Building Blocks of Theoretical Computer Science
Summations

Version 1.1 - August 19, 2019

Introduction

A thought experiment….

Suppose we have a checkerboard and we put 1 grain of rice on the first square, 2 grains of rice on the second square, 3 grains of rice on the third square, etc. How many grains of rice do we need?

1 + 2 + 3 + 4 + …. + 61 + 62 + 63 + 64 = 2080.

How can we figure this out?
Pair the first and last and add: 1+64 = 65
Pair the second and second to last: 2+63 = 65
And keep going. All add up to the same value. There are 64 numbers, which results in 64/2=32 pairs. The sum will be 65 times 32.

Probably apocryphal story. One time in grade school, Carl Friedrich Gauss was causing a distraction. His teacher assigned Carl the task of adding up the integers from 1 to 1000, thinking that this would keep him occupied for a while. Carl came back within a few minutes with answer which he obtained using the method we outlined above.

One grain of rice is about 1/64 of a gram so we need about 32 grams of rice (0.07 pounds).

Suppose that we put 1 grain of rice on the first square, 2 on the second, 4 on the third, 8 on the fourth square, etc. How many grains of rice do we need?

$$1+2+4+8+\ldots+2^{60}+2^{61}+2^{62}+2^{63}=18446744073709551615$$

How can we figure this out? Pairing doesn't work. We need another trick.

$$2S = \quad 2+4+8+\ldots+2^{60}+2^{61}+2^{62}+2^{63}+2^{64}$$
$$S = 1+2+4+8+\ldots+2^{60}+2^{61}+2^{62}+2^{63}$$

Subtract and we get

$S = -1 + 2^{64}$

Similarly this is about 288230376151711744 grams. (635439207568045 pounds) (317719603784 tons) (12 million average size cargo ships worth)

Sequences and summation notation (Fleck 1)

Sequence – a finite or infinite list of numbers a_i indexed by an integer value i usually starting at 0 or 1

Example:
$a_i = 2i + 1$

is the sequence 1, 3, 5, 7, …. When I starts at 1.

Summation notation – a shorthand way to express sums that is often easier to work with $\sum_{i=lower}^{upper} a_i$ - i is the index variable, lower is the smallest value that i will assume and upper is the largest value that i will take on, a_i is some function of i.

Examples:

$$\sum_{i=l}^{64} i = 1 + 2 + \ldots + 64$$

$$\sum_{i=l}^{64} 3i = 3 + 6 + 9 \ldots + 192$$

$$\sum_{i=l}^{64} 2^{i-1} = 1 + 2 + 4 \ldots + 2^{63}$$

$$\sum_{i=0}^{63} 2^{i} = 1 + 2 + 4 \ldots + 2^{63}$$

$$\sum_{i=-2}^{2} \left(i^2 + 1\right) = \left(\left(-2\right)^2 + 1\right) + \left(\left(-1\right)^2 + 1\right) + \left(0^2 + 1\right) + \left(1^2 + 1\right) + \left(2^2 + 1\right)$$

$$\sum_{i=3}^{5} \left(ai - 1\right)^2 = \left(3a - 1\right)^2 + \left(4a - 1\right)^2 + \left(5a - 1\right)^2$$

Arithmetic sequence – a sequence where each term is the previous term plus a constant.

$$a_i = a_0 + ki$$

Example

$$a_i = 10 + 2i$$

$$a_0 = 10,\ a_1 = 12,\ a_2 = 14,\ a_3 = 16, \ldots$$

Geometric sequence – a sequence where each term is the previous term times a constant.

$$a_i = a_0 r^i$$

Example

$$a_i = 3 \cdot 2^i$$

$$a_0 = 3 \cdot 2^0 = 3,\ a_1 = 3 \cdot 2^1 = 6,\ a_2 = 3 \cdot 2^2 = 12,\ a_3 = 3 \cdot 2^3 = 24, \ldots$$

Manipulations of summations

Pulling off a term (we can do this from the front or back)

$$\sum_{i=l}^{u} a_i = a_l + \sum_{i=l+1}^{u} a_i$$

$$\sum_{i=l}^{u} a_i = a_u + \sum_{i=l}^{u-1} a_i$$

Multiplicative constants… Pulling in/out a constant

$$k\sum_{i=l}^{u} a_i = \sum_{i=l}^{u} k a_i$$

Separating a sum

$$\sum_{i=l}^{u} (a_i + b_i) = \sum_{i=l}^{u} a_i + \sum_{i=l}^{u} b_i$$

Examples

$$\sum_{i=0}^{10} (3i + 2) = \sum_{i=0}^{10} 3i + \sum_{i=0}^{10} 2 = 3\sum_{i=0}^{10} i + 2\sum_{i=0}^{10} 1 = 3 \cdot 55 + 2 \cdot 11 = 187$$

$$\sum_{i=0}^{10} 1 = 11$$

$$\sum_{i=0}^{10} i = (0 + 10)\left(\frac{11}{2}\right) = 55$$

Linear shift of index variable. We introduce a new index variable j that is offset from i by some integer amount j = i+k (use in the limits) or i =j-k (use in the formula)

$$\sum_{i=l}^{u} a_i = \sum_{j=l+k}^{u+k} a_{j-k}$$

Question: What problem do we run into if we try to do a transformation like j=2i?

Example:

$$\sum_{i=10}^{100}(2i+1) = \sum_{j=1}^{91}(2(j+9)+1) \quad \text{where} \quad j = i-9 \quad \text{and} \quad i = j+9$$

$$= \sum_{j=1}^{91}(2j+19)$$

$$= \sum_{j=1}^{91} 2j + \sum_{j=1}^{91} 19$$

$$= 2\sum_{j=1}^{91} j + 19\sum_{j=1}^{91} 1$$

$$= 2\left(\frac{(91+1)91}{2}\right) + 19(91)$$

$$= (92)91 + 19(91)$$

$$= (111)91$$

Alternate solution using our trick:

$$\sum_{i=10}^{100}(2i+1) = 21 + 23 + 25 + \ldots + 197 + 199 + 201$$

we pair them up and get 21+201= 222

The number of values is 100-10+1 = 91, so the number of pairs is 91/2

The product is $(222)\frac{91}{2} = (111)91$ which is the same!

Example:

$$\sum_{i=5}^{20} 2^i = \sum_{j=1}^{16} 2^{j+4} \text{ where } j = i - 4 \text{ and } i = j + 4$$
$$= \sum_{j=1}^{16} 2^4 2^j$$
$$= 2^4 \sum_{j=1}^{16} 2^j$$
$$= 2^4 \left(2^{17} - 2\right)$$
$$= 2^5 \left(2^{16} - 1\right)$$

Alternate solution where we pull off the first four terms of a larger summation. This is cleaner than a transformation.

$$\sum_{i=1}^{20} 2^i = \sum_{i=1}^{4} 2^i + \sum_{i=5}^{20} 2^i \text{ which can be rearranged to give}$$
$$\sum_{i=5}^{20} 2^i = \sum_{i=1}^{20} 2^i - \sum_{i=1}^{4} 2^i$$
$$= \left(2^{21} - 2\right) - \left(2^5 - 2\right)$$
$$= \left(2^{21} - 2^5\right)$$
$$= 2^5 \left(2^{16} - 1\right)$$

Example: In some cases we can use a linear shift to simplify the expression inside the summation.

$$\sum_{i=1}^{n} (i+1)^4$$

We can do this sum by multiplying out the expression or we can use a linear shift. I want a new index variable j = i+1. The transformation results in:

$$\sum_{j=1+1}^{n+1} (j)^4 = \sum_{j=2}^{n+1} j^4$$

Closed form expressions for some common summations. You are required to memorize the highlighted formulas for the exam.

$$\sum_{i=1}^{n} 1 = n$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$ (Arithmetic)

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

$$\sum_{i=0}^{n} r^i = \frac{(r^{n+1}-1)}{r-1}$$ (Geometric)

$$\sum_{i=1}^{n} \frac{1}{i} = 1 + 1/2 + 1/3 + 1/4 + \ldots + 1/n \approx \ln(n) + 0.577$$ (Harmonic)

Nested summations.
You can treat the other variable as a constant factor and pull it out of an inner summation

$$\sum_{i=1}^{n}\sum_{j=1}^{m}(3i+j) = \sum_{i=1}^{n}\left(\sum_{j=1}^{m}(3i)+\sum_{j=1}^{m}(j)\right) = \sum_{i=1}^{n}\sum_{j=1}^{m}(3i)+\sum_{i=1}^{n}\sum_{j=1}^{m}(j)$$

$$= \sum_{i=1}^{n}\left(3i\sum_{j=1}^{m}1\right)+\sum_{i=1}^{n}\sum_{j=1}^{m}(j)$$

$$= \sum_{i=1}^{n}(3im)+\sum_{i=1}^{n}\frac{(m+1)m}{2}$$

$$= 3m\sum_{i=1}^{n}i+\frac{(m+1)m}{2}\sum_{i=1}^{n}1$$

$$= 3m\frac{(n+1)n}{2}+\frac{(m+1)m}{2}n$$

$$= 3m\frac{(n+1)n}{2}+\frac{(m+1)m}{2}n$$

$$\sum_{i=1}^{n}\sum_{j=1}^{m}ij = \sum_{i=1}^{n}\left(i\sum_{j=1}^{m}j\right)$$ note that this expression is not the same as $\left(\sum_{i=1}^{n}i\right)\left(\sum_{j=1}^{m}j\right)$

$$= \sum_{i=1}^{n}\left(i\frac{(m+1)m}{2}\right)$$

$$= \frac{(m+1)m}{2}\sum_{i=1}^{n}(i)$$

$$= \frac{(m+1)m}{2}\frac{(n+1)n}{2}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{i}ij = \sum_{i=1}^{n}\left(i\sum_{j=1}^{i}j\right)$$

$$= \sum_{i=1}^{n}\left(i\frac{(i+1)i}{2}\right)$$

$$= \sum_{i=1}^{n}\frac{(i^3+i^2)}{2}$$

$$= \frac{1}{2}\sum_{i=1}^{n}(i^3+i^2)$$

$$= \frac{1}{2}\sum_{i=1}^{n}i^3+\frac{1}{2}\sum_{i=1}^{n}i^2$$

Example: The twelve days of Christmas came and went. Gifts were given! But how many total? The following expression tells us the answer.

$$Gifts = \sum_{i=1}^{1} i + \sum_{i=1}^{2} i + \sum_{i=1}^{3} i + \sum_{i=1}^{4} i + \sum_{i=1}^{5} i + \sum_{i=1}^{6} i + \ldots + \sum_{i=1}^{12} i$$

But we can combine these into a double summation

$$Gifts = \sum_{k=1}^{12} \sum_{i=1}^{k} i$$

Now we can use the common summations to get the value. The inner sum can be replaced by

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$$

Giving

$$Gifts = \sum_{k=1}^{12} \frac{k(k+1)}{2}$$

Multiply out the formula and then separate the sum

$$Gifts = \sum_{k=1}^{12} \frac{k^2+k}{2}$$

$$= \sum_{k=1}^{12} \frac{k^2}{2} + \sum_{k=1}^{12} \frac{k}{2}$$

Pull out the constant factor

$$= \frac{1}{2}\sum_{k=1}^{12} k^2 + \frac{1}{2}\sum_{k=1}^{12} k$$

and then apply the common summations

Given $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

We see that

$$\sum_{k=1}^{12} k = \frac{12(13)}{2} = 6(13) = 78$$

$$\sum_{k=1}^{12} k^2 = \frac{12(13)(25)}{6} = 2(13)(25) = 650$$

So we have

$$Gifts = \frac{1}{2}78 + \frac{1}{2}650 = 364$$

Which coincidentally is close to the number of days in a year… Looks like someone stole a gift.