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$$= \left| \frac{-3}{80}\left(e^{6}-1\right) \right| = \frac{3}{80}\left(e^{6}-1\right)$$

- Vector Space:

Let V be a set, whose elements are called vectors and F be a field, whose elements are called scalars. The set V is called vector space over the field F.

if V(F) satisfy the following conditions:

- Vector Space:

1) Internal Composition $(A, +)$

$A = \{\alpha, \beta, \gamma, \theta, \delta\}$

$\alpha + \beta = \gamma$

Internal composition

2) External Composition $(B, +, \cdot)$

$B = \{a, b, \ldots\}$

$\alpha \cdot a = \beta$

External composition

1) $(V, +)$ is abelian group with respect to internal composition

2) $V(F)$ is closed under scalar multiplication with respect to external composition

$\forall a \in F, \quad \forall \alpha \in V$

$$\Rightarrow \quad a \cdot \alpha = \alpha \cdot a \in V$$

3) $(V, +, \cdot)$ Satisfy

a) $a \cdot \alpha \in V, \quad \forall a \in F$ and $\forall \alpha \in V$

b) $a \cdot (\alpha + \beta) = a \cdot \alpha + a \cdot \beta \quad \forall a \in F$ and $\alpha, \beta \in V$

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$$\left.\begin{aligned} &c)\ a(b\alpha) = (ab)\alpha \\ &d)\ 1.\alpha = \alpha \end{aligned}\right\} \forall\ 1, a, b \in F,\ \alpha \in V$$

Q) $V = R^2 = ((x,y) : x, y \in R)$

$f = R =$ Set of real numbers.

(i) $(V,+) \rightarrow$ abelian group $\quad V = \{0, 1, 2, 3, 4\}$

① Algebric Structure

② Monoid - identity $\quad 0 + 1 = 1$

③ Group - Inverse element $\quad 3 + 0 = 3$

(i) $(V,+) \longrightarrow$ abelian group

$$(x,y) + (a,b) = (\overset{R}{x+a}, \overset{R}{y+b})$$

$$(0,0) + (x,y) = (0+x, 0+y) = (x,y)$$

To check Inverse.

Identity

$$(x,y) + (-x,-y) = (0,0)$$

(ii) $V \longrightarrow (x,y) \in V,\ a \in F$

$$a.(x,y) = (ax, ay) \in V$$

$$4.(2,3) = (4.2, 4.3) = (8, 12)$$

(iii) $\alpha, \beta \in V \quad a, b \in F$

$(x_1, y_1)\ (x_2, y_2)$

$$a(\alpha + \beta) = a((x_1,y_1) + (x_2,y_2))$$

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$$\cancel{= (ax_1, ay_1) + (ax_2, ay_2)}$$

$$= a((x_1+x_2), (y_1+y_2)) = a(x_1, y_1) + a(x_2, y_2)$$

$$= (ax_1 + ax_2), (ay_1 + ay_2)$$

$$= (ax_1, ay_1) + (ax_2, ay_2) = a\alpha + a\beta$$

$$(a+b)\alpha = (a+b)(x_1, y_1)$$

$$= ((a+b).x_1, (a+b).y_1)$$

$$= (ax_1 + bx_1), (ay_1 + by_1)$$

$$= (ax_1, ay_1) + (bx_1, by_1)$$

$$= a(x_1, y_1) + b(x_1, y_1)$$

$$= a\alpha + b\alpha$$

3 conditions satisfied.

- General Property of Vector Space :-

Let $V(F)$ be a vector space and $\bar{0}$ be a zero vector of V and 0 be the zero scalar of F then

Property:

1) $a\bar{0} = \bar{0}$, $\forall a \in F$
2) $0\alpha = \bar{0}$, $\forall \alpha \in V$
3) $a(-\alpha) = -(a\alpha)$, $\forall a \in F$, $\forall \alpha \in V$
4) $(-a)\alpha = -(a\alpha)$, $\forall a \in F$, $\forall \alpha \in V$
5) $a(\alpha - \beta) = a\alpha - a\beta$, $\forall a \in F$, $\forall \alpha, \beta \in V$

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6) $a\alpha = \bar{0} \Rightarrow \alpha = \bar{0}$ or $a = 0$, $\forall a \in F$, $\forall \alpha \in V$

• Proof :

1) $a\bar{0} = \bar{0} \quad \forall a \in F$

$$\bar{0} + \bar{0} = \bar{0}$$

$$a(\bar{0} + \bar{0}) = a\bar{0} \qquad \left[a\bar{0} + \bar{0} = a\bar{0}\right]$$

$$a\bar{0} + a\bar{0} = a\bar{0} + \bar{0}$$

$$\boxed{a\bar{0} = \bar{0}}$$

2) $0\alpha = \bar{0}$

$$0 + 0 = 0$$

$$(0+0)\alpha = 0\alpha$$

$$0\alpha + 0\alpha = 0\alpha + \bar{0} \qquad \boxed{\because 0\alpha + \bar{0} = 0\alpha}$$

$$\boxed{0\alpha = \bar{0}}$$

3)

$$\alpha + (-\alpha) = \bar{0}$$

$$a(\alpha + (-\alpha)) = a\bar{0}$$

$$a\alpha + a(-\alpha) = a\bar{0}$$

$$a\alpha + a(-\alpha) = \bar{0}$$

$$-(a\alpha) + a\alpha + a(-\alpha) = -(a\alpha) + \bar{0}$$

$$0 + a(-\alpha) = -(a\alpha)$$

$$\boxed{a(-\alpha) = -(a\alpha)}$$

4)

$$a + (-a) = 0$$

$$[a + (-a)]\alpha = 0\alpha$$

$$a\alpha + (-a)\alpha = 0\alpha \qquad \text{by } 2$$

$$a\alpha + (-a)\alpha = \bar{0}$$

$$-(a\alpha) + a\alpha + (-a)\alpha = -(a\alpha) + \bar{0}$$

$$0 + (-a)\alpha = -(a\alpha)$$

$$\boxed{(-a)\alpha = -(a\alpha)}$$

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2) $a(\alpha - \beta) = a(\alpha + (-\beta))$

$= a\alpha + a(-\beta)$

$\boxed{a(\alpha - \beta) = a\alpha - a\beta}$ by 3

4) a) let $a \neq 0$

$a\alpha = \bar{0}$

$a = 0 \quad , \alpha = \bar{0}$

Now $a\alpha = \bar{0}$

$a^{-1}(a\alpha) = a^{-1}\bar{0}$

$(a^{-1}a)\alpha = \bar{0}$

$\boxed{\alpha = \bar{0}}$

b) Let $\alpha \neq \bar{0}$

Start with contradiction $a \neq 0$

$a^{-1} \in F \quad , a\alpha = \bar{0}$

$a^{-1}(a\alpha) = a^{-1}\bar{0}$

$\alpha = \bar{0}$

Our assumption is false.

then

$\boxed{a = 0}$

- Pseudo Inverse :-

SVD

E.g. $C = \begin{bmatrix} 5 & 5 \\ -1 & 7 \end{bmatrix}$

$$\boxed{C = U \Sigma V^T}$$

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- $C^T.C = V\Sigma^T\Sigma V^T$

- $CV = U\Sigma$

$$\Rightarrow C^TC = \begin{bmatrix} 5 & -1 \\ 5 & 7 \end{bmatrix}\begin{bmatrix} 5 & 5 \\ -1 & 7 \end{bmatrix} = \begin{bmatrix} 26 & 18 \\ 18 & 74 \end{bmatrix}$$

- Let $(C^TC - \lambda I) = 0$

$$\begin{vmatrix} 26-\lambda & 18 \\ 18 & 74-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 100\lambda + 1600 = 0$$

$$\lambda = 20, 80$$

- $\lambda = 20$

$$CC^T - 20I = \begin{bmatrix} 6 & 18 \\ 18 & 54 \end{bmatrix}$$

$$V_1 = \begin{bmatrix} -3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}$$

$$CC^T - 80Z = \begin{bmatrix} -54 & 18 \\ 18 & -6 \end{bmatrix}$$

$$V_2 = \begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{bmatrix}$$

$$\begin{pmatrix} 5 & 5 \\ -1 & 7 \end{pmatrix} \begin{pmatrix} -3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{pmatrix} = \begin{pmatrix} -\sqrt{10} & 2\sqrt{10} \\ \sqrt{10} & 2\sqrt{10} \end{pmatrix}$$

$$C \quad V \qquad = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{5} & 0 \\ 0 & 4\sqrt{5} \end{bmatrix}$$

$$= U\Sigma = \boxed{\begin{matrix} 2\sqrt{5} & 0 \\ 0 & 4\sqrt{5} \end{matrix}}$$

$\boxed{U, V, \Sigma}$

- $A = \underbrace{U \Sigma V^T}$ → Orthogonal matrix $\left(\begin{matrix} V^T = V^{-1} \\ U^T = U^{-1} \end{matrix} \right)$

$$A^+ = (U \Sigma V^T)^{-1}$$

$$= (V^{-1})^{-1} \Sigma^{-1} U^{-1}$$

Peudo Inverse $\boxed{A^+ = V \Sigma^{-1} U^T}$

$$\Sigma^{-1} = \begin{bmatrix} 1/\sigma_1 & & \\ & 1/\sigma_2 & \\ & & 1/\sigma_3 \end{bmatrix}$$

$\sigma_1, \sigma_2, \sigma_3 \ldots$ Singular values $\sigma_1 = \sqrt{\lambda_1}$

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- $C = \begin{bmatrix} 5 & 5 \\ -1 & 7 \end{bmatrix}$

- $U = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

$V = \begin{bmatrix} -3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix}$ $\quad V^T = \begin{bmatrix} -3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix}$

$\Sigma = \begin{bmatrix} 2\sqrt{5} & 0 \\ 0 & 4\sqrt{5} \end{bmatrix}$ $\quad \Sigma^{-1} = \begin{bmatrix} 1/2\sqrt{5} & 0 \\ 0 & 1/4\sqrt{5} \end{bmatrix}$

$C = \begin{bmatrix} 5 & 5 \\ -1 & 7 \end{bmatrix}$

$C^T = V \Sigma^{-1} U^T$

Q) $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix}$ Find A^+ ?

Soln) $A^T A = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$ $\rightarrow \lambda_1$, $\rightarrow \lambda_2$, $\rightarrow \lambda_3$

$\sigma_1 = \sqrt{\lambda_1}$

$\sigma_2 = \sqrt{\lambda_2}$

$\sigma_3 = \sqrt{\lambda_3}$

$\Sigma = \begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{bmatrix}$

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$$\boxed{\begin{aligned}\lambda_1 &= 54.1693\\ \lambda_2 &= 0.8303\\ \lambda_3 &= 0\end{aligned}}$$

$$X = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{1^2+3^2+2^2}} \end{bmatrix} = \begin{matrix} \frac{1}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \end{matrix}$$

Higher Unit vector

$$X_1 = \begin{bmatrix} -0.675 \\ -0.6067 \\ -0.7921 \end{bmatrix}, \quad X_2 = \begin{bmatrix} -0.9521 \\ 0.2764 \\ -0.1306 \end{bmatrix}$$

$$V = \begin{bmatrix} -0.675 & -0.9521 & -0.2981 \\ -0.6067 & 0.2764 & -0.7454 \\ -0.7921 & -0.1306 & 0.5963 \end{bmatrix}$$

can ask this

$$CV = U\Sigma \qquad CV = \begin{bmatrix} \\ \\ \end{bmatrix}$$

$$U = \begin{bmatrix} \\ \\ \end{bmatrix}$$

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$$U = \begin{bmatrix} -0.4969 & -0.8678 \\ -0.8678 & 0.4969 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 7.36 & 0 & 0 \\ 0 & 0.914 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ can ask this

$A^+ =$ can ask this

Q) ① Show that the set $W = \{(a, b, 0) : a, b \in F\}$ is a subspace of $V_3(F)$.

Soln) Let $\alpha = (a_1, b_1, 0)$, $\beta = (a_2, b_2, 0)$

i.e. $\alpha, \beta \in W$, and $a, b \in F$

Now $a\alpha + b\beta = a(a_1, b_1, 0) + b(a_2, b_2, 0)$

$= (aa_1, ab_1, 0) + (ba_2, bb_2, 0)$

$= (aa_1 + ba_2, ab_1 + bb_2, 0)$

Since, $\forall a_1, b_1, a_2, b_2, a, b \in F$

$a, b \to$ scalars

$\Rightarrow aa_1 + ba_2 \in F$ $\quad ab_1 + bb_2 \in F$

$\Rightarrow a\alpha + b\beta = (aa_1 + ba_2, ab_1 + bb_2) \in W$

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$\Rightarrow$ W is a subspace of V(F).

Q2) $W = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} ; a, b \in F \right\}$ is a sub-space of $M_2(F)$

Sol) Let $\alpha = \begin{bmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{bmatrix}$, $\beta = \begin{bmatrix} a_2 & b_2 \\ -b_2 & a_2 \end{bmatrix}$

i.e. $\alpha, \beta \in W$, and, $a, b \in F$

Now $a\alpha + b\beta = a \begin{bmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{bmatrix} + b \begin{bmatrix} a_2 & b_2 \\ -b_2 & a_2 \end{bmatrix}$

$$= \begin{bmatrix} aa_1 & ab_1 \\ -ab_1 & aa_1 \end{bmatrix} + \begin{bmatrix} ba_2 & bb_2 \\ -bb_2 & ba_2 \end{bmatrix}$$

$$= \begin{bmatrix} aa_1 + ba_2 & ab_1 + bb_2 \\ -ab_1 - bb_2 & aa_1 + ba_2 \end{bmatrix}$$

Since, $\forall\ a_1, b_1, a_2, b_2, a, b \in F$ → scalars.

$\Rightarrow aa_1 + ba_2, \ldots\ldots \in F$

$\Rightarrow a\alpha + b\beta = \begin{bmatrix} \ldots \end{bmatrix} \in W.$

$\Rightarrow$ W is a subspace of V(F).