Lines and Planes in $\mathbb{R}^3$: Span, Subspaces, and Equations

Big picture (one sentence)

In $\mathbb{R}^3$, a *line through the origin* is the span of one nonzero vector, and a *plane through the origin* is the span of two non-collinear vectors; both are *subspaces*.

Reminder: what "span" means

Given vectors $v_1, \ldots, v_k \in \mathbb{R}^3$, their span is

$$\operatorname{span}\{v_1, \ldots, v_k\} = \{a_1 v_1 + \cdots + a_k v_k : \ a_1, \ldots, a_k \in \mathbb{R}\}.$$

Interpretation: you can scale each vector and add them.

1. Lines through the origin = span of one vector

Let $v \in \mathbb{R}^3$ be nonzero. The set

$$L = \{tv : \ t \in \mathbb{R}\} = \operatorname{span}\{v\}$$

is a **line through the origin**.

Why is it a line? Because you can only move in one direction (along v) and its opposite (along $-v$).

Why must $v \neq 0$? If $v = 0$, then $tv = 0$ for all t, so you get only the single point $\{0\}$, not a line.

Example 1 (compute a parametric form)

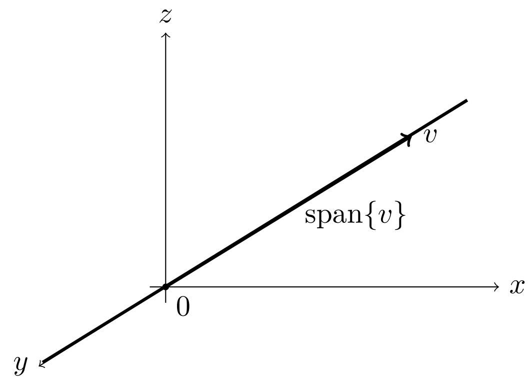
Let $v = \langle 2, -1, 3 \rangle$. Then

$$L = \{t\langle 2, -1, 3 \rangle : t \in \mathbb{R}\}.$$

Writing x, y, z coordinates:

$$(x, y, z) = (2t, -t, 3t), \quad t \in \mathbb{R}.$$

Figure: a line through the origin (schematic)



2. Planes through the origin = span of two non-collinear vectors

Let $v, w \in \mathbb{R}^3$.

2.1 Non-collinear means "not multiples"

Vectors v and w are **collinear** if $w = cv$ for some scalar c. If **not** collinear, then they point in genuinely different directions, so you can move in a 2D way.

2.2 Plane through the origin

If v and w are not collinear, then

$$P = \{sv + tw : \ s, t \in \mathbb{R}\} = \operatorname{span}\{v, w\}$$

is a **plane through the origin**.

Interpretation: sv moves you along direction v, tw moves you along direction w; together you can reach any point in the plane.

Example 2 (plane as span)

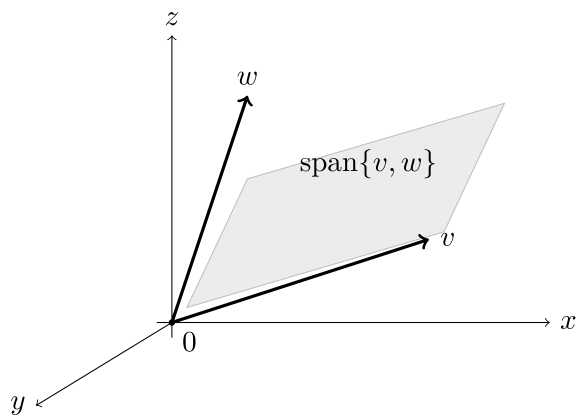
Take $v = \langle 1, 0, 1\rangle$ and $w = \langle 0, 2, 1\rangle$. Then

$$P = \operatorname{span}\{v, w\} = \{s\langle 1, 0, 1\rangle + t\langle 0, 2, 1\rangle\}.$$

So every point in the plane has the form

$$(x, y, z) = \langle s,\ 2t,\ s + t\rangle.$$

Figure: a plane through the origin spanned by two vectors



3. Why "through the origin" is the subspace condition

A subset $S \subseteq \mathbb{R}^3$ is a **subspace** if:

- $0 \in S$,
- if $a, b \in S$ then $a + b \in S$,
- if $a \in S$ and $c \in \mathbb{R}$ then $ca \in S$.

Key fact: $\mathrm{span}\{\cdots\}$ automatically satisfies these conditions, so it is always a subspace.

4. Shifted lines/planes (affine sets): not subspaces

If you take a subspace and shift it away from the origin, you get a **parallel translate**.

4.1 Line through a point a

A line through point $a \in \mathbb{R}^3$ with direction $v \neq 0$ is

$$\boxed{a + \mathrm{span}\{v\} = \{a + tv : \ t \in \mathbb{R}\}}.$$

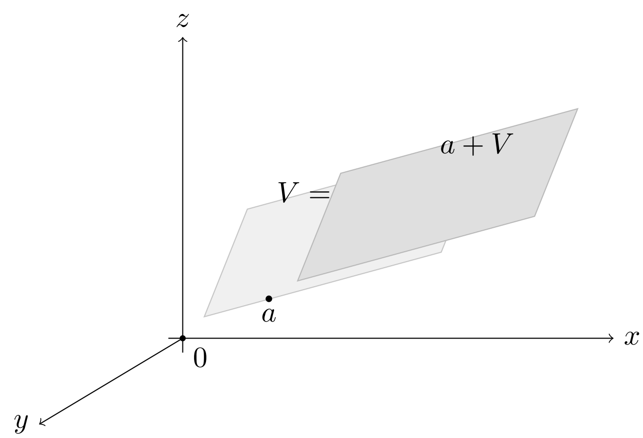
This is **not** a subspace unless $a = 0$ (because it does not contain the origin).

4.2 Plane through a point a

A plane through point a parallel to the plane $\mathrm{span}\{v, w\}$ is

$$\boxed{a + \mathrm{span}\{v, w\} = \{a + sv + tw : \ s, t \in \mathbb{R}\}}.$$

Figure: shifting a plane (schematic)



5. Connecting to equations (how you recognize a plane)

A plane through a point $P_0(x_0, y_0, z_0)$ with normal vector $n = \langle A, B, C\rangle \neq 0$ is

$$\boxed{n \cdot \langle x - x_0,\ y - y_0,\ z - z_0\rangle = 0}$$

which expands to

$$\boxed{A(x - x_0) + B(y - y_0) + C(z - z_0) = 0}$$

and equivalently

$$\boxed{Ax + By + Cz = D, \quad \text{where } D = Ax_0 + By_0 + Cz_0.}$$

Example 3 (plane through origin from two spanning vectors)

If a plane is $P = \operatorname{span}\{v, w\}$ with non-collinear v, w, then a normal is

$$n = v \times w.$$

So the plane through the origin can be written as

$$n \cdot \langle x, y, z\rangle = 0.$$

(That is exactly "plane through the origin" in equation form.)

6. Tiny checklist

- Given $v \neq 0$, write the line through 0 as $\{tv\}$ and in parametric coordinates.
- Given v, w not multiples, write the plane through 0 as $\{sv + tw\}$.
- Given a point a, shift: $a + \operatorname{span}\{\cdots\}$.
- Given a normal n, write plane equation via dot product.