Complexity Theory 2004 – Paper 5 Question 12 (AD)

Recall that a *simple path* in a graph is a path with no repeated nodes. Consider the following two decision problems:

- Given a graph $G = (V, E)$, a positive integer k, source $s \in V$ and a target $t \in V$, is there a simple path from s to t of length *at least* k?
- Given a graph $G = (V, E)$, a positive integer k, source $s \in V$ and a target $t \in V$, is there a simple path from s to t of length *at most* k?

One of these problems is known to be in P while the other one is known to be NP-complete

(a) Which of the two problems is in P and which is NP-complete? [2 marks]

The first problem is NP-complete and the second problem is in P.

(b) Describe a polynomial time algorithm for the problem that is in P. [5 marks]

The second problem can be solved by performing a depth-first search of the graph starting at node s. The search is cut-off and forced to backtrack at depth k. If n is the number of nodes in V, the complexity of the depth-first search algorithm is $O(n^2)$. The cut-off at k can only reduce this complexity. Thus, the algorithm is in polynomial time.

(c) Give a proof of NP-completeness for the problem that is NP-complete. You may assume the NP-completeness of any problem, such as *Hamiltonian Cycle* mentioned in the lecture course. [13 marks]

First, we show that the problem is in NP. A witness to membership is a simple path from s to t of length at least k. Since the path contains no repeated nodes, the length of the path is bounded by the size of V. Thus, the witness is of polynomial size. To verify the witness, one just needs to check whether, for each pair of successive nodes in the purported path, there is indeed an edge in the graph. This is easily done in linear time.

To show NP-hardness, we proceed by giving a reduction from the Hamiltonian Cycle problem. Suppose we are given a graph $G = (V, E)$, we construct a new graph which has nodes $V' = V \cup \{s, t\}$ where s and t are new. The edges include all edges in E. In addition, there is an edge from s to one node (say v) in V. There are also new edges from all nodes $u \in V$ which have an edge to v to t. We set $k = |V| + 1$. We claim that this new graph has a path of length at least k from s to t if, and only if, G has a Hamiltonian cycle. In one direction, suppose G has a Hamiltonian cycle. Starting at s, proceeding to v and following this cycle, in

n steps we end up at a node u that must have an edge to v. Thus, there is an edge from u to t and we have found a path of length k. In the other direction, suppose there is a simple path of length at least k from s to t. Such a path must have $k - 1 = |V|$ intermediate nodes. That is, it visits every node in V. The path through V must start at v and end at a node u that has an edge to v. So, we can take this edge and complete a Hamiltonian cycle.

This question relates to one of the key objectives of the course, to be able to identify NP-complete problems. The theory of NP-completeness forms a core part of the course, being covered in Lectures 4–7.