Paper 10, Q8 Solution notes [3

The question is shamelessly lifted from CST IA, 1996 Paper 1, though the detail is somewhat different.

The material on which it is based derives from lectures 4 and 5.

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a) $\leq$ is REFLEXIVE, i.e $\Delta_S \subseteq \leq$

$\leq$ is ANTI-SYMMETRIC,

i.e $\leq \cap \leq^{-1} = \Delta_S$ (or $\subseteq$)

$\leq$ is TRANSITIVE,

i.e $\leq \circ \leq = \leq$ (or $\subseteq$)

COULD rate this individually

[2 marks]

Computer Science Tripos Part II (General) 2005

Paper 10 Question 8

JKMM — Mathematics for Computation Theory

b) for a TOTAL ordering, the extra condition is TRICHOTOMY (!)

$$\leq \; \cup \; \leq^{-1} \; = \; (S \times S)$$

COULD rate this individually

[1 mark]

c) a well-ordered set is a totally ordered set in which every descending sequence is ultimately constant.

COULD give the notes definition of well-founded, and say "total + w-f".

COULD rate this individually

[2 marks]

The product ordering $\leq_p$ on $(\mathbb{N} \times \mathbb{N})$ is defined as follows:

$$(x_1, y_1) \leq_p (x_2, y_2) \quad \text{iff}$$

$$(x_1 \leq x_2) \ \wedge \ (y_1 \leq y_2).$$

This is a partial but NOT a total ordering. For that reason, it is NOT well-ordered.

[3 marks]

On the other hand, $\leq_p$ IS well-founded, according to the definition given in the notes. Anyone who draws attention to this may be liable for

[1 bonus mark]

$f : (S, \leq) \to (T, \prec)$ is monotonic

if whenever $s_1 \leq s_2$ in S,

$f(s_1) \prec f(s_2)$ in T.

[2 marks]

One reason for setting this question is to place topological sort on record. I've no idea how they'll handle it – I expect few if any of them to have looked at 9-year old part 1A questions, and if they have, good luck to them!

They should immediately think of the lexicographic ordering as one topological sort.

Define $\leq_L$ on $(\mathbb{N} \times \mathbb{N})$ as follows:

$(x_1, y_1) \leq_L (x_2, y_2)$ iff

$(x_1 < x_2)$ OR

$(x_1 = x_2)$ AND $(y_1 \leq y_2)$.

This defines a total ordering, indeed a well-ordering, on $(\mathbb{N} \times \mathbb{N})$. Further,

$(x_1, y_1) \leq_P (x_2, y_2)$

$\iff (x_1 \leq x_2)$ AND $(y_1 \leq y_2)$

$\implies (x_1 < x_2)$ OR

$(x_1 = x_2)$ AND $(y_1 \leq y_2)$

i.e $(x_1, y_1) \leq_L (x_2, y_2)$

Hence $\leq_L$ is a topological sort.

$$X = \{ x_n = (0, n) \mid n \in \mathbb{N} \}$$

is an increasing sequence, bounded above by $(1, 0)$ in $\leq_L$. Hence this sort cannot be isomorphic to $\mathbb{N}$ in the standard ordering

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Alternatively, we can enumerate the points of $(\mathbb{N} \times \mathbb{N})$ diagonally.

Define $\leq_D$ on $(\mathbb{N} \times \mathbb{N})$ as follows:

$(x_1, y_1) \leq_D (x_2, y_2)$ iff

$(x_1 + y_1) < (x_2 + y_2)$ OR

$(x_1 + y_1) = (x_2 + y_2)$

AND $x_1 \leq x_2$.

This defines a total ordering, indeed a well-ordering, on $(\mathbb{N} \times \mathbb{N})$. Further,

$$(x_1, y_1) \leq_p (x_2, y_2)$$

$$\Longleftrightarrow \quad (x_1 \leq x_2) \text{ AND } (y_1 \leq y_2)$$

$$\Longrightarrow \quad (x_1 + y_1) < (x_2 + y_2) \text{ OR}$$

$$(x_1 + y_1) = (x_2 + y_2) \text{ AND } (x_1 \leq x_2)$$

i.e. $(x_1, y_1) \leq_D (x_2, y_2)$.

Hence $\leq_D$ is also a topological sort.

Further, the diagonal enumeration of the points of $((\mathbb{N} \times \mathbb{N}), \leq_D)$ is evidently an order isomorphism with $(\mathbb{N}, \leq)$.

[10 marks]