Line Integrals of Vector Fields and the Fundamental Theorem of Line Integrals

Calculus 3 / Vector Calculus Notes (Thomas/Stewart style)

Line integral of a vector field

Let

$$\mathbf{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle$$

be a vector field, and let C be a smooth curve parametrized by

$$\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle, \quad a \le t \le b.$$

What does $d\mathbf{r}$ mean?

Along a curve, a tiny displacement is

$$d\mathbf{r} = \langle dx, dy, dz\rangle.$$

Using the parametrization,

$$dx = x'(t)\,dt, \quad dy = y'(t)\,dt, \quad dz = z'(t)\,dt,$$

so

$$d\mathbf{r} = \mathbf{r}'(t)\,dt.$$

Definition

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

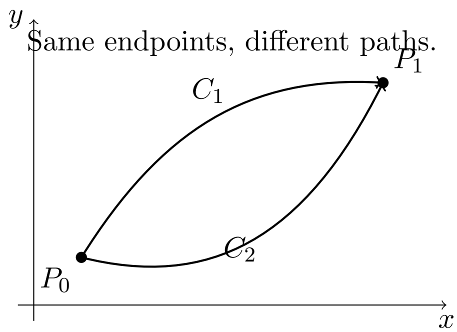
is called the line integral of $\mathbf{F}$ along C.

Computation formula

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt.$$

Path independence (what it means)

Let C be any smooth curve connecting P_0 to P_1. We say the line integral is *path independent* if its value is the same for every curve connecting P_0 to P_1 (as long as all curves stay inside the region where $\mathbf{F}$ is defined).



Conservative vector fields (quick reminder)

A vector field $\mathbf{F}$ is called *conservative* if there exists a scalar function ϕ such that

$$\nabla\phi = \mathbf{F}.$$

The function ϕ is called a *potential function*.

The Fundamental Theorem of Line Integrals

Theorem

If $\mathbf{F} = \nabla\phi$ on a region and C is any smooth curve from P_0 to P_1, then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \phi(P_1) - \phi(P_0).$$

Proof (chain rule)

Let C be parametrized by $\mathbf{r}(t)$, $a \le t \le b$. Then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt.$$

If $\mathbf{F} = \nabla\phi$, then

$$\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t) = \nabla\phi(\mathbf{r}(t))\cdot\mathbf{r}'(t).$$

By the multivariable chain rule,

$$\frac{d}{dt}\big(\phi(\mathbf{r}(t))\big) = \nabla\phi(\mathbf{r}(t))\cdot\mathbf{r}'(t).$$

So

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \frac{d}{dt}\big(\phi(\mathbf{r}(t))\big)\,dt = [\phi(\mathbf{r}(t))]_a^b = \phi(\mathbf{r}(b)) - \phi(\mathbf{r}(a)).$$

Since $\mathbf{r}(a) = P_0$ and $\mathbf{r}(b) = P_1$,

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \phi(P_1) - \phi(P_0).$$

Direct consequences

Path independence

If $\mathbf{F} = \nabla\phi$, then for any two curves C_1 and C_2 with the same endpoints,

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}.$$

Closed curves

If C is closed (starts and ends at the same point), then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

How to decide if F is conservative

In $\mathbb{R}^2$

Let

$$\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle.$$

If P and Q have continuous first partial derivatives on a simply connected region, then

$$\mathbf{F} \text{ is conservative } \iff \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

In $\mathbb{R}^3$

Let

$$\mathbf{F}(x, y, z) = \langle P, Q, R \rangle.$$

If P, Q, R have continuous first partial derivatives on a simply connected region, then

$$\mathbf{F} \text{ is conservative } \iff \nabla \times \mathbf{F} = \mathbf{0}.$$

Finding a potential function (algorithm)

In $\mathbb{R}^3$

Given $\mathbf{F} = \langle P, Q, R \rangle$, we want ϕ such that

$$\phi_x = P, \quad \phi_y = Q, \quad \phi_z = R.$$

1. Integrate P with respect to x:

$$\phi(x, y, z) = \int P(x, y, z)\, dx + g(y, z).$$

2. Differentiate this with respect to y and match Q to determine $g_y(y,z)$.
3. Use $\phi_z = R$ to determine the remaining part of $g(y,z)$.

Worked example

Evaluate

$$\int_C y\,dx + x\,dy + 4\,dz$$

from $A(1,1,1)$ to $B(2,3,-1)$.

Step 1: identify the field

$$\int_C y\,dx + x\,dy + 4\,dz = \int_C \langle y, x, 4\rangle \cdot \langle dx, dy, dz\rangle = \int_C \mathbf{F}\cdot d\mathbf{r},$$

so

$$\mathbf{F}(x,y,z) = \langle y, x, 4\rangle.$$

Step 2: find a potential

We want $\nabla\phi = \langle y, x, 4\rangle$.

Start from $\phi_x = y$:

$$\phi(x,y,z) = \int y\,dx = xy + g(y,z).$$

Match $\phi_y = x$:

$$\phi_y = x + g_y(y,z) = x \quad\Rightarrow\quad g_y(y,z) = 0,$$

so $g(y,z) = h(z)$. Match $\phi_z = 4$:

$$\phi_z = h'(z) = 4 \quad\Rightarrow\quad h(z) = 4z + C.$$

Thus we can take

$$\phi(x,y,z) = xy + 4z.$$

Step 3: apply the theorem

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \phi(B) - \phi(A).$$

Compute

$$\phi(2,3,-1) = 2\cdot 3 + 4(-1) = 2, \qquad \phi(1,1,1) = 1\cdot 1 + 4(1) = 5,$$

so

$$\int_C y\,dx + x\,dy + 4\,dz = 2 - 5 = -3.$$