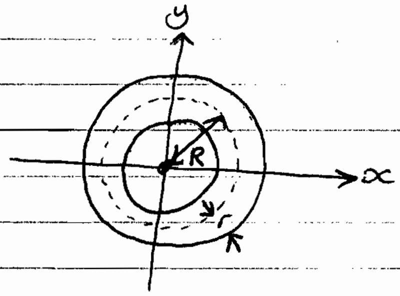
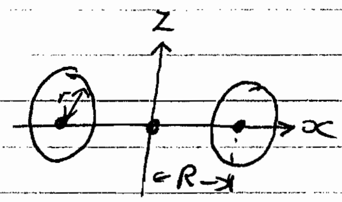
(a) a circle in the xy plane is: $x^2+y^2=r^2$

a torus aligned so that the z-axis passes through the hole is:

$$\left(\sqrt{x^2+y^2}-R\right)^2+z^2=r^2$$

where r is the radius of the toroidal cross-section and R is the distance from the origin to the centre line of the torus





(b) given the ray $\underset{\sim}{R}(t)=\underset{\sim}{O}+t\underset{\sim}{P}$, $0\le t$, where $\underset{\sim}{R}$, $\underset{\sim}{O}$, and $\underset{\sim}{P}$ are 3D vectors

we put $\underset{\sim}{R}(t)$ into the torus equation and solve for t

$$\left(\sqrt{(x_0+tx_p)^2+(y_0+ty_p)^2}-R\right)^2+(z_0+tz_p)^2-r^2=0$$

$$\Rightarrow (x_0+tx_p)^2+(y_0+ty_p)^2+(z_0+tz_p)^2+R^2-r^2=2R\sqrt{(x_0+tx_p)^2+(y_0+ty_p)^2}$$

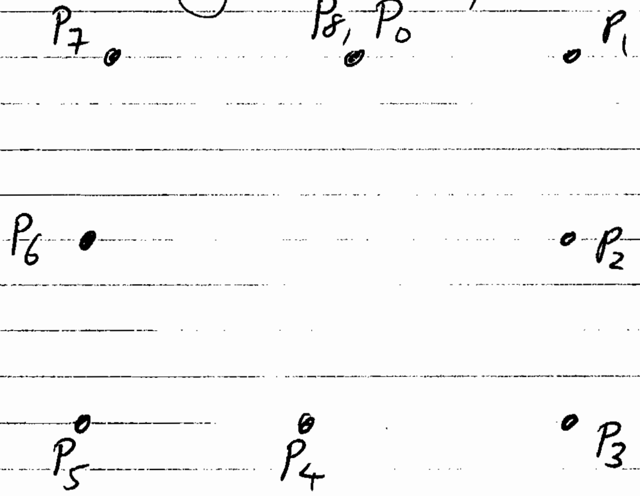
Squaring both sides and rearranging terms will lead to a quartic equation in t:

$$At^4+Bt^3+Ct^2+Dt+E=0$$

for example $A=x_p^4+y_p^4+z_p^4$

Putting this into the Quartic root finder (provided) will give the four roots of the equation. The smallest, real, non-negative value, if it exists, is the desired intersection point. If it does not exist then there is no intersection point.

(c) Construct the eight ~~knot~~ control points shown:



P_1, P_3, P_5, P_7 are the vertices of a square

P_0, P_2, P_4, P_6, P_8 are the midpoints of the sides

$P_0 = P_8$

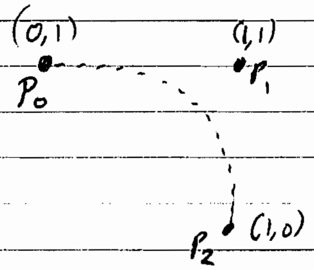
Use a quadratic B-spline basis.

Use the knot vector [0 0 0 1 1 2 2 3 3 4 4 4]

Use the homogeneous co-ordinate vector $[1 \alpha 1 \alpha 1 \alpha 1 \alpha 1]$

where $\alpha = \frac{1}{\sqrt{2}}$

Given that this was a worked example in lectures the entire construction can be simply remembered. Given only the layout of control points, the order of the B-spline, the knot vector, and the form of the homogeneous co-ordinate vector can all be worked out in reasonable time. The value of α requires some long algebra in its derivation. It can be found quickly if one is willing to take a clever short-cut, as follows.



$$P(t) = \frac{(1-t)^2 P_0 + 2\alpha t(1-t) P_1 + t^2 P_1}{(1-t)^2 + 2\alpha t(1-t) + t^2}$$

to fit a circle require $|P(t)| = 1, \; 0 \le t \le 1$

$$\Rightarrow \frac{\left((1-t)^2 + 2\alpha t(1-t)\right)^2 + \left(2\alpha t(1-t) + t^2\right)^2}{\left((1-t)^2 + 2\alpha t(1-t) + t^2\right)^2} = 1, \quad 0 \le t \le 1$$

Let this be: $\frac{N}{D} = 1$;

$N = a_N t^4 + b_N t^3 + c_N t^2 + d_N t + e_N$

$D = a_D t^4 + b_D t^3 + c_D t^2 + d_D t + e_D$

this is $(x(t))^2 + (y(t))^2 = 1$

$\therefore$ we require $a_N = a_D$, $b_N = b_D$, $c_N = c_D$, $d_N = d_D$, $e_N = e_D$

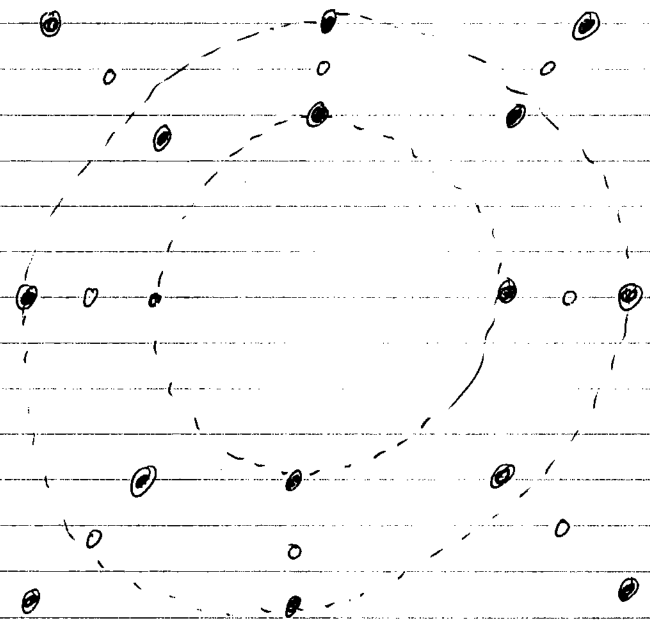
Taking just the first of this we can quickly work out the co-efficients of t^4 by inspection:

$$a_N = 1 + 4\alpha^2 - 4\alpha + 4\alpha^2 - 4\alpha + 1 = 2 - 8\alpha + 8\alpha^2$$
$$a_D = 1 + 4\alpha^2 + 1 + 2 - 4\alpha - 4\alpha = 4 - 8\alpha + 4\alpha^2$$

setting $a_N = a_D$ gives $\alpha = \frac{1}{\sqrt{2}}$ as required

The equations for b + c give the same result; in those for d and e, α cancels out giving the equation $0=0$.

(d) To make a torus you would use a bi-quadratic B-spline basis and the circle definition to position ~~f~~ control points in space:



at each control point ~~for~~ the big circle we place a copy of the control points for the smaller circle oriented as shown in this plan view.

~~[illegible]~~ this set of 64 points controls the biquadratic surface

Instead of (2D): $P(t) = \sum_{i=0}^{n} P_i N_{i,3}(t)$

we get (3D): $P(s,t) = \sum_{i=0}^{m} \sum_{j=0}^{n} P_{i,j} M_{i,3}(s) . N_{i,3}(t)$

where $P_{i,j}$ ~~is a~~ are the homogeneous co-ordinate for the ~~defining~~ control points of $P_{i,j}$.