Three-Dimensional Coordinate System

A **3D Cartesian coordinate system** is a way to label every point in space by an ordered triple

$$(x, y, z) \in \mathbb{R}^3,$$

measured along three mutually perpendicular axes.

Key idea. In 2D you need two independent directions (x, y). In 3D you add a third independent direction z ("height").

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1. $\mathbb{R}^3$ as a mathematical object

Definition. $\mathbb{R}^3$ is the set of all ordered triples of real numbers:

$$\mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}.$$

Geometrically, we represent (x, y, z) as a point in space.

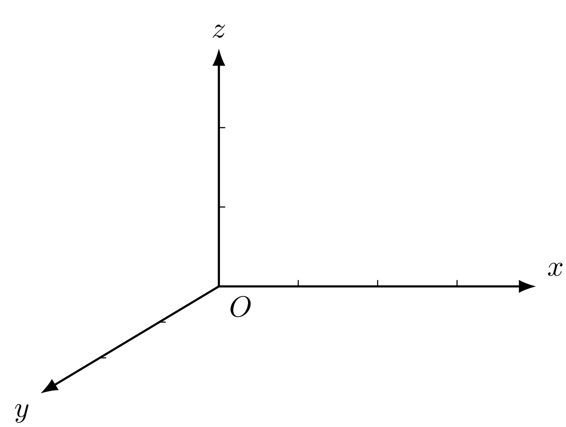
1.1 Origin and axes

The **origin** is

$$O = (0, 0, 0).$$

The coordinate axes are the three special lines:

$$x\text{-axis: } (x, 0, 0), \qquad y\text{-axis: } (0, y, 0), \qquad z\text{-axis: } (0, 0, z).$$



2. What the triple (x, y, z) *means* (movement interpretation)

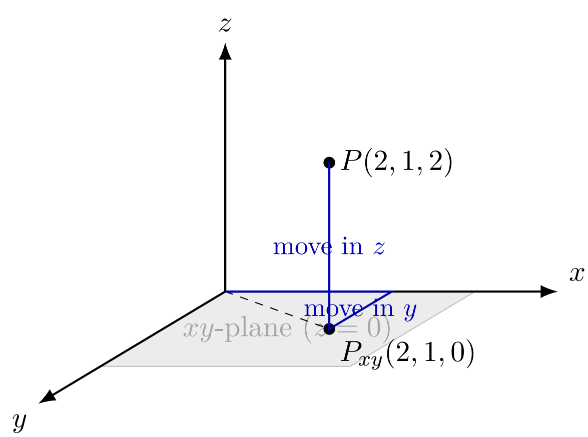
To reach $P = (x, y, z)$ from the origin:

1. Move along the x direction to $(x, 0, 0)$.
2. Move parallel to the y axis to $(x, y, 0)$.
3. Move parallel to the z axis to (x, y, z).

Key idea. The point $(x, y, 0)$ is the **projection** (shadow) of (x, y, z) on the xy-plane. Then z tells you how far above/below that plane you are.

2.1 Picture: plotting a point and its projection

Example point: $P = (2, 1, 2)$. Its projection on the xy-plane is $P_{xy} = (2, 1, 0)$.



3. Coordinate planes and octants

Setting one coordinate to zero gives a fundamental plane:

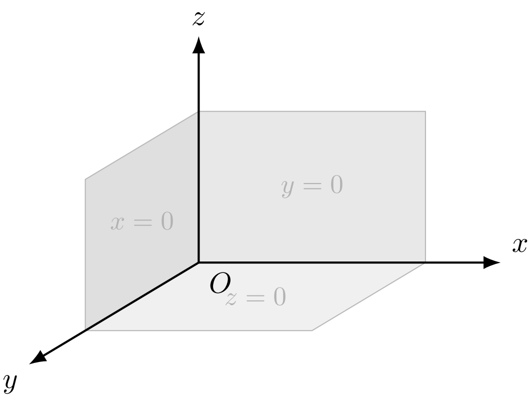
$$xy\text{-plane: } z = 0, \qquad xz\text{-plane: } y = 0, \qquad yz\text{-plane: } x = 0.$$

These three planes cut space into 8 sign-regions called **octants**.

3.1 Quick sign test

$$(-2, 5, 1) \in (-, +, +), \qquad (3, -1, -4) \in (+, -, -).$$

3.2 Picture: the three coordinate planes



4. Points vs vectors (language for physics and calculus)

Point $P = (x, y, z)$ is a location.

Vector $\vec{v} = \langle a, b, c \rangle$ is a displacement (direction + length).

4.1 Standard basis and components

$$\mathbf{i} = \langle 1, 0, 0 \rangle, \qquad \mathbf{j} = \langle 0, 1, 0 \rangle, \qquad \mathbf{k} = \langle 0, 0, 1 \rangle.$$

Any vector can be written as

$$\vec{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}.$$

The **position vector** of P is

$$\overrightarrow{OP} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Key idea. In multivariable calculus, you attach vectors to points: a **vector field** is a function $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ that assigns a vector to each point.

5. Distance in 3D

5.1 The distance formula

For $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ define the difference vector

$$\overrightarrow{P_1P_2} = \langle x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1 \rangle.$$

Then the distance is the length of this vector:

$$d(P_1, P_2) = \|\overrightarrow{P_1P_2}\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

5.2 Why this formula is true (one clean argument)

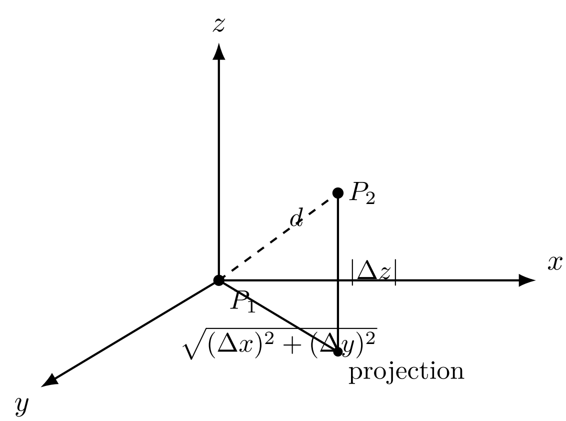
Let $\Delta x = x_2 - x_1$, $\Delta y = y_2 - y_1$, $\Delta z = z_2 - z_1$. First apply Pythagoras in the xy-plane to get the horizontal length

$$\sqrt{(\Delta x)^2 + (\Delta y)^2}.$$

Then apply Pythagoras again using Δz as "vertical":

$$d^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2.$$

5.3 Picture: two-step Pythagoras idea



6. Lines in 3D (parametric form is the default)

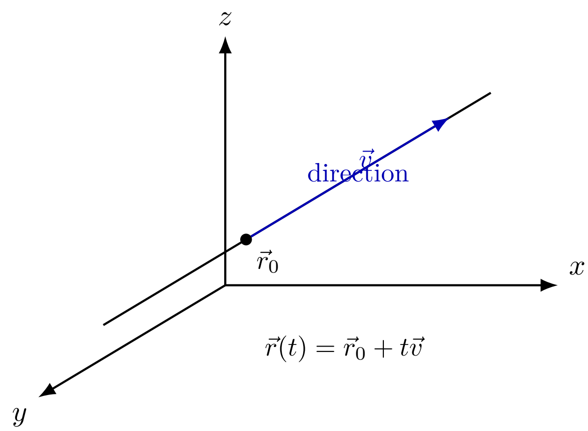
A line is determined by:

- one point $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$ on the line,
- one direction vector $\vec{v} = \langle a, b, c \rangle$.

Parametric equation:

$$\vec{r}(t) = \vec{r}_0 + t\vec{v}, \qquad \text{i.e.} \quad x = x_0 + at,\ y = y_0 + bt,\ z = z_0 + ct.$$

6.1 Picture: a point plus a direction



7. Planes in 3D (normal vector viewpoint)

7.1 Standard form

A plane is often written as

$$Ax + By + Cz = D.$$

The vector $\vec{n} = \langle A, B, C \rangle$ is perpendicular to the plane (it is a **normal**).

7.2 Point–normal form (conceptually clean)

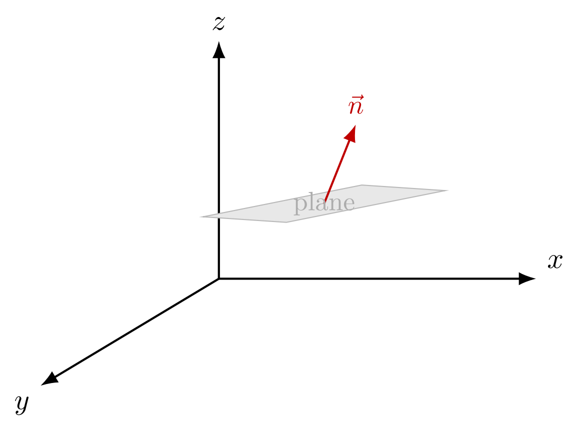
If the plane passes through $P_0 = (x_0, y_0, z_0)$:

$$\vec{n} \cdot (\langle x, y, z \rangle - \langle x_0, y_0, z_0 \rangle) = 0,$$

i.e.

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

7.3 Picture: plane patch + normal vector



8. Spheres and "constant distance"

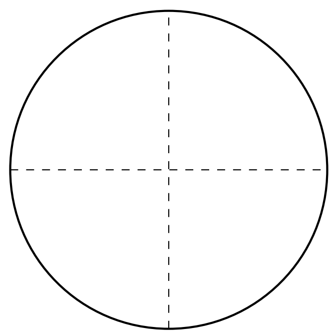
Definition. A sphere with center $C = (h, k, \ell)$ and radius r is the set

$$\{(x, y, z) \in \mathbb{R}^3 : \ (x-h)^2 + (y-k)^2 + (z-\ell)^2 = r^2\}.$$

This is exactly "distance from C equals r".

8.1 Picture: sphere idea (cross-section)

A 3D sphere is hard to draw on a 2D page, but a key fact is: *every slice $z = $ constant is a circle.*



9. Interpreting equations in 3D (the fastest way to think)

Key idea. Ask: **which variables are free** and which are constrained? Free variables usually mean "you can slide" in those directions.

- $x = 2$: y, z are free $\Rightarrow$ a **plane** parallel to the yz-plane.
- $z = 0$: x, y free $\Rightarrow$ the xy**-plane**.
- $x^2 + y^2 = 9$: z is free but (x, y) stay on a circle $\Rightarrow$ a **cylinder** of radius 3 around the z-axis.
- $x^2 + y^2 + z^2 = 9$: all three linked $\Rightarrow$ a **sphere** of radius 3.

10. Mini-exercises (to verify understanding)

Try these quickly (no calculator):

1. Which plane contains $(7, -3, 0)$?
2. Describe the set $x^2 + z^2 = 4$.

3. Give parametric equations of the line through $(1, 2, 3)$ in direction $\langle 2, 0, -1\rangle$.
4. Find a normal vector to the plane $3x - 2y + z = 5$.
5. What is the center and radius of $(x - 1)^2 + (y + 2)^2 + (z - 3)^2 = 16$?