Building Blocks Homework 8
Solution

Question A) Are the following functions onto?

a) $f: A \rightarrow B$ where A={1, 2, 3}, B={2, 4, 6} and f(1)=6, f(2)=4, f(3)=2
b) $f: A \rightarrow B$ where A={1, 2, 3, 4}, B={2, 4, 6} and f(1)=4, f(2)=2, f(3)=2, f(4)=4
c) $f: \mathbb{N} \rightarrow \mathbb{N}$ where f(n) = 2n+2
d) $f: \mathbb{R} \rightarrow \mathbb{R}$ where f(x) = (3/2)x+2
e) $f: \mathbb{Z} \rightarrow \mathbb{N}$ where f(k) = $\lfloor 2^k \rfloor$

a) Is onto since each value in B is mapped into.
b) Is not onto. The value 6 in B does not have a map into it.
c) Is not onto. There is no mapping from the natural numbers into 0 and any odd value
d) Is onto. Pick any real number y. Can we find a value x that maps into y? Solve: $y = (3/2)x + 2$ and we get $(y\text{-}2)(2/3) = x$, so yes.
e) Is not onto. The function only maps into integer powers of 2, so it misses values like 3 and 27.

Question B) Are the following functions one-to-one?

a) $f: A \rightarrow B$ where A={1, 2, 3}, B={2, 4, 6} and f(1)=6, f(2)=4, f(3)=2
b) $f: A \rightarrow B$ where A={1, 2, 3, 4}, B={2, 4, 6} and f(1)=4, f(2)=2, f(3)=2, f(4)=4
c) $f: \mathbb{N} \rightarrow \mathbb{N}$ where f(n) = 2n+2
d) $f: \mathbb{R} \rightarrow \mathbb{R}$ where f(x) = (3/2)x+2
e) $f: \mathbb{Z} \rightarrow \mathbb{N}$ where f(k) = $\lfloor 2^k \rfloor$

a) Is one-to-one since each value in B is mapped once.
b) Is not one-to-one. The value 2 in B is mapped twice.
c) Is one-to-one. If f(a) = f(b) then we know that 2a+2 = 2b+2 or solving that a=b.
d) Is one-to-one. If f(a) = f(b) then we know that (3/2)a+2 = (3/2)b+2. This solves to a=b.
e) Is not one-to-one: Consider f(-1) and f(-2). These are both 0 after the floor is applied.

Question C) Are the following functions one-to-one correspondences (bijections)?

- **a)** $f: A \rightarrow B$ where A={1, 2, 3}, B={2, 4, 6} and f(1)=6, f(2)=4, f(3)=2
- **b)** $f: A \rightarrow B$ where A={1, 2, 3, 4}, B={2, 4, 6} and f(1)=4, f(2)=2, f(3)=2, f(4)=4
- **c)** $f: \mathbb{N} \rightarrow \mathbb{N}$ where f(n) = 2n+2
- **d)** $f: \mathbb{R} \rightarrow \mathbb{R}$ where f(x) = (3/2)x+2
- **e)** $f: \mathbb{Z} \rightarrow \mathbb{N}$ where f(k) = $\lfloor 2^{k} \rfloor$

Any correspondence must be both onto and one-to-one.

a) Is a correspondence.
b) Is not one-to-one so not a correspondence.
c) Is not onto so not a correspondence.
d) Is a correspondence.
e) Is not one-to-one so not a correspondence.

Question D) How many different functions can we create from A={a, b, c, d, e} to B = {a, b, c, d, e, f, g}? How many different functions can we create if the function is one-to-one? (Hint: Use counting tasks where each task fixes the mapping of one of the values in set A.)

Different functions allow reuse for each of the 5 mappings from A into 7 items in B.
This is a sequence with $7 \times 7 \times 7 \times 7 \times 7 = 7^{5}$ possibilities.

One-to-One functions will also be a sequence, but we do not allow reuse so we get $7 \times 6 \times 5 \times 4 \times 3$ possibilities.

Question E) How many different functions can we create from A={a, b, c, d, e, f} to B = {1, 2, 3}? How many different functions can we create if the function is one-to-one?

Different functions allow reuse for each of the 6 mappings from A into 3 items in B.
This is a sequence with $3 \times 3 \times 3 \times 3 \times 3 \times 3 = 3^{6}$ possibilities.

One-to-One functions will also be a sequence, but we do not allow reuse so we get $3 \times 2 \times 1 \times 0 = 0$ possibilities.

Question F) Suppose that a store can make the following kinds of cupcakes: Chocolate, Strawberry, Red Velvet, Pumpkin, Praline and Butterscotch. You order a random assortment of 15 cupcakes packed in a box. Without looking inside the box:

a) What can you conclude about your order? (Use the general PHP.)
b) What can you conclude about the number of Chocolate cupcakes in your order?
c) Can you conclude that there is a cupcake of each kind?

a) In PHP we always start with identifying the objects and the value each object is mapped to. In this case, each cupcake has a color. We have n=15 objects and k=6 kinds. According to the GPHP there is at least one flavor that has at least $\lceil 15/6 \rceil = 3$ cupcakes. Or equivalently, we have at least 3 cupcakes of the same flavor.
b) Basically nothing. We know that the number of chocolate cupcakes is in the range of 0 to 15.
c) No. It might be the case that there are some flavors that did not show up.

Note: for these numbers it is pretty likely that a random distribution of cupcakes would be missing a flavor. It is not too hard to count the number of mappings which is 6^{15}. We compare to the number of mappings that have a single void. Select the void flavor (6 possible) and then distribute over the remaining flavors as 5^{15}. Combining, we have 6×5^{15} mappings with exactly one skipped flavor. The ratio of $\frac{6\times 5^{15}}{6^{15}} = \frac{5^{15}}{6^{14}} \approx 0.39$. So at least 39% of the time we would not get a flavor. We didn't count the number of mappings with more than one void, so the probability is a lower bound, but the number of mappings with two or more voids is much smaller and contribute little.

Question G) How many cupcakes must there be to guarantee that there is at least 4 cupcakes for some kind?

We need values of n such that $\lceil n/6 \rceil = 4$. These values are 19, 20, 21, 22, 23, and 24. We get the guarantee at 19.

Question H) Suppose that you have a pile of colored papers (red, yellow, purple, green, blue, and orange.) 31 people walk up and take away one paper each. What can you conclude about the number of the different kinds of colored papers that have been taken using the general PHP?

Here we are assigning people to a color. N=31 and k=6.
By general PHP, there is at least one color that has at least $\lceil 31/6 \rceil = 6$ people.
Or, some color was claimed by at least 6 people.

Question I) Sammy the rat is very confused. He is in a rat maze and is trying to escape. The maze is 15 cells wide by 15 cells deep. Currently he is in the middle of the maze. Every minute, he is going to randomly move to a neighboring cell.

a) How many minutes are required so that we can guarantee Sammy has been at some location more than once?

b) How many minutes are required so that we can guarantee Sammy has been at some location 3 or more times?

c) Can we guarantee after a certain number of minutes that Sammy will be back at his starting cell?

A) We have 15x15=225 locations. Over M minutes we have times, 0, 1, 2, … M. (M+1) We are mapping times to locations. So if we $\lceil (M+1)/225 \rceil = 2$ we can guarantee a repeated location. M=225 does it.
B) Here we need $\lceil (M+1)/225 \rceil = 3$. M=450 works.
C) We can not guarantee this.

You may choose to solve one (and only one) of the following Extra Credit Problems. If you submit more than one, only the first will be graded.

Extra Credit 1) We want to get a feel for how far away Sammy gets from the starting point. Write a program that records the positions Sammy visits.
Get as input the number of walks and the length of the walk. Start Sammy in position 0, 0. At each walk step, generate a random direction left, right, forward, back. If Sammy would leave the maze, just have him remain in that cell. (The edges are at 0 and 14. Keep a count for each cell. Compute the average visits per walk for each cell.

Extra Credit 2) Suppose that we are interested in creating a strictly decreasing function f with domain **N**.

a) Can we find an f with co-domain **N**?
b) Can we find an f with co-domain **Q**?

a) In a strictly decreasing function as x increases, f(x) decreases. If we pick some value k in N, we would have f(k) > f(k+1) > f(k+2) … each of which is a value from N. But this would require an infinite number of values less than f(k), which is not possible.
b) If the domain is the rational numbers, then we can find many such sequences including things like 0, -1, -2, -3, and 1, ½, ¼, …