Given $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, Evaluate A^{1+i}.

(1) If $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ find A^{1+i}.

(2) Find whether $\begin{bmatrix} 1-i \\ 2 \\ 20+i \end{bmatrix}$ exists in $C(A)$, where

$$A = \begin{bmatrix} 1+i & 2-i & 1 \\ 2-i & 1-i & 2+i \\ 1 & 2+i & 1+i \end{bmatrix}$$

(3) Evaluate $\int_2^3 L\left[\tfrac{1}{3}, x\right] dx$ where

$$L\left[\tfrac{1}{3}, x\right] = e^{(\ln x)^{1/3}(\ln(\ln x))^{2/3}}.$$

use R_c and $n = 7$

(4) Find orthogonal vectors corresponding to

$$\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

(5) (a) Find shortest vector from origin for a given lattice

(b) Find the closest vector from a given vector for a lattice L.

(6) Given $L = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ Find shortest vector

(7) Given $f(x) = \sin x + \tan x + \sqrt{x}$. Find roots.

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$= A \cdot A^2$

(8) If matrix A is +ve definite. B is a projection matrix of A and projects vectors into nullspace of A and projects vectors into nullspace of A. B is negative definite. Then what can you say about A-B?

Also find for if B is semi-negative definite.

(9) Given an iterative function $x_{n+1} = x_n - c f(x_n)$. Find the optimum value of 'c' to minimize the corresponding error function.

(10) Find pseudoinverse of

$$A = \begin{bmatrix} i & 0 & i \\ 0 & -i & 0 \\ 0 & 0 & i \end{bmatrix}$$

Find e^{At} for $t = 2$?

(11) Find eigen values of

$$\begin{bmatrix} 1 & 3 & 17 & i \\ 2 & 1 & 13 & 1 \\ 20 & 5 & 1 & 0 \\ 40 & 6 & 7 & 1 \end{bmatrix}$$

(12) Given a vector space $M = \begin{bmatrix} a & f(x) \\ g(x) & c \end{bmatrix}$ $a \in \mathbb{R}$.

$f(x)$ = any polynomial function of degree n.

$g(x)$ = any trignometric function, where x is radians

c = a complex number.

$A = e^{\ln A^i}$ [$\because A = I_{2\times 2}$].
$= e^{i \ln A}$

What will be the corresponding dimensions on a real vector space?

Date : Page :

(13). If binary digit 0 is represented by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and 1 by $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Then the number of dimensions in real vector space to represent 2-bit numbers and 3-bit numbers are ?

1) Given $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, Evaluate A^{1+i}.

$$A^{1+i} = A \cdot A^{i} = A^{i} \quad \left[\because A = I_{2\times2}\right].$$

$$A^{i} = e^{\ln A^{i}} = e^{i \ln A} \quad \text{—— (1)}.$$

Let, $\ln A = B$ then $e^{B} = A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ —— (2)

Method - 1 : to evaluate eqn - (2) :

- Using series for e^{B}.

$$e^{B} = \sum_{n=0}^{\infty} \frac{1}{n!} B^{n}.$$

or, $e^{B} = I_{2\times2} + \frac{1}{1!} B + \frac{1}{2!} B^{2} + \cdots$

we want matrix B to be such that it is a non-zero matrix but B^2 and other higher powers of B becomes zero matrix i.e $B^2 = B^3 = B^4 = \cdots = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Let's take $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then $B^2 = B^3 = \cdots = O$.

Such 'B' is called nilpotent matrix.

So, $e^{B} = A$ or $\ln A = B$ gives

$$B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{—— (3)}$$

Substitute value of B in eqn. (1) :

$$e^{iB} = e^{i\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}} = e^{\begin{bmatrix} 0 & i \\ 0 & 0 \end{bmatrix}} = \begin{bmatrix} 1 & i \\ 0 & 1 \end{bmatrix}$$ is the answer.

P.T.O.

of $C(A)$.

Method - 2 to evaluate eqn (2):

Using Cayley Hamilton theorem:

$$e^{\lambda_i t} = \sum_{k=0}^{n-1} \alpha_k \lambda_i^k \quad \& \quad e^{At} = \sum_{k=0}^{n-1} \alpha_k A^k \quad \text{——} (4)$$

Here, $t = 1$. & e^{Bt} must be evaluated to get $I_{2\times 2}$.

So, $e^B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Using eqns. (4) we get following relationships:

$$\left.\begin{aligned} e^{\lambda_1} &= \alpha_0 + \alpha_1\lambda_1 . \\ e^{\lambda_2} &= \alpha_0 + \alpha_1 \lambda_2 . \end{aligned}\right\} \text{for matrix } B . \text{——} (5)$$

$$e^B = \alpha_0 I + \alpha_1 B = \begin{bmatrix} \alpha_0 & 0 \\ 0 & \alpha_0 \end{bmatrix} + \begin{bmatrix} \alpha_1 b_{11} & \alpha_1 b_{12} \\ \alpha_1 b_{21} & \alpha_1 b_{22} \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_0 + \alpha_1 b_{11} & \alpha_1 b_{12} \\ \alpha_1 b_{21} & \alpha_0 + \alpha_1 b_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

for $\alpha_1 b_{12}$ & $\alpha_1 b_{21}$ to have corresponding entries to be zero, b_{12} & b_{21} has to be zero, since α_1 is not known and can possibly be non-zero. This gives following relationships

$$\left.\begin{aligned} \alpha_0 + \alpha_1 b_{11} &= 1 \ \& \\ \alpha_0 + \alpha_1 b_{22} &= 1 . \end{aligned}\right\} \text{——} (6)$$

from eqn (5) & (6), subtraction gives:

$$\alpha_1 = \frac{e^{\lambda_1} - 1}{\lambda_1 - b_{11}} \quad \& \quad \alpha_0 = e^{\lambda_1} - \left(\frac{e^{\lambda_1} - 1}{\lambda_1 - b_{11}}\right)\lambda_1$$

and all other possible relations can also be derived in this way.

Since

tracted

n.

From these relations, we are interested in finding the eigenvalues of B instead of finding α_0 & α_1. Since by finding eigenvalues of B, it can be constructed.

After analyzing the relationships it can be infered that $B = \begin{bmatrix} 0 & e^{-n} \\ 0 & 0 \end{bmatrix}$ for very large +ve integer n.

Then e^B gives $I_{2\times2}$.

Substituting this values of B in eqn-(1):

$$e^{iB} = e^{\begin{bmatrix} 0 & ie^{-n} \\ 0 & 0 \end{bmatrix}} = \begin{bmatrix} 1 & ie^{-n} \\ 0 & 1 \end{bmatrix}.$$

The accurate soln is this

However, method 1 is the most preferred method.

2) $A = \begin{bmatrix} 1+i & 2-i & 1 \\ 2-i & 1-i & 2+i \\ 1 & 2+i & 1+i \end{bmatrix}$

$$\det(A) = |A| = \begin{vmatrix} 1+i & 2-i & 1 \\ 2-i & 1-i & 2+i \\ 1 & 2+i & 1+i \end{vmatrix}$$

$$= (1+i)\left[(1-i)(1+i)-(2+i)^2\right] - (2-i)\left[(2-i)(1+i)-(2+i)\right] + \left[(2-i)(2+i)-(1-i)\right]$$

$$= (1+i)\left[(1-i^2) - 4 - i^2 \mp 4i\right] - (2-i)\left[2+2i-i-i^2-2-i\right] + \left[4-i^2-1+i\right]$$

$$= (1+i)\left[1+1-4+1 \mp 4i\right] - (2-i)\left[2+\cancel{i}-\cancel{i}+1-2\right] + \left[4+1-1+i\right]$$

$$= (1+i)\left[-1 \mp 4i\right] - (2-i)\left[1\right] + (4+i)$$

$$= -1 \mp 4i - i \mp 4i^2 - 2 + i + 4 + i$$

$$= -1 \mp 3i + 4 - 2 + 4$$

$$= \cancel{3+5i} \ \ 5-3i.$$

Since, $|A| \neq 0$. Hence, the ~~reduced~~ row Echelon form will have all the pivots entries to be non-zero. This justifies that the column space of A will cover the whole 3-dimensional space and any vector in 3-dimensional complex space can be represented by linear combination of independent vectors of $C(A)$.

3) $\int_2^3 e^{(\ln x)^{1/3}(\ln(\ln(x)))^{2/3}} dx$, $n=7$

Left Rectangle approach taking left rectangles only then

$L_n = \Delta x(y_0 + y_1 + \cdots + y_{n-1})$, $\Delta x = \frac{b-a}{n}$.

$\Delta x = \frac{3-2}{7} = 1/7$.

	x_0	x_1	x_2	x_3	x_4	x_5
x_i	2	$2+\frac{1}{7} = \frac{15}{7}$	$2+\frac{2}{7} = \frac{16}{7}$	$17/7$	$18/7$	$19/7$
$y_i =$	$0.73659 + 0.30495i$	$0.78064 + 0.26895i$	$0.8254 + 0.227498i$	$0.87184 + 0.17852i$	$0.92243 + 0.116895i$	$0.99349 + 0.0111298i$

x_6	x_7
$20/7$	$21/7 = 3$
$1.14498 + 0i$	$0.61894 + 0i$

$L_n = \Delta x[(|y_0| + |y_1| + |y_2| + \cdots + |y_6|)$ since Last rectangle has height y_{n-1} i.e. y_6.

$= \frac{1}{7}(\ldots)$

$= \frac{1}{7}(6.27537 + 1.1079438\, i)$

$= 0.89648143 + 0.1582777 i$.

4) Given $\begin{array}{ccc} a & b & c \\ \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}, & \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, & \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \end{array}$

Applying Gram-schmidt to find orthogonal vectors (not orthonormal)

1) $A = a$ so, $A = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$

2) $$B = b - \frac{A^T b}{A^T A} A = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \left(\frac{\begin{bmatrix} 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}}{\begin{bmatrix} 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}} \right) \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

3) $$C = c - \frac{A^T c}{A^T A} A - \frac{B^T c}{B^T B} B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \left(\frac{\begin{bmatrix} 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}} \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} \right) - \left(\frac{\begin{bmatrix} 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}.$$

Orthogonal vectors are: $\left\{ \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \end{bmatrix} \right\}$

Orthonormal vectors are: $\left\{ \frac{A}{\|A\|}, \frac{B}{\|B\|}, \frac{C}{\|C\|} \right\}.$

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5(6) Lattices are not in syllabus.

If interested to solve the problem based on it then read Lenstra–Lenstra–Lovász algorithm (LLL algo).

7) $f(x) = \sin x + \tan x + \sqrt{x}$, x is in radians for trigonometric functions.

Considering $x = 4$.

$$f'(x) = \cos x + \sec^2 x + \frac{1}{2\sqrt{x}}.$$

Applying Newton Raphson's method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = x_0 - \frac{\sin x_0 + \tan x_0 + \sqrt{x_0}}{\cos x_0 + \sec^2 x_0 + \frac{1}{2\sqrt{x_0}}}$$

Solve it considering x as radians.

the x_{n+1} approaches 0.

Hence, the answer is 0.

8) A is +ve definite.

$P = A(A^TA)^{-1}A^T$ is a projection matrix ~~that~~ corresponding to A such that it projects column vectors into column Space of A, $C(A)$.

But B is a projection matrix that projects column vectors into nullspace of A.

Since, B is a projection matrix it will have only ~~two~~ two possible eigenvalues, either 0 or 1.

Hence, B is not -ve definite, it is ~~...~~ +ve semidefinite.

Given that A is +ve definite then, eigenvalues of A will be +ve and can also be ~~...~~ $0 < \lambda_A < 1$. So,

a) when $\lambda_A \geq 1$ then $A-B$ is +ve semidefinite

b) when $0 < \lambda_A < 1$ then $A-B$ is -ve semidefinite

c) when both a) & b) then $A-B$ ~~is~~ can be indefinite.

9) Read Derivation of Newton Raphson's method.

10) pseudoinverse of $A = \begin{bmatrix} i & 0 & i \\ 0 & -i & 0 \\ 0 & 0 & i \end{bmatrix}$.

SVD(A) is : $A = U\Sigma V^T = \overset{U}{\begin{bmatrix} -0.8507i & 0 & 0.5257i \\ 0 & i & 0 \\ -0.5257i & 0 & -0.8507i \end{bmatrix}}$

$\underset{\Sigma}{\begin{bmatrix} 1.618 & 0 & 0 \\ 0 & 1.0 & 0 \\ 0 & 0 & 0.618 \end{bmatrix}} \underset{V^T}{\begin{bmatrix} -0.5257 & 0 & -0.8507 \\ 0 & -1 & 0 \\ +0.8507 & 0 & -0.5257 \end{bmatrix}}$

$$A = \begin{bmatrix} i & 0 & i \\ 0 & -i & 0 \\ 0 & 0 & i \end{bmatrix} \quad e^{At} \text{ for } t=2 \text{ is}$$

Apply Cayley-Hamilton theorem to find e^{At}.

$$e^{2A} = \begin{bmatrix} -0.4161+0.9093i & 0 & -1.8186-0.8323i \\ 0 & -0.4161-0.9093i & 0 \\ 0 & 0 & -0.4161+0.9093i \end{bmatrix}.$$

11) $$A = \begin{bmatrix} 1 & 3 & 17 & i \\ 2 & 1 & 13 & 1 \\ 20 & 5 & 1 & 0 \\ 40 & 6 & 7 & 1 \end{bmatrix}$$

Eigen values : $|A-\lambda I| = 0$

or, $$\begin{vmatrix} 1-\lambda & 3 & 17 & i \\ 2 & 1-\lambda & 13 & 1 \\ 20 & 5 & 1-\lambda & 0 \\ 40 & 6 & 7 & 1-\lambda \end{vmatrix} = 0$$

or, $\lambda^4 - 4\lambda^3 - 411\lambda^2 - 40i\lambda^2 - 275\lambda - 72i\lambda - 1091 + 1082i = 0.$

$\lambda = -1.1637 - 1.8075i,\ -18.0833 - 0.8248i,\ 0.5087 + 1.6587i,$
$22.7383 + 0.9736i.$

12) $M = \begin{bmatrix} a & f(x) \\ g(x) & c \end{bmatrix}$

a = real number
f(x) = polynomial of degree n.
g(x) = trigonometric funct^n.
c = complex number.

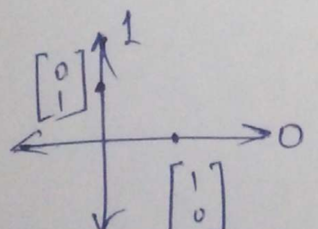
Number of dimensions in real vector space required to locate such matrix M of corresponding matrix vector-space is:

a = need 1-D real vector space.
f(x) = needs 2D — " — ,
g(x) = — " — .
c = — " — .

Sum = 1+2+2+2 = 7.

Hence, 7-dimensions are needed in real vector-space

13) 0 and 1 are represented using 2D space.



To represent 2-bit numbers we need 2^2 real dimensional space,

To represent 3-bit numbers we need 2^3 real dimensional space.

To represent n-bit numbers we need 2^n real dimensional space.

Logic/Rationale was explained in the class.