Fleck Chapter 7-8 – Functions and PHP

Problem 1) Is the following function onto, one-to-one, one-to-one correspondence.

$f: A \rightarrow B$ where A={a, b, c}, B={10, 11, 12, 13} and f(a)=10, f(b)=12, f(c)=13

Onto: **No**. There is no mapping into the value 11 from B.
One-to-one: **Yes**. We don't have two mappings into any value in B.
One-to-one Correspondence. **No**. Not both onto and one-to-one.

Problem 2) Is the following function onto, one-to-one, one-to-one correspondence.

$f: A \rightarrow B$ where A={a, b, c, d, e}, B={10, 11, 12, 13} and f(a)=10, f(b)=12, f(c)=13, f(d)=11, f(e)=10

Onto: **Yes**. Every value from B is mapped to.
One-to-one: **No**. f(a) = 10 and f(e) = 10 so two values map into 10.
One-to-one Correspondence. **No**. Not both onto and one-to-one.

Problem 3) Is the following function onto, one-to-one, one-to-one correspondence.

$f: \mathbb{N} \rightarrow \mathbb{N}$ where f(n) = n^2

Onto: **No**. f(n) does not map into 3. (There is no natural number that is a solution to $n^2 = 3$)
One-to-one: **Yes**. Every value from N maps into a unique value.
One-to-one Correspondence. **No**. Not both onto and one-to-one.

Problem 4) Is the following function onto, one-to-one, one-to-one correspondence.

$f: \mathbb{Z} \rightarrow \mathbb{N}$ where f(n) = n^2

Onto: **No**. f(n) does not map into 3. (There is no integer that is a solution to $n^2 = 3$)
One-to-one: **No**. f(-2) = 4 and f(2) = 4.
One-to-one Correspondence. **No**. Not both onto and one-to-one.

Problem 5) Is the following function onto, one-to-one, one-to-one correspondence.

$f: \mathbb{R} \rightarrow \mathbb{R}$ where f(x) = x-3

Onto: **Yes**. Pick an arbitrary real value d. Can we solve f(x) = x-3 =d? Yes. The solution is x=d+3 which is a real value.
One-to-one: **Yes**. Suppose that there are two different values a and b, where f(a) = f(b)=k. Or, a-3 = b-3. But we can solve this to get a=b, so there is only one arrow into k.
One-to-one Correspondence. **Yes**. It is both onto and one-to-one.

Problem 6) Is the following function onto, one-to-one, one-to-one correspondence.

$f: \mathbb{R} \rightarrow \mathbb{N}$ where f(x) = $\lfloor x/3 \rfloor$

Onto: **Yes**. Pick an arbitrary natural number n. Can we solve f(x) = $\lfloor x/3 \rfloor$ =n? Yes. One solution is x=3n.
One-to-one: **No**. Consider the values a=4 and b=5. We know that f(4) = $\lfloor 4/3 \rfloor = 1$ and also that f(5) = $\lfloor 5/3 \rfloor = 1$. We have two values that f maps into 1. (In fact, there are an infinite number of real values in the interval [3, 6) that f will map into 1)
One-to-one Correspondence. **No**. Not both onto and one-to-one.

Problem 7) Suppose we have a function $f: A \rightarrow B$ where the cardinality of A is 10 and the cardinality of B 6. Can f be onto? Can f be one-to-one?

Onto. It is **possible** that the function is onto, but not guaranteed.
One-to-One. **Not possible**. We have 10 objects in A that map into 6 categories from B. By the PHP we know that there at least two objects from A that are assigned to some category from B, so the function can not be one-to-one.

Problem 8) Suppose we have a function $f: A \rightarrow B$ where the cardinality of A is 4 and the cardinality of B is 10. How many different one-to-one functions are there?

Task 1: Map first value of A into B. 10 choices
Task 2: Map second value of A into B. 9 choices (One less because we can not map two values from A into the same value of B.)
Task 3: Map third value of A into B. 8 choices
Task 4: Map fourth value of A into B. 7 choices

Multiply to get 10 x 9 x 8 x 7 = 5040

Problem 9) Suppose that there is a group of 11 students. Every student will pick their favorite color from {red, green, blue}. What does the GPHP allow us to conclude?

We need to decide what the objects and categories are.
Objects: 11 students
Categories students are assigned to: 3 colors

By the GPHP we know that there is some category with at least $\lceil 11/3 \rceil = 4$ objects. So there are at least 4 students that have the same favorite color.

Problem 10) Suppose that there is a group of 11 students. Every student will pick their favorite color from {red, green, blue}. What can you conclude about the number of students that have red as their favorite color?

We can only draw a very weak conclusion. The number of students that chose red is at least zero and at most 11.

Problem 11) Suppose that there is a group of 11 students. Every student will pick their favorite color from {red, green, blue}. Can you conclude that every category has at least one student?

No. We know that at least one category must have 4 students, but it could be the case that there is some category without any students.

Problem 12) Suppose that there is a group of students. Every student will pick their favorite color from {red, green, blue, yellow, orange}. How many students do you need to guarantee that at least 9 students have the same favorite color?

We have 5 categories. We need to find the smallest value n where $\lceil n/5 \rceil = 9$. This value is 41.

While we don't have to spread out the people evenly, we can use it to help find the smallest value. So we evenly fill out each of the 5 buckets until we are just short of the required amount. Each of the 5 buckets has 8 people. If we add one more person, we have to have 9 (or more) people in some bucket. So the solutions is 5x8 + 1 = 41.

Problem 12) Suppose that there is a party where 7 people attend. Each person will shake hands with some number of other people. Show that there are at least 2 people that shook hands with the same number of people.

We need to decide what the objects and categories are. The objects are the people and the categories are the number of hands they shook.

The number of other people that each person shook hands with comes from the set {0, 1, 2, 3, 4, 5, 6}. This gives us 7 categories, which isn't less than the number of people. But, observe that two of these categories are mutually exclusive. If there is a person that shook hands with no-one, it cannot be the case that someone shook hands with everyone else.Case A: No one shook hands with everyone else.

Categories = {0, 1, 2, 3, 4, 5}.

There are 7 people and 6 categories – There are at least 2 people in the same category.

Case B: No one shook hands with nobody.

Categories = {1, 2, 3, 4, 5, 6}.

There are 7 people and 6 categories – There are at least 2 people in the same category.