Discrete Math Homework 3
Solution

Question A) Rosen 1.1 Exercise 28 (p. 15)

Give the converse, contrapositive, and inverse of each of these statements:

Implication: $p \to q$

Converse: $q \to p$

Contrapositive: $\neg q \to \neg p$

Inverse: $\neg p \to \neg q$

In each statement, the premise (p) is marked in green and the conclusion (q) is marked in blue.

a) If it snows tonight, then I will stay home

Converse: $q \to p$ If I stay home, then it will snow tonight

Contrapositive: $\neg q \to \neg p$ If I don't stay home, then it will not snow tonight

Inverse: $\neg p \to \neg q$ If it does not snow tonight, then I will not stay home.

b) I go the beach whenever it is a sunny summer day.

Converse: $q \to p$ If I go to the beach, then it is a sunny summer day

Contrapositive: $\neg q \to \neg p$ If I don't go to the beach, then it is not a sunny summer day.

Inverse: $\neg p \to \neg q$ If it is not sunny summer day, then I won't go to the beach

c) When I stay up late it is necessary that I sleep until noon

Converse: $q \to p$ If I stay up late, then I sleep until noon

Contrapositive: $\neg q \to \neg p$ If I don't stay up late, then I don't sleep until noon

Inverse: $\neg p \to \neg q$ If I don't sleep until noon, then I don't stay up late.

Question B) Rosen 1.1 Exercise 30 (p. 15).

How many rows appear in a truth table for each of the following compound propositions?

Count the number of propositional variables n, and then compute 2 to n.

a) $(q \rightarrow \neg p) \vee (\neg p \rightarrow \neg q)$ p, q – 2 variables, 2^2=4 rows

b) $(p \vee \neg t) \wedge (p \vee \neg s)$ p, t, s – 3 variables, 2^3=8 rows (Side note: This is in conjunctive normal form and would be an instance of 2-Satisfiability)

c) $(p \rightarrow r) \vee (\neg s \rightarrow \neg t) \vee (\neg u \rightarrow v)$ p, r, s, t, u, v – 6 variables, 2^6=64 rows

d) $(p \wedge r \wedge s) \vee (q \wedge t) \vee (r \wedge \neg t)$ p, r, s, q, t – 5 variables, 2^5=32 rows (Side note: This is in disjunctive normal form. It is trivial to decide if this is satisfiable… It is… For example, p, r, and s = true will work.)

Rosen section 1.3.

Question C) Use a truth table to verify the logical equivalence

$$(p \rightarrow q) \vee r \equiv p \rightarrow (q \vee r)$$

p	q	r	$(p \rightarrow q)$		$q \vee r$	
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	F	T	T	T
T	F	F	F	F	F	F
F	T	T	T	T	T	T
F	T	F	T	T	T	T
F	F	T	T	T	T	T
F	F	F	T	T	F	T

The two columns in the truth table have the same values so the logical equivalence holds.

Question D) Use the logical equivalences in Table 6 and $p \rightarrow q \equiv \neg p \vee q$ to show that the following is true

$$(p \rightarrow q) \vee r \equiv p \rightarrow (q \vee r)$$

We replace equivalent by equivalent.

LHS: $(p \to q) \vee r$

$\equiv (\neg p \vee q) \vee r$ by def of implication

by associativity

RHS: $p \to (q \vee r)$

by def of implication

Both sides are equivalent to the same propositional formula marked in blue.

Question E) Use the logical equivalences in Table 6 and $p \to q \equiv \neg p \vee q$ to show that the following is true

$$(T \to (q \vee \neg r)) \wedge \neg(p \to F) \equiv (r \to q) \wedge p$$

We replace equivalent by equivalent.

LHS: $(T \to (q \vee \neg r)) \wedge \neg(p \to F)$

$(\neg T \vee (q \vee \neg r)) \wedge \neg(\neg p \vee F)$ by def of implication (twice)

$(F \vee (q \vee \neg r)) \wedge \neg(\neg p \vee F)$ by def of negation

$(q \vee \neg r) \wedge \neg(\neg p)$ by identity (twice)

by double negation

RHS: $(r \to q) \wedge p$

$(\neg r \vee q) \wedge p$ by def of implication

by commutativity

Both sides are equivalent to the same propositional formula marked in blue

Question F) Rosen 1.3 Exercise 8 c, d (p. 35).

Use De Morgan's law to find the negation of each of the statements.

Each of the propositions is noted, then the negation is written.
c) James is young and strong - James is young, James is strong
James is not young or James is not strong. We would also accept James is not young or not strong.

d) Rita will move to Oregon or Washington – Rita will move to Oregon, Rita will move to Washington.
Rita will not move to Oregon and Rita will not move to Washington.

Question G) Rosen 1.3 Exercise 10 a, d (p. 35).

Show that each compound proposition is a tautology using a truth table.

a) $(\neg p \wedge (p \vee q)) \rightarrow q$

p	q	$(p \vee q)$	$\neg p$	$(\neg p \wedge (p \vee q))$	
T	T	T	F	F	T
T	F	T	F	F	T
F	T	T	T	T	T
F	F	F	T	F	T

All rows are true, so it is a tautology

d) $[(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)] \rightarrow r$

p	q	r	$(p \vee q)$	$(p \rightarrow r)$	$(q \rightarrow r)$	$[(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)]$	
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	T	T	T	T	T
T	F	F	T	F	T	F	T
F	T	T	T	T	T	T	T
F	T	F	T	T	F	F	T
F	F	T	F	T	T	F	T
F	F	F	F	T	T	F	T

All rows are true, so it is a tautology

Rosen section 1.4.

Question H) Rosen 1.4 Exercise 8 a, b (p. 53)

Translate these statements into English where R(x) is "x is a rabbit" and H(x) is "x hops" and the domain is all animals.

a) $\forall x \left(R(x) \rightarrow H(x)\right)$

All rabbits hop

b) $\forall x \left(R(x) \wedge H(x)\right)$

All animals are rabbits and hop

Question I) Rosen 1.4 Exercise 2 (p. 53)

Let P(x) denote the statement "the word x contains the letter a". What is the truth value of the following

a) P(orange) "the word orange contains the letter a" - True
b) P(lemon) "the word lemon contains the letter a" - False
c) P(true) "the word true contains the letter a" - False
d) P(false) "the word false contains the letter a" - True

Question J) Rosen 1.4 Exercise 6 a, e, f (p. 53)

Let N(x) be the statement "x has visited North Dakota" where the domain consists of students in your school. Express these in English.

a) $\exists x.N(x)$ Some student in your school has visited North Dakota.
e) $\neg\forall x.N(x)$ Not all of the students in your school have visited North Dakota.
f) $\forall x.\neg N(x)$ Every student in your school has not visited North Dakota

You may choose to solve one (and only one) of the following Extra Credit Problems. If you submit more than one, only the first will be graded.

Extra Credit 1) Prove or disprove that the following two compound propositions are logically equivalent.

$$\left(p \rightarrow (q \vee \neg r)\right) \wedge q$$

$$(q \wedge s) \vee \left((p \leftrightarrow q) \oplus \neg p\right)$$

We have 4 propositional variables, so we need 16 rows.

P	Q	R	S	$(q \vee \neg r)$	$(p \to (q \vee \neg r))$	
T	T	T	T	T	T	T
T	T	T	F	T	T	T
T	T	F	T	T	T	T
T	T	F	F	T	T	T
T	F	T	T	F	F	F
T	F	T	F	F	F	F
T	F	F	T	T	T	F
T	F	F	F	T	T	F
F	T	T	T	T	T	T
F	T	T	F	T	T	T
F	T	F	T	T	T	T
F	T	F	F	T	T	T
F	F	T	T	F	T	F
F	F	T	F	F	T	F
F	F	F	T	T	T	F
F	F	F	F	T	T	F

P	Q	R	S	$(q \wedge s)$	$p \leftrightarrow q$	$\neg p$	$(p \leftrightarrow q) \oplus \neg p$	
T	T	T	T	T	T	F	T	T
T	T	T	F	F	T	F	T	T
T	T	F	T	T	T	F	T	T
T	T	F	F	F	T	F	T	T
T	F	T	T	F	F	F	F	F
T	F	T	F	F	F	F	F	F
T	F	F	T	F	F	F	F	F
T	F	F	F	F	F	F	F	F
F	T	T	T	T	F	T	T	T
F	T	T	F	F	F	T	T	T
F	T	F	T	T	F	T	T	T
F	T	F	F	F	F	T	T	T
F	F	T	T	F	T	T	F	F
F	F	T	F	F	T	T	F	F
F	F	F	T	F	T	T	F	F
F	F	F	F	F	T	T	F	F

When we compare the two columns, we see that they are the same for all rows.

Extra Credit 2) Rosen 1.2 Exercise 46 (p. 16) - Hint: See problem 45 for the meaning of Not in this particular Fuzzy Logic system.

The truth value of a conjunction of two propositions in fuzzy logic is the minimum of the truth values of the two propositions. What are the truth values of the statements "Fred and John are happy" and "Neither Fred nor John is happy"?

Known truth values:

"Fred is happy" is 0.8
"John is happy" is 0.4

"Fred and John are happy" has the truth value Minimum(0.8, 0.4) = 0.4
"Neither Fred nor John is happy" is the same as "Fred is not happy and John is not happy" has the truth value Minimum(1-0.8, 1-0.4) = Minimum(0.2, 0.6) = 0.2

Extra Credit 3) Rosen 1.3 Exercise 50 (p. 36) – Hint: Use the result of question 45 to show part c.

We are going to show that the Pierce arrow is functionally complete collection of logical operators. (We can express any possible compound proposition using just the Pierce arrow.)

a) Show that $p \downarrow p$ is logically equivalent to $\neg p$.

We are given that $p \downarrow q$ is true when both p and q are false. It is false other wise

p	q	$p \downarrow q$
T	T	F
T	F	F
F	T	F
F	F	T

p		
T	F	F
F	T	T

So we see that these are logically equivalent.

b) Show that $(p \downarrow q) \downarrow (p \downarrow q)$ is logically equivalent to $p \vee q$.

p	q	$p \downarrow q$	$p \downarrow q$		
T	T	F	F	T	T
T	F	F	F	T	T
F	T	F	F	T	T
F	F	T	T	F	F

So we see that these are logically equivalent.

c) Conclude from parts (a) and (b) and Exercise 49 that Pierce is functionally complete set of operators.

From (a) $p \downarrow p \equiv \neg p$

From (b) $(p \downarrow q) \downarrow (p \downarrow q) \equiv p \vee q$

From 45 we know that that NOT and OR form a functionally complete collection of logical operators. This means that we can express any compound proposition using just propositional variables and the connectives NOT and OR. To show that the Pierce arrow is complete, we first express the compound proposition using just NOR and OR. We can then replace every OR using the logical equivalence from part (b). We can replace every NOT using the logical equivalence from part (a). This leaves a formula with just the Pierce arrow. QED. Any formula can be expressed with just a Pierce arrow.