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CST IB Semantics of Programming Languages 2002, Paper 6, question 9 AMP

(a)

2 Consider $\Omega = (\lambda x. xx)(\lambda x. xx)$.

Suppose, for a contradiction that $\Omega \Downarrow V$

1 holds for some V, and consider the <u>smallest</u> proof of this — it has to look like this:

$$\dfrac{\dfrac{}{(\lambda x.xx) \Downarrow (\lambda x.xx)}(1) \quad \dfrac{}{(\lambda x.xx) \Downarrow (\lambda x.xx)}(1) \quad \overset{P}{\bigvee}\; xx[(\lambda x.xx)/x] \Downarrow V}{\Omega \Downarrow V}(2)$$

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Since $xx[(\lambda x.xx)/x] = \Omega$, P is a strictly smaller

4 proof of $\Omega \Downarrow V$ — contradiction. So no such V exists.

(b) By part (a), we have that for any closed M

1 $\{(\Omega, M)\}$ trivially satisfies the condition to be

2 1 an applicative simulation, and hence $\Omega \leq M$.

(c)

1 Consider $Id \triangleq \{(M, M) \mid M \text{ closed}\}$. Clearly

1 it satisfies the condition to be an applicative

2 simulation. Hence $M \leq M$, all closed M.

(d) Let $\mathcal{S} \triangleq \{(M[V/x], (\lambda x.M)V) \mid M \text{ closed}, V \text{ closed } \lambda\text{-abs}^n.\}$

3 $\cup\ Id$

where Id is as in part (c). Claim that $\mathcal{S}$ is an applicative simulation.

For if $(M_1, M_2) \in \mathcal{S}$, then

1 <u>either</u> $M_1 = M_2$ and the required condition holds trivially (as in (c))

<u>or</u> $M_1 = M[V/x]$ & $M_2 = (\lambda x.M)V$ for some M, x, V. If $M_1 \Downarrow V_1$, then

$$\dfrac{\dfrac{}{\lambda x.M \Downarrow \lambda x.M}(1) \quad \dfrac{}{V \Downarrow V}(1) \quad \overset{\text{hypothesis}}{M[V/x] \Downarrow V_1}}{(\lambda x.M)V \Downarrow V_1}(2)$$

So $M_2 \Downarrow V_2$ with $V_2 = V_1$: so for all V'

2 $(V_1 V', V_2 V') = (V_1 V', V_1 V') \in \mathcal{S}$.

Thus $\mathcal{S}$ is indeed an applicative simulation

(6) and hence $M[V/x] \leq (\lambda x.M)V$, all M, V, x.

(e)

1 No.

Consider the case when $N = \Omega$ and

1 $M = \lambda y.y$.

If $M[N/x] \leq (\lambda x.M)N$, then

$$(M[N/x], (\lambda x.M)N) \in \mathcal{S}$$

for some applicative simulation $\mathcal{S}$. But $M[N/x] = \lambda y.y\,[\Omega/x] = \lambda y.y \Downarrow \lambda y.y$ by (1).

So must have $(\lambda x.M)N \Downarrow V_2$ for some V_2 such that for all V $((\lambda y.y)V, V_2 V) \in \mathcal{S}$. But

$(\lambda x.M)N \Downarrow V_2$ can only hold if it was deduced using (2), so in particular we must have $N \Downarrow V'$ for some V'. Since $N = \Omega$, by (a) this is impossible.

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So no such $\mathcal{I}$ can exist, and hence $(\lambda y.y)[\Omega/x] \not\leq (\lambda x.(\lambda y.y))\Omega$.

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