The Cross Product

Big idea (one sentence)

The **cross product** takes two vectors in $\mathbb{R}^3$ and returns a **new vector perpendicular to both**, with magnitude equal to the **area of the parallelogram** they span.

1. What the cross product outputs (the goal)

Given $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$, the vector $\mathbf{a} \times \mathbf{b}$ satisfies:

- **Perpendicularity:** $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = 0$ and $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0$.
- **Magnitude:** $\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \ \|\mathbf{b}\| \ \sin\theta$ where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.
- **Direction (orientation):** determined by the **right-hand rule**.

So it is the perfect tool when you need a **normal vector** (perpendicular direction) and an **area**.

2. Geometric meaning: area + right-hand rule

2.1 Magnitude = parallelogram area

Let θ be the angle between nonzero vectors $\mathbf{a}$ and $\mathbf{b}$. Then

$$\boxed{\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \ \|\mathbf{b}\| \ \sin\theta}.$$

Why this equals area: think of the parallelogram with sides $\mathbf{a}$ and $\mathbf{b}$. Its area is

$$\text{Area}_{\text{para}} = (\text{base})(\text{height}) = \|\mathbf{a}\| \left(\|\mathbf{b}\| \sin\theta \right).$$

So:

$$\boxed{\text{Area}_{\text{para}} = \|\mathbf{a} \times \mathbf{b}\|}, \qquad \boxed{\text{Area}_{\triangle} = \tfrac{1}{2} \ \|\mathbf{a} \times \mathbf{b}\|}.$$

2.2 Direction = right-hand rule

Right-hand rule: point your right-hand fingers along $\mathbf{a}$, curl toward $\mathbf{b}$ (through the smaller angle), and your thumb points in the direction of $\mathbf{a} \times \mathbf{b}$.

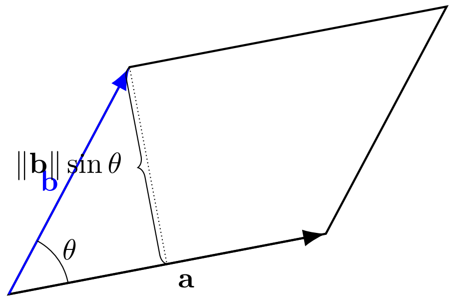

Figure 1: Parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$. The height is $\|\mathbf{b}\| \sin\theta$, so area is $\|\mathbf{a}\| \, \|\mathbf{b}\| \sin\theta = \|\mathbf{a} \times \mathbf{b}\|$.

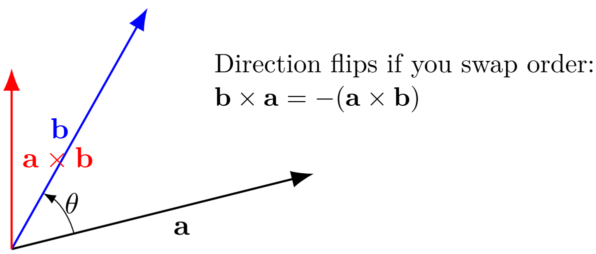

Figure 2: Orientation matters: $\mathbf{a} \times \mathbf{b}$ points according to the right-hand rule, and swapping order reverses direction.

3. Computational formula (what you actually do)

Let $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$. Then

$$\boxed{\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, \;\; a_3b_1 - a_1b_3, \;\; a_1b_2 - a_2b_1 \rangle}.$$

A common mnemonic is the determinant pattern:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

Warning (sign mistake trap): the middle component carries a minus sign in the cofactor expansion:

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}.$$

4. Properties (the ones you actually use)

For $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ and scalar λ:

1. **Anti-commutative:** $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$.
2. **Distributive:** $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$.
3. **Scalars pull out:** $(\lambda\mathbf{a}) \times \mathbf{b} = \lambda(\mathbf{a} \times \mathbf{b})$.

4. **Parallel test:** $\mathbf{a} \times \mathbf{b} = \mathbf{0} \iff \mathbf{a}, \mathbf{b}$ are linearly dependent (parallel or one is zero).
5. **Orthogonality:** $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = 0$ and $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0$.

5. MIT-style "why": why the magnitude is $\|\mathbf{a}\|\,\|\mathbf{b}\| \sin\theta$

This is the geometric fact that makes the cross product useful.

Think of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$. Let θ be the angle between them. The height relative to base $\mathbf{a}$ is $\|\mathbf{b}\| \sin\theta$, so the area is

$$\text{Area} = \|\mathbf{a}\| \cdot (\|\mathbf{b}\| \sin\theta).$$

The cross product is *defined* to be the unique vector perpendicular to both with this magnitude and right-hand direction. So the magnitude formula is not an accident: it is **built into the definition**.

6. Worked examples (fully detailed)

Example 1: compute a cross product

Let $\mathbf{a} = \langle 1, 2, 3\rangle$ and $\mathbf{b} = \langle 4, 0, -1\rangle$. Compute each component carefully:

$$\mathbf{a} \times \mathbf{b} = \langle 2(-1) - 3(0),\ \ 3(4) - 1(-1),\ \ 1(0) - 2(4)\rangle = \langle -2,\ \ 12 + 1,\ \ -8\rangle = \langle -2, 13, -8\rangle.$$

$$\boxed{\mathbf{a} \times \mathbf{b} = \langle -2, 13, -8\rangle.}$$

Example 2: area of a triangle

With the same vectors, the triangle area is

$$\text{Area}_\triangle = \frac{1}{2}\,\|\mathbf{a} \times \mathbf{b}\| = \frac{1}{2}\sqrt{(-2)^2 + 13^2 + (-8)^2} = \frac{1}{2}\sqrt{4 + 169 + 64} = \frac{1}{2}\sqrt{237}.$$

$$\boxed{\text{Area}_\triangle = \frac{\sqrt{237}}{2}.}$$

Example 3: check perpendicularity using dot product

Verify $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = 0$:

$$\langle -2, 13, -8\rangle \cdot \langle 1, 2, 3\rangle = (-2)(1) + 13(2) + (-8)(3) = -2 + 26 - 24 = 0.$$

Similarly, $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0$:

$$\langle -2, 13, -8\rangle \cdot \langle 4, 0, -1\rangle = (-2)(4) + 13(0) + (-8)(-1) = -8 + 0 + 8 = 0.$$

So $\mathbf{a} \times \mathbf{b}$ is perpendicular to both.

Example 4: a normal vector to a plane

A plane contains direction vectors $\mathbf{u} = \langle 1, -1, 0\rangle$ and $\mathbf{v} = \langle 2, 1, 3\rangle$. A normal vector is $\mathbf{n} = \mathbf{u} \times \mathbf{v}$:

$$\mathbf{u} \times \mathbf{v} = \langle (-1)3 - 0 \cdot 1,\ \ 0 \cdot 2 - 1 \cdot 3,\ \ 1 \cdot 1 - (-1)2\rangle = \langle -3,\ \ -3,\ \ 3\rangle = 3\langle -1, -1, 1\rangle.$$

Any nonzero scalar multiple works, so a clean normal is $\boxed{\mathbf{n} = \langle -1, -1, 1\rangle}$.

If the plane passes through $P_0 = (1, 0, 2)$, then its equation is

$$\mathbf{n} \cdot \langle x - 1,\ y - 0,\ z - 2\rangle = 0 \Rightarrow -(x-1) - (y) + (z-2) = 0 \Rightarrow \boxed{-x - y + z - 1 = 0.}$$

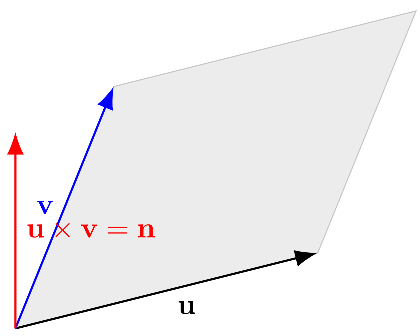

Figure 3: Two non-parallel directions in a plane determine a normal: $\mathbf{n} = \mathbf{u} \times \mathbf{v}$.

Example 5: torque / moment (physics)

If $\mathbf{r}$ is the position vector from the pivot and $\mathbf{F}$ is the force, torque is

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}.$$

Let $\mathbf{r} = \langle 0, 2, 0\rangle$ (2 m up the y-axis) and $\mathbf{F} = \langle 5, 0, 0\rangle$ (5 N in $+x$). Then

$$\boldsymbol{\tau} = \langle 0, 2, 0\rangle \times \langle 5, 0, 0\rangle = \langle 2 \cdot 0 - 0 \cdot 0,\ \ 0 \cdot 5 - 0 \cdot 0,\ \ 0 \cdot 0 - 2 \cdot 5\rangle = \langle 0, 0, -10\rangle.$$

$$\boxed{\boldsymbol{\tau} = \langle 0, 0, -10\rangle \text{ N·m.}}$$

Magnitude 10 equals $\|\mathbf{r}\|\,\|\mathbf{F}\| \sin 90^\circ = 2 \cdot 5 \cdot 1$.

7. Cross product in vector calculus (preview, why you will see it later)

7.1 Parametric surface element

If a surface is parameterized by $\mathbf{r}(u, v)$, then the tangent vectors are

$$\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u}, \qquad \mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v}.$$

Their cross product gives an **oriented area element**:

$$\boxed{d\mathbf{S} = (\mathbf{r}_u \times \mathbf{r}_v)\, du\, dv}.$$

Its magnitude $\|\mathbf{r}_u \times \mathbf{r}_v\|$ is the area-scaling factor.

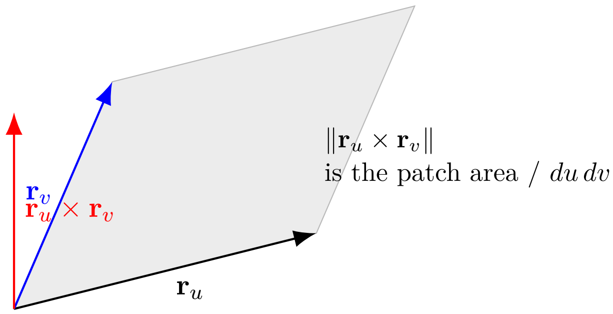

Figure 4: For a parametric surface, $\mathbf{r}_u \times \mathbf{r}_v$ gives the oriented normal and the area scaling factor.

7.2 Connection to *orientation*

If you swap the order, the normal flips:

$$\mathbf{r}_v \times \mathbf{r}_u = -(\mathbf{r}_u \times \mathbf{r}_v).$$

This is why orientation choices matter in surface integrals and Stokes' theorem.

8. Quick check (you should be able to answer instantly)

1. What does $\|\mathbf{a} \times \mathbf{b}\|$ represent geometrically?
2. What happens to $\mathbf{a} \times \mathbf{b}$ if you swap $\mathbf{a}$ and $\mathbf{b}$?
3. When is $\mathbf{a} \times \mathbf{b} = \mathbf{0}$?
4. How do you get a normal vector to a plane spanned by $\mathbf{u}$ and $\mathbf{v}$?
5. What is the area of the triangle with sides $\mathbf{a}$ and $\mathbf{b}$?