# Recursive Functions

**Problem 1) What are the first 5 terms of the recursively defined sequence**

$$\boldsymbol{a_1 = 1}$$
$$\boldsymbol{a_n = a_{n-1} + 2n - 1}$$

$$\boldsymbol{a_1 = 1}$$
$$\boldsymbol{a_2 = a_{2-1} + 2(2) - 1 = a_1 + 3 = 1 + 3 = 4}$$
$$\boldsymbol{a_3 = a_{3-1} + 2(3) - 1 = a_2 + 5 = 4 + 5 = 9}$$
$$\boldsymbol{a_4 = a_{4-1} + 2(4) - 1 = a_3 + 7 = 9 + 7 = 16}$$
$$\boldsymbol{a_5 = a_{5-1} + 2(5) - 1 = a_4 + 9 = 16 + 9 = 25}$$

We replaced n by 2, 3, 4, and 5 to get each of the values after the base case.

**Problem 2) Make a guess for the closed form solution for the recursively defined sequence and then prove that it is correct.**

My guess is that $a_n = n^2$

Proof:
Base case: $a_1 = 1^2 = 1$ ✓

Recursive case:

Take $a_n = n^2$

$a_{n-1} = (n-1)^2$

And put them in

$$\boldsymbol{a_n = a_{n-1} + 2n - 1}$$

and verify that it is correct.

Show

$$n^2 = (n-1)^2 + \boldsymbol{2n - 1}$$

Or

$$n^2 = n^2 \boldsymbol{- 2n + 1 + 2n - 1}$$
$$n^2 = n^2 \checkmark$$

**Problem 3) What are the first 5 terms of the recursively defined sequence**

$$\boldsymbol{a_1 = 1}$$
$$\boldsymbol{a_n = 2a_{n-1} + 1}$$

$$\boldsymbol{a_1 = 1}$$
$$\boldsymbol{a_2 = 2a_{2-1} + 1 = 2a_1 + 1 = 2(1) + 1 = 3}$$
$$\boldsymbol{a_3 = 2a_{3-1} + 1 = 2a_2 + 1 = 2(3) + 1 = 7}$$
$$\boldsymbol{a_4 = 2a_{4-1} + 1 = 2a_3 + 1 = 2(7) + 1 = 15}$$
$$\boldsymbol{a_5 = 2a_{5-1} + 1 = 2a_4 + 1 = 2(15) + 1 = 31}$$

We replaced n by 2, 3, 4, and 5 to get each of the values after the base case.

**Problem 4) Make a guess for the closed form solution for the recursively defined sequence and then prove that it is correct.**

My guess is that $a_n = 2^n - 1$

Proof:
Base case: $a_1 = 2^1 - 1 = 1$ ✓

Recursive case:

Take $a_n = 2^n - 1$

$a_{n-1} = 2^{(n-1)} - 1$

And put them in

$\boldsymbol{a_n = 2a_{n-1} + 1}$

and verify that it is correct.

Show

$2^n - 1 = \mathbf{2}(\, 2^{(n-1)} - 1) + \mathbf{1}$

Or

$$2^n - 1 = \mathbf{2}(\, 2^{(n-1)} - 1) + \mathbf{1}$$
$$2^n - 1 = (2 \times \, 2^{(n-1)} - 2) + \mathbf{1}$$
$$2^n - 1 = (\, 2^{(n-1)+1} - 1)$$
$$2^n - 1 = (\, 2^n - 1) \checkmark$$

**Problem 5) What are the first 5 values of the recursively defined function**

$$\boldsymbol{T(1) = 7}$$
$$\boldsymbol{T(n) \ = \ 3T(n-1)+2}$$

$$\boldsymbol{T(1) = 7}$$
$$\boldsymbol{T(2) \ = \ 3T(2-1)+2 = 3T(1)+2 = 3(7)+2 = 23}$$
$$\boldsymbol{T(3) \ = \ 3T(3-1)+2 = 3T(2)+2 = 3(23)+2 = 71}$$
$$\boldsymbol{T(4) \ = \ 3T(4-1)+2 = 3T(3)+2 = 3(71)+2 = 215}$$
$$\boldsymbol{T(5) \ = \ 3T(5-1)+3 = 3T(4)+2 = 3(215)+2 = 647}$$

We replaced n by 2, 3, 4, and 5 to get each of the values after the base case.

**Problem 6) Make a guess for the closed form solution for the recursively defined sequence and then prove that it is correct.**

We repeatedly apply the recursive definition

$$\boldsymbol{T(n) \ = \ 3T(n-1)+2}$$
$$\boldsymbol{T(n) \ = \ 3[3T(n-2)+2]+2}$$
$$\boldsymbol{T(n) \ = \ 3[3[3T(n-3)+2]+2]+2}$$
$$\boldsymbol{T(n) \ = \ 3[3[3[3T(n-4)+2]+2]+2]+2}$$
$$\boldsymbol{\ldots}$$
$$\boldsymbol{T(n) \ = \ 3[3[3[3[3T(1)+2]...+2]+2]+2]+2}$$

Eventually we hit T(0) which is replaced by 7.

Each time we reduce the argument of T by 1, we get a factor of 3. When we hit T(0), there will be n-1 factors of 3. So part of my solution is $3^{n-1} \times 7$.

The other part of my solution is composed of a bunch of twos. Notice that as we get deeper in working from the right, the 2's get multiplied by a progressively larger number of 3's. The leftmost two picks up one fewer factor than the 7 did. So we get the sequence $(3^{n-2} \times 2) + (3^{n-3} \times 2) + (3^{n-4} \times 2) + \ldots + (3 \times 2) \ + \ (2)$.

This is the same as $2(1 + 3 + \ 3^2 \ + \ \ldots \ + 3^{n-3} + 3^{n-2})$ and the piece inside the parentheses is a geometric sequence.

$$2\left(\frac{3^{n-1}-1}{3-1}\right)=3^{n-1}-1$$

My guess is $T(n) \ = \ 3^{n-1} \times 7 + \ 3^{n-1} - 1$
Or finally

$$T(n) = 3^{n-1} \times 8 - 1$$

Proof:
Base case: $T(1) = 3^{1-1} \times 8 - 1 = 3^0 \times 8 - 1 = 7$ ✓

Recursive case:

Take

$T(n) = 3^{n-1} \times 8 - 1$

$T(n-1) = 3^{n-2} \times 8 - 1$

And put them in the recursive defintion

$\boldsymbol{T(n) = 3T(n-1) + 2}$

and verify that it is correct.

Show

$3^{n-1} \times 8 - 1 = \mathbf{3}(3^{n-2} \times 8 - 1) + \mathbf{2}$

$3^{n-1} \times 8 = \mathbf{3}(3^{n-2} \times 8 - 1) + \mathbf{3}$

$3^{n-1} \times 8 = (3^{n-1} \times 8 - 3) + \mathbf{3}$

$3^{n-1} \times 8 = (3^{n-1} \times 8)$

✓

**Even though we have a proof, Lets use the formula to compute T(5) and verify it matches what we did earlier.**

$$T(5) = 3^{5-1} \times 8 - 1 = 3^4 \times 8 - 1 = 81 \times 8 - 1 = 647$$

**Problem 7) What are the first 7 values of the recursively defined sequence**

$\boldsymbol{f_0 = 0}$
$\boldsymbol{f_1 = 1}$
$\boldsymbol{f_n = f_{n-1} + f_{n-2}}$

$\boldsymbol{f_0 = 0}$
$\boldsymbol{f_1 = 1}$
$f_2 = f_1 + f_0 = 0 + 1 = 1$
$f_3 = f_2 + f_1 = 1 + 1 = 2$
$f_4 = f_3 + f_2 = 1 + 2 = 3$
$f_5 = f_4 + f_3 = 2 + 3 = 5$
$f_6 = f_5 + f_4 = 3 + 5 = 8$
$f_7 = f_6 + f_5 = 5 + 8 = 13$
We replaced n by 2, 3, 4, 5, and 6 to get each of the values after the base cases.

**Problem 8) Use a proof by induction to show that $f_n > 2f_{n-2}$ for all n greater than 3.**

We will use strong as we may have two or more chains of logic that we want to guarantee will terminate. We will start with 2 base cases and add more if needed.

P(n) is "$f_n > 2f_{n-2}$"

**Base:**

Cases: n=4 and n=5

Show: P(4) is "$f_4 > 2f_{4-2}$" **or**

$f_4 > 2(f_2)$

$3 > 2(1)$ ✓

Show: P(5) is "$f_5 > 2f_{5-2}$" **or**

$f_5 > 2(f_3)$

$5 > 2(2)$ ✓

**Induction:**

Assume: P(5), P(6), … P(k) $k \geq 4$

Goal: P(k+1) is "$f_{k+1} > 2f_{k+1-2}$"

**Or**

$f_{k+1} > 2f_{k-1}$

**By the recursive definition of Fibonacci numbers, we know that.**

$f_{k+1} = f_k + f_{k-1}$ **or** $f_k = f_{k+1} - f_{k-1}$

$f_k = f_{k-1} + f_{k-2}$ **or** $f_{k-1} = f_k - f_{k-2}$

$f_{k-1} = f_{k-2} + f_{k-3}$ **or** $f_{k-2} = f_{k-1} - f_{k-3}$

**By the induction hypothesis P(k) we know**

$f_k > 2f_{k-2}$

**Use the recursive def on the LHS**

$f_{k+1} - f_{k-1} > 2f_{k-2}$

$f_{k+1} > f_{k-1} + 2f_{k-2}$

**Use the recursive def on the RHS**

$f_{k+1} > f_{k-1} + 2(f_{k-1} - f_{k-3})$

$f_{k+1} > 2f_{k-1} + (f_{k-1} - 2f_{k-3})$

**If $(f_{k-1} - 2f_{k-3}) > 0$ then we know the highlighted part is true also.**

$f_{k-1} > 2f_{k-3}$

**But this is P(k-1) which is in our list so true by IH.**

**QED**