Building Blocks Homework 6
Solution

Question A) For each of the following pairs of sets, determine the following

I. A is a subset B (true/false)
II. If A is not a subset of B, give an element in set A that is not in set B.
III. B is a subset of A (true/false)
IV. A is a proper subset of B (true/false)

a) A = {4, 2, 3, 4, 1, 4} B={4, 5, 2, 3, 2}
Subset is false, 1 is in A and not in B, B is not a subset of A, A is not a proper subset (not a subset)

b) A = {2, 7, 4} B={ 5, 4, 7, 2}
Subset is true, B is not a subset of A, A is a proper subset of B

c) A = $\{x \in Z | \ x \ is \ a \ factor \ of \ 80\}$
B= $\{x \in Z | \ x \ is \ a \ multiple \ of \ 2\}\}$
A is not a subset of B, 5 is in A and not in B. B is not a subset of A, A is not a proper subset

d) A = $\{x | \ x = a/b \ for \ integer \ a, b \neq 0, \ and \ a \ divisible \ by \ 3\}$
B ={x| x is a rational number}
A is a subset of B. B is a subset of A (can always rewrite a rational number so that its numerator is divisible by 3. Example: x=5/7 is the same as 15/21. A and B are the same set, so not proper.

a) A = {} B={{a},{b}}
A is a subset of B. (trivially). B is not a subset of A. A is a proper subset of B.

Question B) For each of the following pairs of sets, compute

I. The union
II. The intersection.

a) A = {1, 7, a, f, g} B={f, 4, 9, a, b}
Union is {1, 7, a, f, g, f, 4, 9, a, b} crossing out duplicates leaves {1, 7, a, f, g, 4, 9, b}
Intersection is {a, f}

b) A = {1, 2, 3} B={c, d, e}
Union is {1, 2, 3, c, d, e}
Intersection is {} – The sets are disjoint.

c) A = {{2}, {1, 2}, {2, {}}, 3, 4, {{}}} B={{}, {{},{}}, {2, 2}, {3,4} }
Union is {{{2}, {1, 2}, {2, {}}, 3, 4, {{}}, {}, {{},{}}, {2, 2}, {3,4} } or after the duplicates are removed. {{{2}, {1, 2}, {2, {}}, 3, 4, {{}}, {}, {2}, {3,4}}
Intersection is {{{}}}

Question C) What is the cardinality of the following sets:

a) {two, four, eight, sixteen}
b) { a, b, {}, {b}, {a,b}, {b,b,b}, {{{}}}, b }
c) { {2}, 0, {3, 1, 2}, {2, 3}, {1, 2, 3}, ∅, {2, 1}, {1, 3, 2} }

I have marked duplicates in red and individual elements in blue

a) {two, four, eight, sixteen} cardinality is 4
b) { a, b, {}, {b}, {a,b}, {b,b,b}, {{{}}}, b } cardinality is 6
c) { {2}, 0, {3, 1, 2}, {2, 3}, {1, 2, 3}, ∅, {2, 1}, {1, 3, 2} } cardinality is 6

Question D) Suppose that you are going to plan a party by making selections from the following categories. **Theme**: Pirates, Zombies, Egypt, Toga. **Punch Flavor**: Berry, Mint, Watermelon, Orange. **Game**: Twister, Life, or Battleships. **Snack**: Cookies, Pop Corn, Ice Cream, Cake, Flan, or Fudge. How many ways are there to plan the provided that you must make exactly one choice in each category?

Task 1 – Select theme – 4 ways
Task 2 – Select punch – 4 ways
Task 3 – Select game – 3 ways
Task 4 – Select snack - 6 ways
Total is 4 x 4 x 3 x 6 = 16 x 18 = 288

Question E) Over 5 days you will select a single CD to listen to each day. You have 20 CDs to select from.

a) How many ways are there to select CDs if you can listen to a CD more than once?

b) How many ways are there to select CDs if you can listen to a CD only once?

c) How many ways are there to select CDs if at least one CD is listened to more than once? (Hint: the counts for the three parts are related.)

a) This is a sequence

___	___	___	___	___
20	20	20	20	20

Multiply together to get

$20 \times 20 \times 20 \times 20 \times 20 = 20^5$

Using a calculator we have 3,200,000 ways

b) This is a sequence

___	___	___	___	___
20	19	18	17	16

Multiply together to get

$20 \times 19 \times 18 \times 17 \times 16$

Using a calculator we have 1,860,480 ways

c) Take the total from part a and subtract out those from part b.

3200000- 1860480 = 1,339,520

Question F) How many license plates are there that have the following patterns:

a) DDDLL, where L is any letter except "O" and D is a digit.
b) LDDDDD, where L is any letter and D is any digit except "0" and you cannot reuse a digit.
c) SLLSDS, where L is any letter, D is any digit, and S is a star, circle, triangle or square.

a) This is a sequence.

D	D	D	L	L	
10	10	10	25	25	(25 for L since O not allowed)

Total is $10^3 \times 25^2$
By calculator we get 625,000

b) More sequencing

L	D	D	D	D	D
26	9	8	7	6	5

Total is 26 x 9 x 8 x 7 x 6 x 5
By calculator we get 393,120

c) Sequence

S	L	L	S	D	S
3	26	26	3	10	3

Total is 3 x 26 x 26 x 3 x 10 x 3
By calculator we get 182,520

Question G) For the following counting problems a genetic sequence is a sequence of the letters A, T, C, and G.

a) How many sequences of length 10 are there where letters can be reused?
b) How many sequences of length 10 are there where letters can not be reused?
c) How many sequences of length 8 are there that contain exactly 1 A. (Hint: position the A, then fill in the rest of the sequence. (You may reuse the other letters.)
d) How many sequences of length 8 are where the first 5 characters are the same as the last 5 characters. IE. "CGACGACG" and "TAATAATA" would both be counted.
e) How many sequences of length 8 are there that start with GG or end in C.

b) There are 10 slots to fill, each one has 4 selections so we get
$4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 = 4^{10}$
By calculator this is 1,048,576

c) When we try to fill the slots we get
$4 \times 3 \times 2 \times 1 \times 0$
So there are no such sequences.

d) We know there is an A, so our first task is to put the A in one of the eight slots. (8 selections)
Now we can work from left to right and fill each of the 7 remaining slots with C, G, or T. Each contributes a factor of 3 so we get
8×3^7
Which by calculator is 17,496

e) I need a way to characterize the relations, so I will use the positions
We have 1 2 3 4 5 6 7 8
And we need to match the first and last 5 up
1 2 3 4 5
4 5 6 7 8
So 1st and 4th match up, 2nd and 5th, etcetera
We are free to put anything into the first location, but that forces positions 4 and 7 to be the same. Position 2 forces 5 and 8. Position 3 forces 6.

__	__	__	__	__	__	__	__
Free	Free	Free	X	X	X	X	X
4	4	4	1	1	1	1	1

And we get $4 \times 4 \times 4 \times 1 \times 1 \times 1 \times 1 \times 1 = 64$

a) Start with GG is

G	G	4	4	4	4	4	4

Is $4^6 = 4{,}096$

End in C is

4	4	4	4	4	4	4	C

Is $4^7 = 16{,}384$

Both is

G	G	4	4	4	4	4	C

Is $4^5 = 1024$

Using inclusion exclusion we add the individual and subtract out the overlap
Total is 4096 + 16384 – 1024 = 19,456

Question H) We are going to create a password using four types of characters. There are 10 digits, 26 lowercase letters, 26 uppercase letters and 36 symbols we can choose from.

a) How many passwords can you create of length 11 that have exactly one digit, one special symbol and one uppercase letter. The rest of the characters will be lowercase letters. (Hint, for tasks first select the three characters and their positions, finally fill in the remaining slots with lower case letters.)
b) An ordinary desktop computer can try about 100,000,000 passwords in one second. How long will it take to try all the passwords from part a.
c) You want to make your password more secure. Propose a minimum length of time to try all possible passwords and then determine how many additional upper/lowercase characters would be needed.

a) Task 1: position of digit 11
Task 2: position of special 10
Task 3: position of UC 9
Task 4: select digit 10
Task 5: select special 36
Task 6: select UC 26
Task 7: Fill in remaining 8 positions from the lower case options and we can reuse. 26^8

Total is $11 \times 10 \times 9 \times 10 \times 36 \times 26 \times 26^8$
By calculator this is 1.935075111187046e18

b) So 1.935075111187046e18 / 100,000,000 will be the number of seconds required. Or approximately 19,350,751,111
Divide by 60 x 60 to get from seconds to hours: 5,375,208
Divide by 24 x 365 to get from hours to years: 613

c) 600 years seems pretty good. The question is what level of effort you want to protect against. Hackers can create ad hoc networks of machines that they have taken control of. Lets suppose that they have 100,000 computers. Which means that the time is reduced by a factor of 100,000 or about 53 hours. To get back to 600 years, I would need to increase the number of passwords by a factor of 100,000. Lets be paranoid and give ourselves a cushion of an extra 1000. So we want $26^k = 100{,}000{,}000$ where k is the number of extra characters we add. Take a log of both sides and we have

$$\log 26^k = \log 100{,}000{,}000$$

Or

$$k \log 26 = \log 100{,}000{,}000$$

$$k = \frac{\log 100{,}000{,}000}{\log 26} = \frac{8}{\log 26} \approx \frac{8}{1.4} \approx 5.6 \text{ extra characters}$$

Note: Clearly a better option to adding 6 more characters is to use the entire set of 26 + 26 + 36 = 98 characters for each position.

If I assume 100,000,000 tries/sec times 10,000,000 machines and a 600 year target

I need to solve

$log 98^k = log(2x10^{26})$

Or

$$k = \frac{\log 2x10^{26}}{\log 98} = \frac{26}{\log 98} \approx \frac{26}{2} \approx 13 \text{ characters total}$$

Question I) You will roll a 8 -sided dice twice in a row.

a) How many possible sequences are there?
b) How many sequences are there where the first roll is 3 or 5?
c) How many sequences are there where the sum is 6?
d) How many sequences are there where the first roll is 3 or 5 or the sum is 6?

(Hint: When counting in parts b, c, and d use a table.)

a) $\underset{8}{\_\_}\ \underset{8}{\_\_}$ Multiply to get 64.

b) Table time. First die roll on the side. Second die roll on top. 64 spaces we can fill in. Mark first is 3 or 5 with X
Mark sum to 6 with an S.

	1	2	3	4	5	6	7	8
1					S			
2				S				
3	X	X	XS	X	X	X	X	X
4								
5	X	XW	X	X	X	X	X	X
6	S							
7								
8								

Counting we see that there are 16 squares marked for first roll is a 3 or 5.

c) Sum of 6 has 6 square marked.
d) The total number of square with either mark is 16 + 6 – 2 = 20. (We used inclusion-exclusion and noted that there were 2 squares that had both marks and were counted twice.)

You may choose to solve one (and only one) of the following Extra Credit Problems. If you submit more than one, only the first will be graded.

Extra Credit 1) Write a program that will generate a list of all digit sequences of length 5 where the sum of the digits is divisible 9 . Print the number of sequences and then print them out in increasing numerical order.

Trick we can use. These are the numbers that are divisible by 9. So its relatively easy to print them except that we need to print with leading zeros.

```
for val in range(0, 100000, 9):
     asChars = str(val)
     while len(asChars)<5: asChars = “0” + asChars
     print(asChars)
```

Extra Credit 2) We visit the opera and leave our coat at the coat check. The person working the counter forgets to give out receipts. When the patrons come to get their coat, they are given a random coat from the rack. Suppose there are n patrons each of who has a coat. Lets label the coats and patrons from 1 to n. We can think of the coats that are picked up as sequence where there is no repetition. If the number of the coat matches the position in the sequence we will count that as the patron getting back their own coat. Write a program that counts the number of sequences where exactly 1 patron gets back their own coat. Take the number of patrons as an input value. Examples with n=4.

Sequence: 1 2 3 4 (Everybody gets back their coat)
Sequence: 1 4 2 3 (Patron 1 only gets back their coat)
Sequence: 4 2 3 1 (Patrons 2 and 3 get back their own coat)

```
def goodPerm(perm):
    asSet = set(perm)
    return len(asSet) == len(perm)

def oneExact(perm):
    indices = range(len(perm))
    sum = 0;
    for (val, pos) in zip(perm, indices):
        if val==pos: sum +=1
    return sum == 1;

def generator(n, count):
    if n==0: return [[]]
    else:
        result = []
        listOfList = generator(n-1, count)
        #print(listOfList, n-1)
        for val in range(count):
            for sub in listOfList:
                #print(val, sub)
                newsub = sub + [val]
                result.append(newsub)
        return result

print(generator(3, 3))

oneBack = 0
for perm in generator(4,4):
    if goodPerm(perm) and oneExact(perm):
        oneBack += 1
        print(perm)
print("Count of exactly one back is ", oneBack)
```