Discrete Math Homework 1 Solution
Revised Answer for Problem C – Jan 20

General instructions:

- Use standard size paper (8.5 by 11).
- Answer each question in order using a single column.
- Be neat. If we cannot read your solution it is wrong.
- Show your work. If you just write an answer, you will get minimal credit even if the answer is correct.

Question A) Compute the value of the following arithmetic sum using Gauss's method that we discussed in class.

A = 1,000 + 2,000 + 3,000 + 4,000 + ... + 20,000 + 21,000 + 22,000 + 23,000

We pair the first and last term and get the value 24,000. There are 23 values, so the number of pair is 23/2. We multiply to get (24,000)(23/2). Evaluation gives the final answer of 276,000.

Question B) Compute the value of the following geometric sum using the method that we discussed in class.

$$G = 1 + 3 + 9 + 27 + \ldots + 3^{40} + 3^{41} + 3^{42}$$

We multiply the sum by the common ration of 3 to get:

$$3G = 3 + 9 + 27 + \ldots + 3^{41} + 3^{42} + 3^{43}$$

Subtracting gives (Most of the terms cancel.)

$$G - 3G = 1 - 3^{43}$$

Solving for G

$$G = \frac{1-3^{43}}{1-3} = \frac{3^{43}-1}{3-1} = \frac{3^{43}-1}{2}$$

Evaluating results in

$$G = 164128483697268538813$$

Question C) What are the first 10 values in the following sequence where i starts at 0?

$$a_i = (i-1)(i+1)$$

$$a_0 = (0-1)(0+1) = (-1)(1) = -1$$
$$a_1 = (1-1)(1+1) = (0)(2) = 0$$
$$a_2 = (2-1)(2+1) = (1)(3) = 3$$
$$a_3 = (3-1)(3+1) = (2)(4) = 8$$
$$a_4 = (4-1)(4+1) = (3)(5) = 15$$
$$a_5 = (5-1)(5+1) = (4)(6) = 24$$
$$a_6 = (6-1)(6+1) = (5)(7) = 35$$
$$a_7 = (7-1)(7+1) = (6)(8) = 48$$
$$a_8 = (8-1)(8+1) = (7)(9) = 63$$
$$a_9 = (9-1)(9+1) = (8)(10) = 80$$

Alternatively, we could observe that

$$a_i = (i-1)(i+1) = i^2 - 1$$

and use that formula instead.

a4 should have been 15

Question D) Expand the following summation and then compute its value.

$$S = \sum_{i=1}^{5} \frac{1}{i^2}$$

$$S = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2}$$
$$= 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25}$$

We can compute this exactly as a rational, or use a calculator to get a decimal approximation.

$$S \approx 1.463611111111111$$

or

$$S = \frac{4 \cdot 9 \cdot 16 \cdot 25}{4 \cdot 9 \cdot 16 \cdot 25} + \frac{9 \cdot 16 \cdot 25}{4 \cdot 9 \cdot 16 \cdot 25} + \frac{4 \cdot 16 \cdot 25}{4 \cdot 9 \cdot 16 \cdot 25} + \frac{4 \cdot 9 \cdot 25}{4 \cdot 9 \cdot 16 \cdot 25} + \frac{4 \cdot 9 \cdot 16}{4 \cdot 9 \cdot 16 \cdot 25}$$
$$= \frac{14400+3600+1600+900+576}{14400}$$
$$= \frac{21076}{14400}$$

Cancelling common factors, we get the rational number in canonical form.

$$= \frac{21076}{14400} = \frac{10538}{7200} = \frac{5269}{3600}$$

Question E) Give a formula for the arithmetic sequence a_i, that starts at 10 and increases by 5 each time.

$$a_i = 10 + 5i \quad \text{where i starts at 0}$$

or

$$a_i = 5 + 5i \quad \text{where i starts at 1}$$

would be acceptable

Question F) Express the sum of the following geometric sequence using summation notation. Use i as the index variable with a lower limit of one. (*You do not have the evaluate the sum, just write it using summation notation.*)

5, 25, 125, 625, 3125, 15625, 78125, 390625

We count 8 terms and since the lower limit is 1, the upper limit must be 8.

To see the pattern, lets pick the third term. The value of i is 3 and the value of the term is 125, which is 5 cubed. The formula is

$$a_i = 5^i$$

Putting it together, we have

$$\sum_{i=1}^{8} 5^i$$

Question G) Compute the value of the following summation by multiplying out the expression inside the summation and then separating the result. Apply the closed form expressions of the common sequences to get the final answer.

$$\sum_{i=1}^{n} (i+1)^2$$

Multiply the expression

$$\sum_{i=1}^{n}(i+1)^2 = \sum_{i=1}^{n}\left(i^2+2i+1\right)$$

Separate

$$= \sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} 2i + \sum_{i=1}^{n} 1$$

Factor out constants

$$= \sum_{i=1}^{n} i^2 + 2\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1$$

Apply known

$$= \frac{n(n+1)(2n+1)}{6} + 2\frac{n(n+1)}{2} + n$$

Combine and simplify

$$= \frac{n(n+1)(2n+1)}{6} + 2\frac{n(n+1)}{2}\left(\frac{3}{3}\right) + n\left(\frac{6}{6}\right)$$

$$= \frac{n(n+1)(2n+1)+6n(n+1)+6n}{6}$$

$$= \frac{2n^3+3n^2+n+6n^2+6n+6n}{6}$$

$$= \frac{2n^3+9n^2+13n}{6}$$

While it does not guarantee correctness for all values of n, we can verify that the formula does work when n=1. The sum and the formula both give 4.

Question H) Suppose that we have n people sitting in a room numbered from 1 to n. We will award cash to each pair based on the sum of their values. (For example, pair (1, 4) will receive $5.

We will use a double summation to figure out the total amount of cash that will be paid out. The following summation generates the pairs (i, j). [*The limits guarantee that i<j. This means that a person is not allowed to pair with themselves so no (2,2). We also consider that the pair (1,5) is the same as pair(5,1). We will only count one of them.*]

Give a closed form solution for Total.

$$Total = \sum_{j=1}^{n}\sum_{i=1}^{j-1}(i+j)$$

Separate

$$Total = \sum_{j=1}^{n}\left(\sum_{i=1}^{j-1} i + \sum_{i=1}^{j-1} j\right)$$

Factor out constants (index variables for the outer summation can be treated as constant!)

$$Total = \sum_{j=1}^{n}\left(\sum_{i=1}^{j-1} i + j\sum_{i=1}^{j-1} 1\right)$$

Apply known

$$Total = \sum_{j=1}^{n}\left(\frac{(j-1+1)(j-1)}{2} + j(j-1)\right)$$

Simplify

$$Total = \sum_{j=1}^{n}\left(\frac{(j)(j-1)}{2} + j(j-1)\right)$$

$$Total = \sum_{j=1}^{n}\left(\frac{3(j)(j-1)}{2}\right)$$

$$Total = \sum_{j=1}^{n}\left(\frac{3(j^2-j)}{2}\right)$$

Pull out constant

$$Total = \frac{3}{2}\sum_{j=1}^{n}(j^2-j)$$

Separate

$$Total = \frac{3}{2}\left(\sum_{j=1}^{n} j^2 - \sum_{j=1}^{n} j\right)$$

Apply known

$$Total = \frac{3}{2}\left(\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}\right)$$

Factor out common term

$$Total = \frac{3}{2}\frac{n(n+1)}{2}\left(\frac{(2n+1)}{3} - 1\right)$$

Simplify

$$Total = \frac{3}{2}\frac{n(n+1)}{2}\left(\frac{(2n-2)}{3}\right)$$

Cancel common factors

$$Total = \frac{n(n+1)}{2}(n-1)$$

Multiply out

$$Total = \frac{n^3-n}{2}$$

For small values of n=1,2 and 3, we expect results of 0, 3, 12 and the formula is consistent

Question I) Decide whether the following are true or false..

a) $2 \bullet 3^2 < 3 \bullet 2^2$

Evaluate both sides $2 \bullet 9 < 3 \bullet 4$ $18 < 12$ is false

b) $5^3 < 3^5$

Evaluate both sides $125 < 243$ is true

c) $3\log(5) = \log(125)$

Apply rules to left side $3\log(5) = \log(5^3)$ is true

d) $\log(10) - \log(2) = \log(8)$

Apply rules to left side $\log(10) - \log(2) = \log(10/2) = \log(5)$ $\log(5) = \log(8)$ is false (log is a strictly increasing function, or you can evaluate it.)

e) $\lg\left(\lg\left(10^{100}\right)\right) < 10$ reminder $\lg(x) \equiv \log_2(x)$

Apply rules to left side

$$\lg\left(\lg\left(10^{100}\right)\right) = \lg\left(100\lg(10)\right)$$

$$= 2\lg(10) + \lg\left(\lg(10)\right)$$

$$= 2\frac{\log(10)}{\log(2)} + \frac{\log\left(\frac{\log(10)}{\log(2)}\right)}{\log(2)}$$

$$= 2\frac{1}{\log(2)} + \frac{\log\left(\frac{1}{\log(2)}\right)}{\log(2)}$$

Using a calculator we find that $\log(2) \approx 0.3$

$$\approx 2\frac{1}{0.3} + \frac{\log\left(\frac{1}{0.3}\right)}{0.3}$$

$$\approx 2(3.3) + \frac{\log(3.3)}{0.3}$$

and we can then calculate $\log(3.3) \approx 0.52$

$$\approx 6.6 + \frac{.52}{0.3} \approx 6.6 + 1.7 \approx 8.3$$

Since this is less than 10, the expression is true.

Note: If we have access to a calculator that will compute logs to base 2, we can directly evaluate at the highlighted expression to get an answer of 8.375877035419344. (Which is, of course, an approximation.)

Question J) Evaluate the following expressions. No calculator should be needed.

a) $\lfloor 7.5 \rfloor + \lceil 7.5 \rceil$

$$7 + 8 = 15$$

b) $\lfloor -1.4 \rfloor + \lceil 3 \rceil$

$$-2 + 3 = 1$$

c) $\lfloor -4 \rfloor + \lceil -3 \rceil$

$$-4 + -3 = -7$$

d) $\lfloor n+\varepsilon \rfloor + \lceil n+\varepsilon \rceil$ where n is an integer and $0<\varepsilon<1$

$$n + n + 1 = 2n+1$$

e) $\lfloor n-\varepsilon \rfloor + \lceil n+\varepsilon \rceil$ where n is an integer and $0<\varepsilon<1$

$n-1+n+1$ =2n

Extra Credit 1) Suppose we can make our checkerboard larger. In version 1 of the checkerboard problem we increase the number of grains of rice by one in each square. In version 2, we double the number of grains of rice in each square. For both versions, determine how many squares would be required so that the total number of grains of rice will exceed the number of atoms in the universe. (The number of atoms is estimated to be approximately 10^{80}.)

Version 1) The summation is one we know. For n squares, the number of grains of rice is $\frac{n(n+1)}{2}$. We solve

$$\frac{n(n+1)}{2} = 10^{80}$$

Rearranging this we get

$$n^2 + n - 2 \cdot 10^{80} = 0$$

Which we can solve with the quadratic formula

$$n = \frac{-1 \pm \sqrt{1^2 - 4(1)(-2\cdot 10^{80})}}{2}$$

$$\approx \sqrt{2\cdot 10^{80}} \approx 1.4\times 10^{40}$$

Version 2) The summation is one we know. For n squares, the number of grains of rice is $2^n - 1$. We solve

$$2^n - 1 = 10^{80}$$

which is approximately

$$2^n \approx 10^{80}$$

Taking the log of both sides results in

$$\log(2^n) \approx \log(10^{80})$$

or

$$n \log(2) \approx 80$$

and finally

$$n \approx \frac{80}{\log(2)} \approx \frac{80}{0.3} \approx 266$$

Extra Credit 2) Consider the following summation. Show that the value of the sum converges as n approaches infinity. Hint: Bound the sum by an integral and then show that the integral converges.

$$S_n = \sum_{i=1}^{n} \frac{1}{i^2}$$

If we plot each of the terms in the sum and extend a green rectangle to the left with width one, we see that the summation is bounded by the integral of the function

$$f(x) = \frac{1}{x^2}$$

Unfortunately, if we start the integration from zero, the integral is unbounded at x=0. Instead, we will start the integration from 1 and separate out the first term

$$S_n = \sum_{i=1}^{n} \frac{1}{i^2} < 1 + \int_1^n \frac{1}{x^2} dx$$

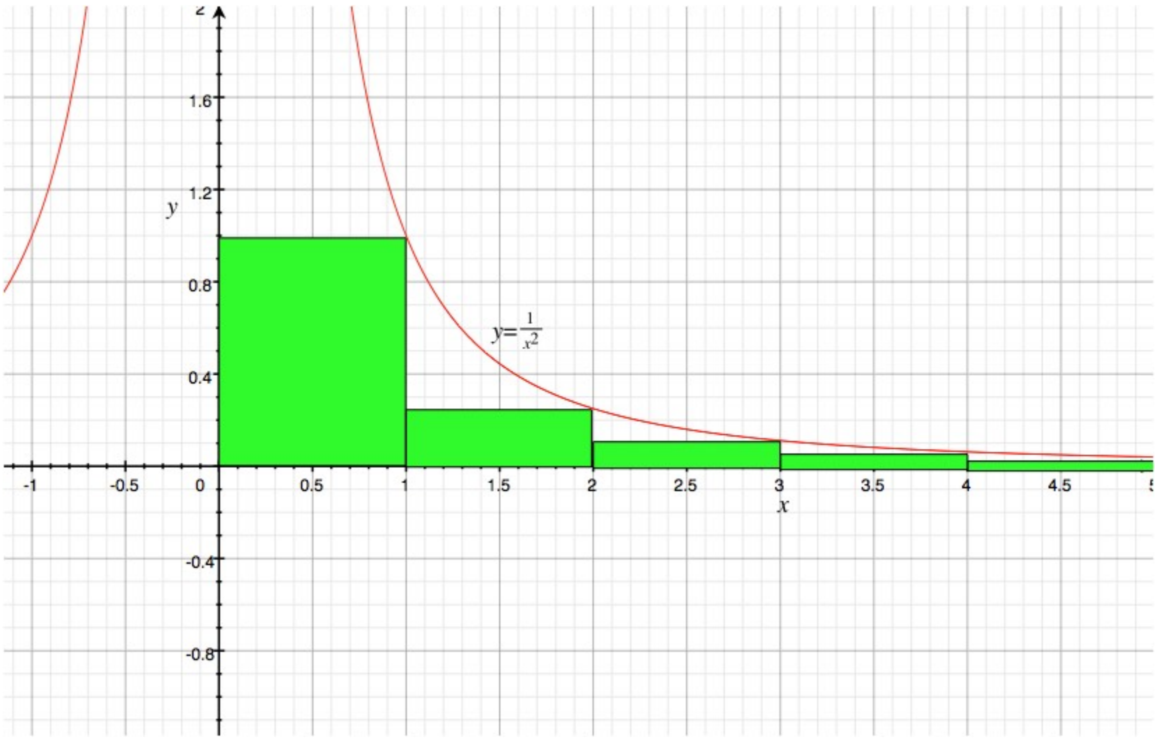
or

$$S_n < 1 + \left[-\frac{1}{x} \right]_1^n$$

$$S_n < 1 + \left[\left(-\frac{1}{n} \right) - \left(-\frac{1}{1} \right) \right]$$

$$S_n < 2 - \frac{1}{n}$$

We know that Sn is strictly increasing, since every term is positive. We know that Sn is strictly bounded by 2 as n gets larges. Therefore, we know that the value of Sn converges to some value between 1 and 2.



Extra Credit 3) Consider the following summation. Evaluate the sum to result in a closed form.

$$T_n = \sum_{i=1}^{n} \log(i)$$

We write the sum in expanded form

$$T_n = \log(1) + \log(2) + \log(3) + \ldots \log(n)$$

By the rules of logs, this is

$$T_n = \log(1 \cdot 2 \cdot 3 \cdot \ldots \cdot n)$$

or

$$T_n = \log(n!)$$

Side note: We can show that

$$\left(\frac{n}{2}\right)^{\frac{n}{2}} \leq n! \leq n^n$$

Which means that

$$\frac{n}{2}\log\left(\frac{n}{2}\right) \leq \log(n!) \leq n\log(n)$$