Numerical Analysis II 2002 p8 q11

MRO'D

Solution Notes – Question A

Question concerns: Elementary approximation theory – best approximations, range reduction, square roots.

(a) $M = n+1$. [2 marks]

(b) If $m \times 2^k$ is normalised, $m \in [1,2)$.

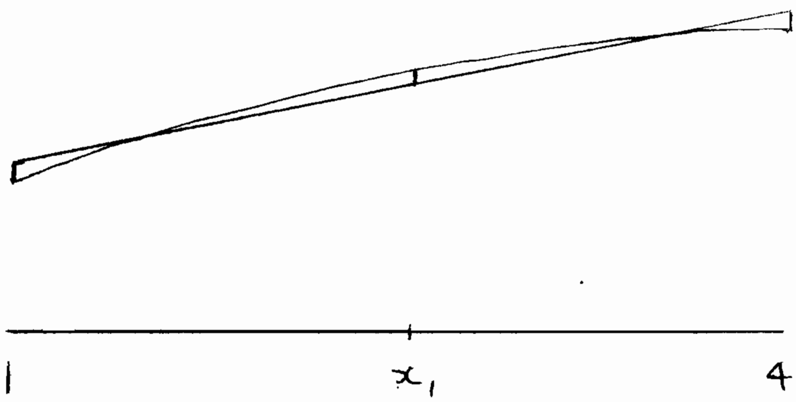
If k is even $\sqrt{x} = \sqrt{m} \times 2^{k/2}$; $\sqrt{m} \in [1,\sqrt{2})$.

If k is odd $\sqrt{x} = \sqrt{2m} \times 2^{(k-1)/2}$; $\sqrt{2m} \in [\sqrt{2},2)$.

Note that $[1,\sqrt{2}) \cup [\sqrt{2},2) = [1,2)$.

Now write x for the significand.

We only need to compute $\sqrt{x} \in [1,2)$, i.e. $x \in [1,4)$.



From the graph the extrema are at $1, x_1, 4$ where x_1 is a turning point of $e(x)$.

Set

$$\frac{de}{dx} = a - \frac{1}{2\sqrt{x}} = 0$$

so $x_1 = \frac{1}{4a^2}$.

From the Chebyshev characterisation theorem,

$$e(1) = -e(x_1) = e(x_4)$$

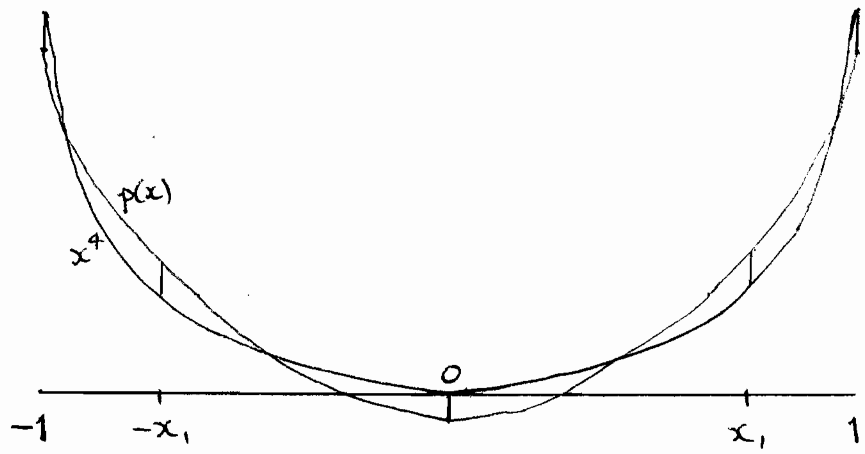
$$\therefore a + b - 1 = -\frac{1}{4a} - b + \frac{1}{2a} = 4a + b - 2$$

$$\therefore a = \frac{1}{3}, \quad b = \frac{17}{24}.$$ [8 marks]

(c) As x^4 is an even function, the best approximation must also be an even function, so it has the form

$$p(x) = ax^2 + b.$$

In the Chebyshev characterisation theorem $n=4$ so we expect 5 extrema:



Write $e(x) = x^4 - ax^2 - b$.

The extrema are at ± 1, $\pm x_1$ and 0, where $\pm x_1$ and 0 are turning points of $e(x)$.

Set

$$\frac{de}{dx} = 4x^3 - 2ax = 0$$

$$\text{so } x_1 = \sqrt{\frac{a}{2}}.$$

From the Chebyshev characterisation theorem,

$$1 - a - b = -\frac{a^2}{4} + \frac{a^2}{2} + b = -b$$

$$\therefore a = 1, \quad b = -\frac{1}{8},$$

$$\text{i.e. } p(x) = x^2 - \frac{1}{8}.$$

[10 marks]