Solving Linear Homogeneous Recurrence Relations w/ constant coeff

Form (degree k)

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots c_k a_{n-k}$ all constants are real and ck is not zero.

Linear because the RHS is sum of functions of n times a terms
Homogeneous because all terms have a
Constant coef because all coef are constant

$F_n = F_{n-1} + F_{n-2}$ is one of these of degree 2. Coefficients are both 1.

$H_n = 2H_{n-1} + 1$ is not homogenous

$a_n = 2a_{n-1} + n a_{n-2}$ is not constant coef

$b_n = 3b_{n-1} + 4\left(b_{n-2}\right)^2$ is not linear

There is a standard trick to solve these (We see a similar trick with Differential equations.)

Assume that we have a solution that looks like $a_n = r^n$
We plug it in and get

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \dots c_k r^{n-k}$$

or dividing by r^{n-k} we get

$$r^k = c_1 r^{k-1} + c_2 r^{k-2} + \dots c_k$$

$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \dots - c_k = 0$ is the characteristic equation.
Solutions for r are characteristic roots.

If we have distinct roots, then we form a linear combination to get the solution

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \alpha_3 r_3^n + \cdots + \alpha_k r_k^n$$

To find the alphas, we use the initial conditions (there must be k of them.)

If there are roots that have multiplicity m greater than one, then you need m terms with r^n prefixed by $\left\{ 1, \quad n, \quad n^2, \quad \cdots \quad n^{m-1} \right\}$ which give

$$\left\{ r^n, \quad nr^n, \quad n^2r^n, \quad \cdots \quad n^{m-1}r^n \right\}$$

Example: Solving Fibonacci

$$F_n = F_{n-1} + F_{n-2}$$
$$F_0 = 0$$
$$F_1 = 1$$

Characteristic equation is

$$r^2 = r + 1$$

$$r^2 - r - 1 = 0$$

This is quadratic and we can solve it by using the quadratic formula with a=1, b=-1, c=-1

$$r = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)} = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

Our solution is

$$F_n = \alpha\left(\frac{1+\sqrt{5}}{2}\right)^n + \beta\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$F_0 = 0 = \alpha\left(\frac{1+\sqrt{5}}{2}\right)^0 + \beta\left(\frac{1-\sqrt{5}}{2}\right)^0 = \alpha + \beta$$

$$F_1 = 1 = \alpha\left(\frac{1+\sqrt{5}}{2}\right)^1 + \beta\left(\frac{1-\sqrt{5}}{2}\right)^1$$

And now we have to solve for alpha and beta.

$$\alpha\left(\frac{1+\sqrt{5}}{2}\right) + \beta\left(\frac{1-\sqrt{5}}{2}\right) = 1$$
$$\alpha + \beta = 0$$

Write $\alpha = -\beta$ and substitute

$$(-\beta)\left(\frac{1+\sqrt{5}}{2}\right)+\beta\left(\frac{1-\sqrt{5}}{2}\right)=1$$

$$-\beta\sqrt{5}=1$$

$$\beta=\frac{-1}{\sqrt{5}}$$

and then

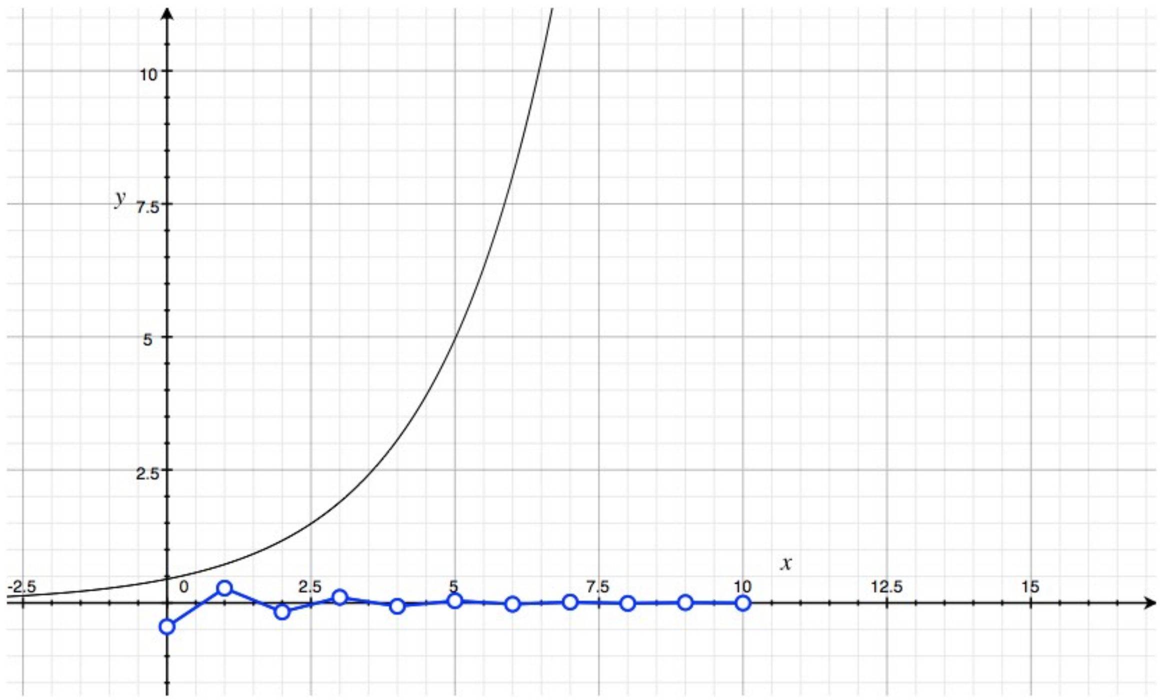
$$\alpha=\frac{1}{\sqrt{5}}$$

$$F_n=\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n-\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$\left(\frac{1+\sqrt{5}}{2}\right)\approx 1.618033988749895$ Is the golden ratio.

$\left(\frac{1-\sqrt{5}}{2}\right)\approx -0.618033988749895$ Negative with an absolute value less than 1.

Our solution has two terms, one for each value of r. The first term is exponential and is the solid line in the graph below. The second term flips sign as n increases and gets closer and closer to zero. Those values are given by the open circles in the graph.



Asymptotically we find that Fibonacci numbers grow exponentially.

$$F_n = \Theta\left(\left(\frac{1+\sqrt{5}}{2}\right)^n\right)$$

Example: Double root

$$a_n = 4a_{n-1} - 4a_{n-2}$$
$$a_0 = 0$$
$$a_1 = 1$$

$$\boxed{\begin{aligned} a_0 &= 0 \\ a_1 &= 1 \\ a_2 &= 4(1) - 4(0) = 4 \\ a_3 &= 4(4) - 4(1) = 12 \\ a_4 &= 4(12) - 4(4) = 32 \\ a_5 &= 4(32) - 4(12) = 80 \end{aligned}}$$

The characteristic equation is
$$r^2 = 4r - 4$$

$$r^2 - 4r + 4 = 0$$

This is quadratic and we can solve it by using the quadratic formula with a=1, b=-4, c=4

$$r = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(4)}}{2(1)} = \frac{4 \pm \sqrt{16-16}}{2} = 2$$

This is a double root.

Our solution needs to prefix 1 and n to the terms.

$$a_n = \alpha(2)^n + \beta n(2)^n$$

Solve for the coeficients

$$a_0 = \alpha(2)^0 + \beta(0)(2)^0 = 0 \quad \text{or} \quad \alpha = 0$$
$$a_1 = \alpha(2)^1 + \beta(1)(2)^1 = 1$$

$$2\beta = 1$$
$$\beta = \frac{1}{2}$$

$$a_n = 0(2)^n + \frac{1}{2}n(2)^n = n(2)^{n-1}$$

And you can check it out against the values.