Fleck Chapter 5 – Sets and some counting

Problem 1 Given the sets

A = { x | x is an integer and x mod2 = 0}

B = { x | x is an integer and x is a multiple of 6}

C = { x | x is an integer and x =2k for some integer k}

a) Is A a subset of B?
b) Is A a subset of C?
c) Is B a subset of A?
d) is C a subset of A?
e) Are A and B in a proper subset relation?
f) Are A and C in a proper subset relation?

To help decide the questions, it will be useful to list some of the values in the sets

A = {…-4, -2, 0, 2, 4, …}

B = {-12, -6, 0, 6, 12, …}

C = {…-4, -2, 0, 2, 4, …}

a) This is false. -4 is in A, it is not a multiple of 6 and is not in B.
b) This is true. Pick some value from A and call it q. By definition we know that q mod2 = 0. By division we know this means that q =2m +0 for some integer m. Therefore it is also in C.
c) This is true. Pick some value from B and call it q. By definition we know that q is a multiple of 6. This means that q =6n for some integer n. qmod2 = (6n)mod2 = 0mod2 = 0 is also in A.
d) This is true. Pick some value from C and call it q. By definition we know that q =2k for some integer k. qmod2 = (2k)mod = 0 is also in A.
e) Yes. B is a proper subset of A, because B is a subset of A and is not equal to A.
f) No. A is a subset of C and C is a subset of A and therefore they are equal to each other. Sets that are equal are not proper subsets.

Problem 2 What are the union and intersections of the following two sets.
the sets:

A = {{1, 2}, {3, 4, {5}}, 1, 6}
B = { {}, 1, {5}, {2, 1}, {6} }

Union: To find the union, first add everything from the first set, then add the values from the second set.

$A \cup B = \{ \{1, 2\}, \{3, 4, \{5\}\}, 1, 6, \{\}, 1, \{5\}, \{2, 1\}, \{6\} \}$

Remove any duplicates (marked in yellow) to get the final answer.

$A \cup B = \{ \{1, 2\}, \{3, 4, \{5\}\}, 1, 6, \{\}, \{5\}, \{6\} \}$

Intersection: To find the union, mark the elements that are in both sets. (The yellow items from the previous problem.) and put them into a set.

$A \cap B = \{ \{1, 2\}, 1\}$

Problem 3 What are the cardinalities of the following sets?

$A = \{1, \{1, 2\}, \{3, 4, \{5\}\}, 1, 6, \{2, 1\} \}$
$B = \{\{\}\}$
$C = \varnothing$
$D = \{ \{1, \{5\}, \{\}, 2, 3, 4\}, \{\}\}$

I will mark each of the unique elements in the set

$A = \{\underbrace{1}_{1}, \underbrace{\{1, 2\}}_{2}, \underbrace{\{3, 4, \{5\}\}}_{3}, 1, \underbrace{6}_{4}, \{2, 1\} \}$ has cardinality 4

$B = \{\underbrace{\{\}}_{1}\}$ has cardinality 1

$C = \varnothing$ has cardinality 0

$D = \{ \underbrace{\{1, \{5\}, \{\}, 2, 3, 4\}}_{1}, \underbrace{\{\}}_{2}\}$ has cardinality 2.

Problem 4 We are going to select an outfit to wear to the party. We select a hat, shirt, pants and shoes. Each item comes in white, black, red, yellow, blue, orange, and purple. How many different outfits can we select if we are allowed to use colors more than once?

This is a product rule problem. We have four tasks. In each case the number of colors we can select from is 7.

Task 1: Select a hat	7 ways
Task 2: Select a shirt	7 ways
Task 3: Select a pair of pants	7 ways
Task 4: Select a pair of shoes	7 ways

By the product rule, we multiply the counts together to get 7x7x7x7 = 7^4.

Using a calculator we get the final answer of 2,401.

Problem 5 We are going to select an outfit to wear to the party. We select a hat, shirt, pants and shoes. Each item comes in white, black, red, yellow, blue, orange, and purple. How many different outfits can we select if we are only allowed to use a color once?

Again, this is a product rule problem. We have four tasks. As we make selections, the number of remaining valid selections decreases by 1.

Task 1: Select a hat	7 ways
Task 2: Select a shirt	6 ways
Task 3: Select a pair of pants	5 ways
Task 4: Select a pair of shoes	4 ways

By the product rule, we multiply the counts together to get 7x6x5x4.

Using a calculator we get the final answer of 840.

Problem 6 We are going to select an outfit to wear to the party. We select a hat, shirt, pants and shoes. Each item comes in white, black, red, yellow, blue, orange, and purple. How many different outfits can we select if we know that exactly one color is used twice?

This is more complicated. It is no longer a straightforward product rule. We will break the problem up into ***mutually exclusive*** selections and **add** all the results.

Way one: Hat/Shirt are the same color.

Task 1: Select a hat/shirt	7 ways
Task 2: Select a pair of pants	6 ways
Task 3: Select a pair of shoes	5 ways

Way two: Hat/Pants are the same color.

Task 1: Select a hat/pants	7 ways
Task 2: Select a pair of shirt	6 ways
Task 3: Select a pair of shoes	5 ways

Way three: Hat/Shoes are the same color.

Task 1: Select a hat/shoes	7 ways
Task 2: Select a pair of shirt	6 ways
Task 3: Select a pair of pants	5 ways

Way four: Shirt/Pants are the same color.

Task 1: Select a shirt/pants	7 ways
Task 2: Select a pair of hat	6 ways
Task 3: Select a pair of shoes	5 ways

Way five: Shirt/Shoes are the same color.

Task 1: Select a shirt/shoes	7 ways
Task 2: Select a pair of hat	6 ways
Task 3: Select a pair of pants	5 ways

Way six: Pants/Shoes are the same color.

Task 1: Select a pants/shoes	7 ways
Task 2: Select a pair of hat	6 ways
Task 3: Select a pair of shirt	5 ways

Each way is 7x6x5 = 210

Add together the mutually exclusive ways to get
210 + 210 + 210 + 210 + 210 + 210 = 1260

Quick check. The results from problems 5 and 6 are mutually exclusive. They also do not cover all the possible ways of picking outfits. For example, every item the same color is not covered by either 5 or 6. We expect that if we add the results from 5 to 6, that the result should be less that what we got for problem 4.

840 + 1260 = 2100 is less than 2401 ✔

Problem 7 We are going to select an outfit to wear to the party. We select a hat, shirt, pants and shoes. Each item comes in white, black, red, yellow, blue, orange, and purple. How many different outfits can we select if we know that some color is used more than once?

We could break it up into mutually exclusive cases:

1) One color used twice (AABC pattern) which we did in the previous part
2) One color used three times (AAAB pattern)
3) One color used four times (AAAA pattern)
4) Two colors used twice (AABB pattern)

But in this case it is easier to use the complement. If we look at all the possible outfits where colors can be reused and subtract out those that only use colors once, we are left with all the possible outfits that used a color more than once.

2401 – 840 = 1561

Problem 8 How many palindromes are there of length 9 using letters from the alphabet? If it takes 10 seconds to write down a single palindrome, how long will it take to write down all of the length 9 palindromes.

This will be a sequence of nine letters, where the sequence reads the same forward and backwards. This means that the first and last letters depend on each other and must be the same.

$$\underline{26}\quad\underline{26}\quad\underline{26}\quad\underline{26}\quad\underline{26}\quad\underline{\ 1\ }\quad\underline{\ 1\ }\quad\underline{\ 1\ }\quad\underline{\ 1\ }$$

By the product rule, we get 26^5 .
Using a calculator we get 11,881,376.

The number seconds to write these all down would be 118813760 seconds.
Divide by 60 to get minutes. ($\cong$ 1980229)
Divide by 60 to get hours. ($\cong$ 33004)
Divide by 24 to get days. ($\cong$ 1375)
Divide by 365 to get years. ($\cong$ 3.75)

Problem 9

a) How many length 5 strings of digits are there?
b) How many strings of digits of length 5 are there where the initial digit is non-zero?
c) How many strings of digits are there of length 5 where the initial digit is non-zero or the string represents an even number?

a) This is a sequence where we can reuse the digits.

$\underline{10}\quad \underline{10}\quad \underline{10}\quad \underline{10}\quad \underline{10}$

Using the product rule, the answer is 100000.

b) We have only 9 choices for the first digit, but all the rest are still 10.

$\underline{9}\quad \underline{10}\quad \underline{10}\quad \underline{10}\quad \underline{10}$

And the answer is 90000.

c) To determine the number of strings that are even numbers, note that we only have to look at the last digit to determine if the number is even. Anything that ends in 0, 2, 4, 6 or 8 will be even.

$\underline{10}\quad \underline{10}\quad \underline{10}\quad \underline{10}\quad \underline{\ 5\ }$

And the answer is 50000. (Which makes sense… half of the values should be even and the other half should be odd.)

So we have two sets:

A: initial digit is non-zero = 90000
B: even number = 50000

Clearly, we can not just add the values because the result would be larger than the possible number of strings.

We must use inclusion-exclusion and subtract out the strings that are common to A and B.

Initial digit is non-zero and the number is even.

$\underline{9}\quad \underline{10}\quad \underline{10}\quad \underline{10}\quad \underline{\ 5\ }$

Has 45000

Final answer is 90000 + 50000 – 45000 = 95000