Paper 10

Solution

a) i) $\prec$ is a well-founded relation on set if every $\prec$-descending chain $\{x_n\}$ (i.e. s.t $(x_{n+1} \prec x_n)$ for all $n \in \mathbb{N}$) is ultimately constant.

ii) $y \in S$ is a minimal element for $\prec$ if $x \in S$, $x \prec y \Rightarrow x = y$.

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b) suppose $\exists$ a non-empty subset $X \subseteq S$ having no minimal element. $\exists$ $x_0 \in X$.

by induction, since $x_n \in X$ is <u>NOT</u> minimal, can find $x_{n+1} \neq x_n$ such that $(x_{n+1} \prec x_n)$. $\{x_n\}$ is $\prec$-descending,

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Solution ctd) but NOT wlt. constant.

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hence every non-empty s/s has a minimal element

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c) Let $\mathbb{P}$ be the set of all finite partially ordered sets $(P, \leq)$, and let the size of a p.o. set be $|P|$, the number of elements in set P.

Either the result holds for all $(P, \leq)$,

or $\exists$ a counterexample having a minimal number of elements.

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Solution ctd)

Choose such a counterexample $(Q, \leq)$.

Let $M \subseteq Q$ be the set of all minimal elements of Q, certainly non-empty.

$m, m' \in M$, $m \leq m' \Rightarrow m = m'$, by definition of minimality. Hence certainly M is an antichain of Q.

Consider now the p.o. set $((Q \setminus M), \leq)$. It has smaller cardinality than the minimal counterexample Q, hence it satisfies the theorem. Take a cover of $(Q \setminus M)$, $\bigcup_{i=1}^{k} Y_i$, containing a minimum # of antichains.

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Solution ctd) Any non-minimal element of Q must be in a separate member of the cover from some element of M, hence the cover $\bigcup_{i=1}^{k} Y_i$ say can be extended by the single antichain M to form a minimal cover of Q.

Consider a chain in Q of maximal length, say C. C contains a unique minimal element $m \in M$; $(C \setminus \{m\})$ is a chain of maximal length in $(Q \setminus M)$ hence by hypothesis has length k.

$\therefore$ C has length $(k+1)$, the ~~length~~ size of a minimal cover of Q //