Continuous Mathematics Paper 4 q5 2002 RJG

[Relate to Fourier transforms]

(a) Fourier transform [2 marks]

$$F(\mu) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-i\mu x}\, dx$$

Inverse Fourier transform

$$f(x) = \int_{-\infty}^{\infty} F(\mu)\, e^{i\mu x}\, d\mu$$

(b) Shift rule [3 marks]

Fourier transform of $f(x-\alpha)$ is

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x-\alpha)\, e^{-i\mu x}\, dx$$

Put $u = x - \alpha$
$du = dx$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\, e^{-i\mu(u+\alpha)}\, du$$

$$= \frac{e^{-i\mu\alpha}}{2\pi}\int_{-\infty}^{\infty} f(u)\, e^{-i\mu u}\, du$$

$$= e^{-i\mu\alpha}\, F(\mu)$$

(c) Scale rule, $\alpha \neq 0$ [3 marks]

Fourier transform of $f(\alpha x)$ is

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\alpha x)\, e^{-i\mu x}\, dx$$

$$= \begin{cases} \dfrac{1}{2\pi\alpha}\displaystyle\int_{-\infty}^{\infty} f(u)\, e^{-i(\mu/\alpha)u}\, du & (\alpha > 0) \\ \dfrac{1}{2\pi\alpha}\displaystyle\int_{\infty}^{-\infty} f(u)\, e^{-i(\mu/\alpha)u}\, du & (\alpha < 0) \end{cases}$$

$$= \frac{1}{|\alpha|} F(\mu/\alpha) \qquad (\alpha \neq 0)$$

(d) $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ has F.T. $F(\mu)$ [6 marks]

$\frac{1}{\sqrt{2\pi}} e^{-x^2/2\sigma^2} = \frac{1}{\sqrt{2\pi}} e^{-(x/\sigma)^2/2}$ has F.T. $\sigma F(\sigma\mu)$
(scale rule)

$\frac{1}{\sqrt{2\pi}\,\sigma} e^{-x^2/2\sigma^2}$ has F.T. $\frac{1}{\sigma}\left(\sigma F(\sigma\mu)\right) = F(\sigma\mu)$
(multiplication by constant)

$\frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\delta)^2}{2\sigma^2}}$ has F.T. $e^{-i\mu\delta} F(\sigma\mu)$
(shift rule)

[6 marks]

(e)

$$G(\mu) = A \int_{-\infty}^{\infty} g(x)\, e^{-ia\mu x}\, dx$$

So, $$\frac{G(\mu)}{2\pi A} = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(x)\, e^{-ia\mu x}\, dx$$

So, $$\frac{G(\mu/a)}{2\pi A} = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(x)\, e^{-i\mu x}\, dx$$

— special case $a = 1$, $A^{-1} = 2\pi$

So, $$g(x) = \int_{-\infty}^{\infty} \left(\frac{G(\mu/a)}{2\pi A}\right) e^{i\mu x}\, dx \qquad \left(\frac{|a|}{2\pi A} = \right.$$

$$= \frac{1}{2\pi A} \int_{-\infty}^{\infty} G(\mu/a)\, e^{i\mu x}\, dx$$

$$= \frac{|a|}{2\pi A} \int_{-\infty}^{\infty} G(\mu)\, e^{i\mu a x}\, dx$$

(by scale rule