Errata for
Data-driven Modeling of Structured Populations: A Practical Guide to the Integral Projection Model

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Page 12: We think that the statement about "piecewise continuous" in the footnote is true, but at least one proof in the book isn't valid regardless of what the curves are that divide $\mathbf{Z}^2$ into subregions. A slightly less general definition, which should be sufficient for any applications, is as follows. For the basic model where the individual-level state space $\mathbf{Z}$ is a bounded interval $[L, U]$, a partition of $\mathbf{Z}$ is a set of intervals

$$\mathbf{Z}_1 = (z_0, z_1), \mathbf{Z}_2 = (z_1, z_2), \cdots, \mathbf{Z}_m = (z_{m-1}, z_m) \tag{1}$$

where

$$L = z_0 < z_1 < z_2 < \cdots < z_M = U. \tag{2}$$

A partition breaks $\mathbf{Z}^2$ into a set of open rectangles

$$\mathbf{Z}_{ij} = \mathbf{Z}_i \times \mathbf{Z}_j = \{(z', z) : z' \in \mathbf{Z}_i, z \in \mathbf{Z}_j\}.$$

Define the kernels K_{ij} to be K restricted to $\mathbf{Z}_{ij}$.

We say that K is *piecewise continuous* if there exists a partition such that each of the kernels K_{ij} is continuous on $\mathbf{Z}_{ij}$, and can be defined on the boundary of $\mathbf{Z}_{ij}$ so that it is continuous on the closed rectangle $\bar{\mathbf{Z}}_{ij}$ consisting of $\mathbf{Z}_{ij}$ and its boundary.

The reason this definition works is the general theory in section 6.9 (originally in the Appendices to Ellner & Rees 2006), applied to the closed intervals $\bar{\mathbf{Z}}_i = [z_0, z_1]$ as a set of continuous components, with continuous component-to-component kernels K_{ij}. In terms of the general theory, this is an IPM with continuous kernels. Formally, there is a separate state distribution function $n_i(z, t)$ defined on each $\bar{\mathbf{Z}}_i$. But we can also think of there being a single distribution function $n(z, t)$ on all of $[L, U]$, consisting of $n_1, n_2, \cdots n_M$ side-by-side.

Each of the boundary points z_i is in two adjacent components, but this doesn't matter because single points contribute nothing to the subsequent population. As an example consider $\mathbf{Z} = [0, 2]$ and the (contrived) kernel $K(z', z) = 1$ if $z' > 1$, and 0 otherwise. The partition is $\mathbf{Z}_1 = (0, 1), \mathbf{Z}_2 = (1, 2)$ and the kernels are

$$K_{11} = K_{12} \equiv 0, K_{21} = K_{22} \equiv 1. \tag{3}$$

The population dynamics are

$$\begin{aligned} n_1(z', t+1) &= \int_0^1 K_{11}(z', z) n_1(z, t) dz + \int_1^2 K_{12}(z', z) n_2(z, t) dz = 0 \\ n_2(z', t+1) &= \int_0^1 K_{21}(z', z) n_1(z, t) dz + \int_1^2 K_{22}(z', z) n_2(z, t) dz \\ &= \int_0^1 n_1(z, t) dz + \int_1^2 n_2(z, t) dz. \end{aligned} \tag{4}$$

So $n_1(1, t+1) = 0$, while $n_2(1, t+1) > 0$ unless there were no individuals at time t. However, the values of n_1 and n_2 at the one point $z = 1$ have no effect on the integrals in the population dynamics, so we can regard $n(1, t+1)$ as being undefined, or give it an arbitrary value such the average $n_1(1, t+1)$ and $n_2(1, t+1)$.

The contrived kernel (4) is a counter-example to the claim in section 6.9 of the book that a piecewise continuous kernel, as defined in that section, maps $L_1(\mathbf{Z})$ into $C(\mathbf{Z})$. The gap in the proof is the assertion that the functions f_n converge almost everywhere, which is not necessarily true regardless of how the partitioning into sets $\mathcal{U}_k$ is done. But on the set of domains $\bar{\mathbf{Z}}_i$ the component kernels are all continuous and the f_n converge pointwise, which is sufficient for the rest of the proof. In example (4), n_1 is continuous on $\bar{\mathbf{Z}}_1$ and n_2 is continuous on $\bar{\mathbf{Z}}_2$, and that is exactly what it means to be continuous on the state space with domains $\bar{\mathbf{Z}}_1$ and $\bar{\mathbf{Z}}_2$.

Please see the related correction below about **Page 181**.

Page 15: Equation (2.3.6), C_1 should be C_0 as it is in the life-cycle diagram, Figure 2.2.

Page 29: 9 lines from the bottom, "the the" → "the"

Page 181: As we wrote regarding **Page 12**, "piecewise continuous" needs to be defined more narrowly, so that all the claims in this section are valid. Even if we ignore the fact that "continuous curve" was conveniently undefined, eqn. (4) remains a counterexample to the claim that K maps $L_1(\mathbf{Z})$ into $C(\mathbf{Z})$.

Fortunately, the same approach works in the general setting. A definition of "piecewise continuous" that actually works is (informally): we can break up $\mathbf{Z}$ into pieces $\mathbf{Z}_j$, and thereby break up the kernel into pieces K_{ij} for transitions from $\mathbf{Z}_j$ to $\mathbf{Z}_i$, in such a way that each K_{ij} is continuous.

Reading the rest of this item requires some familiarity with the basics of measure theory and functional analysis. Sorry, there's no way around this.

We are considering an IPM where the individual state-space $\mathbf{Z}$ is a compact metric space with metric d, and the kernels are defined relative to a Borel measure μ on $\mathbf{Z}$. The population dynamics are

$$n(z', t+1) = \int_{\mathbf{Z}} K(z', z) n(z, t) d\mu(z); \tag{5}$$

in the book this is eqn. (6.9.1). The kernel is a function on $\mathbf{Z}^2 = \mathbf{Z} times \mathbf{Z}$, which we give the L_2 product metric and resulting product topology. Subsets of $\mathbf{Z}$ and $\mathbf{Z}^2$ are given the relative metric and measure.

Following the definition for the basic model, a partition of $\mathbf{Z}$ is defined to be a collection of disjoint open (and therefore measurable) sets $\mathbf{Z}_1, \mathbf{Z}_2, \cdots, \mathbf{Z}_m$ whose boundaries all have μ-measure 0, such that the union of their closures $\bar{\mathbf{Z}}_j$ is $\mathbf{Z}$. Again, define the kernels K_{ij} to be K restricted to $\mathbf{Z}_{ij} = \mathbf{Z}_i \times \mathbf{Z}_j$.

We say that a kernel K is piecewise continuous if there is a partition such that each K_{ij} is continuous on each $\mathbf{Z}_{ij} = \mathbf{Z}_i \times \mathbf{Z}_j$ and can be defined on the boundary of $\mathbf{Z}_{ij}$ so as to be continuous on its closure $\bar{\mathbf{Z}}_{ij}$. At a point (z', z) that is on the boundary of $\mathbf{Z}_{ij}$ and of $\mathbf{Z}_{kl}$, the values chosen for $K_{ij}(z', z)$ and for $K_{kl}(z', z)$ to ensure continuity of K_{ij} and K_{kl} do not need to be the same.

The reason this works is (again) that the set of kernels K_{ij} (extended to be continuous on their domain $\bar{\mathbf{Z}}_{ij}$) defines a general IPM in the sense of section 6.9 in which the overall kernel is continuous. The individual-level state space for the general IPM is the collection of sets $\bar{\mathbf{Z}}_j, j = 1, 2, \cdots m$ considered as distinct sets. Formally, define $\bar{\mathbf{Z}}_j^*$ to be the set of ordered pairs (z, j) where $z \in \bar{\mathbf{Z}}_j$, and let $\mathbf{Z}^*$ be the collection of all $\bar{\mathbf{Z}}_j^*$ with the metric

$$d^*((z_1, i), (z_2, j)) = \sqrt{d(z_1, z_2)^2 + (i-j)^2}. \tag{6}$$

The overall kernel is

$$K^*((z', i), (z, j)) = K_{ij}(z', z). \tag{7}$$

A convergent sequence $x_n^* \to x_0^*$ in $\mathbf{Z}^* \times \mathbf{Z}^*$ must eventually consist of points whose z coordinates all lie in the same $\bar{\mathbf{Z}}_{ij}$, so the sequence is convergent in $\bar{\mathbf{Z}}_{ij}$. So for sufficiently large n, $K^*(x_n^*) = K_{ij}(x_n^*)$ which converges to $K_{ij}(x_0^*) = K^*(x_0^*)$ because K_{ij} is continuous. The overall kernel is therefore continuous. All results in section 6.9 for continuous kernels therefore also hold for piecewise continuous kernels under the definition of piecewise continuity given here.

Page 313: 2nd line after eqn. (10.7.2), $Cov(Y_1, Y_2) \to Cov[X_2, Y_2]$

Page 313: line before eqn. (10.7.5) should refer to eqn. (10.7.4)

Page 314: penultimate line, alá $\to$ à la