Discrete Math Homework 7 Solution
Due Wednesday, March 1 at the beginning of class

General instructions:
- Use standard size paper (8.5 by 11).
- Answer each question in order using a single column.
- Be neat. If we cannot read your solution it is wrong.
- Show your work. If you just write an answer, you will get minimal credit even if the answer is correct.

Rosen section 2.3.

Question A) Rosen 2.3 Exercise 4 a, b, c (p. 152)

Find domain and range for each
a) the function that assigns to each nonnegative integer its last digit.
  Domain = {0, 1, 2, 3, …} or nonnegative integers
  Range = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}
b) the function that assigns to the next largest integer to a positive integer
  Domain = {1, 2, 3, …} or positive integers or Z^+
  Range = {2, 3, 4, 5, 6, 7, 8, 9, …}
c) the function that assigns to a bit string the number of one bits in the string
  Domain = {e, 0, 1, 00, 10, 01, 11, 000, 001, ..} the set of bit strings (the empty string is a bit string)
  Range = {0, 1, 2, 3, …} or nonnegative integers

Question B) Rosen 2.3 Exercise 10 (p. 153)

Determine if each of the following functions from {a, b, c, d} to itself is one-to-one
a) f(a)=b; f(b) = a; f(c)=c; f(d)=d
  Is one-to-one (and onto)
b) f(a)=b; f(b) = b; f(c)=d; f(d)=c
  Is not one-to-one f(a)=f(b)
c) f(a)=d; f(b) = b; f(c)=c; f(d)=d
  Is not one-to-one f(a)=f(d)

Question C) Rosen 2.3 Exercise 12 (p. 153)

Determine if each of the functions from Z to Z is one-to-one

a) $f(n) = n - 1$

This function is one-to-one. We need to show that f(a) = f(b) implies that a=b

$$f(a) = f(b)$$
$$a - 1 = b - 1$$
$$a = b$$

QED

b) $f(n) = n^2 + 1$

This function is not one-to-one. We demonstrate this by finding different values a and b where f(a) = f(b). In this case a=-1 and b=1 give f(-1)=2=f(1).

QED

c) $f(n) = n^3$

This function is one-to-one. We need to show that f(a) = f(b) implies that a=b. Use a proof by contradiction.

Given: f(a) = f(b)

Assume: a not equal to b.

Without loss of generality, we may assume that a is less than b. This means we can write b=a+k, where k is some positive integer value.

$$f(a) = f(a+k)$$
$$a^3 = (a+k)^3 = a^3 + 3a^2k + 3ak^2 + k^3$$
$$0 = k\left(3a^2 + 3ak + k^2\right)$$

For this to have a solution, one of the factors must be zero. By assumption, k is not zero.

We can solve the second term for a using the quadratic formula

$$a = \frac{-(3k) \pm \sqrt{(3k)^2 - 4(3)(k^2)}}{2(3)}$$
$$a = \frac{-3k \pm \sqrt{9k^2 - 12k^2}}{6}$$

If we look at the value under the radical, it must be negative and there are no real solutions for a. Therefore, a must equal b.

QED

d) $f(n) = \lceil n/2 \rceil$

This function is not one-to-one. We demonstrate this using the values a=1 and b=2.

$$f(1) = \lceil 1/2 \rceil = 1$$
$$f(2) = \lceil 2/2 \rceil = 1$$

Question D) For each of the functions from Exercise 12 , determine if the function is onto.

Determine if each of the functions from Z to Z is onto

a) $f(n) = n - 1$

This function is onto. We need to show that for every b there is some a, where f(a) = b.

$f(a) = b$

$a - 1 = b$

$a = b + 1$

Note that a is an integer, so we found a value.

QED

b) $f(n) = n^2 + 1$

This function is not onto. We demonstrate this by finding a value of b, without a map into it. Take b = 3.

$f(a) = 3$

$a^2 + 1 = 3$

$a^2 = 2$

The only possible solutions for a are non-integer. Therefore there is no mapping into the value 3. (There are many such values.)

QED

c) $f(n) = n^3$

This function is not onto. Take b = 3.

$f(a) = 3$

$a^3 = 3$

The only possible solutions for a are non-integer. Therefore there is no mapping into the value 3. (There are many such values.)

QED

d) $f(n) = \lceil n/2 \rceil$

This function is onto.

$f(a) = b$

$\lceil a/2 \rceil = b$

if a= 2b then we have

$\lceil 2b/2 \rceil = \lceil b \rceil = b$

Note that a=2b is an integer, so we found a value.

QED

Question E) Rosen 2.3 Exercise 22 (p. 153)
Determine if each of the following is a bijection from **R** to **R. (A function is a bijection (one-to-one correspondence) if it is an injection(one-to-one) and a surjection(onto)**

a) $f(x) = -3x + 4$

This is a bijection:

One-to-One

$f(a) = f(b)$

$-3a + 4 = -3b + 4$

$-3a = -3b$

$a = b$

Onto

$f(a) = b$

$-3a + 4 = b$

$-3a = b - 4$

$a = (b-4)/-3$

For any b, we can find a real a.
QED

b) $f(x) = -3x^2 + 7$

This is not a bijection:

Not One-to-One

If a=1 and b=-1, then f(a)=f(b)

Not Onto

(We don't have to show this one we know its not one-to-one)

Pick b= 10 and solve for a

$f(a) = b$

$-3a^2 + 7 = 10$

$-3a^2 = 3$

$a^2 = -1$

Has no real solutions.
QED

c) $f(x) = (x+1)/(x+2)$

This is **not a function** from R to R so it isn't a bijection
if x=-2, this function has no real value.

Aside from this, we can ask if there are two values of x that map to the same value. The mappings are unique.

Is every value in R mapped to? No. There is no solution for f(a) = 1.

d) $f(x) = x^5 + 1$

This is a bijection:
One-to-One
First note that if a function is one-to-one, we can add any constant to that function and still have a One-to-One function. (We just shift the graph up.) So we will show that g(x) = x^5 is one-to-one.

Given: g(a) = g(b)
Assume: a is not b
WLG, we can assume that a is less than b
We can write b = a+q , where q is a positive real.
We can see that for negative values of x g(x)<0 and for positive value of x g(x)>0. This means that if we find two values that map to the same value they must either both be positive or both be negative. Furthermore, we know that g(x) has odd symmetry so g(x) = g(-x) and so if I can find a pair of negative values, there must be a corresponding pair of positive values and I can further restrict a to be positive.

$$g(a) = g(a+q)$$
$$a^5 = (a+q)^5$$
$$a^5 = a^5 + 5a^4q + 10a^3q^2 + 10a^2q^3 + 5aq^4 + q^5$$
$$0 = 5a^4q + 10a^3q^2 + 10a^2q^3 + 5aq^4 + q^5$$
$$0 = q\left(5a^4 + 10a^3q + 10a^2q^2 + 5aq^3 + q^4\right)$$

For the last expression to be zero, one of its factors must be zero. q is not zero by assumption. The right factor has only positive coefficients, so there can not be any positive roots. Therefore g(x) is one-to-one and f(x) is one-to-one as well.
QED
Onto
$$f(a) = b$$
$$a^5 + 1 = b$$
$$a^5 = b - 1$$
Fifth roots behave well and for any b, we can find a real a.
QED

Rosen section 2.4

Question F) Rosen 2.4 Exercise 10 a, c, e (p. 168)
Find the first 6 terms in the sequence.

a) $\begin{array}{l} a_n = -2a_{n-1} \\ a_0 = -1 \end{array}$

$$\boxed{\begin{array}{l} a_0 = -1 \\ a_1 = -2a_0 = -2\cdot -1 = 2 \\ a_2 = -2a_1 = -2\cdot 2 = -4 \\ a_3 = -2a_2 = -2\cdot -4 = 8 \\ a_4 = -2a_3 = -2\cdot 8 = -16 \\ a_5 = -2a_4 = -2\cdot -16 = 32 \end{array}}$$

c) $\begin{array}{l} a_n = 3a_{n-1}^2 \\ a_0 = 1 \end{array}$

$$\boxed{\begin{array}{l} a_0 = 1 \\ a_1 = 3a_0^2 = 3\cdot(1)^2 = 3\cdot 1 = 3 \\ a_2 = 3a_1^2 = 3\cdot(3)^2 = 3\cdot 9 = 27 \\ a_3 = 3a_2^2 = 3\cdot(27)^2 = 3\cdot 729 = 2187 \\ a_4 = 3a_3^2 = 3\cdot(2187)^2 = 3\cdot 4782969 = 4782972 \\ a_5 = 3a_4^2 = 3\cdot(4782972)^2 = 3\cdot 22876821152784 = 68630463458352 \end{array}}$$

e) $\begin{array}{l} a_n = a_{n-1} - a_{n-2} + a_{n-3} \\ a_0 = 1 \\ a_1 = 1 \\ a_2 = 2 \end{array}$

$$\boxed{\begin{array}{l} a_0 = 1 \\ a_1 = 1 \\ a_2 = 2 \\ a_3 = a_2 - a_1 + a_0 = 2 - 1 + 1 = 2 \\ a_4 = a_3 - a_2 + a_1 = 2 - 2 + 1 = 1 \\ a_5 = a_4 - a_3 + a_2 = 1 - 2 + 2 = 1 \end{array}}$$

Question G) Rosen 2.4 Exercise 12 (p. 168)
Show that the sequence is a solution of the recurrence relation $a_n = -3a_{n-1} + 4a_{n-2}$

a) $a_n = 0$

Plug the closed form into the recurrence relation and we get

$$0 = -3(0) + 4(0)$$
$$0 = 0$$

Check

b) $a_n = 1$

Plug the closed form into the recurrence relation and we get

$$1 = -3(1) + 4(1)$$
$$1 = -3 + 4$$
$$1 = 1$$

Check

c) $a_n = (-4)^n$

Plug the closed form into the recurrence relation and we get

$$(-4)^n = -3\left((-4)^{n-1}\right) + 4\left((-4)^{n-2}\right)$$
$$(-4)^n = \left[(-4)^{n-2}\right]\left\{-3\left((-4)^1\right) + 4\right\}$$
$$(-4)^2 = \left\{-3\left((-4)^1\right) + 4\right\}$$
$$16 = \{12 + 4\}$$
$$16 = 16$$

Check

d) $a_n = 2(-4)^n + 3$

$$2(-4)^n + 3 = -3\left(2(-4)^{n-1} + 3\right) + 4\left(2(-4)^{n-2} + 3\right)$$
$$2(-4)^n + 3 = \left((-6)(-4)^{n-1} - 9\right) + \left(8(-4)^{n-2} + 12\right)$$
$$2(-4)^n = (-6)(-4)^{n-1} + 8(-4)^{n-2} \qquad \Leftarrow \text{Divide by } (-4)^{n-2}$$
$$2(-4)^2 = (-6)(-4)^1 + 8(-4)^0$$
$$32 = 24 + 8$$
$$32 = 32$$

Check

Question H) Rosen 2.4 Exercise 14 a, c, g (p. 168)
For each of the following sequences find a recurrence relation that they satisfy. It will help to write out some terms in the sequences

a) $a_n = 3$ is the sequence 3, 3, 3, 3, 3, 3… and satisfies

$$a_n = a_{n-1}$$

c) $a_n = 2n+3$ is the sequence 5, 7, 9, 11, … and satisfies

$$a_n = a_{n-1} + 2$$

g) $a_n = n + (-1)^n$ is the sequence 0, 3, 2, 5, 3, 6, 4, 7, 5, … and satisfies

$$a_n = a_{n-2} + 2$$

Question I) Rosen 2.4 Exercise 16 a, d, g (p. 168) Find solutions using an iterative approach.

a) $\begin{aligned} a_n &= -a_{n-1} \\ a_0 &= 5 \end{aligned}$

$$\begin{aligned} a_n &= -a_{n-1} && \textit{use} \quad a_{n-1} = -a_{n-2} \\ a_n &= -(-a_{n-2}) && \textit{use} \quad a_{n-2} = -a_{n-3} \\ a_n &= -(-(-a_{n-3})) \end{aligned}$$

We see that the number of negatives matches the value we are subtracting from n. Also, we will continue until we get to a0. We deduce that the formula is

$$a_n = 5(-1)^n$$

d) $a_n = 2a_{n-1} - 3$
$a_0 = -1$

$$a_n = 2a_{n-1} - 3 \qquad use \quad a_{n-1} = 2a_{n-2} - 3$$
$$a_n = 2(2a_{n-2} - 3) - 3 \qquad use \quad a_{n-2} = 2a_{n-3} - 3$$
$$a_n = 2(2(2a_{n-3} - 3) - 3) - 3$$

We see that the number of twos matches the value we are subtracting from n. We also see that we can rearrange the terms to get

$$a_n = 2^3 a_{n-3} - 2^2 3 - 2^1 3 - 2^0 3$$

or

$$a_n = 2^3 a_{n-3} - 3(2^2 + 2^1 + 2^0)$$

and we see that we have a geometric sequence

$$a_n = 2^3 a_{n-3} - 3(2^3 - 1)$$

Generalizing we get

$$a_n = 2^n(-1) - 3(2^n - 1)$$

or

$$a_n = 2^n(-4) + 3$$

g) $a_n = -a_{n-1} + n - 1$
$a_0 = 7$

$$a_n = -a_{n-1} + n - 1 \qquad use \quad a_{n-1} = -a_{n-2} + (n-1) - 1$$

$$a_n = -\left(-a_{n-2} + (n-1) - 1\right) + n - 1 \qquad use \quad a_{n-2} = -a_{n-3} + (n-2) - 1$$

$$a_n = -\left(-\left(-a_{n-3} + (n-2) - 1\right) + (n-1) - 1\right) + n - 1$$

Lets rearrange that last term

$$a_n = -\left(-\left(-a_{n-3} + (n-3)\right) + (n-2)\right) + n - 1$$

$$a_n = -\left(-\left(-a_{n-3}\right)\right) + (n-3) - (n-2) + (n-1)$$

$$a_n = -\left(-\left(-a_{n-3}\right)\right) + (n-3) + 1$$

$$a_n = -\left(-\left(-\left(-a_{n-4}\right)\right)\right) - (n-4) + (n-3) + 1$$

$$a_n = -\left(-\left(-\left(-a_{n-4}\right)\right)\right) + 1 + 1$$

This can be written as

$$a_n = (-1)^n \cdot 7 + \lfloor n/2 \rfloor$$

You may choose to solve one (and only one) of the following Extra Credit Problems. If you submit more than one, only the first will be graded.

Extra Credit 1) Consider the sequence defined by the following recurrence relation:

$$a_n = a_{n-1} + 3n^2 - 3n + 1$$
$$a_0 = 1$$

a) Write the first 5 terms of the sequence.

$$a_0 = 1$$
$$a_1 = a_0 + 3(1)^2 - 3(1) + 1 = 2$$
$$a_2 = a_1 + 3(2)^2 - 3(2) + 1 = 9$$
$$a_3 = a_2 + 3(3)^2 - 3(3) + 1 = 28$$
$$a_4 = a_3 + 3(4)^2 - 3(4) + 1 = 65$$

b) Propose a closed form solution for a_n

$$a_n = n^3 + 1$$

c) Verify that your solution is correct

$$a_0 = 0^3 + 1 = 1$$

Check

$$\begin{aligned} a_{n-1} + 3n^2 - 3n + 1 &= (n-1)^3 + 1 + 3n^2 - 3n + 1 \\ &= \left(n^3 - 3n^2 + 3n - 1\right) + 1 + 3n^2 - 3n + 1 \\ &= n^3 + 1 \\ &= a_n \end{aligned}$$

Check

Extra Credit 2) Consider the sequences defined by the following recurrence relations.

$$a_n = 2b_{n-1}$$
$$a_0 = 1$$
$$b_n = a_{n-1} + 1$$
$$b_0 = 1$$

a) Write the first 5 terms of the sequences.

$a_0 = 1$	$b_0 = 1$
$a_1 = 2b_0 = 2$	$b_1 = a_0 + 1 = 2$
$a_2 = 2b_1 = 4$	$b_2 = a_1 + 1 = 3$
$a_3 = 2b_2 = 6$	$b_3 = a_2 + 1 = 5$
$a_4 = 2b_3 = 10$	$b_4 = a_3 + 1 = 7$

b) Come up with a recurrence relation for a that does not use b.

We know that $b_{n-1} = a_{n-2} + 1$ because of the recursive definition of b and we can put that into the definition for a. Because we are going back by two, we need a second initial condition.

$$a_n = 2(a_{n-2} + 1)$$
$$a_0 = 1$$
$$a_1 = 2$$

c) Come up with a recurrence relation for b that does not use a.

We know that $a_{n-1} = 2b_{n-2}$ because of the recursive definition of a and we can put that into the definition for b. Because we are going back by two, we need a second initial condition.

$$b_n = (2b_{n-2}) + 1$$
$$b_0 = 1$$
$$b_1 = 2$$