Solution Notes

p8q12
MROD

Numerical Analysis II – Question A 2004

Context: Chebyshev polynomials, best approximations, economisation.

(a) $$T_{k+1}(x) = 2x\,T_k(x) - T_{k-1}(x)$$

where $T_0(x) = 1$, $T_1(x) = x$.

Using this formula

$$T_2(x) = 2x^2 - 1$$
$$T_3(x) = 4x^3 - 3x$$
$$T_4(x) = 8x^4 - 8x^2 + 1$$
$$T_5(x) = 16x^5 - 20x^3 + 5x.$$

[4 marks]

(b) The best L_∞ approximation to x^k by a polynomial of lower degree is

$$x^k - \overline{T}_k(x)$$

where $$\overline{T}_k(x) = T_k(x)/2^{k-1}.$$

The degree of this approximation is $k-2$.

The method of economisation consists of constructing the best L_∞ approximation of lower degree to a truncated Taylor series, i.e. if

$$P_k(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_k x^k$$

then the best L_∞ approximation of lower degree is

$$P_k(x) - a_k\,\overline{T}_k(x)$$

which is of degree $k-1$.

The error in this approximation is given by

$$\|a_k \overline{T}_k(x)\|_\infty = \frac{a_k}{2^{k-1}}\|T_k(x)\|_\infty = \frac{a_k}{2^{k-1}}.$$

[7 marks]

over.

(c) For 2 decimal places we require a total absolute error of 0.5×10^{-2}.

$$f(x) = x + \frac{x^2}{4} + \frac{x^3}{18} + \frac{x^4}{96} + \frac{x^5}{600} + \dots$$

The approximate L_∞ error in the approximation

$$f(x) \simeq P_4(x)$$

is then $\frac{1}{600} < 0.5 \times 10^{-2}$.

If we economise $P_4(x)$ then the additional error is

$$\frac{1}{8 \times 96} = \frac{1}{768}.$$

The total error is therefore

$$\frac{1}{600} + \frac{1}{768} < 0.5 \times 10^{-2}$$

which is acceptable.

The economised polynomial is then

$$P_4(x) - \frac{T_4(x)}{96}$$

$$= x + \frac{x^2}{4} + \frac{x^3}{18} + \cancel{\frac{x^4}{96}} - \frac{1}{8 \times 96}(\cancel{8x^4} - 8x^2 + 1)$$

$$= -\frac{1}{768} + x + \left(\frac{1}{4} + \frac{1}{96}\right)x^2 + \frac{x^3}{18}$$

$$= -\frac{1}{768} + x + \frac{25}{96}x^2 + \frac{x^3}{18}. \qquad (1)$$

Economising again would add an error of

$$\frac{1}{4 \times 18} > 0.5 \times 10^{-2}$$

which is too large, so (1) is the required approximation.

[9 marks]