Numerical Analysis II 2000

p9 q10
p13 q10
MROD

Acceptable Answer B

(a) A is positive definite if $\underline{x}^T A \underline{x} > 0$ for any vector $\underline{x} \neq \underline{0}$.
(b) A is positive semidefinite if $\underline{x}^T A \underline{x} \geq 0$ for any vector $\underline{x}$.

$B^T B$ is symmetric. If B is non-singular then $B^T B$ is non-singular and

$$\underline{x}^T B^T B \underline{x} = (B\underline{x})^T (B\underline{x}) > 0 \quad \text{for any } \underline{x} \neq \underline{0}$$

so $B^T B$ is positive definite. If B is singular then $B^T B$ is singular and positive semidefinite. [4 marks]

Write $\|.\|_M$, $\|.\|_V$ for compatible matrix and vector norms. Schwarz's inequality for AB is

$$\|AB\|_M \leq \|A\|_M \|B\|_M .$$

If B is replaced by $\underline{x}$ then

$$\|A\underline{x}\|_V \leq \|A\|_M \|\underline{x}\|_V .$$

If $\lambda_1, \lambda_2, \ldots \lambda_n$ are the real, non-negative eigenvalues of $A^T A$ arranged in non-increasing order of magnitude then the singular values $\sigma_1, \sigma_2, \ldots \sigma_n$ are given by

$$\sigma_j = \sqrt{\lambda_j} .$$

$\|A\|_2 = \max_j \sqrt{\lambda_j} = \sigma_1$. $W = \mathrm{diag}\{\sigma_1, \sigma_2, \ldots \sigma_n\}$. [5 marks]

$$A\underline{e} = A\underline{x} - A\underline{\hat{x}} = \underline{b} - A\underline{\hat{x}} = \underline{r}$$

so
$$\underline{e} = A^{-1}\underline{r}$$
$$\|\underline{e}\| = \|A^{-1}\underline{r}\|$$
$$\leq \|A^{-1}\| . \|\underline{r}\| \quad \text{by Schwarz}$$

so
$$\frac{\|\underline{e}\|}{\|\underline{x}\|} \leq \|A^{-1}\| . \frac{\|\underline{r}\|}{\|\underline{x}\|}$$

PTO.

But $\|\underline{b}\| = \|A\underline{x}\| \leq \|A\| \, \|\underline{x}\|$ by Schwarz

so
$$\frac{\|\underline{e}\|}{\|\underline{x}\|} \leq \|A\| \, \|A^{-1}\| \, \frac{\|\underline{r}\|}{\|\underline{b}\|}$$

where the right-hand side is now computable.
If the l_2 norm is used, the condition number

$$K = \|A\|_2 \, \|A^{-1}\|_2 = \frac{\sigma_1}{\sigma_n}.$$

[8 marks]

Machine epsilon $= 10^{-15}$ and

$$W = \mathrm{diag}\,\{1, 10^{-6}, 10^{-10}, 10^{-17}, 0\}.$$

The singular value $10^{-17} <$ machine epsilon so represents noise and should be ignored. The effective rank of the system is 3, so the appropriate generalised inverse is

$$W^{+} = \mathrm{diag}\,\{1, 10^{6}, 10^{10}, 0, 0\}.$$

[3 marks]