Papers 4, 11 Solution

a) i) Primitive Recursive functions are obtained by closure under the application of the schemes of substitution and (primitive) recursion to an (infinite) set of initial functions, which are themselves PR:

α) the 0-ary constant 0;

β) the unary successor function $S(x) = x+1$ (written x');

γ) projection functions $U_n^i(\underline{x}) = x_i$ for all $\underline{x} = (x_1, \dots x_n)$, where $n \in \mathbb{N}^+$, $1 \leq i \leq n$.

The substitution scheme works as follows:

Papers 4, 11 Solution (ctd)

if $\{f_i(x_1, \dots x_n)\}_{i=1}^{r}$ are n-ary PR fns

and $g(y_1, \dots y_r)$ is an r-ary PR fn

then $h(x_1, \dots x_n) = g(f_1(x_1 \dots x_n), \dots, f_r(x_1 \dots x_n))$

is an n-ary PR function.

The recursion scheme works as follows:

if $g(x_1, \dots x_{n-1})$ is an $(n-1)$-ary PR fn

and $h(x_1, \dots x_{n+1})$ is an $(n+1)$-ary PR fn

the

$$\begin{cases} f(0, x_2 \dots x_n) = g(x_2 \dots x_{n-1}) \\ f(x_1', x_2 \dots x_n) = h(f(x_1, \dots x_n), x_1, \dots x_n) \end{cases}$$

is the PR fn defined by recursion on base function g and inductive step fn h.

Paper 4.11 Solution (ctd)

a) ii) $S \subseteq \mathbb{N}$ is recursively enumerable if and only if it is the range of an n-ary Total Recursive (Computable) function $f(x_1, x_2 \ldots x_n)$. Conventionally the empty set is recursively enumerable.

(I'm not quite sure why this is here — except that I think the q. is a bit lightweight)

b) suppose that $g_n(y) = f(n, y)$.

Fix $n \geq 0$, and consider $y = 0$.

$$g_{n+1}(0) = f(n+1, 0) = f(n, 1)$$
$$= g_n(1) = g_n^{(0+1)}(1).$$

Now suppose $g_{n+1}(y) = g_n^{(y+1)}(1)$.

Papers 4, 11 Solution ctd)

b) ctd) then $g_{n+1}(y+1) = f(n+1, y+1)$

$$= f(n, f(n+1, y)) = f(n, g_{n+1}(y))$$

$$= g_n\{g_n^{(y+1)}(1)\} = g_n^{(y+2)}(1).$$

Hence by induction on y,

$$g_{n+1}(y) = g_n^{(y+1)}(1) \quad \text{for all } y.$$

This result holds for all $n \geq 0$

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c) for a primitive recursive fn $g(y)$,

define

$$\begin{cases} h(0) = g(1) \\ h(y+1) = g(h(y)) \end{cases}$$

then $h(y) = g^{(y+1)}(1)$ is PR by the recursion scheme (mod details of arity)

Papers 4,11 Solution ctd)

c) ctd) Now $g_0(y) = f(0,y) = S(y)$ is PR.

Define
$$\begin{cases} g_{n+1}(0) = g_n(1) \\ g_{n+1}(y+1) = g_n\{g_{n+1}(y)\} \end{cases}$$
as above. Hence by induction on n, each fn $g_n(y)$ is PR.

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d) PR functions are TOTAL, hence each value $g_n(y) = f(n,y)$ is defined. Further, PR functions are computable. $\therefore$ $f(x,y)$ is computable and total, hence a PR fn.

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e) YES.