Discrete Math Homework 4 Solution

Revised 2-08-2017 Problem A, part c. Problem statement incorrectly transcribed.

Exam 1 is Friday, February 10 and covers all the material up to (and including) Rosen 1.5

Rosen section 1.4.

Question A) Rosen 1.4 Exercise 10 a, c, e (p. 53)

C(x) is "x has a Cat", D(x) is "x has a Dog", F(x) is "x has a Ferret".

a) A student in your class has a cat, a dog, and a ferret.

$\exists x[C(x) \wedge D(x) \wedge F(x)]$ where x is a student in your class

c) Some student in your class has a cat and a dog, but not a ferret.

$\exists x[C(x) \wedge D(x) \wedge \neg F(x)]$ where x is a student in your class

Corrected statement from part c: 'Some student in your class has a cat and a ferret, but not a dog."

$\exists x.[C(x) \wedge F(x) \wedge \neg D(x)]$ where x is a student in your class

e) For each of the three animals, there is some student in your class that has this animal as a pet.

$\exists x.\exists y.\exists z.[C(x) \wedge D(y) \wedge F(z)]$ where x,y,z is a student in your class

another solution is

$\exists x.C(x) \wedge \exists x.D(x) \wedge \exists x.F(x)$ where x is a student in your class

(Note that each x can be different.)

Question B) Rosen 1.4 Exercise 12 (p. 53)

Q(x) is "x + 1 > 2x" and the domain is all integers.

a) $Q(0)$ is "$0+1>2\cdot 0$" or "$1>0$" which has the value true
b) $Q(-1)$ is "$-1+1>2\cdot(-1)$" or "$0>-2$" which has the value true
c) $Q(1)$ is "$1+1>2\cdot 1$" or "$2>2$" which has the value false
d) $\exists x.Q(x)$ has the value true (x=0 is an example)
e) $\forall x.Q(x)$ has the value false (x=1 is a counter example)
f) $\exists x.\neg Q(x)$ has the value true (x=1 is an example)
g) $\forall x.\neg Q(x)$ has the value false (x=0 is a counter example)

Question C) Show that $\forall x.(P(x) \vee Q(x))$ and $\forall x.P(x) \vee \forall x.Q(x)$ are not logically equivalent. (Hint: Come up with a counter example consisting of a domain and truth values for the predicates P(x) and Q(x) over that domain.)

For the counter example, the domain will be {a, b}. Define P and Q as
P(a) = true Q(a) = false
P(b) = false Q(b) = true

$$\begin{aligned}\forall x.(P(x) \vee Q(x)) &= [P(a) \vee Q(a)] \wedge [P(b) \vee Q(b)] \\ &= [\mathit{true} \vee \mathit{false}] \wedge [\mathit{false} \vee \mathit{true}] \\ &= [\mathit{true}] \wedge [\mathit{true}] \\ &= \mathit{true}\end{aligned}$$

$$\begin{aligned}\forall x.P(x) \vee \forall x.Q(x) &= [P(a) \wedge P(b)] \vee [Q(a) \wedge Q(b)] \\ &= [\mathit{true} \wedge \mathit{fase}] \vee [\mathit{false} \wedge \mathit{true}] \\ &= [\mathit{false}] \vee [\mathit{false}] \\ &= \mathit{false}\end{aligned}$$

These have a different truth value, so they are not logically equivalent.

Rosen section 1.5.

Question D) Rosen 1.5 Exercise 6 c,e (p. 64)

C(x,y) is "Student x is enrolled in class y" with domain x is all students at your school and y is all classes at your school.

c) $\exists y.C(CarolSitea, y)$ in English is "Carol Sitea is enrolled in some class at NWMSU"

e) $\exists x.\exists y.\forall z\left[(x \neq y) \wedge \left(C(x,z) \rightarrow C(y,z)\right)\right]$ in English is "There are two different students at NWMSU such that if x is in a class then y is also in the class"

Question E) Rosen 1.5 Exercise 12 d, j, l, o (p. 65)
I(X) is "x has an internet connection"; C(x,y) is "x and y have chatted over the internet"; The domain is students in your class.

d) No one in the class has chatted with Bob.
$\forall x.\neg C(x, Bob)$ or $\neg\exists x.C(x, Bob)$

j) Everyone in your class with an internet connection has chatted over the internet with at least one other student in your class.
$\forall x.\left[I(x) \rightarrow \exists y.\left[(x \neq y) \wedge C(x,y)\right]\right]$

l) There are two students in your class who have not chatted with each other over the Internet.
$\exists x \exists y.\left[(x \neq y) \wedge \neg C(x,y)\right]$

o) There are two students in your class who between them have chatted with everyone else in the class.
$\exists x \exists y.\left[(x \neq y) \wedge \forall z.\left[C(x,z) \vee C(y,z)\right]\right]$

Question F) Rosen 1.5 Exercise 20 (p. 67)

a) The product of two negative integers is positive.
$\forall x \forall y.\left[(x>0)\wedge(y>0)\rightarrow xy>0\right]$ where x, y are integers

b) The average of two positive integers is positive.
$\forall x \forall y.\left[(x>0)\wedge(y>0)\rightarrow \frac{x+y}{2}>0\right]$ where x, y are integers

c) The difference of two negative integers is not necessarily negative.
$\exists x \exists y.\left[(x<0)\wedge(y<0)\wedge\neg(x-y<0)\right]$ where x, y are integers

or

$\exists x \exists y.\left[(x<0)\wedge(y<0)\wedge(x-y\geq 0)\right]$

d) The absolute value of the sum of two integers does not exceed the sum of the absolute values of these integers.
$\forall x \forall y.\neg\left[|x+y|>|x|+|y|\right]$ where x, y are integers

Question G) Rosen 1.5 Exercise 28 b, e, i (p. 67)

The domain is all reals.

b) $\forall x \exists y\left(x=y^2\right)$ has a truth value of ***false***.

We show this by giving a counter example of x = -1 which results in $\exists y\left(-1=y^2\right)$ but we know if you square any real number the result must be non-negative, so no y exists that can make it true.

e) $\forall x\left(x\neq 0\rightarrow \exists y(xy=1)\right)$ has a truth value of ***true***.
Let x be a real number… call it D. If D is zero, the implication is trivially true. If D is non-zero we examine $\exists y(Dy=1)$ We can find a y by computing 1/D. This value is real for non-zero values.

i) $\forall x \exists y(x+y=2\wedge 2x-y=1)$ has a truth value of ***false***.
We show this by giving a counter example of x = 0 which results in $\exists y(0+y=2\wedge 2\cdot 0-y=1)$ which is $\exists y(y=2\wedge y=-1)$. There is no value of y that can equal to both 2 and -1.

Question H) Rosen 1.5 Exercise 30 (p. 67)

a) $\neg\exists y\exists x.P(x,y)$ is logically equivalent to $\forall y\neg\exists x.P(x,y)$
is logically equivalent to $\forall y\forall x.\neg P(x,y)$

b) $\neg\forall x\exists y.P(x,y)$ is logically equivalent to $\exists x.\neg\exists y.P(x,y)$
is logically equivalent to $\forall y\forall x.\neg P(x,y)$

c) $\neg\exists y.(Q(y)\wedge\forall x.\neg R(x,y))$ is logically equivalent to $\forall y.\neg(Q(y)\wedge\forall x.\neg R(x,y))$
is logically equivalent to $\forall y.(\neg Q(y)\vee\neg\forall x.\neg R(x,y))$
is logically equivalent to $\forall y.(\neg Q(y)\vee\exists x.\neg\neg R(x,y))$
is logically equivalent to $\forall y.(\neg Q(y)\vee\exists x.R(x,y))$

d) $\neg\exists y.(\exists x.R(x,y)\vee\forall x.S(x,y))$ is logically equivalent to
$\forall y.\neg(\exists x.R(x,y)\vee\forall x.S(x,y))$
is logically equivalent to $\forall y.(\neg\exists x.R(x,y)\wedge\neg\forall x.S(x,y))$
is logically equivalent to $\forall y.(\forall x.\neg R(x,y)\wedge\exists x.\neg S(x,y))$

e) $\neg\exists y.(\forall x.\exists z.T(x,y,z)\vee\exists x.\forall z.U(x,y,z))$ is logically equivalent to
$\forall y.\neg(\forall x.\exists z.T(x,y,z)\vee\exists x.\forall z.U(x,y,z))$
is logically equivalent to $\forall y.(\neg\forall x.\exists z.T(x,y,z)\wedge\neg\exists x.\forall z.U(x,y,z))$
is logically equivalent to $\forall y.(\exists x\neg.\exists z.T(x,y,z)\wedge\forall x.\neg\forall z.U(x,y,z))$
is logically equivalent to $\forall y.(\exists x\forall z.\neg T(x,y,z)\wedge\forall x.\exists z.\neg U(x,y,z))$

Question I) Rosen 1.5 Exercise 40 (p. 68) (Hint: There is a counter example for all of them. When you state the counter example in parts a, b, give a value for x and then show that the resulting predicate after substitution cannot be true. For part c, give a value for both x and y.

The domain is all integers.

a) $\forall x.\exists y.(x = 1/y)$ Counter example: Let x = 2. This leaves the statement $.\exists y.(2 = 1/y)$. The only possible solution for y is ½ which is not an integer.

b) $\forall x.\exists y.(y^2 - x < 100)$ Counter example: x=-100. This leaves the statement $.\exists y.(y^2 - (-100) < 100)$ or $.\exists y.(y^2 < 0)$, but we know that every integer value when squared is non-negative.

c) $\forall x.\forall y.(x^2 \neq y^3)$ Counter example: x=0 and y=0. This leaves $(0^2 \neq 0^3)$, which is false.

You may choose to solve one (and only one) of the following Extra Credit Problems. If you submit more than one, only the first will be graded.

Extra Credit 1) Rosen 1.4 Exercise 62 (p. 57)

P(x) is "x is a duck", Q(x) is "x is one of my poultry", R(x) is "x is an officer", S(x) is "x is willing to waltz".

a) No ducks are willing to waltz.

$$\forall x.[P(x) \rightarrow \neg S(x)]$$

b) No officers ever decline to waltz.

$$\forall x.[R(x) \rightarrow S(x)]$$

c) All my poultry are ducks.

$$\forall x.[Q(x) \rightarrow P(x)]$$

d) My poultry are not officers

$$\forall x.[Q(x) \rightarrow \neg R(x)]$$

e) Does d follow from a,b, and c?

Part a leaves us with three possibilities, where a, b and c are arbitrary elements in the domain.
P(a) is false and S(a) is true
P(b) is false and S(b) is false
P(c) is true and S(c) is false

We can summarize this in a truth table with red lines ruled out.

P(x)	Q(x)	R(x)	S(x)
T	T	T	T
T	T	T	F
T	T	F	T
T	T	F	F
T	F	T	T
T	F	T	F
T	F	F	T
T	F	F	F
F	T	T	T
F	T	T	F
F	T	F	T
F	T	F	F
F	F	T	T
F	F	T	F
F	F	F	T
F	F	F	F

Part b rules out R is true and S is false.

P(x)	Q(x)	R(x)	S(x)
T	T	T	T
T	T	T	F
T	T	F	T
T	T	F	F
T	F	T	T
T	F	T	F
T	F	F	T
T	F	F	F
F	T	T	T
F	T	T	F
F	T	F	T
F	T	F	F
F	F	T	T
F	F	T	F
F	F	F	T
F	F	F	F

Part c rules out Q is true and P is false

P(x)	Q(x)	R(x)	S(x)
T	T	T	T
T	T	T	F
T	T	F	T
T	T	F	F
T	F	T	T
T	F	T	F
T	F	F	T
T	F	F	F
F	T	T	T
F	T	T	F
F	T	F	T
F	T	F	F
F	F	T	T
F	F	T	F
F	F	F	T
F	F	F	F

For part d to be true, it must be that on any remaining line that Q and R are not both true. Since this is the case, part d follows from a/b/c.

~~**Extra Credit 2)** Rosen 1.4 Exercise 12 (p. 53)~~

Extra Credit 3) Show that $\exists x.P(x)$ and $\forall x.P(x)$ are not logically equivalent.

We need a domain and definition for P such that the two statements have different truth values. There are many that will work. Here are two.

Domain is {}
$\exists x.P(x)$ is trivially false
$\forall x.P(x)$ is trivially true

Domain is {a, b} and P(a)=true, P(b)=false
$\exists x.P(x) = P(a) \vee P(b) = true \vee false = true$
$\forall x.P(x) = P(a) \wedge P(b) = true \wedge false = false$