Building Blocks Homework 11

Solution

Question A) Use a proof by induction to show that $n^n > n!$ for all integers greater than 3.

P(n) is $n^n > n!$
n_0 is 4
Base: P(4) is $4^4 > 4!$ Or $256 > 24$ Check
Assume: P(k) is true: $k^k > k!$ k>3
Goal: Show P(k+1) is true: $(k+1)^{k+1} > (k+1)!$

Start with the induction hypothesis and multiply both sides by k+1 which is guaranteed to be positive.

$$(k+1)k^k > (k+1)k!$$

Or

$$(k+1)k^k > (k+1)!$$

We can achieve the goal if we prove that

$$(k+1)^{k+1} > (k+1)k^k$$

Dividing both sides by k+1 (safe since positive) we get

$$(k+1)^k > k^k$$

If we multiply out the LHS, all of the terms will be positive since k>3. The leading term will be k^k.

$$k^k + \textit{other positive terms} > k^k$$

$$\textit{other positive terms} > 0$$

Check
QED

Question B) Use strong induction to prove that a non-empty, connected, acyclic graph with n nodes will have n-1 edges. (No loops or multi edges allowed.)

Hint: Every such graph with 2 or more nodes has at least one cut edge that disconnects the graph into two non-empty subgraphs.

P(n) is "a non-empty, connected, acyclic graph with n nodes will have n-1 edges"
n_0 is 1 (non-empty)
Base: P(1) is a non-empty, connected, acyclic graph with 1 nodes will have 1-1 edges.
This is just a graph with a single node, so there are no edges. Check

Assume: P(1) … P(k) are true
Goal: Show P(k+1) is true: a non-empty, connected, acyclic graph with k+1 nodes will have k+1-1 edges

We know that our graph of size k+1 can be divided into two pieces by removing a cut edge. Lets say that the size of the first piece is R. We know that R is at least 1 (non-empty cut) and no more than k (both sides of cut are non empty.)
So we can apply IH P(R) and we know that the number of edges is R-1.
Similarly for the second piece we know that the number of nodes is k+1 – R. We know that its value is at least 1 and not more than k. So we can apply IH P(k+1-R) and the number of edges is k+1-R-1.
Total edges is (R-1) for first piece plus (k+1-R-1) for the second piece plus the cut edge. R - 1 + k + 1 – R – 1 + 1 = k. Check.
QED. (Note: This is basically the candy bar problem pattern.)

Question C) Prove that a graph as defined in problem B will be 2-colorable using a proof by strong induction.

P(n) is "a non-empty, connected, acyclic graph with n nodes is 2-colorable"
We will color using red and green.
n_0 is 1 (non-empty)
Base: P(1) is a non-empty, connected, acyclic graph with 1 is 2-colorable.
This is just a graph with a single node, so we color it red. Check.

Assume: P(1) … P(k) are true
Goal: Show P(k+1) is true: a non-empty, connected, acyclic graph with k+1 nodes is 2-colorable.

We know that our graph of size k+1 can be divided into two pieces by removing a cut edge. Lets say that the size of the first piece is R. We know that R is at least 1 (non-empty cut) and no more than k (both sides of cut are non empty.)
So we can apply IH P(R) and we know that the first piece is 2-colorable.

Similarly for the second piece we know that the number of nodes is k+1 – R. We know that its value is at least 1 and not more than k. So we can apply IH P(k+1-R) and second piece is 2-colorable. To 2-color the entire graph we just need to make sure that the nodes on either side of the cut edge are not the same color.
Case 1: They are red and green. Perfect!
Case 2: They are both red. Take the second piece and swap the colors. Perfect!
Case 3: They are both green. Same argument as case 2. Perfect!

QED. (Note: This is basically the candy bar problem pattern again.)

Question D) Prove that for a sufficiently large amount n, postage can always be composed of 3 and 7 cent stamps.

a) Find the smallest value where every value greater or equal can be composed of 3 and 7 cent stamps.

b) Use Strong induction to prove this.

n	postage
3	1 x 3
4	fail
5	fail
6	2 x 3
7	1x7
8	fail
9	3 x 3
10	1 x 3 and 1 x 7
11	fail
12	4 x 3
13	2 x 3 and 1 x 7
14	2 x 7

Conjecture: 12

(not a total guess, once I see 3 in a row that work, we should be able to get all the rest by adding in 3 cent stamps.)

P(n) is "postage of n cents can be composed of 3 and 7 cent stamps."

We will color using red and green.

n_0 is 12

Base:

P(12) is postage of 12 cents can be composed of 3 and 7 cent stamps.

P(13) is postage of 13 cents can be composed of 3 and 7 cent stamps.

P(14) is postage of 14 cents can be composed of 3 and 7 cent stamps.

Check on all (See table)

Assume: P(12), P(13), P(14) ... P(k) are true for k>=14

Goal: Show P(k+1) is true: postage of k+1 cents can be composed of 3 and 7 cent stamps.

If we can make postage for the value k+1-3 = k-2, then we can add a single 3 cent stamp to complete.

Because k>= 14, we know that k-2>=12. Therefore P(k-2) is in my list of know facts (Not less than 12 and not more than k.)

QED

Question E) Consider the following strong inductive proof.
Every student will get the same grade.

Formally: For every non-empty set of students, the students will all get the same grade.
Base: The smallest non-empty set has size 1. Obviously, every student in a set of size 1 will have the same grade.
Inductive: We assume that for sets of students of size 1, 2, 3, … k, that every student will get the same grade. We need to show that in a set of students of size k+1 all students will get the same grade. We are going to construct two subsets A and B. Pick some student and put them in both subsets. Split up the remaining students and distribute them between A and B. Since A is non-empty, we know by the induction hypotheses that every student in subset A will get the same grade. Similarly, by the induction hypotheses we know that every student in subset B will get the same grade. Since the two sets share a share a student, the grades must be the same in the larger set of students too. QED.

Clearly this proof is wrong. Identify the flaw in the proof.

Is it true for P(1)? Yes.
Is it true for P(2)? Maybe not.

So there is a flaw in going from P(1) -> P(2) when k=1.
Lets say that tom and bob are in my set. We construct two sets A and B and pick bob as the special student in both sets. When we go to split up the remaining students and distribute we get something like A={bob, tom} and B={bob} and set A is not covered by what I know which is just P(1).

Question F)

1) What are the first 5 terms of the following recursively defined sequence.
   $a_1 = 4$
   $a_n = a_{n-1} + 3$

2) Guess a solution for the sequence and then verify that it is correct.

$a_1 = 4$
$a_2 = a_{2-1} + 3 = a_1 + 3 = 4 + 3 = 7$
$a_3 = a_{3-1} + 3 = a_2 + 3 = 7 + 3 = 10$
$a_4 = a_{4-1} + 3 = a_3 + 3 = 10 + 3 = 13$
$a_5 = a_{5-1} + 3 = a_4 + 3 = 13 + 3 = 16$

Guess: $a_n = 3n + 1$

Base: Show $a_1 = 4$
$a_1 = 3(1) + 1 = 4$ Check!

Recurence: Show $a_n = a_{n-1} + 3$

By guess
$a_n = 3n + 1$
$a_{n-1} = 3(n-1) + 1$

And plug in to get:
$3n + 1 = 3(n-1) + 1 + 3$

Or
$3n + 1 = 3n - 3 + 1 + 3$
$3n + 1 = 3n + 1$
Check!

Question G)

1) What are the first 5 terms of the following recursively defined sequence.

$a_1 = 5$
$a_n = 3a_{n-1}$

2) Guess a solution for the sequence and then verify that it is correct.

$a_1 = 5$
$a_2 = 3a_{2-1} = 3a_1 = 3 \times 5 = 15$
$a_3 = 3a_{3-1} = 3a_2 = 3 \times 15 = 45$
$a_4 = 3a_{4-1} = 3a_3 = 3 \times 45 = 135$
$a_5 = 3a_{5-1} = 3a_4 = 3 \times 135 = 405$

Guess: $a_n = 5 \times 3^{n-1}$

Base: Show $a_1 = 5$
$a_1 = 5 \times 3^{1-1} = 5$ Check!

Recurence: Show $a_n = 3 \times a_{n-1}$

By guess
$a_n = 5 \times 3^{n-1}$
$a_{n-1} = 5 \times 3^{n-1-1}$

And plug in to get:
$5 \times 3^{n-1} = 3 \times 5 \times 3^{n-1-1}$

Or dividing by 5
$3^{n-1} = 3 \times 3^{n-1-1}$
$3^{n-1} = 3^1 \times 3^{n-1-1} = 3^{n-1}$
Check!

Question H)

1) Consider the following recursively defined function over the positive integers. Compute the value of the function for n=1, 2, 3, 4, and 5.

$$f(1) = 1$$
$$f(n) = f(n-1)\left(\frac{n-1}{n}\right)$$

2)Guess a solution for the function and then verify that it is correct.

$f(1) = 1$

$f(2) = f(2-1)\left(\frac{2-1}{2}\right) = 1\left(\frac{2-1}{2}\right) = \frac{1}{2}$

$f(3) = f(3-1)\left(\frac{3-1}{3}\right) = \frac{1}{2}\left(\frac{3-1}{3}\right) = \frac{1}{3}$

$f(4) = f(4-1)\left(\frac{4-1}{4}\right) = \frac{1}{3}\left(\frac{4-1}{4}\right) = \frac{1}{4}$

$f(5) = f(5-1)\left(\frac{5-1}{5}\right) = \frac{1}{4}\left(\frac{5-1}{5}\right) = \frac{1}{5}$

Guess: $f(n) = \frac{1}{n}$

Base: Show $\mathrm{f}(1) = 1$

$\mathrm{f}(1) = \frac{1}{1} = 1$ Check!

Recurence: Show $f(n) = f(n-1)\left(\frac{n-1}{n}\right)$

By guess

$f(n) = \frac{1}{n}$

$f(n-1) = \frac{1}{n-1}$

And plug in to get:

$$\frac{1}{n} = \frac{1}{n-1}\left(\frac{n-1}{n}\right)$$

Or

$$\frac{1}{n} = \frac{1}{n}$$

Check!

You may choose to solve one (and only one) of the following Extra Credit Problems. If you submit more than one, only the first will be graded.

Extra Credit 1)

Prove or disprove that $G_n \leq 3^n$ ($G\ is\ upper\ bounded$) when n is positive.

$G_1 = 1$

$G_n = 2G_{n-1} + 1$

$f(1) = 1$

$f(2) = f(2-1)\left(\frac{2-1}{2}\right) = 1\left(\frac{2-1}{2}\right) = \frac{1}{2}$

$f(3) = f(3-1)\left(\frac{3-1}{3}\right) = \frac{1}{2}\left(\frac{3-1}{3}\right) = \frac{1}{3}$

$f(4) = f(4-1)\left(\frac{4-1}{4}\right) = \frac{1}{3}\left(\frac{4-1}{4}\right) = \frac{1}{4}$

$f(5) = f(5-1)\left(\frac{5-1}{5}\right) = \frac{1}{4}\left(\frac{5-1}{5}\right) = \frac{1}{5}$

Guess: $f(n) = \frac{1}{n}$

Base: Show f(1) = 1

f(1) = $\frac{1}{1}$ = 1 Check!

Recurence: Show $f(n) = f(n-1)\left(\frac{n-1}{n}\right)$

By guess

$f(n) = \frac{1}{n}$

$f(n-1) = \frac{1}{n-1}$

And plug in to get:

$$\frac{1}{n} = \frac{1}{n-1}\left(\frac{n-1}{n}\right)$$

Or

$$\frac{1}{n} = \frac{1}{n}$$

Check!

Extra Credit 2) Consider the following recursive definition and python function that implements this definition.

$$g_1 = 1$$
$$g_n = g_{n-1} \times g_{n-1}$$

```
def g (n):
  if == 1:
       return 1
  else:
       return g(n-1) * g(n-1)
```

a) What is the value of g_{200}?
b) How many function calls will be made by the recursive function?
c) How does this compare with the age of the universe in seconds?

a) Take the previous value in the sequence and square. The value after 1 will be 1 again. Every value in the sequence is 1.
b) If we start at g(10) we make 2 calls to g(9). Each of those calls make 2 calls to g(8).
   g(10) – 1 call
   g(9) - 2 calls
   g(8) – 4 calls
   g(7) – 8 calls
   ...
   g(1) – 2^9 calls
   So starting at g(200) we get
   $1 + 2^1 + 2^2 + \cdots 2^{199}$
   We know this summation and can solve directly to $2^{200} - 1$

c) Which is approximately $\mathbf{10^{60}}$. The age of the universe in seconds is about $\mathbf{4.3 \times 10^{17}}$. Comparatively, this value is much, much larger.