Fibonacci sequence: Named after the mathematician Fibonacci (alsoLeonardo Bonacci or Leonardo De Pisa), it was used to solve a problem he posed in Liber Abaci about the population growth of rabbits on an island. There are earlier examples of the sequence that show up in India.

Leonardo is the one of the seminal mathematicians from the 1200s that very few people today recognize. He traveled to Northern Africa and was exposed to mathematical ideas that originated in the Middle East and India. He brought back algebra and the decimal number system from Northern Africa to Italy. In addition to original contributions, he wrote a series of math texts that revolutionized mathematics through out Europe.

Fibonacci sequence

$$f_n = f_{n-1} + f_{n-2}$$
$$f_0 = 0$$
$$f_1 = 1$$

Values in the sequence: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, …

Finding a new recursive definition for the Fibonacci sequence.

Take the recurrence can replace n by n-1 to get

$$f_{n-1} = f_{(n-1)-1} + f_{(n-1)-2}$$

which is just

$$f_{n-1} = f_{n-2} + f_{n-3}$$

Plug that back into the original recurrence

$$f_n = f_{n-1} + f_{n-2} = \left(f_{n-2} + f_{n-3}\right) + f_{n-2}$$

or

$$f_n = 2f_{n-2} + f_{n-3}$$

We see that this relation holds for the values 21, 34, 55, and 89.

$$2(34) + 21 = 68 + 21 = 89$$

Because this reaches back three terms, we need to have three initial conditions.

Fibonacci sequence

$$f_n = 2f_{n-2} + f_{n-3}$$
$$f_0 = 0$$
$$f_1 = 1$$
$$f_2 = 1$$

This alternate form allows us to discover some important information about the growth of the numbers in the Fibonacci sequence. (Please note, that in the following arguments, claims are true provided you are far enough along in the sequence, but may not be true for small values of n.)

Lower Bounding the sequence:
Consider values in the sequence that are two apart ($A = f_n$ and B=f_{n-2}). Because the values in my sequence are non-negative, A is more than twice as big as B.

$$f_n = 2f_{n-2} + f_{n-3}$$

$$\boxed{\text{twice}}\ \boxed{\text{and a little bit more}}$$

Restating: This means than every time I add 2 to n, the value of the Fibonacci number more than doubles. It does not take long to get to a point in the sequence where

$$f_n > 2^{n/2} = \sqrt{2}^n$$

or using asymptotic notation $\boxed{f_n = \Omega\left(\sqrt{2}^n\right)}$

Upper Bounding the sequence:
Again, consider values in the sequence that are two apart ($A = f_n$ and B=f_{n-2}). The values in my sequence are increasing. This means that $f_{n-3} < f_{n-2}$

$$f_n = 2f_{n-2} + f_{n-3}$$

$$\boxed{\text{twice}}\ \boxed{\text{but not three times}}$$

Restating: This means than every time I add 2 to n, the value of the Fibonacci number less than triples. It does not take long to get to a point in the sequence where

$$f_n < 3^{n/2} = \sqrt{3}^n$$

or using asymptotic notation $\boxed{f_n = O\left(\sqrt{3}^n\right)}$

Tight Bound:
We will show later that the ratio between values in the Fibonacci sequence approaches $\frac{1+\sqrt{5}}{2} \approx 1.618$ which is known as the golden ratio.

$$\boxed{f_n = \Theta\left(\left(\frac{1+\sqrt{5}}{2}\right)^n\right)}$$