Building Blocks Homework 3
Solution

Question A)

a) Give a regular expression over the alphabet {0, 1} (bit strings) that represents strings that have 2 consecutive 0's.

Examples in the language: 1000, 001001000, 00000, 0001100111, 0011111111000110

Examples not in the language: the empty string, 1, 010, 01010, 0111110101, 011100, 011, 11111

b) Give a regular expression that represents the strings that do not have two consecutive zeros. (This is the complement of the regular expression from part a. Any word that is in the expression from a will not be in the expression for b and vice-versa)

a) Anything followed by 00 followed by anything.
   (0|1)*00(0|1)*
b) This is harder to represent. Any zero must be followed by at least one 1 unless the zero is at the end of the string. So we start with an expression that tries to encapsulate that part of it where e is the empty string.
   (011*)*(e|0)
   This expression fails to capture strings that start with 1s and we can have any number of them
   1*(011*)*(e|0)

Question B) Which of the following are propositions. Give the truth value of any proposition if known.

a) "10 > 11"
b) "The dinosaurs were wiped out by a meteor."
c) "What is a mollusk?"
d) "x+y = y+x"
e) "There are farms in Kansas."

a) Is a proposition. Under standard interpretation this is false.
b) Is a proposition. Good evidence for this, but not certain.
c) Not a proposition because not a statement.
d) Not a proposition. Variables not allowed.
e) Is a proposition. True.

Question C) How many rows appear in a truth table for each of the compound propositions.

a) $(r \rightarrow p) \vee (p \rightarrow \neg q) \vee (p \rightarrow \neg s)$

b) $(p \vee \neg t) \wedge (p \vee \neg p) \wedge t$

c) $((q \wedge t) \vee (r \wedge \neg t)) \vee (p \wedge u \wedge \neg w) \vee (\neg u \rightarrow v)$

d) $(\neg s \rightarrow \neg t) \vee (p \wedge r \wedge s) \vee (t \wedge s \wedge s)$

a) Variables are r, p, q, s. (4) so the number of rows is $2^4 = 16$
b) Variables are p, t. (2) so the number of rows is $2^2 = 4$
c) Variables are q, t, r, p u w, v. (7) so the number of rows is $2^7 = 128$
d) Variables are s, t, p, r. (4) so the number of rows is $2^4 = 16$

Question D) Construct a truth table for the following formula. Identify if the formula is satisfiable.

$(p \rightarrow q) \wedge \neg q$

Two variables require 4 lines.

p	q	$(p \rightarrow q)$	$\neg q$	$(p \rightarrow q) \wedge \neg q$
T	T	T	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

The line marked in green has a value of true for the formula, so it is satisfiable.

Question E) Are the following two formulas logically equivalent?

$(r \rightarrow \neg p) \vee \neg q$ and $\neg(\neg p \wedge r) \vee \neg q$

Three variables requires 8 lines.

p	q	r	$\neg p$	$(r \rightarrow \neg p)$	$\neg q$	$(r \rightarrow \neg p) \vee \neg q$	$\neg p \wedge r$	$\neg(\neg p \wedge r)$	$\neg(\neg p \wedge r) \vee \neg q$
T	T	T	F	F	F	F	F	T	T
T	T	F	F	T	F	T	F	T	T
T	F	T	F	F	T	T	F	T	T
T	F	F	F	T	T	T	F	T	T
F	T	T	T	T	F	T	T	F	F
F	T	F	T	T	F	T	F	T	T
F	F	T	T	T	T	T	T	F	T
F	F	F	T	T	T	T	F	T	T

Because the formulas differ on at least one row (marked in pink) they are not logically equivalent.

Question F) Are the following two formulas logically equivalent?

$(r \leftrightarrow p) \wedge q$ and $(q \wedge p) \vee ((r \rightarrow q) \wedge p)$

Three variables require 8 lines.

p	q	r	$(r \leftrightarrow p)$	$(r \leftrightarrow p) \wedge q$	$(q \wedge p)$	$(r \rightarrow q)$	$((r \rightarrow q) \wedge p$	$(q \wedge p) \vee ((r \rightarrow q) \wedge p)$
T	T	T	T	T	T	T	T	T
T	T	F	F	F	T	T	T	T
T	F	T	T	F	F	F	F	F
T	F	F	F	F	F	T	T	T
F	T	T	F	F	F	T	F	F
F	T	F	T	T	F	T	F	F
F	F	T	F	F	F	F	F	F
F	F	F	T	F	F	T	F	F

Because the formulas differ on at least one row (marked in pink) they are not logically equivalent.

Question G) Is the following formula a tautology, a contradiction, or neither.

$$(p \rightarrow q) \rightarrow (q \vee \neg p)$$

Two variables require 4 lines.

p	q	(p → q)	$\neg p$	$(q \vee \neg p)$	$(p \rightarrow q) \rightarrow (q \vee \neg p)$
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

In all lines the formula is true so this is a tautology. (It is also satisfiable.)

Question H) What is the truth value of the following prepositional formulae with a universal quantifier. Give examples/counter examples as needed.

a) $\forall x, y \in N. [2xy \neq 3]$
For this statement to be true, the integers 2, x, and y would all need to be factors of 3. But 2 is not a factor of 3. This is false.

b) $\forall x, \exists y \in R. [2x = y]$
Pick an arbitrary real value d. We need to find a value of y that makes the statement true. Solving, we see that y=2d. This is true.

c) $\forall x \in R. [x < 0 \rightarrow x^3 > 0]$
Consider an arbitrary real number d which is less than zero. If we cube d, the result will also be negative and the implication is false.

d) $\exists x, \exists y \in N. [x^3 + y^3 = 10]$
If I pick two natural numbers there are only a few pairs of values that could make this true. (For example if either value is greater than 2, then the sum will be greater than 10. I can look at the remaining pairs and verify that none of them work. This is false.

Question I) Consider the following prepositional formulae using the predicates
Pet(x) = "x owns a pet" where x is a person
Drives(x) = "x drives a car" where x is a person

a) $\forall x \in S. [Pet(x) \rightarrow Drives(x)]$ where S is the non-empty domain {Fred, Jack}.
   i) Can this quantified statement be true if Fred does not own a pet?
   ii) Can this quantified statement be true if Jack does not drive a car?
   iii) Can this quantified statement be true if no one drives a car?

> i) It can be true. If Pet(Fred) is false, the implication is true and the truth value depends on Jack.
> ii) It can be true. If Pet(Jack) is false, the implication is true and the truth value depends on Fred.
> iii) If neither Fred or Jack own a pet, both implication will be true and the universal will be true.

b) $\forall x \in S. [Pet(x) \wedge Drives(x)]$ where S is the non-empty domain {Fred, Jack}.
   i) Can this quantified statement be true if Fred does not own a pet?
   ii) Can this quantified statement be true if Jack does not drive a car?
   iii) Can this quantified statement be true if no one drives a car?

> iv) It can not be true. If Pet(Fred) is false the AND is also false and the universal is false.
> v) It can not be true. Because of the AND if Drives is false for anyone the universal fails.
> vi) It can not be true. Because of the AND if Drives is false for anyone the universal fails.

Question J) Give a logically equivalent formula to each of the following where negations do not apply to compound formulae and formulas use only the binary operations $\wedge$ and $\vee$.

a) $\neg\big(p \rightarrow (r \wedge q)\big)$

$\neg\big(\neg p \vee (r \wedge q)\big)$ by def of implication

$\big(\neg\neg p \wedge \neg(r \wedge q)\big)$ by DeMorgan

$\big(p \wedge \neg(r \wedge q)\big)$ by double negation

$\big(p \wedge (\neg r \vee \neg q)\big)$ by DeMorgan

b) $\neg\exists x. Happy(x)$

$\forall x. \neg Happy(x)$ by negation of existential

c) $\neg\big((p \leftrightarrow \neg r) \vee (p \wedge \neg q)\big)$

$\neg\Big(\big((p \rightarrow \neg r) \wedge (\neg r \rightarrow p)\big) \vee (p \wedge \neg q)\Big)$ by def of $\leftrightarrow$

$\neg\Big(\big((\neg p \vee \neg r) \wedge (\neg\neg r \vee p)\big) \vee (p \wedge \neg q)\Big)$ by def of $\rightarrow$ (twice)

$\Big(\neg\big((\neg p \vee \neg r) \wedge (\neg\neg r \vee p)\big) \wedge \neg(p \wedge \neg q)\Big)$ by DeMorgan on highlighted

$\Big(\big(\neg(\neg p \vee \neg r) \vee \neg(\neg\neg r \vee p)\big) \wedge (\neg p \vee \neg\neg q)\Big)$ by DeMorgan twice

$\Big(\big((\neg\neg p \wedge \neg\neg r) \vee (\neg\neg\neg r \wedge \neg p)\big) \wedge (\neg p \vee \neg\neg q)\Big)$ by DeMorgan twice again

$\Big(\big((\neg\neg p \wedge \neg\neg r) \vee (\neg\neg\neg r \wedge \neg p)\big) \wedge (\neg p \vee \neg\neg q)\Big)$ by double negation

$\Big(\big((p \wedge r) \vee (\neg r \wedge \neg p)\big) \wedge (\neg p \vee q)\Big)$ by double negation

d) $\neg\forall \mathrm{y}. \exists \mathrm{x}. [\mathrm{Like}(\mathrm{x},\mathrm{y}) \wedge \mathrm{Tiny}(\mathrm{y})]$

$\neg\forall \mathrm{y}. \exists \mathrm{x}. [\mathrm{Like}(\mathrm{x},\mathrm{y}) \wedge \mathrm{Tiny}(\mathrm{y})]$

$\exists \mathrm{y}. \neg\exists \mathrm{x}. [\mathrm{Like}(\mathrm{x},\mathrm{y}) \wedge \mathrm{Tiny}(\mathrm{y})]$

$\exists \mathrm{y}. \forall \mathrm{x}. [\neg\mathrm{Like}(\mathrm{x},\mathrm{y}) \vee \neg\mathrm{Tiny}(\mathrm{y})]$

e) $\neg\exists y. \big[\big(\forall x. P(x)\big) \vee Q(y)\big]$

$\forall y. \neg\big[\big(\forall x. P(x)\big) \vee Q(y)\big]$

$\forall y. \big[\neg\big(\forall x. P(x)\big) \wedge \neg Q(y)\big]$

$\forall y. \big[\big(\exists x. \neg P(x)\big) \wedge \neg Q(y)\big]$

You may choose to solve one (and only one) of the following Extra Credit Problems. If you submit more than one, only the first will be graded.

Extra Credit 1)
There is an island on which there are three kinds of people: knights who always tell the truth, knaves who always lie, and spies (called normals by Smullyan) who can either lie or tell the truth. You encounter three people, A, B, and C. You know one of these people is a knight, one is a knave, and one is a spy. Each of the three people knows the type of person each of other two is. For the following set of statements, determine whether there is a unique solution and determine who the knave, knight, and spy are. When there is no unique solution, list all possible solutions or state that there are no solutions.

A says "C is the spy"
B says "A is not the spy"
C says "A is the spy"

Because there is one of each type, we have six possible situations each is a row in the table. We then mark each statement as consistent (OK) or inconsistent (X). Every statement made by a knave is consistent (we gain no information) so lets make those with a green OK. Now lets look at the statements for the knight which must be true. Mark with Blue. Finally, we look at the statements for the spies which must be false. We mark those in yellow

A is	B is	C is	A says C is spy	B says A is not spy	C says A is spy
spy	knave	knight	OK	OK	OK
spy	knight	knave	OK	X	OK
knave	spy	knight	OK	X	X
knave	knight	spy	OK	OK	OK
knight	spy	knave	X	X	OK
knight	knave	spy	OK	OK	OK

We have 3 lines where all of the statements are consistent marked in purple.

Extra Credit 2) Consider the statements S =”Statement T is false.” and T= “Statement S is true”. Show that if the S is false, you have a contradiction. Show that if S is true, you have a contradiction.

Case: S is false.

So “Statement T is false” would be false, meaning that statement T is true.

Continuing, “Statement S is true” would be true

And finally S is true. Which is a contradiction.

Case: S is true.

So “Statement T is false” would be true, meaning that statement T is false.

Continuing, “Statement S is true” would be false

And finally S is false. Which is a contradiction.

While S and T look like propositions, because of this paradox, we say that they are in fact not valid propositions and have no truth values. Self reference is a problem.