Divergence Theorem (Gauss' Theorem)

Goal. This theorem converts a **flux surface integral** over a **closed surface** into a **triple integral** over the **volume inside**. It is one of the most powerful shortcuts in vector calculus.

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1) Flux (what the left side means).

Let S be an oriented surface in $\mathbb{R}^3$ with outward unit normal $\vec{n}$. The **flux** of a vector field $\vec{F}$ through S is

$$\iint_S \vec{F} \cdot \vec{n}\, dS.$$

Interpretation: $\vec{F} \cdot \vec{n}$ is the component of $\vec{F}$ *perpendicular* to the surface.

- If $\vec{F}$ points outward, $\vec{F} \cdot \vec{n} > 0$ (positive outflow).
- If $\vec{F}$ points inward, $\vec{F} \cdot \vec{n} < 0$ (negative outflow = inflow).
- If $\vec{F}$ is tangent to S, then $\vec{F} \cdot \vec{n} = 0$ (no flow through).

2) Divergence (what the right side measures locally).

If $\vec{F} = \langle P(x, y, z),\, Q(x, y, z),\, R(x, y, z)\rangle$, then

$$\nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Interpretation: $\nabla \cdot \vec{F}$ measures **local net outflow density** (source/sink strength). A precise intuition is:

$$(\nabla \cdot \vec{F})(\text{point}) = \lim_{\Delta V \to 0} \frac{(\text{outward flux through the boundary of a tiny box})}{\Delta V}.$$

So divergence is like "flux per unit volume" in the infinitesimal limit.

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3) Statement of the Divergence Theorem.

Let E be a solid region in $\mathbb{R}^3$ with **closed, piecewise smooth boundary** ∂E. Orient ∂E with the **outward** unit normal $\vec{n}$. If $\vec{F}$ has continuous first partial derivatives on a region containing E, then

$$\iint_{\partial E} \vec{F} \cdot \vec{n}\, dS = \iiint_E (\nabla \cdot \vec{F})\, dV.$$

In words:

Total outward flux through the boundary = Total divergence inside the volume.

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4) Why it is true (clean proof idea).

First prove it for a rectangular box $[x, x+\Delta x] \times [y, y+\Delta y] \times [z, z+\Delta z]$.
Flux through the two faces perpendicular to the x-axis:

$$\text{(outward flux)} \approx P(x+\Delta x, y, z)\,\Delta y\,\Delta z - P(x, y, z)\,\Delta y\,\Delta z$$
$$= \big(P(x+\Delta x, y, z) - P(x, y, z)\big)\Delta y\,\Delta z \approx \frac{\partial P}{\partial x}\Delta x\,\Delta y\,\Delta z.$$

Do the same for y and z, then add:

$$\text{flux through the box} \approx \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)\Delta V = (\nabla \cdot \vec{F})\Delta V.$$

Now chop any region E into many tiny boxes:

- Flux through **interior faces cancels** (the same face is counted twice with opposite normals).
- Only flux through the **outer boundary** survives.

Taking the limit gives the theorem.

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5) When you cannot use it blindly (important!).

- **The surface must be closed.** If S is open (like a cone without a cap), the theorem does not apply directly. You must **close the surface** first.
- **$\vec{F}$ must be well-behaved inside E.** If $\vec{F}$ has a singularity inside (e.g. $\vec{F} = \frac{\langle x,y,z\rangle}{r^3}$ at $r = 0$), extra care is needed (you often remove a tiny ball around the singularity).

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6) Problem-solving template (use this every time).

1. Confirm ∂E is **closed** and oriented **outward**.
2. Compute $\nabla \cdot \vec{F}$.
3. Describe the volume E with clean bounds (Cartesian / cylindrical / spherical).
4. Compute $\iiint_E (\nabla \cdot \vec{F})\,dV$.

Example 1 (your screenshot): cone + top disk.
Vector field:

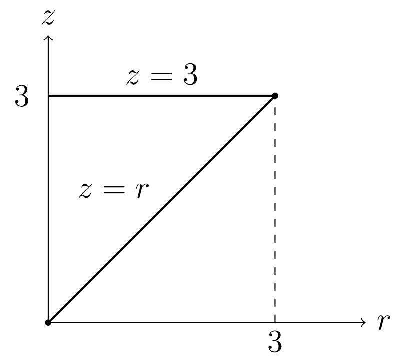
$$\vec{F} = \langle x^2,\ 2y^2,\ 3z^2\rangle.$$

Closed surface:

$$S_1 :\ z = 3 \quad \text{(top disk)}, \qquad S_2 :\ z = \sqrt{x^2 + y^2} \quad \text{(cone)}.$$

This is closed (cone + cap), outward orientation.

Cross-section picture in the (r, z)-plane:



Step 1: divergence.

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(2y^2) + \frac{\partial}{\partial z}(3z^2) = 2x + 4y + 6z.$$

Step 2: bounds (cylindrical coordinates). Use $x = r\cos\theta$, $y = r\sin\theta$, $dV = r\,dz\,dr\,d\theta$. Cone: $z = r$. Top: $z = 3$. Intersection: $r = 3$.

$$0 \le \theta \le 2\pi, \qquad 0 \le r \le 3, \qquad r \le z \le 3.$$

Step 3: set up the triple integral. Substitute $x = r\cos\theta$, $y = r\sin\theta$:

$$\iiint_E (\nabla \cdot \vec{F})\,dV = \int_0^{2\pi}\int_0^3\int_r^3 \big(2r\cos\theta + 4r\sin\theta + 6z\big)\,r\,dz\,dr\,d\theta.$$

Key simplification: over $[0, 2\pi]$,

$$\int_0^{2\pi} \cos\theta\,d\theta = 0, \qquad \int_0^{2\pi} \sin\theta\,d\theta = 0,$$

so the $2r\cos\theta$ and $4r\sin\theta$ parts vanish after integrating in θ. Thus,

$$= \int_0^{2\pi}\int_0^3\int_r^3 6z\,r\,dz\,dr\,d\theta.$$

Step 4: compute.

$$\int_r^3 6z\,r\,dz = 6r\left[\frac{z^2}{2}\right]_r^3 = 3r(9 - r^2).$$

$$\int_0^{2\pi}\int_0^3 3r(9 - r^2)\,dr\,d\theta = \int_0^{2\pi}\left(\left[\frac{243}{4}\right]\right)d\theta = \frac{243\pi}{2}.$$

Answer:

$$\iint_{\partial E} \vec{F} \cdot \vec{n}\,dS = \frac{243\pi}{2}.$$

Example 2 (sphere, classic).

Let E be the ball $x^2 + y^2 + z^2 \le a^2$ and $\vec{F} = \langle x, y, z \rangle$.

$$\nabla \cdot \vec{F} = 1 + 1 + 1 = 3.$$

So by Divergence Theorem:

$$\iint_{\partial E} \vec{F} \cdot \vec{n}\, dS = \iiint_E 3\, dV = 3 \cdot \mathrm{Vol}(E) = 3 \cdot \frac{4}{3}\pi a^3 = 4\pi a^3.$$

Example 3 (cube: one triple integral instead of 6 surface integrals).
Let $E = [0,1] \times [0,1] \times [0,1]$ and $\vec{F} = \langle xy,\ yz,\ zx \rangle$.

$$\nabla \cdot \vec{F} = \frac{\partial(xy)}{\partial x} + \frac{\partial(yz)}{\partial y} + \frac{\partial(zx)}{\partial z} = y + z + x.$$

Hence

$$\iint_{\partial E} \vec{F} \cdot \vec{n}\, dS = \iiint_E (x + y + z)\, dV = \int_0^1 \int_0^1 \int_0^1 (x + y + z)\, dx\, dy\, dz.$$

Compute by linearity:

$$\iiint_E x\, dV = \frac{1}{2}, \qquad \iiint_E y\, dV = \frac{1}{2}, \qquad \iiint_E z\, dV = \frac{1}{2},$$

so

$$\iint_{\partial E} \vec{F} \cdot \vec{n}\, dS = \frac{3}{2}.$$

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7) Fast corollary (very useful).
If $\nabla \cdot \vec{F} = 0$ everywhere inside E, then

$$\iint_{\partial E} \vec{F} \cdot \vec{n}\, dS = \iiint_E 0\, dV = 0.$$

So **divergence-free fields have zero net flux through any closed surface** (under the smoothness conditions).