f 14.8
PR

Discrete Mathematics 2003

Question B

Define the terms injective, surjective and bijective, and state the Schröder-Bernstein theorem concerning the existence of a bijection between two sets. [4 marks]

What is a countable set? [2 marks]

Prove the following assertions:

1. If C is a countable set and $f: A \to C$ is an injection, then A is countable. [2 marks]
2. If A and B are countable sets, then $A \times B$ is countable. [2 marks]
3. $\mathbb{Z}$ and $\mathbb{Q}$, the sets of integer and rational numbers, are both countable. [4 marks]
4. $\mathcal{P}(\mathbb{N})$, the set of all subsets of the natural numbers, is not countable. [2 marks]
5. If U is an uncountable set and $f: U \to V$ is an injection, then V is not countable. [2 marks]
6. $\mathbb{R}$, the set of real numbers, is not countable. [2 marks]

Solution B

$f: A \to B$ is injective iff $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$. $f: A \to B$ is surjective iff $\forall\, b \in B\ \exists\, a \in A$ s.t. $f(a) = b$. A function is bijective if it is both injective and surjective.

If there are injections $A \to B$ and $B \to A$, then there is a bijection $A \to B$.

A set is countable if it is finite or in 1-1 correspondence with the natural numbers.

1. Either A is finite or there is an injection $\mathbb{N} \to A$. Use S-B.
2. Given bijections $p: A \to \mathbb{N}$ and $q: B \to \mathbb{N}$, consider the injection $r(a,b) = 2^{p(a)}3^{q(b)}$. Use (1).
3. Consider the injection $f(z) = 2z + 1$ for $z \geq 0$, $-2z$ otherwise.
   Consider the injection $g(a/b) = 2^{f(a)}3^{b}$. Use (1).
4. Use contradiction. Suppose $f: \mathbb{N} \to \mathcal{P}(\mathbb{N})$ is a bijection. Let $S = \{n \in \mathbb{N} \mid n \notin f(n)\}$ and $s = f^{-1}(S)$. Consider $s \in S$ to get a contradiction.
5. Use contradiction. Suppose V is countable. Then U is countable by (1).
6. Consider the injection $f: \mathcal{P}(\mathbb{N}) \to \mathbb{R}$ using decimal expansion.