Logic and Proof 2005 – Paper 6 Question 9 (LCP)

(a) The propositional formula ϕ contains four propositional letters: P, Q, R and S. This formula evaluates to true in every valuation but one, namely when Q and R are false while S is true.

(i) What is the BDD for ϕ? [2 marks]

(ii) What is the BDD for $\neg P \to \phi$? [3 marks]

(iii) What is the BDD for the formula $P \wedge S \to R$? [2 marks]

(iv) What is the BDD for the formula $(P \wedge S \to R) \wedge \phi$? [4 marks]

(Use alphabetic ordering for all BDDs.)

(b) Use the DPLL procedure to determine whether or not the following set of clauses is satisfiable.

$\{P, Q, R\}$ $\{\neg P, Q, R\}$ $\{P, \neg Q, \neg R\}$ $\{\neg P, \neg Q, \neg R\}$ $\{\neg Q, R\}$ $\{\neg P, Q, \neg R\}$

[5 marks]

(c) Prove the formula $\forall x[\neg P(x) \to Q(x)] \wedge \exists x\, \neg Q(x) \to \exists x\, P(x)$ using the tableau calculus (with Skolemization). [4 marks]

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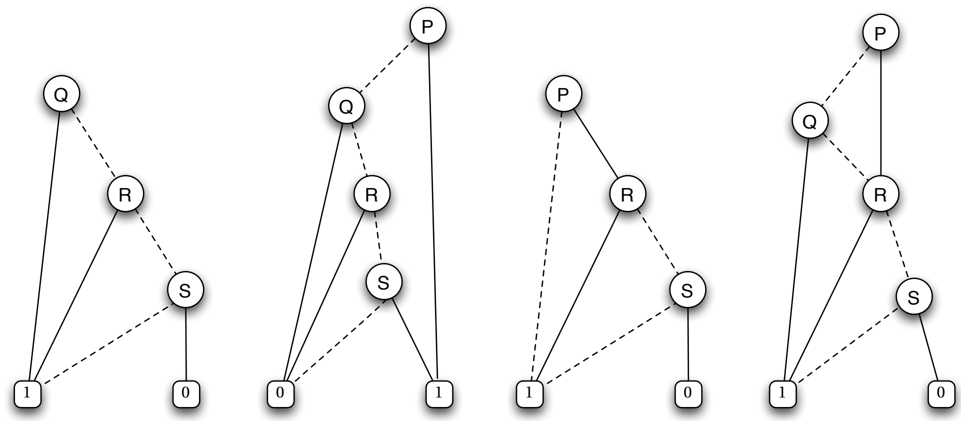
(a) Here are the BDDs. In most cases they can be written down directly from a knowledge of the truth tables. The last one must exhibit sharing, as shown.



(b) Here is an execution. Students should be able to see that a case split on the shortest clause will involve them in the least work. So, we begin with a case split on Q:

$Q =$ **true** leaves us with $\{P, \neg R\}$ $\{\neg P, \neg R\}$ $\{R\}$. In this case $R =$ **true** by the unit rule, leaving $\{P\}$ $\{\neg P\}$. Now since $P =$ **true** we derive the empty clause.

$Q =$ **false** leaves us with $\{P, R\}$ $\{\neg P, R\}$ $\{\neg P, \neg R\}$. A further case split is required, say on P.

$P =$ **true** leaves us with $\{R\}$ $\{\neg R\}$. Now since $R =$ **true** by the unit rule, we derive the empty clause.

$P =$ **false** leaves us with simply $\{R\}$. Applying the pure literal rule (or the unit rule) leaves the empty set of clauses. Therefore, the original set is satisfiable when $P = Q =$ **false** and $R =$ **true**.

(c) The formula must first be negated, Skolemized, and moved to the left side of the arrow. The proof is essentially as follows (apologies for primitive typesetting):

$$\begin{array}{c}
P(a),\ \neg Q(a),\ \neg P(a) \ \Rightarrow \qquad Q(a),\ \neg Q(a),\ \neg P(a) \Rightarrow \\ \hline
P(a) \lor Q(a),\ \neg Q(a),\ \neg P(a) \ \Rightarrow \\ \hline
\forall x[P(x) \lor Q(x)],\ \neg Q(a),\ \neg P(a) \ \Rightarrow \\ \hline
\forall x[P(x) \lor Q(x)],\ \neg Q(a),\ \forall x\, \neg P(x) \ \Rightarrow \\ \hline
\forall x[P(x) \lor Q(x)] \land \neg Q(a) \land \forall x\, \neg P(x) \ \Rightarrow
\end{array}$$