1

1999 CST Part II Denotational Semantics, paper 9, q 10

AMP

Closed PCF terms $M, M' : \tau$ are contextually equivalent, $M_1 \cong_{ctx} M_2 : \tau$, if:

for all contexts $\mathcal{C}[-]$ satisfying $\mathcal{C}[M_1], \mathcal{C}[M_2] : \gamma$ for $\gamma = bool$ or $\gamma = nat$ and for all values $V : \gamma$, it is the case that

$$\mathcal{C}[M_1] \Downarrow_\gamma V \iff \mathcal{C}[M_2] \Downarrow_\gamma V.$$

5

<u>Failure of full abstraction</u>: for all PCF types τ, and closed terms $M_1, M_2 : \tau$, whilst one does have

$$[\![M_1]\!] = [\![M_2]\!] \in [\![\tau]\!] \implies M_1 \cong_{ctx} M_2 : \tau$$

2 the converse implication does not necessarily hold.

For example, define

$$M_i \stackrel{def}{=} \text{fn } f : bool \to (bool \to bool).$$
$$\quad \text{if } f \text{ true } \Omega \text{ then}$$
$$\quad\quad \text{if } f\, \Omega \text{ true then}$$
$$\quad\quad\quad \text{if } f \text{ false false then } \Omega \text{ else } B_i$$
$$\quad\quad \text{else } \Omega$$
$$\quad \text{else } \Omega$$

2 where $B_1 \stackrel{def}{=} true$, $B_2 \stackrel{def}{=} false$ & $\Omega \stackrel{def}{=} fix\, x : bool.\, x$.

Then claim:

(1) $[\![M_1]\!] \neq [\![M_2]\!] \in (\mathbb{B}_\perp \to (\mathbb{B}_\perp \to \mathbb{B}_\perp)) \to \mathbb{B}_\perp$.

(2) $M_1 \cong_{ctx} M_2 : (bool \to (bool \to bool)) \to bool$.

Proof of (1)

We use the parallel-or function $por \in \mathbb{B}_\perp \to (\mathbb{B}_\perp \to \mathbb{B}_\perp)$, the unique continuous (indeed, unique monotone) function satisfying

$$por\ true\ \perp = true$$
$$por\ \perp\ true = true$$
2 $$por\ false\ false = false$$

(all the other values of $por\ b\ b'$ can be deduced from these by monotonicity).

By definition of M_1, M_2 we can calculate that

$$\begin{cases} [\![M_1]\!](por) = [\![B_1]\!] = true \\ [\![M_2]\!](por) = [\![B_2]\!] = false \end{cases}$$ 2

and hence $[\![M_1]\!] \neq [\![M]\!]_2$.

Proof of (2)

We use the following extensionality properties of $\cong_{ctx}$:

(3) For all $F_1, F_2 : \tau \to \tau'$

$$(\forall A : \tau .\ F_1 A \cong_{ctx} F_2 A : \tau') \Rightarrow F_1 \cong F_2 : \tau \to \tau'$$

(4) For all $B_1, B_2 : bool$

2 $$(\forall\ V \in \{true, false\}.\ B_1 \Downarrow_{bool} V \Leftrightarrow B_2 \Downarrow_{bool} V)$$
$$\Rightarrow B_1 \cong_{ctx} B_2 : bool.$$

So to prove (2), by (3)&(4) it suffices to show that for all $A : bool \to (bool \to bool)$ and for $V \in \{true, false\}$ that

$$(5) \qquad M_1 A \Downarrow_{bool} V \iff M_2 A \Downarrow_{bool} V$$

This holds because in fact for all such A $M_i A$ diverges $(i = 1, 2)$.
For, by definition of M_i and of evaluation $M_i A \Downarrow_{bool} V$ holds only if

$$(6) \quad \begin{cases} M_i\, true\, \Omega \Downarrow_{bool} true \\ M_i\, \Omega\, true \Downarrow_{bool} true \\ M_i\, false\, false \Downarrow_{bool} false \end{cases}$$

Apply the <u>Soundness property</u> of $[\![\ ]\!]$ w.r.t $\Downarrow$:

2 $$M \Downarrow_\tau V \Rightarrow [\![M]\!] = [\![V]\!] \in [\![\tau]\!]$$

to (6) to deduce

$$\begin{cases} [\![M_i]\!]\, true\, \perp = true \\ [\![M_i]\!] \perp true = true \\ [\![M_i]\!]\, false\, false = false \end{cases}$$

(using fact that $[\![\Omega]\!] = \perp \in \mathbb{B}_\perp$)

1 So by the property uniquely defining por, we have $[\![M_i]\!] = por$. But

2 <u>Fact</u>: $\nexists\, M : bool \to (bool \to bool)$ with $[\![M]\!] = por$.

15 Hence can't have $M_i A \Downarrow_{bool} V$; so (5) does indeed hold.

(20)