p9q10
PNB

Solution Notes. Part II Types 2004 (PNB)

1. Mutually inductive definitions. (2 marks for broad structure.)

   - (2 marks.) Since $\alpha[\alpha_i/\beta] = \alpha$

$$\begin{aligned} P_\alpha &\stackrel{\text{def}}{=} \Lambda\alpha_1,\alpha_2.\lambda f:\alpha_1\to\alpha_2.\lambda x:\alpha.x \\ N_\alpha &\stackrel{\text{def}}{=} \Lambda\alpha_1,\alpha_2.\lambda f:\alpha_1\to\alpha_2.\lambda x:\alpha.x \end{aligned}$$

   - (2 marks.) Since $\beta[\alpha_i/\beta] = \alpha_i$

$$P_\beta \stackrel{\text{def}}{=} \Lambda\alpha_1,\alpha_2.\lambda f:\alpha_1\to\alpha_2.f$$

   Alternatively, one could eta-expand, thus:

$$P_\beta \stackrel{\text{def}}{=} \Lambda\alpha_1,\alpha_2.\lambda f:\alpha_1\to\alpha_2.\lambda x:\alpha_1.fx$$

   - (2 marks.) Since $(\forall\alpha.\tau)[\alpha_i/\beta] = \forall\alpha.(\tau[\alpha_i/\beta])$ (by α-converting if necessary):

$$\begin{aligned} P_{\forall\alpha.\tau} &\stackrel{\text{def}}{=} \Lambda\alpha_1,\alpha_2.\lambda f:\alpha_1\to\alpha_2.\lambda x:\forall\alpha.\tau[\alpha_1/\beta].\Lambda\alpha.P_\tau\alpha_1\alpha_2 f(x\alpha) \\ N_{\forall\alpha.\nu} &\stackrel{\text{def}}{=} \Lambda\alpha_1,\alpha_2.\lambda f:\alpha_1\to\alpha_2.\lambda x:\forall\alpha.\nu[\alpha_2/\beta].\Lambda\alpha.N_\nu\alpha_1\alpha_2 f(x\alpha) \end{aligned}$$

   - (4 marks.) Since $(\nu\to\tau)[\alpha_i/\beta] = \nu[\alpha_i/\beta]\to\tau[\alpha_i/\beta]$ and similarly the other way around:

$$\begin{aligned} P_{\nu\to\tau} &\stackrel{\text{def}}{=} \Lambda\alpha_1,\alpha_2.\lambda f:\alpha_1\to\alpha_2.\lambda g:\nu[\alpha_1/\beta]\to\tau[\alpha_1/\beta]. \\ &\qquad \lambda x:\nu[\alpha_2/\beta].P_\tau\ \alpha_1\ \alpha_2\ f\ (g\ (N_\nu\ \alpha_1\ \alpha_2\ f\ x)) \\ N_{\tau\to\nu} &\stackrel{\text{def}}{=} \Lambda\alpha_1,\alpha_2.\lambda f:\alpha_1\to\alpha_2.\lambda g:\nu[\alpha_2/\beta]\to\tau[\alpha_2/\beta]. \\ &\qquad \lambda x:\nu[\alpha_1/\beta].N_\nu\ \alpha_1\ \alpha_2\ f\ (g\ (P_\tau\ \alpha_1\ \alpha_2\ f\ x)) \end{aligned}$$

2. Normal form. (8 marks total, roughly one per line of the following.)

   $P_{\forall\alpha.(\beta\to\alpha)\to\alpha} = \Lambda\alpha_1,\alpha_2.\lambda f:\alpha_1\to\alpha_2.\lambda p:\forall\alpha.(\alpha_1\to\alpha)\to\alpha.\Lambda\alpha.P_{(\beta\to\alpha)\to\alpha}\ \alpha_1\ \alpha_2\ f\ (p\,\alpha)$

   Now

	$P_{(\beta\to\alpha)\to\alpha}\ \alpha_1\ \alpha_2\ f\ (p\ \alpha)$
$\stackrel{\text{def}}{=}$	$\lambda x:\alpha_2\to\alpha.P_\alpha\ \alpha_1\ \alpha_2\ f\ (p\ \alpha\ (N_{\beta\to\alpha}\alpha_1\ \alpha_2\ f\ x))$
$\to$	$\lambda x:\alpha_2\to\alpha.p\ \alpha\ (N_{\beta\to\alpha}\alpha_1\ \alpha_2\ f\ x)$
(by definition of P_α)	
$\stackrel{\text{def}}{=}$	$\lambda x:\alpha_2\to\alpha.p\ \alpha\ (\lambda y:\alpha_1.N_\alpha\alpha_1\ \alpha_2\ f\ (x\ (P_\beta\alpha_1\ \alpha_2\ fy)))$
$\to$	$\lambda x:\alpha_2\to\alpha.p\ \alpha\ (\lambda y:\alpha_1.x\ (P_\beta\alpha_1\ \alpha_2\ fy))$
(by definition of N_α)	
$\to$	$\lambda x:\alpha_2\to\alpha.p\ \alpha\ (\lambda y:\alpha_1.x\ (fy))$
(by definition of P_β)	

   Hence $P_{\forall\alpha.(\beta\to\alpha)\to\alpha}$ beta-reduces to

   $\Lambda\alpha_1,\alpha_2.\lambda f:\alpha_1\to\alpha_2.\lambda p:\forall\alpha.(\alpha_1\to\alpha)\to\alpha.\Lambda\alpha.\lambda x:\alpha_2\to\alpha.p\,\alpha\,(\lambda y:\alpha_1.x\,(fy))$

   which is a normal form, as required.