Scalar Line Integrals (with ds) and Special Line Integrals (with dx, dy, dz)

Calculus 3 / Vector Calculus Notes (Thomas/Stewart style)

Recall from Calculus 1: what an integral is

In Calculus 1, the definite integral

$$\int_a^b f(x)\,dx$$

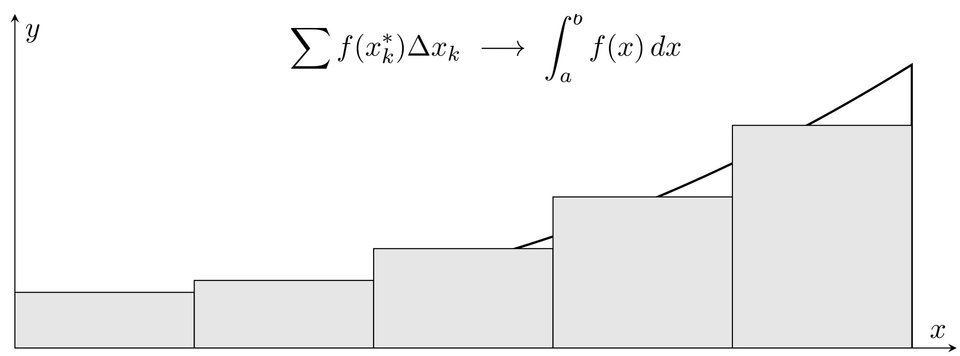
is defined as a limit of *Riemann sums*. We split $[a, b]$ into small subintervals, choose a sample point x_k^* in each subinterval, and form the sum

$$\sum_{k=1}^{n} f(x_k^*)\,\Delta x_k.$$

If the limit exists as the partition gets finer (meaning $\max_k \Delta x_k \to 0$), then

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\,\Delta x_k.$$

Picture: area as a limit of rectangles



The main idea that we will reuse in line integrals is: *take small pieces, multiply "value" × "size of piece", and pass to a limit.*

From area in 2D to "curtain area" in 3D

Now we move from a 1D interval $[a, b]$ to a *curve* C in the plane or in space.

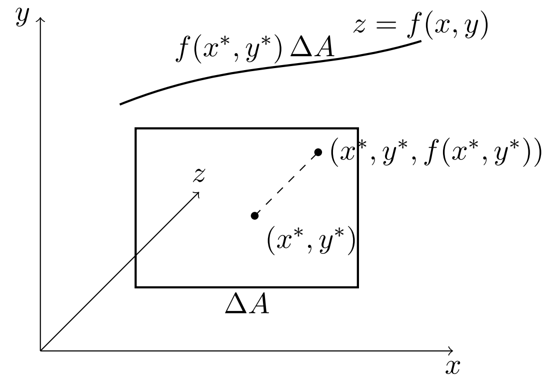
In 2D (functions of two variables), you also met Riemann sums for double integrals. Over a rectangle

$$R = [a, b] \times [c, d],$$

we split R into small rectangles of area ΔA_{ij}, pick a sample point (x_{ij}^*, y_{ij}^*), and define

$$\iint_R f(x, y)\,dA = \lim_{\max \Delta A_{ij} \to 0} \sum f(x_{ij}^*, y_{ij}^*)\,\Delta A_{ij}.$$

Picture: x, y, z, a rectangle in the xy-plane, and a surface



That picture is the same logic as Riemann sums: "height" $f(x^*, y^*)$ times "base area" ΔA.

For line integrals of a *scalar* function, we will do a similar thing: "value" times *arc length piece* Δs.

Scalar line integrals with respect to arc length

What is a curve C?

A *curve* C is a path (a set of points) in the plane or in space.

To *compute* with a curve, we usually describe it by a *parametrization*:

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle, \quad a \le t \le b.$$

New word: parameter. The variable t is the parameter. As t runs from a to b, the point $\mathbf{r}(t)$ moves along the curve.

Why do we need the limits a and b? They tell us which part of the curve we travel, and in which direction:

- $t = a$ gives the starting point $\mathbf{r}(a)$,
- $t = b$ gives the ending point $\mathbf{r}(b)$.

What is arc length s and where does ds come from?

Arc length is the ordinary geometric length of the curve, like the length of a bent wire.

In Calculus 1, for a graph $y = y(x)$ you learned

$$ds = \sqrt{1 + (y')^2}\, dx.$$

In space, if $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, the derivative

$$\mathbf{r}'(t) = \big\langle x'(t), y'(t), z'(t) \big\rangle$$

is the *velocity / tangent vector*. Its magnitude is the *speed*:

$$\|\mathbf{r}'(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}.$$

Over a tiny time step dt, distance $\approx$ speed times time, so the small arc length piece is

$$ds = \|\mathbf{r}'(t)\|\, dt.$$

Definition (scalar line integral with ds)

Let $f(x, y, z)$ be a scalar function, and let C be a smooth curve. Split C into small pieces of length Δs_k, pick a point on each piece, and form

$$\sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\, \Delta s_k.$$

If the limit exists as $\max \Delta s_k \to 0$, we define

$$\int_C f(x, y, z)\, ds = \lim_{\max \Delta s_k \to 0} \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\, \Delta s_k.$$

Important notation fix. In $\int_C$, the C does *not* mean "limit". It means *integrate along the curve* C. The *limit* is the one hidden in the definition above (mesh $\to 0$).

Computation formula

If C is parametrized by $\mathbf{r}(t)$, $a \le t \le b$, then

$$\int_C f(x, y, z)\, ds = \int_a^b f(\mathbf{r}(t))\, \|\mathbf{r}'(t)\|\, dt.$$

Example 1 (step by step): $\int_C x\, y^2\, ds$ on a quarter circle

Step 0: understand the curve and its limits

We are given

$$C: \ \mathbf{r}(t) = \langle \cos t,\ \sin t \rangle, \quad 0 \le t \le \frac{\pi}{2}.$$

What is C? It is a quarter of the unit circle $x^2 + y^2 = 1$ in the first quadrant.

Why $0 \le t \le \pi/2$? Because:

$$\mathbf{r}(0) = \langle 1, 0 \rangle, \qquad \mathbf{r}\left(\frac{\pi}{2}\right) = \langle 0, 1 \rangle,$$

and as t increases from 0 to $\pi/2$, the point moves counterclockwise along that quarter arc.

Step 1: write the function along the curve

Here

$$f(x,y) = x\,y^2.$$

Along C we substitute $x = \cos t$, $y = \sin t$:

$$f(\mathbf{r}(t)) = (\cos t)\,(\sin t)^2.$$

Step 2: compute ds

Compute the derivative:

$$\mathbf{r}'(t) = \langle -\sin t,\ \cos t\rangle.$$

Magnitude:

$$\|\mathbf{r}'(t)\| = \sqrt{\sin^2 t + \cos^2 t} = 1.$$

Therefore

$$ds = \|\mathbf{r}'(t)\|\,dt = dt.$$

Step 3: build the integral

$$\int_C x\,y^2\,ds = \int_0^{\pi/2} (\cos t)(\sin t)^2\,dt.$$

Step 4: compute

Let $u = \sin t$, so $du = \cos t\,dt$. When $t = 0$, $u = 0$; when $t = \pi/2$, $u = 1$. Then

$$\int_0^{\pi/2} (\cos t)(\sin t)^2\,dt = \int_0^1 u^2\,du = \left[\frac{u^3}{3}\right]_0^1 = \frac{1}{3}.$$

Answer:

$$\int_C x\,y^2\,ds = \frac{1}{3}.$$

Example 2 (step by step): $\int_C xe^{yz}\,ds$ along a line segment

Compute

$$\int_C xe^{yz}\,ds$$

where C is the line segment from $P(0,0,0)$ to $Q(1,2,3)$.

Step 1: parametrize the line segment

$$\mathbf{r}(t) = P + t(Q-P) = \langle t,\ 2t,\ 3t\rangle, \quad 0 \le t \le 1.$$

Step 2: substitute into the integrand

$$x = t, \quad y = 2t, \quad z = 3t \quad \Rightarrow \quad xe^{yz} = t\,e^{(2t)(3t)} = t\,e^{6t^2}.$$

Step 3: compute ds

$$\mathbf{r}'(t) = \langle 1, 2, 3 \rangle, \qquad \|\mathbf{r}'(t)\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}, \qquad ds = \sqrt{14}\,dt.$$

Step 4: integrate

$$\int_C xe^{yz}\,ds = \int_0^1 \left(te^{6t^2}\right)\sqrt{14}\,dt = \sqrt{14}\int_0^1 te^{6t^2}\,dt.$$

Let $u = 6t^2$, so $du = 12t\,dt$ and $t\,dt = \frac{1}{12}du$. When $t = 0$, $u = 0$. When $t = 1$, $u = 6$. Then

$$\sqrt{14}\int_0^1 te^{6t^2}\,dt = \sqrt{14}\int_0^6 e^u \cdot \frac{1}{12}\,du = \frac{\sqrt{14}}{12}\left[e^u\right]_0^6 = \frac{\sqrt{14}}{12}\left(e^6 - 1\right).$$

Answer:

$$\int_C xe^{yz}\,ds = \frac{\sqrt{14}}{12}\left(e^6 - 1\right).$$

Special line integrals: integrating with respect to x, y, or z

Sometimes we integrate not with respect to ds, but with respect to dx, dy, or dz.

Why not ds?

- ds measures length and is always nonnegative: $ds \geq 0$.
- dx, dy, dz measure change in coordinates. They can be positive or negative depending on direction.

So $\int_C f\,ds$ is "value times length", while $\int_C f\,dx$ is sensitive to orientation.

Definition and computation

Let $w = f(x, y, z)$ and let C be parametrized by $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, $a \leq t \leq b$.

Because

$$dx = x'(t)\,dt, \quad dy = y'(t)\,dt, \quad dz = z'(t)\,dt,$$

we define

$$\int_C f(x, y, z)\,dx = \int_a^b f(\mathbf{r}(t))\,x'(t)\,dt,$$

$$\int_C f(x,y,z)\,dy = \int_a^b f(\mathbf{r}(t))\,y'(t)\,dt,$$
$$\int_C f(x,y,z)\,dz = \int_a^b f(\mathbf{r}(t))\,z'(t)\,dt.$$

Combining them: $P\,dx + Q\,dy + R\,dz$ and vector fields

A very common combination is

$$\int_C P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz.$$

Using $dx = x'(t)dt$ etc., we get

$$\int_C P\,dx + Q\,dy + R\,dz = \int_a^b \big(P(\mathbf{r}(t))x'(t) + Q(\mathbf{r}(t))y'(t) + R(\mathbf{r}(t))z'(t)\big)\,dt.$$

Now define a vector field

$$\mathbf{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle,$$

and the differential vector

$$d\mathbf{r} = \langle dx, dy, dz\rangle.$$

Then

$$\mathbf{F}\cdot d\mathbf{r} = \langle P,Q,R\rangle \cdot \langle dx,dy,dz\rangle = P\,dx + Q\,dy + R\,dz,$$

so

$$\int_C P\,dx + Q\,dy + R\,dz = \int_C \mathbf{F}\cdot d\mathbf{r}.$$

Mini-example

Let

$$\mathbf{F}(x,y) = \langle y,\ x\rangle$$

and

$$\mathbf{r}(t) = \langle t,\ t^2\rangle, \quad 0 \le t \le 1.$$

Then

$$\mathbf{F}(\mathbf{r}(t)) = \langle t^2,\ t\rangle, \qquad \mathbf{r}'(t) = \langle 1,\ 2t\rangle,$$

so

$$\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t) = t^2\cdot 1 + t\cdot 2t = 3t^2.$$

Therefore

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 3t^2\,dt = \big[t^3\big]_0^1 = 1.$$

What comes next

In the next file we will focus on path independence and the Fundamental Theorem of Line Integrals.