Dot Product (Scalar Product)

Big idea (one sentence)

The **dot product** takes two vectors and produces a **number** that measures **how strongly they point in the same direction**.

1. Definition in coordinates (the rule you actually compute)

1.1 In 2D

If $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$, then

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2.$$

1.2 In 3D

If $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, then

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

Memory trick: multiply matching coordinates and add.

1.3 Unit-vector form

In $\mathbb{R}^3$, if $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, then it is the same:

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

(Reason: $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$ and mixed dot products are 0.)

2. First consequences: length and orthogonality

2.1 The dot product gives length

Take $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$. Then

$$\mathbf{u} \cdot \mathbf{u} = u_1^2 + u_2^2 + u_3^2 = \|\mathbf{u}\|^2, \qquad \Rightarrow \qquad \|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}}.$$

So dot product *contains* the length formula.

2.2 Perpendicular test

Two nonzero vectors are perpendicular (orthogonal) exactly when

$$\mathbf{u} \cdot \mathbf{v} = 0.$$

Why? Because "no alignment" means the cosine of the angle is 0 (we make this precise next).

3. Geometric meaning: the angle formula

For nonzero vectors, the dot product satisfies

$$\boxed{\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \ \|\mathbf{v}\| \ \cos\theta}$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$ (between 0 and π).

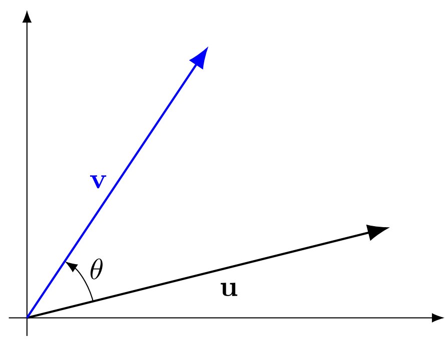

Figure 1: Angle θ between two vectors. The dot product measures alignment via $\cos\theta$.

Immediate interpretation:

- If $\theta < \frac{\pi}{2}$ (acute), then $\cos\theta > 0$ so $\mathbf{u} \cdot \mathbf{v} > 0$.
- If $\theta = \frac{\pi}{2}$, then $\cos\theta = 0$ so $\mathbf{u} \cdot \mathbf{v} = 0$ (perpendicular).
- If $\theta > \frac{\pi}{2}$ (obtuse), then $\cos\theta < 0$ so $\mathbf{u} \cdot \mathbf{v} < 0$.

4. Why the angle formula is true (MIT-style proof idea)

This is the key identity that connects the *coordinate definition* to *geometry*.

Let $\mathbf{A}$ and $\mathbf{B}$ be vectors with angle θ between them, and define

$$\mathbf{C} = \mathbf{A} - \mathbf{B}.$$

Compute $\|\mathbf{C}\|^2$ **in two ways**.

Way 1: use dot product algebra

$$\|\mathbf{C}\|^2 = \mathbf{C}\cdot\mathbf{C} = (\mathbf{A}-\mathbf{B})\cdot(\mathbf{A}-\mathbf{B}) = \mathbf{A}\cdot\mathbf{A} - 2\mathbf{A}\cdot\mathbf{B} + \mathbf{B}\cdot\mathbf{B}.$$

But $\mathbf{A}\cdot\mathbf{A} = \|\mathbf{A}\|^2$ and $\mathbf{B}\cdot\mathbf{B} = \|\mathbf{B}\|^2$, so

$$\|\mathbf{C}\|^2 = \|\mathbf{A}\|^2 + \|\mathbf{B}\|^2 - 2(\mathbf{A}\cdot\mathbf{B}).$$

Way 2: use the Law of Cosines

Geometrically, $\mathbf{A}, \mathbf{B}, \mathbf{C}$ form a triangle, and the law of cosines says

$$\|\mathbf{C}\|^2 = \|\mathbf{A}\|^2 + \|\mathbf{B}\|^2 - 2\,\|\mathbf{A}\|\,\|\mathbf{B}\|\cos\theta.$$

Compare the two formulas

Both right-hand sides equal $\|\mathbf{C}\|^2$, so the "$-2(\cdots)$" terms must match:

$$2(\mathbf{A}\cdot\mathbf{B}) = 2\,\|\mathbf{A}\|\,\|\mathbf{B}\|\cos\theta \quad\Rightarrow\quad \boxed{\mathbf{A}\cdot\mathbf{B} = \|\mathbf{A}\|\,\|\mathbf{B}\|\cos\theta.}$$

5. Angle between two vectors (how you compute it)

If $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{v} \neq \mathbf{0}$:

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\ \|\mathbf{v}\|}, \qquad \theta = \arccos\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\ \|\mathbf{v}\|}\right).$$

6. Projection (this is the dot product's superpower)

6.1 Scalar projection

From the geometric formula:

$$\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta$$

and $\|\mathbf{u}\|\cos\theta$ is exactly the **signed length** of the shadow of $\mathbf{u}$ onto the direction of $\mathbf{v}$.

So the **scalar projection of u onto v** is

$$\boxed{\mathrm{comp}_{\mathbf{v}}(\mathbf{u}) = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|}} \qquad (\mathbf{v}\neq\mathbf{0}).$$

6.2 Vector projection

The actual projected vector must point in the direction of $\mathbf{v}$, so it is a scalar times $\mathbf{v}$:

$$\mathrm{proj}_{\mathbf{v}}(\mathbf{u}) = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}.$$

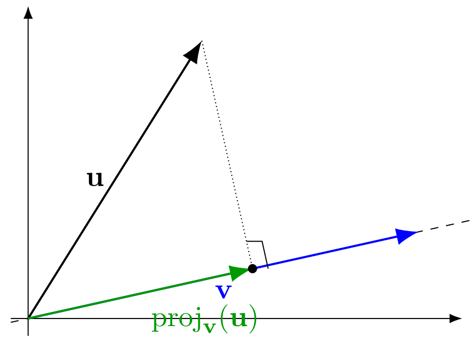

Figure 2: Projection: $\mathbf{u} = \text{proj}_{\mathbf{v}}(\mathbf{u}) + (\text{perpendicular part})$.

7. Main algebra rules (so you can simplify expressions)

For vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ and scalar c:

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} \quad \text{(commutative)},$$

$$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w} \quad \text{(distributive)},$$

$$(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) \quad \text{(pull scalars out)}.$$

8. Worked examples (fully solved)

Example 1: compute a dot product (2D)

Let $\mathbf{u} = \langle 3, -2 \rangle$ and $\mathbf{v} = \langle -2, 5 \rangle$. Then

$$\mathbf{u} \cdot \mathbf{v} = 3(-2) + (-2)(5) = -6 - 10 = -16.$$

$$\boxed{\mathbf{u} \cdot \mathbf{v} = -16.}$$

Meaning: negative dot product $\Rightarrow$ angle is obtuse (they point more opposite than same).

Example 2: perpendicular test

Are $\langle 1, 2 \rangle$ and $\langle 2, -1 \rangle$ perpendicular?

$$\langle 1, 2 \rangle \cdot \langle 2, -1 \rangle = 1 \cdot 2 + 2 \cdot (-1) = 2 - 2 = 0.$$

$$\boxed{\text{Yes. Dot product } 0 \Rightarrow \text{perpendicular.}}$$

Example 3: find an angle

Let $\mathbf{a} = \langle 1, 2, 3 \rangle$ and $\mathbf{b} = \langle 4, 0, -1 \rangle$.

$$\mathbf{a} \cdot \mathbf{b} = 1 \cdot 4 + 2 \cdot 0 + 3 \cdot (-1) = 4 - 3 = 1.$$

Lengths:

$$\|\mathbf{a}\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}, \qquad \|\mathbf{b}\| = \sqrt{4^2 + 0^2 + (-1)^2} = \sqrt{17}.$$

So

$$\cos\theta = \frac{1}{\sqrt{14}\sqrt{17}} = \frac{1}{\sqrt{238}}, \qquad \boxed{\theta = \arccos\left(\frac{1}{\sqrt{238}}\right).}$$

Example 4: scalar projection

Let $\mathbf{u} = \langle 2, 3\rangle$ and $\mathbf{v} = \langle 4, 0\rangle$.

$$\text{comp}_{\mathbf{v}}(\mathbf{u}) = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|} = \frac{(2)(4) + (3)(0)}{4} = \frac{8}{4} = 2.$$

$$\boxed{\text{comp}_{\mathbf{v}}(\mathbf{u}) = 2.}$$

Meaning: along the direction of $\mathbf{v}$ *(the* x*-axis),* $\mathbf{u}$ *has component* 2*.*

Example 5: vector projection

With the same vectors,

$$\text{proj}_{\mathbf{v}}(\mathbf{u}) = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} = \left(\frac{8}{16}\right)\langle 4, 0\rangle = \frac{1}{2}\langle 4, 0\rangle = \langle 2, 0\rangle.$$

$$\boxed{\text{proj}_{\mathbf{v}}(\mathbf{u}) = \langle 2, 0\rangle.}$$

Example 6: decomposition into parallel + perpendicular parts

Let $\mathbf{u} = \langle 2, 3\rangle$ and $\mathbf{v} = \langle 4, 0\rangle$. We already found $\text{proj}_{\mathbf{v}}(\mathbf{u}) = \langle 2, 0\rangle$. So the perpendicular remainder is

$$\mathbf{u} - \text{proj}_{\mathbf{v}}(\mathbf{u}) = \langle 2, 3\rangle - \langle 2, 0\rangle = \langle 0, 3\rangle.$$

Check perpendicularity:

$$\langle 0, 3\rangle \cdot \langle 4, 0\rangle = 0\cdot 4 + 3\cdot 0 = 0.$$

$$\boxed{\mathbf{u} = \langle 2, 0\rangle + \langle 0, 3\rangle \text{ (parallel part + perpendicular part).}}$$

Example 7: work (physics meaning)

A force $\mathbf{F}$ moving an object through displacement $\mathbf{d}$ does work

$$W = \mathbf{F}\cdot\mathbf{d}.$$

Let $\mathbf{F} = \langle 10, 0\rangle$ (10 N to the right) and $\mathbf{d} = \langle 3, 4\rangle$ (meters).

$$W = \langle 10, 0\rangle \cdot \langle 3, 4\rangle = 10\cdot 3 + 0\cdot 4 = 30.$$

$$\boxed{W = 30 \text{ J.}}$$

Only the component of force in the displacement direction contributes.

Example 8: "closest point" idea (projection in disguise)

Find the point on the x-axis closest to $P = (2, 3)$. The x-axis direction is $\mathbf{v} = \langle 1, 0\rangle$, so the closest point is the projection of $\langle 2, 3\rangle$ onto $\mathbf{v}$:

$$\mathrm{proj}_{\mathbf{v}}(\langle 2,3\rangle) = \left(\frac{\langle 2,3\rangle \cdot \langle 1,0\rangle}{\|\langle 1,0\rangle\|^2}\right)\langle 1,0\rangle = (2)\langle 1,0\rangle = \langle 2,0\rangle.$$

$$\boxed{(2,0) \text{ is closest.}}$$

9. Quick check (you should be able to answer instantly)

1. What does $\mathbf{u} \cdot \mathbf{u}$ equal?
2. If $\mathbf{u} \cdot \mathbf{v} < 0$, is the angle acute or obtuse?
3. If $\mathbf{u} \cdot \mathbf{v} = 0$ and both nonzero, what does that mean geometrically?
4. Write $\mathrm{proj}_{\mathbf{v}}(\mathbf{u})$ using only dot products and $\mathbf{v}$.