Fleck Chapter 6 – Relations

Determine if each of the following relations is reflexive, irreflexive, symmetric, anti-symmetric and transitive. Based on the results is the relation a partial order? Is the relation a total order? Is the relation an equivalence relation?

Problem 1) R is the relation over strings of characters where (s,t) is in the relationship provided that every letter in s is also in t.
EG. s="cat" and t= "cantaloupe" is in the relation, s= "cat" and t="antelope" is not in the relation.

Reflexive: $\forall s.\ \boldsymbol{sRs}$

True. Every letter in s is also in s. ✓

Irreflexive: $\forall s.\ \neg\ \boldsymbol{sRs}$

False. s="cat" is in the relation.

Symmetric: $\forall s,t.\ \boldsymbol{sRt \rightarrow tRs}$

False. Counter example cat R cantaloupe, but not cantaloupe R cat.

Anti-Symmetric: For all pairs of unique s,t, at most one of (s,t) and (t,s) is in the relation. (Another way of saying this is that if sRt and sRt are both true, then s=t.)
Consider s="trap" and t="part"

"trap" R "part" is true since every letter in "trap" is also in "part"

"part" R "trap" is true since every letter in "part" is also in "trap"

The relation is not anti-symmetric.

Transitive:

$$\forall a,b,c.\ (aRb\ \wedge\ bRc) \rightarrow aRc$$

True. We know that every letter in a is also in b, and every letter in b is also in c, therefore we know that every letter in a is also in c. ✓

For this to be a **partial order**, it must be reflexive, anti-symmetric and transitive. It only meets two of the conditions. This is not a partial order.

For this to be a **total order**, it must be a partial order where every pair of elements is ordered. This is not a total order.

For this to be an **equivalence relation**, it must be Reflexive, Symmetric, and Transitive, but it fails on Symmetry. This is not an equivalence relation.

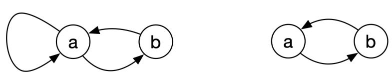
To prove **reflexive** – every value must be related to itself.
To prove **irreflexive** – every value must not be related to itself.

To prove **symmetry** - $\forall(a, b)$. $aRb \rightarrow bRa$. Between any pair of values a, b if there is an arrow in one direction, there must also be an arrow in the other direction.

> We can safely make the assumption when proving symmetry that a and b are unique. This is because the presence or absence of loops does not affect symmetry. If a=b, then we need to show $aRa \rightarrow aRa$ which is always true.

To prove **anti-symmetry**, - $\forall(a, b)$. aRb and $bRa \rightarrow a = b$. Between any pair of values a, b, there can be at most one arrow.

> Again, the presence or absence of loops does not affect anti-symmetry.

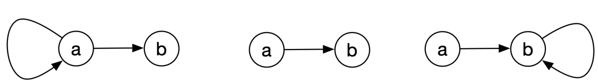


Both Symmetric

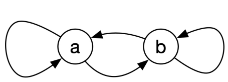
Both Anti-Symmetric

To prove **transitivity**, - $\forall(a, b, c)$. aRb and $bRc \rightarrow aRc$.

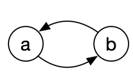
> When showing this, we can safely assume that a and b are different. If they are the same, then we need to show that
>
> $\forall(a, c)$. aRa and $aRc \rightarrow aRc$, but this is trivially true.
>
> Similarly, we can assume that b and c are different. If they are the same, then we need to show that
>
> $\forall(a, c)$. aRc and $cRc \rightarrow aRc$, which is also trivially true.
>
> We cannot assume that a and c are different.
>
> - $\forall(a, c)$. aRc and $cRa \rightarrow aRa$.



All Transitive



Transitive



Not Transitive

Problem 2) R is the relation over integers where (a,b) is in the relation provided that a=2b.
EG. a=2 and b=1 is in the relation, but a= 1 b=3 is not in the relation.

Reflexive: $\forall a. aRa$

False. Consider a=2, $\mathbf{2 \neq 2(2)}$. In fact, the only value for which this holds is a=0.

Irreflexive: $\forall \boldsymbol{a}. \neg \boldsymbol{aRa}$

False. (0,0) is in the relation.

Symmetric: $\forall \boldsymbol{a}, \boldsymbol{b}. \boldsymbol{aRb} \rightarrow \boldsymbol{bRa}$

False. Counter example 2 R 1, but not 1 R 2.

Anti-Symmetric: For all pairs of unique a,b, at most one of (a,b) and (b,a) is in the relation. Suppose that we have a pair of values where both are in the relation. This means that a=2b and b=2a. Solving we see that a = 4a which only has one possible solution with a=0 and b=0. Therefore, for every pair of distinct values at most one of (a,b) and (b,a) is in the relation.

The relation is anti-symmetric ✓

Transitive: $\forall a,b,c. (aRb \wedge bRc) \rightarrow aRc$

False. (4,2) and (2,1) are in the relation, but (4,1) is not in the relation.

For this to be a **partial order**, it must be reflexive, anti-symmetric and transitive. It only meets one of the conditions. This is not a partial order.

For this to be a **total order**, it must be a partial order where every pair of elements is ordered. This is not a total order.

For this to be an **equivalence relation**, it must be Reflexive, Symmetric, and Transitive, but it fails on all three. This is not an equivalence relation.

Problem 3) R is the relation over positive integers where (a,b) is in the relation provided that a is a factor of b.
EG. a=2 and b=8 is in the relation, but a= 3 b=7 is not in the relation.

Reflexive: $\forall a.\ aRa$

True. Every positive integer is a factor of itself. ✓

Irreflexive: $\forall a. \neg aRa$

False. (1,1) is in the relation.

Symmetric: $\forall a, b. aRb \rightarrow bRa$

False. Counter example 3 R 9, but not 9 R 3. (Three is a factor of nine, but not vice-verse.

Anti-Symmetric: For all pairs of unique a,b, at most one of (a,b) and (b,a) is in the relation. Suppose that we have a pair of values where both are in the relation. This means that a is a factor of b and b is a factor of a. For a to be a factor of b, b=am for some positive integer m. Similarly, a=bn for some integer n. Putting these together we have b=mnb and mn must be 1. The only positive integers which can satify this equation are m=1 and m=1. But then a=b. Therefore, for every pair of distinct values at most one of (a,b) and (b,a) is in the relation. The relation is anti-symmetric ✓

Transitive: $\forall a,b,c.\ (aRb \wedge bRc) \rightarrow aRc$

True. If a is a factor of b and b is a factor of c, then a is also a factor of c.

b=ma for some positive integer m, and c=nb for some positive integer n. But then c=mna and mn is a positive integer, so a is a factor of c. ✓

For this to be a **partial order**, it must be reflexive, anti-symmetric and transitive. This is a partial order.

For this to be a **total order**, it must be a partial order where every pair of elements is ordered. Consider (3,7) and (7, 3). Neither one of these is in the relation. This is not a total order.

For this to be an **equivalence relation**, it must be Reflexive, Symmetric, and Transitive, but it fails on symmetry. This is not an equivalence relation.

Problem 4) R is the relation over finite sets where (s,t) is in the relation if s has the same cardinality as t.
EG. s={a, b} and t ={2, 5} is in the relation, but s={a, b, 2, x, y} and t ={1, 0, 2, 5} is not in the relation.

Reflexive: $\forall \boldsymbol{s}.\, \boldsymbol{sRs}$

True. Set s has the same cardinality as itself. ✓

Irreflexive: $\forall \boldsymbol{s}.\, \neg\, \boldsymbol{sRs}$

False. s={1, 2, 3} is in the relation.

Symmetric: $\forall \boldsymbol{s}, \boldsymbol{t}.\, \boldsymbol{sRt} \rightarrow \boldsymbol{tRs}$

True. If s has the same cardinality as t, then t has the same cardinality as s. ✓

Anti-Symmetric: For all pairs of unique s,t, at most one of (s,t) and (t,s) is in the relation. (Another way of saying this is that if sRt and sRt are both true, then s=t.)
Consider s={1, 2, 3} and t ={a, b, c}. Both orders are in the relation
The relation is not anti-symmetric.

Transitive:

True. If a has the same cardinality as b, and b has the same cardinality as c, then a must have the same cardinality as c. ✓

For this to be a **partial order**, it must be reflexive, anti-symmetric and transitive. This is not a partial order because it isn't anti-symmetric.

For this to be a **total order**, it must be a partial order. This is not a total order.

For this to be an **equivalence relation**, it must be Reflexive, Symmetric, and Transitive. This is an equivalence relation.

Note: If we have an equivalence order, it divides it's universe into disjoint equivalence classes. For this relation, the equivalence classes are associated with the cardinality of the set. While each of the sets is finite, there are an infinite number of equivalence classes.
Class 0: {}
Class 1: {1}, {2}, {a}, ..
Class 2: {1, b}, {love, 2}, …
Class 3: {1, 2, 3}, {fred, 2.4, x}, …