Logic & Proof 2

(a) The formula A is converted to clauses, omitting the initial step of negating the formula. Resolution is attempted, but the empty clause cannot be derived. What can we conclude about A? [3 marks]

(b) The formula A is converted to clauses by the usual procedure, except that Skolemization is performed before negating the formula instead of afterward. Resolution is attempted and the empty clause is derived. What can we conclude about A? [3 marks]

(c) The formula A is converted to clauses by the usual procedure. The Davis-Putnam method is applied. In some of the case splits the empty clause is derived, but in others it is not. What can we conclude about A? [3 marks]

(d) For each of the following sequents, present either a formal proof or a falsifying interpretation. The modal logic is S4.

$$((\exists x\, P(x)) \to Q) \to \forall x\, (P(x) \to Q)$$
$$\Diamond\Box A \to \Diamond\Box\Diamond\Box A$$

[5+6 marks]

Model Answer

Parts (a)–(c) concern clause methods, which are presented in lectures 7–10.

(a) A is satisfiable: all the steps in the transformation to clause form preserve consistency. No empty clause means the clauses are consistent.

(b) We can't conclude anything, in general. Skolemizing before negating reverses the roles of $\forall$ and $\exists$, so consistency is not preserved. Take any theorem of first-order logic and replace each $\forall$ by $\exists$ and vice versa; the result in most cases will not be a theorem, but it will be "proved" by this procedure. *

(c) A is not valid, and the cases where the empty clause cannot be derived describe counterexamples. But neither is A inconsistent: those cases giving rise to the empty clause give satisfying interpretations.

For (d), sequent calculus proofs refer to lecture 3 (general), lecture 6 (first-order logic) and lecture 11 (modal logic). Both sequents have formal proofs:

* We do however learn that A is consistent, since if the skolemized formula is valid then it is consistent too. Note that if the universe has just one element, $\forall$ and $\exists$ coincide.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{P(x) \Rightarrow Q,\ P(x)}}{P(x) \Rightarrow Q,\ \exists x\, P(x)}\,(\exists r) \qquad \overline{Q,\ P(x) \Rightarrow Q}}{\exists x\, P(x) \to Q,\ P(x) \Rightarrow Q}\,(\to l)}{\exists x\, P(x) \to Q \Rightarrow P(x) \to Q}\,(\to r)}{\exists x\, P(x) \to Q \Rightarrow \forall x\,(P(x) \to Q)}\,(\forall r)}{\Rightarrow (\exists x\, P(x) \to Q) \to \forall x\,(P(x) \to Q)}\,(\to r)
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{\Box A \Rightarrow \Box A}}{\Box A \Rightarrow \Diamond\Box A}\,(\Diamond r)}{\Box A \Rightarrow \Box\Diamond\Box A}\,(\Box r)}{\Box A \Rightarrow \Diamond\Box\Diamond\Box A}\,(\Diamond r)}{\Diamond\Box A \Rightarrow \Diamond\Box\Diamond\Box A}\,(\Diamond l)}{\Rightarrow \Diamond\Box A \to \Diamond\Box\Diamond\Box A}\,(\to r)
$$

In the modal logic proof, the applications of $(\Diamond l)$ and $(\Box r)$ respect the condition that all left-side formulas must start with a $\Box$ and all right-side formulas with a $\Diamond$. Probably there are no valid proofs other than the one given above. However, there are a number of alternative proofs for the first-order sequent.