x, y, z) satisfies it for *all* z. So the curve in the xy-plane is copied upward/downward.

1.2 Core example: circular cylinder $x^2 + y^2 = 1$

$$x^2 + y^2 = 1$$

This is the circle of radius 1 in the xy-plane, extruded along the z-direction.

How to parametrize (useful for intersections). A standard parametrization is

$$x = \cos t, \quad y = \sin t, \quad z = s,$$

where t runs around the circle and s is free (height).

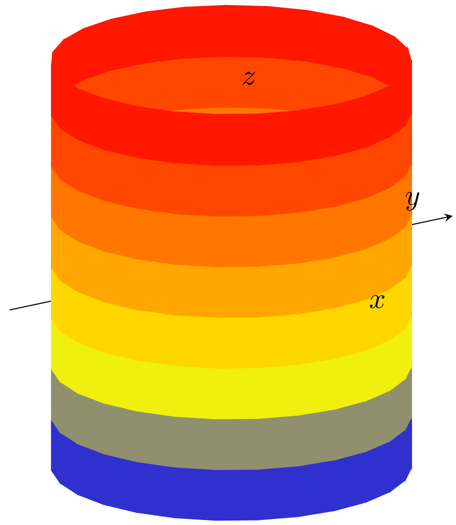

Figure 1: The cylinder $x^2 + y^2 = 1$ (independent of z).

1.3 Other common cylinders you actually meet

- **Elliptic cylinder:** $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ (ellipse extruded).
- **Parabolic cylinder:** $y = x^2$ (parabola extruded).
- **"Sideways" cylinder:** $y^2 + z^2 = 1$ (missing $x \Rightarrow$ axis is x).

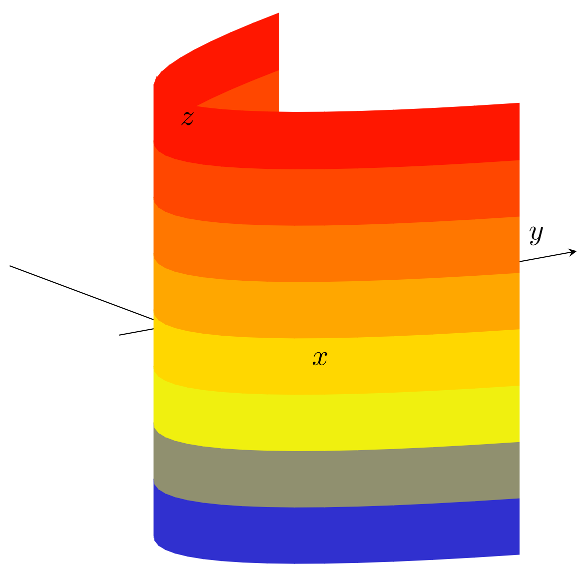

Figure 2: Parabolic cylinder $y = x^2$ (independent of z).

2. Quadric surfaces (in $\mathbb{R}^3$)

2.1 Definition (what "quadric" means)

A **quadric surface** is given by a polynomial equation of **total degree 2** in x, y, z. In this course, we typically work with the standard named families (below) and recognize them by their forms and by cross-sections.

A practical rule: If the highest power you see is a square (x^2, y^2, z^2) and there are no cubes etc., you are in "quadric world."

2.2 The sketching method we actually use

To understand a 3D surface from its equation, you do **traces** and **level curves**:

- **Coordinate plane traces:** set $x = 0$, $y = 0$, $z = 0$ (intersections with the coordinate planes).
- **Parallel-plane traces:** set $z = k$ (or $x = k$, $y = k$) and see what curves you get as k changes.
- **Interpret the family:** circles/ellipses vs hyperbolas vs parabolas tells you the "type."

This method is the right level here: it is geometric, computational, and does not require linear algebra machinery.

3. The main quadric families you meet (with intuition and traces)

3.1 Sphere

$$x^2 + y^2 + z^2 = R^2.$$

Meaning: all points at distance R from the origin.

Traces: setting $z = k$ gives

$$x^2 + y^2 = R^2 - k^2,$$

a circle (if $|k| < R$), a point ($|k| = R$), or empty ($|k| > R$).

3.2 Ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Meaning: a "stretched sphere" with semi-axes a, b, c.

Traces:

- $z = 0 \Rightarrow \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ (ellipse).

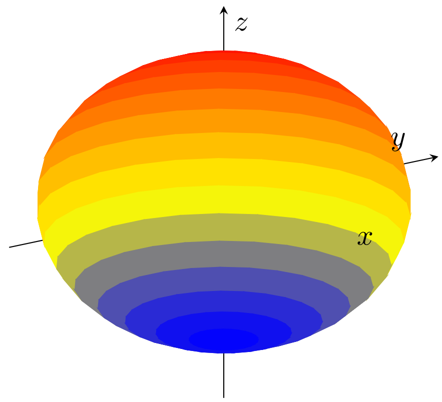

Figure 3: Sphere $x^2 + y^2 + z^2 = 1$ (a quadric).

- $x = 0 \Rightarrow \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ (ellipse).
- $y = 0 \Rightarrow \frac{x^2}{a^2} + \frac{z^2}{c^2} = 1$ (ellipse).

3.3 Elliptic paraboloid ("bowl")

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}.$$

Why it is a bowl: x^2 and y^2 are always ≥ 0, so $z \geq 0$ and grows as you move away from the origin.

Traces:

- $y = 0 \Rightarrow z = \frac{x^2}{a^2}$ (parabola in the xz-plane).
- $x = 0 \Rightarrow z = \frac{y^2}{b^2}$ (parabola in the yz-plane).
- $z = k > 0 \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = k$ (ellipse).

3.4 Hyperbolic paraboloid ("saddle")

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}.$$

Core idea: in the x-direction it curves up, in the y-direction it curves down.

Traces:

- $y = 0 \Rightarrow z = \frac{x^2}{a^2}$ (upward parabola).

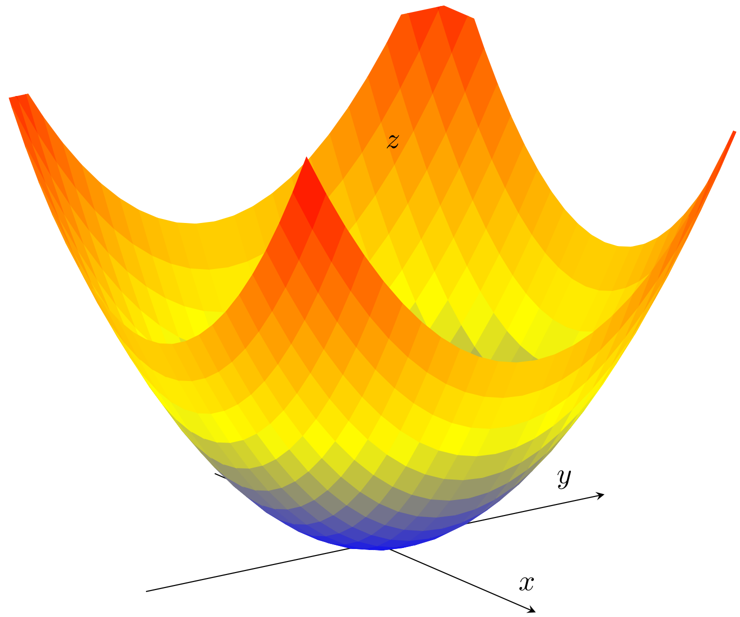

Figure 4: Elliptic paraboloid $z \propto x^2 + y^2$ (typical "bowl").

- $x = 0 \Rightarrow z = -\dfrac{y^2}{b^2}$ (downward parabola).
- $z = k \Rightarrow \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = k$ (hyperbolas for $k \neq 0$; pair of lines when $k = 0$ in the simplified $z = x^2 - y^2$ model).

3.5 Elliptic cone

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}.$$

Meaning: "double cone" with vertex at the origin.

Traces:

- $z = k$ gives $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{k^2}{c^2}$ (ellipse), radius grows linearly with $|k|$.
- $z = 0$ gives only the point $(0, 0, 0)$ (the vertex).

3.6 Hyperboloid of one sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

Meaning: one connected "tube" (a waist in the middle).

Traces:

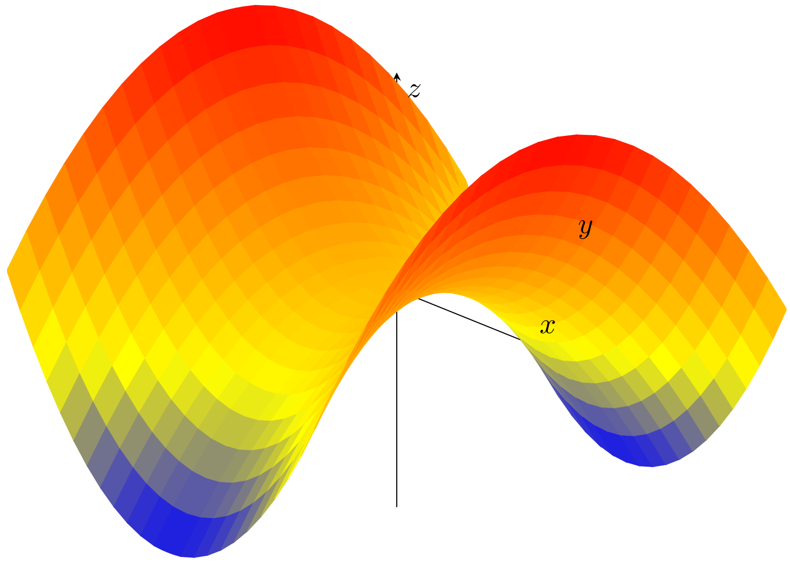

Figure 5: Hyperbolic paraboloid $z \propto x^2 - y^2$ (saddle).

- $z = 0 \Rightarrow \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ (ellipse).
- $z = k \Rightarrow \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 + \dfrac{k^2}{c^2}$ (ellipse that grows as $|k|$ increases).
- $x = 0$ or $y = 0$ gives hyperbolas (because of the minus sign).

3.7 Hyperboloid of two sheets

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \left(\text{equivalently } \frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\right).$$

Meaning: two disconnected parts (one for large positive z, one for large negative z).

Trace test: set $z = 0$ and you get

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

which is impossible (*no points near the origin*) $\Rightarrow$ the surface splits into two sheets away from 0.

4. "Cylinder $\cap$ surface" gives a space curve (a pattern you see a lot)

A very common construction is:

$$\text{cylinder } (x^2 + y^2 = 1) \quad \cap \quad \text{surface } (z = f(x, y)).$$

Then you move around the circle in the xy-plane and lift to the corresponding height.

Standard parametrization trick:

$$x = \cos t, \quad y = \sin t, \quad z = f(\cos t, \sin t).$$

This is exactly the level of technique you need for later line/surface integral setups.

5. Quick "recognize it fast" checklist

1. **Is a variable missing?** If yes $\Rightarrow$ cylinder parallel to that axis.
2. **Do you see only degree ≤ 2?** If yes $\Rightarrow$ you are in quadric territory.
3. **Check signs in x^2, y^2, z^2:**
   - all plus with $= 1$ style $\Rightarrow$ ellipsoid/sphere.
   - plus-plus equals $z \Rightarrow$ elliptic paraboloid.
   - plus-minus equals $z \Rightarrow$ hyperbolic paraboloid.
   - plus-plus-minus equals $1 \Rightarrow$ hyperboloid of one sheet.
   - minus-minus-plus equals $1 \Rightarrow$ hyperboloid of two sheets.
   - equals $z^2 \Rightarrow$ cone.
4. **Confirm by traces:** set $z = 0$, then $z = k$, and see whether you get circles/ellipses, hyperbolas, or parabolas.