Discrete Math Homework 10 Solution
Due Wednesday, March 29 at the beginning of class

General instructions:

- Use standard size paper (8.5 by 11).
- Answer each question in order using a single column.
- Be neat. If we cannot read your solution it is wrong.
- Show your work. If you just write an answer, you will get minimal credit even if the answer is correct.

Rosen section 5.2

Question A) Rosen 5.2 Exercise 4 (p. 341)

Let P (n) be the statement that a postage of n cents can be formed using just 4-cent stamps and 7-cent stamps. The parts of this exercise outline a strong induction proof that P(n) is true for$n \geq 18$.

a) Show statements P(18), P(19), P(20), and P(21) are true, completing the basis step of the proof.

P(18) – Use two 7-cent and one 4-cent stamp.

P(19) – Use one 7-cent and three 4-cent stamps.

P(20) – Use five 4-cent stamps.

P(21) – Use three 7-cent stamps.

b) What is the inductive hypothesis of the proof?

Postage of j cents can be formed using 4-cent and 7-cent j where 18 <= j <= k.

c) What do you need to prove in the inductive step?

Postage of k+1 cents can be formed using 4-cent and 7-cent for $k \geq 21$

d) Complete the inductive step for k ≥ 21.

To make postage of k+1 cents, use a 4-cent stamp. The remaining postage is k-3. The remaining postage k-3 ≥ 18 and less than k, so must be covered by one of the inductive hypotheses. By the IH we can form k-3 postage from 4 and 7-cent stamps.

e) Explain why these steps show that this statement is true whenever n ≥ 18.

This is a valid application of strong induction. We have completed the basis and inductive steps.

Question B) Rosen 5.1 Exercise 10 (p. 342)

Assume that a chocolate bar consists of n squares arranged in a rectangular pattern. The entire bar, a smaller rectangular piece of the bar, can be broken along a vertical or a horizontal line separating the squares. Assuming that only one piece can be broken at a time, determine how many breaks you must successively make to break the bar into n separate squares. Use strong induction to prove your answer.

If you look at a chocolate with

1 square – it requires 0 breaks.
2 squares – it requires 1 breaks.
3 square – it requires 2 breaks.

Let P(n) be the statement that "a chocolate bar of n squares arranged in a rectangular pattern can be broken into separate squares using n-1 breaks."

Basis: n=1

1 square requires 1-1 = 0 breaks. **Check.**

Inductive Hypothesis:

"a chocolate bar of j squares arranged in a rectangular pattern can be broken into separate squares using j-1 breaks." For all j where $k \geq j \geq 1$

Prove for P(k+1).

If we break a rectangle with k+1 squares our first break will split the bar into rectangular pieces. Each piece must have at least one square. Each piece will have k or fewer squares. Lets say that the sizes are X and k+1-X. Each piece will be covered by the induction hypotheses. So the required number of breaks is

1 for the original break
X-1 for the first piece
k+1 – X – 1 for the second piece.

Total is 1 + X – 1 + k + 1 – X - 1 = k.

We have shown that a bar with k+1 pieces requires k breaks.
QED

Rosen section 5.3

Question C) Rosen 5.3 Exercise 22 (p. 358)
Show that the set S defined by $1 \in S$ and $s+t \in S$ whenever $s \in S$ and $t \in S$ is the set of positive integers.

We must prove that everything in S is a positive integer. Use a proof by structural induction.

Basis: 1 is in S. 1 is a positive integer. **Check**.

Ind: Assume that s and t are positive integers. By the inductive definition s+t is also in the set. The sum of two positive integers is also a positive integer. **Check.**

We also need to show that every positive integer is in the set S. Lets suppose that this is not the case. There must be a set F containing positive integers that are not in S. By the well ordering property, there must be a minimum value m in F. We know that m is not 1 because of the basis. We also know that m-1 is at least 1 and less than m, so it must also be in S. Both values 1 and m-1 are in S and by the recursive definition, their sum is also in S. We have a contradiction.

Question D) Rosen 5.3 Exercise 24 (p. 358)
(For part c, you only have to generate polynomials using the variable x. Here are some examples of things that should be in the set.

$3, \quad 8x, \quad 9x^2+8x, \quad 9x^{10}-3x^2+x$

Give a recursive definition of

a) the set of positive integers.

There are lots of possibilities including the definition of S from the previous problem. Here is another.

$1 \in S$ and $s+1 \in S$ whenever $s \in S$

b) the set of positive integer powers of 3.

There are lots of possibilities here are a couple.

$3 \in S$ and $st \in S$ whenever $s \in S$ and $t \in S$

$3 \in S$ and $3t \in S$ whenever $t \in S$

c) the set of polynomials with integer coefficients.

Here is a possibility:

$1 \in S$

$-1 \in S$

$x \in S$

$s+t \in S$ whenever $s \in S$ and $t \in S$

$st \in S$ whenever $s \in S$ and $t \in S$

Question E) Rosen 5.3 Exercise 26 a c (p. 358)
(The terminology "5 | a + b" means that "5 divides a+b" or equivalently "a+b is a multiple of 5".

Let S be the subset of the set of ordered pairs of integers defined recursively by

Basis step: $(0, 0) \in S$.

Recursive step: If $(a,b) \in S$, then $(a + 2, b + 3) \in S$ and $(a+3,b+2) \in S$.

a) List the elements of S produced by the first five applications of the recursive definition.

Start with	(0,0)
1st Application	(2,3) (3,2)
2nd Application	(4,6) (5,5) (5,5) (6,4) just the new stuff
3rd Application	(6,9) (7,8) (8,7) (9,6)
4th Application	(8,12) (9,11) (10,10) (11,9) (12, 8)
5th Application	(10,15) (11,14) (12,14) (13,12) (14, 11) (15, 10)

c) Use structural induction to show that $5 \mid a + b$ when $(a,b) \in S$.

Basis: (0,0) is in S. 0+0 = 0 is divisible by 5. **Check**

Inductive step: Given that (a,b) is in S, we can assume that a+b is divisible by 5. We have two recursive rules for adding values into S. The first adds in the pair (a+2, b+3). The sum is a+2+b+3 = a+b + 5. By induction a+b is divisible by 5 and therefore, a+b+5 must also be divisible by 5. Similarly for the second inductive rule.

Question F) Rosen 5.3 Exercise 32 (p. 359)
(For part a, look at example 7 on page 350. For part b, look at example 12 on page 355.)

Recursive definition of ones:

ones(λ) = 0;

ones(wx)=ones(w)+1 if $w \in \{0,1\}^*$ and $x \in \{1\}$.

ones(wx)=ones(w) if $w \in \{0,1\}^*$ and $x \in \{0\}$.

Use structural induction to prove that ones(xy) = ones(x) + ones(y), where x and y belong to $\{0,1\}^*$ the set of bit strings.

Recursive definition of bit strings:

The set $\{0,1\}^*$ of *strings* over the alphabet $\{0,1\}$ is defined recursively by

BASIS STEP: $\lambda \in \{0,1\}^*$ (where λ is the empty string containing no symbols).

RECURSIVE STEP: If $w \in \{0,1\}^*$ and $x \in \{0,1\}$,then $wx \in \{0,1\}^*$.

Let P (y) be the statement that ones(xy) = ones(x) + ones(y) whenever x belongs to $\{0,1\}^*$.

BASIS STEP: To complete the basis step, we must show that P (λ) is true. That is, we must show that ones(xλ) = ones(x) + ones(λ) for all x $\in$ $\{0,1\}^*$. Because ones(xλ) = ones(x) = ones(x) + 0 = ones(x) + ones(λ) for every string x , it follows that P (λ) is true.

RECURSIVE STEP: To complete the inductive step, we assume that P(y) is true and show that this implies that P (ya) is true whenever a $\in$ {0,1} . What we need to show is that ones(xya) = ones(x) + ones(ya) for every a $\in$ {0,1}.

CASE: a=0:

To show this, note that by the recursive definition of ones(*w*), we have ones(xy0) = ones(xy) + 0 and ones(y0) = ones(y) + 0. And, by the inductive hypothesis, ones(xy) = ones(x) + ones(y).

CASE: a=1:

To show this, note that by the recursive definition of ones(*w*), we have ones(xy1) = ones(xy) + 1 and ones(y1) = ones(y) + 1. And, by the inductive hypothesis, ones(xy) = ones(x) + ones(y).

We conclude that in both cases ones(xya) = ones(x) + ones(y) = ones(x) + ones(ya).

You may choose to solve one (and only one) of the following Extra Credit Problems. If you submit more than one, only the first will be graded.

Extra Credit 1) Rosen 5.2 Exercise 16 (p. 342)
Prove that the first player has a winning strategy for the game of Chomp, introduced in Example 12 in Section 1.8, if the initial board is two squares wide, that is, a 2 × n board. [*Hint:* Use strong induction. The first move of the first player should be to chomp the cookie in the bottom row at the far right.]

P(n) is "Player 1 has a winning strategy for chomp with a 2 x n board."

P(1) – Board is as shown. Player one eats S and everything below and to the right, leaving just the poison cookie for Player two.

P
S

IH: Player 1 has a winning strategy for chomp with a 2 x j board. Where j is goes from 1 to k.

Show: Player 1 has a winning strategy for chomp with a 2 x (k+1) board.

P	S	S	…	S
S	S	S	…	S

Player 1 eats the cookie in the lower left corner.

If Player 2 eats any cookie in the top row, they must eat every cookie below and to the right and we are left with a board that is 2xj and the IH applies so player 1 has a winning strategy.

So the only possible choice for Player 2 is to eat a cookie in the bottom row. In this case, Player 1, should eat the cookie in the top row just to the right of the cookie that Player 2 ate. We are again in a situation where the only choice is for player two to eat a cookie in the bottom row. Eventually, player 2 would eat the last cookie in the bottom row and there would still be at least two cookies in the top row. Player one eats all but the poison cookie and Player 2 looses.

Extra Credit 2) Rosen 5.2 Exercise 11 (p. 342)
Consider this variation of the game of Nim. The game begins with n matches. Two players take turns removing matches, one, two, or three at a time. The player removing the last match loses. Use strong induction to show that if each player plays the best strategy possible, the first player wins if n=4j, 4j+2,or 4j+3for some nonnegative integer j and the second player wins in the remaining case when n = 4j + 1 for some nonnegative integer j.

Basis:

n=1: The first player looses because they must take the remaining match (second player wins)

n=2: The first player wins because they take 1 match and the second player must take the remaining match.

n=3: The first player wins because they take 2 matches and the second player must take the remaining match.

n=4: The first player wins because they take 3 matches and the second player must take the remaining match.

Inductive Hypotheses:

For all values of j from 1 to k, with optimal play the first player loses if the remainder mod 4 is 1, otherwise the first player wins.

Inductive Step:

For k+1 matches, show that the first player looses if the remainder of k mod 4 is 1, otherwise the first player wins.

Case: k+1 mod 4 = 1.

Sub case: Player 1 takes 1 match. This leaves us with k matches where k mod 4 = 0. Player 2 is the first player now and by IH wins with optimal play.

Sub case: Player 1 takes 2 matches. This leaves us with k-1 matches where k-1 mod 4 = 3. Player 2 is the first player now and by IH wins with optimal play.

Sub case: Player 1 takes 3 matches. This leaves us with k-2 matches where k-2 mod 4 = 2. Player 2 is the first player now and by IH wins with optimal play.

No matter what player 1 does, player 2 can always force a win.

Case: k+1 mod 4 = 0.

Sub case: Player 1 takes 1 match. This leaves us with k matches where k mod 4 = 3. Player 2 is the first player now and by IH wins with optimal play.

Sub case: Player 1 takes 2 matches. This leaves us with k-1 matches where k-1 mod 4 = 2. Player 2 is the first player now and by IH wins with optimal play.

Sub case: Player 1 takes 3 matches. This leaves us with k-2 matches where k-2 mod 4 = 1. Player 2 is the first player now and by IH looses with optimal play.

Player 1 has an optimal strategy. They must take 3 matches.

Case: k+1 mod 4 = 2.

Sub case: Player 1 takes 1 match. This leaves us with k matches where k mod 4 = 1. Player 2 is the first player now and by IH looses with optimal play.

Sub case: Player 1 takes 2 matches. This leaves us with k-1 matches where k-1 mod 4 = 0. Player 2 is the first player now and by IH wins with optimal play.

Sub case: Player 1 takes 3 matches. This leaves us with k-2 matches where k-2 mod 4 = 3. Player 2 is the first player now and by IH looses with optimal play.

Player 1 has an optimal strategy. They must take 1 match.

Case: k+1 mod 4 = 3.

Sub case: Player 1 takes 1 match. This leaves us with k matches where k mod 4 = 2. Player 2 is the first player now and by IH wins with optimal play.

Sub case: Player 1 takes 2 matches. This leaves us with k-1 matches where k-1 mod 4 = 1. Player 2 is the first player now and by IH looses with optimal play.

Sub case: Player 1 takes 3 matches. This leaves us with k-2 matches where k-2 mod 4 = 0. Player 2 is the first player now and by IH wins with optimal play.

Player 1 has an optimal strategy. They must take 2 matches.