Digital Signal Processing 2004 – Paper 7 Question 10 (MGK)

(*a*) (*i*) This is a single-tap infinite impulse response low-pass filter (also known as exponential averaging filter).

(*ii*) With $y_i = 0.8y_{i-1} + 0.2x_i$, we have

$$y_i = 0.2 \cdot \sum_{j=0}^{i} 0.8^j \cdot x_{i-j}.$$

Recalling that if A and B are independent random variables then $\mathrm{Var}(\alpha A + \beta B) = \alpha^2 \mathrm{Var}(A) + \beta^2 \mathrm{Var}(B)$, we get

$$\mathrm{Var}(y_i) = 0.2^2 \cdot \sum_{j=0}^{i} 0.8^{2j} \cdot \mathrm{Var}(x_{i-j})$$

and looking only at the noise variance $\mathrm{Var}(x_i) = (30\ \mathrm{mm})^2$ and the limit $i \to \infty$, we obtain

$$\mathrm{Var}(y_i) = (30\ \mathrm{mm})^2 \cdot 0.2^2 \cdot \sum_{j=0}^{\infty} 0.8^{2j} = (30\ \mathrm{mm})^2 \cdot 0.2^2 \cdot \frac{1}{1-0.8^2}$$

$$= (30\ \mathrm{mm})^2 \cdot \frac{(1-0.8)^2}{1-0.8^2} = (30\ \mathrm{mm})^2 \cdot \frac{1-0.8}{1+0.8} = (30\ \mathrm{mm})^2 \cdot \frac{1}{9} = (10\ \mathrm{mm})^2$$

The standard deviation caused by small waves in the displayed values will be 10 mm.

(*b*) Shifting a spectrum by $f_\mathrm{s}/2$ is equivalent to multiplying a signal with a cosine function of frequency $f_\mathrm{s}/2$, therefore $h'_i = h_i \cdot \cos(2\pi \frac{f_\mathrm{s}}{2} \frac{i}{f_\mathrm{s}}) = h_i \cdot \cos(\pi i) = h_i \cdot (-1)^i$.

(*c*) (*i*) The DFT will represent the power spectrum of a signal accurately only for tones whose period divides the block length of the DFT evenly, that is whose frequency matches exactly one of the discrete frequencies in the DFT output. Sine waves of other frequencies suffer a discontinuity at the DFT block boundary, which leads to spectral energy leakage into neighbour frequencies. Cutting a finite DFT block out of a periodic signal is equivalent to multiplying the signal in the time domain with a rectangular function, or convolving it in the frequency domain with the corresponding $\sin(x)/x$-shaped spectrum.

(*ii*) The signal block can be multiplied with a windowing function (e.g., raised-cosine/Hanning window $0.5 - 0.5\cos(2\pi n/(N-1))$ for $0 \le n < N$) before applying the DFT. This smooths the block boundary to zero, eliminating any discontinuity there. A good windowing function has a compacter spectrum than the equivalent rectangular function and the convolution with it in the time domain will leak less energy to distant neighbour frequencies.

[The questions relate to the sections on (a) IIR filters and averaging, (b) design of FIR filters and spectral inversion, and (c) spectral estimation.]