Solution Notes

p9 q12
MROD

Numerical Analysis II – Question 8 2004

Context: quadrature, Riemann sum, composite rules, product rules.

(a) A sum of the form

$$\sum_{i=1}^{n} (\xi_i - \xi_{i-1}) f(x_i)$$

where $a = \xi_0 < \xi_1 < \dots < \xi_n = b$ is a Riemann sum if $x_i \in [\xi_{i-1}, \xi_i]$ for $i = 1, 2, \dots n$.

The mesh norm is given by

$$\Delta\xi = \max_i |\xi_i - \xi_{i-1}| .$$ [4 marks]

(b) If $[a, b] = [-1, 1]$ then $h = \frac{2}{3}$ and

$$Qf = \tfrac{1}{4} f(-1) + \tfrac{3}{4} f(-\tfrac{1}{3}) + \tfrac{3}{4} f(+\tfrac{1}{3}) + \tfrac{1}{4} f(+1) - \frac{2 f^{(4)}(\lambda)}{405}$$

$\xi_0 = -1,\ \xi_1 = -\frac{3}{4},\ \xi_2 = 0,\ \xi_3 = \frac{3}{4},\ \xi_4 = 1.$

The abscissae are

$$-1 \in [-1, -\tfrac{3}{4}],\quad -\tfrac{1}{3} \in [-\tfrac{3}{4}, 0],$$
$$+\tfrac{1}{3} \in [0, \tfrac{3}{4}],\quad +1 \in [\tfrac{3}{4}, 1],$$

so it is a Riemann sum. [3 marks]

(c)

$$(n \times R) f = \frac{b-a}{2n} \sum_{i=1}^{n} \sum_{j=1}^{m} w_j f(x_{ij})$$

where x_{ij} is the jth abscissa of the ith subinterval.

R integrates constants exactly, so

$$R.1 = \sum_{j=1}^{m} w_j = \int_{-1}^{1} dx = 2 .$$

Reversing summations and taking the limit

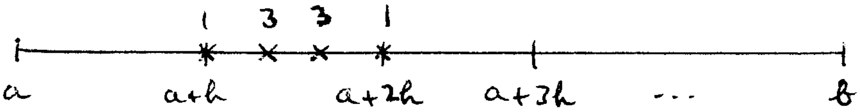
$$\lim_{n\to\infty} (n \times R) f = \frac{1}{2} \sum_{j=1}^{m} w_j . \lim_{n\to\infty} \left\{ \frac{b-a}{n} \sum_{i=1}^{n} f(x_{ij}) \right\}$$
$$= \cancel{\tfrac{1}{2}} \int_a^b f(x)\, dx . \cancel{\sum_{j=1}^{m} w_j}$$

since the limit on the right-hand side is a Riemann sum.

[6 marks]

over.

(d) Divide $[a,b]$ into n subintervals of width $h = \frac{b-a}{n}$.



Either

$$(n \times Q) f = \frac{b-a}{2n} \sum_{i=1}^{n} \left\{ \tfrac{1}{4} f[a+(i-1)h] + \tfrac{3}{4} f[a+(i-\tfrac{2}{3})h] + \tfrac{3}{4} f[a+(i-\tfrac{1}{3})h] + \tfrac{1}{4} f[a+ih] \right\}$$

or

$$(n \times Q) f = \frac{b-a}{8n} \left\{ f(a) + f(b) + 2 \sum_{i=1}^{n-1} f(a+ih) + 3 \sum_{i=1}^{n} \left[f[a+(i-\tfrac{2}{3})h] + f[a+(i-\tfrac{1}{3})h] \right] \right\}.$$

[4 marks]

(e) Qf integrates cubic polynomials exactly since its error term is $O(h^5)$.

$(Q \times Q) F(x,y)$ will therefore integrate exactly any linear combination of the monomials

1	x	x^2	x^3
y	xy	x^2y	x^3y
y^2	xy^2	x^2y^2	x^3y^2
y^3	xy^3	x^2y^3	x^3y^3

[3 marks]