Discrete Math Homework 8
Due Monday, March 6 at the beginning of class

General instructions:

- Use standard size paper (8.5 by 11).
- Answer each question in order using a single column.
- Be neat. If we cannot read your solution it is wrong.
- Show your work. If you just write an answer, you will get minimal credit even if the answer is correct.

Exam 1 is Friday, March 10 and covers all the material from Rosen sections: 1.6, 1.7, 1.8, 2.1, 2.2, 2.3, 2.4, 2.5, 3.2

Rosen section 2.5.

Question A) Rosen 2.5 Exercise 2 a, d, f (p. 176)
Determine whether each of these sets is finite, countably infinite or uncountable. If countably infinite give a one-to-one correspondence to the positive integers.

a) the integers greater than 10
This is countably infinite. F(n) = n+9 is the one-to-one correspondence.
d) the real numbers between 0 and 2
This is uncountably infinite. We know that it is not countable since the real numbers from 0 to 1 is a subset and is uncountable. The superset must be at least as large.
f) the integers that are multiples of 10
This is countably infinite. F(n) = 10n.

Question B) Rosen 2.5 Exercise 10 (p. 176)

Give an example of two uncountable sets A and B such that A-B is
a) finite
if A=[0, 1] and B=[0, 1) then A-B ={1}
b) countably infinite
A = Reals and B=Irrationals then A-B=Rationals
c) uncountable
If A = [0, 2) and B=[0, 1) then A-B=[1, 2)

Question C) Show that the union of two uncountable sets is uncountable. (Do a proof by contradiction.)

Let A and B be uncountable sets. This means that there is no one-to-one correspondence to the positive integers.
Assume that the union of A and B is not uncountable.
Therefore the union of A and B is countable.
This means that we can list out the elements of A union B
Therefore we can list out the elements of A by putting them in the same order as for the union.
This means that there is a one-to-one correspondence from the positive integers to A.
This is a contradiction.
QED.

Question D) Show that the set of finite bit strings is countable. (Finite bit strings are things like "01", "101001", and the empty string "". Show that you can list them all.)

List the strings by length. If strings are equal length, order them by interpreting them as integers.
Here is part of the list to illustrate the correspondence:

Empty, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, 100, 101, 110, 111, 0000, …

Question E) Show that the set of infinite bit strings is uncountable.

So first note that none of the strings from the previous example is in this set. We will do a diagonalization proof. Assume that the set of infinite bit string is countable. That means we can list them out

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S1 = b b b b b b b b b b …
S2 = b b b b b b b b b b …
S3 = b b b b b b b b b b …
S4 = b b b b b b b b b b …
S5 = b b b b b b b b b b …
…
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We can construct a new bit string T that is not in the list. Bit i of T is found by looking at bit i of Si. If the bit is 0, use 1. (And vice verse.) T is an infinite bit string. T differs in at least one bit with every string in the list, so T is not in the list.

This is a contradiction. Therefore the set of infinite bit strings is uncountable. QED.

Rosen section 3.2

Question F) Show that $x+3 \in O\left(x^3\right)$

I will find a C and k that work so that

$$|x+3| \leq C\left|x^3\right| \text{ for all x>k}$$

As long as k is positive, the terms are positive and I can drop the absolute value signs.

$$x+3 \leq Cx^3 \text{ for all x>k}$$

I choose C=1 and k=3

$$x+3 \leq x^3 \text{ for all x>3}$$

We start with

$$x > 3$$

Add x to both sides

$$x+x > x+3$$

Or

$$2x > x+3$$

If x>3 then x>2 as well

$$x > 2$$

Multiply both sides by x

$$x^2 > 2x$$

If x>3 then x>1 as well

$$x > 2$$

Multiply both sides by x squared

$$x^3 > x^2$$

Combining these, we have

$$x^3 > x^2 > 2x > x+3$$

Which is what we needed to show when x>3.

Question G) Show that $x^3 \notin O(x+3)$

I will show that no values for C and k will work so that

$$|x^3| \le C|x+3| \text{ for all x>k}$$

We can restrict k and C to be positive. If I can find values of k and C that work, I can always find larger values that will work as well.

the terms are positive and I can drop the absolute value signs.

$$x^3 \le C(x+3) \text{ for all x>k}$$

Divide both sides by x+3 (positive so the inequality does not flip)

$$\frac{x^3}{x+3} \le C \text{ for all x>k}$$

Multiply top and bottom by 1/x

$$\frac{x^3}{x+3}\left(\frac{1/x}{1/x}\right) \le C \text{ for all x>k}$$

and then manipulate

$$\frac{x^2}{\left(\frac{x}{x}+\frac{3}{x}\right)} \le C \text{ for all x>k}$$

$$\frac{x^2}{\left(1+\frac{3}{x}\right)} \le C \text{ for all x>k}$$

If we look at the LHS, we see that as x increases the denominator gets closer and closer to 1 and the numerator is unbounded. No matter what value I pick for C, I can always find a value a value of x such that the LHS is larger than C.
QED.

Question H) Show that $2x^4 \in \Theta\left(x^4\right)$ (Note: you have to show it is both upper and lower bounded.)

1) Show that $2x^4 \in O\left(x^4\right)$

I will find a C and k that work so that

$$\left|2x^4\right| \le C\left|x^4\right| \text{ for all x>k}$$

As long as k is positive, the terms are positive and I can drop the absolute value signs.

$$2x^4 \le Cx^4 \text{ for all x>k}$$

I choose C=2 and k=1

$$2x^4 \le 2x^4 \text{ for all x>1}$$

Divide both side by $2x^4$

$$1 \le 1 \text{ for all x>1}$$

Is true regardless of the value of x.

2) Show that $2x^4 \in \Omega\left(x^4\right)$

I will find a C and k that work so that (Note that k and C must be positive for this proof. If we don't make this restriction, then negative values of C would always work.)

$$\left|2x^4\right| \ge C\left|x^4\right| \text{ for all x>k}$$

As long as k is positive, the terms are positive and I can drop the absolute value signs.

$$2x^4 \ge Cx^4 \text{ for all x>k}$$

I choose C=1 and k=1

$$2x^4 \ge x^4 \text{ for all x>1}$$

Divide both side by $2x^4$

$$2 \ge 1 \text{ for all x>1}$$

Is true regardless of the value of x.

Since it is upper and lower bounded, we know that $2x^4 \in \Theta\left(x^4\right)$

QED.

Question I) Rosen 3.2 Exercise 4 (p. 216)
Show that $2^x + 17 \in O\left(3^x\right)$

I will find a C and k that work so that

$$\left|2^x + 17\right| \le C\left|3^x\right| \text{ for all x>k}$$

As long as k is positive, the terms are positive and I can drop the absolute value signs.

$$2^x + 17 \le C \cdot 3^x \text{ for all x>k}$$

I choose C=2 and k=10

$$2^x + 17 \le 2 \cdot 3^x \text{ for all x>3}$$

or

$$2^x + 17 \le 3^x + 3^x \text{ for all x>3}$$

This will be true if both of the following are true

a) $2^x \le 3^x$ for all x>3
b) $17 \le 3^x$ for all x>3

For (a) we can divide both sides by 2^x

$$1 \le \left(\frac{3}{2}\right)^x \text{ for all x>3}$$

Since b^x is increasing provided that b is greater than 1, this will be true for x>3.

For (b)

$$17 \le 3^x \text{ for all x>3}$$

We again note that the RHS is an increasing function. Since $3^3 = 27$, this again is true for all x>3.
QED.

Question J) Rosen 3.2 Exercise 22 (p. 217) (You are not required to prove the BigO relation between the functions, but you should think carefully about the order you list the functions. Looking at the value of the functions for small values of x may not give you the correct order.)

Original list: $(1.5)^n \quad n^{100} \quad (\log n)^3 \quad \sqrt{n}\log n \quad 10^n \quad (n!)^2 \quad n^{99} + n^{98}$

Order: $(\log n)^3 \quad \sqrt{n}\log n \quad n^{99} + n^{98} \quad n^{100} \quad (1.5)^n \quad 10^n \quad (n!)^2$

To note:
logn is O(n to any power)
b^n is O(n!) for any base b.

You may choose to solve one (and only one) of the following Extra Credit Problems. If you submit more than one, only the first will be graded.

Extra Credit 1) Rosen 2.5 Exercise 9 (p. 176)

Countably infinite number of busses with countably infinite passengers arrive at Hilberts fully occupied Grand Hotel. Show that all the new guests can be accommodated.

For each bus Bi we have a list of passengers. (i=1, 2, 3, 4, 5, 6, …)

For ease in wording, we will assume that the original guests are in bus zero (B0).

We just need to show that we can list out all the passengers

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B0: p1   p2   p3   p4   p5 …
B1: q1   q2   q3   q4   q5 …
B2: r1   r2   r3   r4   r5 …
B3: s1   s2   s3   s4   s5 …
…
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We will assign people to rooms by going down diagonals in the following order.

p1
p2 q1
p3 q2 r1
p4 q3 r2 s1
…

Everybody is assigned a room.

Extra Credit 2) Rosen 2.5 Exercise 26 (p. 177)

Use the result of exercise 25 to give an alternate proof that the set of rational numbers is countably infinite.

Exercise 25: If it is possible to label each element of an infinite set S with a finite string of keyboard characters from a finite list of characters where no two elements of S have the same label, then S is a countably infinite set.

Every positive rational number can be expressed as p/q where p and q are positive integers that share no common factors.
Every negative rational number can be expressed as –p/q where p and q are positive integers that share no common factors.
Zero can be expressed as 0

Every positive integer can be expressed uniquely as a finite string of digits that does not start with zero.
Therefore we can label each rational number with a unique string composed of characters from the set {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, /, -}

By the result of Exercise 25, this means that the rational numbers are countable.