

Structural Induction

Let S be the subset of the set of ordered pairs of integers defined recursively by

Basis step: $(2, 4) \in S$.

Recursive step: If $(a, b) \in S$, then $(a+1, b+1) \in S$ and $(a-1, b-1) \in S$.

b) [6 pts] Use structural induction to show for any pair $(x, y) \in S$ that $x+2=y$

Base Case: $(2, 4) \in S$

We need to show that $2+2=4$. **Check**

Induction Case: If $(a, b) \in S$, then $(a+1, b+1) \in S$ and $(a-1, b-1) \in S$.

Assume: The property holds for the pair (a, b) that we get from S .

Induction hypothesis is: **$a+2 = b$**

Show: The property holds for the pairs $(a+1, b+1)$ and $(a-1, b-1)$ that we put into the set.

For $(a+1, b+1)$ need to show that $(a+1) + 2 = (b+1)$. If we subtract one from both sides we get $a+2 = b$, which is true by the IH. **Check**

For $(a-1, b-1)$ need to show that $(a-1) + 2 = (b-1)$. If we add one to both sides we get $a+2 = b$, which is true by the IH. **Check**

QED

Recursively Defined Sets [5 pts each]

Give a recursive definition of the set that contains integer powers of three with an even exponent. $\{3^0, 3^2, 3^4, 3^6, 3^8, \dots\}$

Basis step: $1 \in S$.

Recursive step: If $x \in S$, then $9x \in S$

Basis step: $3^0 \in S$.

Recursive step: If $3^x \in S$, then $3^{x+2} \in S$

b) Give a recursive definition of the set that contains all bit strings

$\{\lambda, 0, 1, 00, 01, 10, 11, 000, \dots\}$

(Note: λ is the empty string and you can indicate the concatenation of x with y by xy)

Basis step: $\lambda \in S$.

Recursive step: If $x \in S$, then $x0 \in S$ and $x1 \in S$

Basis step: $\lambda, 0, 1 \in S$.

Recursive step: If $x \in S$, and $y \in S$ then $xy \in S$

Question 6) Strong Induction [10 pts]

Suppose that you have a rope that is n feet long and you are going to cut it into 1-foot long segments. You can cut the rope at any foot marker. So for example, if you have a 10-foot rope, you could cut it into a 4-foot and 6-foot length piece. (Along with other options as well.) Use strong induction to show that the number of cuts required to cut an n -foot rope into 1-foot segments is $n-1$ for all positive n .

a) What is $P(n)$?

"The number of cuts required to cut an n -foot rope into 1-foot segments is $n-1$ "

b) What is n_0 ?

1

c) Show $P(n_0)$ is true.

"The number of cuts required to cut an 1-foot rope into 1-foot segments is $1-1$ "
No cuts are needed. Check

d) What can you assume?

$P(1), P(2), P(3), \dots, P(k)$

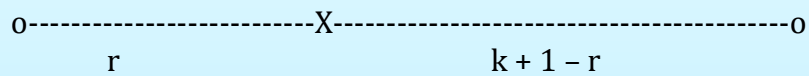
e) What do you need to show?

$P(k+1)$

f) Show it!

We have a rope that is $k+1$ feet and we need to show that it will require k cuts to get it into 1-foot long pieces.

If we think about our first cut it can be anywhere. We get two pieces, one of length r and the other of length $k+1 - r$.



We know that both of the pieces must be at least 1-foot and can not be longer than k -feet. That means we can apply the induction hypothesis on each of the smaller pieces of rope. The total cuts is:

Original cut: 1

Cuts for left by IH: $r-1$

Cuts for right by IH: $k+1-r-1$

Adding we get $1 + r - 1 + k + 1 - r - 1 = k$. Check.

QED