[A question with broad coverage from the notes on Den. Sem.]

Denotational semantics (2) 2002 p9q1 GW

(6) (a) $R \subseteq \mathrm{Fin}(A) \times A$.

$$\hat{R}(X) = \{a \in A \mid \exists X_0 \subseteq X.\ (X_0, a) \in R\}.$$

Clearly $\hat{R} : \mathcal{P}(A) \to \mathcal{P}(A)$.

$\hat{R}$ is monotonic: Clearly,
$X \subseteq Y \Rightarrow \hat{R}(X) \subseteq \hat{R}(Y)$.

$\hat{R}$ is continuous: Let

$$X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n \subseteq \cdots$$

be a chain of sets in $\mathcal{P}(A)$. By monotonicity,

$$\hat{R}(X_m) \subseteq \hat{R}(\bigcup_n X_n), \text{ all } m.$$

$$\therefore \quad \bigcup_n \hat{R}(X_n) \subseteq \hat{R}(\bigcup_n X_n).$$

To show the converse inclusion, let $a \in \hat{R}(\bigcup_n X_n)$.

Then $(X_0, a) \in R$ and $X_0 \subseteq \bigcup_n X_n$ for some $X_0 \in \mathrm{Fin}(A)$. As X_0 is finite, there exists m s.t. $X_0 \subseteq X_m$. Hence $a \in \hat{R}(X_m)$.

~~(4)~~ (b) The function cpo $(D \to E, \sqsubseteq)$ consists ~~of~~ of all continuous functions $D \to E$ ordered pointwise i.e. $f \sqsubseteq g$ iff $\forall x \in D.\ f(x) \sqsubseteq g(x)$.

To be a cpo we need lubs of chains

$$f_0 \sqsubseteq f_1 \sqsubseteq \cdots \sqsubseteq f_n \sqsubseteq \cdots$$

Claim: The chain has lub f where

$$f(x) = \bigsqcup_n f_n(x).$$

We need f to be a continuous function $D \to E$.

Let $x \sqsubseteq y$. Then $f(x) = \bigsqcup_n f_n(x) \sqsubseteq \bigsqcup_n f_n(y) \sqsubseteq f(y)$.
So f is monotone.

Let $x_0 \sqsubseteq \cdots \sqsubseteq x_m \sqsubseteq \cdots$ ~~be~~ in D. Then

$$f(\bigsqcup_m x_m) = \bigsqcup_n f_n(\bigsqcup_m x_m)$$

$$= \bigsqcup_n \left(\bigsqcup_m f_n(x_m)\right) \quad \text{as ea. } f_n \text{ is cts.}$$

$$= \bigsqcup_m \left(\bigsqcup_n f_n(x_m)\right) \quad \text{by lemma below}^{*}$$

$$= \bigsqcup_m f(x_m)$$

Lemma: Let $e_{n,m} \in E$, a cpo. st.

$$n \leq n', m \leq m' \Rightarrow e_{n,m} \sqsubseteq e_{n',m'}.$$

Then $$\bigsqcup_n \left(\bigsqcup_m e_{n,m} \right) = \bigsqcup_n e_{n,n} = \bigsqcup_m \left(\bigsqcup_n e_{n,m} \right).$$

~~(2) (c) Two ~~terms~~ closed terms t_1, t_2 of PCF are contextually equivalent iff for all ground contexts $\mathcal{C}[-]$ (of type bool, ~~int~~ nat) $\mathcal{C}[t_1] \Downarrow v$ iff $\mathcal{C}[t_2] \Downarrow v$ for all values v.~~

Two terms T_1, T_2 where:

~~(4)~~ ~~(d)~~ $T_i \equiv$ fn f: bool $\to$ (bool $\to$ bool).

if (f true Ω) then

if (f Ω true) then

if (f false false) then Ω else B_i

else Ω

else Ω with $B_1 =$ true, $B_2 =$ false.

where $\Omega \equiv$ fix(λx:bool. x)

~~The~~ T_1, T_2 differ on parallel-or:

	$\perp$	true	false
$\perp$	$\perp$	true	$\perp$
true	true	true	true
false	$\perp$	true	false.