pb9/10
LCP

1999

Logic & Proof 2

Describe the role of Herbrand models in mechanical theorem proving. What may we infer when a set of clauses has no Herbrand model? [3 marks]

This question concerns using clause methods to determine whether or not the formula

$$\exists x\,[P(x) \wedge Q(x)] \rightarrow \exists x\,[P(f(x,x)) \vee \forall y\, Q(y)]$$

is a theorem.

Convert the problem into clause form. Justify each step you take and explain how the set of clauses is equivalent to the original problem. [4 marks]

Describe the Herbrand universe for your clauses. [3 marks]

Produce a resolution proof from your clauses, or give reasons why none exists. [4 marks]

Exhibit a Herbrand model for your clauses, or give reasons why none exists. [6 marks]

Model Answer

A set of clauses is consistent if and only if it has a Herbrand model. Thus, if it has no Herbrand models, then it is inconsistent. Clause-based theorem provers are based on the Skolem-Gödel-Herbrand theorem, which in effect says that if a set of clauses is inconsistent, then there is a finite proof of this fact. (More precisely, there is a finite inconsistent set of ground instances of those clauses.) Terms formed during a resolution proof (or during the execution of a Prolog program) are terms of the clauses' Herbrand universe.

To convert the problem to clause form, first we negate the formula, getting

$$\exists x\,[P(x) \wedge Q(x)] \wedge \neg(\exists x\,[P(f(x,x)) \vee \forall y\, Q(y)])$$

and then

$$\exists x\,[P(x) \wedge Q(x)] \wedge \forall x\,[\neg P(f(x,x)) \wedge \exists y\, \neg Q(x)]$$

Skolemizing the first conjunct yields the clauses $\{P(a)\}$ and $\{Q(a)\}$. Skolemizing and dropping the universal quantifier in the second conjunct yields two clauses $\{\neg P(f(x,x))\}$ and $\{\neg Q(g(x))\}$. The clauses are equivalent to the original problem in the sense that if they clauses are inconsistent then the original formula is a theorem.

The Herbrand universe consists of all ground terms that can be built from the constant a and the functions f and g. Thus it is the set of terms

$$\{a, f(a,a), g(a), f(f(a,a),a), f(g(a),a), f(a,f(a,a)), f(a,g(a)), g(f(a,a)), \ldots\}$$

There is obviously no resolution proof because there are no valid resolution steps at all. The complementary literals $P(a)$ and $\neg P(f(x,x))$ are not unifiable because a is a constant symbol and f is a function symbol. The complementary literals $Q(a)$ and $\neg Q(g(x))$ are similarly not unifiable. Moreover, the original formula can easily be falisfied. Here is a countermodel over the integers: ~~let $a = 0$ and~~ let $P(x)$ and $Q(x)$ be both equivalent to $x = 0$. Let the function f satisfy $f(x, y) = 1$ for all integers x, y.

Since there is no contradiction, the clauses are consistent and must have a Herbrand model. Its universe is the Herbrand universe. It interprets the constant a by itself. It interprets f as the function that maps the ground terms x and y to the ground term $f(x, y)$; for example, the result of applying f to the arguments a and $g(a)$ is just $f(a, g(a))$. This interpretation is standard for all Herbrand models. Finally we must specify the Herbrand base, interpreting predicates P and Q. Here we set $P(a)$ and $Q(a)$ to true. We set $\neg P(f(x, x))$ and $\neg Q(g(x))$ to false for every ground term x. So the model assigns $P(f(a, a))$, $Q(g(a))$, $P(f(g(a), g(a)))$, $Q(g(g(a)))$, etc. to false. This interpretation is sensible because a is distinct from all terms of the form $f(x)$: we are dealing with ground terms. Predicate instances not set true or false above can be assigned arbitrarily.