SOLUTION NOTES

Logic and Proof 2002 Paper 5 Question 11 (LCP)

(a) Satisfiability and validity were defined in the first lecture, and again more formally in the second lecture (on propositional logic) and in the fifth lecture (on first-order logic). Methods for solving these questions were presented in the sixth lecture (on formal reasoning).

$((Q \to R) \to Q) \land \neg Q$ is unsatisfiable. If Q is true it obviously evaluates to false, and if Q is false the result is the same.

$((P \leftrightarrow Q) \leftrightarrow P) \leftrightarrow Q$ is valid. If Q is true then it simplifies to $P \leftrightarrow P$, while if Q is false it simplifies to $\neg(\neg P \leftrightarrow P)$.

$\exists xy\,[P(x,y) \to \forall xy\,P(x,y)]$ can be rewritten as $\exists xy\,[\neg P(x,y) \lor \forall xy\,P(x,y)]$ and then (reducing the quantifier scope) to $\exists xy\,\neg P(x,y) \lor \forall xy\,P(x,y)$ and finally to $\neg(\forall xy\,P(x,y)) \lor \forall xy\,P(x,y)$, which is obviously valid.

$[\forall x\,(P(x) \to Q(x)) \land \exists x\,P(x)] \to \forall x\,Q(x)$ is satisfiable: it is true in any interpretation in which $Q(x)$ is true for all arguments. It is not valid: if P and Q have the same meaning, then it collapses to $\exists x\,P(x) \to \forall x\,P(x)$, which obviously is false in general.

(b) This comes from the fifth lecture (on first-order logic). "Briefly outline" means candidates should present the main points. They are not expected to reproduce the full Tarski truth definition. They should state that an interpretation consists of a non-empty universe and assigns meanings to the constants, function symbols and predicates. To handle quantifiers, the semantics can also take a variable interpretation. The meaning of a formula is defined by recursion, which for quantifiers involves considering all possible assignments to the bound variable. Equality denotes itself, while the propositional connectives are evaluated by truth tables.

The formula $\forall xy\,f(x,y) = f(y,x)$ will be true in a model in which the symbol f denotes a commutative function. Formally, it asserts that f denotes a function over the chosen universe (say U) such that for each x and y in U, the value of $f(x,y)$ equals that of $f(y,x)$.

(c) This question again refers to the fifth lecture. The universe can be the set of natural numbers (or integers, reals, etc.) The constant a can denote 0, or any particular number. The function $g(x)$ denotes $x + 1$.