Example

Suppose we have the following recursive definition of a set S

Basis: 3 is element of S
Induction: if x and y are in S, then x + y is in S.

We can look at this set and see that the values we get are {3, 6, 9, 12, 15, 18, 21, …}.

We make the conjecture that this is the set of all positive multiples of 3. (Lets call it P3.)

We will prove that S=P3 by showing $S \subseteq P3$ and $S \supseteq P3$.

Part 1: $S \subseteq P3$
We are going to show that every value in S is a multiple of three. In other words, we will show that the definition of S is ***correct***.

Proof by structural induction:
Base Case: 3 is a positive multiple of 3
    Check.
Induction: Assume x and y are positive multiples of 3:
    x=3k and y=3j for some positive integers k,j
    Then x+y= 3k+3j = 3(k+j)
    k+j is positive so x+y is a positive multiple of 3.
Check.

Part 1: $S \supseteq P3$
We are going to show that every value in P3 is in S. Rephrasing, we will show that we didn't miss any positive multiples of 3. In other words, the definition of S is ***complete***.

Proof by contradiction.
Assume that there are values that are positive multiples of 3, but are not in S. Let E be the set of these values. Because E is a subset of positive integers, we know that the well ordering property applies and there must be a smallest value x in E.

We know that x cannot be 3 because we explicitly put it into S.

So x must be at least 6. We know that we can write x as the sum of (x-3) and 3. Both of these must be positive multiples of 3 and both of them are in S. (Whatever x-3 is, we know that it is in S, because it is at least 3 and it is smaller than x (the smallest value that was in P3, but not in S).

Since both values are in S, the induction rule can choose them and put the sum (x) into S.   This contradicts the assumption that x is not in S.
QED

Alternative definitions and their relation to P3

Basis: 6 is element of S
Induction: if x and y are in S, then x + y is in S.
Would be correct for P3, but not complete

Basis: 2 and 3 are elements of S
Induction: if x and y are in S, then x + y is in S.
Would be incorrect for P3, but complete