Fleck Chapter 2 (Prepositional Logic)

Problem 1 Given that Contains(x,y) = "the word x contains the letter y" where x is a word in the English language and y is an alphabetic character, what is the truth value of the following prepositional formulas?

Contains("Hello", "a")
Contains("Hello", "e")
$\exists x \in S.Contains(x,"a")$

where S={"joy", "luck", "club"}

$\forall x \in T.(Contains(x,"e") \vee Contains(x,"i"))$

where T={"bye", "miss", "american", "pie"}

Contains("Hello", "a")
is False

Contains("Hello", "e")
Is True

$\exists x \in S.Contains(x,"a")$

where S={"joy", "luck", "club"}
is false, nothing in S contains an a.

$\forall x \in T.(Contains(x,"e") \vee Contains(x,"i"))$

where T={"bye", "miss", "american", "pie"}
is True. Each word in T contains an i or an e or both.

Problem 2 What is the truth value of the following formulae. Give examples/counterexamples as needed.

a) $\forall x \in Z.\left[x^2 > x\right]$

b) $\forall x, y \in R.\left[\left|x+y\right| \leq \left|x\right| + \left|y\right|\right]$

c) $\forall x, \exists y \in R.\left[x + y = 0\right]$

d) $\exists x, \forall y \in R.\left[x + y = 0\right]$

a) $\forall x \in Z.\left[x^2 > x\right]$

This is false. A counter example is x=0.

b) $\forall x, y \in R.\left[\left|x+y\right| \leq \left|x\right| + \left|y\right|\right]$

This is true. It is called the triangle inequality.

c) $\forall x, \exists y \in R.\left[x + y = 0\right]$

This is true. Given any x, we can find a y to make x + y = 0. The value is y=-x

d) $\exists x, \forall y \in R.\left[x + y = 0\right]$

This is false. Suppose it is true, then there is some x=a for which it is true.

My counter example is y = 1-a.

x+y = a + (1-a) = 1 is not 0

Problem 3 Show an assignment of truth values to the predicate P(x,y) where x, and y are elements of the set {a, b, c} where the statements evaluate to different truth values.

$\forall x, \exists y, P(x,y)$ and $\exists y, \forall x, P(x,y)$

P(a,a) is True	P(b,a) is False	P(c,a) is False
P(a,b) is False	P(b,b) is True	P(c,b) is False
P(a,c) is False	P(b,c) is False	P(c,c) is True

$\forall x, \exists y, P(x,y)$ requires that the assignment of truth values for P in our table must have a true value in each column. For the given assignment this is true.
$\exists y, \forall x, P(x,y)$ requires that the assignment of truth values for P in our table must have some row that is all true. For the given assignment this is false

Clearly there are some situations where the quantified formulas would agree (e.g. if all the value in the table are true), but in general they are different.

Problem 3 Give a logically equivalent formula to each of the following where negations do not apply to compound formulae.

a) $\neg \forall x.P(x)$

b) $\neg \forall x.P(x) \vee \exists x.P(x)$

c) $\neg \forall x, \exists y, L(x,y)$

d) $\neg \forall x, \forall y, \left(L(x,y) \vee \neg Q(y)\right)$

a) $\neg \forall x.P(x)$

$\equiv \exists x.\neg P(x)$

b) $\neg \forall x.P(x) \vee \exists x.P(x)$

$\equiv \exists x.\neg P(x) \vee \exists x.P(x)$ the negation only applies to the for all.

c) $\neg \forall x, \exists y, L(x,y)$

$\equiv \exists x, \neg \exists y, L(x,y)$ push over the first quantifier

$\equiv \exists x, \forall y, \neg L(x,y)$ push over the second quantifier

d) $\neg \forall x, \forall y, \left(L(x,y) \vee \neg Q(y)\right)$

$\equiv \exists x, \neg \forall y, \left(L(x,y) \vee \neg Q(y)\right)$

$\equiv \exists x, \exists y, \neg\left(L(x,y) \vee \neg Q(y)\right)$

$\equiv \exists x, \exists y, \left(\neg L(x,y) \wedge \neg\neg Q(y)\right)$ push over the formula inside the quantifier with DeMorgan

$\equiv \exists x, \exists y, \left(\neg L(x,y) \wedge Q(y)\right)$ use double negation

Problem 4 Given the following predicates where domains are people.

Likes(x,y) = "x likes y"
Drives(x) = "x drives a car"
City(x) = "x lives in the city"

Translate the following formulae into English statements.

a) $\forall x, (Likes(x, Carol) \rightarrow Drives(x))$

b) $\exists x, (Drives(x) \wedge City(x))$

c) $\neg \exists y, \forall x, Likes(x, y)$

d) $\forall x, \exists y, (Likes(x, y) \vee Drives(x))$

a) $\forall x, (Likes(x, Carol) \rightarrow Drives(x))$

"For all people, if they like Carol then they drive."

b) $\exists x, (Drives(x) \wedge City(x))$

"There exists some person that either drives a car or lives in a city."

c) $\neg \exists y, \forall x, Likes(x, y)$

"It isn't the case there they is a person y such that for all people x, that x likes y. " This reads better if you push the negation inside the quantifiers.

d) $\forall x, \exists y, (Likes(x, y) \vee Drives(x))$

"For every person x, there is some person y such that either x likes y or x likes to drive."

Note: When we quantify with an existential, we can pick any value in the set. This includes values that have been chosen before. So if we have $\exists x, \exists y, (Likes(x, y))$ this would be true if Likes(a,a) is true. x and y get the same value. If we want the values to be different we would add in a condition to mask out all cases where the values are the same.

$$\exists x, \exists y, ((x \neq y) \wedge Likes(x, y))$$

By the way, an implication would not work

$$\exists x, \exists y, ((x \neq y) \rightarrow Likes(x, y))$$

This would be true for any domain with two or more values. If we can find any x,y pair that are not equal the implication is true and we have our values.

Problem 5 In the following formula indicate for each variable if it is bound or free. If it is bound indicate which quantifier it is bound to.

$$\exists z\Big(\big(\forall x,\big(P(x,y)\big)\wedge\exists x,y,S(x,y)\big)\vee\forall y,Q(x,y,z)\Big)$$

$$\underbrace{\exists z\left(\left(\underbrace{\forall x,\left(P(x,\overbrace{y}^{free})\right)}_{x\text{ bound}}\wedge\underbrace{\exists x,y,S(x,y)}_{\text{x,y bound}}\right)\vee\underbrace{\forall y,Q(\overbrace{x}^{free},y,z)}_{\text{y bound}}\right)}_{z\text{ bound}}$$

Problem 6 What is the truth value of each of the following quantified formulas in predicate logic?

a) $\forall x\in N.3x>2x$

b) $\forall x,y\in N.x+y=y+x$

c) $\forall x\exists y\in R.xy=1$

d) $\forall x\exists y\in R.x\neq 0\rightarrow xy=1$

e) $\exists y\forall x\in R.x\neq 0\rightarrow xy=1$

a) $\forall x\in N.3x>2x$ We can manipulate the inequality to get an equivalent formula by subtracting 2x from both side.

$$\forall x\in N.x>0$$

To decide if this is true we need to show that it is true for every value in the set of natural numbers. It is true except for x=0, but that one value is enough to make the universally quantified statement false.

b) $\forall x,y\in N.x+y=y+x$ We can manipulate the equality to get an equivalent formula by subtracting y from both sides of the equation.

$$\forall x,y\in N.x=x$$

This is clearly true for all values of x and y.

c) $\forall x\exists y\in R.xy=1$ This says that for every value there is another value such that when they are multiplied, the result is 1. But if x=0, we cannot find a value of y where the product is 1. This is false.

d) $\forall x \exists y \in R . x \neq 0 \rightarrow xy = 1$ This is similar to the previous formula, except that the implication will automatically be true if x=0. So for every other value, we need to show that there is a multiplicative inverse. Solving, we see that y = 1/x exists for those values.
This is true

e) $\exists y \forall x \in R . x \neq 0 \rightarrow xy = 1$ We are looking for a value of y, such that for every value of x not equal to zero, x times y is equal to 1. Suppose there is such a value of y. We know that y is not zero. Lets take two ***different*** values for x, call them a and b.
We know that ay=1 and by=1 and therefore ay=by. Divide both sides by y and we have a=b. But this is a problem, because we chose a and b to be different. So there cannot be any value of y that works. This is false.

Note: Our proofs were relatively easy to prove or disprove. In general though, it is more complicated. Unlike with a propositional formula, we cannot list out all the values from an infinite set. As a consequence, there are statements that we believe are true, but they have not been shown to be true. Showing a formula is true for any finite number of values is not sufficient. Even worse, the mathematician Godel showed that in any sufficiently complicated theory there are true statements that cannot be proven true. It doesn't take much, addition and multiplication with natural numbers is enough.