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Model Answer, Continuous Mathematics, Question 1.

(A1): When numerically computing the solution to an ODE that involves higher-than first-order derivatives, the higher-order terms must be reduced to 1st-order terms. This can be accomplished by introducing new substituted variables such as

$$z(x) = \frac{dy(x)}{dx}$$

which adds another equation into the coupled system to be solved, but now we have terms like $\frac{dz(x)}{dx}$ instead of $\frac{dy^2(x)}{dx^2}$.

(A2) Thus the example ODE,

$$\frac{d^2y}{dx^2} + q(x)\frac{dy}{dx} = r(x)$$

would be replaced by the following 1st-order system:

$$\frac{dy(x)}{dx} = z(x)$$

$$\frac{dz(x)}{dx} = r(x) - q(x)z(x)$$

(B1) The incrementing rule for the Euler method of numerical integration is:

$$f(x_{n+1}) = f(x_n) + hf'(x_n)$$

(B2) If the stepsize h is too large, this piecewise-linear approximation to the solution will be inaccurate where the true solution has significant second derivatives (which this method ignores). The solution may "blow up."

(B3) If the stepsize h is made too small, the computation may take too long and the round-off error (truncation of floating-point numbers depending on the word length of the machine) will accumulate too much, making the solution inaccurate or even unstable.

(B4) The primary advantage of the Runge-Kutta method over the Euler method for numerical integration of ODEs is that its accumulated error depends on the stepsize h as its 4th power, h^4, whereas in the Euler method the accumulated error is simply linear in h. Thus, for example, reducing the stepsize h by half would only reduce the accumulated error by half in the Euler method, but it would reduce it 16-fold in the Runge-Kutta method.

(B5) The stepsize h should be made adaptive if the solution varies greatly in its rates of change. When the rate of change is low, the stepsize can be made large, but when the rate of change is rapid, the stepsize should be made small. Thus h should be linked inversely to $f'(x_n)$.