Example: Prove that $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + 1/2 \rfloor$

Write $x = n + \varepsilon$ where n is an integer and $0 \le \varepsilon < 1$
We rewrite the left hand side (LHS) as

$\lfloor 2(n+\varepsilon) \rfloor = \lfloor 2n + 2\varepsilon \rfloor$

$= 2n + \lfloor 2\varepsilon \rfloor$ (Table $\lfloor x + n \rfloor = \lfloor x \rfloor + n$)

We know that $0 \le \varepsilon < 1$ so if we multiply by 2 we get

$0 \le 2\varepsilon < 2$

Therefore the LHS can have one of two values depending on whether the floor results in 0 or 1. We are going to create a case for each of these possibilities.
Case 1: $0 \le 2\varepsilon < 1$ gives $\lfloor 2\varepsilon \rfloor = 0$

So: $0 \le \varepsilon < 1/2$

Case 2: $1 \le 2\varepsilon < 2$ gives $\lfloor 2\varepsilon \rfloor = 1$

So: $1/2 \le \varepsilon < 1$

We now proceed to show each case
Case 1: $0 \le 2\varepsilon < 1$ gives $\lfloor 2\varepsilon \rfloor = 0$

The LHS is $\lfloor 2x \rfloor = 2n + \lfloor 2\varepsilon \rfloor = 2n + 0$

$= 2n$

The RHS is $\lfloor x \rfloor + \lfloor x + 1/2 \rfloor = \lfloor n + \varepsilon \rfloor + \lfloor n + \varepsilon + 1/2 \rfloor$

$= n + n + \lfloor \varepsilon + 1/2 \rfloor$

$= n + n + 0$

$= 2n$

The LHS and RHS match.

> We know that $0 \le \varepsilon < 1/2$
> Adding in ½ we get
> $1/2 \le (\varepsilon + 1/2) < 1$
> Therefore $\lfloor \varepsilon + 1/2 \rfloor = 0$

Case 2: $1 \le 2\varepsilon < 2$ gives $\lfloor 2\varepsilon \rfloor = 1$

The LHS is $\lfloor 2x \rfloor = 2n + \lfloor 2\varepsilon \rfloor = 2n + 1$

$= 2n + 1$

The RHS is $\lfloor x \rfloor + \lfloor x + 1/2 \rfloor = \lfloor n + \varepsilon \rfloor + \lfloor n + \varepsilon + 1/2 \rfloor$

$= n + n + \lfloor \varepsilon + 1/2 \rfloor$

$= n + n + 1$

$= 2n + 1$

The LHS and RHS match.

> We know that $1/2 \le \varepsilon < 1$
> Adding in ½ we get
> $1 \le (\varepsilon + 1/2) < 1 + 1/2$
> Therefore $\lfloor \varepsilon + 1/2 \rfloor = 1$

The statement is true in all cases.
QED