Numerical Analysis II   Question B   1999   f9 q14 ~~p13 q13~~ MFOD

Model Answer

Given initial conditions defined at x_0, the numerical solution is evaluated at $x_1, x_2, \ldots$. For constant step length h, $x_n = x_0 + nh$. By convention y_n is an approximation to the true solution $y(x_n)$. f_n is shorthand for $f(x_n, y_n)$.

[3 marks]

Local error is the error over a single step.
Global error is the error over n steps.
If the local error of the method is $O(h^{k+1})$ then the method is said to have order k. Typically, the global error of such a method will be $O(h^k)$.

[3 marks]

If $y' = f(x, y(x))$ then multistep formulae are derived by applying quadrature rules to

$$\int_{x_{n-p}}^{x_{n+1}} y'\,dx = \int_{x_{n-p}}^{x_{n+1}} f(x, y(x))\,dx$$

i.e.

$$y_{n+1} = y_{n-p} + \int_{x_{n-p}}^{x_{n+1}} f(x, y(x))\,dx$$

where $p \geq 0$, using abscissae possibly outside the interval of integration.
If the point x_{n+1} is an abscissa then a term f_{n+1} arises, based on the value y_{n+1} which has not yet been calculated. This initially unknown value is usually written $\tilde{f}_{n+1}$. A suitable starting procedure would be a one step method of the same order, e.g. RK4.
The Milne formulae are called the predictor and the corrector, respectively. The latter is more accurate but is implicit because it contains the initially unknown term $\tilde{f}_{n+1}$.

For each step the predictor is used typically once, and the corrector is then used iteratively to improve y_{n+1} and hence $\tilde{f}_{n+1}$. The starting procedure is required for the first few steps and thereafter only if the step size needs to be changed.

[8 marks]

The first application of the predictor is for $n=3$, so

$$y_4 = y_0 + \frac{4h}{3}(2f_3 - f_2 + 2f_1)$$

$$= 0.4(2\times 3.4 - 2.1 + 2\times 1.3)$$

$$= 2.92$$

$$\tilde{f}_4 = f(1.2, 2.92)$$

$$= 5.3$$

[3 marks]

A comparable one step method is RK4. One step methods are unconditionally stable for sufficiently small h, whereas multistep methods may be unstable. RK4 requires four function evaluations per step, but Milne's method requires only two. It is easier to estimate a suitable step size for Milne's method, whereas a step size control technique has to be used with RK4. On the other hand, changing step size is harder with multistep methods because a starting procedure is required.

[3 marks]