p 19.8
PR

Discrete mathematics – long question B

Suppose that A is a finite set with a bijection: $A \to A \times A$. Calculate $|A|$. [2 marks]

Given an example of a countably infinite set B with a bijection: $B \to B \times B$, proving the result carefully. [4 marks]

Let $M = \{ n \in \mathbb{N} \mid 2|n \}$, the even numbers,
$O = \mathbb{N} \setminus M$, the odd numbers,
$P = \mathcal{P}(\mathbb{N})$, the set of subsets of $\mathbb{N}$,
$Q = \mathcal{P}(M)$,
and $R = \mathcal{P}(O)$.

Show that P, Q and R are uncountable, and construct a bijection: $P \to Q \times R$. [12 marks]

Hence show that there is an uncountable set C with a bijection: $C \to C \times C$. [2 marks]

Solution

$|A| = |A|^2$ [1]
which has the solutions $|A| = 0$ and 1. [1]

$B = \mathbb{N}$. [1]
Map $\frac{1}{2}(m+n-1)(m+n-2) + n \leftrightarrow (m, n)$. [1]
Show injective and surjective. [1+1]

Use contradiction. Suppose there is a bijection $f\colon \mathbb{N} \leftrightarrow \mathcal{P}(\mathbb{N})$. [1]
Let $S = \{ n \in \mathbb{N} \mid n \notin f(n) \}$. [1]
Let $s = f^{-1}(S)$ so $S = f(s)$. [1]
Is $s \in S$? [1]

Bijection $\mathbb{N} \leftrightarrow M$ using $n \leftrightarrow 2n$. [1]
Induce bijection $P \leftrightarrow Q$. [2]
R similar. [1]

Given $X \in P$, let $Y = X \cap M$ and $Z = X \cap O$. Map $X \leftrightarrow (Y, Z)$. [2]
Bijection. [2]

Observe bijections $P \leftrightarrow Q$ and $P \leftrightarrow R$. Now $C = P$. [2]