1999 Probability

p2 q5 answers
FHK

Notes on the Model Answer to Paper 2 Question 5

The probability of obtaining heads at turn n depends on whether or not the result at turn $n-1$ was heads. If the result was heads the probability is p, if not it is $\frac{1}{2}$. Taking u_{n-1} as the probability of obtaining heads at turn $n-1$ gives:

$$u_n = u_{n-1}.p + (1 - u_{n-1}).\tfrac{1}{2}$$

Hence:

$$2u_n - 2p\,u_{n-1} + u_{n-1} = 1$$
$$2u_n + (1-2p)u_{n-1} = 1$$

[4 marks]

If u_0 is taken as zero then the difference equation for the case $n = 1$ gives:

$$2u_1 = 1 \qquad \text{so} \qquad u_1 = \frac{1}{2}$$

This is the correct value for u_1 since it is stated that the fair coin is used at the first turn. Accordingly, if $u_0 = 0$, the equation holds for $n = 1$.

[2 marks]

To solve the equation first take the homogeneous case where the right-hand side is zero and guess $u_n = Aw^n$. This leads to the auxiliary equation:

$$2w + (1-2p) = 0 \qquad \text{so} \qquad w = p - \tfrac{1}{2}$$

Accordingly, the general solution to the original inhomogeneous equation is given by:

$$u_n = A(p - \tfrac{1}{2})^n + k$$

where k is some constant which is required to ensure that the right-hand side is 1 when this expression for u_n is fed into the original equation. This requires that k satisfies:

$$2k + (1-2p)k = 1 \qquad \text{so} \qquad k = \frac{1}{3-2p}$$

The general solution is therefore:

$$u_n = A(p - \tfrac{1}{2})^n + \frac{1}{3-2p}$$

Consider the case when $n = 0$ and note that $u_0 = 0$ to obtain:

$$0 = A.1 + \frac{1}{3-2p} \qquad \text{so} \qquad A = -\frac{1}{3-2p}$$

Accordingly:

$$u_n = \frac{1 - (p - \frac{1}{2})^n}{3-2p}$$

[14 marks]