[Relates to p496 ① RJG Fourier Transform

2003 Paper 11
Continuous Mathematics

(1.) $$F(\mu) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-i\mu x}\, dx$$ Fourier Transform

and
$$f(x) = \int_{-\infty}^{\infty} F(\mu)\, e^{i\mu x}\, d\mu$$ Inverse Fourier Transform

(2). If $$f(x) = \begin{cases} e^{-ax} & x \geq 0 \\ 0 & x < 0 \end{cases} \qquad (a > 0)$$

Then
$$F(\mu) = \frac{1}{2\pi}\int_{0}^{\infty} e^{-ax}\, e^{-i\mu x}\, dx$$

$$= \frac{1}{2\pi}\int_{0}^{\infty} e^{-(a+i\mu)x}\, dx$$

$$= \frac{1}{2\pi}\left[-\frac{e^{-(a+i\mu)x}}{(a+i\mu)}\right]_0^{\infty} = \frac{1}{2\pi(a+i\mu)}$$

③ If $f(x) = e^{-a|x|}$

then

$$F(\mu) = \frac{1}{2\pi}\left(\int_{-\infty}^{0} e^{ax} e^{-i\mu x} dx + \int_{0}^{\infty} e^{-ax} e^{-i\mu x} dx \right)$$

$$= \frac{1}{2\pi}\left(\left[\frac{e^{(a-i\mu)\,dx}}{(a-i\mu)} \right]_{-\infty}^{0} + \frac{1}{(a+i\mu)} \right)$$

$$= \frac{1}{2\pi}\left(\frac{1}{(a-i\mu)} + \frac{1}{a+i\mu} \right)$$

$$= \frac{a}{\pi(a^2+\mu^2)}$$

④ Set $a=1$ then $e^{-|x|}$ has F.T. $\frac{1}{\pi(1+\mu^2)}$

$$\therefore \quad e^{-|x|} = \int_{-\infty}^{\infty} \frac{1}{\pi(1+\mu^2)} e^{i\mu x} d\mu$$

Replace x by $-x$ $\Rightarrow$ $e^{-|x|} = \int_{-\infty}^{\infty} \frac{1}{\pi(1+\mu^2)} e^{-i\mu x} d\mu$

Now swap x and μ

$$e^{-|\mu|} = \int_{-\infty}^{\infty} \frac{1}{\pi(1+x^2)} e^{-i\mu x} dx$$

$$\therefore \quad \frac{1}{2} e^{-|\mu|} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{(1+x^2)} e^{-i\mu x} dx$$

$\therefore$ $\frac{1}{1+x^2}$ has F.T. $\frac{1}{2} e^{-|\mu|}$