Numerical Analysis & Linear Algebra.

CSE – 216

(1) X. Higher Engg. Mathematics – Dr. B.S. Grewal.
(2) Intro. to Linear Algebra – Gilbert Strang.
(3) Intro. to Numerical Analysis – Hildebrand.

$\boxed{\text{Syllabus}}$:- CSE-216.

→ Bisection Method, Regular-Falsi, Secant, Newton-Raphson Method, Trapezoidal Rule, Simpson's $\left(\frac{1}{3}\right)$rd Rule, Simpson's $\left(\frac{3}{8}\right)$th Rule,

→ Determinant of Matrix, Inverse, Rank of Matrix, Solution of System of Linear Equation (Homogenous/ Non-Homogenous)

→ Eigen Value, Eigen Vector, Caley Hamilton Theorem.

→ LU, LL^T – Decomposition, SVD, Pseudo Inverse, Vector Space, Sub-space.

$\Rightarrow \quad \rho(A) = 3$

$\Rightarrow \quad \rho(A|B) = 3$.

Now, we know that $AX = B$

$x + 2y + 3z = 6 \quad \text{——} \; (1)$

~~x + 3y + ...~~

$y + z = 2 \quad \text{——} \; (2)$

$-z = 4 \quad \text{——} \; (3) \quad \boxed{z = -4}$

In (2): (Back Substitution)

$\boxed{y = 6}$

$x = 6 - 2y - 3z = 6 - 2(6) - 3(-4)$

$= 6 - \cancel{12} + \cancel{12}$

$\boxed{x = 6}$

Ans:

$X = \begin{bmatrix} 6 \\ 6 \\ -4 \end{bmatrix}$ $\therefore$ soln is unique.

26th sept '17
Tuesday

Solution of System of Linear Equations :-

For Homogenous Eqns :-

$AX = B$

$AX = 0 \quad (\because |B| = 0)$.

* $AX = B$ is called homogenous system if it's denoted by $\boxed{AX = 0}$.

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NOTE 1 : $AX=0$ is always consistent.

because $\rightarrow$ $\boxed{\rho(A) = \rho(A|B)}$.
$= \rho(A|0)$.

$\therefore$ consistent.

2 $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ ~~if all be a~~.

is always a solution of every $AX=0$ & it's called trivial solution (Zero Solution)
Unique solution / Finite Solution.

3 If a solution exists other than trivial soln, for $AX=0$, then such solution is called non-trivial solution / non-zero solution / Infinitely many solutions.

Method :- Step 1 Consider $AX=0$.

2 Consider A & reduce it to its echelon form, by applying row operation

3 Find $\rho(A)$:- rank of A

4 (A) If $\rho(A) = n$, then, $AX=0$ will have only 1 trivial solution.

(B) If $\rho(A) \neq n$ then, $AX=0$; will have trivial & non-trivial solution.

5 If the non-Trivial soln exists, Then find

a soln by rewriting system of eqns from echelon form of coeff matrix.
(By backward substitution).

$\boxed{\text{NOTE}}$: (1) (Imp. Properties)

$$AX = 0$$
$$\downarrow$$
$$A_{n \times n}$$

$|A| \neq 0$.
$\downarrow$
Only Trivial solutions

$|A| = 0$ rank(A) < n.

Trivial | Non-Trivial solutions.

(2) If $\rho(A) = r$ (rank of A) then, n = number of variables,

then, $AX = 0$ will have $(n-r)$ independent solutions.

(Q.1) If $A_{3\times3}$ has rank 2 then, $AX = 0$ has:

(a) Only Trivial.
✓ (b) 1 Indep soln(s).
(c) 2 Indep soln(s)
(d) 3 Indep soln.

$r = 2$.
$n = 3$.
$(3-2) = 1$ indep.

(Q) $A_{n\times n} \longrightarrow \lambda$ eigen value. (λ = scalar)

* $A^n \longrightarrow \lambda^n$.

* $A + K \longrightarrow$ has $\lambda + K$ eigen value

$\ast \quad \lambda_1 + \lambda_2 + \lambda_3 = tr(A_{n\times n})$

$= a_{11} + a_{22} + a_{33}$.

$\ast \; |\lambda_1 \cdot \lambda_2 \cdot \lambda_3| = |A|_{n\times n}$.

3rd Oct'17 | Tuesday

Errors in Trapezoidal Rule.

① $\boxed{\text{Trapezoidal Rule}}$:

$\Rightarrow$ Error = $\left| \frac{-h^3}{12} f''(\xi_i) \right|$

$\xi_i \to$ 2ai (xi).

② $\boxed{\text{Simpson's Rule}}$:

$\Rightarrow$ Error = $\left| \frac{-h^5}{180} f^{IV}(\xi_i) \right|$.

③ $\boxed{\text{Simpson's 3/8th Rule}}$:

$\Rightarrow$ Error = $\left| \frac{-3h^5}{80} f^{IV}(\xi_i) \right|$.

(Q.1) Find error using Simpson's Rule (1/3rd).

$f(x) = \int_0^{\pi/2} \sin x \, dx$ with 11 ordinates

$$\int_a^b f^{iv}(\xi) = \left[f'''(\xi)\right]_a^b .$$

soln →

$b-a = 2nh$

$\frac{\pi}{2} = 10h$

$\boxed{h = \pi/20}$

$$\text{error} = \left| \frac{-\left(\frac{\pi}{20}\right)^5}{180} \left[f'''(x)\right]_0^{\pi/2} \right|.$$

$f'(x) = +\cos x$

$f''(x) = -\sin x$

$f'''(x) = -\cos x$.

$$= \left| \frac{-\left(\frac{\pi}{20}\right)^5}{180} \times -(0-1) \right|.$$

$$\text{Ans:} = \frac{\left(\frac{\pi}{20}\right)^5}{180} .$$

(Q.2)

Find error using Simpson's $\frac{3}{8}$th Rule.

$f(x) = \int_0^6 e^x dx$; using 6 subintervals.

$6-0 = 3n \times h$. $n = 1$.

$6-0 = 6 \times h$.

$$= \left| \frac{-3}{80} \left[e^x \right]_0^6 \right|$$

$$= \left| \frac{-3}{80} \left[e^6 - e^0 \right] \right|$$

$$= \left| \frac{-3}{80} \left(e^6 - 1 \right) \right|$$

Vector sub-spaces.

4th Oct'17
Wednesday.

$\boxed{\text{VECTOR SPACE}}$

① Internal composition.
② External composition.

$(A, +)$ $(B, +, \cdot)$

$A = \{\alpha, \beta, \gamma, \theta, \delta\}$ $B = (a, b, c, d, e, \ldots)$

$\alpha \oplus \beta = \gamma$. internal composition. (when output of set A itself using elem of A itself).

$\alpha \odot a = \beta$ external composition at set A. (when elem from different sets are taken).

Definition → Let V be a set, whose elements are called vectors and F be ~~F~~ a field, $F = \{a, b, c\}$

whose elements're called scalars. The set V is called Vector space over the field F.

If V(F) satisfies the following conditions → is

$V = (\alpha, \beta, \gamma)$; $\alpha = \{a, b, c\}$, $\beta = \{\alpha_1, \alpha_2, \alpha_3\}$.

① $(V, +)$ is abelian group with respect to (w.r.t.) internal composition.

② V(F) is closed under scalar multiplication w.r.t. external composition.

$\forall a \in F$, $\forall \alpha \in V \Rightarrow a.\alpha = \alpha.a \in V$.

External composition.

③ $(V, +, \cdot)$ satisfies :

(a) $a \cdot \alpha \in V$, $\forall a \in F$ and, $\forall \alpha \in V$.

(b) $a \cdot (\alpha + \beta) = a \cdot \alpha + a \cdot \beta$ $\forall a \in F$ and $\alpha, \beta \in V$.

(c) $a(b\alpha) = (ab)\alpha$

(d) $1 \cdot \alpha = \alpha$

$\forall\ 1, a, b \in F$, $\alpha \in V$.

*(Q.1) $V = R^2 = ((x, y) : x, y \in R)$, (Field) $F = R$ = set of real numbers.

Soln 1: Condn (i) $(V, +) \rightarrow$ abelian group,

$(x, y) + (a, b) = (x+a, y+b)$

(a) Algebraic str. checked : ✓. Real Real $\rightarrow \in V$.

$a, b \in R$

Abelian group condition →

① Algebraic structure.

② Monoid – identity element present.

Eg: $V = \{0, 1, 2, 3, 4\}$

$0 + 1 = 1$. (algebraic structure)

0 → identity element present. $3 + 0 = 3$.

③ Group → Inverse Element. (Operation b/w 2 elements to get finally 0) addition / multipli

$1 + (-1) = 0$.

$2 + (-2) = 0$ + : 0 identity elem.

$2 \times \frac{1}{2} = 1$ (Inverse) * : 1. identity element.

Abelian Group: if it satisfies commutative law : $ab = ba$.

Solu^n:

$(0,0) + (x, y) = (0+x, 0+y) = (x, y)$

Identity element present.

(b) Monoid checked.

$(x, y) + (-x, -y) = (0, 0)$

Corresponding inverse present.

(c) Group checked.

$x + y = y + x$: commutative.

$(2, 3) + (5, 6) = (7, 8)$ ↳ $\in R$.

$(2, 3) + (0, 0) = (2, 3)$

↳ Identity element.

$(2, 3) + (-2, -3) = (0, 0)$.

↳ Inverse element.

Cond^n(ii)

$a \cdot (x, y) = (ax, ay) \in V$.

eg: $4 \cdot (2, 3) = (8, 12)$ ↳ $\in V$.

Cond^n(iii)

$\alpha, \beta \in V$ $\quad a, b \in F$

$\alpha \to (x_1, y_1)$ $\quad \beta \to (x_2, y_2)$

$a(\alpha + \beta) = a\left((x_1, y_1) + (x_2, y_2)\right)$

$a(\alpha + \beta) = \cancel{(ax_1, ay_1) + (ax_2, ay_2)}$

We have to Prove distributive law.

$= a\left((x_1 + x_2), (y_1 + y_2)\right)$.

$= (ax_1 + ax_2), (ay_1 + ay_2)$

$$= (ax_1, ay_1) + (ax_2, ay_2)$$

$$= a(x_1, y_1) + a(x_2, y_2)$$

$$= \underline{a\alpha + a\beta}.$$

$$(a+b)\alpha = (a+b)(x_1, y_1)$$

$\alpha \downarrow (x_1, y_1)$

$$= ((a+b)x_1, (a+b)y_1)$$

$$= ((ax_1 + bx_1), (ay_1 + by_1))$$

$$= (ax_1, ay_1) + (bx_1, by_1).$$

$$= a(x_1, y_1) + b(x_1, y_1)$$

$$= \underline{a\alpha + b\alpha}.$$

25th Oct'17
Wednesday

General Property of Vector space :-

⇒ Let V(F) be a vector space & $\vec{O}$ be a zero vector of V, and 0 be the zero scalar of F then,

~~Let V.~~

Property ① $a\bar{o} = \bar{o} \quad \forall\, a \in F.$

Property ② $0\alpha = \bar{o} \quad \forall\, \alpha \in V.$

Property ③ $a(-\alpha) = -(a\alpha) \quad \forall\, a \in F,\ \forall\, \alpha \in V.$

Property ④ $(-a)\alpha = -(a\alpha),$

Property ⑤ $a(\alpha - \beta) = a\alpha - a\beta, \quad \forall\, a \in F, \forall\, \alpha, \beta \in V.$

Property ⑥ $a\alpha = \bar{o} \Rightarrow \alpha = \bar{o}$ or $a = 0, \forall\, a \in F, \forall\, \alpha \in V.$

① Proof ⇒ $a\bar{o} = \bar{o} \quad \forall\, a \in F.$

$\bar{o} + \bar{o} = \bar{o}.$

$a(\bar{o} + \bar{o}) = a\bar{o}$ } Adding $\bar{o}$ to RHS.

$\cancel{a\bar{o}} + a\bar{o} = \cancel{a\bar{o}} + \bar{o}$ $\quad (\because a\bar{o} + \bar{o} = a\bar{o}).$

$\boxed{a\bar{o} = \bar{o}}$

Hence, proved

② Proof ⇒ $0\alpha = \bar{o}.$

$0 + 0 = 0$

$(0+0)\alpha = 0\alpha$ } Adding $\bar{o}$.

$\cancel{0\alpha} + 0\alpha = \cancel{0\alpha} + \bar{o}$ $\quad (\because 0\alpha + \bar{o} = 0\alpha)$

$\boxed{0\alpha = \bar{0}}$.

H.P.

~~8lost prop proof~~

(3) ~~$\alpha + (-\alpha) = \bar{0}$~~. $a(-\alpha) = -(a\alpha)$

$\alpha + (-\alpha) = \bar{0}$

$a(\alpha + (-\alpha)) = a\bar{0}$

$a\alpha + a(-\alpha) = a\bar{0}$

$a\alpha + a(-\alpha) = \bar{0}$

Adding $-(a\alpha)$ on both the sides $\Rightarrow$

$\Rightarrow -\cancel{(a\alpha)} + \cancel{(a\alpha)} + a(\alpha) = -(a\alpha) + \bar{0}$

$\Rightarrow 0 + a(-\alpha) = -(a\alpha)$

$\boxed{a(-\alpha) = -(a\alpha)}$

Hence, proved

(4) $(-a)\alpha = -(a\alpha)$

$\Rightarrow a + (-a) = 0$

$(a + (-a))\alpha = 0\alpha$

~~$\Rightarrow a\alpha + (-a)\alpha = 0\alpha$~~

$\Rightarrow a\alpha + (-a)\alpha = \bar{0}$ $\rightarrow$ by (1)

$\Rightarrow -(a\alpha) + (a\alpha) + (-a)\alpha = -(a\alpha) + \bar{0}.$

$\Rightarrow 0 + (-a)\alpha = -(a\alpha).$

$$\Rightarrow \boxed{(-a)\alpha = -(a\alpha)}$$

Property ⑤ $\Rightarrow$ $a(\alpha - \beta) = a\alpha - a\beta, \quad \forall a \in F, \forall \alpha, \beta \in V$

Proof:

$= a(\alpha + (-\beta)).$

$= a\alpha + a(-\beta).$ by ③.

$= (a\alpha - a\beta).$

Hence, proved

Property - ⑥ $\Rightarrow$ $a\alpha = \bar{0}. \Rightarrow \alpha = \bar{0}$ or $a = 0,$

$\forall a \in F, \forall \alpha \in V$

Proof:

ⓐ Let $a \neq 0$

Now, $a\alpha = \bar{0}.$

$a^{-1}(a\alpha) = a^{-1}\bar{0}.$

$= \bar{0}.$

$(a^{-1}a)\alpha = \bar{0}.$

$$\boxed{\alpha = \bar{0}}$$

ⓑ Let $\alpha \neq \bar{0}.$

$a\alpha = \bar{0}. \quad a^{-1} \in F$

$a^{-1}(a\alpha) = a^{-1}\bar{0}.$

$$\boxed{\alpha = \bar{0}}$$

Our assumption's false then,

$$\boxed{a = 0}$$

Pseudo Inverse

for finding Inverse of $m \times n$ matrix also other than square matrix

How To find Pseudo Inverse? (Note)

SVD.

eg: $C = \begin{bmatrix} 5 & 5 \\ -1 & 7 \end{bmatrix}$.

$$\boxed{C = V \Sigma V^T}$$

(*) ① $C^T C = V \Sigma^T \Sigma V^T$.

(*) ② $C V = U \Sigma$.

$$C^T C = \begin{bmatrix} 5 & -1 \\ 5 & 7 \end{bmatrix} \begin{bmatrix} 5 & 5 \\ -1 & 7 \end{bmatrix} = \begin{bmatrix} 26 & 18 \\ 18 & 74 \end{bmatrix}.$$

(*) ①' $\det(C^T C - \lambda I) = 0 \Rightarrow \begin{vmatrix} 26-\lambda & 18 \\ 18 & 74-\lambda \end{vmatrix} = 0$

$\lambda^2 - 100\lambda + 1600 = 0$

$(\lambda - 20)(\lambda - 80) = 0$ $\boxed{\lambda = 20, 80}$

$\lambda = 20$

$$CC^T - 20I = \begin{bmatrix} 6 & 18 \\ 18 & 54 \end{bmatrix}$$

$$V_1 = \begin{bmatrix} -3/\sqrt{10} \\ -1/\sqrt{10} \end{bmatrix}$$

$$C^TC - 80I = \begin{bmatrix} -54 & 18 \\ 18 & -6 \end{bmatrix}$$

$$V_2 = \begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{bmatrix}$$

vector space :-

$$V = \begin{bmatrix} -3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 2\sqrt{5} & 0 \\ 0 & 4\sqrt{5} \end{bmatrix}$$

(2) $$CV = \begin{bmatrix} 5 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} -3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix} = \begin{bmatrix} -\sqrt{10} & 2\sqrt{10} \\ \sqrt{10} & 2\sqrt{10} \end{bmatrix}$$

CV

$$U\Sigma = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \underbrace{\begin{bmatrix} \cancel{2\sqrt{5}}\ 2\sqrt{5} & 0 \\ 0 & 4\sqrt{5} \end{bmatrix}}_{\Sigma}$$

$$= \begin{bmatrix} 2\sqrt{5} & 0 \\ 0 & 4\sqrt{5} \end{bmatrix}$$

$$\boxed{U, V, \Sigma}$$

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if eigen ~~values~~ vector's > 1.

$$\boxed{\begin{aligned} \lambda_1 &= 54.1693 \\ \lambda_2 &= 0.8303 \\ \lambda_3 &= 0 \end{aligned}}$$

$\rightarrow \sigma_1 = \sqrt{\lambda_1} = 7.36$

$\rightarrow \sigma_2 = \sqrt{\lambda_2} = 0.9114$

$\rightarrow \sigma_3 = \sqrt{\lambda_3} = 0$.

$x_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ $\rightarrow$ suppose if values in this > 1.

$$= \begin{bmatrix} 1/\sqrt{1+9+4} \\ 3/\sqrt{1+9+4} \\ 2/\sqrt{1+9+4} \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{14} \\ 3/\sqrt{14} \\ 2/\sqrt{14} \end{bmatrix}.$$

In this case, eigen vector's $\Rightarrow$

$x_1 = \begin{bmatrix} -0.675 \\ -0.6067 \\ -0.7921 \end{bmatrix}$, $x_2 = \begin{bmatrix} -0.9521 \\ 0.2764 \\ -0.1306 \end{bmatrix}$.

$x_3 = \begin{bmatrix} -0.2981 \\ -0.7454 \\ 0.5963 \end{bmatrix}$ Here, already values in eigen vector are less than < 1.

$$V = \begin{bmatrix} -0.675 & -0.9521 & -0.2981 \\ -0.6067 & 0.2764 & -0.7454 \\ -0.7921 & -0.1306 & 0.5963 \end{bmatrix}$$
$\quad x_1 \uparrow \quad x_2 \uparrow \quad x_3 \uparrow$

Exam Q can come till finding SBD i.e Σ.
or till A^+.

Using ② $\underline{\underline{CV = U\Sigma}}$

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix} V = U\Sigma$$

We can find U

$\boxed{U = \Sigma^{-1}CV}$

$$U = \begin{bmatrix} -0.4969 & -0.8678 \\ -0.8678 & 0.4969 \end{bmatrix}$$

$(\because A^+ = V\Sigma^{-1}U^T)$

find Σ^{-1}

$$\Sigma = \begin{bmatrix} 7.36 & 0 & 0 \\ 0 & 0.9114 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ Course till P. 1.