Governing Equations

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$$
\begin{cases}
\dfrac{du}{dt} = \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + w\dfrac{\partial u}{\partial z} \quad = -\dfrac{\partial}{\partial x}(C_p \theta_{v0} \pi^{'}) & (1) \\
\dfrac{dw}{dt} = \dfrac{\partial w}{\partial t} + u\dfrac{\partial w}{\partial x} + w\dfrac{\partial w}{\partial z} \; = -\dfrac{\partial}{\partial z}(C_p \theta_{v0} \pi^{'}) + \underbrace{g\dfrac{\theta^{'}}{\theta_0}}_{B} & (2) \\
\dfrac{d\theta}{dt} = \dfrac{\partial \theta}{\partial t} + u\dfrac{\partial \theta}{\partial x} + w\dfrac{\partial \theta}{\partial z} \; = 0 & (3) \\
\dfrac{\partial(\rho_0 u)}{\partial x} + \dfrac{\partial(\rho_0 w)}{\partial z} = 0 & (4)
\end{cases}
$$

$\zeta = \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}$

$$\frac{\partial(2)}{\partial x} : \frac{\partial^2 w}{\partial x \partial t} + \frac{\partial u}{\partial x}\frac{\partial w}{\partial x} + u\frac{\partial^2 w}{\partial x^2} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial z} + w\frac{\partial^2 w}{\partial x \partial z} = -\frac{\partial^2}{\partial z \partial x}(C_p \theta_{v0} \pi^{'}) + \frac{\partial B}{\partial x}$$

$$\frac{\partial(1)}{\partial z} : \frac{\partial^2 u}{\partial z \partial t} + \frac{\partial u}{\partial z}\frac{\partial u}{\partial x} + u\frac{\partial^2 u}{\partial z \partial x} + \frac{\partial w}{\partial z}\frac{\partial u}{\partial z} + w\frac{\partial^2 u}{\partial z^2} = -\frac{\partial^2}{\partial x \partial z}(C_p \theta_{v0} \pi^{'})$$

$$\frac{\partial(2)}{\partial x} - \frac{\partial(1)}{\partial z} : \frac{\partial \zeta}{\partial t} + \frac{\partial(u\zeta)}{\partial x} + \frac{\partial(w\zeta)}{\partial z} = \frac{\partial B}{\partial x}$$

$$\frac{\partial \zeta}{\partial t} = -(\frac{\partial(u\zeta)}{\partial x} + \frac{\partial(w\zeta)}{\partial z}) + \frac{\partial B}{\partial x}$$

$$From\ (4):\ \frac{\partial u}{\partial x} + \frac{1}{\rho_0}\frac{\partial(\rho_0 w)}{\partial z} = 0 \tag{5}$$

$$\frac{\partial \zeta}{\partial x} = \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial z}, \quad Substitute\ (5)\ into\ it \Rightarrow \frac{\partial \zeta}{\partial x} = \frac{\partial}{\partial z}(\frac{1}{\rho_0}\frac{\partial(\rho_0 w)}{\partial z}) + \frac{\partial^2 w}{\partial x^2}$$

$$
\begin{cases}
\dfrac{\partial \zeta}{\partial t} = -(\dfrac{\partial(u\zeta)}{\partial x} + \dfrac{\partial(w\zeta)}{\partial z}) + \dfrac{\partial B}{\partial x} & (6) \\
\dfrac{\partial \theta}{\partial t} = -(\dfrac{\partial u\theta}{\partial x} + \dfrac{1}{\rho_0}\dfrac{\partial \rho_0 w\theta}{\partial z}) & (7) \\
\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial}{\partial z}(\dfrac{1}{\rho_0}\dfrac{\partial(\rho_0 w)}{\partial z}) = \dfrac{\partial \zeta}{\partial x} & (8) \\
\dfrac{\partial u}{\partial x} + \dfrac{1}{\rho_0}\dfrac{\partial(\rho_0 w)}{\partial z} = 0 & (9)
\end{cases}
$$

$u = u_\chi, \quad u_\chi = \frac{\partial \chi}{\partial x}$

$$\frac{\partial^2 \chi}{\partial x^2} = -\frac{1}{\rho_0}\frac{\partial(\rho_0 w)}{\partial z} \tag{10}$$

$$\bar{u} = const. \tag{11}$$

$u_T = \bar{u} + u_\chi$

$$u = \int_{z_T}^{z}(\frac{\partial w}{\partial x} - \zeta)dz + u_T \tag{12}$$

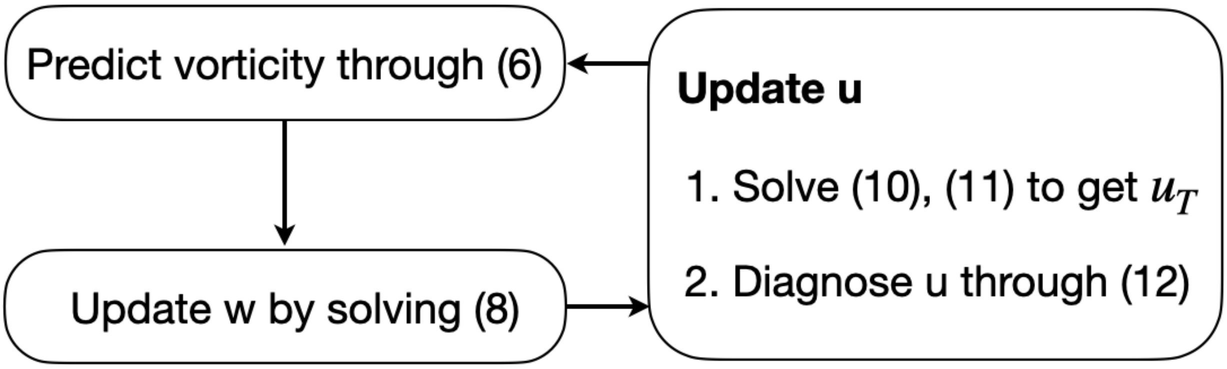

Figure 1: Flowchart

Discretization

$$(6): \frac{\partial \zeta}{\partial t} = -(\frac{\partial (u\zeta)}{\partial x} + \frac{\partial (w\zeta)}{\partial z}) + \frac{\partial B}{\partial x}$$

$$u\_zeta^n_{i,k} = \frac{u_{i,k-1}+u_{i,k}}{2},\ w\_zeta^n_{i,k} = \frac{w_{i-1,k}+w_{i,k}}{2},\ \theta'\_zeta^n_{i,k} = \frac{\theta'_{i-1,k-1}+\theta'_{i-1,k}+\theta'_{i,k-1}+\theta'_{i,k}}{4}$$

[_zeta means which variable's location is at zeta's place]

$$\frac{\partial \zeta}{\partial t} : \frac{\zeta^{n+1}_{i,k} - \zeta^{n-1}_{i,k}}{2\Delta t}$$

$$\frac{\partial (u\zeta)}{\partial x} : \frac{u\_zeta^n_{i+1,k} \times \zeta^n_{i+1,k} - u\_zeta^n_{i-1,k} \times \zeta^n_{i-1,k}}{2\Delta x}$$

$$\frac{\partial (w\zeta)}{\partial z} : \frac{w\_zeta^n_{i,k+1} \times \zeta^n_{i,k+1} - w\_zeta^n_{i,k-1} \times \zeta^n_{i,k-1}}{2\Delta z}$$

$$\frac{\partial B}{\partial x} = \frac{g}{\theta_0}\frac{\partial \theta'}{\partial x} = \frac{g}{\theta_0} \times \frac{\theta'\_zeta^n_{i+1,k} - \theta'\_zeta^n_{i-1,k}}{2\Delta x}$$

$$(8): \frac{\partial^2 w}{\partial x^2} + \frac{\partial}{\partial z}(\frac{1}{\rho_0}\frac{\partial(\rho_0 w)}{\partial z}) = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2} + \frac{\partial}{\partial z}(\frac{w}{\rho_0})\frac{\partial \rho_0}{\partial z} + \frac{w}{\rho_0}\frac{\partial^2 \rho_0}{\partial z^2} = \frac{\partial \zeta}{\partial x}$$

$\rho_0\_zeta^n_k = \rho_0\_w^n_k =$ the average density at zeta and w's height ,

$\zeta\_w^n_{i,k} = \frac{\zeta^n_{i,k+1}+\zeta^n_{i,k}}{2} =$ value of zeta at w's location

$$\frac{\partial^2 w}{\partial x^2} : \frac{w^n_{i+1,k} - 2w^n_{i,k} + w^n_{i-1,k}}{\Delta x^2}$$

$$\frac{\partial^2 w}{\partial z^2} : \frac{w^n_{i,k+1} - 2w^n_{i,k} + w^n_{i,k-1}}{\Delta z^2}$$

$$\frac{\partial}{\partial z}(\frac{w}{\rho_0})\frac{\partial \rho_0}{\partial z} : \frac{\frac{w^n_{i,k+1}}{\rho_0\_w^n_{i,k+1}} - \frac{w^n_{i,k-1}}{\rho_0\_w^n_{i,k-1}}}{2\Delta z} \times \frac{\rho_0\_w^n_{i,k+1} - \rho_0\_w^n_{i,k-1}}{2\Delta z} = \frac{(\rho_0\_w^n_{i,k+1} - \rho_0\_w^n_{i,k-1})(w^n_{i,k-1} - w^n_{i,k+1} + 1)}{4\Delta z^2 \rho_0\_w_{i,k+1}\rho_0\_w_{i,k-1}}$$

$$\frac{w}{\rho_0}\frac{\partial^2 \rho_0}{\partial z^2} : \frac{w^n_{i,k}}{\rho_0\text{\_}w^n_{i,k}}\frac{\rho_0\text{\_}w^n_{i,k+1} - 2\rho_0\text{\_}w^n_{i,k} + \rho_0\text{\_}w^n_{i,k-1}}{\Delta z^2}$$

$$\frac{\partial \zeta}{\partial x} : \frac{\zeta^n_{i+1,k} - \zeta^n_{i-1,k}}{2\Delta x}$$

$\because \Delta x = \Delta z$

$$\Rightarrow w^n_{i,k}(4 - \overbrace{\frac{\rho_0\text{\_}w^n_{i,k+1} - 2\rho_0\text{\_}w^n_{i,k} + \rho_0\text{\_}w^n_{i,k-1}}{\rho^n_{0i,k}}}^{P}) - w^n_{i+1,k} - w^n_{i-1,k} - w^n_{i,k+1}(1 + \overbrace{\frac{\rho_0\text{\_}w_{i,k+1} - \rho_0\text{\_}w_{i,k-1}}{4\rho_0\text{\_}w_{i,k+1}}}^{Q})$$
$$-w^n_{i,k-1}(1 - \underbrace{\frac{\rho_0\text{\_}w_{i,k+1} - \rho_0\text{\_}w_{i,k-1}}{4\rho_0\text{\_}w_{i,k-1}}}_{O}) = -\frac{\Delta x}{2}(\underbrace{\zeta^n_{i+1,k} - \zeta^n_{i-1,k}}_{R}) \quad (13)$$

$$A\vec{w} = \vec{b}$$

$$1 \le i \le m-2, 2 \le k \le n-2, \vec{w} = \left[w_{12}, w_{22}, \dots, w_{(m-2)2}, w_{13}, w_{23}, \dots, w_{(m-2)3}, \dots, w_{(m-2)(n-2)}\right]^T$$

$A : (m-2)(n-3) \times (m-2)(n-3), D : (m-2) \times (m-2), E : (m-2) \times (m-2)$

$F : (m-2) \times (m-2), w : (m-2)(n-3) \times 1, b : (m-2)(n-3) \times 1$

$$A = \begin{bmatrix} D & E & 0 & 0 & 0 & \dots & 0 \\ F & D & E & 0 & 0 & \dots & 0 \\ 0 & F & D & E & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & F & D & E & 0 \\ 0 & \dots & \dots & 0 & F & D & E \\ 0 & \dots & \dots & \dots & 0 & F & D \end{bmatrix}, D = \begin{bmatrix} 4-P & -1 & 0 & 0 & 0 & \dots & 0 \\ -1 & 4-P & -1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 4-P & -1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & -1 & 4-P & -1 & 0 \\ 0 & \dots & \dots & 0 & -1 & 4-P & -1 \\ 0 & \dots & \dots & \dots & 0 & -1 & 4-P \end{bmatrix}$$

$$E = \begin{bmatrix} -1-Q & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & -1-Q & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & -1-Q & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & -1-Q & 0 & 0 \\ 0 & \dots & \dots & 0 & 0 & -1-Q & 0 \\ 0 & \dots & \dots & \dots & 0 & 0 & -1-Q \end{bmatrix}$$

$$F = \begin{bmatrix} -1+O & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & -1+O & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & -1+O & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & -1+O & 0 & 0 \\ 0 & \dots & \dots & 0 & 0 & -1+O & 0 \\ 0 & \dots & \dots & \dots & 0 & 0 & -1+O \end{bmatrix}$$

$$\vec{b} = \begin{bmatrix} -\frac{\Delta x}{2}R_{12} \\ -\frac{\Delta x}{2}R_{22} \\ -\frac{\Delta x}{2}R_{32} \\ \vdots \\ -\frac{\Delta x}{2}R_{(m-2)2} \\ -\frac{\Delta x}{2}R_{13} \\ -\frac{\Delta x}{2}R_{23} \\ -\frac{\Delta x}{2}R_{33} \\ \vdots \\ -\frac{\Delta x}{2}R_{(m-2)3} \\ \vdots \\ -\frac{\Delta x}{2}R_{(m-2)(n-2)} \end{bmatrix} + \begin{bmatrix} w_{02} & + w_{11}(1-O_{12}) \\ & w_{21}(1-O_{22}) \\ & w_{31}(1-O_{32}) \\ & \vdots \\ w_{(m-1)2} & + w_{(m-2)1}(1-O_{12}) \\ w_{03} & \\ 0 & \\ 0 & \\ & \vdots \\ w_{(m-1)3} & \\ & \vdots \\ w_{(m-1)(n-2)} & + w_{(m-2)(n-1)}(1+Q_{(m-2)(n-2)}) \end{bmatrix},$$

$$\begin{cases} i=1: & w_{0k} = w_{(m-2)_k} \\ i=m-2: & w_{(m-1)k} = w_{1k} \\ k=2: & w_{i1}(1-O_{i2}) = 0 \\ k=n-2: & w_{i(n-1)}(1+Q_{i(n-2)}) = 0 \end{cases}$$

$$A^{'}\vec{w} = \vec{c}$$

$$A' = \begin{bmatrix} D' & E & 0 & 0 & 0 & \dots & 0 \\ F & D' & E & 0 & 0 & \dots & 0 \\ 0 & F & D' & E & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & F & D' & E & 0 \\ 0 & \dots & \dots & 0 & F & D' & E \\ 0 & \dots & \dots & \dots & 0 & F & D' \end{bmatrix}, D' = \begin{bmatrix} 4-P & -1 & 0 & 0 & 0 & \dots & -1 \\ -1 & 4-P & -1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 4-P & -1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & -1 & 4-P & -1 & 0 \\ 0 & \dots & \dots & 0 & -1 & 4-P & -1 \\ -1 & \dots & \dots & \dots & 0 & -1 & 4-P \end{bmatrix}$$

$$\vec{c} = \begin{bmatrix} -\frac{\Delta x}{2}R_{11} \\ -\frac{\Delta x}{2}R_{21} \\ -\frac{\Delta x}{2}R_{31} \\ \vdots \\ -\frac{\Delta x}{2}R_{(m-2)1} \\ -\frac{\Delta x}{2}R_{12} \\ -\frac{\Delta x}{2}R_{22} \\ -\frac{\Delta x}{2}R_{32} \\ \vdots \\ -\frac{\Delta x}{2}R_{(m-2)2} \\ \vdots \\ -\frac{\Delta x}{2}R_{(m-2)(n-2)} \end{bmatrix}$$

$$(10): \frac{\partial^2\chi}{\partial x^2} = -\frac{1}{\rho_0}\frac{\partial(\rho_0 w)}{\partial z}$$

$$w\_zeta^n_{i,k} = \frac{w^n_{i-1,k} + w^n_{i,k}}{2}, \quad w\_u^n_{i,k} = \frac{w\_zeta^n_{i,k+1} + w\_zeta^n_{i,k}}{2}$$

$$\frac{\partial^2\chi}{\partial x^2} : \frac{\chi^n_{i+1,k} - 2\chi^n_{i,k} + \chi^n_{i-1,k}}{\Delta x^2}$$

$$-\frac{1}{\rho_0}\frac{\partial(\rho_0 w)}{\partial z} : -\frac{1}{\rho^n_{0i,k}}\frac{\rho^n_{0i,k+1} w\_u^n_{i,k+1} - \rho^n_{0i,k-1} w\_u^n_{i,k-1}}{2\Delta z}$$

$$\Rightarrow 2\chi^n_{i,k} - \chi^n_{i+1,k} - \chi^n_{i-1,k} = \frac{\Delta x}{2} \underbrace{\frac{\rho^n_{0i,k+1} w\_u^n_{i,k+1} - \rho^n_{0i,k-1} w\_u^n_{i,k-1}}{\rho^n_{0i,k}}}_{S} \tag{14}$$

$$G\vec{\chi} = \vec{h}$$

$$1 \le i \le m-2, k = nz-2, \vec{\chi} = \left[\chi_1, \chi_2, \ldots, \chi_{m-2}\right]^T$$

$$G : (m-2)\times(m-2), \chi : (m-2)\times 1, h : (m-2)\times 1$$

$$G = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & \ldots & 0 \\ -1 & 2 & -1 & 0 & 0 & \ldots & 0 \\ 0 & -1 & 2 & -1 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & -1 & 2 & -1 & 0 \\ 0 & \ldots & \ldots & 0 & -1 & 2 & -1 \\ 0 & \ldots & \ldots & \ldots & 0 & -1 & 2 \end{bmatrix}, \vec{h} = \begin{bmatrix} \frac{\Delta x}{2}S_1 \\ \frac{\Delta x}{2}S_2 \\ \frac{\Delta x}{2}S_3 \\ \vdots \\ \frac{\Delta x}{2}S_{(m-2)} \end{bmatrix} + \begin{bmatrix} \chi_0 \\ 0 \\ 0 \\ \vdots \\ \chi_{(m-1)} \end{bmatrix} if \begin{cases} i = 1 : & w_0 = w_{(m-2)} \\ i = m-2 : & w_{(m-1)} = w_1 \end{cases}$$

$$G'\vec{\chi} = \vec{h'}$$

$$G' = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & \ldots & -1 \\ -1 & 2 & -1 & 0 & 0 & \ldots & 0 \\ 0 & -1 & 2 & -1 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & -1 & 2 & -1 & 0 \\ 0 & \ldots & \ldots & 0 & -1 & 2 & -1 \\ -1 & \ldots & \ldots & \ldots & 0 & -1 & 2 \end{bmatrix}, \quad h' = \begin{bmatrix} \frac{\Delta x}{2}S_1 \\ \frac{\Delta x}{2}S_2 \\ \frac{\Delta x}{2}S_3 \\ \vdots \\ \frac{\Delta x}{2}S_{(m-2)} \end{bmatrix}$$

$$u_\chi = \frac{\partial \chi}{\partial x} = \frac{\chi^n_{i+1,k} - \chi^n_{i-1,k}}{2\Delta x}$$

$$u = \int_{z_T}^{z} (\frac{\partial w}{\partial x} - \zeta) dz + u_T$$

$$u^n_{i,k} = (\frac{w^n_{i+1,k} - w^n_{i-1,k}}{2\Delta x} - \zeta_{i,k}) \times z - (\frac{w^n_{i+1,nz-1} - w^n_{i-1,nz-1}}{2\Delta x} - \zeta_{i,nz-1}) \times z + u_\chi$$

$$\frac{\partial \theta}{\partial t} = -(u\frac{\partial \theta}{\partial x} + \frac{1}{\rho_0}\frac{\partial \rho_0 w \theta}{\partial z})$$

$$\frac{\partial \theta}{\partial t} : \frac{\theta^{n+1}_{i,k} - \theta^{n-1}_{i,k}}{2\Delta t}$$

$$u\frac{\partial \theta}{\partial x} : u\_th^n_{i,k}\frac{\theta^n_{i+1,k} - \theta^n_{i-1,k}}{2\Delta x}$$

$$\frac{1}{\rho_0}\frac{\partial \rho_0 w \theta^{'}}{\partial z} : \frac{1}{\rho_{0k}}\frac{\rho_{0k+1} w\_th^n_{i,k+1}\theta^{'n}_{i,k+1} - \rho_{0k-1} w\_th^n_{i,k-1}\theta^{'n}_{i,k-1}}{2\Delta z}$$