Building Blocks Homework 2 Solution

Question A) Write a program in any language that will compute the value of the following sum and report the value when n=10.

$$K_n \approx \sum_{i=1}^{n} \frac{1}{i^2} = \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \ldots$$

I will use python

```
def k(n):
    sum = 0
    for i in range (1,n+1):
        sum += 1/(i**2)
    return sum

print(k(10))

1.5497677311665408
```

Question B) Consider the sum of the first 3000 values in an **arithmetic** sequence that starts 3, 9, 15, … Express the sum using summation notation where the index variable starts at 1.

We start by creating the summation. Since we started at 1and there are 3000 values in the sequence, the upper value will be 3000:

$\sum_{i=1}^{3000}(some formula using i)$

We see that the value increases by 6 each time, so our formula will need to use 6i. Looking at the values of i and 6i and comparing with the actual values, we discover that 6i is too big by 3

i=1	i=2	i=3
6i = 6	6i=12	6i=18

$$\sum_{i=1}^{1000}(6i-3)$$

Question C) Consider the sum of the first 3000 values in a **geometric** sequence that starts 5, 10, 20, … Express the sum using summation notation where the index variable starts at 0.

We start by creating the summation. Since we start at 0and there are 3000 values in the sequence, the upper value will be 2999: $\sum_{i=0}^{2999}(some\ formula\ using\ i)$

We see that the value increases by a factor of 2 each time, so our formula will need to use 2^i. Looking at the values of i and 2^i and comparing with the actual values, we discover that 2^i is off by a factor of 5.

i=0	i=1	i=2
$2^0 = 1$	$2^1 = 2$	$2^2 = 4$

$$\sum_{i=0}^{2999} 5(2^i)$$

Question D) What is the value of the summation

$$\sum_{i=1}^{4} (2i)!$$

Since the number of terms is small, we will use i in the formula and compute the result directly. I am alternating the color so each of the four terms is easier to identify.

$$\sum_{i=1}^{4} (2i)! = (2 \times 1)! + (2 \times 2)! + (2 \times 3)! + (2 \times 4)!$$

$= (2)! + (4)! + (6)! + (8)!$

by calculator

$= 2 + 24 + 720 + 40320$

$= 41{,}066$

Question E) Manipulate the following summation so that you can apply the known sums to get its value.

$$S = \sum_{i=1}^{14} (1 - i)^2$$

First, expand

$$S = \sum_{i=1}^{14} (1 - 2i + i^2)$$

Next split

$$S = \sum_{i=1}^{14} 1 + \sum_{i=1}^{14} (-2i) + \sum_{i=1}^{14} i^2$$

Factor constant -2

$$S = \sum_{i=1}^{14} 1 + -2\sum_{i=1}^{14} (\mathrm{i}) + \sum_{i=1}^{14} i^2$$

Apply known formulas and we have our happy end point for exams

$$S = 14 + -2\frac{(15)(14)}{2} + \frac{(29)(15)(14)}{6}$$

Compute value

$$S = 14 - (15)(14) + (29)(7)(5)$$
$$= 819$$

Question F) Manipulate the following summation so that you can apply the known sums to get a closed form solution.

$$S = \sum_{i=1}^{n} (i+1)(3i-1)$$

First, expand

$$S = \sum_{i=1}^{n} (3i^2 + 2i - 1)$$

Next split

$$S = \sum_{i=1}^{n} (3i^2) + \sum_{i=1}^{n} (2i) + \sum_{i=1}^{n} (-1)$$

Factor constants

$$S = 3\sum_{i=1}^{n} (i^2) + 2\sum_{i=1}^{n} (\mathrm{i}) - \sum_{i=1}^{n} (1)$$

Apply known formulas and we have our happy end point for exams

$$S = 3\frac{(2\mathrm{n}+1)(\mathrm{n}+1)(\mathrm{n})}{6} + 2\frac{(\mathrm{n}+1)(\mathrm{n})}{2} - n$$

Simplifications

$$S = \frac{(2\mathrm{n}+1)(\mathrm{n}+1)(\mathrm{n})}{2} + (\mathrm{n}+1)(\mathrm{n}) - n$$

$$S = \frac{(2\mathrm{n}+1)(\mathrm{n}+1)(\mathrm{n})}{2} + n^2$$

This is a good place to stop.

Question G) Manipulate the following summation so that you can apply the known sums to get a closed form solution. (Notice that the lower limit is 10.)

$$S = \sum_{i=10}^{n} (2i^2 + 1)$$

First we add and subtract extra terms from i=1 to 9. (Completely legal since the extra bits combine to zero.)

$$S = \sum_{i=10}^{n} (2i^2 + 1) + \sum_{i=1}^{9} (2i^2 + 1) - \sum_{i=1}^{9} (2i^2 + 1)$$

The first two sums can be combined to get

$$S = \sum_{i=10}^{n} (2i^2 + 1) - \sum_{i=1}^{9} (2i^2 + 1)$$

And now we can separate and apply knowns

$$S = 2\sum_{i=10}^{n} (i^2) + \sum_{i=10}^{n} (1) - 2\sum_{i=1}^{9} (i^2) \text{-} \sum_{i=1}^{9} (1)$$

$$S = 2\left[\frac{(2n+1)(n+1)n}{2}\right] + [n] - \left[\frac{(2(9)+1)(9+1)9}{2}\right] - [9]$$

We can leave the answer here or do some simplifications

$$S = (2n+1)(n+1)n + [n] - [864]$$

Question H) Manipulate the following summation so that you can apply the known sums to get its value.

$$S = \sum_{i=0}^{20} 3\left(\frac{1}{3}\right)^{i}$$

There are no terms to multiply, so the first thing we will do is to factor out the constant 3.

$$S = 3\sum_{i=0}^{20} \left(\frac{1}{3}\right)^{i}$$

Apply the known with n=20

$$S = 3\left[\frac{1-\left(\frac{1}{3}\right)^{20+1}}{1-\left(\frac{1}{3}\right)}\right]$$

And simplify

$$S = 3\left[\frac{1-\left(\frac{1}{3}\right)^{21}}{\left(\frac{2}{3}\right)}\right] = 3\left[1-\left(\frac{1}{3}\right)^{21}\right]\left(\frac{3}{2}\right) = \left(\frac{9}{2}\right)\left[1-\left(\frac{1}{3}\right)^{21}\right]$$

If we look at this, we see that the value marked in green is close to 0, so the result will be slightly less than 9/2.

Using a calculator we get the value 4.499999999569804

Question I) Manipulate the following double summation so you can apply the known sums to get a closed form solution.

$$S = \sum_{i=1}^{n} \sum_{j=1}^{n} (2i + j)$$

We work the nested summation from inside to out. First split the inner sum.

$$S = \sum_{i=1}^{n} \left[\sum_{j=1}^{n} (2i) + \sum_{j=1}^{n} (j) \right]$$

The 2i in the first sum can be treated as a constant with respect to j so we pull it out. The second sum can be evaluated using a known form.

$$S = \sum_{i=1}^{n} \left[(2\mathrm{i}) \sum_{j=1}^{n} 1 + \sum_{j=1}^{n} (j) \right]$$

$$S = \sum_{i=1}^{n} \left[(2\mathrm{i})\mathrm{n} + \frac{(\mathrm{n}+1)\mathrm{n}}{2} \right]$$

Split this and then factor out constants (including n)

$$S = \sum_{i=1}^{n} [(2\mathrm{i})\mathrm{n}] + \sum_{i=1}^{n} \left[\frac{(\mathrm{n}+1)\mathrm{n}}{2} \right]$$

$$S = 2\mathrm{n} \sum_{i=1}^{n} [\mathrm{i}] + \frac{(\mathrm{n}+1)\mathrm{n}}{2} \sum_{i=1}^{n} [1]$$

Apply knowns and we have

$$S = 2\mathrm{n} \left[\frac{(\mathrm{n}+1)\mathrm{n}}{2} \right] + \frac{(\mathrm{n}+1)\mathrm{n}}{2} [\mathrm{n}]$$

Which is

$$S = 3\mathrm{n} \left[\frac{(\mathrm{n}+1)\mathrm{n}}{2} \right]$$

Question J) For each of the following strings decide if it is a member of each of the regular expressions.
Strings: ε, a, b, ab, aaaa, aabb, bababa, abbabbb
Regular Expressions: ab*ab*, b*a*b, (b| aa)b*, (a|b)*a

We mark strings that are a member of the regular expression in green.

ab*ab* These strings must start with a and there must be exactly two of them

ε, a, b, ab, aaaa, aabb, bababa, abbabbb

b*a*b These strings must end in b. All a's must be contiguous with single b

ε, a, b, ab, aaaa, aabb, bababa, abbabbb

(b| aa)b* These strings start with b or aa. Followed by 0 or more b

ε, a, b, ab, aaaa, aabb, bababa, abbabbb

(a|b)*a These strings are anything that ends with a

ε, a, b, ab, aaaa, aabb, bababa, abbabbb

You may choose to solve one (and only one) of the following Extra Credit Problems. If you submit more than one, only the first will be graded.

Extra Credit 1) Consider two summations. One is the sum of the squares (S) and the other is the square of the sums (T). As n gets big what can you say about the ratio of T over S?

$$S_n = \sum_{i=1}^{n} i^2$$

We can solve for both of these

$$S_n = \sum_{i=1}^{n} i^2 = \frac{(2\mathrm{n}+1)(\mathrm{n}+1)\mathrm{n}}{6}$$

$$T_n = \left(\sum_{i=1}^{n} i\right)^2 = \left[\frac{(\mathrm{n}+1)\mathrm{n}}{2}\right]^2$$

Dividing

$$\frac{T_n}{S_n} = \left[\frac{(\mathrm{n}+1)\mathrm{n}}{2}\right]^2 \Bigg/ \frac{(2\mathrm{n}+1)(\mathrm{n}+1)\mathrm{n}}{6}$$

Dividing by a fraction is the same as multiplying by the reciprocal

$$\frac{T_n}{S_n} = \left[\frac{(\mathrm{n}+1)\mathrm{n}}{2}\right]^2 \times \frac{6}{(2\mathrm{n}+1)(\mathrm{n}+1)\mathrm{n}}$$

Cancelling gives

$$\frac{T_n}{S_n} = \frac{(\mathrm{n}+1)\mathrm{n}}{4} \times \frac{6}{(2\mathrm{n}+1)}$$

$$\frac{T_n}{S_n} = \frac{3(\mathrm{n}+1)\mathrm{n}}{2(2\mathrm{n}+1)}$$

When n is large, we can ignore the +1

$$\frac{T_n}{S_n} \approx \frac{3(\mathrm{n})\mathrm{n}}{2(2\mathrm{n})} \approx \frac{3(\mathrm{n})}{2(2)} = \frac{3\mathrm{n}}{4}$$

$$T_n = \left(\sum_{i=1}^{n} i\right)^2$$

Extra Credit 2) Give a regular expression for strings over the alphabet {a,b,c} that have the sequence **abc** somewhere and either start with **b** or ends with a **b**. Example: The strings **aabbcc** and **abca** are not in the language. But the strings **bababcabbc** and **abcab** are in the language.

If we just wanted strings that contain abc, we would use

(a|b|c)*abc(a|b|c)*

If we just add b to front and back, that would require both. So we have a choice

b(a|b|c)*abc(a|b|c)* | (a|b|c)*abc(a|b|c)*b

Notice that the string babcb is covered by both options, but the expressions before and after the bar represent sets of strings and the or results in the union. We do not require the sets to be disjoint.