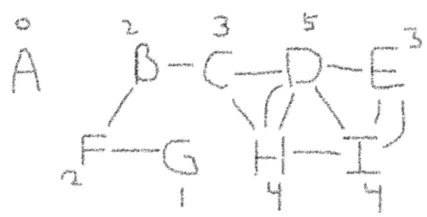
Building Blocks Homework 10

Solution

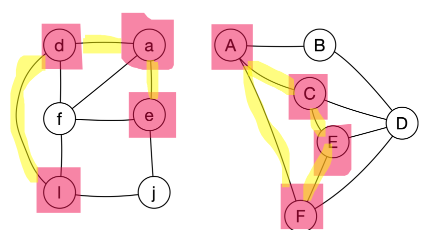
Question A) Can we construct a graph where the list of degrees is the following? Explain. If so, sketch such a graph. (You may use loops and multi-edges.)

a) {0, 2, 1, 2, 3, 5, 3, 4, 4}
b) {2, 4, 2, 4, 1, 2, 1, 2, 2, 1}

a) We have 4 nodes of odd degree. We can build a solution. It will have 9 vertices and (0+2+1+2+3+5+3+4+4)/2 = 12 edges
b) We have 3 nodes of odd degree. By the hand shaking theorem, no such graph exists.

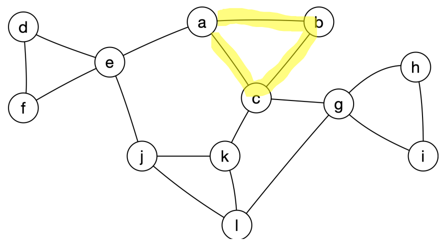


Question B) Show that the following graphs are not isomorphic.
(Hint: Look at the degrees of the nodes and subgraphs)

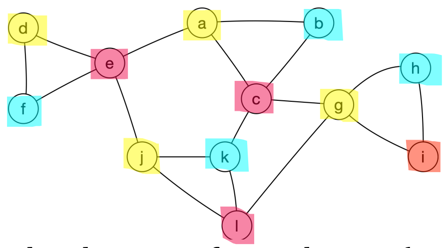


If we look at just the nodes that have degree 3 (Colored in red) and the edges (colored in yellow) we see that one subgraph has a cycle and the other does not. These two graphs are not isomorphic.

Question C) Show that the following graph cannot be 2 colored. Show that it can be 4 colored.



The a,b,c subgraph cannot be two colored. Once you determine a color for a, then b and c cannot be that color and they can not match. Three collors are needed.



The above is a four coloring. (In fact, it is also a 3 coloring.)

Question D) Use a proof by induction to show that

$0 + 2 + 4 + 6 + \cdots + 2n = n^2 + n$ for all natural numbers.

1) **P(n) is** "$0 + 2 + 4 + 6 + \cdots + 2n = n^2 + n$"**.**
2) $\boldsymbol{n_0}$ **is 0**
3) **Base case: Show P(0)**
   is "$0 = 0^2 + 0$" **or** "$0 = 0$" **is true**
4) **Assume P(k) is** "$0 + 2 + 4 + 6 + \cdots + 2\mathrm{k} = \mathrm{k}^2 + \mathrm{k}$" **is true.**
5) **Show P(k+1)**
   "$0 + 2 + 4 + 6 + \cdots + 2\mathrm{k}\ + 2(\mathrm{k} + 1) = (\mathrm{k} + 1)^2 + (\mathrm{k} + 1)$"
6) **Proof:**
   Start with what we are trying to show and notice that the terms in red are the same as the LHS of the induction hypothesis. By the IH, we can replaces the terms in red with the blue terms to get
   "$0 + 2 + 4 + 6 + \cdots + 2\mathrm{k}\ + 2(\mathrm{k} + 1) = (\mathrm{k} + 1)^2 + (\mathrm{k} + 1)$"
   "$\mathrm{k}^2 + \mathrm{k}\ + 2(\mathrm{k} + 1) = (\mathrm{k} + 1)^2 + (\mathrm{k} + 1)$"
   We can do mathematical manipulations to get
   $\mathrm{k}^2 + \mathrm{k}\ + 2\mathrm{k} + 2 = \mathrm{k}^2 + 2\mathrm{k} + 1\ + (\mathrm{k} + 1)$"
   $\mathrm{k}^2\ + 3\mathrm{k} + 2 = \mathrm{k}^2 + 3\mathrm{k} + 2$"
   Which is true!
7) **QED**

Question E) Use a proof by induction to show that

$2^1 + 2^2 + \cdots + 2^n = 2^{n+1} - 2$ for all n greater than zero.

1) **P(n) is** "$2^1 + 2^2 + \cdots + 2^n = 2^{n+1} - 2$"**.**
2) $\boldsymbol{n_0}$ **is 1**
3) **Base case: Show P(1)**
   is "$2^1 = 2^{1+1} - 2$" **or** "$2 = 4 - 2$" **which is true**
4) **Assume P(k) is** "$2^1 + 2^2 + \cdots + 2^k = 2^{k+1} - 2$ " **is true.**
5) **Show P(k+1)**
   "$2^1 + 2^2 + \cdots + 2^k + 2^{k+1} = 2^{k+1+1} - 2$ "
6) **Proof:**

Start with what we are trying to show and notice that the terms in red are the same as the LHS of the induction hypothesis. By the IH, we can replaces the terms in red with the blue terms to get

"$2^1 + 2^2 + \cdots + 2^k + 2^{k+1} = 2^{k+1+1} - 2$ "

"$2^{k+1} - 2 + 2^{k+1} = 2^{k+1+1} - 2$ "

We can do mathematical manipulations to get

"$2 \times 2^{k+1} - 2 = 2^{k+2} - 2$ "

"$2 \times 2^{k+1+1} - 2 = 2^{k+2} - 2$ "

Which is true!

7) **QED**

Question F) Use a proof by induction to show that $n^4 - n^2$ is divisible by 3 for all positive integers.

1) **P(n) is** "$n^4 - n^2$ is divisible by 3"**.**
2) $\boldsymbol{n_0}$ **is 1**
3) **Base case: Show P(1)**
   is "$1^4 - 1^2$ is divisible by 3"
   or "0 is divisible by 3" **which is true**
4) **Assume P(k)** "$k^4 - k^2$ is divisible by 3" **is true.**
5) **Show P(k+1)**
   "$(k+1)^4 - (k+1)^2$ is divisible by 3"
6) **Proof:**
   Start with what we are trying to show. First we multiply out the terms.
   "$(k^4 + 4k^3 + 6k^2 + 4k + 1) - (k^2 + 2k + 1)$ is divisible by 3"
   "$(k^4 + 4k^3 + 6k^2 + 4k + 1) + (-k^2 - 2k - 1)$ is divisible by 3"
   Notice that we can see the terms from the induction hypothesis which are marked in red. By the induction hypothesis we know that those are divisible by 3. If the remaining terms are also divisible by 3, then we are done.
   $(4k^3 + 6k^2 + 4k + 1) + (-2k - 1)$
   Combining like terms gives
   $(4k^3 + 6k^2 + 2k)$
   Lets mod this by 3 and see what we get
   $(4k^3 + 6k^2 + 2k)mod_3$
   $= (1 \times k^3 + 0 \times k^2 + 2k)mod_3$
   $= (k^3 + 2k)mod_3$
   Applying cases: kmod3= 0, 1, 2
   Case 0: $(0^3 + 0)mod_3$= 0 Check
   Case 1: $(1^3 + 2)mod_3$= $(3)mod_3 = 0$ Check
   Case 2: $(2^3 + 2 \times 2)mod_3$= $(12)mod_3 =$ 0 Check

   (We could have done another proof by induction here.)
7) QED

You may choose to solve one (and only one) of the following Extra Credit Problems. If you submit more than one, only the first will be graded.

Extra Credit 1)

Use a proof by induction to show that for any positive real value a, that $a^n < (a+1)^n$ for all positive integers.

1. P(n) is "$a^n < (a+1)^n$"
2. n_0 is 1
3. Show P(1) is true. "$a^1 < (a+1)^1$" or subtracting a from both sides 0<1 Check
4. Assume P(k) is true. "$a^k < (a+1)^k$ "
5. Show P(k+1) is true.
   a. "$a^{k+1} < (a+1)^{k+1}$ "
6. In this case, it is probably easier to start from the induction hypothesis.
   By IH we know $a^k < (a+1)^k$

Since a is positive we can multiply both sides by a to get

$$a^{k+1} < a(a+1)^k$$

This is close, but not quite there. I know that

$a < a+1$ from P(1) and I can multiply both sides by $(a+1)^k$.

Since a is positive, a+1 is also positive and I don't have the flip the inequality.

$$a(a+1)^k < (a+1)(a+1)^k$$

Or

$$a(a+1)^k < (a+1)^{k+1}$$

But now we can combine the inequalities marked in green to get

$$a^{k+1} < (a+1)^{k+1}$$

7. QED

Extra Credit 2)

a) Consider the proposition that $2^n > n^5$. Clearly it is not true for all natural numbers. What is the smallest value of n for which it is true?
b) Use a proof by induction to show that it is true for all integers greater than or equal to your answer from part a.

Lets take a log base 2 of both sides of the inequality and see if we can find an n that is nearly equal.

$$log_2 2^n > log_2 n^5$$

Or

$$\text{n} > 5log_2 n$$

We try values until we pin down a transition. So n=23

n	$5log_2 n$
10	16.6
15	19.5
20	21.6
25	23.2
21	21.9
22	22.3
23	22.6
24	22.9

P(n) is "$2^n > n^5$" for all n greater than or equal to 23

Base case: P(23) is

8,388,608 > 1,952,265 is true.

Assume P(k): "$2^k > k^5$" is true. k>=23

Show P(k+1) is true: $2^{k+1} > (k+1)^5$

We can multiply terms

$$2^k + 2^k > k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1$$

By the induction hypothesis we know that red terms satisfy the inequality, so if we can show the remaining terms also work, we are done. (k>= 23)

$$2^k > 5k^4 + 10k^3 + 10k^2 + 5k + 1$$

By the induction hypothesis I know that $2^k > k^5$

So if I can show that $k^5 > 5k^4 + 10k^3 + 10k^2 + 5k + 1$. I am done.

Lets divide both sides by k^4 (Safe because this is positive)

$$k > 5 + \frac{10}{k} + \frac{10}{k^2} + \frac{5}{k^3} + \frac{1}{k^4}$$

The function on the right is a decreasing function with a limiting value of 5. If this is true for k = 23 it will be true for larger values of k as well.

$$23 > 5 + \frac{10}{23} + \frac{10}{23^2} + \frac{5}{23^3} + \frac{1}{23^3} \approx 5.45 \text{ is true!}$$