p8q13 RJG 1

2003 Paper 8
Computer Systems Modelling

[Relates to generation of random variables and confidence intervals for simulation estimates]

(1.) Multiplicative congruential method

$$x_n = a\, x_{n-1} \mod m$$

x_0 seed, a, m parameters

Mixed congruential method

$$x_n = (a x_{n-1} + c) \mod m$$

x_0 seed, a, c, m parameters

(2.) Let F_X be the distribution function of $X = F^{-1}(U)$ then

$$F_X(x) = P(X \le x)$$

$$= P(F^{-1}(U) \le x)$$

$$= P(F(F^{-1}(U)) \le F(x))$$ F increasing

$$= P(U \le F(x))$$ $F(F^{-1}(U)) = U$

$$= F(x)$$ since U is uniform on (0,1).

So, X has distribution function $F(x)$.

3) $X_1, X_2, \ldots, X_n$ mean μ
variance σ^2

Sample mean, $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$

Sample variance, $S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$

4.) $\mathbb{E}(\bar{X}) = \mathbb{E}\left(\frac{\sum_{i=1}^{n} X_i}{n} \right) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}(X_i) = \frac{n\mu}{n} = \mu$

$Var(\bar{X}) = Var\left(\frac{1}{n} \sum_{i=1}^{n} X_i \right) = \frac{1}{n^2} \sum_{i=1}^{n} Var(X_i) = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$

5.) Set

$$Z = \frac{\bar{X} - \mu}{\sqrt{\sigma^2/n}} = \frac{\sqrt{n}(\bar{X} - \mu)}{\sigma}$$

Then by central limit theorem

$$Z \sim N(0,1) \quad \text{for large } n$$

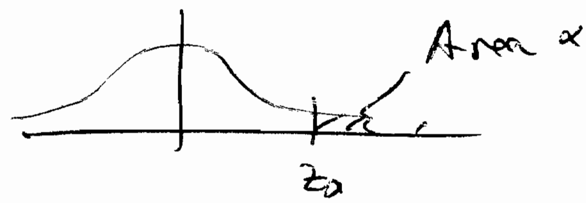
The variance, σ^2, is unknown so we replace it by S^2. Hence, we approximately have

$$\frac{\sqrt{n}(\bar{X} - \mu)}{S} \sim N(0,1)$$

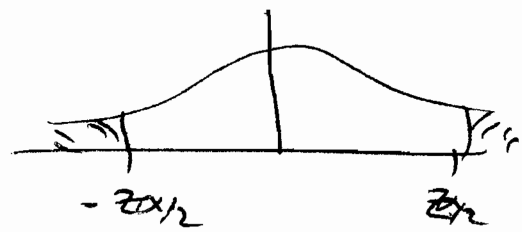
Now if $Z \sim N(0,1)$ and $0 < \alpha < 1$

let z_α be given such that

$$P(Z > z_\alpha) = \alpha$$



Then, $P(-z_{\alpha/2} < Z < z_{\alpha/2}) = 1 - \alpha$



i.e. $$P\left(-z_{\alpha/2} < \frac{\sqrt{n}(\bar{X}-\mu)}{S} < z_{\alpha/2}\right) = 1-\alpha$$

i.e $$P\left(\bar{X} - z_{\alpha/2}\frac{S}{\sqrt{n}} < \mu < \bar{X} + z_{\alpha/2}\frac{S}{\sqrt{n}}\right) = 1$$

gives an approximate $100(1-\alpha)$ percent confidence interval for μ.

(b) Algorithm

1. Generate an initial set of data points to remove transients say 100 data points

2. Continue generating data values in the simulation updating $\bar{X}$, S and n until

$$2 z_{\alpha/2} \frac{S}{\sqrt{n}} < l.$$