SOLUTION NOTES

Complexity Theory 2002 Paper 6 Question 12 (AD)

(a) **Time Hierarchy Theorem**
For any constructible function f with $f(n) \geq n$, $\textsc{Time}(f(n))$ is properly contained in $\textsc{Time}(f(2n+1)^3)$.

Proof
Define the language:

$$H_f = \{[M], x \mid M \text{ accepts } x \text{ in } f(|x|) \text{ steps}\}$$

First, we observe that:

$$H_f \in \textsc{Time}(f(n)^3).$$

A machine for recognising H_f would first compute $f(|x|)$, and on a separate work tape, write out 0, $f(|x|)$ times, to use as a clock for the rest of the computation. It would then simulate machine M on input x for $f(|x|)$ many steps, at each step looking through the description of M given for the appropriate transition. This can be done within the required time bounds.

Secondly,

$$H_f \notin \textsc{Time}(f(\lfloor n/2 \rfloor)).$$

For, suppose $H_f \in \textsc{Time}(f(\lfloor n/2 \rfloor))$. Then, we can construct a machine N which accepts $[M]$ if, and only if, $[M], [M] \notin H_f$. The machine simply copies $[M]$, inserting a comma between the two copies, and then runs the machine that accepts H_f. Moreover, the running time of N on an input of length n is $f(\lfloor (2n+1)/2 \rfloor) = f(n)$. We can now as whether N accepts the input $[N]$, and we see that we get a contradiction either way.

From these two observations, the Time Hierarchy Theorem immediately follows.

(b) (i) This is not a consequence of the Time Hierarchy Theorem as stated above, because it is not the case that $(n \log n)^3 = O(n^2)$. However, it could be derived from a tighter version of the theorem, which is known to be true.

(ii) This does follow from the theorem as given above. The theorem implies, for instance that there is a language in $\textsc{Time}(2^n)$ which is not in $\textsc{Time}(2^{n/3})$. However, every language that is decidable in polynomial time is decidable in time $O(2^{n/3})$.

(iii) This does not follow from the theorem, as the theorem is stated for deterministic machines. While a similar theorem could be formulated for nondeterministic classes, it would only separate nondeterministic polynomial time from nondeterministic exponential time, not NP from deterministic exponential time.