Advanced Graphics

(a) Both are based on a set of control points specified in a definite order. B-splines are made by specifying a curve defined by the points and a set of basis functions, one for each point:

$$f(t) = \sum_{i=1}^{N} P_i \cdot N_{i,k}(t)$$

Subdivision is defined as a mechanism for generating a more dense set of points from the original points. A subdivision curve is the limit of recursively applying this mechanism. Both methods can produce the same curve from the same set of points but each can produce results not possible with the other.

B-splines are able to produce a wide range of behaviour, including reducing the continuity of the curve at a point and exactly reproducing conics.

Subdivision is extremely simple to implement but has much higher memory requirements than B-splines owing to the need to store all the generated points.

Subdivision better matches the way in which we actually draw curves in computer graphics: as a sequence of straight line segments approximating the actual curve.

[Any four salient points would get the marks]

(b) Given a knot vector $[t_1\ t_2\ t_3\ t_4 \ldots]$ the B-spline basis functions are recursively defined:

$$N_{i,1}(t) = \begin{cases} 1, & t_i \le t < t_{i+1} \\ 0, & \text{otherwise} \end{cases}$$

$$N_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t)$$

provided that we define $\frac{0}{0} = 0$

(c) We want $N_{1,3}(t)$ given knots $[0, 1, 2, 3]$

$$N_{1,1}(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & \text{otherwise} \end{cases}; \quad N_{2,1}(t) = \begin{cases} 1, & 1 \le t < 2 \\ 0, & \text{otherwise} \end{cases}; \quad N_{3,1}(t) = \begin{cases} 1, & 2 \le t < 3 \\ 0, & \text{otherwise} \end{cases}$$

$$N_{1,2}(t) = \begin{cases} t, & 0 \le t < 1 \\ 2-t, & 1 \le t < 2 \\ 0, & \text{otherwise} \end{cases} \qquad N_{2,2}(t) = \begin{cases} t-1, & 1 \le t < 2 \\ 3-t, & 2 \le t < 3 \\ 0, & \text{otherwise} \end{cases}$$

$$N_{1,3}(t) = \frac{t}{2} \cdot N_{1,2}(t) + \frac{3-t}{2} N_{2,2}(t)$$

$$= \begin{cases} \frac{1}{2} t^2, & 0 \le t < 1 \\ \frac{1}{2} t(2-t) + \frac{1}{2}(3-t)(t-1), & 1 \le t < 2 \\ \frac{1}{2}(3-t)^2, & 2 \le t < 3 \\ 0, & \text{otherwise} \end{cases}$$

(d) Given a ray: $\underset{\sim}{P}(t) = \underset{\sim}{E} + t\underset{\sim}{D}, \quad t \ge 0$ $\quad (*)$

and a cylinder: $x^2 + y^2 = r^2, \quad z_{min} \le z \le z_{max}$ $\quad (+)$

(1) Find the intersections (if any) of the line $\underset{\sim}{E} + t\underset{\sim}{D}$ with the infinite cylinder $x^2 + y^2 = r^2$

Subititute equation $(*)$ into equation $(+)$

$$(x_E + t x_D)^2 + (y_E + t y_D)^2 = r^2$$

rearrange to get a quadratic equation in t and find the real roots. There are either two, call them t_1 and t_2, which may be equal, or none.

(2) If t_1 and t_2 exist, check whether they lie on the finite length cylinder. Calculate $z_i = z_E + t_i z_D$, $i \in \{1,2\}$.
Check whether $z_{min} \le z_i \le z_{max}$. If so, t_i is a valid intersection point.

(3) Find the intersection of the ray with the two end caps, call these t_3 and t_4.

$$t_3 = \frac{z_{min} - z_E}{z_D} \; ; \; t_4 = \frac{z_{max} - z_E}{z_D}$$

If $z_D = 0$, this cannot be done and t_3 and t_4 do not exist.
Otherwise check:

$$(x_E + t_i x_D)^2 + (y_E + t_i y_D)^2 \le r^2 \qquad i \in \{3,4\}$$

If true, t_i is a valid intersection point.

(4) Reject any remaining, valid, t_i $i \in \{1,2,3,4\}$ if $t_i < 0$

(5) If any valid t_i remain, $i \in \{1,2,3,4\}$, then the smallest value is the intersection point and this point is at $\underline{P} = \underline{E} + t_i \underline{D}$

If no valid t_i remain, then there is no ~~valid~~ intersection between the ray and the finite length closed cylinder.

[This algorithm does what is required. It is somewhat inefficient and there are a number of possible optimisations.]

Context

This question covers material from 3 of the 8 lectures

- ray tracing
- NURBS
- subdivision

(b), (c) and (d) are bookwork

(a) requires a little more understanding

Past experience shows that a minority of students will be stumped by (b) and (c), and that a majority of the students will miss out some of the fiddly details in (d)

Preliminary Marking Scheme

(a) one mark for each salient points, up to a total of four 4

(b) $N_{i,1}(t)$

$N_{i,k}(t)$ $\begin{cases} \text{correct form} \\ \text{correct subscripts on } N_{i,k-1} \ \& \ N_{i+1,k-1} \\ \text{correct factors to be multiplied by these} \end{cases}$ 4

(c) $N_{1,2}$ and $N_{2,2}$

$N_{1,3}$ $\begin{cases} \frac{1}{2}t^2 \text{ and } \frac{1}{2}(3-t)^2 \\ \frac{1}{2}t(2-t)+\frac{1}{2}(3-t)(t-1) \\ \text{ranges correct and } 0 \text{, otherwise} \end{cases}$ 4

(d) ray and cylinder equations

substitute one in the other & get t_1, t_2

check $z_{min} \le t_i \le z_{max}$ $i \in \{1, 2\}$

get t_3 & t_4

check $(x_E + t_i x_D)^2 + (y_E + t_i y_D)^2 \le r^2$ $i \in \{3, 4\}$

check $t_i < 0$

return smallest, if it exists

if it doesn't, no intersection 8

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E T Z C R O S N