2003 Paper 7 P7q5 ①

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Computer Systems Modelling

[Relate to Poisso Process]

①. The necessary conditions are

(i) $N(0) = 0$

(ii) The numbers of events in disjoint time intervals are independent

(iii) The distribution of the number of events in a given interval depends only on the interval's length (& not its location).

(iv) $P(N(h) = 0) = 1 - \lambda h + o(h)$ as $h \to 0$

(v) $P(N(h) = 1) = \lambda h + o(h)$ as $h \to 0$

(vi) $P(N(h) > 1) = o(h)$ as $h \to 0$

② Divide $[0, t]$ into n equal length sub-intervals – each of length t/n

Now for large n

P(a sub-interval contains 1 event) $\simeq \lambda t/n$ & P(a sub-interval contains no events) $\simeq 1 - \lambda t/n$

& numbers of events in sub-intervals are independent

$\therefore$ Total number of events occurring is given by a Binomial random variable with parameters n and $p = \lambda t/n$

So, for large n, the number of events in $[0,t)$, $N(t)$, follows a Poisson distribution with mean $np = n\lambda t/n = \lambda t$. So $P(N(t) = j) = \frac{e^{-\lambda t}(\lambda t)^j}{j!}$ $j = 0,1,2,\ldots$

(3) We have that

$$P(X_1 > t) = P(N(t) = 0) = e^{-\lambda t}$$

So $X_1 \sim Exp(\lambda)$

and

$$P(X_2 > t \mid X_1 = s) = P(0 \text{ events in } (s, t+s) \mid X_1 = s)$$
$$= P(0 \text{ events in } (s, t+s))$$
$$= e^{-\lambda t}$$

So, X_1 and X_2 are independent with $X_2 \sim Exp(\lambda)$.

(4) Set $S_n = \sum_{i=1}^{n} X_i$

Then,

$$P(S_n \le t) = P(N(t) \ge n)$$

$$= \sum_{j=n}^{\infty} e^{-\lambda t} \frac{(\lambda t)^j}{j!}$$

Differentiate w.r.t. t to find density, $f_n(t)$, of $S_n(t)$ as

$$f_n(t) = \sum_{j=n}^{\infty} (j\lambda) e^{-\lambda t} \frac{(\lambda t)^{j-1}}{j!} - \sum_{j=n}^{\infty} \lambda e^{-\lambda t} \frac{(\lambda t)^j}{j!}$$

$$= \sum_{j=n}^{\infty} \lambda e^{-\lambda t} \frac{(\lambda t)^{j-1}}{(j-1)!} - \sum_{j=n}^{\infty} \lambda e^{-\lambda t} \frac{(\lambda t)^j}{j!}$$

$$= \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}$$

(5.) Algorithm

1. Set $t = 0$, $I = 0$
2. Generate (pseudo) random number, U
3. $t = t - \frac{1}{\lambda} \log U$, if $t > T$, stop
4. $I = I + 1$, $S(I) = t$
5. Go to step (2)

I is number of event

$S(I)$ is time of I^{th} event.