Coupled Three-Mode Squeezed Vacuum

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Multipartite entanglement is a key resource for various quantum information tasks. Here, we present a scheme for generating genuine tripartite entanglement via nonlinear optical processes. We derive, in the Fock basis, the corresponding output state which we termed the *coupled three-mode squeezed vacuum*. We find unintuitive behaviors arise in intensity squeezing between two of the three output modes due to the coupling present. We also show that this state can be genuinely tripartite entangled.

Quantum entanglement has been a topic of extensive research ever since Einstein, Podolsky, and Rosen proposed the nonlocal nature of quantum mechanics [1]. More recently, entanglement theory has been generalized to systems with more than two subsystems [2–4]. Such multipartite entanglement, integral to the generation of future quantum networks, has also been observed experimentally [5, 6].

Quantum networks enable quantum communication between arbitrary network users [7]. Multipartite entanglement serves as a fundamental resource for various quantum information tasks in such networks [8]. For example, information exchange in a network may be done through quantum teleportation [9], which requires prior generation and distribution of entanglement between multiple parties. Multipartite entanglement also allows parties in the network to perform distributed information-processing tasks that exhibit quantum advantages [10]. One of the many obstacles in realizing quantum networks is precisely the generation and distribution of robust multipartite entanglement.

Nonlinear optics provides a number of promising experimental tools for realizing multipartite entanglement, one of which is four-wave mixing (FWM). Four-wave mixing is a general term used to describe the parametric interaction between four coherent fields mediated by a nonlinear medium. Theoretical studies on using FWM for the generation of a two-mode squeezed vacuum (TMSV) or other two-mode squeezed states have been well-presented [11–16]. FWM in hot atomic vapor has been demonstrated to be a reliable source of squeezed light experimentally [17–20]. Recent work has utilized FWM as an efficient source of multimode quantum correlated states using either multiple pump beams [21–23], spatially structured pump beams [24, 25], or cascading setups [26–29]. Here, we show that such processes (as well as those like spontaneous parametric down conversion) may be used to generate genuine tripartite entangled states and derive the generated quantum state.

In this letter, we consider the setup in Fig. 1 (a). The pump photons interact with the nonlinear medium (⁸⁵Rb

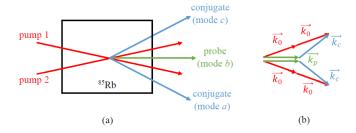


FIG. 1. Dual-pump coupled FWM. (a) Schematic of setup. (b) Phase-matching diagram. $\overrightarrow{k_0}$, $\overrightarrow{k_p}$, $\overrightarrow{k_c}$ are the wavevectors of the pump, probe, and conjugate photons, respectively.

atomic vapor) through a FWM process. The two pump beams of the same frequency are aligned such that their output probe beams overlap in the middle (mode b). In this configuration, two photons from pump 1 are annihilated to create a probe photon in mode b and a conjugate photon in mode a, and two photons from pump 2 are annihilated to create a probe photon in mode b and a conjugate photon in mode b. Figure 1 (b) shows the phase-matching diagram.

The time-independent interaction Hamiltonian for this setup can be written as

$$\hat{H}_{I} = i\hbar (\eta_{1}^{*} \hat{a} \hat{b} - \eta_{1} \hat{a}^{\dagger} \hat{b}^{\dagger} + \eta_{2}^{*} \hat{b} \hat{c} - \eta_{2} \hat{b}^{\dagger} \hat{c}^{\dagger}), \tag{1}$$

where \hat{a} , \hat{b} , \hat{c} are the bosonic annihilation operators for modes a, b, c, respectively, η_1 and η_2 are complex numbers that depend on the respective states of pumps 1 and 2 and the third-order nonlinearity of the medium. Here we assume undepleted and coherent pumps, and make use of the rotating wave approximation [30]. The unitary time evolution operator associated with this Hamiltonian is what we call the *coupled three-mode squeezing operator*:

$$\hat{S}_3(\xi_1, \xi_2) = \exp(\xi_1^* \hat{a} \hat{b} + \xi_2^* \hat{b} \hat{c} - \xi_1 \hat{a}^{\dagger} \hat{b}^{\dagger} - \xi_2 \hat{b}^{\dagger} \hat{c}^{\dagger}), \quad (2)$$

$$\xi_1 = \eta_1 t = r_1 e^{i\theta_1}, \ \xi_2 = \eta_2 t = r_2 e^{i\theta_2},$$
 (3)

where t is the interaction time. This interaction can be characterized by the two complex squeezing parameters ξ_1 and ξ_2 , whose respective magnitudes are r_1 , r_2 , and

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respective phases are θ_1 , θ_2 . We note that this squeezing operator can also be realized using other nonlinear optical processes such as spontaneous parametric down-conversion.

The coupled three-mode squeezed vacuum (C3MSV) is the output state when \hat{S}_3 acts on vacuum inputs. The result is, in the Fock basis,

$$|\text{C3MSV}\rangle \equiv \hat{S}_3 |0, 0, 0\rangle = \frac{1}{\cosh r} \sum_{n, l=0}^{\infty} (-1)^{n+l} e^{i(n\theta_1 + l\theta_2)} \left(\frac{r_1}{r} \tanh r\right)^n \left(\frac{r_2}{r} \tanh r\right)^l \sqrt{\frac{(n+l)!}{n! \, l!}} |n, n+l, l\rangle,$$
 (4)

where $r = \sqrt{r_1^2 + r_2^2}$, and the set of states $\{|n,m,l\rangle \equiv |n\rangle_a \otimes |m\rangle_b \otimes |l\rangle_c\}$ is the basis of the three-mode Fock space. The C3MSV becomes a TMSV in modes a, b and vacuum in mode c when $r_1 > 0$ and $r_2 = 0$, and a TMSV in modes b, c and vacuum in mode a when $r_1 = 0$ and $r_2 > 0$.

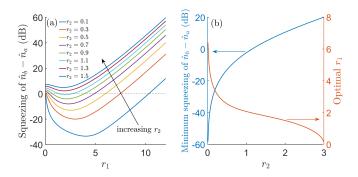


FIG. 2. (a) Squeezing, in decibels, of the photon number operator $\hat{n}_b - \hat{n}_a$ with changing r_1 and r_2 . (b) Minimum value of squeezing of $\hat{n}_b - \hat{n}_a$ and the value of r_1 where the minimum occurs w.r.t r_2 .

Having found the output state, we now look at the intensity squeezing properties of said state, namely the squeezing of combinations of photon number operators $\hat{n}_a \equiv \hat{a}^{\dagger}\hat{a}$, $\hat{n}_b \equiv \hat{b}^{\dagger}\hat{b}$, and $\hat{n}_c \equiv \hat{c}^{\dagger}\hat{c}$. Squeezing of a Hermitian operator \hat{O} is defined as the ratio $\langle (\Delta \hat{O})^2 \rangle / \langle (\Delta \hat{O})^2 \rangle_{\rm coh}$, expressed in decibels, where $\langle (\Delta \hat{O})^2 \rangle$ denotes the variance of \hat{O} w.r.t the C3MSV, and $\langle (\Delta \hat{O})^2 \rangle_{\rm coh}$ denotes the standard or shot noise limit (SNL) of \hat{O} , which is calculated to be the variance of the operator w.r.t coherent states in each mode with the same respective mean photon number as the C3MSV. A physical quantity is squeezed whenever the squeezing of the corresponding operator is below 0 dB.

The C3MSV is an eigenstate of the photon number operator $\hat{n}_b - \hat{n}_a - \hat{n}_c$ with eigenvalue 0. Hence, the variance of the difference between the intensity of mode b and the sum of intensities of mode a and c is ideally 0. This is expected by simple examination of the setup. Interesting behaviors arise when we look at the intensity squeezing between two modes, namely $\hat{n}_b - \hat{n}_a$ and $\hat{n}_b - \hat{n}_c$. Since r_1 is the squeezing parameter that characterizes the interaction between pump 1 and the medium,

which generates photons in modes a and b, one would expect the strength of the squeezing of $\hat{n}_b - \hat{n}_a$ to increase (i.e. the ratio $\langle [\Delta(\hat{n}_b - \hat{n}_a)]^2 \rangle / \langle [\Delta(\hat{n}_b - \hat{n}_a)]^2 \rangle_{\text{coh}}$ decreases) as r_1 increases, as it would in a TMSV [16]. However, Fig. 2 (a) shows that a maximum exists for the strength of squeezing of $\hat{n}_b - \hat{n}_a$ (minimum of the ratio $\langle [\Delta(\hat{n}_b-\hat{n}_a)]^2 \rangle / \langle [\Delta(\hat{n}_b-\hat{n}_a)]^2 \rangle_{\rm coh} \rangle$ for any value of $r_2 \neq 0$, which means that the intensity difference between modes b and a will always become not squeezed for sufficiently large r_1 . Fig. 2 (b) shows how the minimum of $\langle [\Delta(\hat{n}_b - \hat{n}_a)]^2 \rangle / \langle [\Delta(\hat{n}_b - \hat{n}_a)]^2 \rangle_{\text{coh}}$ and the optimal r_1 (the value of r_1 at the minimum) changes with r_2 . Each mode in the C3MSV is in a thermal state individually, which is similar to those of a TMSV [16, 31]. Thus the interaction between modes a and b is effectively being seeded by a thermal state in mode b, which is being generated from the interaction between modes b and c.

The intensity squeezing results shown above only depend on the magnitudes r_1 and r_2 , but not on the phases θ_1 and θ_2 . Experimentally, r_1 and r_2 depend on factors such as temperature of the medium, pump powers, and interaction time, so their tunability is limited. To investigate the presence of entanglement, we would like to look at the quadrature squeezing properties of the C3MSV, which do depend on the phases θ_1 and θ_2 . Here, we introduce the quadrature operators

$$\hat{U} = h_1 \hat{X}_a + h_2 \hat{X}_b + h_3 \hat{X}_c, \tag{5}$$

$$\hat{V} = h_1 \hat{P}_a + h_2 \hat{P}_b + h_3 \hat{P}_c, \tag{6}$$

where $\hat{X}_j = (\hat{j} + \hat{j}^{\dagger})/2$, $\hat{P}_j = (\hat{j} - \hat{j}^{\dagger})/(2i)$ for j = a, b, c, and h_1, h_2, h_3 are real numbers. The uncertainty principle gives us

$$\langle (\Delta \hat{U})^2 \rangle \langle (\Delta \hat{V})^2 \rangle \ge \frac{1}{16} (h_1^2 + h_2^2 + h_3^2)^2,$$
 (7)

and the SNLs of \hat{U} and \hat{V} are

$$\langle (\Delta \hat{U})^2 \rangle_{\rm coh} = \langle (\Delta \hat{V})^2 \rangle_{\rm coh} = \frac{1}{4} (h_1^2 + h_2^2 + h_3^2).$$
 (8)

Figure 3 and Fig. 4 show the squeezing of \hat{U} and \hat{V} for $h_1 = h_2 = h_3 = 1$ and various values of r_1 and r_2 . Figure 3 shows the results for symmetric magnitudes of the squeezing parameters $(r_1 = r_2)$, and Fig. 4 shows the results for asymmetric magnitudes $(r_1 \neq r_2)$. In both cases,

there are values of θ_1 and θ_2 where \hat{U} and \hat{V} are both not squeezed, but never were they both squeezed. This is similar to the result of a joint quadrature measurement on a TMSV. For the asymmetric case shown in Fig. 4 $(r_1 > r_2)$, there are certain values of θ_1 where the squeezing is independent of θ_2 .

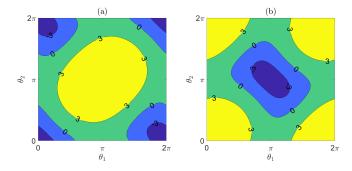


FIG. 3. (Color online) Contour plot of squeezing, in decibels, of (a) \hat{U} , (b) \hat{V} , for $h_1 = h_2 = h_3 = 1$, $r_1 = r_2 = 0.5$. The blue regions are where the operators are squeezed.

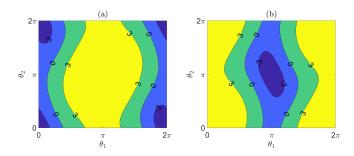


FIG. 4. (Color online) Contour plot of squeezing, in decibels, of (a) \hat{U} , (b) \hat{V} , for $h_1 = h_2 = h_3 = 1$, $r_1 = 0.7$, $r_2 = 0.3$. The blue regions are where the operators are squeezed.

We would also like to show that the C3MSV is genuine tripartite entangled. Genuine tripartite entanglement is a stronger criterion than, and is not equivalent to full tripartite inseparability in general, the details of which are explained in Ref. [32]. A criterion based on quadrature operators for demonstrating genuine tripartite entanglement has been presented and proven in Ref. [33]. The criterion is stated as follows: the violation of any of the inequalities,

$$\langle (\Delta \hat{U}_1)^2 \rangle \langle (\Delta \hat{V}_1)^2 \rangle \ge 1,$$
 (9a)

$$\langle (\Delta \hat{U}_2)^2 \rangle \langle (\Delta \hat{V}_2)^2 \rangle \ge 1,$$
 (9b)

$$\langle (\Delta \hat{U}_3)^2 \rangle \langle (\Delta \hat{V}_3)^2 \rangle \ge 1,$$
 (9c)

is sufficient to confirm genuine tripartite entanglement, where

$$\hat{U}_1 = 2\hat{X}_a - \sqrt{2}(\hat{X}_b + \hat{X}_c), \ \hat{V}_1 = 2\hat{P}_a + \sqrt{2}(\hat{P}_b + \hat{P}_c),$$

$$\hat{U}_2 = 2\hat{X}_b - \sqrt{2}(\hat{X}_a + \hat{X}_c), \ \hat{V}_2 = 2\hat{P}_b + \sqrt{2}(\hat{P}_a + \hat{P}_c),$$

$$\hat{U}_3 = 2\hat{X}_c - \sqrt{2}(\hat{X}_a + \hat{X}_b), \ \hat{V}_3 = 2\hat{P}_c + \sqrt{2}(\hat{P}_a + \hat{P}_b).$$

Notice that $[\hat{U}_j, \hat{V}_j] = 0$, so the product $\langle (\Delta \hat{U}_j)^2 \rangle \langle (\Delta \hat{V}_j)^2 \rangle$ is lower bounded by 0 for j = a, b, c. We find that the inequalities in Eqs. (9a) and (9c) are never violated, but the inequality in Eq. (9b) can be violated.

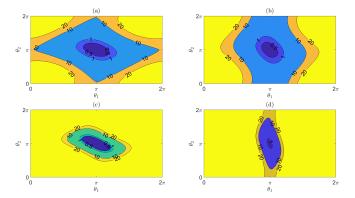


FIG. 5. (Color online) Contour plot of the product $\langle (\Delta \hat{U}_2)^2 \rangle \langle (\Delta \hat{V}_2)^2 \rangle$ for (a) $r_1 = r_2 = 0.5$, (b) $r_1 = 0.7$, $r_2 = 0.3$, (c) $r_1 = r_2 = 1.0$, (d) $r_1 = 1.7$, $r_2 = 0.3$. The region inside the line labeled "1" is where the inequality in Eq. (9b) is violated.

Figure 5 shows contour plots of the product $\langle (\Delta \hat{U}_2)^2 \rangle \langle (\Delta \hat{V}_2)^2 \rangle$ for symmetric and asymmetric values of r_1 and r_2 . In all four cases in Fig. 5, the region where we can confirm genuine tripartite entanglement is centered at the point $\theta_1 = \theta_2 = \pi$. The violation of the inequality in Eq. 9b becomes stronger at the expense of a smaller range of θ_1 and θ_2 values as r_1 and r_2 increase. This region becomes smallest in the most asymmetric case (Fig. 5 (d)) since the C3MSV reduces to a TMSV in modes a and b and vacuum in mode c when $r_1 \gg r_2$, which is not genuine tripartite entangled.

In conclusion, we presented a scheme for generating tripartite entanglement via nonlinear optical processes, such as FWM, using two crossed pump beams. We derived the output state of this setup for vacuum inputs, which we termed the coupled three-mode squeezed vacuum. We investigated the intensity squeezing and quadrature squeezing properties of the C3MSV state, and showed that some unintuitive behaviors arise due to the coupling. We also showed that C3MSV can be genuinely tripartite entangled. We would like to point out that the violation of inequalities in Eq. (9) is only sufficient in demonstrating genuine tripartite entanglement, thus the state may still be genuine tripartite entangled even in regimes where none of the inequalities in Eq. (9) are violated. The setup presented here can be modified and expanded to generate more than three output modes [24, 34]. Such setups provide us with a platform to study multipartite entanglement, and will further the development of quantum information protocols and quantum networks.

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