Discrete Math Homework 5
Due Wednesday, February 15 at the beginning of class

General instructions:

- Use standard size paper (8.5 by 11).
- Answer each question in order using a single column.
- Be neat. If we cannot read your solution it is wrong.
- Show your work. If you just write an answer, you will get minimal credit even if the answer is correct.

Rosen section 1.6.

Question A) Rosen 1.6 Exercise 4 (p. 78)

What kind of inference rule is used in each argument. I will color the various propositions

a) Kangaroos live in Australia and are marsupials. Therefore Kangaroos are marsupials.

We need to rewrite this one slightly.

Kangaroos live in Australia and kangaroos are marsupials. Therefore Kangaroos are marsupials.

P and Q. Conclusion Q… this is **simplification**

b)

It is either hotter than 100 degrees today or the pollution is dangerous. It is less than 100 degrees outside today. Therefor, the pollution is dangerous.

P or Q. P. Conclusion Q. this is **disjunctive syllogism**.

c)

Linda is an excellent swimmer. If Linda is an excellent swimmer, then she can work as a lifeguard. Therefore, Linda can work as a lifeguard.

P If P then Q Conclusion Q This is **Modus Ponens**

d)

Steve will work at a computer company this summer. There, this summer Steve will work at a computer company or he will be a beach bum.

P, Conclusion P or Q. This is **addition**.

e)

If I work all night on this homework, then I can answer all the exercises. If I answer all the exercises, I will understand the material. Therefore, if I work all night on this homework, then I will understand the material.
If P then Q. If Q then R. Conclusion If P then R. This is **Hypothetical Syllogism**.

Question B) Rosen 1.6 Exercise 10 a, c, e (p. 79)
For each of the sets of premises what relevant conclusions can be drawn? Explain the rule of inference used to generate each conclusion.
a) If I play hockey, then I am sore the next day. I use the whirlpool if I am sore. I did not use the whirlpool.

Given: If hockey then sore. If sore then whirlpool. Not whirlpool. We can conclude:
1) I am not sore (Modus Tollens)
2) I did not play hockey. (Modus Tollens)

c) All insects have six legs. Dragonflies are insects. Spiders do not have six legs. Spiders eat dragonflies.

Write these as

$$\forall x[Insect(x) \to 6Leg(x)]$$
$$\forall x[DF(x) \to Insect(x)]$$
$$\forall x[Spider(x) \to \neg 6Leg(x)]$$
$$\forall x \forall y[(Spider(x) \wedge DF(y)) \to Eats(x,y)]$$

1) All dragonflies have six legs.

$DF(a) \to Insect(a)$ for arbitrary a (Universal instantiation)

$Insect(a) \to 6Leg(a)$ for arbitrary a (Universal instantiation)

$DF(a) \to 6Leg(a)$ for arbitrary a (Hypothetical syllogism)

$\forall x[DF(x) \to 6Leg(x)]$ (Universal generalization)

2) Spiders are not insects. (Universal instantiation, Modus Tollens, Universal generalization.)

e) All foods that are healthy to eat do not taste good. Tofu is healthy to eat. You only eat what tastes good. You do not eat tofu. Cheeseburgers are not healthy to eat.

Write these as

$$\forall x\left[Healthy(x) \rightarrow \neg TG(x)\right]$$

$$Healthy(Tofu)$$

$$\forall x\left[\left(TG(x) \rightarrow Eat(x)\right) \wedge \left(Eat(x) \rightarrow TG(x)\right)\right]$$

$$\neg Eat(Tofu)$$

$$\neg Healthy(CheeseBurger)$$

1) Tofu does not taste good. (Universal instantiation, Modus Ponens)
2) If it wasn't stated directly you could infer
You don't eat Tofu (Universal instantiation, Simplification, Modus Ponens)

You can not conclude whether cheeseburgers taste good or not

Question C) Rosen 1.6 Exercise 16 (p. 79) Determine if the following arguments are valid or not.

a) Everyone enrolled in the university has lived in a dormitory. Mia has never lived in a dormitory. Therefore. Mia is not enrolled in the university.
This is a valid application of universal instantiation and Modus Tollens.

b) A convertible car is fun to drive. Isaac's car is not a convertible. Therefore, Isaac's car is not fun to drive.
This is not valid. It is an example of the fallacy of denying the hypothesis.

c) Quincy likes all action movies. Quincy likes the movie Eight Men Out. Therefore, Eight Men out is an action movie.
This is not valid. The first statement equivalent to "If a movie is an action movie, then Quincy likes it." The argument is an example of the fallacy of affirming the conclusion.

d) All lobstermen set at least a dozen taps. Hamilton is a lobsterman. Therefore, Hamilton sets a least a dozen traps.
This is a valid application of universal instantiation and Modus Ponens.

Question D) Rosen 1.6 Exercise 30 (p. 80)

Use resolution to show the hypothesis
"Allen is a bad boy or Hillary is a good girl"
and "Allen is a good boy or David is happy" imply the conclusion
"Hillary is a good girl or David is happy."

Has the form
Not P or Q
P or R

Q or R (via resolution)

Rosen section 1.7.

Question E) Rosen 1.7 Exercise 6 (p. 91)
Use a direct proof to show that the product of two odd numbers is odd.

Premises: x is odd and y is odd
Goal: xy is odd.

Either of the two following proofs is a valid direct proof

We can write x as 2j+1 for some integer j
We can write y as 2k+1 for some integer k
The product of xy = (2j+1)(2k+1)
= 4jk + 2j + 2k + 1
= 2(2jk + j + k) + 1
V= 2jk + j + k is an integer
Therefore xy can be written as 2V+1 which is odd.

We know that x mod 2 = 1
We know that y mod 2 = 1
(xy) mod 2 = {(xmod2)(ymod2)} mod 2 (By definition of mod)
= [(1)(1)]mod2
= [1]mod2
= 1
Therefore xy is odd.

Question F) Rosen 1.7 Exercise 12 (p. 91)

Prove or disprove that the product of a non-zero rational number and an irrational number is irrational.

Premises, x is non zero rational, y is irrational
Goal: xy is irrational

Thinking about this it seems that it could be true, so we will look for a proof.

If we go with a direct proof we know we can write x as p/q, but we can not do the same for y….

Lets try a proof by contradiction.

Assume xy is not irrational
xy is rational
we can write xy = P/Q for some integers P and Q (not 0).
By premise we can write x as S/T for some integers S (not 0) and T (not 0)

So, we know that (S/T)y = P/Q
Solving for y (we are allowed to divide by S since it is non-zero)

y = (P/Q)(T/S)

or

y= PT/QS

PT is an integer, QS is an integer, QS is not zero
So y is rational.
This contradicts our premises.

Therefore, xy is irrational.
QED

Question G) Show that if n is an integer and $n^3 - 7$ is odd, then n is even using a proof by contradiction.

Premises: n is an integer and $n^3 - 7$ is odd
Goal: n is even

Assume that n is not even
So n is odd
We can write n=2k+1 for some integer k
So we can write

$$\begin{aligned} n^3 - 7 &= (2k+1)^3 - 7 \\ &= (4k^2 + 4k + 1)(2k+1) - 7 \\ &= (8k^3 + 10k^2 + 6k + 1) - 7 \\ &= (8k^3 + 10k^2 + 6k - 6) \\ &= 2(4k^3 + 5k^2 + 3k - 3) \end{aligned}$$

is 2 times some integer. Therefore this is even
This contradicts our premise.
There fore n must be even.
QED

Alternate proof using mod
Assume that n is not even
So n is odd
We can write n mod 2 = 1
So we can write

$$(n^3 - 7)\bmod 2 = \left((n^3)\bmod 2 - 7\bmod 2\right)\bmod 2$$
$$= \left((n \bmod 2)^3 - 7\bmod 2\right)\bmod 2$$
$$= \left((1)^3 - 1\right)\bmod 2$$
$$= (0)\bmod 2$$
$$= 0$$

So we know that $n^3 - 7$ is even.
This contradicts our premise.
There fore n must be even.
QED

Question H) Rosen 1.7 Exercise 26 (p. 91). This is a bi-conditional, so you must prove both directions.

If n is a positive integer, then n is even iff 7n+4 is even.

Forward direction: if n is even then 7n+4 is even.
n is even means we can write n=2k for some integer k.
We can then write 7n+4 = 7(2k) +4
= 2(7k + 2)
This is 2 times some integer so we know that
7n+4 is even

Backward direction: if 7n+4 is even then n is even
Lets do a proof by contradiction
Assumes that n is not even
N is odd means we can write n=2k+1 for some integer k

7n+4 = 7(2k+1) + 4
= 14k + 11
=2(7k + 5) + 1
7k+5 is an integer so we know that 7n+4 is odd.
This is a contradiction.
Therefore n must be even.

Both directions have been shown.
QED

Question I) Consider our proof that $\sqrt{2}$ is irrational.
Proof:
Any rational number can be written as p/q, where p and q are integers where $q \neq 0$ and p and q share no common factors (aside from 1).

Proof by contradiction:
Assume that $\sqrt{2}$ is rational. This means that we can write

$$\sqrt{2} = \frac{p}{q}$$

where p and q are integers that share no common factors and $q \neq 0$.
Square both sides to get

$$2 = \frac{p^2}{q^2}$$

We can then multiply both sides by q squared

$$2q^2 = p^2$$

Both sides of this equation are integers. If we list out the factors of p squared, 2 must be there. Therefore, we can conclude that 2 is also a factor of p. Thus we can write p as p=2k for some integer k. Replacing p, we get

$$2q^2 = (2k)^2$$

Dividing both side by 2 results in

$$q^2 = 2k^2$$

Again, both sides are integers and we know that 2 must be a factor of q squared. And we can then conclude that 2 is a factor of q.

We have a contradiction because p and q share 2 as a common factor.
Therefore, $\sqrt{2}$ must be irrational.
QED.

We could also prove that $\sqrt{3}$ is irrational by replacing 2 by 3 in our argument above (except in the exponents).

The argument must fail if we try to show that $\sqrt{9}$ is irrational by replacing 2 by 9. At which step does the proof fail?

Fallacious Proof:
Any rational number can be written as p/q, where p and q are integers where $q \neq 0$ and p and q share no common factors.

Proof by contradiction:
Assume that $\sqrt{9}$ is rational. This means that we can write

$$\sqrt{9} = \frac{p}{q}$$

where p and q are integers that share no common factors and $q \neq 0$.
Square both sides to get

$$9 = \frac{p^2}{q^2}$$

We can then multiply both sides by q squared

$$9q^2 = p^2$$

Both sides of this equation are integers. If we list out the factors of p squared, 9 must be there. Therefore, we can conclude that 9 is also a factor of p. Thus we can write p as p=9k for some integer k. Replacing p, we get

$$9q^2 = (9k)^2$$

Dividing both side by 9 results in

$$q^2 = 9k^2$$

Again, both sides are integers and we know that 9 must be a factor of q squared. And we can then conclude that 9 is a factor of q.

We have a contradiction because p and q share 9 as a common factor.
Therefore, $\sqrt{9}$ must be irrational.
QED.

At the red line, the most that we can conclude is that 3 is also a factor of p
Continuing the proof we have
Thus we can write p as p=3k for some integer k. Replacing p, we get

$$9q^2 = (3k)^2$$

Dividing both side by 9 results in

$$q^2 = k^2$$

From here we cannot derive a contradiction.

In fact, if we let k = 1, we get the rational value 3/1 and
$\sqrt{9}$ is rational.
QED.

You may choose to solve one (and only one) of the following Extra Credit Problems. If you submit more than one, only the first will be graded.

Extra Credit 1) Rosen 1.7 Exercise 22 (p. 91)

Show that if you pick three socks from a drawer containing just blue socks and black socks, you must get either a pair of blue socks or a pair of black socks.

Exhaustive proof:
List all possible ways of drawing 3 socks
1st sock is blue/black
2nd sock is blue/black
3rd sock is blue/black
There are 8 possibilities, examine each one

Direct proof:
The first sock must be blue or black. If the second sock matches the first sock, we have a pair. In the case the socks don't match, we have one blue and one black sock. The third sock draw must match one of them and we have a pair.

Proof by contradiction:
Assume that we do not get "a pair of blue sock or a pair of black socks". By DeMorgan we do not have a pair of blue socks and we do not have a pair of black socks. This means that the number of blue socks is 0 or 1. Similarly, the number of black socks is 0 or 1. The maximum number of socks that we can have is 1+1=2. This contradicts the premise that we picked 3 socks.
QED.

Extra Credit 2) Consider the following proof that 1=0: Let S be the infinite sum of terms where 1 and -1 alternate

$$S = 1 + (-1) + 1 + (-1) + 1 + (-1) + \dots$$

We can compute this sum by pairing each of the 1's with a -1.

$$S = \mathbf{[}1 + (-1)\mathbf{]} + \mathbf{[}1 + (-1)\mathbf{]} + \mathbf{[}1 + (-1)\mathbf{]} + \dots$$

Which is just

$$S = 0 + 0 + 0 + \dots$$

$$S = 0$$

But we could have paired up the values as follows:

$$S = 1 + \mathbf{[}(-1) + 1\mathbf{]} + \mathbf{[}(-1) + 1\mathbf{]} + \mathbf{[}(-1) + 1\mathbf{]} + \dots$$

Which is just

$$S = 1 + 0 + 0 + 0 + \dots$$

$$S = 1$$

We can therefore conclude that 1 = 0.

If this argument is correct, show that we can prove every number x would be equal to every other number y.

We have

$$1 = 0$$

Multiply both sides by x

$$x = 0$$

Similarly

$$1 = 0$$

Multiply both sides by y

$$y = 0$$

Therefore, $x=0=y$