Structural Induction

Let S be the subset of the set of ordered pairs of integers defined recursively by

Basis step: (2, 4) ∈ S.

Recursive step: If (a,b) ∈ S, then (a+1, b+1) ∈ S and (a-1,b-1) ∈ S.

b) [6 pts] Use structural induction to show for any pair (x,y) ∈ S that x+2=y

Base Case*:* (2, 4) ∈ S

We need to show that 2+2=4. **Check**

Induction Case: If (a,b) ∈ S, then (a+1, b+1) ∈ S and (a-1,b-1) ∈ S.

Assume: The property holds for the pair (a,b) that we get from S.

Induction hypothesis is: **a+2 = b**

Show: The property holds for the pairs (a+1, b+1) and (a-1, b-1) that we put into the set.

For (a+1, b+1) need to show that (a+1) + 2 = (b+1). If we subtract one from both sides we get a+2 = b, which is true by the IH. **Check**

For (a-1, b-1) need to show that (a-1) + 2 = (b-1). If we add one to both sides we get a+2 = b, which is true by the IH. **Check**

QED

Recursively Defined Sets [5 pts each]

Give a recursive definition of the set that contains integer powers of three with an even exponent. $\{3^0, 3^2, 3^4, 3^6, 3^8, \cdots\}$

Basis step: $1 \in S$.

Recursive step: If $x \in S$, then $9x \in S$

Basis step: $3^0 \in S$.

Recursive step: If $3^x \in S$, then $3^{x+2} \in S$

b) Give a recursive definition of the set that contains all bit strings

$\{\lambda, 0, 1, 00, 01, 10, 11, 000, \cdots\}$

(Note: λ is the empty string and you can indicate the concatenation of x with y by xy)

Basis step: $\lambda \in S$.

Recursive step: If $x \in S$, then $x0 \in S$ and $x1 \in S$

Basis step: $\lambda, 0, 1 \in S$.

Recursive step: If $x \in S$, and $y \in S$ then $xy \in S$

Question 6) Strong Induction [10 pts]

Suppose that you have a rope that is n feet long and you are going to cut it into 1-foot long segments. You can cut the rope at any foot marker. So for example, if you have a 10-foot rope, you could cut it into a 4-foot and 6-foot length piece. (Along with other options as well.) Use strong induction to show that the number of cuts required to cut an n-foot rope into 1-foot segments is n-1 for all positive n.

a) What is P(n)?

"The number of cuts required to cut an n-foot rope into 1-foot segments is n-1"

b) What is n0?

1

c) Show P(n0) is true.

"The number of cuts required to cut an 1-foot rope into 1-foot segments is 1-1"
No cuts are needed. Check

d) What can you assume?

P(1), P(2), P(3), ...P(k)

e) What do you need to show?

P(k+1)

f) Show it!

We have a rope that is k+1 feet and we need to show that it will require k cuts to get it into 1-foot long pieces.

If we think about our first cut it can be anywhere. We get two pieces, one of length r and the other of length k+1 – r.

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o--------------------------X----------------------------------------o
        r                                k + 1 – r
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We know that both of the pieces must be at least 1-foot and can not be longer than k-feet. That means we can apply the induction hypothesis on each of the smaller pieces of rope. The total cuts is:

Original cut:	1
Cuts for left by IH:	r-1
Cuts for right by IH:	k+1-r-1

Adding we get 1 + r – 1 + k + 1 – r – 1 = k. Check.

QED