SOLUTION NOTES

Logic and Proof 2003 Paper 6 Question 9 (LCP)

This question covers Lecture 1 (on the general notions of validity and interpretations) and Lectures 3 and 6 (on the sequent calculus).

(a) The sequent $A_1, \ldots, A_m \Rightarrow B_1, \ldots, B_n$ is true provided that if A_1, …, A_m are all true then at least one of B_1, …, B_n is true. A sequent is valid provided it is true in all interpretations. A basic sequent has some formula both on its right side and its left, and thus is trivially valid.

(b) The first rule expresses that if some proposition follows from A and the same proposition follows from B then it must also follow from $A \vee B$, since if the disjunction is true then either A or B must be true. The second rule is subject to the proviso that x must not be free in the conclusion, i.e. in Γ or Δ. The intuition is that if some proposition follows from $A(x)$, where x is not mentioned in the conclusion, then the same proposition must also follow from $\exists x\, A(x)$, since this formula implies that $A(x)$ is true for some x.

(c) Both proofs are by contradiction. First, take the disjunction rule. We assume that $A \vee B, \Gamma \Rightarrow \Delta$ is not valid and prove that either $A, \Gamma \Rightarrow \Delta$ or $B, \Gamma \Rightarrow \Delta$ is not valid. If $A \vee B, \Gamma \Rightarrow \Delta$ is not valid then some interpretation makes $A \vee B$ and the formulæ in Γ true while making every formula in Δ false. Since $A \vee B$ true in this interpretation, either A is true or B is true. If A is true then the sequent $A, \Gamma \Rightarrow \Delta$ is not valid, since this interpretation makes all the left-side formulæ true and all the right-side formulæ false. If B is true then the sequent $B, \Gamma \Rightarrow \Delta$ is similarly invalid.

For the other rule, suppose that $\exists x\, A, \Gamma \Rightarrow \Delta$ is not valid. Then some interpretation makes $\exists x\, A$ and the formulæ in Γ true while making every formula in Δ false. Thus A is true under this interpretation for at least one valuation of x, and therefore $A, \Gamma \Rightarrow \Delta$ is not valid. (Proving soundness of this rule may be too difficult for some candidates, but they can opt for the other rule.)

(d) This question covers Lecture 9 (Unifcation). It is pure bookwork. Candidates can get partial credit with an intuitive answer, saying e.g. that a unifier is a way of making two terms identical by instantiating variables, and that an MGU is a unifier that makes no unnecessary instantiations.

Full credit requires a more precise answer, defining a substitution as a map from variables to terms. Then a unifier of two terms t and u is a substitution θ such that $t\theta = u\theta$. The composition $\theta_1 \circ \theta_2$ of two substitutions, θ_1 and θ_2, is a substitution that combines the effect of both: $t(\theta_1 \circ \theta_2) = t\theta_1\theta_2$. Now θ is a most general unifier of t and u if $t\theta = u\theta$ and moreover for all ϕ such that $t\phi = u\phi$ there exists some θ' such that $\phi = \theta \circ \theta'$.