Solution Notes

p499
MROD

Numerical Analysis I   2001

Question B

A matrix A is symmetric if $A = A^T$. A symmetric matrix A is positive definite if $\underline{x}^T A \underline{x} > 0$ for any $\underline{x} \neq \underline{0}$. [3 marks]

Let $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Then $A\underline{x} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 + x_2 \\ x_1 + 2x_2 \end{bmatrix}$.

$\underline{x}^T A \underline{x} = 2x_1^2 + 2x_1x_2 + 2x_2^2 = x_1^2 + x_2^2 + (x_1+x_2)^2 > 0$ if $\underline{x} \neq \underline{0}$. [4 marks]

$L(DL^T\underline{x}) = \underline{b}$ so $DL^T\underline{x} = \underline{y}$ or $L^T\underline{x} = D^{-1}\underline{y}$.

$$D^{-1}\underline{y} = \begin{bmatrix} \frac{1}{2} & \\ & \frac{2}{3} \end{bmatrix}\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \end{bmatrix}$$

so the upper triangular system is

$$\begin{bmatrix} 1 & \frac{1}{2} \\ & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \end{bmatrix}$$ [4 marks]

Hence, by back substitution, $x_2 = \frac{1}{3}$ and $x_1 = \frac{1}{3}$. [2 marks]

If ϵ_n is the absolute error in the nth iteration, and there exist constants $p \geqslant 1$, C such that

$$\lim_{n\to\infty} \left| \frac{\epsilon_{n+1}}{\epsilon_n^p} \right| = C$$

then the process has order of convergence p.
[Also acceptable: $|\epsilon_{n+1}| \simeq C|\epsilon_n|^p$, or $|\epsilon_{n+1}| = O(|\epsilon_n|^p)$.] [1 mark]

The Newton-Raphson formula is

$$x_{n+1} = x_n - \frac{f(x)}{f'(x)}$$

and has order of convergence 2. [2 marks]

Observe that the solution is $x=2$, so $\epsilon_3 \simeq 6\times10^{-4}$, $\epsilon_4 \simeq 9\times10^{-8}$. Estimate $C \simeq \epsilon_4/\epsilon_3^2 = \frac{1}{4}$. Hence $\epsilon_5 \simeq \frac{1}{4}\epsilon_4^2 \simeq 2\times10^{-15}$, so there are about 14 decimal digits of accuracy.
[Also acceptable: Observe that $C = O(1)$ so $\epsilon_5 \simeq \epsilon_4^2 \simeq 10^{-14}$. Any reasoned answer in the range 13–15 digits could be acceptable.] [4 marks]