Numerical Analysis I Question B

p4 q9

1999 Model Answer

p11 q10

$$e_n = I_n - f(n) = \frac{f''(\theta_n)}{24}$$

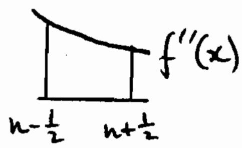
$$S_{1,\infty} = S_{1,N} + \sum_{n=N+1}^{\infty} (I_n - e_n)$$

$$= S_{1,N} + \underbrace{\int_{N+\frac{1}{2}}^{\infty} f(x)\,dx}_{\text{integral remainder}} \underbrace{- \frac{1}{24}\sum_{n=N+1}^{\infty} f''(\theta_n)}_{\text{error } E_N}$$

[6 marks]

If $f''(x)$ is positive decreasing then

$$|e_n| < \frac{f''(n-\frac{1}{2})}{24}$$



so

$$|E_N| < \frac{1}{24}\sum_{n=N+1}^{\infty} f''(n-\tfrac{1}{2})$$

$$\simeq \frac{1}{24}\int_N^{\infty} f''(x)\,dx = -\frac{f'(N)}{24}$$

[5 marks]

Now

$$f(x) = \frac{1}{x(x+2)} \simeq \frac{1}{x^2} \quad \text{for } x \text{ large}$$

so

$$f'(x) \simeq -\frac{2}{x^3} \quad \text{for } x \text{ large}$$

$$f''(x) \simeq \frac{6}{x^4} \quad \text{for } x \text{ large}$$

so $f''(x)$ is positive decreasing for x sufficiently large.

Integral remainder is

$$\left. -\tfrac{1}{2}\log_e\left(1 + \frac{2}{x}\right)\right]_{N+\frac{1}{2}}^{\infty}$$

$$= \tfrac{1}{2}\log_e\left(1 + \frac{2}{N+\frac{1}{2}}\right)$$

$$\simeq \frac{1}{2}\cdot\frac{2}{N+\frac{1}{2}}, \quad \text{for } N \text{ large}$$

$$\simeq \frac{1}{N}.$$

[6 marks]

Now

$$f'(x) = -\frac{2(x+1)}{x^2(x+2)^2}$$

so

$$|E_N| \simeq \frac{2(N+1)}{24N^2(N+2)^2} = 1.8 \times 10^{-11}$$

$$\therefore\ N^3 \simeq \frac{10^{12}}{12 \cdot 18}$$

$$N \simeq \frac{10000}{6} \simeq 1700.$$

[3 marks]