SOLUTION NOTES

Logic and Proof 2003 Paper 5 Question 9 (LCP)

(a) This relates to Lecture 4, on ordered binary decision diagrams. A good ordering is $P_1 < Q_1 < P_2 < Q_2 < \cdots P_i < Q_i \cdots < R$. Note that $\phi(k)$ is true unless R is false and $P_i \leftrightarrow Q_i$ is true for all $i > 0$. The corresponding OBDD will start with an OBDD for $P_1 \leftrightarrow Q_1$, modified such that the final link to false instead leads to true, and the final link for true instead leads to a similarly modified OBDD for $P_2 \leftrightarrow Q_2$. The OBDD ends with the trivial one for R. Ideally, the candidate should supply a diagram such as the one attached (see the file logic-figure.pdf). That one is for the case $k = 2$, but the general case is easy to sketch using pen and paper.

(b) One possible ordering is $P_1 < P_2 < \cdots < Q_1 < Q_2 < \cdots < R$. The OBDD will be exponential in k because the outcome for Q_i depends upon the value of P_i. Thus, the OBDD below Q_1 will have to depend upon the values of $P_1, \ldots P_k$. There are 2^k possible combinations those values and therefore at least 2^k decision nodes for Q_1.

(c) This relates to Lecture 7, resolution. Negating the formula gives

$$(P_1 \leftrightarrow Q_1) \land \ldots \land (P_k \leftrightarrow Q_k) \land \neg R.$$

Since $P \leftrightarrow Q \simeq (P \to Q) \land (Q \to P)$, the clauses are

$$\{\neg P_1, Q_1\} \quad \{\neg Q_1, P_1\} \quad \ldots \{\neg P_k, Q_k\} \quad \{\neg Q_k, P_k\} \quad \{\neg R\}.$$

(d) The Davis-Putnam procedure would reduce it to the empty set of clauses, indicating that $\neg\phi(k)$ is satisfiable and that $\phi(k)$ is not valid. This is the correct outcome, as the formula is obviously falsifiable. The first step of Davis-Putnam would act on the pure literal $\neg R$ by deleting the clause $\{\neg R\}$. Exponentially many case splits then would be necessary. For example, a case split on P_1 would, in the true case, delete the clause $\{\neg Q_1, P_1\}$ and reduce the clause $\{\neg P_1, Q_1\}$ to $\{Q_1\}$, which would then be deleted as Q_1 would be pure. Further case splits on the remaining variables would eliminate all the clauses. The false case would be similar.