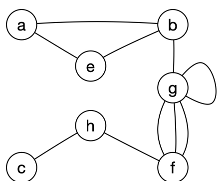
Fleck Chapter 9-10 – Graphs and Bounding

Graph A:



Problem 1) What are the degrees of the nodes in graph A?

Deg(a) = 2	Deg(b) = 3	Deg(c) = 1
Deg(e) = 2	Deg(f) = 4	Deg(g) =6
Deg(h) = 2		

Problem 2) Does graph A have an Euler Circuit? (A path through the graph where every edge is used exactly once and we start and end at the same node.)

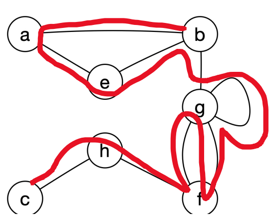
The graph has an Euler circuit provided it is connected and every node has even degree.

Graph A is connected. ✓
Graph A has only nodes of even degree. ×
Graph A does not have an Euler Circuit.

Problem 3) Does graph A have an Euler Path? (A path through the graph where every edge is used exactly once.)

The graph has an Euler path provided it is connected and either 0 or 2 nodes of odd degree.

Graph A is connected. ✓
Graph A has 2 nodes of odd degree. ✓
Graph A has an Euler path.
The path must start and end on nodes b and c (the two odd degree nodes.)



Problem 4) Does graph A have a Hamiltonian circuit? (A path through the graph where every edge is used at most one time and every node is visited exactly once, except that the starting node is the same as the end node.)

For graph to have a Hamiltonian circuit it must be connected. Another requirement is that if you remove any node or edge, the graph must stay connected. Unfortunately, if a graph meets these requirements, it still may not have a Hamiltonian

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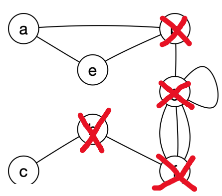
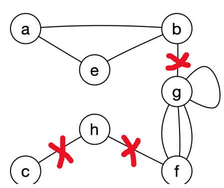
Graph A is connected. ✓
Graph A is disconnected if you remove any of the marked nodes/edges. ✗
Graph A does not have a Hamiltonian Circuit.

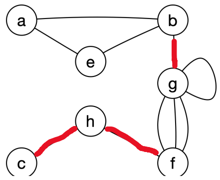


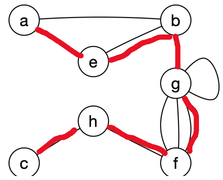


Problem 5) Does graph A have a Hamiltonian path? (A path through the graph where every edge is used at most one time and every node is visited exactly once.

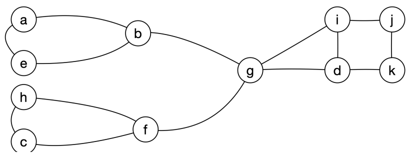
For graph to have a Hamiltonian circuit it must be connected. If there is a node of degree 1, it must be an end of the path. The edges that disconnect the graph are critical and must be in the Hamiltonian if there is one.

Graph A is connected. ✓
Lets see if we can create a Hamiltonian Path.
The edges that disconnect the Graph A are marked. Then we connect them up to make the path.



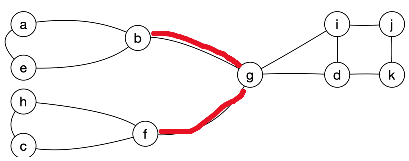


Graph B:

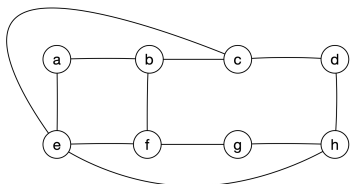


Problem 6) Does graph B have a Hamiltonian path? (A path through the graph where every edge is used at most one time and every node is visited exactly once.

We know that edges which disconnect the graph must be a part of the Hamiltonian. We have marked those edges in red.
In this case, we see that there is no way we can visit the nodes to the right of g without visiting g a second time. No Hamiltonian path exists.

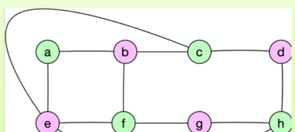


Graph C:



Problem 6) Is graph C bipartite? If so, give a 2 coloring of the graph and then partition the nodes into two sets where there are only edges between the set of nodes.

Yes, it is bipartite as the following 2 coloring shows. Every edge has a pink end and a green end.



The partition is {a, c, f, h} and {b, d, e, g}.

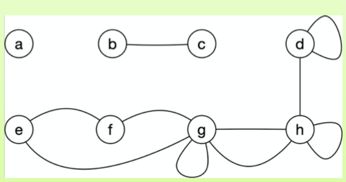
Note: While this partition had the same number of nodes in each partition, this need not be the case in general.

Problem 7) Suppose that I tell you there is a graph where the nodes have degrees: 0, 1, 1, 3, 5, 6, 2, 2.

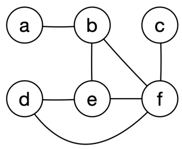
a) **Use the handshaking theorem to determine the number of edges in the graph.**
b) **Draw such a graph. (The answer is not unique.)**

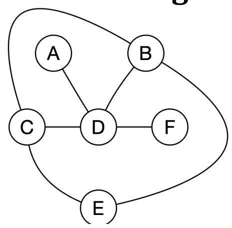
a) The handshaking theorem says that the sum of the degrees of the nodes in a graph must be twice the number of edges.
$0 + 1 + 1 + 3 + 5 + 6 + 2 + 2 = 20$
So the graph has 10 edges.

b) Since there are 8 values in the list, we have 8 nodes.



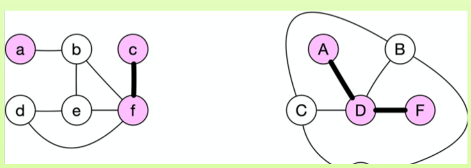
Problem 8) Show that the following two graphs are not isomorphic.





Checklist:

1) **Same number of nodes? Both 6. Yes.**
2) **Same number of edges? Both 7. Yes.**
3) **Degrees the same? Listed in ascending order both are 1, 1, 2, 3, 3, 4. Yes.**
4) **Same subgraphs? NO. Form a subgraph of each of the 1 degree nodes with the degree 4 node (in pink). Keep only the edges between the pink nodes (darkened). We see that the two subgraphs are not the same. There is a path of length 2 between the 1 degree nodes in the second graph, but no such path in the first graph.**



Note: If it passes the tests, look to see if there is a way to map the nodes in the first graph to the nodes in the second graph.

Problem 9) Show that 2^n is lower bounded by $\sqrt{2}^n$ and upper bounded by $\sqrt{5}^n$ where n is a positive integer.

To show this claim we are going to rely on the following little proof.

Lemma: If I have four real numbers that are positive, where a<b and c<d, then ac<bd.

Proof:

Since a<b, we can multiply both sides by c (positive) to get $ac < bc$

Since c<d, we can multiply both sides by b (positive) to get $bc < bd$

Using transitivity, we know that ac<bd.

Lower bound: Show $\sqrt{2}^n < 2^n$

$$\sqrt{2}^n = \underbrace{\sqrt{2} \times \sqrt{2} \times \sqrt{2} \ldots \times \sqrt{2}}_{\text{n times}} < 2^n = \underbrace{2 \times 2 \times 2 \ldots \times 2}_{\text{n times}}$$

Since $\sqrt{2} < 2$ we can repeatedly apply our lemma to show that

$$\sqrt{2} \times \sqrt{2} < 2 \times 2$$

$$\sqrt{2} \times \sqrt{2} \times \sqrt{2} < 2 \times 2 \times 2$$

...

$$\underbrace{\sqrt{2} \times \sqrt{2} \times \sqrt{2} \ldots \times \sqrt{2}}_{\text{n times}} < \underbrace{2 \times 2 \times 2 \ldots \times 2}_{\text{n times}}$$

Upper bound: Show $2^n < \sqrt{5}^n$

Similarly to the previous argument, we know that $2 < \sqrt{5}$

So we can conclude that

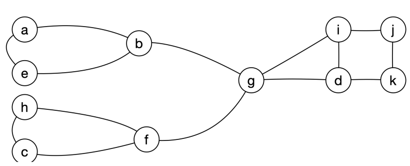
$$2 \times 2 < \sqrt{5} \times \sqrt{5}$$

$$2 \times 2 \times 2 < \sqrt{5} \times \sqrt{5} \times \sqrt{5} <$$

...

$$\underbrace{2 \times 2 \times 2 \ldots \times 2}_{\text{n times}} < \underbrace{\sqrt{5} \times \sqrt{5} \times \sqrt{5} \ldots \times \sqrt{5}}_{\text{n times}}$$

Graph B:



Problem 10) Determine the number of colors required to color graph B by upper and lower bounding the value.

Clearly, graph B is not two colorable. The nodes a, b, and e must all be different colors.

Lets try 3 coloring B. We succeed!

Recap: We found a 3 coloring of the graph (upper bound) and showed that a 2 coloring failed (lower bound.)