Logic and Proof 2004 – Paper 6 Question 9 (JEH)

Context: This question tests familiarity of several major sections of the course, by asking simple questions that test whether the student understands essential details. The sections covered are: conversion to CNF; the DPLL method; OBDDs; Skolemization; Herbrand Universes; unification; resolution with factoring; modal logic.

For each of the following statements, briefly justify whether it is true or false. In the following x, y, z are variables, and a, b, c are constants.

(a) Given any propositional logic formula ϕ that is a tautology, converting ϕ to CNF will result in $\mathtt{t}$.

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True. The other CNF formulas are F (obviously not a tautology) or
(L_1 \/ ... \/ L_n) /\ ... where all the literals L_i are distinct
propositional letters. This is also not a tautology, because it can
be falsified by setting all the L_i to false.
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(b) Executing the DPLL method on the clauses

$$\{P, Q, \neg S\} \quad \{\neg P, Q, \neg R\} \quad \{P\} \quad \{\neg Q, R\} \quad \{S, \neg Q\}$$

produces a result without needing any case split steps.

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False. After propagating the unit clause {P} and then deleting the
pure literal S, we are left with the clauses {Q,~R} and {~Q,R} which
now requires a case split step.
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(c) The OBDD corresponding to the propositional logic formula $(P \vee Q) \wedge \neg P$ does not have any decision nodes for the propositional letter P.

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True. Either construct the OBDD, or observe that the formula is
logically equivalent to Q, and the OBDD for Q will obviously not
have any decision nodes for P.
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(d) Skolemizing the first order logic formula $\exists x(\phi(x))$ results in a logically equivalent formula $\phi(a)$ (where a is a fresh constant).

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False. Skolemization only preserves satisfiability, not logical
equivalence. For example, ?x. P(x) is true and P(a) is false in the
interpretation ({0,1}, [a |-> 1, P |-> {0}]).
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(*e*) The Herbrand Universe that is generated from the clauses $\{P(a)\}$, $\{Q(x, b), \neg P(x)\}$ and $\{\neg Q(a, y)\}$ contains two elements.

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True. The Herbrand universe is {a,b} and nothing else, since there are
no function symbols.
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(*f*) The two terms $f(x, y, z)$ and $f(g(y, y), g(z, z), g(a, a))$ can be unified.

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True. The unifier is [z |-> g(a,a)] o [y |-> g(z,z)] o [x |-> g(y,y)],
which becomes rather long if the substitution compositions are expanded.
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(*g*) It is not possible to resolve the clauses $\{P(x)\}$ and $\{\neg P(f(x))\}$ because the *occurs check* prevents the literals being unified.

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False. We are allowed to rename clauses before trying to resolve them,
the first clause can become {P(y)} and now the unification trivially
succeeds.
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(*h*) The clause $\{P(x, x), P(x, a)\}$ can be factored to give the new clause $\{P(x, a)\}$.

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False. The only way to factor the clause is to unify P(x,x) and P(x,a)
which results in the substitution [x |-> a], so the factored clause is
{P(a,a)}.
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(*i*) The empty clause can be derived from the clauses $\{P(x), P(a)\}$, $\{P(x), \neg P(a)\}$, $\{\neg P(b), Q\}$ and $\{\neg P(c), \neg Q\}$ using resolution.

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True. Resolve the first two clauses to get the clause {P(x)}. Resolve
this new clause with both the last two clauses to get {Q} and {~Q},
and now derive the empty clause from these.
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(*j*) Because in the modal logic S4 the equivalence $\Box\Box\phi \simeq \Box\phi$ holds for every formula ϕ, it follows that $\Diamond\Diamond\phi \simeq \Diamond\phi$.

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True. DDp = DD~~p = D~B~p = ~BB~p = ~B~p = D~~p = Dp
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[2 marks each]