Building Blocks of Theoretical Computer Science
Summations Example Problems

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Introduction

Problem 1 What are the first 20 terms of the arithmetic sequence that starts 1, 3, 5, 7, … ?

The terms are 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37, 39

Problem 2 What is the sum of the first 20 terms of the arithmetic sequence that starts 1, 3, 5, 7, … ? Use our trick from class (Gausses method.)

Add the first and last value in the sequence and multiply by the number of pairs.

First = 1
Last = 39
Pairs = 20/2

Sum = (1+39) (10) = 400

Problem 3 Express the sum in the previous problem using summation notation with a starting index of 1.

The general term in the sequence looks something like

$$a_i = k + id$$

k is related to a starting value and id gets you an arithmetic increase by d from term to term.

We know that

$$a_1 = k + 1 \cdot 2 = 1$$

Solving for k, we find k = -1.

Our sum has a lower index limit of 1 and an upper limit of 20 (number of terms since the low was 1.)

$$S = \sum_{i=1}^{20} \left(-1 + 2i\right)$$

Problem 4 What are the first 7 terms of the geometric sequence that starts 2, 6, 18, …

We have to determine the constant we multiply by each time. In this case, it is 3.

Using a calculator, we find the terms are 2, 6, 18, 54, 162, 486, 1458

Problem 5 What is the sum of the first 7 terms of the geometric sequence that starts 2, 6, 18… ? (Use our trick from class where you multiply by the common factor then subtract and solve.)

We list out the terms in order and then multiply by our constant ratio.

S = 2 + 6 + 18 + 54 + 162 + 486 + 1458

3S = 3x2 + 3x6 + 3x18 + 3x54 + 3x162 + 3x486 + 3x1458

or

3S = 6 + 18 + 54 + 162 + 486 + 1458 +4374

Notice that there are common terms that cancel when we subtract

3S – S = 4374 – 2

And now we can solve for S

S(3-1) = 4374 – 2

S = (4374 – 2)/(3-1) = 4372/2 = 2186

Problem 6 Express the sum in the previous problem using summation notation with a starting index of 1.

The general term in the sequence looks something like

$$a_i = k \times r^i$$

k is related to a starting value and r^i r gets you a geometric ratio of r from term to term.

We know that

$$a_1 = k \times 3^1 = 2$$

Solving for k, we find k = 2/3.

Our sum has a lower index limit of 1 and an upper limit of 7 (number of terms since the low was 1.)

$$S = \sum_{i=1}^{7} \left(\frac{2}{3} \times 3^i \right)$$

Problem 6 Manipulate the following summation so that you can apply the known sums to get a closed form solution.

$$S = \sum_{i=1}^{500} i\left(i^2 + 3\right)$$

First, multiply out the expression in the summation

$$S = \sum_{i=1}^{500} \left(i^3 + 3i\right)$$

Now split it into one summation for each term.

$$S = \sum_{i=1}^{500} i^3 + \sum_{i=1}^{500} 3i$$

Factor out constants.

$$S = \sum_{i=1}^{500} i^3 + 3\sum_{i=1}^{500} i$$

Each of these is a known sum, so apply them using n=500

$$S = \left[\frac{500^2\left(500+1\right)^2}{4}\right] + 3\left[\frac{500\left(500+1\right)}{2}\right]$$

We do the easy arithmetic to get

$$S = \left[250^2\left(501\right)^2\right] + 3\left[250\left(501\right)\right]$$

Using a calculator for homework, we would complete the evaluation.

$$S = \left[125250^2\right] + 3\left[125250\right]$$
$$S = 15{,}687{,}938{,}250$$

Problem 6 Manipulate the following summation so that you can apply the known sums to get a closed form solution.

$$S = \sum_{i=1}^{n} (3i - 2)(i + 5)$$

First, multiply out the expression in the summation

$$S = \sum_{i=1}^{n} (3i^2 + 13i - 10)$$

Now split it into one summation for each term.

$$S = \sum_{i=1}^{n} 3i^2 + \sum_{i=1}^{n} (13i) + \sum_{i=1}^{n} (-10)$$

Factor out constants.

$$S = 3\sum_{i=1}^{n} i^2 + 13\sum_{i=1}^{n} (i) - 10\sum_{i=1}^{n} 1$$

Each of these is a known sum, so apply them

$$S = 3\left[\frac{n^2(n+1)^2}{4}\right] + 13\left[\frac{n(n+1)}{2}\right] - 10[n]$$

We can simplify, but this is an acceptable stopping point for our class on both homework and exam.

To simplify factor out n/4 to get

$$S = \left(\frac{n}{4}\right)\left\{3\left[n(n+1)^2\right] + 13\left[2(n+1)\right] - 40\right\}$$

expand to get

$$S = \left(\frac{n}{4}\right)\left\{3\left[n\left(n^2+2n+1\right)\right] + 26(n+1) - 40\right\}$$

$$S = \left(\frac{n}{4}\right)\left\{3\left(n^3+2n^2+n\right) + 26(n+1) - 40\right\}$$

$$S = \left(\frac{n}{4}\right)\left\{\left(3n^3+6n^2+3n\right) + \left(26n+26\right) - 40\right\}$$

$$S = \left(\frac{n}{4}\right)\left(3n^3+6n^2+29n-14\right)$$

$$S = \left(\frac{3n^4+6n^3+29n^2-14n}{4}\right)$$

Problem 7 Manipulate the following summation so that you can apply the known sums to get a closed form solution.

$$S = \sum_{i=100}^{n} \left(3i^2 - 2\right)$$

Before we split the sum, we will manipulate this so that the index starts from 1 instead of 100.

Approach 1: (My preferred approach.)

Add in the missing terms so that the sum starts from 1 and then subtract them out.

$$S = \sum_{i=1}^{99} \left(3i^2 - 2\right) + \sum_{i=100}^{n} \left(3i^2 - 2\right) - \sum_{i=1}^{99} \left(3i^2 - 2\right)$$

$$S = \sum_{i=1}^{n} \left(3i^2 - 2\right) - \sum_{i=1}^{99} \left(3i^2 - 2\right)$$

And now it is straight forward. Separate each sum and factor out the constants.

$$S = 3\sum_{i=1}^{n} i^2 - 2\sum_{i=1}^{n} 1 - \left[3\sum_{i=1}^{99} i^2 - 2\sum_{i=1}^{99} 1\right]$$

Apply knowns

$$S = 3\left[\frac{n(n+1)(2n+1)}{6}\right] - 2n - \left[3\left[\frac{99(99+1)(2(99)+1)}{6}\right] - 2(99)\right]$$

Simple arithmetic

$$S = \left[\frac{n(n+1)(2n+1)}{2}\right] - 2n - \left[\left[99(50)(199)\right] - 2(99)\right]$$

Where we stop.

Approach 2: Shift the index variable so that it starts at 1. We will replace I by a new index J. The relation between I and J is that J = I -99. Inside the formula, we will replace I by J+99. The upper limit is smaller now, too. The new sum is:

$$S = \sum_{j=1}^{n-99} \left(3\left(j+99\right)^2 - 2 \right)$$

Multiply this out

$$S = \sum_{j=1}^{n-99} \left(3\left(j^2 + 198j + 9801\right) - 2 \right)$$

$$S = \sum_{j=1}^{n-99} \left(\left(3j^2 + 594j + 29403\right) - 2 \right)$$

$$S = \sum_{j=1}^{n-99} \left(3j^2 + 594j + 29401 \right)$$

And now we can separate and factor and apply knowns

$$S = 3\sum_{j=1}^{n-99} j^2 + 594\sum_{j=1}^{n-99} j + 29401\sum_{j=1}^{n-99} 1$$

$$S = 3\left[\frac{\left(n-99\right)\left(n-99+1\right)\left(2\left(n-99\right)+1\right)}{6} \right] + 594\left[\frac{\left(n-99\right)\left(n-99+1\right)\left(2\left(n-99\right)+1\right)}{2} \right] +$$

Remember that when we apply the closed form we use whatever the upper limit is as n in the formula (which is n-99 for this problem.)

Problem 8 Manipulate the following summation so that you can apply the known sums to get a closed form solution.

$$S=\sum_{i=3}^{64} 2\times 3^i$$

This is geometric. We have two issues, the known formula we have for a geometric sum starts at zero and it does not allow for a constant factor.

Factor out the 2

$$S=2\sum_{i=3}^{64} 3^i$$

Add in terms starting at zero and ending at 2

$$S+2\sum_{i=0}^{2} 3^i=2\sum_{i=0}^{2} 3^i+2\sum_{i=3}^{64} 3^i=2\sum_{i=0}^{64} 3^i$$

or

$$S=2\sum_{i=0}^{64} 3^i-2\sum_{i=0}^{2} 3^i$$

The second sum only has 3 terms so we can just compute them directly

$$S=2\sum_{i=0}^{64} 3^i-2\left[1+3+9\right]$$

And we can apply the known formula to get

$$S=2\left[\frac{3^{64+1}-1}{3-1}\right]-2\left[1+3+9\right]$$

Simplifying gives

$$S=\left(3^{65}-1\right)-2\left[13\right]$$

$$S=3^{65}-27$$

On an exam, this where we will leave it. In homework, we us python to get an exact answer (my calculator will only give an approximate answer.)

$S=$ 3433683820292512484657849089281-27

$S=$ 3433683820292512484657849089254

Problem 9 Manipulate the following double summation so you can apply the known sums to get a closed form solution.

$$S = \sum_{i=1}^{n}\sum_{j=1}^{m}(ij)$$

If you expand the outer sum, we see that the value of I is constant for each of the inner sums.

$$S = \sum_{i=1}^{n}\sum_{j=1}^{m}(ij) = \sum_{j=1}^{m}(1\times j) + \sum_{j=1}^{m}(2\times j) + \sum_{j=1}^{m}(3\times j) + \ldots$$

So we can factor it out of the inner sum

$$S = \sum_{i=1}^{n}\sum_{j=1}^{m}(ij) = \sum_{i=1}^{n} i \sum_{j=1}^{m}(j)$$

Apply the know sum to get

$$= \sum_{i=1}^{n} i\left(\frac{m(m+1)}{2}\right)$$

We can factor out the constant

$$= \left(\frac{m(m+1)}{2}\right)\sum_{i=1}^{n} i$$

Apply the known sum to get

$$= \left(\frac{m(m+1)}{2}\right)\left(\frac{n(n+1)}{2}\right)$$

Problem 10 You have an N by N checkerboard where you put zero token on each of the squares on the diagonal or below the diagon . On each square above the diagonal, you will put one token, two on the squares above those, etc. Express this as a double summation and then solve.

Example for n=5

	J=1	J=2	J=3	J=4	J=5
I=1	0	1	2	3	4
I=2		0	1	2	3
I=3			0	1	2
I=4				0	1
I=5					0

As we create the summation, we want to only consider values of J that are greater than I. (This pattern is common when we want to consider pairs items where the first and second value are not ordered.)

$$S = \sum_{i=1}^{n} \sum_{j=i}^{n} \ldots$$

Now that we have a summation that covers each of the values above the diagonal we need a formula. Examining the values we see that J-I does the job.

$$S = \sum_{i=1}^{n} \sum_{j=i}^{n} (j - i)$$

Separate the inner sum

$$S = \sum_{i=1}^{n} \left(\sum_{j=i}^{n} j - \sum_{j=i}^{n} i \right)$$

I is a constant in the second sum.

$$S = \sum_{i=1}^{n} \left(\sum_{j=i}^{n} j - i \sum_{j=i}^{n} 1 \right)$$

We use our trick of adding in the missing values from 1 to i-1 and then subtracting them out so we can apply known sums.

$$S = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} j - \sum_{j=1}^{i-1} j - i \left(\sum_{j=1}^{n} 1 - \sum_{j=1}^{i-1} 1 \right) \right)$$

Known sums give

$$S=\sum_{i=1}^{n}\left(\frac{n(n+1)}{2}-\frac{(i-1)(i-1+1)}{2}-i\left(n-(i-1)\right)\right)$$

$$S=\sum_{i=1}^{n}\left(\frac{n^2+n}{2}-\frac{(i^2-i)}{2}-(ni-i^2+i)\right)$$

Separate and factor out the constants

$$S=\left(\frac{n^2+n}{2}\right)-\frac{1}{2}\sum_{i=1}^{n}(i^2-i)-\sum_{i=1}^{n}(ni-i^2+i)$$

$$S=\left(\frac{n^2+n}{2}\right)-\frac{1}{2}\sum_{i=1}^{n}i^2+\frac{1}{2}\sum_{i=1}^{n}i-n\sum_{i=1}^{n}i+\sum_{i=1}^{n}i^2-\sum_{i=1}^{n}i$$

Apply knowns

$$S=\left(\frac{n^2+n}{2}\right)[n]-\frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6}\right]+\frac{1}{2}\left[\frac{n(n+1)}{2}\right]-n\left[\frac{n(n+1)}{2}\right]+\left[\frac{n(n+1)(2n+1)}{6}\right]-\left[\frac{n(n+1)}{2}\right]$$

All of the terms contain n(n+1)/2, so factor that out

$$S=\left(\frac{n^2+n}{2}\right)\left\{[n]-\left[\frac{(2n+1)}{6}\right]+\frac{1}{2}-n+\left[\frac{(2n+1)}{3}\right]-1\right\}$$

To add the fractions we factor out 1/6

$$S=\left(\frac{n^2+n}{2}\right)\frac{1}{6}\left\{6n-(2n+1)+3-6n+2(2n+1)-6\right\}$$

And then add the resulting terms

$$S=\left(\frac{n^2+n}{2}\right)\frac{1}{6}\left\{(2n-2)\right\}$$

$$S=\left(\frac{n^2+n}{2}\right)\frac{1}{3}(n-1)$$

We have done a lot of manipulations, so to give us some confidence in our result we try n=5 and check to see if the formula gives us the value 20.

$$\left(\frac{5^2+5}{2}\right)\frac{1}{3}(5-1)=\left(\frac{30}{2}\right)\frac{1}{3}(4)=20$$