Discrete Math Homework 6
Due Wednesday, February 22 at the beginning of class

General instructions:

- Use standard size paper (8.5 by 11).
- Answer each question in order using a single column.
- Be neat. If we cannot read your solution it is wrong.
- Show your work. If you just write an answer, you will get minimal credit even if the answer is correct.

Rosen section 1.8.

Question A) Show that $n^4 + 2n^3 + n^2$ is divisible by 4 for all integer n.

Hint: Note that a number x is divisible by 4 iff xmod4=0.
Do a proof by cases for each of the possible values of nmod4.
Remember our rules for mods:

$$(x+y) \bmod m = (x \bmod m + y \bmod m) \bmod m$$

$$(x \cdot y) \bmod m = ((x \bmod m) \cdot (y \bmod m)) \bmod m$$

You can look at home work 5 solutions to see manipulations of mods.

To solve this we will do a proof by cases. In particular, every integer value will be covered by one of the following four cases:

1) $n \bmod 4 = 0$
2) $n \bmod 4 = 1$
3) $n \bmod 4 = 2$
4) $n \bmod 4 = 3$

Case 1) $n \bmod 4 = 0$

We compute $\left(n^4 + 2n^3 + n^2\right) \bmod 4$

$$= \left(\left(n \bmod 4\right)^4 + 2\left(n \bmod 4\right)^3 + \left(n \bmod 4\right)^2\right) \bmod 4$$
$$= \left(\left(0\right)^4 + 2\left(0\right)^3 + \left(0\right)^2\right) \bmod 4$$
$$= \left(0\right) \bmod 4$$
$$= 0$$

So the expression is divisible by 4

Case 2) $n \bmod 4 = 1$

We compute $\left(n^4 + 2n^3 + n^2\right) \bmod 4$

$$= \left(\left(n \bmod 4\right)^4 + 2\left(n \bmod 4\right)^3 + \left(n \bmod 4\right)^2\right) \bmod 4$$
$$= \left(\left(1\right)^4 + 2\left(1\right)^3 + \left(1\right)^2\right) \bmod 4$$
$$= \left(1 + 2 + 1\right) \bmod 4$$
$$= \left(4\right) \bmod 4$$
$$= 0$$

So the expression is divisible by 4

Case 3) $n \bmod 4 = 2$

We compute $\left(n^4 + 2n^3 + n^2\right) \bmod 4$

$$= \left(\left(n \bmod 4\right)^4 + 2\left(n \bmod 4\right)^3 + \left(n \bmod 4\right)^2\right) \bmod 4$$
$$= \left(\left(2\right)^4 + 2\left(2\right)^3 + \left(2\right)^2\right) \bmod 4$$
$$= \left(16 + 16 + 4\right) \bmod 4$$
$$= \left(36\right) \bmod 4$$
$$= 0$$

So the expression is divisible by 4

Case 4) $n \bmod 4 = 3$

We compute $\left(n^4 + 2n^3 + n^2\right) \bmod 4$

$$= \left(\left(n \bmod 4\right)^4 + 2\left(n \bmod 4\right)^3 + \left(n \bmod 4\right)^2\right) \bmod 4$$
$$= \left(\left(3\right)^4 + 2\left(3\right)^3 + \left(3\right)^2\right) \bmod 4$$

The value inside the mod is potentially larger than I want to compute, so lets apply mod4 inside to reduce it to something more manageable.

$$= \left(9 \cdot 9 + 6 \cdot 9 + 9\right) \bmod 4$$
$$= \left(\left(9 \bmod 4\right) \cdot \left(9 \bmod 4\right) + \left(6 \bmod 4\right) \cdot \left(9 \bmod 4\right) + \left(9 \bmod 4\right)\right) \bmod 4$$
$$= \left(1 \cdot 1 + 2 \cdot 1 + 1\right) \bmod 4$$
$$= \left(4\right) \bmod 4$$
$$= 0$$

So the expression is divisible by 4

In all cases, the expression is divisible by 4.

QED

Question B) Rosen 1.8 Exercise 10 (p. 108) Hint: Think about the spacing of perfect squares.

Prove that either $2 \cdot 10^{500} + 15$ or $2 \cdot 10^{500} + 16$ is not a perfect square.
Think about two consecutive perfect squares n^2 and $(n+1)^2 = n^2 + 2n + 1$. The difference between them is 2n+1 which increases. The two numbers listed in the problem have a difference of 1 and therefore can not both be perfect squares.
QED.
Since we did not determine if either of the values in particular is not a perfect square, this is a non-constructive proof.

Rosen section 2.1.

Question C) Rosen 2.1 Exercise 6 (p. 125)
Suppose that A={2, 4, 6} B={2, 6} C={4, 6} and D={4, 6, 8}. Determine which of these are subsets of the others.

Note that a set cannot be a subset of a smaller set, So we start with the smallest sets and work our way up in size.
1) Is $B \subseteq C$? No. B has element 2 and C does not
2) Is $C \subseteq B$? No. C has element 4 and B does not
3) Is $B \subseteq A$? Yes
4) Is $B \subseteq D$? No. B has element 2 and D does not
5) Is $C \subseteq A$? Yes
6) Is $C \subseteq D$? Yes
7) Is $A \subseteq D$? No. A has element 2 and D does not
8) Is $D \subseteq A$? No. D has element 8 and A does not

Question D) Rosen 2.1 Exercise 10 (p. 125)

Determine whether these statements are true for false
a) $\varnothing \in \{\varnothing\}$ True. The empty set is an element of the set on the right.
b) $\varnothing \in \{\varnothing,\{\varnothing\}\}$ True. The empty set is the first element of the set on the right
c) $\{\varnothing\} \in \{\varnothing\}$ False. The set containing the empty set is not in the set on the right
d) $\{\varnothing\} \in \{\{\varnothing\}\}$ True. The set on the right has a single element, which is the set containing the empty set.
e) $\{\varnothing\} \subset \{\varnothing,\{\varnothing\}\}$ True, every value in the set on the left (which is just the empty set) is in the set on the right. Also, these are not the same sets.
f) $\{\{\varnothing\}\} \subset \{\varnothing,\{\varnothing\}\}$ True. The only value in the set on the left is the set containing the empty set which is the second value in the set on the right. Also, these are not the same sets.
g) $\{\{\varnothing\}\} \subset \{\{\varnothing\},\{\varnothing\}\}$ False. While the set on the left is a subset of the set on the right, they are in fact the same set. The set on the right lists its single value twice. So we do not have a proper subset relation.

Question E) Rosen 2.1 Exercise 20 (p. 126)

What is the cardinality of each of the following sets
a) $\varnothing$ The empty set has cardinality 0
b) $\{\varnothing\}$ This set has 1 element
c) $\{\varnothing,\{\varnothing\}\}$ This set has 2 elements
d) $\{\varnothing,\{\varnothing\},\{\varnothing,\{\varnothing\}\}\}$ This set has 3 elements $\varnothing$ $\{\varnothing\}$ $\{\varnothing,\{\varnothing\}\}$

Question F) Suppose that $A_n = \{1,2,3,\cdots n\}$ and $B = \{true, false\}$. What are the following?

a) $A_2 \times B$

$$\{(1,true),(1,false),(2,true),(2,false)\}$$

b) $P(B)$

$$\{\{\ \},\{true\},\{false\},\{true,false\}\}$$

c) $|P(A_5)|$

A5 is {1, 2, 3, 4, 5}. Each value is either in or out of the subset. So we get 2x2x2x2x2 = 32

d) $|A_{10} \times A_{20}|$

The number of elements is the cardinality of the first set times the cardinality of the second set. That is 10 in the first and 20 in the second for a product of 200.

Rosen section 2.2.

Question G) Rosen 2.2 Exercise 4 (p. 136).

A={a, b, c, d, e} and B={a, b, c, d, e, f, g, h}

a) $A \cup B = \{a,b,c,d,e,f,g,h\}$

b) $A \cap B = \{a,b,c,d,e\}$

c) $A - B = \{\ \}$

d) $B - A = \{f,g,h\}$

Question H) Rosen 2.2 Exercise 12 (p. 136). (Use a membership table.)

Prove that $A \cup (A \cap B) = A$

A	B	$A \cap B$	$A \cup (A \cap B)$
1	1	1	1
1	0	0	1
0	1	0	0
0	0	0	0

These columns are the same, so the set identity holds.

Question I) Rosen 2.2 Exercise 24 (p. 136).

Show that $(A-B)-C=(A-C)-(B-C)$. There are lots of ways that we can do this… Application of know identities, Venn diagrams, membership tables…

This solution will use a membership table. Membership table for A-B is 1 if and only if A is 1 and B is 0. My strategy is to find the cells with a one and then fill in after with zeros.

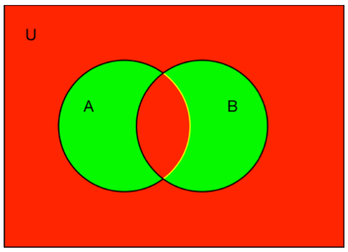
A	B	C	$(A-B)$	$(A-B)-C$	$A-C$	$B-C$	$(A-C)-(B-C)$
1	1	1	0	0	0	0	0
1	1	0	0	0	1	1	0
1	0	1	1	0	0	0	0
1	0	0	1	1	1	0	1
0	1	1	0	0	0	0	0
0	1	0	0	0	0	1	0
0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0

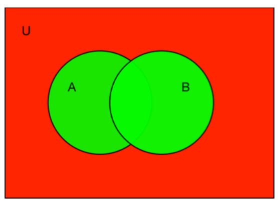
The two columns contain the same values, so they are equal.

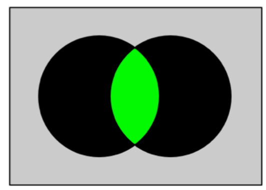
Question J) Rosen 2.2 Exercise 34 and Exercise 35 (p. 137). (Use Venn diagrams)

Draw a Venn diagram for the symmetric difference of A and B. ($A \oplus B$) and then show that $A \oplus B = (A \cup B) - (A \cap B)$.

The symmetric difference is defined as the set containing values in A or B, but not in both. That is represented by the green areas in this Venn diagram



The union is 

and the intersection is 

When you subtract the green areas on the right from the green areas on the left, you are left with the symmetric difference .

You may choose to solve one (and only one) of the following Extra Credit Problems. If you submit more than one, only the first will be graded.

Extra Credit 1) Rosen 1.8 Exercise 38 (p. 109)

Prove or disprove that if you have an 8 gallon jug of water and two empty jugs with capacities of 5 gallons and 3 gallons, then you can measure 4 gallons by successively pouring some or all of the water in a jug into another jug.

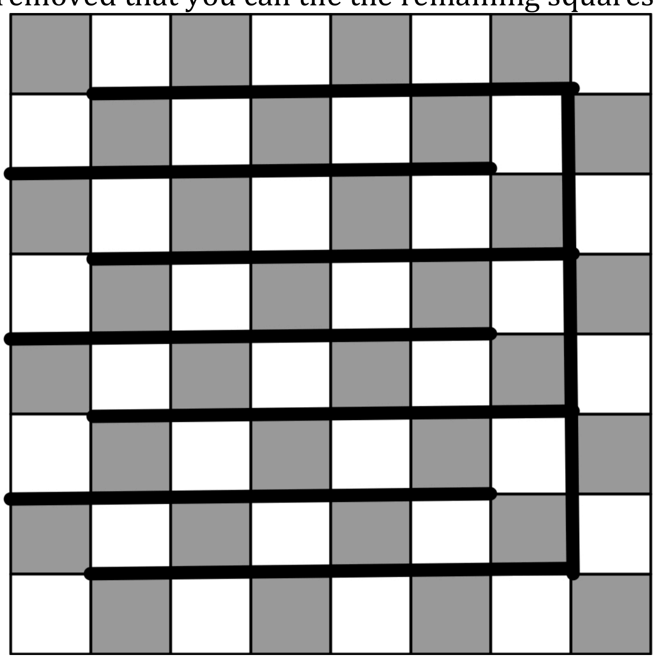
You need to end up with 4 gallons in the 8 or 5 gallon jug.

What we did	Amount in 8	Amount in 5	Amount in 3
Started with	8	0	0
Pour from 8 to 5	3	5	0
Pour from 5 to 3	3	2	3
Pour from 3 to 8	6	2	0
Pour from 5 to 3	6	0	2
Pour from 8 to 5	1	5	2
Pour from 5 to 3	1	4	3

We are done!

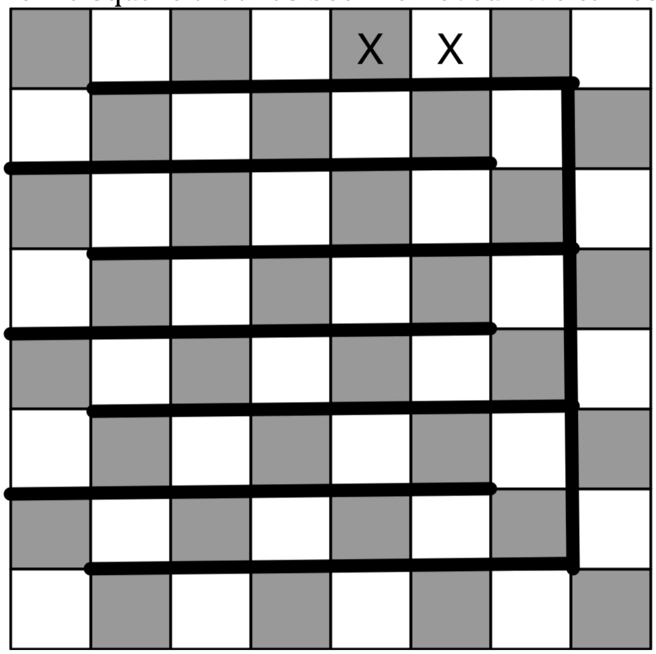
Extra Credit 2) Rosen 1.8 Exercise 46 (p. 109) (See p. 103 for definition of domino and checkerboard coloring)

Prove that when a white/black checkerboard has one white and one black square removed that you can tile the remaining squares with dominos.

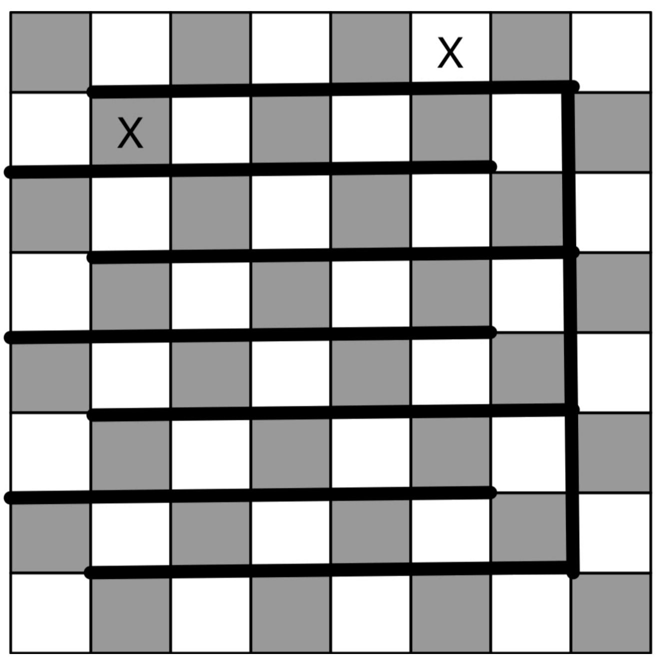


If you look at this, you see that we have a chain of squares that alternate black and white. Furthermore, any two consecutive squares in the chain can be covered by a 1x2 domino.

Case 1: The white and the black square are adjacent on the chain. We start tiling from a square that has been removed. We can continue tiling until all is covered.



Case 2: The white an black squares are not adjacent on the chain. This will divide the chain into two parts. Note that each of the parts must have an even number of squares and we can tile each part of the chain.



Extra Credit 3) Rosen 2.2 Exercise 40 (p. 137)

Determine if the symmetric difference is associative… Is $(A \oplus B) \oplus C = A \oplus (B \oplus C)$

A	B	C	$(A \oplus B)$	$(A \oplus B) \oplus C$	$(B \oplus C)$	$A \oplus (B \oplus C)$
1	1	1	0	1	0	1
1	1	0	0	0	1	0
1	0	1	1	0	1	0
1	0	0	1	1	0	1
0	1	1	1	0	0	0
0	1	0	1	1	1	1
0	0	1	0	1	1	1
0	0	0	0	0	0	0

The two columns contain the same values, so they are equal.