Maths for Computation Theory 2003 p10q8 JKMM

4

Paper 10 Solution

This question refers to Part A section 4, "Relations on a set". Part of the answer requires familiarity with Part A section 3, "Algebra of relations".

i) In order that R be a partial order on A:

" $(a,b) \in R$, $(b,a) \in R \Rightarrow a = b$ "

ii) In order that R be a total order on A:

iii), and " for all $a, b \in A$, $(a,b) \in R$ or $(b,a) \in R$ (or both!) "

Algebraic formulation of the conditions:

i) $\Delta_A \subseteq R$;

ii) $R \circ R \subseteq R$;

iii) $R \cap R^{-1} \subseteq \Delta_A$ (equivalently, $= \Delta_A$);

iv) $R \cup R^{-1} = (A \times A)$.

In this formalism $S = R \cap R^{-1}$.

Paper 10 Solution (ctd.)

Proof: We must show that S is reflexive, symmetric and transitive.

a) $\Delta_A \subseteq R \Rightarrow \Delta_A = \Delta_A^{-1} \subseteq R^{-1}$

$\Rightarrow \Delta_A \subseteq (R \cap R^{-1}) = S$. REFLEXIVE

b) $S^{-1} = (R \cap R^{-1})^{-1} = R^{-1} \cap R = S$

SYMMETRIC

c) $(S \circ S) = (R \cap R^{-1}) \circ (R \cap R^{-1})$

$\subseteq (R \circ R) \cap (R^{-1} \circ R^{-1})$

$= (R \circ R) \cap (R \circ R)^{-1}$

$\subseteq R \cap R^{-1} = S$

TRANSITIVE

Hence S is indeed an equivalence relⁿ

Paper 10 Solution (ctd)

HIGHLY DESIRABLE to show first that $\leq$ is well-defined, but I don't expect them to do so.

Suppose $[a] \leq [b]$, $\alpha \in [a]$, $\beta \in [b]$.
By definition aRb, and by definition of S;
$\alpha R a$ and $a R \alpha$; $\beta R b$ and $b R \beta$.

BUT $\alpha R a$, $a R b$, $b R \beta \Rightarrow \alpha R \beta$.
Hence the definition of $\leq$ on A/S is independent of the choices $a \in [a]$, $b \in [b]$.

THIS PAGE IS WHAT PRODUCES THE EXTRA 2 MARKS.

Hence given $[a] \in A/S$, we may represent it by a. But aRa.
$\therefore [a] \leq [a]$, and i) holds.

Suppose $[a] \leq [b]$ and $[b] \leq [a]$.
By definition, aRb and bRa, hence $(a,b) \in S$. $\therefore [a] = [b]$.
Hence condition iii) also holds

Suppose $[a] \leq [b]$ and $[b] \leq [c]$, so that by definition aRb and bRc.
Condition ii) holds for R, hence aRc.
By definition, $[a] \leq [c]$.
Hence condition ii) holds for $\leq$, and $(A/S, \leq)$ is a poset.

Paper 10 Solutions (ctd)

i) $1 \in \mathbb{Z}$, so if $x \in \mathbb{Z}$, then by taking $q = 1$ we may show $(x,x) \in R$.

ii) suppose $(x,y) \in R$ and $(y,z) \in R$, say $y = px$, $z = qy$, for $p, q \in \mathbb{Z}$. Then $z = (pq)x$, so that $(x,z) \in R$.

$\therefore$ R is a pre-order on $\mathbb{Z}$.

HOWEVER $1 = -1 \cdot -1$, $-1 = 1 \cdot -1$,

so $(1,-1) \in R$, $(-1,1) \in R$.

$\therefore$ R is NOT a partial order on $\mathbb{Z}$.

Paper 10 Solution (ctd)

Let S be the equivalence relation derived from R. Then

$(x,y) \in S$ iff $|x| = |y|$.

$(\mathbb{Z}/_S, \leq)$ behaves in exactly the same way as $(\mathbb{N}, R_{\mathbb{N}})$, where

$R_{\mathbb{N}} = R \cap (\mathbb{N} \times \mathbb{N})$ is the restriction of R to the natural numbers.

In $(\mathbb{Z}/_S, \leq)$ there is a single maximal element $[0] = \{0\}$, and a single minimal element $[1] = \{1, -1\}$.