P4Q2

JGD

Model Answer, Continuous Mathematics, Question 2.

(A) By generalization of the Differentiation Theorem of Fourier Analysis, if a function $f(x)$ has Fourier Transform $F(\mu)$, then $f^{(\beta)}(x)$, the β^{th} derivative of $f(x)$ (where β is not necessarily an integer) has Fourier Transform: $(i\mu)^{\beta}F(\mu)$.

The inverse Fourier Transform of this expression then gives precise quantitative meaning to Fractional Differentiation.

Algorithm:

1. Go into the Fourier domain by taking the Fourier Transform of $f(x)$:

$$f(x) \stackrel{FT}{\Longrightarrow} F(\mu)$$

2. Multiply $F(\mu)$ by the desired β power of $(i\mu)$, in this case 1.5:

$$F(\mu) \stackrel{\beta}{\Longrightarrow} (i\mu)^{1.5}F(\mu)$$

which in this case amounts to a soft high-pass filtering plus a phase rotation.

3. Compute the inverse Fourier Transform of the result to obtain the desired new function:

$$(i\mu)^{1.5}F(\mu) \stackrel{FT^{-1}}{\Longrightarrow} \frac{d^{(1.5)}x}{dx^{(1.5)}}$$

(B)

Complex exponential form: $\sqrt{i} = \exp(i\pi/4)$	[1 mark]
This complex number lies in the first quadrant	[1 mark]
The real part of $\sqrt{i}$ is $\sqrt{2}/2$	[1 mark]
The imaginary part of $\sqrt{i}$ is also $\sqrt{2}/2$	[1 mark]
The length of this complex number is 1.	[1 mark]

(C) For partial differential equations expressing boundary value problems, there is no unique "starting point" nor direction of solution propagation. Such equations require a solution to satisfy a constraint along a perimeter. Simply propagating a solution in different directions will not lead to a consistent solution over the whole domain.

One general class of numerical methods for solving PDEs such as Poisson's Equation are <u>relaxation methods</u>. They proceed iteratively, until successive changes become vanishingly small. Divide up the domain into finite elements; impose the boundary value constraint; calculate the differential forces across each pair of finite elements according to the PDEs; and iteratively relax their differences with decreasing stepsizes until a stable and consistent solution is reached over all the finite elements and the boundary values.