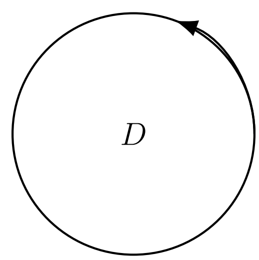
Green's Theorem

(Notes + Proof idea + Examples + Flux/Divergence)

Green's Theorem (Circulation–Curl Form)

Setup. Let C be a simple closed, piecewise-smooth curve in the xy-plane. Let D be the region enclosed by C. Assume $P(x, y)$ and $Q(x, y)$ have continuous first partial derivatives on an open region containing D.

Positive orientation (IMPORTANT). "C is positively oriented" means: when you move along C, the region D stays on your **left**. For the outer boundary, this is the usual **counterclockwise** direction.



positive orientation ⇒ counterclockwise (outer boundary)

Theorem (Green).

$$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

Meaning.

- Left side is a **line integral around the boundary** (circulation / work along C).
- Right side is a **double integral over the inside** of the quantity $Q_x - P_y$ (local rotation density).

Curl notation in 2D (why $Q_x - P_y$?). For a planar field $\mathbf{F} = \langle P, Q\rangle$, define the scalar (2D) curl by

$$\mathrm{curl}_2(\mathbf{F}) = Q_x - P_y.$$

Reason: if we embed the field in 3D as $\langle P, Q, 0\rangle$, then

$$\nabla \times \langle P, Q, 0\rangle = \langle 0, 0, Q_x - P_y\rangle,$$

so $Q_x - P_y$ is exactly the k-component (rotation out of the plane).

Quick consequence (with an important condition). If $Q_x - P_y = 0$ everywhere in a **simply connected** region (no holes) and P, Q are smooth there, then for any closed curve C in that region,

$$\oint_C P\,dx + Q\,dy = 0.$$

This matches the idea: "zero curl ⇒ conservative" (but the no-holes condition matters).

Example (VERY detailed): $\mathbf{F} = \langle x+y,\ y \rangle$, $C: x^2+y^2=4$ (positive orientation)

Goal. Compute $\oint_C \mathbf{F}\cdot d\mathbf{r}$ where

$$\mathbf{F}(x,y) = \langle x+y,\ y\rangle, \qquad C:\ x^2+y^2=4,$$

and C is **positively oriented**.

Step 1: What does $x^2+y^2=4$ mean? This is a circle centered at $(0,0)$. Since $x^2+y^2=r^2$, here $r^2=4$ so $r=2$. So C is the circle of radius 2.

Step 2: What does "positively oriented" mean here? It means you traverse the circle **counterclockwise** so the disk $D:\ x^2+y^2\le 4$ stays on your left.

Step 3: Rewrite as $P\,dx+Q\,dy$. Because $\mathbf{F}=\langle P,Q\rangle$, we have

$$P(x,y)=x+y, \qquad Q(x,y)=y.$$

So

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C P\,dx+Q\,dy = \oint_C (x+y)\,dx + y\,dy.$$

Step 4: Use Green's Theorem (this is why it's easy). Compute partial derivatives:

$$Q_x = \frac{\partial}{\partial x}(y) = 0, \qquad P_y = \frac{\partial}{\partial y}(x+y) = 1.$$

So

$$Q_x - P_y = 0-1 = -1.$$

Step 5: Convert to a double integral over the disk D.

$$\oint_C P\,dx+Q\,dy = \iint_D (-1)\,dA = -\operatorname{Area}(D).$$

Step 6: Area of the disk. Radius $2 \Rightarrow$ area $=\pi(2)^2=4\pi$. Therefore

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = -4\pi.$$

$$\boxed{\oint_C \mathbf{F}\cdot d\mathbf{r} = -4\pi.}$$

Example: $\oint_C (2y+\sqrt{9+x^3})\,dx+(5x+\arctan y)\,dy$, $C: x^2+y^2=4$ (CCW)

Goal. Compute

$$\oint_C \Big((2y+\sqrt{9+x^3})\,dx+(5x+\arctan y)\,dy\Big),\quad C:\ x^2+y^2=4,$$

with C counterclockwise (that is the positive orientation for the disk).

Step 1: Identify P and Q.

$$P(x,y)=2y+\sqrt{9+x^3},\qquad Q(x,y)=5x+\arctan y.$$

Step 2: Compute Q_x and P_y.

$$Q_x=\frac{\partial}{\partial x}(5x+\arctan y)=5,\qquad P_y=\frac{\partial}{\partial y}(2y+\sqrt{9+x^3})=2.$$

So

$$Q_x-P_y=5-2=3.$$

Step 3: Apply Green on the region $D:\ x^2+y^2\le 4$.

$$\oint_C P\,dx+Q\,dy=\iint_D 3\,dA=3\cdot\text{Area}(D).$$

Area of radius-2 disk is 4π, so

$$\oint_C P\,dx+Q\,dy=3\cdot 4\pi=12\pi.$$

$$\boxed{\oint_C \Big((2y+\sqrt{9+x^3})\,dx+(5x+\arctan y)\,dy\Big)=12\pi.}$$

Example: $\mathbf{F}=\langle x+y,\ y\rangle$ on the top semicircle (radius 2)

Important warning. A semicircle arc by itself is **not closed**, so Green's Theorem does **not** directly apply unless you close the curve. So here we compute the line integral by parametrization.

Let C be the **top semicircle** of $x^2+y^2=4$ oriented counterclockwise from $(2,0)$ to $(-2,0)$.

Step 1: Parametrize the arc.

$$x=2\cos t,\quad y=2\sin t,\quad 0\le t\le\pi,$$

$$dx = -2\sin t\,dt, \quad dy = 2\cos t\,dt.$$

Step 2: Write P, Q and substitute. For $\mathbf{F} = \langle P, Q\rangle = \langle x+y,\ y\rangle$:

$$P = x + y = 2\cos t + 2\sin t, \qquad Q = y = 2\sin t.$$

Then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C P\,dx + Q\,dy = \int_0^{\pi} \Big[(2\cos t + 2\sin t)(-2\sin t) + (2\sin t)(2\cos t)\Big]\,dt.$$

Simplify:

$$(2\cos t + 2\sin t)(-2\sin t) = -4\sin t\cos t - 4\sin^2 t,$$
$$(2\sin t)(2\cos t) = 4\sin t\cos t,$$

so the sum is

$$-4\sin^2 t.$$

Step 3: Integrate.

$$\int_0^{\pi} -4\sin^2 t\,dt = -4\cdot\frac{\pi}{2} = -2\pi.$$

$$\boxed{\int_C \mathbf{F}\cdot d\mathbf{r} = -2\pi \text{ (for this CCW top arc orientation).}}$$

Proof idea (clean sketch)

Idea. First prove Green's formula on a simple region where the boundary splits nicely, then extend by cutting and cancellation.

- **Simple region case ("between two graphs").** Assume D can be described as $a \le x \le b$ with $\varphi_1(x) \le y \le \varphi_2(x)$. Then you can show separately:

  $$\oint_C Q\,dy = \iint_D Q_x\,dA, \qquad \oint_C P\,dx = -\iint_D P_y\,dA,$$

  using the Fundamental Theorem of Calculus along vertical and horizontal pieces. Adding gives $\oint_C (P\,dx + Q\,dy) = \iint_D (Q_x - P_y)\,dA$.

- **General region.** Cut D into finitely many simple pieces. When you add the boundary integrals of all pieces, every interior edge appears twice with opposite directions, so those interior contributions cancel. Only the outer boundary remains. The area integrals add up to $\iint_D$.

Regions with holes. Outer boundary is CCW, inner boundaries are clockwise, so D stays on your left along every boundary component.

Flux, divergence, and the 2D Divergence Theorem (Green's flux form)

Let $\mathbf{F} = \langle M(x,y), N(x,y) \rangle$.

Divergence.

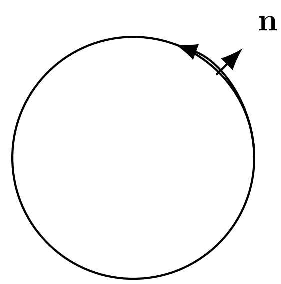
$$\operatorname{div} \mathbf{F} = M_x + N_y.$$

Intuition: positive divergence = source (fluid created), negative divergence = sink (fluid absorbed).

Flux across a closed curve. Outward flux across C is

$$\oint_C \mathbf{F} \cdot \mathbf{n} \, ds,$$

where $\mathbf{n}$ is the outward unit normal.



CCW orientation ⇒ outward normal points outside

Useful identity (turn flux into dx, dy form).

$$\mathbf{F} \cdot \mathbf{n} \, ds = M \, dy - N \, dx,$$

so the flux becomes

$$\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \oint_C (M \, dy - N \, dx).$$

Theorem (Green, flux–divergence form / 2D Divergence Theorem).

$$\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \oint_C (M \, dy - N \, dx) = \iint_D (M_x + N_y) \, dA = \iint_D \operatorname{div} \mathbf{F} \, dA.$$

Flux example

Compute the outward flux of $\mathbf{F} = \langle x, y \rangle$ across $C : \; x^2 + y^2 = 4$ (CCW).

$$\operatorname{div} \mathbf{F} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} = 2.$$

Region is the disk of radius 2, area 4π, so

$$\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_D 2 \, dA = 2 \cdot 4\pi = 8\pi.$$

$$\boxed{8\pi}$$

Exam checklist (super practical)

- If you see $\oint_C P\,dx + Q\,dy$ and C is closed & positively oriented, try Green:

$$\oint_C P\,dx + Q\,dy = \iint_D (Q_x - P_y)\,dA.$$

- If it's a flux problem, try the flux form:

$$\oint_C \mathbf{F} \cdot \mathbf{n}\,ds = \iint_D (M_x + N_y)\,dA.$$

- If the curve is **not closed** (like a semicircle arc), Green does not apply directly: parametrize, or close the curve and account for the added piece.
- Always check orientation: reversing direction flips the sign of the line integral.