Fleck Chapter 1
Example Problems

Problem 1 For each of the following numbers, determine if it is a member of the following sets: **Z**, **N**, $\mathbf{Z}^+$, **R**, **Q**

Note: Originally, the notation for these special sets was published in books using a bold font. On a typed manuscript, this was simulated by a double strike with a small offset. We will use bold face font.

Values: 2, 9/3, 5/3, 2.4, 0, -3, 2π

Anything that is an integer is also a rational.
Anything that is a rational is also a real.

a) 2 is a member of each of the sets **Z**, **N**, $\mathbf{Z}^+$, **R**, **Q**
  Anything that is an integer is also a rational.
  Anything that is a rational is also a real.
b) 9/3 can be rewritten as 3 and is a member of each of the sets **Z**, **N**, $\mathbf{Z}^+$, **R**, **Q**
c) 5/3 is a member of each of the sets **R**, **Q**
d) 2.4 can be rewritten as 24/10, we can reduce by factoring 2 from both numerator and denominator then cancelling to get 12/5 as the lowest form. It is a member of each of the sets **R**, **Q**
e) 0 is a member of each of the sets **Z**, **N**, **R**, **Q**
  Zero is the only real number that is neither positive nor negative.
f) -3 is a member of each of the sets **Z**, **R**, **Q**
g) 2π is a member of **R**.
  Showing that 2π can not be written as the ratio of two integers is tricky.

Problem 2 For each of the following numbers, determine if it is a member of the following sets: [0, 10], (0, 2.4], (-3, 3)

Values: 2, 9/3, 5/3, 2.4, 0, -3, 2π

a) 2 is a member of each of the sets : [0, 10], (0, 2.4], (-3, 3)

We see that each of the following are true

$0 \le 2 \le 10$

$0 < 2 \le 2.4$

$-3 < 2 < 3$

b) 9/3 can be rewritten as 3 and is a member of the set [0, 10]

The set (-3, 3) is open at both ends, so the boundary value 3 is not in the set.

c) 5/3 = 1.6... is a member of each of the sets [0, 10], (0, 2.4], (-3, 3)

d) 2.4 is a member of each of the sets [0, 10], (0, 2.4], (-3, 3)

The set (0, 2.4] is open at the left end, but closed on the right. The boundary value 2.4 is in the set.

e) 0 is a member of each of the sets [0, 10], (-3, 3)

The set [0, 10] is closed at both ends so 0 is a member of the set.

The set (0, 2.4] is open on the left end so 0 is not a member of the set.

f) -3 is a member of none of the sets.

The only interval it might fall into is (-3, 3), but both ends are open.

g) 2π is a member of the set [0, 10],

The value of 2π is approximately 6.28 and is clearly outside the other two intervals.

Problem 3 Evaluate each of the following expressions in decimal form.

Expressions: 2×3^4, $2^4 \times 2^7$, $2^4 \times 3^4$, $\left(2^{10}\right)^4$, $3^{3/5}$, 2^{-10}, $2^2 + 2^4$

a) $2\times 3^4 = 2\times 3\times 3\times 3\times 3 = 162$

We **can not** write this as $(2\times 3)^4$ which is a much larger value. We can only combine if the base or the exponents are the same

b) $2\times 2^4\times 2^7 = 2^1\times 2^4\times 2^7 = 2^{1+4+7} = 2^{12} = 4096$

The bases are the same so we can add the exponents together

c) $2^4\times 3^4 = (2\times 3)^4 = 6^4 = 1296$

The exponents are the same so we can multiply the bases together

d) $(2^{10})^4 = (2^{10})\times(2^{10})\times(2^{10})\times(2^{10}) = 2^{40} = 1{,}099{,}511{,}627{,}776$

A handy approximation is that 2^{10} is close to 1000, so

$(2^{10})^4 \approx (10^3)^4 = 10^{3\times 4} = 10^{12}$

e) $3^{3/5} = 3^{3x1/5} = (3^3)^{1/5} = (27)^{1/5} \approx 1.933$

This is the same as $3^{3x1/5} = (3^{1/5})^3 \approx (1.246)^3 \approx 1.933$

f) $2^{-10} = \left(\frac{1}{2}\right)^{10} = \frac{1}{2^{10}} = \frac{1}{1024} = 0.0009765625$

g) $2^2 + 2^4 = 4 + 16 = 20$

Even though the bases are the same, we cannot combine over the addition. The best that we can do is to factor out a 2 squared.

$2^2 + 2^4 = 2^2(1+2^2) = 4(1+4) = 4\times 5$

Problem 4 Evaluate each of the following expressions in decimal form.

Expressions: $\log_2(10)+\log_2(4)$, $\log(15\pi)$, $\log(23/7)$, $\log_3(324)$, $\ln(\pi^{\pi+1})$, $\log_2(512)$, $\frac{\log(100)}{\log(50)}$, $\log_2(5)+\log_{10}(7)$

a) $\log_2(10)+\log_2(4)$

Because the bases are the same we can rewrite this as a multiplication inside the argument of the log.

$\log_2(10)+\log_2(4)=\log_2(10\times 4)=\log_2(40)$

Alternately, we can factor the arguments

$\log_2(10)+\log_2(4)=\log_2(2\times 5)+\log_2(2^2)=\log_2(2)+\log_2(5)+\log_2(2^2)$

and then evaluate the two logs that in the special form $\log_b(b^x)=x$

$\log_2(2)+\log_2(5)+\log_2(2^2)=1+\log_2(5)+2$

$=3+\log_2(5)$

Both of these would be satisfactory answers on an exam. On homework, use a calculator to arrive at the value 5.321928094887362

b) $\log(15\pi)$

We can factor the argument

$=\log(3\times 5\times \pi)$

And then each factor is wrapped in a log and added together

$=\log(3)+\log(5)+\log(\pi)$

On homework, we use a calculator to arrive at the value

= 0.477121254719662 + 0.698970004336019 + 0.497149872694134

= 1.673241131749815

c) $\log(23/7)$

Division works similarly to multiplication except that we subtract for each factor in the denominator.

$=\log(23)-\log(7)$

On homework, we use a calculator to arrive at the value 0.516629796003336

d) $\log_3(324)$

Factor the argument 324 = $2\times 162 = 2\times 2\times 81 = 2^2\times 3^4$
Now we can split the log

$$\log_3\left(2^2\times 3^4\right)=\log_3\left(2^2\right)+\log_3\left(3^4\right)$$

The first log we see that the argument has an exponent we can pull out and the second is the special form we can evaluate directly.

$$=2\log_3(2)+4$$

On homework, we use a calculator to arrive at the value 5.261859507142915

e) $\ln\left(\pi^{\pi+1}\right)$

We can pull down the exponent

$$=(\pi+1)\ln(\pi)$$

On homework, we use a calculator to arrive at the value

= (4.141592653589793)(1.1447298858494)
= 4.741004885578558

f) $\log_2(512)$

Factor the argument to get

$$=\log_2\left(2^9\right)$$

This is in the special form that we can evaluate directly

= 9

g) $\dfrac{\log(100)}{\log(50)}$

It is tempting to try and combine the two logs into one, but what we know how to do is to take a division inside the argument and convert it into a difference of logs and that doesn't apply here.

We factor the arguments to get

$$=\frac{\log\left(10^2\right)}{\log(5\times 10)}=\frac{2}{\log(5)+\log(10)}=\frac{2}{\log(5)+1}$$

We use a calculator and compute

$$=\frac{2}{1+0.698970004336019}=1.177183820135558$$

h) $\log_2(5)+\log_{10}(7)$

Unless the bases are the same, we cannot directly combine logs. Five is a prime, so factoring the arguments doesn't get us anywhere.

We compute each log and then add.

= 2.321928094887362+0.845098040014257

= 3.167026134901619

Note: The last rule listed allows us to convert from one base into another. So we could apply that rule to convert the first log into a log base 10.

$$\log_2(5) = K\log_{10}(5)$$

The conversion factor K, is a log using both of the bases (2 and 10) in the conversion. We know that $\log_2(5)$ is larger than $\log_{10}(5)$ so the conversion factor must be greater than one. Of the two choices $\log_2(10)$ and $\log_{10}(2)$, the first choice is about 3 and the second choice is less than one.

$$\log_2(5) = \log_{10}(2)\log_{10}(5)$$

So we can rewrite

$$\log_2(5)+\log_{10}(7) = \log_{10}(2)\log_{10}(5)+\log_{10}(7)$$

Unfortunately, the resulting expression can not be combined into a single log just yet. When adding logs of the same base, we don't allow multiplication by other factors outside the log. We can pull it up into the exponent of the argument

$$= \log_{10}\left(5^{\log_{10}(2)}\right)+\log_{10}(7)$$

And then we can combine the logs to get

$$= \log_{10}\left(5^{\log_{10}(2)}\times 7\right)$$

But it is unclear if this expression gains us anything.

Problem 5 Evaluate each of the following expressions in decimal form.

Expressions: $5!$, $\lfloor 5.2 \rfloor, \lfloor -5.2 \rfloor, \lfloor -0.2 \rfloor + \lceil 2+0.3 \rceil$

a) 5!

Five factorial is the product of all the integers from 1 to 5

$5! = 1 \times 2 \times 3 \times 4 \times 5 = 120$

b) $\lfloor 5.2 \rfloor$

The floor of 5.2 is the largest integer value which is less than 5.2.

$\lfloor 5.2 \rfloor$ =5

c) $\lfloor -5.2 \rfloor$

The floor of negative 5.2 is the largest integer value which is less than -5.2. Notice that this value is surrounded by the integers -6 and -5, so the floor will be negative 6.

$\lfloor -5.2 \rfloor$ = -6

d) $\lfloor -0.2 \rfloor + \lceil 2+0.3 \rceil$

The floor of -0.2 is -1. We can write $\lceil 2+0.3 \rceil = 2 + \lceil 0.3 \rceil = 2$.

$\lfloor -0.2 \rfloor + \lceil 2+0.3 \rceil$ = -1 + 2 = 1

'

SEE THE SUMMATIONS HANDOUT FOR EXAMPLES OF SUMMATIONS!

Problem 6 For each of the following strings decide if it is covered by each of the regular expressions.
Strings: ε, a, b, abc, aaaa, aabbcc, bababa, abcabc
Regular Expressions: (ab)*, a(aa)*, a*b*c*, (ab | bc)*

	(ab)*	a(aa)*	a*b*c*	(a | bc)*
ε	Yes	No, must have at least one a	Yes	Yes
a	No, any a must be followed by b	Yes, we take no copies of the aa	Yes, we take 1 a and then no b or c.	Yes, choose a once
b	No, any b must be preceded by an a	No, only strings of at least one a	Yes, we take 1 b and no a or c	No, b must be followed by c
abc	No, c are not part of the regular expression	No, only strings of at least one a	Yes, take one each of a, b and c	Yes, choose a, then choose bc
aaaa	No, any a must be followed by b	No, the number of a's must be odd	Yes, take 4 a's and no b or c	Yes, choose a four times
aabbcc	No, any a must be followed by b	No, only strings of a's	Yes, take 2 of each of a, b and c	No, b must be followed by c
bababa	No, any a must be followed by b	No, only strings of a's	No, a is not allowed after b	No, b must be followed by c
abcabc	No, c not allowed	No, c not allowed	No, a is not allowed after b	Yes, choose a, then bc, then a, then bc.

Problem 7 Give a regular expression over the alphabet {0, 1} (bit strings) that represents strings that end in 0.

This includes strings like 0, 00, 10, 000, 010, 100, 110, ...

We can express this as any bit string (1|0)* concatenated with 0 or (1|0)*0

Problem 8 Give a regular expression over the alphabet {0, 1} (bit strings) that represents positive even binary numbers with no leading zeros.

This is similar to the previous problem. Any even binary number will have a zero in the last place. We are restricted to positive values, so 0 is out. We are left with

10, 100, 110, 1000, 1010, …

We see that all these strings must start with 1 and end in 0 and can have anything in between

1(1|0)*0

Note: If we wanted all even integers, we would have to include the string 0 which is not covered by the previous regular expression. We would add that special case in directly to get:

0 | 1(1|0)*0

Problem 9 Give a regular expression over the alphabet {0, 1} (bit strings) that represents odd binary numbers with no leading zeros.

This is similar to the previous problem. We have strings

1, 11, 101, 111, 1001, 1011, …

We see that all these strings must start with 1 and end in 1 and can have anything in between.

1(1|0)*1

This almost works. Note that the smallest length string we can generate is length 2. So we do not cover "1". We can add that string in

1 | 1(1|0)*1

If have a choice with a common prefix, we can factor that prefix out. For example: adbc* | adcc*c is the same as ad(bc* | cc*c). Using this, we can rewrite the regular expression as

1(ε | (1|0)*1)

Problem 10 Give a regular expression over the alphabet {0, 1} (bit strings) that have an even number of zeros.

An example of the kinds of strings are ε, 00, 111, 010010, 11001110100001110011000

Thought 1: Repeat pairs of zeros!

(00)*

This hits the empty string, and the string 00, but it doesn't get 111.
We can have an arbitrary number of 1s before and after any zeros.

(I*(00)*1*)*

This still doesn't get them all. If we look at the longer strings we see that we don't need to have blocks of zeros that are even length. We solve this problem by noting each zero must belong in a pair. Lets mark those pairs in the string 11001110100001110011000 by adding in a space before and after the matching zeros.

11 00 111 010 00 01110 0110 00

We see that each of the pieces is a 1* or a 01*0 so we write the regular expression as

(1* | 01*0)*