Discrete Math Homework 9
Due Wednesday, March 15 at the beginning of class

General instructions:

- Use standard size paper (8.5 by 11).
- Answer each question in order using a single column.
- Be neat. If we cannot read your solution it is wrong.
- Show your work. If you just write an answer, you will get minimal credit even if the answer is correct.

Rosen section 5.1

Question A) Use a proof by induction to show that $2+4+6+\cdots+2n = n(n+1)$ for all positive integers.

1) P(n) is $2+4+6+\cdots+2n = n(n+1)$
2) n0 is 1
3) P(1) is $2(1) = 1(1+1)$

   or 2=2 **check**
4) Induction Hypothesis P(k) is

   $2+4+6+\cdots+2k = k(k+1)$
5) Show P(k+1)

   $2+4+6+\cdots+2(k+1) = (k+1)((k+1)+1)$

   Rewrite this a little bit

   $\underbrace{2+4+6+\cdots+2k}_{IH}+2(k+1) = (k+1)(k+2)$

   Applying the Induction Hypothesis

   $k(k+1)+2(k+1) = (k+1)(k+2)$

   Multiply out LHS and RHS

   $k^2+k+2k+2 = k^2+3k+2$

   $k^2+3k+2 = k^2+3k+2$

   check
6) QED

Question B) Rosen 5.1 Exercise 6 (p. 329)
Prove that $1\cdot 1!+2\cdot 2!+3\cdot 3!+\cdots+n\cdot n!=(n+1)!-1$ for all positive integer n.

1) P(n) is $1\cdot 1!+2\cdot 2!+3\cdot 3!+\cdots+n\cdot n!=(n+1)!-1$
2) n0 is 1
3) P(1) is $1\cdot 1!=(1+1)!-1$

$1=(2)!-1$

or 1=1 **check**

4) Induction Hypothesis P(k) is

$1\cdot 1!+2\cdot 2!+3\cdot 3!+\cdots+k\cdot k!=(k+1)!-1$

5) Show P(k+1)

$1\cdot 1!+2\cdot 2!+3\cdot 3!+\cdots+(k+1)\cdot(k+1)!=((k+1)+1)!-1$

Rewrite this a little bit

$\underbrace{1\cdot 1!+2\cdot 2!+3\cdot 3!+\cdots+k\cdot k!}_{IH}+(k+1)\cdot(k+1)!=(k+2)!-1$

Applying the Induction Hypothesis

$(k+1)!-1+(k+1)\cdot(k+1)!=(k+2)!-1$

Rearrange and then factor out (k+1)! on LHS

$(k+1)!+(k+1)\cdot(k+1)!-1=(k+2)!-1$

$(k+1)![1+k+1]-1=(k+2)!-1$

$(k+1)![k+2]-1=(k+2)!-1$

$(k+2)!-1=(k+2)!-1$

check

6) QED

Question C) Rosen 5.1 Exercise 10 (p. 330)

a) Find a formula for $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n\cdot(n+1)}$

By examining values of the expression for small n

n=1 value is $\frac{1}{2}$

n=2 value is $\frac{1}{2}+\frac{1}{6}=\frac{3}{6}+\frac{1}{6}=\frac{4}{6}=\frac{2}{3}$

n=3 value is $\frac{1}{2}+\frac{1}{6}+\frac{1}{12}=\frac{2}{3}+\frac{1}{12}=\frac{8}{12}+\frac{1}{12}=\frac{9}{12}=\frac{3}{4}$

Guess: $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n\cdot(n+1)}=\frac{n}{(n+1)}$

b) Prove the formula is correct

1) P(n) is $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n\cdot(n+1)}=\frac{n}{(n+1)}$

2) n0 is 1

3) P(1) is $\frac{1}{1\cdot 2}=\frac{1}{(1+1)}$

$$\frac{1}{2}=\frac{1}{2}$$

check

4) Induction Hypothesis P(k) is

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{k\cdot(k+1)}=\frac{k}{(k+1)}$$

5) Show P(k+1)

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{(k+1)\cdot((k+1)+1)}=\frac{(k+1)}{((k+1)+1)}$$

Rewrite this a little bit

$$\underbrace{\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{k\cdot(k+1)}}_{IH}+\frac{1}{(k+1)\cdot(k+2)}=\frac{(k+1)}{(k+2)}$$

Applying the Induction Hypothesis

$$\frac{k}{(k+1)}+\frac{1}{(k+1)\cdot(k+2)}=\frac{(k+1)}{(k+2)}$$

Rewrite so we have a common denominator and then do algebra

$$\frac{k(k+2)}{(k+1)\cdot(k+2)}+\frac{1}{(k+1)\cdot(k+2)}=\frac{(k+1)}{(k+2)}$$

$$\frac{k(k+2)+1}{(k+1)\cdot(k+2)}=\frac{(k+1)}{(k+2)}$$

$$\frac{k^2+2k+1}{(k+1)\cdot(k+2)}=\frac{(k+1)}{(k+2)}$$

$$\frac{(k+1)^2}{(k+1)\cdot(k+2)}=\frac{(k+1)}{(k+2)}$$

$$\frac{(k+1)}{(k+2)}=\frac{(k+1)}{(k+2)}$$

check

6) QED

Question D) Rosen 5.1 Exercise 20 (p. 330)

Prove that $3^n < n!$ if n is a an integer greater than 6

1) P(n) is $3^n < n!$
2) n0 is 7
3) P(7) is $3^7 < 7!$
   2187 < 5040
   check
4) Induction Hypothesis P(k) is
   $3^k < k!$
5) Show P(k+1) for all k greater than 6
   $3^{k+1} < (k+1)!$
   Rewrite this a little bit
   $3 \cdot \underbrace{3^k}_{IH} < (k+1) \cdot \underbrace{(k)!}_{IH}$
   We know that the second factors obey the IH, so the complete expressions will satisfy the inequality if we can show that
   $3 < (k+1)$
   or k>2
   But k is limited to k>6 which must also satisfy k>2.
6) QED

Question E) Rosen 5.1 Exercise 32 (p. 331)
Prove that 3 divides $n^3 + 2n$ whenever n is a positive integer

1) P(n) is 3 divides $n^3 + 2n$
2) n0 is 1
3) P(1) is 3 divides $(1)^3 + 2(1)$ =1+2=3. 3 divides 3.

check

4) Induction Hypothesis P(k) is

3 divides $k^3 + 2k$

5) Show P(k+1) for all k greater than 0

3 divides $(k+1)^3 + 2(k+1)$

Rewrite this a little bit

3 divides $k^3 + 3k^2 + 3k + 1 + 2k + 2$

3 divides $\underbrace{k^3}_{IH} + 3k^2 + 3k + 1 + \underbrace{2k}_{IH} + 2$

We know that 3 divides the marked terms via the Induction Hypothesis. Now we just need to know if the remaining terms are divisible by 3.

3 divides $3k^2 + 3k + 1 + 2$

3 divides $3k^2 + 3k + 3$

3 divides $3(k^2 + k + 1)$

This value is an integer and it has 3 as a factor, so it is divisible by 3.

6) QED

Question F) Rosen 5.1 Exercise 49 (p. 331)

What is wrong with the following proof that all horses are the same color.
P(n) is the proposition that all horses in a set of size n are the same color.
Basis: P(1) is true
Induction: Assume P(k) is true. Consider a set of k+1 horses. Number them as 1, 2, 3, … k, k+1. By the induction hypothesis, the set of the first k horses have the same color. Similarly, the set of horses from 2 to k +1 have the same color. Because the sets overlap, all the horses must have the same color. This shows that P(k+1) is true. QED.

The problem is that when k=1, we are attempting to show that P(1) -> P(2). In the induction step we split the set of two horses into two sets we do not have any overlap. So, we can not get from P(1) to P(2). After that the induction argument is ok, but we still can not get past P(2).

You may choose to solve one (and only one) of the following Extra Credit Problems. If you submit more than one, only the first will be graded.

Extra Credit 1) Rosen 5.1 Exercise 30 (p. 330)

Prove that $H_1 + H_2 + \cdots H_n = (n+1)H_n - n$

Note that: $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$

1) P(n) is $H_1 + H_2 + \cdots H_n = (n+1)H_n - n$

2) n0 is 1

3) P(1) is $H_1 = (1+1)H_1 - 1$

By definition we replace $H_1 = 1$

$1 = (1+1)1 - 1$

$1 = 2 - 1$

$1 = 1$

check

4) Induction Hypothesis P(k) is

$H_1 + H_2 + \cdots H_k = (k+1)H_k - k$

5) Show P(k+1) for all k greater than 0

$H_1 + H_2 + \cdots + H_{k+1} = ((k+1)+1)H_{k+1} - (k+1)$

Rewrite this a little bit

$H_1 + H_2 + \cdots + H_k + H_{k+1} = (k+2)H_{k+1} - (k+1)$

$\underbrace{H_1 + H_2 + \cdots + H_k}_{IH} + H_{k+1} = (k+2)H_{k+1} - (k+1)$

Use the induction hypothesis on the LHS to get

$(k+1)H_k - k + H_{k+1} = (k+2)H_{k+1} - (k+1)$

Using algebraic manipulations we have

$(k+1)H_k - k + H_{k+1} = (k+2)H_{k+1} - k - 1$

$(k+1)H_k + H_{k+1} = (k+2)H_{k+1} - 1$

$(k+1)H_k = (k+1)H_{k+1} - 1$

$H_k = H_{k+1} - \frac{1}{k+1}$

$H_{k+1} = H_k + \frac{1}{k+1}$

Which matches the definition of Hn

6) QED

Extra Credit 2) Rosen 5.1 Exercise 68 (p. 330)

At a party a celebrity is a person that everyone else knows, but they don't know anyone. Any party has at most one celebrity. A party may not have a celebrity. Your job is to find a celebrity, but you can only ask one guest if they know another particular guest. Prove that if there are n people at the party, you can find the celebrity (if there is one) with 3(n-1) questions.

1) P(n) is if there are n people at the party, you can find the celebrity (if there is one) with 3(n-1) questions
2) n0 is 2
3) P(2) is if there are 2 people at the party, you can find the celebrity (if there is one) with 3(2-1) questions

   With two people we can determine if there is a celebrity with one question to each person and we are allowed up to 3.
   check

4) Induction Hypothesis P(k) is

   if there are k people at the party, you can find the celebrity (if there is one) with 3(k-1) questions

5) Show P(k+1) (for k>= 2)

   if there are k+1 people at the party, you can find the celebrity (if there is one) with 3((k+1)-1) = 3k questions

First note that if a person is a celebrity for the k+1 people, they must also be a celebrity for any subset with at least two members.

If we split one person off, we know that for the remaining k people by the induction hypothesis, we can find a celebrity (if there is one) in 3(k-1) questions. This leaves us with 3 more questions. The problem is that if the person we split off is the celebrity, it will take many questions to find that out. (We need to know that each of the k people knows them and they don't know any of the k people.) We must guarantee that the person we split off is not the celebrity.

Pick two people A and B. Ask if A knows B. (1 Question)

Case 1) A knows B. A cannot be the celebrity. Take all the people with A removed. By the IH we can determine a celebrity in at most 3(k-1) questions. If we get a possible celebrity C, ask that person if they know A (1 Question), if yes C is not a celebrity and there is none. Ask A if they if they know C (1 Question). If no C is not a celebrity and there is none. Otherwise, C is the celebrity!. We ask at most 1 + 1 + 1 + 3(k-1) = 3k questions.

Case 2) A does not know B. B cannot be the celebrity. Take all the people with B removed. By the IH we can determine a celebrity in at most 3(k-1) questions. If we get a possible celebrity C, ask that person if they know B (1 Question), if yes C is not a celebrity and there is none. Ask B if they if they know C (1 Question). If no C is not a celebrity and there is none. Otherwise, C is the celebrity!. We ask at most 1 + 1 + 1 + 3(k-1) = 3k questions.

6) QED