Probability 2002 p 24 FHK

Consider n lengths of wire and the possible outcomes of joining the first pair. After arbitrarily selecting the first end of the pair there are $2n-1$ other ends to which it may be joined. One of these is the other end of the same length of wire. Two possibilities may be identified:

1. There is a probability of $\frac{1}{2n-1}$ that the joined ends belong to the same length of wire. This joining results in the first loop. The probability of ending up with r loops altogether is then the product of $\frac{1}{2n-1}$ and the probability that the remaining $n-1$ lengths result in $r-1$ loops. In this case:
$$\mathrm{P}(X_n = r) = \frac{1}{2n-1}\,\mathrm{P}(X_{n-1} = r-1) \tag{1}$$
2. There is a probability of $\frac{2n-2}{2n-1}$ that the joined ends belong to different lengths of wire. The first joining produces no loops but reduces the number of lengths of wire to $n-1$. The probability of ending up with r loops altogether is then the product of $\frac{2n-2}{2n-1}$ and the probability that the still-remaining $n-1$ lengths result in r loops. In this case:
$$\mathrm{P}(X_n = r) = \frac{2n-2}{2n-1}\,\mathrm{P}(X_{n-1} = r) \tag{2}$$

The two possibilities are disjoint so the addition rule is applicable. Summing (1) and (2) gives:
$$\mathrm{P}(X_n = r) = \frac{1}{2n-1}\,\mathrm{P}(X_{n-1} = r-1) + \frac{2n-2}{2n-1}\,\mathrm{P}(X_{n-1} = r)$$

[8 marks]

This result holds for $1 < r < n$. In the special cases $r = 1$ and $r = n$ the corresponding equations are, by inspection:
$$\mathrm{P}(X_n = 1) = \frac{2n-2}{2n-1}\,\mathrm{P}(X_{n-1} = 1) \quad \text{and} \quad \mathrm{P}(X_n = n) = \frac{1}{2n-1}\,\mathrm{P}(X_{n-1} = n-1)$$

[4 marks]

The distributions of X_1, X_2 and X_3 are conveniently arranged in a triangular array with $\mathrm{P}(X_1 = 1)$ at the apex:
$$\begin{array}{ccccc} & & 1 & & \\ & \frac{2}{3}.1 & & \frac{1}{3}.1 & \\ \frac{4}{5}.\frac{2}{3}.1 & & \frac{1}{5}.\frac{2}{3}.1 + \frac{4}{5}.\frac{1}{3}.1 & & \frac{1}{5}.\frac{1}{3}.1 \end{array}$$

[5 marks]

The Expectation $\mathrm{E}(X_3)$ is derived from the values of $\mathrm{P}(X_3 = r)$ just tabulated:
$$\mathrm{E}(X_3) = 1.\frac{8}{15} + 2.\frac{6}{15} + 3.\frac{1}{15} = \frac{23}{15}$$

[3 marks]