p4q6 ①
RJG

2004 Paper 4
Continuous Mathematics

This question relates to Fourier Series

Writing
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos(nx) + b_n\sin(nx)\right)$$

then $\int_0^{2\pi} f(x)\,dx = \int_0^{2\pi}\frac{a_0}{2}\,dx$ since $0 = \int_0^{2\pi}\cos(nx)\,dx = \int_0^{2\pi}\sin(nx)\,dx$

$$= \left[\frac{a_0 x}{2}\right]_0^{2\pi} = a_0\pi \Rightarrow a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

and, for $m \geq 1$,
$$\int_0^{2\pi} f(x)\cos(mx)\,dx = a_m\int_0^{2\pi}\cos(mx)\cos(mx)\,dx$$
by orthogonality
$$= a_m\pi \Rightarrow a_m = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(mx)\,dx$$

$$\int_0^{2\pi} f(x)\sin(mx)\,dx = b_m\int_0^{2\pi}\sin(mx)\sin(mx)\,dx$$

$$= b_m\pi \Rightarrow b_m = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(mx)\,dx$$

(b) We have $S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N-1}\left(a_n\cos(nx) + b_n\sin(nx)\right)$

— the truncated Fourier series of $f(x)$

and $S_N'(x) = \frac{a_0'}{2} + \sum_{n=1}^{N-1}\left(a_n'\cos(nx) + b_n'\sin(nx)\right)$

for any constants a_0', a_n', b_n'.

Thus,

$\pi a_0 = \int_0^{2\pi} f(x)\,dx$ & $\int_0^{2\pi} S_N(x)\,dx = \int_0^{2\pi}\left(\frac{a_0}{2} + \sum_{n=1}^{N-1} a_n\cos(nx) + b_n\sin(nx)\right)dx$

$= a_0\pi$

and

$\pi a_m = \int_0^{2\pi} f(x)\cos(mx)\,dx$ & $\int_0^{2\pi} S_N(x)\cos(mx)\,dx$

$= a_m\pi$

$\pi b_m = \int_0^{2\pi} f(x)\sin(mx)\,dx$ & $\int_0^{2\pi} S_N(x)\sin(mx)\,dx$

$= b_m\pi$

Hence,

$$\int_0^{2\pi}\left(f(x) - S_N(x)\right)\left(S_N(x) - S_N'(x)\right)dx$$

$$= \int_0^{2\pi}\left(f(x) - S_N(x)\right)\left(\frac{a_0 - a_0'}{2} + \sum_{n=1}^{N-1}\left((a_n - a_n')\cos(nx) + (b_n - b_n')\sin(nx)\right)\right)$$

dx

$$= \frac{a_0 - a_0'}{2} \int_0^{2\pi} \left(f(x) - S_N(x) \right) dx$$

$$+ \sum_{n=1}^{N-1} (a_n - a_n') \int_0^{2\pi} \left(f(x)\cos(nx) - S_N(x)\cos nx \right) dx$$

$$+ \sum_{n=1}^{N-1} (b_n - b_n') \int_0^{2\pi} \left(f(x)\sin(nx) - S_N(x)\sin(nx) \right) dx$$

$$= 0$$

(e) Now,

$$\int_0^{2\pi} \left(f(x) - S_N'(x) \right)^2 dx$$

$$= \int_0^{2\pi} \left(f(x) - S_N(x) + S_N(x) - S_N'(x) \right)^2 dx$$

$$= \int_0^{2\pi} \left(f(x) - S_N(x) \right)^2 + 2\int_0^{2\pi} \left(f(x) - S_N(x) \right)\left(S_N(x) - S_N'(x) \right) dx$$

$$+ \int_0^{2\pi} \left(S_N(x) - S_N'(x) \right)^2 dx$$

$$= \int_0^{2\pi} \left(f(x) - S_N(x) \right)^2 + \int_0^{2\pi} \left(S_N(x) - S_N'(x) \right)^2 dx$$

But the second term is ≥ 0 and only equals zero when $S_N(x) = S_N'(x)$ $\forall x$ ie $a_0' = a_0$, $a_n' = a_n$, $b_n' = b_n$. Thus mean squared error is minimized by Fourier series, $S_N(x)$