Fleck Chapter 11 – Induction

Weak Induction.
Prove P(n) for $n \geq n_0$.

By showing **Base Case**
$P(n_0)$
And **Inductive Case**
$P(k) \rightarrow P(k+1)$ for $k \geq n_0$

Strong Induction.
Prove P(n) for $n \geq n_0$.

By showing **Base Cases**
$P(n_0),\ P(n_1), P(n_2), \ldots\ P(n_d)$

And **Inductive Case**
$P(n_0),\ P(n_1), P(n_2), \ldots\ P(n_d), \ldots, P(k) \rightarrow P(k+1)$ for $k \geq n_d$

Weak induction is conceptually easier to understand. Prove the first thing and then show that and one thing leads to the next in line. We are guaranteed to only need one base case, though it doesn't hurt to put in more.

Strong induction allows us to use all of the things we know in order to prove the next in line. Some arguments are easier to make in strong induction, though technically anything that is provable by strong induction is provable using weak induction and vice-verse. In strong induction, we may need more than one base case depending on the argument made in the inductive case.

Notice in both forms of induction, we only need to show the inductive step for values that are greater than or equal to the last base case.

I am going to demand that you structure your proof in a very rigid way for this course. As beginning proof constructors, these guiderails will make it less likely that you build an incorrect proof. As you gain more experience, you can abbreviate the steps.

Steps

1) What is P(n)?
2) What is the base case n_0?
3) Is the statement true in the base case?
   Prove $\boldsymbol{P(n_0)}$ is true.
4) What are your induction hypotheses?
   Write out $P(n_0)$, $P(n_1), P(n_2)$, … $P(n_d), \ldots, P(k)$
5) What are you trying to show?
   Write out $P(k+1)$
6) Complete the argument for the induction case. You **must identify** where you are using the induction hypotheses.
7) QED

Weak Induction Proofs

Problem 1) Use a proof by induction to show that

$$\mathbf{1 + 2 + 3 + \ldots n = \frac{n(n+1)}{2}}$$

for all positive integer values.

Note: We don't need induction to show that this is true. This can make our inductive argument a bit tricky, so we are going to be especially careful to make sure we use the inductive hypothesis. We will use weak induction.

1) What is P(n)?

   $$\mathbf{1 + 2 + 3 + \ldots n = \frac{n(n+1)}{2}}$$

2) What is the base case n_0?

   The smallest positive integer value is 1.

3) Is the statement true in the base case?

   Prove $\boldsymbol{P(1)}$ is true. Replace n by 1 in our predicate.

   $$\mathbf{1 = \frac{1(1+1)}{2}}$$

   $$\mathbf{1 = 1} \quad \checkmark$$

4) What are your induction hypotheses?

   P(k) is $\mathbf{1 + 2 + 3 + \ldots k = \frac{k(k+1)}{2}}$ for $k \geq 1$

5) What are you trying to show?

   P(k+1) is $\mathbf{1 + 2 + 3 + \ldots + k + (k+1) = \frac{(k+1)((k+1)+1)}{2}}$

6) Complete the argument for the induction case. You **must identify** where you are using the induction hypotheses.

   Lets start with what we want to show

   $$\color{red}{\mathbf{1 + 2 + 3 + \ldots + k}} \mathbf{+ (k+1) = \frac{(k+1)((k+1)+1)}{2}}$$

   We have seen this pattern before in step 4. **By the induction hypothesis** we can replace the red part.

   $$\color{red}{\mathbf{\frac{k(k+1)}{2}}} \mathbf{+ (k+1) = \frac{(k+1)((k+1)+1)}{2}}$$

   Now we just need to show that both sides are the same.

   $$\mathbf{\frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}}$$

   K+1 is not zero, so we can divide both sides

   $$\mathbf{\frac{k}{2} + 1 = \frac{(k+2)}{2}}$$

   or

   $$\mathbf{\frac{k}{2} + 1 = \frac{k}{2} + \frac{2}{2}} \quad \checkmark$$

7) QED

Problem 2) Use a proof by induction to show that

$$\boldsymbol{n^2 \geq n}$$

for all integer n greater than 1.

We will use weak induction.

1) What is P(n)?

   "$\boldsymbol{n^2 \geq n}$"

2) What is the base case n_0?

   $n_0 = 2$

3) Is the statement true in the base case?

   Prove $\boldsymbol{P(2)}$ is true.

   $\boldsymbol{2^2 \geq 2}$

   $\boldsymbol{4 \geq 2}$ ✓

4) What are your induction hypotheses?

   P(k) is "$\boldsymbol{k^2 \geq k}$" for $k \geq 2$

5) What are you trying to show?

   P(k+1) is "$\boldsymbol{(k+1)^2 \geq (k+1)}$"

6) Complete the argument for the induction case.

   Lets start with what we want to show

   $$\boldsymbol{(k+1)^2 \geq (k+1)}$$

   First expand out the terms

   $$\boldsymbol{k^2 + 2k + 1 \geq k + 1}$$

   Subtract one from both sides

   $$\boldsymbol{k^2 + 2k \geq k}$$

   By the induction hypothesis we know that k squared is greater than or equal to k. If we can show that $\boldsymbol{2k \geq 0}$, then we are done**.**

   In step four we restricted the values of k to be

   $$\boldsymbol{k \geq 2.}$$

   Multiplying by 2 we have

   $$\boldsymbol{2k \geq 4.}$$

   Combined with 4>0 we arrive at the desired inequality.

   $$\boldsymbol{2k \geq 0} \checkmark$$

7) QED

Problem 3) Use a proof by induction to show that a complete graph with n nodes has $\frac{n(n-1)}{2}$ edges. (Definition: A complete graph will have exactly one undirected edge between every pair of distinct nodes. So no loops or multi edges are allowed.)

We will use weak induction.

1) What is P(n)?
   "a complete graph with n nodes has $\frac{n(n-1)}{2}$ edges"
2) What is the base case n_0?
   $n_0 = 0$ (we can have a graph that has no nodes)
3) Is the statement true in the base case?
   Prove $P(0)$ is true.
   "a complete graph with 0 nodes has $\frac{0(0-1)}{2}$ edges"
   "a complete graph with 0 nodes has 0 edges" ✓
4) What are your induction hypotheses?
   P(k) is "a complete graph with k nodes has $\frac{k(k-1)}{2}$ edges" for $k \geq 0$
5) What are you trying to show?
   P(k+1) is "a complete graph with k+1 nodes has $\frac{(k+1)((k+1)-1)}{2}$ edges"
6) Complete the argument for the induction case.
   We need to show that a graph with k+1 nodes will have $\frac{(k+1)(k)}{2}$ edges.

   Suppose we start with a complete graph that has k nodes. **By the induction hypothesis** that graph will have $\frac{k(k-1)}{2}$ edges. We add another node.

Complete graph
with k nodes
has $\frac{k(k-1)}{2}$ edges

Extra node
to get to k+1 nodes
Need k more edges.

   Number of edges for a graph with k+1 nodes will be

   $$\frac{k(k-1)}{2} + k$$

   Applying arithmetic manipulations this is equal to

   $$\frac{k(k-1)}{2} + \frac{2k}{2} = \frac{k(k-1)+2k}{2} = \frac{k(k-1+2)}{2} = \frac{k(k+1)}{2} \checkmark$$

7) QED

Problem 4) Use a proof by induction to show that

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

for all positive integer values.

1) What is P(n)?

   $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

2) What is the base case n_0?

   The smallest positive integer value is 1.

3) Is the statement true in the base case?

   Prove $P(1)$ is true. Replace n by 1 in our predicate.

   $\sum_{i=1}^{1} i = \frac{1(1+1)}{2}$

   $1 = \frac{1(2)}{2}$

   $1 = 1$ ✓

4) What are your induction hypotheses?

   P(k) is $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$

5) What are you trying to show?

   P(k+1) is $\sum_{i=1}^{k+1} i = \frac{(k+1)((k+1)+1)}{2}$

6) Complete the argument for the induction case.

   We want to show that

   $\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$

   Focusing on the LHS (left hand side) of the equality we can take the last term out of the summation.

   $\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1)$

   By the induction hypothesis, we can replace the summation to get

   $\sum_{i=1}^{k+1} i = \frac{k(k+1)}{2} + (k+1)$

   Applying standard algebra to the RHS (right hand side)

   $\sum_{i=1}^{k+1} i = (k+1)\left[\frac{k}{2} + 1\right]$

   $\sum_{i=1}^{k+1} i = (k+1)\left[\frac{k+2}{2}\right]$ ✓

7) QED

Problem 5) Use a proof by induction to show that $\boldsymbol{n^3 - n}$ is divisibly by 3 for all natural numbers.

We will use weak induction.

1) What is P(n)?
   " $n^3 - n$ is divisibly by 3"
2) What is the base case n_0?
   $n_0 = 0$
3) Is the statement true in the base case?
   Prove $\boldsymbol{P(0)}$ is true.
   " $0^3 - 0$ is divisibly by 3"
   " 0 is divisibly by 3" ✓
4) What are your induction hypotheses?
   P(k) is " $k^3 - k$ is divisibly by 3" for $k \geq 0$
5) What are you trying to show?
   P(k+1) is "$(k+1)^3 - (k+1)$ is divisibly by 3"
6) Complete the argument for the induction case.
   Lets expand out the expression
   $(k^3 + 3k^2 + 3k + 1) - (\boldsymbol{k} + \mathbf{1})$

   By the induction hypothesis we know that the terms in red are divisible by 3. To complete the proof, we must show that the remaining terms are divisible by 3 as well.
   $(3k^2 + 3k + 1) - (\mathbf{1})$
   Is
   $3k^2 + 3k$
   As long as k is an integer this expression is obviously divisible by 3.

   ✓

7) QED

Strong Induction Proofs

In the game NIM, you start with some piles of matches. Players take turns removing matches from the piles. The first player who is unable to take a match will lose.
Rule 1: You must take at least one match.
Rule 2: You can only take from a single pile.

Problem 6) Use a proof by induction to show if we start a game of NIM with two piles that have n matches each, that the second player can guarantee they will win.

1) What is P(n)?
   "If we start a game of NIM with two piles that have n matches each, the second player can guarantee they will win"
2) What is the base case n_0?
   $n_0 = 0$
   (It seems weird to start a game without any matches, but it makes the recursive argument smoother.) It is possible that we will decide to add more base cases later.
3) Is the statement true in the base case?
   Prove $\boldsymbol{P(0)}$ is true.
   "If we start a game of NIM with two piles that have 0 matches each, the second player can guarantee they will win"
   If there are no matches, the first player is unable to take a match and loses, which results in a win for the second player. ✓
4) What are your induction hypotheses?
   P(0), P(1), P(2) ... P(k) are all true for $k \geq 0$
5) What are you trying to show?
   P(k+1) is "If we start a game of NIM with two piles that have k+1 matches each, the second player can guarantee they will win"
6) Complete the argument for the induction case.
   We start with two piles of objects (k+1) and (k+1). The first player must take at least one match from one of the piles. The most they can take is k+1 matches. We call the amount of matches they take T where $1 \leq T \leq k + 1$. Note that the last match has not yet been removed, so no winner/looser yet.

   There is a perfect play for the second player. They take exactly the same number of matches from the other pile. This results in two piles with
   (k+1-T) and (k+1-T) matches each.
   The number of remaining matches is in the range of 0 to k with the same number in each pile. This means that one of the **induction hypotheses** P(0) through P(k) will **apply** after both players have gone and the second player can guarantee a win no matter how many matches the first play takes.
   ✓

7) QED

Problem 7) Use a proof by induction to show that any amount of postage n over 5 cents can be composed of 3 and 4 cent stamps.

We will use ***strong*** induction.

1) What is P(n)?
   "**postage n can be composed of 3 and 4 cent stamps**"
2) What is the base case n_0?
   $n_0 = 6$ and we might need more…
3) Is the statement true in the base case?
   Prove $\boldsymbol{P(6)}$ is true.
   " postage 6 can be composed of 3 and 4 cent stamps"
   2 x 3cent ✓
4) What are your induction hypotheses?
   P(6), …$P(n_d)$, … P(k) are all true for $k \geq n_d$
5) What are you trying to show?
   P(k+1) is "postage k+1 can be composed of 3 and 4 cent stamps"
6) Complete the argument for the induction case.
   Lets start with a 3cent stamp. The remaining postage we need is
   k+1-3 = k-2 cents.
   So if we can show that P(k-2) is in our list of known facts, we are done.
   Unfortunately, this chain of logic is skipping by three's and we may miss P(6).
   So we need to go back and add in extra base cases to terminate each of the chains.
   Chain: P(6), P(9), P(12),….
   Chain: P(7), P(10), P(13),….
   Chain: P(8), P(11), P(14),….

   Revision to steps:

2) What is the base case n_0?
   $n_0 = 6$, $n_1 = 7$, and $n_2 = 8$
3) Is the statement true in the base case?
   Prove $\boldsymbol{P(6)}$ is true.
   " postage 6 can be composed of 3 and 4 cent stamps"
   2 x 3cent ✓
   Prove $\boldsymbol{P(7)}$ is true.
   " postage 7 can be composed of 3 and 4 cent stamps"
   1 x 3cent and 1x4cent ✓
   Prove $\boldsymbol{P(8)}$ is true.
   " postage 8 can be composed of 3 and 4 cent stamps"
   2 x 4cent ✓

4) What are your induction hypotheses?
   P(6), …$P(8)$, … P(k) are all true for $k \geq 8$
5) What are you trying to show?

P(k+1) is “postage k+1 can be composed of 3 and 4 cent stamps”

6) Start with a 3 cent stamp. The remaining postage we need is

k+1-3 = k-2 cents

If we can show that P(k-2) is in our list of known facts from step 4, then we are done. From step 4, we know that

$k \geq 8$

Subtracting 2 from both sides and we see that

$k - 2 \geq 6$

We also know that

$k - 2 < k$

Therefore

$6 \leq k - 2 < k$

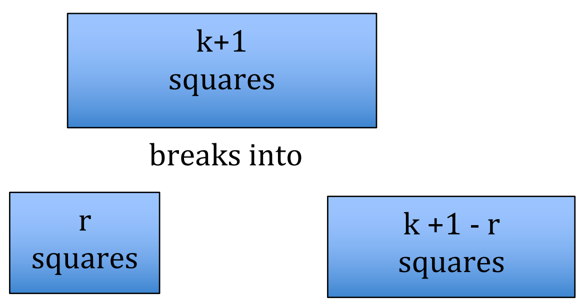
And P(k-2) is in the list P(6), …P(8), … P(k)

By the Induction Hypothesis, k-2 cents can be composed of 3 and 4 cent stamps and we can add a 3cent stamp to get postage k+1.

7) QED

Problem 8) Use a proof by induction to show a rectangular bar of chocolate of size n squares, will take exactly n-1 breaks to turn it into individual squares.

1) What is P(n)?
   "a rectangular bar of chocolate of size n squares, will take exactly n-1 breaks to turn it into individual squares"
2) What is the base case n_0?
   $n_0 = 1$
   (We need at least one square for this predicate to be true.) It is possible that we will decide to add more base cases later.
3) Is the statement true in the base case?
   Prove $\boldsymbol{P(1)}$ is true.
   "a rectangular bar of chocolate of size 1 squares, will take exactly 1-1 breaks to turn it into individual squares"
   Zero breaks are required. ✓
4) What are your induction hypotheses? (May add more later.)
   P(1), P(2) … P(k) are all true for $k \geq 0$
5) What are you trying to show?
   P(k+1) is "a rectangular bar of chocolate of size k+1 squares, will take exactly (k+1) -1 breaks to turn it into individual squares"
   Or Size k+1 requires k breaks
6) Complete the argument for the induction case.
   We start with a rectangular bar that has k+1 pieces. If we break the rectangular bar into two chunks.
   Each chunk
   1) must be a rectangle
   2)have at least 1 and no more than k squares.



The left chunk is of size r where $1 \leq r \leq k$. We know that P(r) is in the list P(1), P(2) … P(k), so **by the induction hypothesis** P(r) is true and the left chunk will take exactly r-1 breaks to turn it into individual squares.

The right chunk is of size k+1-r where $1 \leq \mathrm{k} + 1 - \mathrm{r} \leq k$. We know that P(k+1-r) is in the list P(1), P(2) … P(k), so **by the induction hypothesis** P(k+1-r) is true and the right chunk will take exactly k+1-r-1 breaks to turn it into individual squares.

Recap:

- 1 break into left and right chunks
- r-1 breaks for the left chunk
- k+1-r-1 breaks for the right chunk

Total number of breaks is

$$1 + (r-1) + (k + 1 - r - 1) = k$$

Which is what we needed to show.

7) QED