Surface Integrals (Scalar + Flux)

1) Line integral (quick reminder)

If $C : \vec{\boldsymbol{r}}(t) = \langle x(t), y(t) \rangle$ for $t \in [a, b]$, then

$$\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\tfrac{dx}{dt}\right)^2 + \left(\tfrac{dy}{dt}\right)^2}\, dt.$$

2) Scalar surface integral

Definition. Let S be a smooth surface in $\mathbb{R}^3$ and let $f(x, y, z)$ be a scalar function defined on S. The *scalar surface integral* of f over S is

$$\iint_S f\, dS,$$

meaning: "sum of f over the surface, weighted by tiny surface areas."

3) How to compute dS (two standard cases)

A) Surface is a graph $z = g(x, y)$

Assume S is given by $z = g(x, y)$ over a region R in the xy-plane.

Proposition (Area element for a graph)**.** *For* $S : \ z = g(x, y)$,

$$dS = \sqrt{1 + g_x(x, y)^2 + g_y(x, y)^2}\, dA.$$

Hence

$$\iint_S f\, dS = \iint_R f(x, y, g(x, y)) \sqrt{1 + g_x^2 + g_y^2}\, dA.$$

Proof. Parameterize the surface by (x, y):

$$\vec{\boldsymbol{r}}(x, y) = \langle x,\ y,\ g(x, y) \rangle.$$

Then

$$\vec{\boldsymbol{r}}_x = \langle 1, 0, g_x \rangle, \qquad \vec{\boldsymbol{r}}_y = \langle 0, 1, g_y \rangle.$$

A small surface patch is approximated by the parallelogram spanned by $\vec{\boldsymbol{r}}_x\, dx$ and $\vec{\boldsymbol{r}}_y\, dy$. Its area is $\|\vec{\boldsymbol{r}}_x \times \vec{\boldsymbol{r}}_y\|\, dx\, dy$. Compute the cross product:

$$\vec{\boldsymbol{r}}_x \times \vec{\boldsymbol{r}}_y = \langle -g_x,\ -g_y,\ 1 \rangle, \quad \|\vec{\boldsymbol{r}}_x \times \vec{\boldsymbol{r}}_y\| = \sqrt{g_x^2 + g_y^2 + 1}.$$

So $dS = \sqrt{1 + g_x^2 + g_y^2}\, dA.$ □

B) Parametric surface $\vec{\boldsymbol{r}}(u,v)$

If S is parameterized by

$$\vec{\boldsymbol{r}}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle, \quad (u,v) \in D,$$

then

$$dS = \|\vec{\boldsymbol{r}}_u \times \vec{\boldsymbol{r}}_v\|\, du\, dv, \qquad \iint_S f\, dS = \iint_D f(\vec{\boldsymbol{r}}(u,v))\, \|\vec{\boldsymbol{r}}_u \times \vec{\boldsymbol{r}}_v\|\, du\, dv.$$

4) Flux (surface integral of a vector field)

Definition. Let $\vec{\boldsymbol{F}}(x,y,z)$ be a vector field and let S be an *oriented* surface with chosen unit normal $\vec{\boldsymbol{n}}$. The *flux* of $\vec{\boldsymbol{F}}$ through S is

$$\iint_S \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{n}}\, dS.$$

Flux for a graph $z = g(x,y)$ (no Divergence Theorem)

Proposition (Practical flux formula for graphs)**.** *If S is $z = g(x,y)$ over R and we choose the* upward *orientation (normal with positive z-component), then*

$$\vec{\boldsymbol{n}}\, dS = \langle -g_x,\ -g_y,\ 1\rangle\, dA$$

and therefore

$$\iint_S \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{n}}\, dS = \iint_R \vec{\boldsymbol{F}}(x,y,g(x,y)) \cdot \langle -g_x,\ -g_y,\ 1\rangle\, dA.$$

(Downward orientation just multiplies the answer by -1.)

Proof. From the graph parameterization $\vec{\boldsymbol{r}}(x,y) = \langle x, y, g(x,y)\rangle$,

$$\vec{\boldsymbol{r}}_x \times \vec{\boldsymbol{r}}_y = \langle -g_x,\ -g_y,\ 1\rangle.$$

This vector is normal to the surface and its direction corresponds to "upward" (since the z-component is $+1$). For flux we use the oriented area vector element

$$d\vec{\boldsymbol{S}} = \vec{\boldsymbol{n}}\, dS = (\vec{\boldsymbol{r}}_x \times \vec{\boldsymbol{r}}_y)\, dA.$$

So

$$\iint_S \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{n}}\, dS = \iint_R \vec{\boldsymbol{F}}(\vec{\boldsymbol{r}}(x,y)) \cdot (\vec{\boldsymbol{r}}_x \times \vec{\boldsymbol{r}}_y)\, dA,$$

which is exactly the claimed formula. $\square$

5) Worked examples (with diagrams)

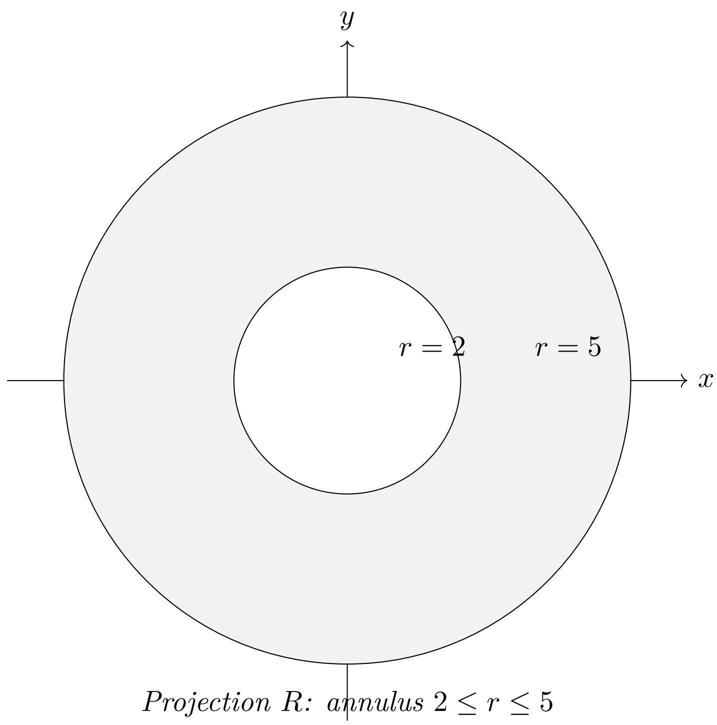
Example (Scalar surface integral on a cone (different numbers))**.** *Let S be the cone*

$$z = \sqrt{x^2 + y^2}$$

between $z = 2$ and $z = 5$ (so it is a cone frustum). Compute

$$\iint_S z\, dS.$$

Step 1: projection region in the xy-plane. *On the cone, $z = \sqrt{x^2 + y^2} = r$. The condition $2 \le z \le 5$ becomes $2 \le r \le 5$. So the projection R is an annulus: $2 \le r \le 5$.*



Projection R: annulus $2 \le r \le 5$

Step 2: compute dS for the graph. *Here $g(x, y) = \sqrt{x^2 + y^2} = r$. Compute partial derivatives (for $(x, y) \neq (0, 0)$):*

$$g_x = \frac{x}{\sqrt{x^2 + y^2}} = \frac{x}{r}, \qquad g_y = \frac{y}{\sqrt{x^2 + y^2}} = \frac{y}{r}.$$

Then

$$g_x^2 + g_y^2 = \frac{x^2 + y^2}{r^2} = 1, \quad \Rightarrow \quad dS = \sqrt{1 + 1}\, dA = \sqrt{2}\, dA.$$

Step 3: set up the integral and switch to polar. *On S, $z = r$. Therefore*

$$\iint_S z\, dS = \iint_R z \cdot \sqrt{2}\, dA = \iint_R r\sqrt{2}\, dA.$$

In polar coordinates, $dA = r\, dr\, d\theta$, so

$$\iint_S z\, dS = \int_0^{2\pi} \int_2^5 r\sqrt{2} \cdot r\, dr\, d\theta = \sqrt{2} \int_0^{2\pi} \int_2^5 r^2\, dr\, d\theta.$$

Step 4: compute.

$$\int_2^5 r^2\, dr = \left[\frac{r^3}{3}\right]_2^5 = \frac{125 - 8}{3} = \frac{117}{3} = 39.$$

So

$$\iint_S z\,dS = \sqrt{2}\,(2\pi)\,39 = 78\pi\sqrt{2}.$$

$$\boxed{\iint_S z\,dS = 78\pi\sqrt{2}.}$$

Example (Scalar surface integral on a parametric surface (parabolic cylinder))**.** *Let S be the part of the surface $y = x^2$ with*

$$1 \le x \le 3, \qquad 0 \le z \le 4.$$

Compute

$$\iint_S (xz)\,dS.$$

Step 1: parameterize the surface. *A natural parameterization is*

$$\vec{\boldsymbol{r}}(x,z) = \langle x,\ x^2,\ z\rangle, \quad (x,z) \in [1,3] \times [0,4].$$

Step 2: compute the area factor.

$$\vec{\boldsymbol{r}}_x = \langle 1,\ 2x,\ 0\rangle, \qquad \vec{\boldsymbol{r}}_z = \langle 0,\ 0,\ 1\rangle.$$

Cross product:

$$\vec{\boldsymbol{r}}_x \times \vec{\boldsymbol{r}}_z = \begin{vmatrix} \vec{\boldsymbol{i}} & \vec{\boldsymbol{j}} & \vec{\boldsymbol{k}} \\ 1 & 2x & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle 2x,\ -1,\ 0\rangle.$$

So

$$dS = \|\vec{\boldsymbol{r}}_x \times \vec{\boldsymbol{r}}_z\|\ dx\,dz = \sqrt{(2x)^2 + (-1)^2}\,dx\,dz = \sqrt{4x^2+1}\,dx\,dz.$$

Step 3: plug the integrand in. *On the surface, the integrand is (xz) (already in parameters), hence*

$$\iint_S (xz)\,dS = \int_{x=1}^{3}\int_{z=0}^{4} (xz)\sqrt{4x^2+1}\,dz\,dx.$$

Step 4: compute (explanation happens here, not earlier). *First integrate in z:*

$$\int_0^4 (xz)\sqrt{4x^2+1}\,dz = x\sqrt{4x^2+1}\int_0^4 z\,dz = x\sqrt{4x^2+1}\left[\frac{z^2}{2}\right]_0^4 = x\sqrt{4x^2+1}\cdot 8.$$

So

$$\iint_S (xz)\,dS = 8\int_1^3 x\sqrt{4x^2+1}\,dx.$$

Let $u = 4x^2+1$, then $du = 8x\,dx$, so $x\,dx = \frac{du}{8}$. When $x = 1$, $u = 5$; when $x = 3$, $u = 37$. Thus

$$8\int_1^3 x\sqrt{4x^2+1}\,dx = 8\int_5^{37} \sqrt{u}\cdot\frac{du}{8} = \int_5^{37} u^{1/2}\,du = \left[\frac{2}{3}u^{3/2}\right]_5^{37}.$$

Therefore

$$\boxed{\iint_S (xz)\,dS = \frac{2}{3}\Big(37^{3/2} - 5^{3/2}\Big) = \frac{2}{3}\Big(37\sqrt{37} - 5\sqrt{5}\Big).}$$

Example (Flux through a plane patch (NO Divergence Theorem)). *Let* $\vec{\boldsymbol{F}} = \langle 0,\ 2y,\ 3z\rangle$ *and let* S *be the part of the plane*

$$z = 2x + 1$$

lying above the disk $x^2 + y^2 \le 9$, *oriented* upward. *Find the flux* $\iint_S \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{n}}\, dS$.

Step 1: use the graph flux formula. *Here* $g(x, y) = 2x + 1$, *so*

$$g_x = 2, \qquad g_y = 0, \qquad \vec{\boldsymbol{n}}\, dS = \langle -g_x,\ -g_y,\ 1\rangle\, dA = \langle -2,\ 0,\ 1\rangle\, dA.$$

Step 2: put $z = g(x, y)$ ***into the field.*** *On the surface,* $z = 2x + 1$, *so*

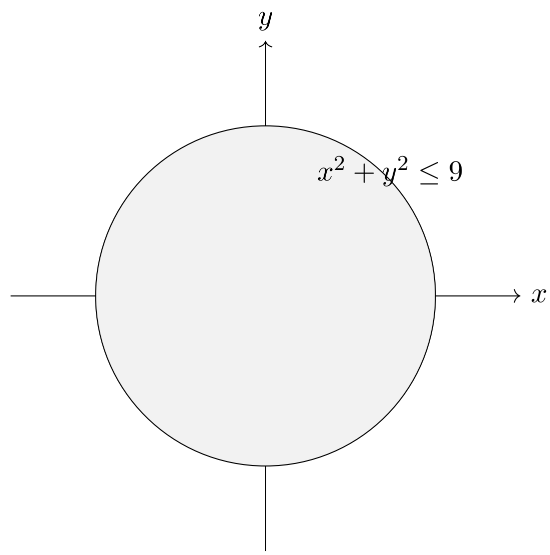
$$\vec{\boldsymbol{F}}(x, y, g(x, y)) = \langle 0,\ 2y,\ 3(2x + 1)\rangle = \langle 0,\ 2y,\ 6x + 3\rangle.$$

Step 3: dot product (this is where the key simplification happens).

$$\vec{\boldsymbol{F}} \cdot (\vec{\boldsymbol{n}}\, dS) = \langle 0,\ 2y,\ 6x + 3\rangle \cdot \langle -2,\ 0,\ 1\rangle\, dA = (6x + 3)\, dA.$$

So the flux is

$$\iint_S \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{n}}\, dS = \iint_{x^2+y^2\le 9} (6x + 3)\, dA.$$



Step 4: compute without any extra theorems. *Split the integral:*

$$\iint (6x + 3)\, dA = 6 \iint x\, dA + 3 \iint 1\, dA.$$

Over a disk centered at the origin, symmetry gives $\iint x\, dA = 0$. *Also* $\iint 1\, dA =$ *Area(disk)* $= \pi(3)^2 = 9\pi$. *Therefore*

$$\boxed{\iint_S \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{n}}\, dS = 3 \cdot 9\pi = 27\pi.}$$

Example (Flux through an upper hemisphere (direct computation, still NO Divergence Theorem))**.** *Let S be the upper hemisphere of radius 3:*

$$x^2 + y^2 + z^2 = 9, \quad z \geq 0,$$

with outward orientation. Let $\vec{\boldsymbol{F}} = \langle x,\ y,\ z \rangle$. Find the outward flux through S.

Step 1: understand the outward unit normal on a sphere. *On the sphere of radius 3, the outward unit normal is*

$$\vec{\boldsymbol{n}} = \frac{1}{3}\langle x,\ y,\ z \rangle.$$

So on S,

$$\vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{n}} = \langle x, y, z \rangle \cdot \frac{1}{3}\langle x, y, z \rangle = \frac{1}{3}(x^2 + y^2 + z^2) = \frac{1}{3} \cdot 9 = 3.$$

That means the integrand is constant: $\vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{n}} = 3$ everywhere on the hemisphere.

Step 2: flux becomes "constant times area".

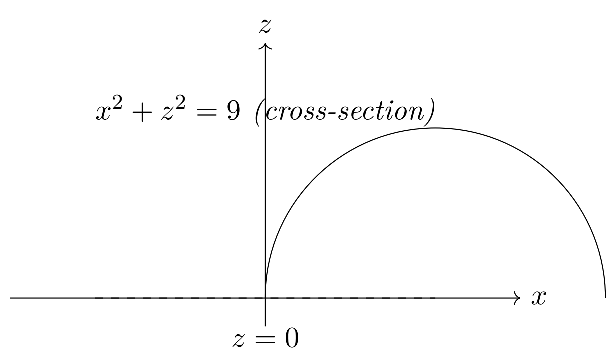
$$\iint_S \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{n}}\, dS = \iint_S 3\, dS = 3 \cdot \textit{Area(upper hemisphere)}.$$

Area of a full sphere is $4\pi r^2$, so the upper hemisphere has area $2\pi r^2$. With $r = 3$:

$$\textit{Area(upper hemisphere)} = 2\pi \cdot 3^2 = 18\pi.$$

Therefore

$$\boxed{\iint_S \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{n}}\, dS = 3 \cdot 18\pi = 54\pi.}$$



Mini-checklist (so you don't get lost)

- If the surface is a graph $z = g(x, y)$: use $dS = \sqrt{1 + g_x^2 + g_y^2}\, dA$.
- Flux on a graph (upward): use $\vec{\boldsymbol{n}}\, dS = \langle -g_x, -g_y, 1 \rangle\, dA$.
- If the surface is not a graph: parameterize $\vec{\boldsymbol{r}}(u, v)$ and use $\|\vec{\boldsymbol{r}}_u \times \vec{\boldsymbol{r}}_v\|$.