Mathematics for Computation Theory

Solution notes 2005 (JKMM)

We haven't had the Pumping Lemma since 2001, when for some reason I quoted a weakened form, which is inadequate for many of the potential applications. This is partly to set the record straight.

The material on which this question is based derives from lectures 11 and 12.

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a)

i) IS straight from the notes. Let

$x = s_1 s_2 \dots s_n y$, where $s_i \in S$, $1 \leq i \leq n$

are the first n characters of the string $x \in S^*$, $\ell(x) \geq n$, which M accepts.

Let $q_0 = 1$, the initial state of M,

and $q_i = f(q_{i-1}, s_i)$, $1 \leq i \leq n$,

where $f: (Q \times S) \to Q$ is transition fn.

M has n states, hence the $(n+1)$ states $\{q_i\}_{i=0}^{n}$ must be such that

$$q_j = q_k, \quad \text{for some } 0 \leq j < k \leq n.$$

If $j = 0$, set $u = \varepsilon$ (NULL string)

else $u = s_1 s_2 \ldots s_j$, $\ell(u) = j$.

Set $v = s_{j+1} s_{j+2} \ldots s_k$, $\ell(v) \geq 1$,
$\ell(uv) \leq n$

Set $w = s_{k+1} \ldots s_n \cdot y$.

Evidently applying $z_k = uv^k w$ to M leaves it in the same state as results from applying $x = z_1 = uvw$, i.e. ACCEPT

[8 marks]

a)

ii) If M accepts some word $y \in S^*$, suppose l is the length of some shortest word x accepted by M.

If $l \geq n$, write $x = uvw$ as in i). M also accepts $z_0 = uw$, $l(z_0) < l = l(z_1)$, CONTRADICTION

If M accepts ANY word x s.t. $l(x) \geq n$, then it accepts an infinite number, by part i).

Conversely, suppose M accepts an infinite set of words. Only a finite number have length $< 2n$, hence it accepts words $l(x) \geq 2n$.

Suppose $l \geq 2n$ is the length of some shortest word y accepted by M, subject to the condition $l(y) \geq 2n$.

Writing $y = uvw$ as in part i), word $z = uw$ is accepted by M, and $n \leq l(z) < 2n$. [5 marks]

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b) I've no idea what they'll make of these examples. The marking scheme is meant to be a hint.

i) evidently $x \in L_1$ iff $3 \mid l(x)$.
Hence $L_1 = \{(a+b)^3\}^*$ is regular

[3 marks]

(b) (ii) here an automaton to recognize L_2 will need to remember unbounded state, which sounds tricky. Use the Pumping Lemma.

Suppose if possible that an n-state DFM can be found to accept L_2.

Write $w = a^n b$, and consider the application of $x = ww = a^n.b.a^n.b$, which must be accepted. Using the Pumping Lemma, M also accepts some word $z_0 = a^m.b.a^n.b$, $m < n$.

But $z_0 \notin L_2$, CONTRADICTION.

Hence L_2 CANNOT be regular

[4 marks]