Building Blocks Homework 4
Solution

Question A) Given the following predicates where the domain of x is people and y is a kind of animal.

Pet(x,y) = "x has y as a pet"
Happy(x) = "x is happy"
Tidy(x) = "x is tidy"

Translate the following formulae into English statements.

a) $\forall x, (Pet(x, dog) \rightarrow Happy(x))$

b) $\forall x, ((\neg Pet(x, cat) \wedge \neg \mathrm{Pet}(\mathrm{x}, \mathrm{dog})) \rightarrow Tidy(x))$

c) $\forall y, \exists x, (Pet(x, \mathrm{y}) \wedge (\mathrm{Happy}(x)))$

d) $\neg \exists x, (Tidy(x) \rightarrow Happy(x))$

a) For all people, if they have a dog as a pet then they will be happy.
b) For all people, if they don't have a cat as a pet and don't have a dog as a pet, then they are tidy.
c) For every kind of animal, there is a person that owns that animal and is happy.
d) There does not exist a person such that if they are tidy then they are happy.

Some of these could be written more succinctly and naturally, but you have to be careful that the meaning is the same.

a) If a person owns a dog then they are happy. (OK - The universal quantification over people is implied.)
a) There is some person that owns a dog if they are happy. (WRONG – This is an existential and the order of implication is backwards.)
a) If every person owns a dog, then they are happy. (WRONG - The if needs to be inside the quantifier.)

Question B) In the following formula indicate for each variable if it is bound or free. If it is bound indicate which quantifier it is bound to.

$$\exists y, (R(x) \rightarrow C(y)) \rightarrow \exists x, \left(R(z) \rightarrow \forall z, (M(x,y,z) \wedge \forall x, \neg \mathrm{T}(\mathrm{x}, 50))\right)$$

Use the parentheses to determine the parts of the formula each quantifier applies to. Work inside out. There are four quantifiers and I will use color to mark them

$$\exists y, (R(x) \rightarrow C(y)) \rightarrow \exists x, \left(R(z) \rightarrow \forall z, (M(x,y,z) \wedge \forall x, \neg \mathrm{T}(\mathrm{x}, 50))\right)$$

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The unbound quantifiers are marked in red. Notice that the last x is bound by the universal quantifier and not the existential. To make the meaning clear we would rename one of the x's to some thing else.

$$\exists y, (R(x) \rightarrow C(y)) \rightarrow \exists x, \left(R(z) \rightarrow \forall z, (M(x,y,z) \wedge \forall x, \neg \mathrm{T}(\mathrm{x}, 50))\right)$$

Question C) Give a direct proof of the following statement.

For all positive rational values x and y, (x+y)/2 (the average) is also rational.

Given: x is rational, y is rational
Show: (x+y)/2 is rational.

Since x is rational we can write x=a/b for integer a, b, b is not zero.
Since y is rational we can write y=c/d for integer c, d, d is not zero.

$$(x+y)/2 = (a/b + c/d)/2 = (ad/bd + cb/bd)/2$$
$$= (ad+cb)/2bd$$

This is rational provided that

1) $(\boldsymbol{ad} + \boldsymbol{cb})$ **is an integer** Check – a, b, c, d are integers and integers are closed under addition and multiplication.
2) $\mathbf{2}\boldsymbol{bd}$ **is an integer.** Check. b, d and 2 are integers and integers are closed under mulitiplication.
3) $\mathbf{2}\boldsymbol{bd}$ **is not zero.** Check. The only way a product can be zero is if one or both of the factors is zero. We know that b and d are both not zero so the product can not be zero either.

QED

Question D) Give a direct proof of the following statement.

If n+1 is odd, then $7n^2$ is an even integer

Given: n+1 is odd
Show: $7n^2$ is even

Since n+1 is odd we can write n+1 = 2k+1 for some integer k.
Or n=2k

Consider $7n^2 = 7(2k)^2 = 2(14k^2)$ will be odd provided that $14k^2$ is an integer.
14 and k are integers and will be closed under multiplication.

QED

Question E) Give a direct proof of the following statement.

For all non-negative integers n, m, if n is odd and m is even, then n +2m is odd.

Given: n is odd, m is even
Show: n+2m is odd

Since n is odd we can write n = 2k+1 for some integer k.
Since m is even we can write m=2j for some integer j

Consider n+2m = 2k+1 + 2(2j) = 2(k+2j) + 1 meets the definition for odd if k+2j is an integer. K, j and 2 are integers and integers are closed under addition and multiplication. Check.

QED

Question F) Give proof by cases (odd/even) of the following statement.

For all integer n, $3n^2 - 3n$ is even.

Case 1: n is even

Show: $3n^2 - 3n$ is even

Since n is even, we can write n=2k for some integer k.

Consider $3n^2 - 3n = 3(2k)^2 - 3(2k) = 2(6k^2 - 3k)$

Is even provided that $6k^2 - 3k$ is an integer. But k and the constants are integers and integers are closed under subtraction and multiplication. Check.

Case 2: n is odd

Show: $3n^2 - 3n$ is even

Since n is odd, we can write n=2k+1 for some integer k.

Consider $3n^2 - 3n = 3(2k+1)^2 - 3(2k+1)$

$$
\begin{aligned}
&= 3(4k^2 + 4k + 1) - (6k + 3) \\
&= (12k^2 + 12k + 3) - (6k + 3) \\
&= (12k^2 + 6k) \\
&= 2(6k^2 + 3k)
\end{aligned}
$$

Is even provided that $6k^2 + 3k$ is an integer. But k and the constants are integers and integers are closed under subtraction and multiplication. Check.

QED

Question G) We want to show one of the cases for the inequality $|x| - |y| \leq |x - y|$ where x and y are real numbers.

Prove that the inequality is true if x and y are both greater than zero. Consider three cases: $x > y$, x=y, and x<y.

Case: x>0, y>0

Subcase: x>y

Show: $|x| - |y| \leq |x - y|$ is true

Because x and y are positive, we can drop the absolute values on the left hand side. (LHS)

$x - y \leq |x - y|$

Because x > y we know that x-y is positive, we can drop the absolute values on the RHS

$x - y \leq x - y$ is true. Check

Subcase: x=y

Show: $|x| - |y| \leq |x - y|$ is true

Because x and y are positive, we can drop the absolute values on the left hand side. (LHS)

$x - y \leq |x - y|$

Because x = y we know that x-y is zero, we can drop the absolute values on the RHS

$0 \leq 0$ is true. Check

Subcase: x<y

Show: $|x| - |y| \leq |x - y|$ is true

Because x and y are positive, we can drop the absolute values on the left hand side. (LHS)

$x - y \leq |x - y|$

Because x < y we know the LHS is negative. The absolute value is always zero or positive and must be larger than any negative value.

$x - y \leq x - y$ is true. Check

Question H) Are the following true or false?

a) 5 | 30	is true since $30 = 5 \times 6$
b) 30 | 5	is false since 30 is not a factor of 5
c) 16 | 848	is true since $848 = 16 \times 53$
d) 9 | 135	is true since $135 = 9 \times 15$

Question I) Give prime factorizations of the following numbers.

a) 280 is $280 = 2 \times 140 = 2^2 \times 70 = 2^3 \times 35$
$= 2^3 \times 5 \times 7$

b) 1234 is $1234 = 2 \times 617$

c) 971 is 971

To check if a value is prime, try possible factors that are prime up to the square root of the value. 23 and 31 are the last values to try in parts b and c respectively.

You may choose to solve one (and only one) of the following Extra Credit Problems. If you submit more than one, only the first will be graded.

Extra Credit 1)
Give a proof that if x is irrational, then 3x is also irrational.
(Hint: Use a proof by contradiction.)

Given: x is irrational
Assume: 3x is not irrational

Since 3x is not irrational it must be rational. Therefore we can write 3x as a/b for integer a and b where b is not 0.
3x = a/b or
x= a/3b. But a and 3b are integers and we know that 3b is not zero, so x is rational. This contradicts the given.
QED

Extra Credit 2) Suppose that k is a positive integer value that ends in n zeros where n is a natural number. (E.g. 55200 ends in two zeros.) Show that the prime factorization of k has at least n factors of 5.

Each 0 at the end of a positive integer value represents a factor of 10. Each factor of 10 results in a factor of 2 and a factor of 5. Therefore there will be at least n factors of 2 and n factors of 5.