Probability 2002 p2q5 FHK

Use the generating functions $G_X(\eta) = e^{\lambda_A \eta} \,.\, e^{-\lambda_A}$ and $G_Y(\eta) = e^{\lambda_B \eta} \,.\, e^{-\lambda_B}$, from which:

$$G_{X+Y}(\eta) = e^{\lambda_A \eta} \,.\, e^{-\lambda_A} \,.\, e^{\lambda_B \eta} \,.\, e^{-\lambda_B} = e^{(\lambda_A + \lambda_B)\eta} \,.\, e^{-(\lambda_A+\lambda_B)}$$

Accordingly, $X + Y$ is distributed Poisson$(\lambda_A + \lambda_B)$. [6 marks]

Each of the n faults noted during the month may be attributed to either of the two channels. This is analogous to having n trials each of which can result in just two possible outcomes. The probabilities of these outcomes must sum to 1 and, given independence, they will be the same for each trial. This exactly describes the arrangements that give rise to a Binomial distribution. [4 marks]

Given n faults altogether of which r are attributable to channel A and $n - r$ to channel B, what is required is the conditional probability:

$$\mathrm{P}(X = r \mid Y = n - r) = \frac{\mathrm{P}(X = r \cap Y = n - r)}{\mathrm{P}(X + Y = n)}$$

Noting independence, the top line on the right-hand side can be re-expressed:

$$\mathrm{P}(X = r \cap Y = n - r) = \mathrm{P}(X = r) \,.\, \mathrm{P}(Y = n - r) = \frac{\lambda_A^r}{r!} e^{-\lambda_A} \,.\, \frac{\lambda_B^{(n-r)}}{(n-r)!} e^{-\lambda_B}$$

From part (a) it is known that $X + Y$ is distributed Poisson$(\lambda_A + \lambda_B)$ so the bottom line can be re-expressed:

$$\mathrm{P}(X + Y = n) = \frac{(\lambda_A + \lambda_B)^n}{n!} e^{-(\lambda_A+\lambda_B)}$$

Hence:

$$\mathrm{P}(X = r \mid Y = n - r) = \frac{\frac{\lambda_A^r}{r!} e^{-\lambda_A} \,.\, \frac{\lambda_B^{(n-r)}}{(n-r)!} e^{-\lambda_B}}{\frac{(\lambda_A+\lambda_B)^n}{n!} e^{-(\lambda_A+\lambda_B)}}$$

Cancel the exponentials and then rearrange the right-hand side noting that $(\lambda_A + \lambda_B)^n$ can be expressed as $(\lambda_A + \lambda_B)^r \,.\, (\lambda_A + \lambda_B)^{n-r}$:

$$\mathrm{P}(X = r \mid Y = n - r) = \frac{n!}{r!\,(n-r)!} \,.\, \left(\frac{\lambda_A}{\lambda_A + \lambda_B}\right)^r \,.\, \left(\frac{\lambda_B}{\lambda_A + \lambda_B}\right)^{n-r}$$

This result is of the form $\binom{n}{r} p^r q^{n-r}$. Accordingly, the distribution is Binomial$(n, \frac{\lambda_A}{\lambda_A+\lambda_B})$. [8 marks]

In the case of a Binomial distribution, Expectation $= np$, so the Expected number of faults attributable to channel A is $n\,\frac{\lambda_A}{\lambda_A+\lambda_B} = 5.\frac{4}{4+6} = 2$. [2 marks]