Question: 0! = ?

We have a closed form definition of n!. Namely, n! is the product of integer values from 1 to n. So we know the following:

Factorials

$$2! = 1 \cdot 2$$
$$3! = 1 \cdot 2 \cdot 3$$
$$4! = 1 \cdot 2 \cdot 3 \cdot 4$$
$$5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5$$
$$6! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6$$
$$\vdots$$

Factorials

$$1! = 1$$

The value of 1! is slightly problematic. By our definition it is the product of the integer values from 1 to 1. How we compute the product of a single value? A reasonable answer is that it is just 1.

It is easy to see that we have a recursive definition for this sequence.

Recursive Definition

$$n! = (n-1)! \cdot n$$
$$1! = 1$$

But have we found the right initial condition for the sequence? We will try to push our definition back.

One step back:

We use the recursive formula to compute 1!

$$1! = (1-1)! \cdot 1$$

or

$$1! = 0! \cdot 1$$

Solving for 0! we get:

Factorials

$$0! = 1$$

And now our recursive definition is

Recursive Definition

$$n! = (n-1)! \cdot n$$
$$0! = 1$$

Two steps back:
If we can do it once, perhaps we can do it again….

We use the recursive formula to compute 0!

$$0! = (0-1)! \cdot 0$$

or

$$1 = 1 \cdot 0$$

This has no solution, so we can not extend the recursive formula back any farther.

Final Note:
While we used the recursive formula to determine the value of 0! as one, there are some good reasons for it as well. We can think of factorial as an example of applying a function (multiplication) to a set of values. An appropriate value to return if the set is empty is the identity value. For multiplication, 1 is the identity. (I.E 1 times x is x.) At a more pragmatic level, later on when we compute combinatorial values, we will run across 0! and the appropriate value to use is 1.