From Probability to Likelihood

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Exams

- Short answer/practical problems
- Calculations and coding (bring a laptop)
  - btw, you can check out laptops for 4 hours from the library
- Write out code (it will be minimal)
- formulae and function cheat-sheet provided.

Textbook Definition of Probabilty

"If an observation is made N times and event A occurs N_A times, then with a high degree of certainty, the relative frequency of N_A/N is close to $P(A)$, the probability of A in a single trial, $P(A) \approx N_A/N$, provided N is sufficiently large."

- "event" = something that does or does not occur
- "trial" = single observation or data point or "experiment" in which the event can occur or not

- P(A) = probability that event (A) will occur in a single trial
- N = total number of trials
- N_A = number of times A occurs

$$P(A) = \lim_{N \to \infty} \left(\frac{N_A}{N} \right)$$

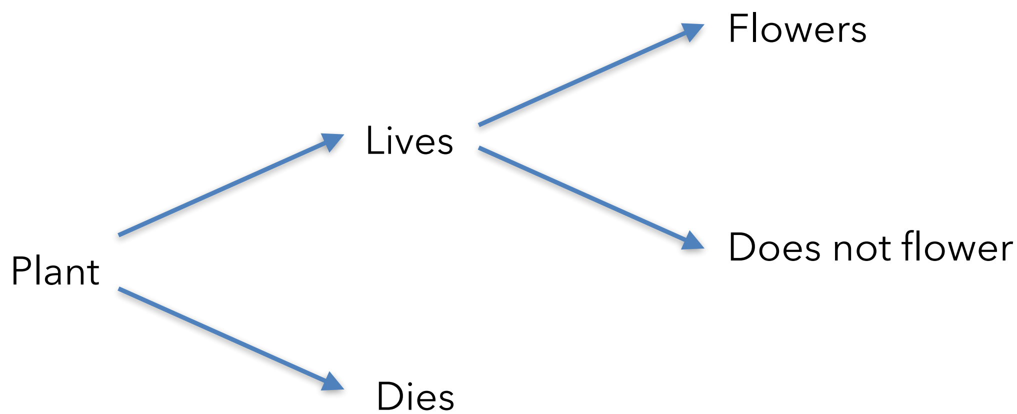
Example application: Demography of a perennial wildflower

$$P(A) = \lim_{N \to \infty} \left(\frac{N_A}{N} \right)$$

P(survival) = 4/5 = 0.8

P(flowering) = 1/4 = 0.25

Plant #	fate
1	Flowers
2	Vegetative
3	Vegetative
4	Vegetative
5	Dead

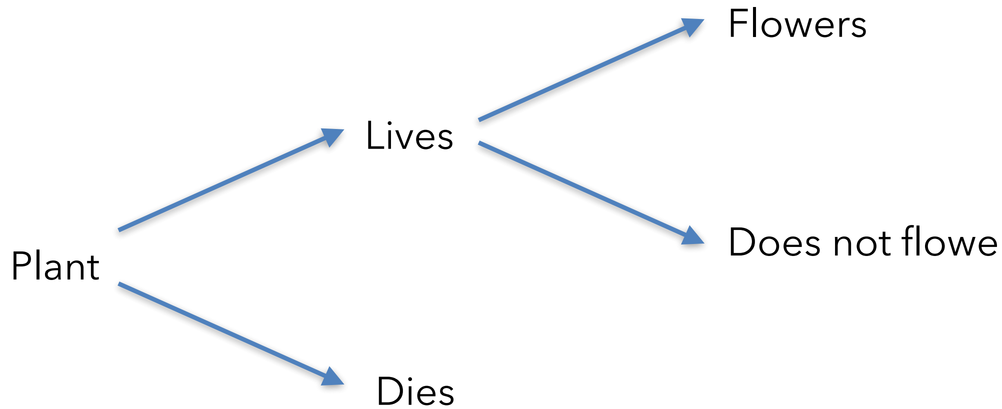


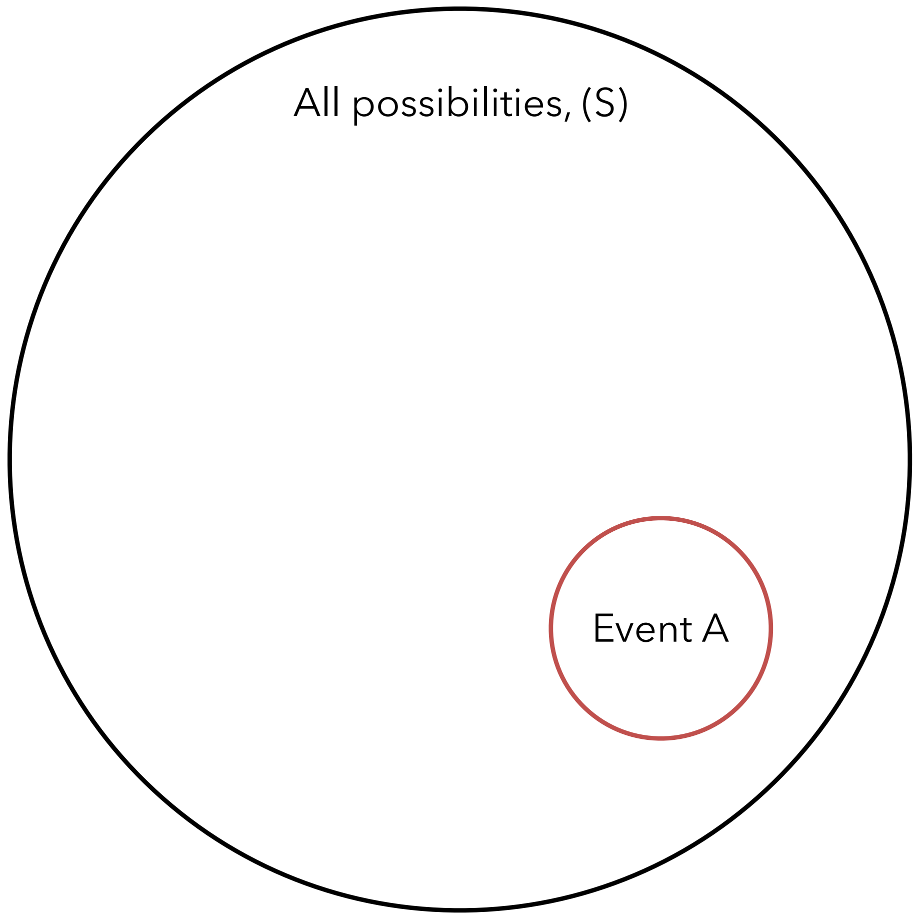
Example application: Demography of a perennial wildflower

What is the probability that a plant survives AND flowers?

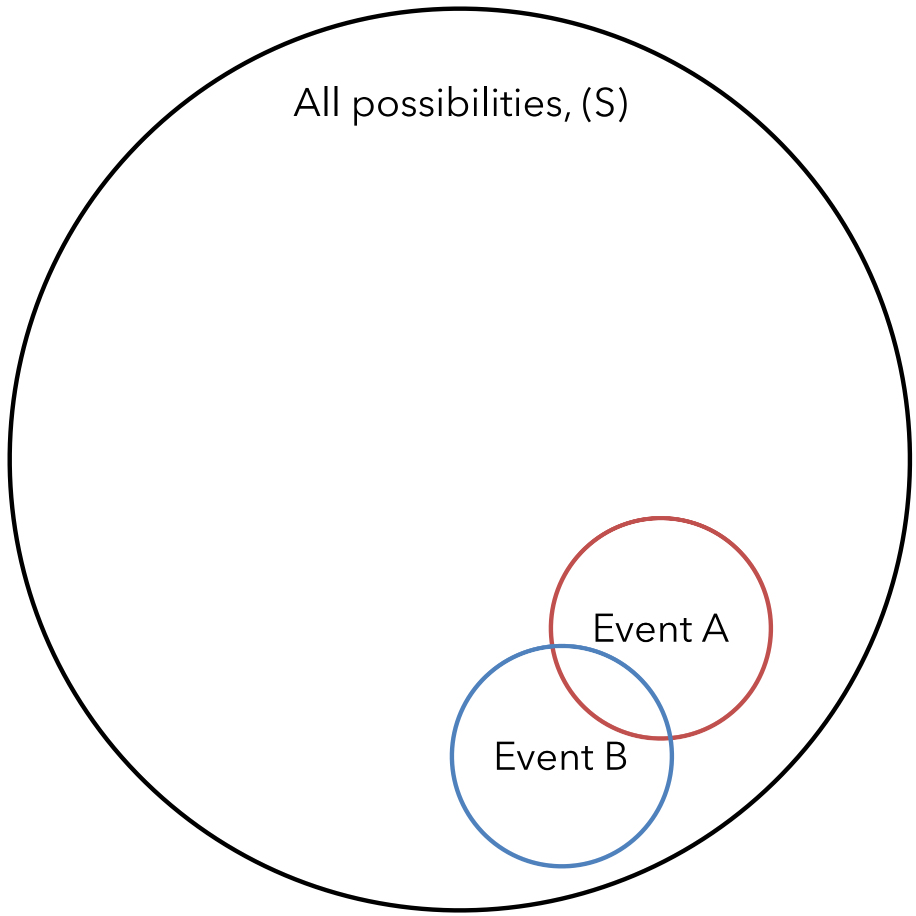
Need to know about the rules of probability…

Plant #	fate
1	Flowers
2	Vegetative
3	Vegetative
4	Vegetative
5	Dead

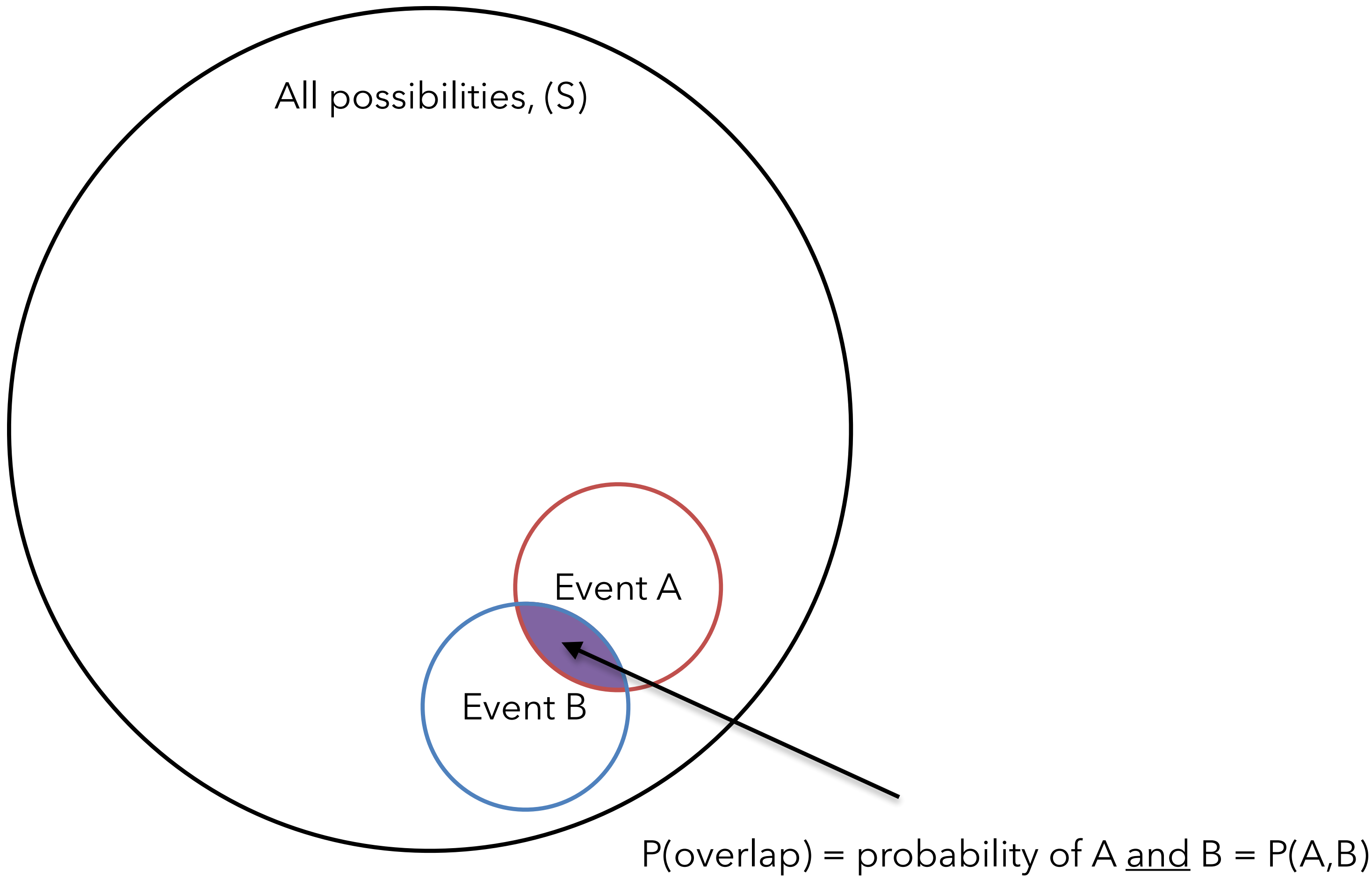




$$P(A) = \frac{\text{area A}}{\text{area S}}$$

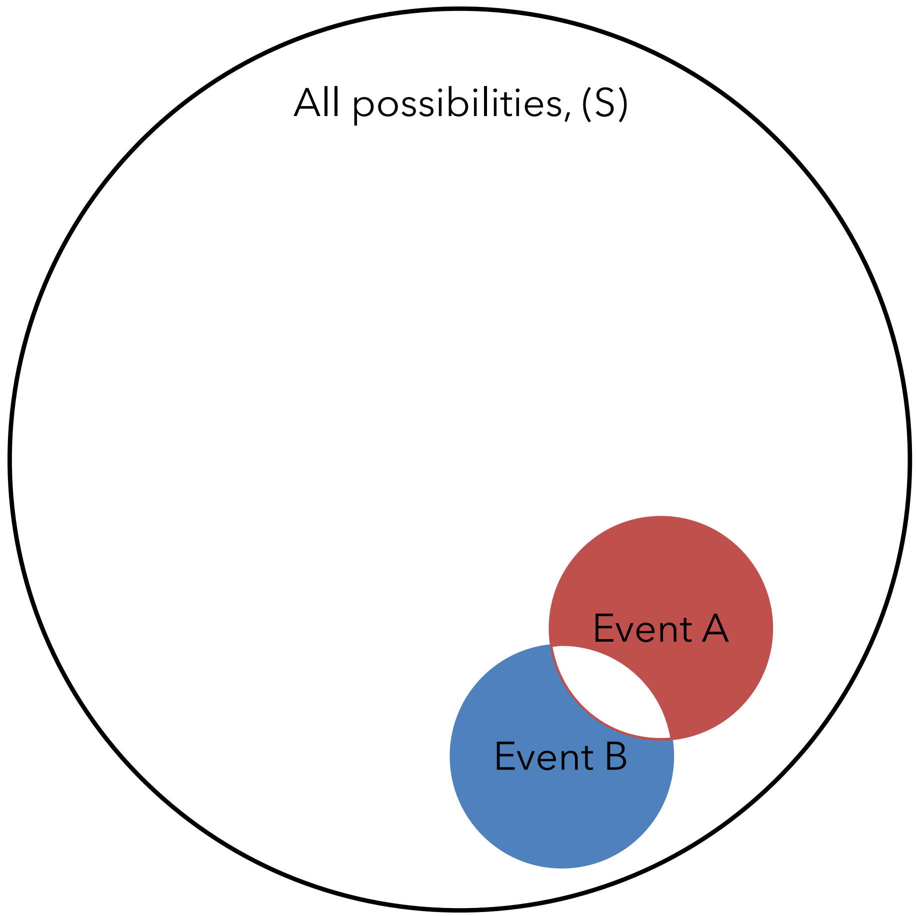


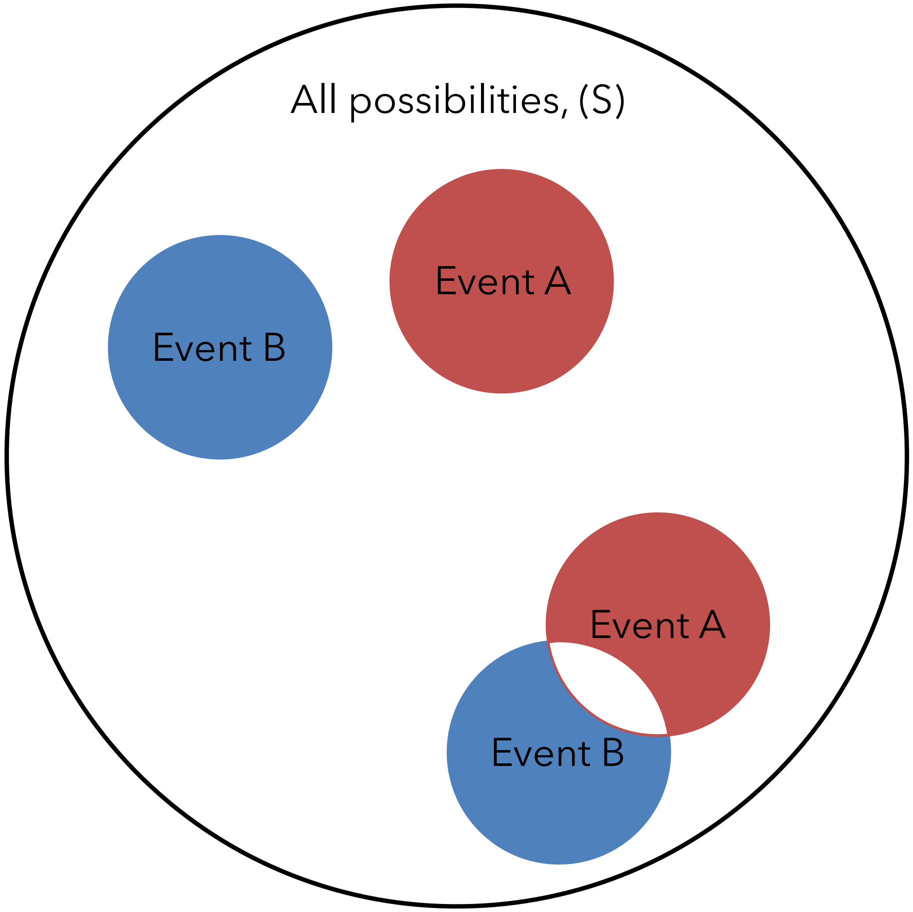
$$P(B) = \frac{\text{area A}}{\text{area S}}$$


All possibilities, (S)
Event A
Event B
P(overlap) = probability of A and B = P(A,B)

Probability of A <u>or</u> B

$P(A+B) = P(A) + P(B)$ - area of overlap

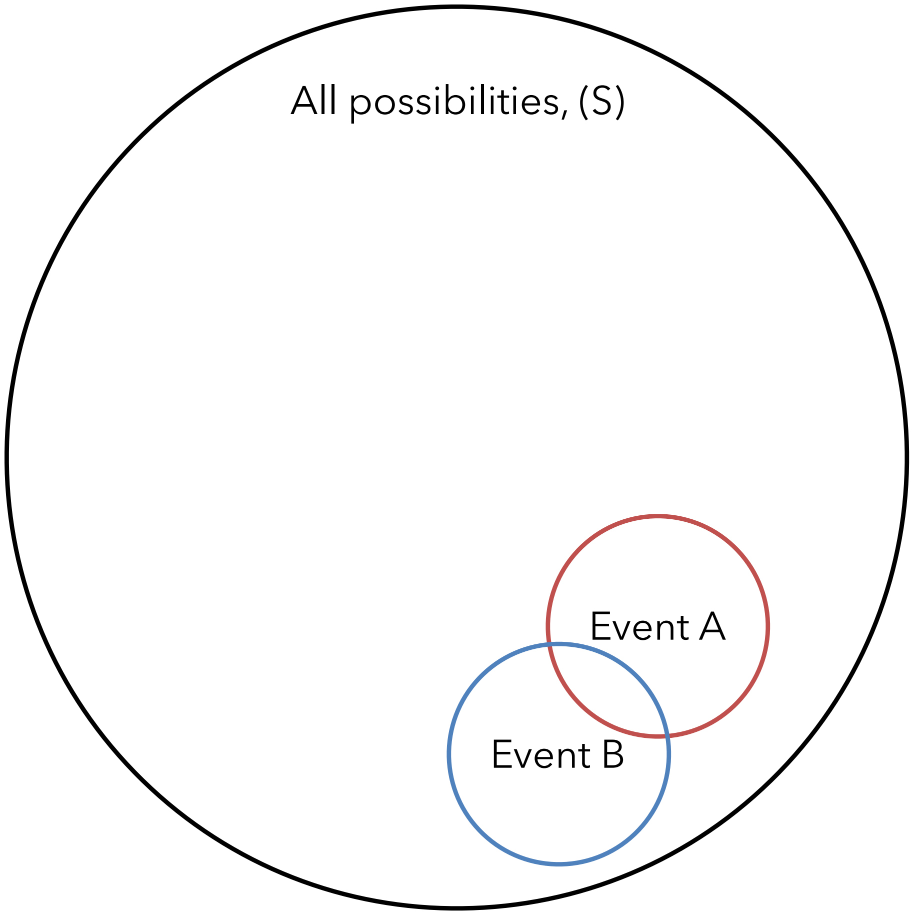




If A and B are mutually exclusive:

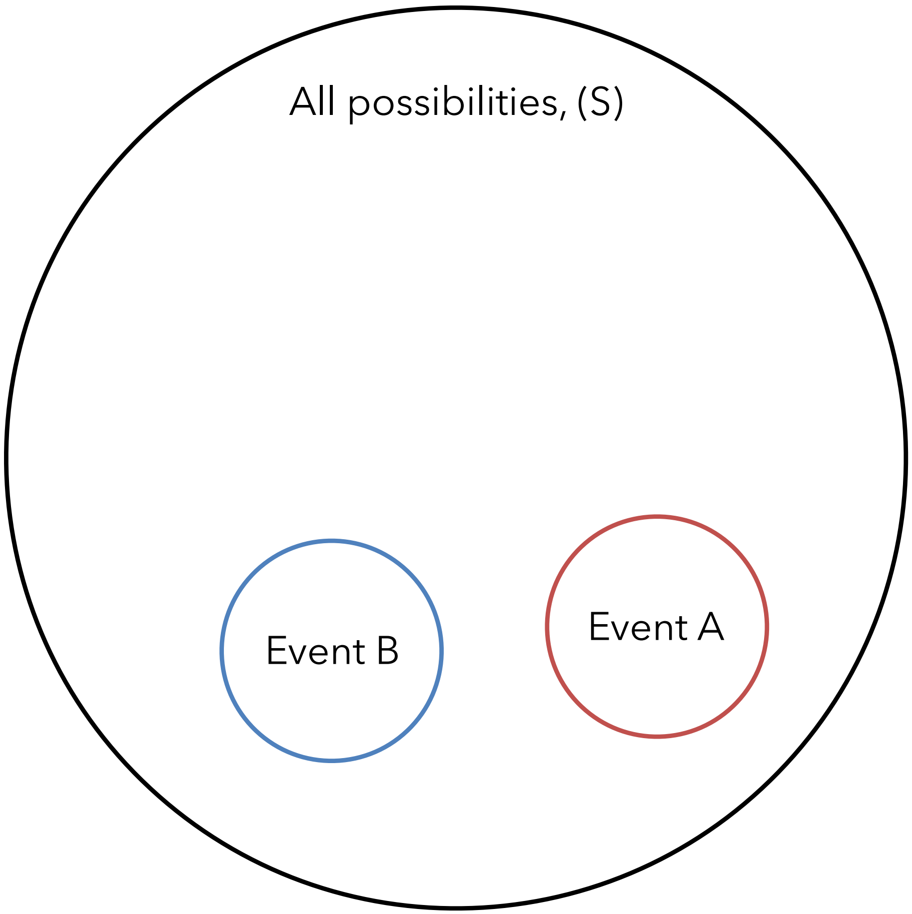
- P(A,B) = 0
- P(A + B) = P(A) + P(B)

P(B given A) = Probability of B and A out of probability of A



$$P(B \mid A) = \frac{P(A, B)}{P(A)}$$

Conditional probability of B



If A and B are independent, then one occurring has no effect on whether or not the other occurs.

P(B|A) = P(B)

Substitute this in to our conditional probability equation

$$P(B\,|\,A) = \frac{P(A,B)}{P(A)}$$

$$P(B) = \frac{P(A,B)}{P(A)}$$

$$P(B) \times P(A) = P(A,B)$$

Two MAJOR axioms of probability

1. Probability that one of two mutually exclusive events occurs:
    - P(A+B) = P(A) + P(B) (OR rule)

2. Probability that (both of) two independent events occur:
    - P(A,B) = P(A)*P(B) (AND rule)

Using these two axioms allow us to do fairly sophisticated statistical analysis:

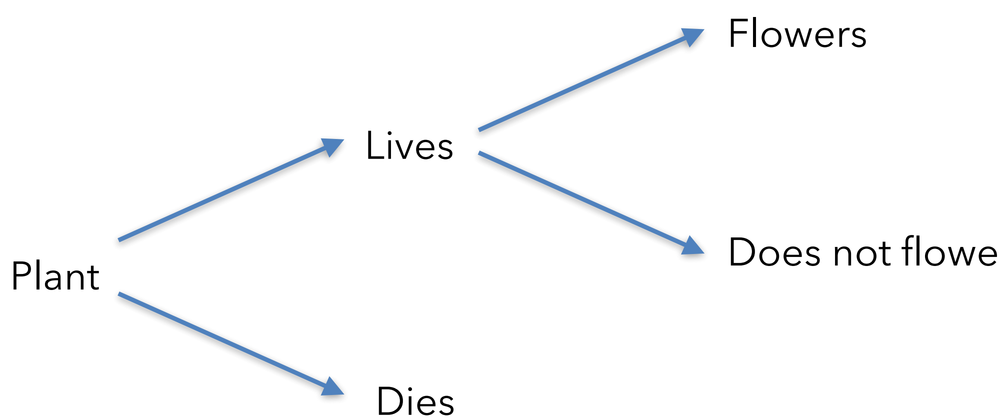
Example application: Demography of a perennial wildflower

How do we know if N is large enough to get a good estimate of our parameters?

Compare "competing models" with different values for *s* and *f*.

$\boxed{\begin{array}{l} s = 0.8 \\ f = 0.25 \end{array}}$ vs. $\boxed{\begin{array}{l} s = 0.6 \\ f = 0.4 \end{array}}$

Plant #	fate
1	Flowers
2	Vegetative
3	Vegetative
4	Vegetative
5	Dead



$s = 0.8$
$f = 0.25$

Plant #	fate	Events leading to fate?	Probability?
1	Flowers		
2	Vegetative		
3	Vegetative		
4	Vegetative		
5	Dead		

$s = 0.8$
$f = 0.25$

Plant #	fate	Events leading to fate?	Probability?
1	Flowers	survives AND flowers	
2	Vegetative	survives AND no flower	
3	Vegetative	survives AND no flower	
4	Vegetative	survives AND no flower	
5	Dead	dies	

s = 0.8
f = 0.25

Plant #	fate	Events leading to fate?	Probability?
1	Flowers	survives AND flowers	0.8 x 0.25 = 0.2
2	Vegetative	survives AND no flower	0.8 x (1 - 0.25) = 0.6
3	Vegetative	survives AND no flower	0.8 x (1 - 0.25) = 0.6
4	Vegetative	survives AND no flower	0.8 x (1 - 0.25) = 0.6
5	Dead	dies	1 - 0.8 = 0.2

Which axiom allows us to calculate the probability of the entire data set?
[assuming flowers live/die and flower independently]

$s = 0.8$
$f = 0.25$

Plant #	fate	Events leading to fate?	Probability?
1	Flowers	survives AND flowers	0.8 x 0.25 = 0.2
2	Vegetative	survives AND no flower	0.8 x (1 - 0.25) = 0.6
3	Vegetative	survives AND no flower	0.8 x (1 - 0.25) = 0.6
4	Vegetative	survives AND no flower	0.8 x (1 - 0.25) = 0.6
5	Dead	dies	1 - 0.8 = 0.2

Which axiom allows us to calculate the probability of the entire data set?
[assuming flowers live/die and flower independently]

0.2 x 0.6 x 0.6 x 0.6 x 0.2 = **0.00864**

Repeat with second model

s = 0.6
f = 0.4

Plant #	fate	Events leading to fate?	Probability?
1	Flowers	survives AND flowers	
2	Vegetative	survives AND no flower	
3	Vegetative	survives AND no flower	
4	Vegetative	survives AND no flower	
5	Dead	dies	

Probability of dataset:

$s = 0.6$
$f = 0.4$

Repeat with second model

Plant #	fate	Events leading to fate?	Probability?
1	Flowers	survives AND flowers	0.6 x 0.4 = 0.24
2	Vegetative	survives AND no flower	0.6 x (1 - 0.4) = 0.36
3	Vegetative	survives AND no flower	0.6 x (1 - 0.4) = 0.36
4	Vegetative	survives AND no flower	0.6 x (1 - 0.4) = 0.36
5	Dead	dies	1 - 0.6 = 0.4

Probability of dataset:

0.24 x 0.36 x 0.36 x 0.36 x 0.4 = **0.00448**

Likelihood

Using probabilities (of data), we can compute the **likelihoods** of different parameters given our dataset.

Likelihood (Edwards 1972) :

> "The likelihood L, of the hypothesis H, given data R, is proportional to $P(R|H)$, the constant of proportionality being arbitrary" [but specific to a particular data set]
>
> – (Note: "Hypothesis" = "model" = "set of parameters")

- Likelihood of first set of parameters (s = 0.8, f = 0.25) is proportional to 0.00864

$$L_1 = L(s = 0.8, f = 0.2 \,|\, R) = k_1 \times 0.00864$$

- Likelihood of 2nd set of parameters (s = 0.6, f = 0.4) is proportional to 0.00448

$$L_2 = L(s = 0.6, f = 0.4 \,|\, R) = k_2 \times 0.00448$$

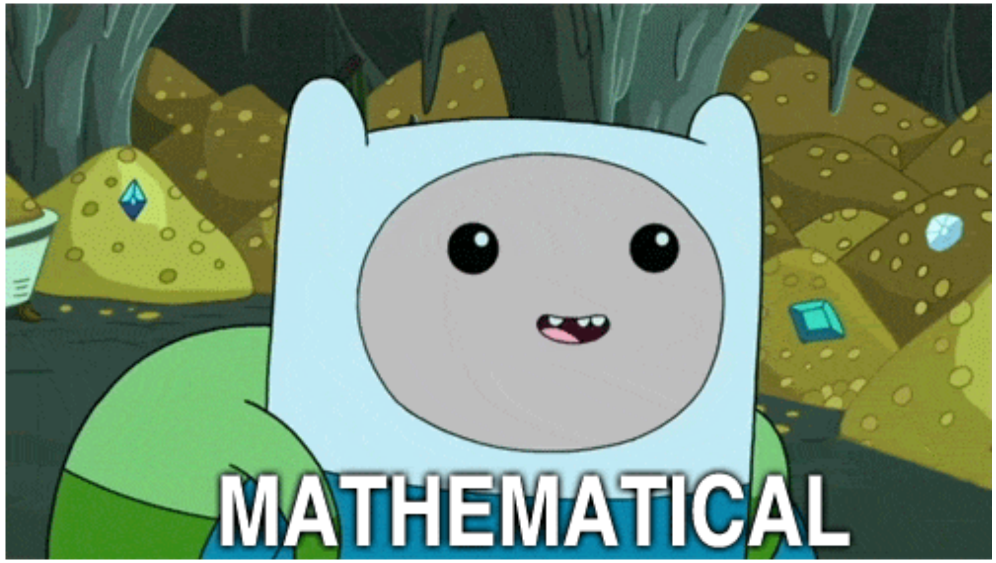
- Because our hypotheses are fit to EXACTLY the same dataset, k1 = k2 = k

Likelihood ratio

The **likelihood ratio** is our primary method of comparing models:

$$\frac{L_1}{L_2} = \frac{0.00864 \times k}{0.00448 \times k} = 1.93$$

First set of parameters is about twice as likely as our second, given this small data set.



We have gone from calculating the probability of our **data**, given fixed parameters, to making inferences about support for different possible values of our **parameters** (AKA models, hypotheses), given a dataset.

Repeat for a luckier ecologist who has 10 plants with the same distribution of fates: 2 flowering, 6 vegetative, 2 dead

- Likelihood of model #1:

$$L(s = 0.8, f = 0.25 \mid R_2) = k_3 \times 0.22 \times 0.66 \times 0.22 = 0.00007465$$

- Likelihood of model #2:

$$L(s = 0.6, f = 0.4 \mid R_2) = k_3 \times 0.242 \times 0.366 \times 0.42 = 0.00002006$$

- Likelihood ratio:

$$\frac{0.00007465 \times k_3}{0.00002006 \times k_3} = 3.72$$

- Same models, different likelihoods AND different likelihood ratio.

3 Features of Likelihood

1. Because it's the product of probabilities (and probabilities are all < 1), adding more data **reduces** the likelihood of any particular model.
    - Why the constant (k) is specific to a particular dataset
    - Also why we can't use likelihoods to compare models fit to different data

2. Adding more data makes the **likelihood ratio** between two models get larger because of the **AND** rule.
    - Larger sample size = greater ability to detect small differences
3. To avoid dealing with tiny numbers for typical (large) data sets, we often use log likelihoods:

$\ln(0.00007465) = -9.50$

$\ln(0.00002006) = -10.82$

Rules of working with exponents and logarithms:

1. Exponential transformation “undoes” (natural) log transformation, and vice versa:
   - ln(exp(A)) = A
   - exp(ln(A)) = A
2. Log transformation converts multiplication & division into addition & subtraction
   - ln(AB) = ln(A) + ln(B)
   - ln(A/B) = ln(A) - ln(B)
   - ln(A^C) = C x ln(A)
   - Therefore ln[P(A,B)] = ln[P(A)] + ln[P(B)]

Logarithms in R

```
A <- 5
log(exp(A))
#> [1] 5
exp(log(A))
#> [1] 5
B <- 6
log(A*B)
#> [1] 3.401197
log(A) + log(B)
#> [1] 3.401197
C <- 3
log(A^C)
#> [1] 4.828314
C * log(A)
#> [1] 4.828314
```

- R uses natural log as default.

Implications for the and axiom of probability

- Log-likelihoods of H1 and H2 [on board]
- log-transformed likelihood ratio become differences in likelihoods
- *log-likelihood ratio* can be back-transformed to calculate relative support in an absolute sense:

Ignoring flowering (for now), explore the likelihood of different values of survival, given the 5-plant data set (1 dies, 4 survive).

a. Calculate the likelihood of survival having each of the following values: 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9.
b. Make a graph of the log-likelihood (y-axis) vs. value of survival (x-axis).
c. Repeat a&b for the 10-plant data set (2 die, 8 survive). How does the shape of the graph change?
d. What is the likelihood that survival is 0? Or 1? What happens to the log-likelihood at these values?

2. Thinking about probabilities…

a. What is P(B|A) for two mutually exclusive events? Is it possible for events to be independent and mutually exclusive?
b. What is P(A+B), the probability that A <u>or</u> B occurs, if they are independent events?