Logic and Proof 2004 – Paper 5 Question 9 (JEH)

Context: This questions tests in-depth knowledge of the semantics and sequent calculus for first order logic.

In this question x, y, z are variables, and a, b, c are constants.

(a) Briefly outline the semantics of first order logic. [5 marks]

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Let L be a first order language. An interpretation of L is a pair
(D,I) consisting of a domain and an interpreting function I that maps
n-ary relation sybols to subsets of D^n, and n-ary function symbols to
functions D^n -> D. Valuations V are functions from variable symbols
to elements of D. Now define the semantic operator |= recursively:
|=I,V R(t_1,...,t_n) iff (I_V(t_1),...,I_V(t_n)) \in I(R)
|=I,V t = u iff I_V(t) = I_V(u)
|=I,V ~A iff |=I,V A doesn't hold
|=I,V A /\ B iff both |=I,V A and |=I,V B hold
|=I,V !x. A iff for every m \in D |=I,V[x|->m] A holds
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(b) Use the semantics of first order logic to justify that the set of formulas

$$\{\forall x(x = c),\ P(a),\ \neg P(b)\}$$

is unsatisfiable. [2 marks]

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Given an interpretation (I,D) |=I,V !x. x = c will only hold if the
domain D has precisely one element. But then I(P) is either {}
which means P(a) fails to hold; or D which means ~P(b) fails to hold.
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(c) For each of the following first order logic formulas: **either** prove it to be valid using the sequent calculus; **or** give an interpretation that makes it false.

$$[\forall x(\exists y(R(x, y)))] \to \exists x(R(x, x))$$

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Invalid. Interpretation is (Z, R |-> {(m,n) | m < n})
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$$[\exists x(\neg P(x))] \to \neg\exists x(P(x))$$

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Invalid. Interpretation is ({0,1}, P |-> {0})
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$$[\neg\exists x(P(x))] \to \exists x(\neg P(x))$$

```
Valid.
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 P a ==> P a
--------------- ~r
 ==> P a, ~P a
------------------- ?r
 ==> P a, ?x. ~P x
----------------------- ?r
 ==> ?x. P x, ?x. ~P x
----------------------- ~l
 ~?x. P x ==> ?x. ~P x
----------------------------- -->r
 ==> (~?x. P x) --> ?x. ~P x
```

$$\exists x(P(x) \to P(a) \land P(b))$$

```
Valid.
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 P a, P b ==> P a      P a, P b ==> P b
---------------------------------------- /\r
 P a, P b ==> P a /\ P b
------------------------------------- weaken r
 P a, P b ==> P a /\ P b, P a /\ P b
---------------------------------------- -->r
 P a ==> P a /\ P b, P b --> P a /\ P b
-------------------------------------------- -->r
 ==> P a --> P a /\ P b, P b --> P a /\ P b
------------------------------------------------ ?r
 ==> ?x. P x --> P a /\ P b, P b --> P a /\ P b
------------------------------------------------ ?r
 ==> ?x. P x --> P a /\ P b
```

[2 marks each]

(d) Consider the following set Γ of first order logic formulas:

$$\left\{ \begin{array}{l} \forall x(\neg R(x,x)), \quad \forall xyz(R(x,y) \land R(y,z) \to R(x,z)), \\ R(a,b), \quad \forall xy(R(x,y) \to \exists z(R(x,z) \land R(z,y))) \end{array} \right\}$$

(i) Find an interpretation that satisfies Γ. [3 marks]

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One interpretation is (Q, [R |-> {(x,y) | x < y}, a |-> 0, b |-> 1])
This uses the rational numbers Q.
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(ii) Can Γ be satisfied by an interpretation with a finite domain? [2 marks]

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No. Proof:
* Given an interpretation (D,I) satisfying Gamma, we will construct
  an infinite subset {b_0,b_1,...} of D.
* At stage n of the construction the following invariant will hold:
    (a,b_n) \in I(R)  AND  (b_n,b_m) \in I(R) for every m < n
* Let b_0 be b. Certainly the invariant is true at stage 0.
* Suppose the invariant is true at stage n.
* Using the last formula in Gamma with a and b_n, we know there is a
  point b_{n+1} \in D satisfying (a,b_{n+1}) \in I(R) and
  (b_{n+1},b_n) \in I(R).
* But the transitivity formula in Gamma implies that (b_{n+1},b_m) \in
  I(R) for every m < n. Thus the invariant holds at stage n+1.
* Thus the invariant holds at every stage. But the second part of
  the invariant implies that all the elements b_i are distinct,
  because of the irreflexivity formula in Gamma. Thus {b_0,b_1,...}
  is an infinite subset of D. QED.
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