Introduction to Discrete Mathematics
Version 1.3 - January 18, 2017

Introduction

A thought experiment….

Suppose we have a checkerboard and we put 1 grain of rice on the first square, 2 grains of rice on the second square, 3 grains of rice on the third square, etc. How many grains of rice do we need?

1 + 2 + 3 + 4 + …. + 61 + 62 + 63 + 64 = 2080.

How can we figure this out?
Pair the first and last and add: 1+64 = 65
Pair the second and second to last: 2+63 = 65
And keep going. All add up to the same value. There are 64 numbers, which results in 64/2=32 pairs. The sum will be 65 times 32.

Probably apocryphal story. One time in grade school, Carl Friedrich Gauss was causing a distraction. His teacher assigned Carl the task of adding up the integers from 1 to 1000, thinking that this would keep him occupied for a while. Carl came back within a few minutes with answer which he obtained using the method we outlined above.

One grain of rice is about 1/64 of a gram so we need about 32 grams of rice (0.07 pounds).

Suppose that we put 1 grain of rice on the first square, 2 on the second, 4 on the third, 8 on the fourth square, etc. How many grains of rice do we need?

$$1+2+4+8+\ldots+2^{60}+2^{61}+2^{62}+2^{63}=18446744073709551615$$

How can we figure this out? Pairing doesn't work. We need another trick.

$$2S = \quad 2+4+8+\ldots+2^{60}+2^{61}+2^{62}+2^{63}+2^{64}$$
$$S = 1+2+4+8+\ldots+2^{60}+2^{61}+2^{62}+2^{63}$$

Subtract and we get
$$S = \ -1 \ +2^{64}$$

Similarly this is about 288230376151711744 grams. (635439207568045 pounds) (317719603784 tons) (12 million average size cargo ships worth)

Sequences and summation notation

Sequence – a finite or infinite list of numbers a_i indexed by an integer value i usually starting at 0 or 1 (Rosen page 156)

Example:
$a_i = 2i + 1$

is the sequence 1, 3, 5, 7, …. When I starts at 1.

Summation notation – a shorthand way to express sums that is often easier to work with $\sum_{i=lower}^{upper} a_i$ - i is the index variable, lower is the smallest value that i will assume and upper is the largest value that i will take on, a_i is some function of i. (Rosen pp163-166)

Examples:

$$\sum_{i=l}^{64} i = 1 + 2 + \ldots + 64$$

$$\sum_{i=l}^{64} 3i = 3 + 6 + 9 \ldots + 192$$

$$\sum_{i=l}^{64} 2^{i-1} = 1 + 2 + 4 \ldots + 2^{63}$$

$$\sum_{i=0}^{63} 2^{i} = 1 + 2 + 4 \ldots + 2^{63}$$

$$\sum_{i=-2}^{2} \left(i^2 + 1\right) = \left(\left(-2\right)^2 + 1\right) + \left(\left(-1\right)^2 + 1\right) + \left(0^2 + 1\right) + \left(1^2 + 1\right) + \left(2^2 + 1\right)$$

$$\sum_{i=3}^{5} \left(ai - 1\right)^2 = \left(3a - 1\right)^2 + \left(4a - 1\right)^2 + \left(5a - 1\right)^2$$

Arithmetic sequence – a sequence where each term is the previous term plus a constant. (Rosen p157)

$$a_i = a_0 + ki$$

Example

$$a_i = 10 + 2i$$

$$a_0 = 10,\ a_1 = 12,\ a_2 = 14,\ a_3 = 16, \ldots$$

Geometric sequence – a sequence where each term is the previous term times a constant. (Rosen p157)

$$a_i = a_0 r^i$$

Example

$$a_i = 3 \cdot 2^i$$

$$a_0 = 3 \cdot 2^0 = 3,\ a_1 = 3 \cdot 2^1 = 6,\ a_2 = 3 \cdot 2^2 = 12,\ a_3 = 3 \cdot 2^3 = 24, \ldots$$

Manipulations of summations

Pulling off a term (we can do this from the front or back)

$$\sum_{i=l}^{u} a_i = a_l + \sum_{i=l+1}^{u} a_i$$

$$\sum_{i=l}^{u} a_i = a_u + \sum_{i=l}^{u-1} a_i$$

Multiplicative constants... Pulling in/out a constant

$$k\sum_{i=l}^{u} a_i = \sum_{i=l}^{u} k a_i$$

Separating a sum

$$\sum_{i=l}^{u} (a_i + b_i) = \sum_{i=l}^{u} a_i + \sum_{i=l}^{u} b_i$$

Examples

$$\sum_{i=0}^{10} (3i + 2) = \sum_{i=0}^{10} 3i + \sum_{i=0}^{10} 2 = 3\sum_{i=0}^{10} i + 2\sum_{i=0}^{10} 1 = 3 \cdot 55 + 2 \cdot 11 = 187$$

$$\sum_{i=0}^{10} 1 = 11$$

$$\sum_{i=0}^{10} i = (0 + 10)\left(\frac{11}{2}\right) = 55$$

Linear shift of index variable. We introduce a new index variable j that is offset from i by some integer amount j = i+k (use in the limits) or i =j-k (use in the formula) Question: Why can't we do a change like j=2i?

$$\sum_{i=l}^{u} a_i = \sum_{j=l+k}^{u+k} a_{j-k}$$

Example:

$$\sum_{i=10}^{100}(2i+1) = \sum_{j=1}^{91}(2(j+9)+1) \quad \text{where} \quad j = i-9 \quad \text{and} \quad i = j+9$$

$$= \sum_{j=1}^{91}(2j+19)$$

$$= \sum_{j=1}^{91} 2j + \sum_{j=1}^{91} 19$$

$$= 2\sum_{j=1}^{91} j + 19\sum_{j=1}^{91} 1$$

$$= 2\left(\frac{(91+1)91}{2}\right) + 19(91)$$

$$= (92)91 + 19(91)$$

$$= (111)91$$

Alternate solution using our trick

$$\sum_{i=10}^{100}(2i+1) = 21+23+25+\ldots+197+199+201$$

we pair them up and get 21+201= 222

The number of values is 100-10+1 = 91, so the number of pairs is 91/2

The product is $(222)\frac{91}{2} = (111)91$ which is the same!

Example:

$$\sum_{i=5}^{20} 2^i = \sum_{j=1}^{16} 2^{j+4} \quad \text{where} \quad j = i-4 \quad \text{and} \quad i = j+4$$

$$= \sum_{j=1}^{16} 2^4 2^j$$

$$= 2^4 \sum_{j=1}^{16} 2^j$$
$$= 2^4 \left(2^{17} - 2\right)$$
$$= 2^5 \left(2^{16} - 1\right)$$

Alternate solution where we pull off the first four terms of a larger summation

$$\sum_{i=1}^{20} 2^i = \sum_{i=1}^{4} 2^i + \sum_{i=5}^{20} 2^i$$ which can be rearranged to give

$$\sum_{i=5}^{20} 2^i = \sum_{i=1}^{20} 2^i - \sum_{i=1}^{4} 2^i$$

$$= \left(2^{21} - 2\right) - \left(2^5 - 2\right)$$
$$= \left(2^{21} - 2^5\right)$$
$$= 2^5 \left(2^{16} - 1\right)$$

Closed form expressions for some common summations. You are required to memorize the highlighted formulas for the exam.

$$\sum_{i=1}^{n} 1 = n$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$ (Arithmetic)

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

$$\sum_{i=0}^{n} r^i = \frac{\left(r^{n+1} - 1\right)}{r - 1}$$ (Geometric)

$$\sum_{i=1}^{n} \frac{1}{i} = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \ldots + \tfrac{1}{n} \approx \ln(n) + 0.577$$ (Harmonic)

Nested summations.
You can treat the other variable as a constant factor and pull it out of an inner summation

$$\sum_{i=1}^{n}\sum_{j=1}^{m}(3i+j) = \sum_{i=1}^{n}\left(\sum_{j=1}^{m}(3i)+\sum_{j=1}^{m}(j)\right) = \sum_{i=1}^{n}\sum_{j=1}^{m}(3i) + \sum_{i=1}^{n}\sum_{j=1}^{m}(j)$$

$$= \sum_{i=1}^{n}\left(3i\sum_{j=1}^{m}1\right) + \sum_{i=1}^{n}\sum_{j=1}^{m}(j)$$

$$= \sum_{i=1}^{n}(3im) + \sum_{i=1}^{n}\frac{(m+1)m}{2}$$

$$= 3m\sum_{i=1}^{n}i + \frac{(m+1)m}{2}\sum_{i=1}^{n}1$$

$$= 3m\frac{(n+1)n}{2} + \frac{(m+1)m}{2}n$$

$$= 3m\frac{(n+1)n}{2} + \frac{(m+1)m}{2}n$$

$\sum_{i=1}^{n}\sum_{j=1}^{m} ij = \sum_{i=1}^{n}\left(i\sum_{j=1}^{m} j\right)$ note that this expression is not the same as $\left(\sum_{i=1}^{n} i\right)\left(\sum_{j=1}^{m} j\right)$

$$= \sum_{i=1}^{n}\left(i\frac{(m+1)m}{2}\right)$$

$$= \frac{(m+1)m}{2}\sum_{i=1}^{n}(i)$$

$$= \frac{(m+1)m}{2}\frac{(n+1)n}{2}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{i} ij = \sum_{i=1}^{n}\left(i\sum_{j=1}^{i} j\right)$$

$$= \sum_{i=1}^{n}\left(i\frac{(i+1)i}{2}\right)$$

$$= \sum_{i=1}^{n}\frac{\left(i^3+i^2\right)}{2}$$

$$= \frac{1}{2}\sum_{i=1}^{n}\left(i^3+i^2\right)$$

$$= \frac{1}{2}\sum_{i=1}^{n}i^3 + \frac{1}{2}\sum_{i=1}^{n}i^2$$

Example: The twelve days of Christmas came and went. Gifts were given! But how many total? The following expression tells us the answer.

$$Gifts = \sum_{i=1}^{1} i + \sum_{i=1}^{2} i + \sum_{i=1}^{3} i + \sum_{i=1}^{4} i + \sum_{i=1}^{5} i + \sum_{i=1}^{6} i + \ldots + \sum_{i=1}^{12} i$$

But we can combine these into a double summation

$$Gifts = \sum_{k=1}^{12} \sum_{i=1}^{k} i$$

Now we can use the common summations to get the value. The inner sum can be replaced by

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$$

Giving

$$Gifts = \sum_{k=1}^{12} \frac{k(k+1)}{2}$$

Multiply out the formula and then separate the sum

$$Gifts = \sum_{k=1}^{12} \frac{k^2+k}{2}$$

$$= \sum_{k=1}^{12} \frac{k^2}{2} + \sum_{k=1}^{12} \frac{k}{2}$$

Pull out the constant factor

$$= \frac{1}{2}\sum_{k=1}^{12} k^2 + \frac{1}{2}\sum_{k=1}^{12} k$$

and then apply the common summations

Given $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

We see that

$$\sum_{k=1}^{12} k = \frac{12(13)}{2} = 6(13) = 78$$

$$\sum_{k=1}^{12} k^2 = \frac{12(13)(25)}{6} = 2(13)(25) = 650$$

So we have

$$Gifts = \frac{1}{2}78 + \frac{1}{2}650 = 364$$

Some special function and notation

We will brush up on some basic notation.

Set – a collection of items
Examples:
{1, 3, 9} – a set containing 3 items.
{3, 1, 9} – the same set

{2, 4, 6, 8, …} the set of even positive integers (infinite)
{0, 1, 2, 3, …} the set of non-negative integers (infinite)

p 116 Rosen 7th ed
$\boldsymbol{N} = \{0, 1, 2, 3, \ldots\}$ the set of natural numbers
$\boldsymbol{Z} = \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$ the set of integers
$\boldsymbol{Z}^+ = \{1, 2, 3, \ldots\}$ the set of positive integers
$\boldsymbol{Q} = \{n/m | n \text{ and } m \text{ are integers}\}$ the set rational numbers
$\boldsymbol{R}$ the set of real numbers

Function – A mapping from one set of values (***domain***) to another set of values (***codomain***)

$f : A \to B$ is a function mapping values from set A into set B. Note that every value in A must be mapped into a value in B. Not every value in B must participate in a mapping. The set of values from B that are mapped into is known as the ***range*** of the function.

Example:

$$f : N \to R$$

$$f(x) = x + 1$$

is a function from the natural numbers to the real numbers. The range is the positive integers.

Polynomial function – a function like kx^n where k is a constant and n is a non-negative integer. If we have two polynomial functions and add or multiply them together, the result is a polynomial function as well.

Examples:

$$f(x) = 2$$

$$f(x) = 3x^2$$

$$f(x) = 3x^5 - 10x^3 + 3.4$$

Exponential function – a function like kb^x where k is a constant and b is positive and not 1. (If b is 1, we get a very boring constant function

Examples:

$$f(x) = 2^x$$

$$f(x) = 2 \cdot 3^x$$

Some rules for dealing with exponents:

$$b^0 = 1$$

$$b^{-x} = \frac{1}{b^x}$$

$$b^{x+y} = b^x b^y$$

$$b^{x-y} = \frac{b^x}{b^y}$$

$$a^x b^x = (ab)^x$$

$$b^{xy} = \left(b^x\right)^y = \left(b^y\right)^x$$

$$b^{\frac{1}{n}} = \sqrt[n]{b}$$

Note $ab^x \neq (ab)^x$

Log function – A function like $k \log_b x$ where k is a constant and b is positive and not one.

By definition, the log function is the inverse of exponentiation. This means that the following two relations must be true. It also means that the argument of a log function must be greater than zero

$$b^{\log_b x} = x$$

$$\log_b b^x = x$$

Examples:

$$f(x) = 2 \ln x \equiv 2 \log_e x$$

$$f(x) = 5 \lg x \equiv 2 \log_2 x$$

$$f(x) = 7 \log x \equiv 2 \log_{10} x$$

$$f(x) = 8 \log_7 x$$

Some rules for dealing with logarithms:

$$\log_b(x) + \log_b(y) = \log_b(xy)$$
$$\log_b(x) - \log_b(y) = \log_b(x/y)$$
$$\log_b(x^y) = y\log_b(x)$$
$$\log_b(x) = \frac{\log_a(x)}{\log_a(b)}$$

The last formula has the important implication that every log function is related via a constant factor.

Examples:

$$\log_3(x) + \log_3(2) = \log_3(2x)$$
$$2\log(x) = \log(x^2)$$
$$\log_2(1000) = \frac{\log(1000)}{\log(2)} = \frac{\log(10^3)}{\log(2)} = \frac{3}{\log(2)} \approx \frac{3}{0.301029995663981} \approx 9.965784284662087$$
$$\log_5 5^{3+2\pi} = 3 + 2\pi$$

Example:

$$\log_3(3^{\sin x}) = \sin x$$
$$3^{\log_3 \sin x} = \sin x \qquad \text{provided sin x is } \textit{positive}$$

Floor/Ceiling

$$floor : R \to Z$$
$$ceiling : R \to Z$$

$$\lfloor x \rfloor \equiv floor(x)$$
$$\lceil x \rceil \equiv ceiling(x)$$

The floor is defined as the largest integer value that is less than or equal to x. Similarly, the ceiling is the smallest integer value that is greater than or equal to x.

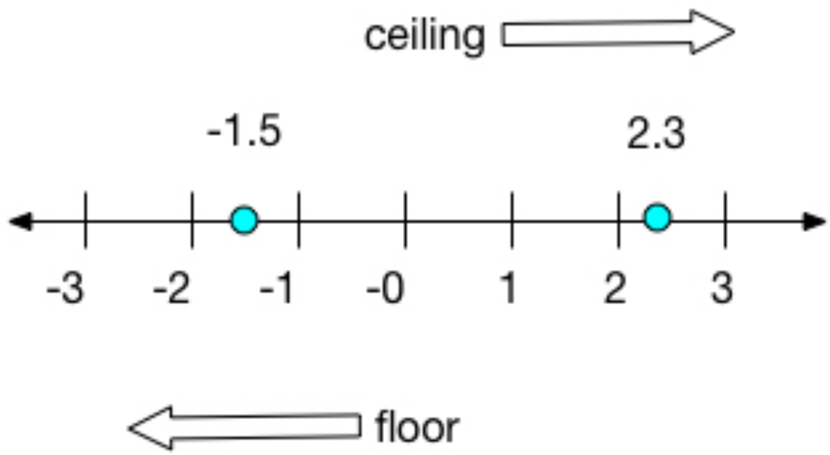
Examples:

$\lfloor 10 \rfloor = 10$ $\quad$ $\lceil 10 \rceil = 10$

$\lfloor 2.3 \rfloor = 2$ $\quad$ $\lceil 2.3 \rceil = 3$

$\lfloor -2 \rfloor = -2$ $\quad$ $\lceil -2 \rceil = -2$

$\lfloor -1.5 \rfloor = -2$ $\quad$ $\lceil -1.5 \rceil = -1$



Factorial

$$factorial : N \to N$$

Written as n!, factorial is defined as the product of all the integers from 1 to n. This definition is somewhat problematic for the values of 0 and 1. (What is the product of all the integers from 1 to 0?) For now, we will take it on faith that 0! And 1! both have the value of 1.

$$0! \equiv 1$$
$$1! \equiv 1$$
$$2! = 1 \times 2 = 2$$
$$3! = 1 \times 2 \times 3 = 6$$
$$4! = 1 \times 2 \times 3 \times 4 = 24$$
$$5! = 1 \times 2 \times 3 \times 4 \times 4 = 120$$

Order of Growth (Asymptotic analysis)

We are going to take a much closer look at this later in the course, but for now we want to get an intuitive idea of what this means and see how to use it in practice. Essentially, what we want to do is to characterize a function f(x) down to its important parts especially when the value of x is large.

What we don't care about

1) Constant factors

2) Added terms that are "small"

3) What the function does for small values of x.

Example:

$f(x) = 10x^2$

As far as we are concerned the 10 is unimportant

$g(x) = 10x^2 + 5$

As far as we are concerned the 5 is unimportant

$h(x) = 3x^2 + 1/x$

As far as we are concerned the 1/x is unimportant

From the point of view of orders of growth all of these functions behave similar to x^2

We will indicate this by using Big-Theta notation

$f(x) = 10x^2 = \Theta(x^2)$

(The use of the = sign here is wrong, but it is such a common abuse of the notation that we use it. Later in the course we will see what this notation really means.)

Big-O, Big-Theta, Big-Omega

- If we write, $g(x) = O(x^2)$ we mean that g(x) is upper bounded by the indicated function
- If we write, $g(x) = \Omega(x^2)$ we mean that g(x) is lower bounded by the indicated function
- If we write, $g(x) = \Theta(x^2)$ we mean that g(x) is both upper and lower bounded by the indicated function

You have to be careful. Some authors are sloppy and will use Big-O and Big-Theta interchangeably.

Example:

Because x squared is a tight bound for f(x) it will be both upper and lower bounded

$$f(x) = 10x^2 = O(x^2) \text{ and } f(x) = 10x^2 = \Omega(x^2)$$

But f(x) it is also upper bounded by x cubed, so

$$f(x) = 10x^2 = O(x^3)$$

Similarly, f(x) lower bounded by x, so

$$f(x) = 10x^2 = \Omega(x)$$

Typical orders of growth (from slow to fast)

Constant - $O(1)$
Double Log - $O(\log\log x)$
Log - $O(\log x)$
Linear - $O(x)$
LogLinear - $O(x \log x)$
Quadratic - $O(x^2)$
Cubic - $O(x^3)$
Polynomial - $O(x^k)$
Exponential - $O(b^x)$
Factorial - $O(x!)$

Example:

Think about the rice problem… When we used the sequence 1, 2, 3, …n the sum is

$$\frac{(n+1)n}{2} = \Theta(n^2)$$

Note that we can rewrite the function as

$$\frac{n^2 + n}{2}$$

Ignore the smaller term that is added in and ignore the constant factor.

Example:

When we used the sequence 1, 2, 4, … 2^n the sum is

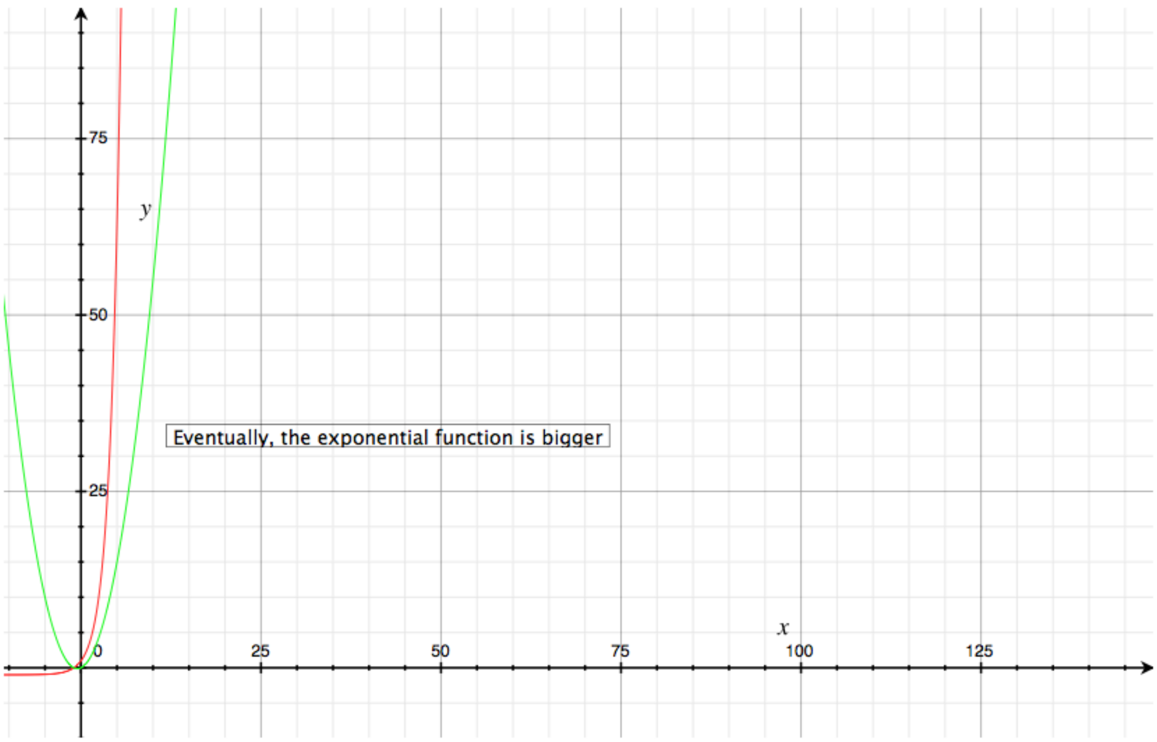
$$2^{n+1} - 1 = \Theta\left(2^{n}\right)$$

Note that we can rewrite the function as

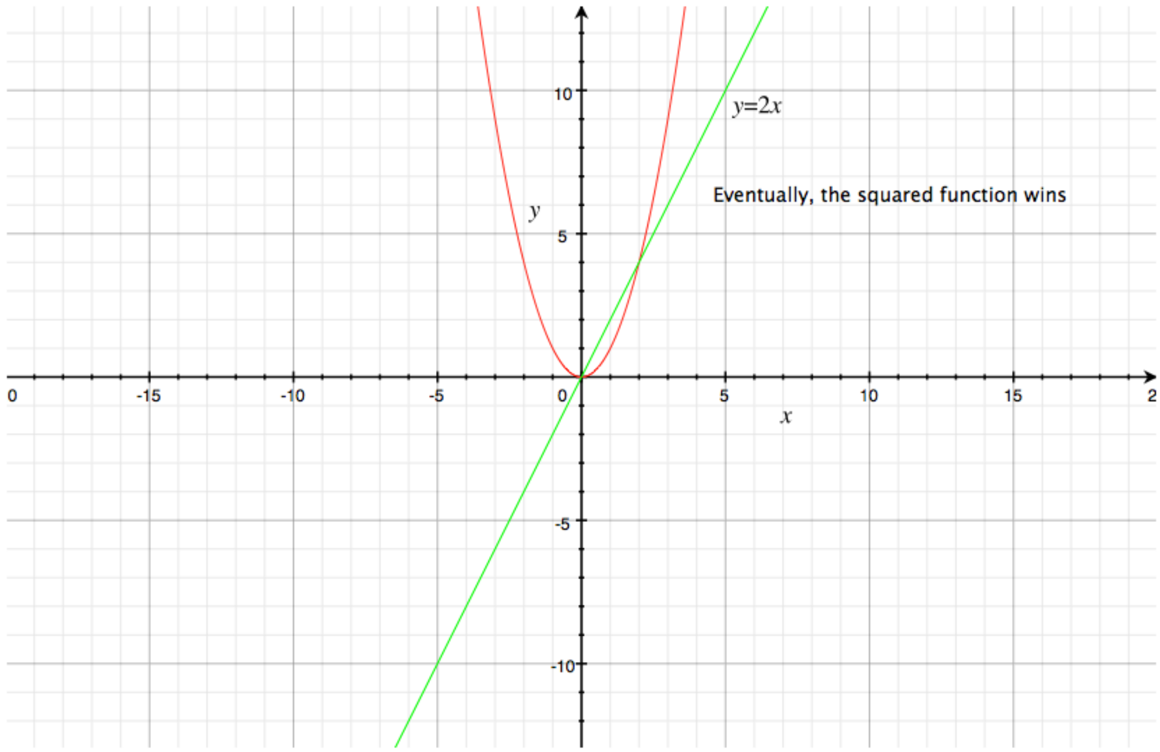
$$2 \times 2^{n} - 1$$

Ignore the smaller term that is subtracted and ignore the constant factor.

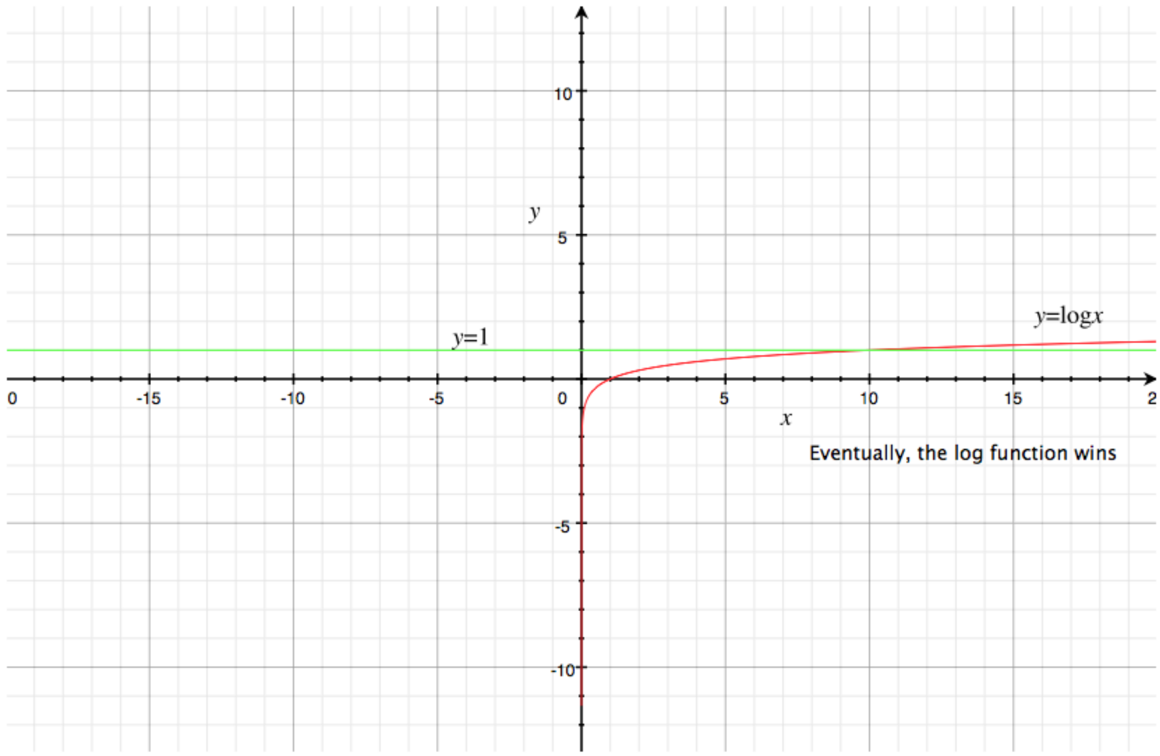
We can plot these two functions



And a couple more to look at



y=2x
Eventually, the squared function wins
y
x
10
5
-5
-10
0
-15
-10
-5
0
5
10
15
2



Using Big-Theta to make predictions.

Suppose that we have a function where we know

$$f(x) = \Theta\left(x^2\right)$$

We are going to say that f is some constant of proportionality times our growth plus smaller terms

$$f(x) = kx^2 + smaller$$

Suppose that we also know the value of the function f, for a given value of x. We will use these values to solve for k. Then we can make predictions about the value of f.

Example 1:

$$f(x) = x^3 + 3x^2 = \Theta\left(x^3\right)$$

Given that f(10) = 1300, approximate the value of f(20).

$$f(x) \approx kx^3$$

Given f(10) = 1300 we know that

$$1300 \approx k(10)^3$$

Solving for k we get

$$k \approx \frac{1300}{10^3}$$

Which means

$$f(x) \approx \frac{1300}{10^3} x^3$$

Now we use this to approximate f(20)

$$f(20) \approx \frac{1300}{10^3} 20^3 = \frac{20^3}{10^3} 1300 = 8 \times 1300 = 10400$$

We can compare this with the true value

$$f(20) = 20^3 + 3(20)^2 = 8000 + 1200 = 9200$$

is 13% off.

Example 2:

Suppose that we have are playing a game where I am thinking of an integer between 1 and n. Your goal is to determine the number. In every unit of time, you are allowed to ask a single Yes/No question about the value. Use Big-Theta notations to express the number of questions you will need in best case, average and worst case.

Process 1: Is the value 1? Is the value 2? Is the value 3? ….

Best case: 1 question

$$\Theta(1)$$

Average case: n/2 questions

$$\Theta(n)$$

Worst case: n questions

$$\Theta(n)$$

Process 2: Is the value greater than n/2? Is the value greater than n/4?... (Each time we reduce the possibilities by a factor of 2.)

Best case: ceiling(lgn) approximately

$$\Theta(\lg n)$$

Average case: Same

Worst case: Same

Follow On question 1:

Suppose that using process 1, on average it takes me 30 minutes to find the value when n=2000. If I can ask the questions twice as fast, how big a problem can I solve now?

First we express the time as a function of n.

$$T(n) = \Theta(n)$$

or

$$T(n) = kn + \text{terms to be ignored}$$

We are given that n=2000 and T(2000)=30… Plug these values in, and solve for k.

$$30 = k \cdot 2000$$

so

$$k = \frac{30}{2000} = \frac{3}{200}$$

and

$$T(n) \approx \frac{3}{200} n$$

When I am allowed to ask the questions twice as fast, we will have a new expression

$$T'(n) \approx k'n$$

where

$$k' = \frac{1}{2} k$$

(Twice as fast would mean half the time.)

Our new formula is

$$T'(n) \approx \frac{1}{2} \cdot \frac{3}{200} n$$

We know the time (30 min) and want to solve for n.

$$30 \approx \frac{1}{2} \cdot \frac{3}{200} n$$

so

$$30 \approx \frac{1}{2} \cdot \frac{3}{200} n = 30 \cdot \frac{2}{1} \cdot \frac{200}{3}$$

$$n = 4000$$

Grand conclusion… if we can ask the questions twice as fast, we can solve a problem that is twice as big.

Follow On question 2:

Suppose that using process 2, on average it takes me 30 minutes to find the value when n=2000. If I can ask the questions twice as fast, how big a problem can I solve now?

First we express the time as a function of n.

$$T(n) = \Theta(\lg n)$$

or

$$T(n) = k \lg n + \text{terms to be ignored}$$

We are given that n=2000 and T(2000)=30… Plug these values in, and solve for k.

$$30 = k \cdot \lg 2000$$

so

$$k = \frac{30}{\lg 2000}$$

and

$$T(n) \approx \frac{30}{\lg 2000} n$$

When I am allowed to ask the questions twice as fast, we will have a new expression

$$T'(n) \approx k' \lg n$$

where

$$k' = \frac{1}{2} k$$

(Twice as fast would mean half the time.)

Our new formula is

$$T'(n) \approx \frac{1}{2} \cdot \frac{30}{\lg 2000} \lg n$$

We know the time (30 min) and want to solve for n.

$$30 \approx \frac{1}{2} \cdot \frac{30}{\lg 2000} \lg n$$

so

$$\lg n = 30 \cdot \frac{2}{1} \cdot \frac{\lg 2000}{30}$$

$$\lg n = 2 \cdot \lg 2000$$

To solve, we will use the inverse of lg

$$2^{\lg n} = 2^{2 \cdot \lg 2000}$$

The left hand side is just n and we can evaluate the right side. lg(1000) is very close to 10, so lg(2000) = lg(2) + lg(1000) is very close to 11.

$$n \approx 2^{2 \cdot 11} \approx 4000000$$

Grand conclusion… if we can ask the questions twice as fast, we can solve a problem that is approximately 2000 times as big.

Example: Suppose we are going to visit n cities. How many different orders are there that we can use to visit all the cities exactly once? Express your answer in Big-Theta notation.

We have n choices for the first, n-1 choices for the second, n-2 choices for the third, etc. We multiply these all together and we get

$$Orders = \Theta(n!)$$