CST II Types

2002, Paper 9, question 6

(a)

1 (var) $\Gamma \vdash x : \tau$ if Γ ok & $(x:\tau) \in \Gamma_{ta}$

1 (fn) $$\frac{\Gamma, x:\tau \vdash M : \tau'}{\Gamma \vdash \lambda x(M) : \tau \to \tau'}$$ if $x \notin dom(\Gamma_{ta})$

1 (app) $$\frac{\Gamma \vdash M_1 : \tau \to \tau' \quad \Gamma \vdash M_2 : \tau}{\Gamma \vdash M_1 M_2 : \tau'}$$

1 (gen) $$\frac{\alpha, \Gamma \vdash M : \tau}{\Gamma \vdash \Lambda\alpha(M) : \forall\alpha(\tau)}$$ if Γ ok & $\alpha \notin \Gamma_{tv}$

1 (spec) $$\frac{\Gamma \vdash M : \forall\alpha(\tau_1)}{\Gamma \vdash M\tau_2 : \tau_1[\tau_2/\alpha]}$$ if $ftv(\tau_2) \subseteq \Gamma_{tv}$

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(b)

Define

2 $$pair \triangleq \Lambda\alpha_1(\Lambda\alpha_2(\lambda x_1 : \alpha_1(\lambda x_2 : \alpha_2(\Lambda\alpha(\lambda f : \alpha_1 \to \alpha_2 \to \alpha(f x_1 x_2))))))$$

2 $$fst \triangleq \Lambda\alpha_1(\Lambda\alpha_2(\lambda p : prod(\alpha_1, \alpha_2)(p\alpha_1(\lambda x_1 : \alpha_1(\lambda x_2 : \alpha_2(x_1))))))$$

1 $$snd \triangleq \Lambda\alpha_1(\Lambda\alpha_2(\lambda p : prod(\alpha_1, \alpha_2)(p\alpha_2(\lambda x_1 : \alpha_1(\lambda x_2 : \alpha_2(x_2))))))$$

Claim:

(1) $\emptyset, \emptyset \vdash \mathrm{pair} : \forall\alpha_1(\forall\alpha_2(\alpha_1 \to \alpha_2 \to \mathrm{prod}(\alpha_1, \alpha_2)))$

(2) $\emptyset, \emptyset \vdash \mathrm{fst} : \forall\alpha_1(\forall\alpha_2(\mathrm{prod}(\alpha_1, \alpha_2) \to \alpha_1))$

(and $\emptyset, \emptyset \vdash \mathrm{snd} : \forall\alpha_1(\forall\alpha_2(\mathrm{prod}(\alpha_1, \alpha_2) \to \alpha_2))$)

For all types τ_1, τ_2 and terms M_1, M_2

(3) $\mathrm{fst}\ \tau_1 \tau_2 (\mathrm{pair}\ \tau_1 \tau_2\ M_1 M_2) =_\beta M_1$

(and $\mathrm{snd}\ \tau_1 \tau_2 (\mathrm{pair}\ \tau_1 \tau_2\ M_1\ M_2) =_\beta M_2$).

Proof of (1):

$\{\alpha_1, \alpha_2, \alpha\}, [x_1 : \alpha_1, x_2 : \alpha_2, f : \alpha_1 \to \alpha_2 \to \alpha] \vdash f x_1 x_2 : \alpha$

holds by (var) & (app) twice. Applying (fn), (gen), (fn) twice & (gen) twice, yields (1). $\square$

Proof of (2):

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$\{\alpha_1, \alpha_2\}, [p : \mathrm{prod}(\alpha_1, \alpha_2)] \vdash \lambda x_1 : \alpha_1 (\lambda x_2 : \alpha_2 (x_1)) : \alpha_1 \to \alpha_2 \to \alpha_1$

holds by (var) & (fn) twice. So by (spec) & (app) we have

$\{\alpha_1, \alpha_2\}, [p : \mathrm{prod}(\alpha_1, \alpha_2)] \vdash p\,\alpha_1 (\lambda x_1 : \alpha_1 (\lambda x_2 : \alpha_2 (x_1))) : \alpha_1$

So applying (fn) & (gen) twice we get (2). $\square$

~~Proof of (3) is similar to that of (2)~~

~~Proof of (4)~~

$\cancel{\mathrm{fst}\ \tau_1 \tau_2 (\mathrm{pair}\ \tau_1 \tau_2\ M_1\ M_2) \to_\beta^*}$

1 Proof of (3):

$$\mathit{fst}\ \tau_1\tau_2\,(\mathit{pair}\ \tau_1\tau_2\ M_1M_2)$$

$$\rightarrow_\beta^* \mathit{fst}\ \tau_1\tau_2(\Lambda\alpha(\lambda f:\tau_1\to\tau_2\to\alpha\,(f\,M_1\,M_2)))$$

$$= (\Lambda\alpha(\lambda f:\tau_1\to\tau_2\to\alpha(f\,M_1\,M_2)))\tau_1\,(\lambda x_1:\tau_1\,(\lambda x_2:\tau_2\,(x_1)))$$

$$\rightarrow_\beta^* (\lambda x_1:\tau_1\,(\lambda x_2:\tau_2\,(x_1)))M_1\,M_2$$

$$\rightarrow_\beta^* M_1$$

1 (Where, without loss of generality, we assume the bound variables $\alpha_1\alpha_2\alpha\ x_1 x_2 f$ are distinct from any free variables of $\tau_1\ \tau_2\ M_1 M_2$). □

10 Proof of (5) is similar to (4).

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1 (c) No, can have

$$\mathit{pair}\ \tau_1\tau_2\,(\mathit{fst}\ \tau_1\tau_2\ M)(\mathit{snd}\ \tau_1\tau_2\ M) \neq_\beta M$$

for typeable M.

To see this we use the fact that β-reduction is strongly normalising & Church-Rosser for typeable PLC terms: so if $\Gamma\vdash M_1:\tau$ & $\Gamma\vdash M_2:\tau$ and M_1 & M_2 reduce to different normal forms

2 (up to α-conversion), then $M_1\neq_\beta M_2$.

Apply this when $\Gamma = \{\alpha_1,\alpha_2\},[x:\mathit{prod}(\alpha_1,\alpha_2)]$

1 and $M_1 = x$, $M_2 = \mathit{pair}\,\alpha_1\alpha_2\,(\mathit{fst}\,\alpha_1\alpha_2\,x)(\mathit{snd}\,\alpha_1\alpha_2\,x)$.

Now $\Gamma\vdash x:\mathit{prod}(\alpha_1,\alpha_2)$, but

M_1 & M_2 reduce to different normal forms.
Indeed $M_1 = x$ is in normal form, whereas

$$\mathrm{fst}\, \alpha_1 \alpha_2\, x \;\to_\beta^*\; \underbrace{x\, \alpha_1 \,(\lambda x_1 : \alpha_1 (\lambda x_2 : \alpha_2 (x_1)))}_{\triangleq\, N_1 \text{, normal form}}$$

$$\mathrm{snd}\, \alpha_1 \alpha_2\, x \;\to_\beta^*\; \underbrace{x\, \alpha_2 \,(\lambda x_1 : \alpha_1 (\lambda x_2 : \alpha_2 (x_2)))}_{\triangleq\, N_2 \text{, normal form}}$$

and so

$$M_2 \;\to_\beta^*\; \mathrm{pair}\, \alpha_1 \alpha_2\, N_1\, N_2$$

$$\;\to_\beta^*\; \Lambda \alpha (\lambda f : \alpha_1 \to \alpha_2 \to \alpha \,(f\, N_1 N_2)$$

5 and the latter is a normal form not equal to x.

To which parts of the lecture course does ~~this~~ question refer?

(a) Definition from lecture 6.

(b) Uses material from lectures 6 – 8.
This particular example was given as an exercise in the lecture notes, but not covered in lectures explicitly.

(c) Requires original thought, based on knowledge of typing & reduction properties of PLC covered in lecture 7.