Fleck Chapter 2 (Proposition Logic)

Problem 1 Which of the following is a proposition? If it is a proposition, what is the truth value?

a) "Eat less meat" - Not a statement, no truth value, not a proposition.
b) "Where is the store?" – Not a statement, no truth value, not a proposition.
c) "The moon is bigger than the earth." - This is a proposition. It is false.
d) "The moon is closer to the earth than the sun." This is a proposition. It is true.
e) "x + x = 2x" This is not a proposition, since we do not allow variables in a proposition.
f) "1+1=10" - This is a proposition. The truth value is problematic… A truth value depends on the interpretation of the symbols in the statement. If we interpret the numbers in binary, this is in fact true.

Note: Consider the following "✖★✖❖✖⭘". Is this a proposition? Does it have a truth value? Our first inclination is to say it isn't, but it has exactly the same pattern as the proposition in part f. Unless, I otherwise specified, you can assume a standard interpretation of the symbols.

Problem 2 How many lines are required for the following propositional formulas in a truth table?

a) $(p \vee q) \rightarrow q$

b) $(p \vee q) \rightarrow (s \wedge t)$

c) $\neg(p \vee q) \leftrightarrow ((r \wedge F) \rightarrow T)$

a) This has two propositional variables, so it requires $2^2 = 4$ lines.
b) This has four propositional variables, so it requires $2^4 = 16$ lines.
c) This has three propositional variables, so it requires $2^3 = 8$ lines. (T and F are constants representing true and false respectively.)

Problem 3 Construct a truth table for the following propositional formula

$(p \wedge q) \rightarrow q$

p	q	$p \wedge q$	$(p \wedge q) \rightarrow q$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

Since this formula is true in all cases, it is a tautology.

Problem 4 Are the following two formulas logically equivalent?

$(p \rightarrow q) \rightarrow r$ and $r \vee (p \wedge \neg q)$

Since there are three propositional variables, we need 8 lines in our truth table.

p	q	r	$(p \rightarrow q)$	$(p \rightarrow q) \rightarrow r$	$\neg q$	$(p \wedge \neg q)$	$r \vee (p \wedge \neg q)$
T	T	T	T	T	F	F	T
T	T	F	T	F	F	F	F
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	F	F	T
F	T	F	T	F	F	F	F
F	F	T	T	T	T	F	T
F	F	F	T	F	T	F	F

Compare the truth values in each of the lines for the two formulas. Since they are all the same, the two formulas are logically equivalent.

Problem 5 Are the following two formulas logically equivalent?

$(p \vee q) \leftrightarrow (r \wedge p)$ and $(\neg p \vee \neg q) \rightarrow r$

Since there are three propositional variables, we need 8 lines in our truth table.

p	q	r	$(p \vee q)$	$(r \wedge p)$	$(p \vee q) \leftrightarrow (r \wedge p)$	$\neg p$	$\neg q$	$(\neg p \vee \neg q)$	$(\neg p \vee \neg q) \rightarrow r$
T	T	T	T	T	T	F	F	F	T
T	T	F	T	F	F	F	F	F	T
T	F	T	T	T	T	F	T	T	T
T	F	F	T	F	F	F	T	T	F
F	T	T	T	F	F	T	F	T	T
F	T	F	T	F	F	T	F	T	F
F	F	T	F	F	T	T	T	T	T
F	F	F	F	F	T	T	T	T	F

Compare the truth values in each of the lines for the two formulas. Since they differ in at least one line, these two formulas are not logically equivalent.

Problem 6 Apply logical equivalences so that the following formula has negations applied only to propositional variables with only the logical connectives AND and OR.

$$\neg\left(\left(\left(p \vee s\right) \rightarrow q\right) \leftrightarrow \left(r \wedge p\right)\right)$$

First, rewrite so that there are no double implications in the formula.

$$\neg\left(\left[\left(\left(p \vee s\right) \rightarrow q\right) \rightarrow \left(r \wedge p\right)\right] \wedge \left[\left(r \wedge p\right) \rightarrow \left(\left(p \vee s\right) \rightarrow q\right)\right]\right)$$

Second, rewrite so that there are no implications in the formula. (I am using a double underscore so it is easier to see where the application is. Remember that you negate the premise and turn the arrow into an or.)

$$\neg\left(\left[\left(\underline{\underline{\left(p \vee s\right)}} \rightarrow \underline{\underline{q}}\right) \rightarrow \left(r \wedge p\right)\right] \wedge \left[\left(r \wedge p\right) \rightarrow \left(\underline{\underline{\left(p \vee s\right)}} \rightarrow \underline{\underline{q}}\right)\right]\right)$$

$$\neg\left(\left[\underline{\underline{\left(\neg\left(p \vee s\right) \vee q\right)}} \rightarrow \underline{\underline{\left(r \wedge p\right)}}\right] \wedge \left[\underline{\underline{\left(r \wedge p\right)}} \rightarrow \underline{\underline{\left(\neg\left(p \vee s\right) \vee q\right)}}\right]\right)$$

$$\neg\left(\left[\neg\left(\neg\left(p \vee s\right) \vee q\right) \vee \left(r \wedge p\right)\right] \wedge \left[\neg\left(r \wedge p\right) \vee \left(\neg\left(p \vee s\right) \vee q\right)\right]\right)$$

Third, Use DeMorgan to push negations inside of any logical connection. I like to work from the outside in. Apply double negation as you go when needed.

$$\neg\left(\underline{\underline{\left[\neg\left(\neg\left(p \vee s\right) \vee q\right) \vee \left(r \wedge p\right)\right]}} \wedge \underline{\underline{\left[\neg\left(r \wedge p\right) \vee \left(\neg\left(p \vee s\right) \vee q\right)\right]}}\right)$$

$$\left(\neg\left[\underline{\underline{\neg\left(\neg\left(p \vee s\right) \vee q\right)}} \vee \underline{\underline{\left(r \wedge p\right)}}\right] \vee \neg\left[\neg\left(r \wedge p\right) \vee \left(\neg\left(p \vee s\right) \vee q\right)\right]\right)$$

$$\left(\left[\neg\neg\underline{\underline{\left(\neg\left(p \vee s\right) \vee q\right)}} \wedge \neg\left(r \wedge p\right)\right] \vee \neg\left[\neg\left(r \wedge p\right) \vee \left(\neg\left(p \vee s\right) \vee q\right)\right]\right)$$

$$\left(\left[\left(\neg\left(p \vee s\right) \vee q\right) \wedge \neg\left(r \wedge p\right)\right] \vee \neg\left[\underline{\underline{\neg\left(r \wedge p\right)}} \vee \underline{\underline{\left(\neg\left(p \vee s\right) \vee q\right)}}\right]\right)$$

$$\left(\left[\left(\neg\left(p \vee s\right) \vee q\right) \wedge \neg\left(r \wedge p\right)\right] \vee \left[\neg\neg\underline{\underline{\left(r \wedge p\right)}} \wedge \neg\left(\neg\left(p \vee s\right) \vee q\right)\right]\right)$$

$$\left(\left[\left(\neg\left(\underline{\underline{p}} \vee \underline{\underline{s}}\right) \vee q\right) \wedge \neg\left(\underline{\underline{r}} \wedge \underline{\underline{p}}\right)\right] \vee \left[\left(r \wedge p\right) \wedge \neg\left(\underline{\underline{\neg\left(p \vee s\right)}} \vee \underline{\underline{q}}\right)\right]\right)$$

$$\left(\left[\left(\left(\neg p \wedge \neg s\right) \vee q\right) \wedge \left(\neg r \vee \neg p\right)\right] \vee \left[\left(r \wedge p\right) \wedge \left(\neg\neg\underline{\underline{\left(p \vee s\right)}} \wedge \neg q\right)\right]\right)$$

$$\left(\left[\left(\left(\neg p \wedge \neg s\right) \vee q\right) \wedge \left(\neg r \vee \neg p\right)\right] \vee \left[\left(r \wedge p\right) \wedge \left(\left(p \vee s\right) \wedge \neg q\right)\right]\right)$$

Note: Formulas in propositional logic can always be converted into one of two kinds of normal forms: Conjuctive and Disjunctive. In both forms, we only use the logical operations AND, OR and NOT. NOT can only be applied to variables. The formulas are flat so that all the AND/OR operations are on the same level.

Conjunctive normal form (CNF): All the ANDs are at the top level and connect clauses containing ORs.
Example:

$$(p \vee r \vee \neg s) \wedge (\neg p \vee r) \wedge (q) \wedge (p \vee \neg r \vee \neg s \vee t)$$

Disjunctive normal form (DNF): All the ORs are at the top level and connect clauses containing ANDs.
Example:

$$(\neg p \wedge r) \vee (\neg p \wedge \neg r \wedge \neg s) \vee (r) \vee (q \wedge t \wedge r \wedge s)$$

Problem 7 Are the following formulas tautologies, contradictions, and satisfiable.

a) $(p \to q) \vee p$

b) $(p \to q) \wedge (q \to r) \wedge (r \to p)$

c) $(p \to q) \wedge (p \vee \neg q)$

Use a truth table.

p	q	$p \to q$	$(p \to q) \vee p$
T	T	T	T
T	F	F	T
F	T	T	T
F	F	T	T

All true so this is satisfiable and a tautology

p	q	r	$(p \rightarrow q)$	$(q \rightarrow r)$	$(r \rightarrow p)$	$(p \rightarrow q) \wedge (q \rightarrow r) \wedge (r \rightarrow p)$
T	T	T	T	T	T	T
T	T	F	T	F	T	F
T	F	T	F	T	T	F
T	F	F	F	T	T	F
F	T	T	T	T	F	F
F	T	F	T	F	T	F
F	F	T	T	T	F	F
F	F	F	T	T	T	T

At least one line is true so it is satisfiable. Some are false so it isn't a tautology.

p	q	$\neg q$	$(p \rightarrow q)$	$(p \wedge \neg q)$	$(p \rightarrow q) \wedge (p \wedge \neg q)$
T	T	F	T	F	F
T	F	T	F	T	F
F	T	F	T	F	F
F	F	T	F	F	F

All are false so this is not satisfiable and is a contradiction.