Fleck Chapter 3

Problem 1 Give a direct proof of the following statement.
For all rational values x and y, x+y is also rational.

Definition: x is rational iff x=a/b for some integer values a, b with b≠0.

Given:
x is rational
y is rational
Show:
x+y is rational
Proof:
X is rational means x=a/b where a is integer, b is integer and b≠0.
Y is rational means x=c/d where c is integer, d is integer and d≠0.

$$x + y = \frac{a}{b} + \frac{c}{d}$$

When adding fractions, we need to put them over the same denominator

$$= \frac{a}{b}\left(\frac{d}{d}\right) + \frac{c}{d}\left(\frac{b}{b}\right)$$

And complete the addition

$$= \frac{ad + cb}{bd}$$

This is rational according to our definition provided that we can show three things

- $ad + cb$ is an integer. All four are integers and the result must be as well. ✔
- bd is an integer. Both values are integers and the result must be as well ✔
- $bd \neq 0$. The only way the product of two numbers can be zero is if one or the other is zero. We know both b and d are not zero. ✔

QED.

Problem 2 Give a direct proof of the following statement.

For any pair of different rational values x and y, there is a rational value z between x and y. Hint: think about the average.

Given:

x is rational

y is rational

x≠y

Show:

There is a rational z where z is between x and y

Proof:

X is rational means x=a/b where a is integer, b is integer and b≠0.

Y is rational means x=c/d where c is integer, d is integer and d≠0.

Without loss of generality, assume that x < y.

I need a value between x and y. Consider the average of the two values

$$z = \frac{1}{2}\left(\frac{a}{b} + \frac{c}{d}\right) = \frac{ad + bc}{2bd}$$

Is x < z < y?

1) Show x <z which is

$$x < \frac{1}{2}(x + y)$$

Multiply both side by a positive, non-negative value

$$2x < (x + y)$$

Subtract x from both sides

$$x < y$$ ✔

$$\frac{a}{b} <$$

2) Show z<y which is

$$\frac{1}{2}(x + y) < y$$

Multiply both side by a positive, non-negative value

$$(x + y) < 2y$$

Subtract y from both sides

$$x < y$$ ✔

Is z rational? ad+bc is an integer, 2bd is an integer, 2bd≠0 (See previous problem.) Yes ✔

QED

Problem 3 Disprove the following statement.

For all integer values x and y, if $x<y$ there is an integer value z, where $x < z < y$.

If $x=5$ and $y=6$, there is no integer value z where $5 < z < 6$.
QED

Problem 4 Give a direct proof of the following statement.

For all integer n, if n is even then $3n+1$ is odd.

Given:

n is an integer
n is even

Show:

$3n+1$ is odd

Proof:

Since n is even, we know that $n=2k$ for some integer k.
Consider $3n+1 = 3(2k) + 1$
We can write this as $2(3k) + 1$
This will be odd, provided that 3k is an integer.
An integer times and integer is an integer. ✔

QED

Problem 5 Give a direct proof of the following statement.
For all integer n, m, if m is odd then m+2n is odd.

Given:
n is an integer
m is an integer
m is odd
Show:
m+2n is odd

Proof:
Since m is odd, we know that m=2k+1 for some integer k.
Consider m+2n = 2k+1 +2n
We can write this as 2(k+n) + 1
This will be odd, provided that k+n is an integer. Both k and n are integers so their sum is also an integer. ✔
QED

Problem 6 Give a direct proof of the following statement.
For all integer n, m. If n and m are odd, then n+m is even.

Given:
n is an integer
m is an integer
n is odd
m is odd
Show:
n+m is even

Proof:
Since n is odd, we know that n=2k+1 for some integer k.
Since m is odd, we know that m=2j+1 for some integer j.
Consider n+m
We can write this as 2k+1 +2j + 1 = 2k +2j+2 = 2(k+j+1)
This will be even, provided that k+j+1 is an integer. Both k, j and 1 are integers, so their sum is also an integer. ✔
QED

Problem 7 Give a proof by cases of the the following statement.

For all integer n, n^2+n is even.

We will have two cases: n is even, n is odd

Case 1:

Given:

n is an integer

n is even

Show:

n^2+n is even

Proof:

Since n is even, we know that n=2k for some integer k.

Consider n^2+n

We can write this as $(2k)^2+2k = 4k^2+2k=2(2k^2+k)$

This will be even, provided $2k^2+k$ is an integer. Since 2 and k are integers, and expression only has addition and multiplication, it will be an integer.

✔

Case 2:

Given:

n is an integer

n is odd

Show:

n^2+n is even

Proof:

Since n is odd, we know that n=2k+1 for some integer k.

Consider n^2+n

We can write this as $(2k+1)^2+2k$

$= 4k^2+4k+1 + 2k +1$

$= 4k^2+6k+2$

$= 2(2k^2+3k+1)$

This will be even, provided $2k^2+3k+1$ is an integer. Since 2, 3 and k are integers, the expression will be an integer. ✔

QED

Problem 8 Give a proof by cases of the following statement.

For all integer n, m, if m+n is even then m-n is even.

We will have 4 cases.

1) n,m both even,
2) n,m both odd
3) n even, m odd
4) n odd, m even

Case 1:
Given:

n is an integer
m is an integer
n is even
m is even

Show:

If m+n is even then m-n is even

Proof:

Since n is even, we know that n=2k for some integer k.
Since m is even, we know that m=2j for some integer j.
Consider m+n
We can write this as 2k+2j = 2(k+j)
Since k+j is an integer, m+n is even

Consider m-n
We can write this as 2k-2j = 2(k-j)
This will be even, provided k-j is an integer. Both k and j are integers and integers are closed under subtraction, so it will be an integer. ✔

Case 2:
Given:

n is an integer
m is an integer
n is odd
m is odd

Show:

If m+n is even then m-n is even

Proof:

Since n is odd, we know that n=2k+1 for some integer k.
Since m is odd, we know that m=2j+1 for some integer j.
Consider m+n
We can write this as 2k+1+2j+1 = 2(k+j+1)
Since k+j+1is an integer, m+n is even

Consider m-n
We can write this as 2k+1-(2j+1) = 2(k-j)
This will be even, provided k-j is an integer. Both k and j are integers and integers are closed under subtraction, so it will be an integer. ✔

Case 3:
Given:

n is an integer
m is an integer
n is even
m is odd

Show:

If m+n is even then m-n is even

Proof:

Since n is even, we know that n=2k for some integer k.
Since m is odd, we know that m=2j+1 for some integer j.
Consider m+n
We can write this as 2k+2j+1 = 2(k+j)+1
Since k+j is an integer, m+n is odd and the implication will be trivially true.

Case 4:
Given:

n is an integer
m is an integer
n is odd
m is even

Show:

If m+n is even then m-n is even

Proof:

This is essentially similar to case 3.

QED