Probability 2003 Paper 2 Question 4 (FHK)

Ducting Problem — Solution Notes

A cube has eight vertices. Label a diagonally opposite pair A and D. Assign the label B to each of the three nodes which are adjacent to A and assign the label C to each of the three nodes which are adjacent to D. Now visualise the ducting thus:

A ——— B ▬▬▬ C ——— D

The thin lines indicate three lengths of ducting and the thick line represents six lengths of ducting.

(a) The device effectively undertakes a random walk along this linear path and since it starts at A (move 0) it necessarily visits A and C on even moves and B and D on odd moves.

(b) Let A_n, B_n, C_n and D_n be the probabilities of visiting A, B, C and D respectively at move n. By inspection:

$$A_{n+1} = \frac{1}{3} B_n \qquad B_{n+1} = A_n + \frac{2}{3} C_n \qquad C_{n+1} = \frac{2}{3} B_n + D_n \qquad D_{n+1} = \frac{1}{3} C_n$$

From this it is easy to draw up the following table:

n	A_n	B_n	C_n	D_n
0	1	0	0	0
1	0	1	0	0
2	$\frac{1}{3}$	0	$\frac{2}{3}$	0
3	0	$\frac{7}{9}$	0	$\frac{2}{9}$
4	$\frac{7}{27}$	0	$\frac{20}{27}$	0
5	0	$\frac{61}{81}$	0	$\frac{20}{81}$
6	$\frac{61}{243}$	0	$\frac{182}{243}$	0
7	0	$\frac{547}{729}$	0	$\frac{182}{729}$

The values $A_2 = \frac{1}{3}$, $A_4 = \frac{7}{27}$ and $A_6 = \frac{61}{24}$ can be taken from the table.

(c) From the table it is fairly clear that A_n is going to settle down at $\frac{1}{4}$ (for even n) but this is obvious by inspection anyway: at even moves at equilibrium, node A and the three nodes labelled C will be equally likely to receive a visit.

(d) From the values quoted for A_2, A_4 and A_6 it can quickly be conjectured that:

$$A_n = \begin{cases} \frac{1}{4}\left(1+\frac{1}{3^{n-1}}\right), & \text{if } n \geqslant 0 \text{ and even} \\ 0, & \text{otherwise} \end{cases}$$