P7 q11 ✓.
AMP

CST II 2000. Paper 7, q 11
Types

1 (var) $A, \Delta \vdash x : \tau$ if $(x:\tau) \in \Delta$

1 (abs) $$\frac{A, \Delta, x:\tau \vdash M : \tau'}{A, \Delta \vdash \lambda x.M : \tau \to \tau'}$$ if $x \notin dom(\Delta)$

1 (app) $$\frac{A, \Delta \vdash M_1 : \tau \to \tau' \quad A, \Delta \vdash M_2 : \tau}{A, \Delta \vdash M_1 M_2 : \tau'}$$

3

(a) Define $S(\tau_1)$ by recursion on its structure
$S(\alpha) \triangleq$ the type in $Typ(A')$ which S maps α to
2 $S(\tau \to \tau') \triangleq S(\tau) \to S(\tau')$

(b) TS is the function mapping $\alpha \in A$ to
1 $T(S(\alpha))$, using (a).

1 (c) S unifies τ_1 & τ_2 if $S(\tau_1) = S(\tau_2)$

(d) S is a mgu for τ_1 & τ_2 if
1 (i) it unifies τ_1 & τ_2
1 (ii) if $S : A \to Typ(A'')$ is any unifier for τ_1 & τ_2
2 then there is some $T : A' \to Typ(A'')$ with
$S' = TS$.

(e) (S, τ') is a <u>typing</u> for $A, \Delta \vdash M : ?$

1 if $\quad A', S\Delta \vdash M : \tau'$

is derivable from (var)+(abs)+(app).

1 Here $S\Delta \stackrel{\text{def}}{=} \{x_1 : S(\tau_1), \ldots, x_n : S(\tau_n)\}$

if Δ is $\{x_1 : \tau_1, \ldots, x_n : \tau_n\}$.

2

(f) (S, τ') is a <u>principal typing</u> for $A, \Delta \vdash M : ?$ if

1 (i) it is a typing, and

(ii) for any typing (S', τ'') of $A, \Delta \vdash M : ?$, with $S' : A \to \mathrm{Typ}(A'')$ & $\tau'' \in \mathrm{Typ}(A'')$ say, then there is some $T : A' \to \mathrm{Typ}(A'')$ with

1 $$S' = TS$$

1 and $$\tau'' = T(\tau')$$

3

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1 $M_1 \triangleq \lambda x.x$ has typing $(S, \alpha \to \alpha)$ where $S : \emptyset \to \mathrm{Typ}(\{\alpha\})$ is unique. For

$$\{\alpha\}, \emptyset \vdash \lambda x.x : \alpha \to \alpha$$

holds by applying (abs) to

$$\{\alpha\}, \{x : \alpha\} \vdash x : \alpha$$

1 which holds by (var).

1 $M_2 \triangleq \lambda x.(x\,x)$ has no typing. For suppose it did — then we'd have

(1) $$A', \emptyset \vdash \lambda x.(x\,x) : \tau'$$

derivable, for some A' and $\tau' \in Typ(A')$. The proof of (1) has to look like

$$\dfrac{\dfrac{\dfrac{}{A', x:\tau_1 \vdash x:\tau_3}\,\text{② (var)} \qquad \dfrac{}{A', x:\tau_1 \vdash x:\tau_4}\,\text{③ (var)}}{A', x:\tau_1 \vdash x\,x : \tau_2}\,\text{④ (app)}}{A', \emptyset \vdash \lambda x.(x\,x) : \tau'}\,\text{⑤ (abs)}$$

for some $\tau_1, \tau_2, \tau_3, \tau_4$ satisfying

② $\tau_1 = \tau_3$

③ $\tau_1 = \tau_4$

④ $\tau_3 = \tau_4 \to \tau_2$

⑤ $\tau' = \tau_1 \to \tau_2$.

Hence $\tau_1 \overset{②}{=} \tau_3 \overset{④}{=} \tau_4 \to \tau_2 \overset{③}{=} \tau_1 \to \tau_2$

2 making τ_1 a proper subexpression of itself — which is impossible.

5 So no such typing exists.

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1 Yes, if a partial typing judgement has a typing, it has a principal one (special case of the Hindley-Damas-Milner theorem).