p1q8 2005
GW

Discrete Maths.

(a) A bijection is a function $f : A \to B$ which is

both injective : $f(a) = f(a') \Rightarrow a = a'$ for all $a, a' \in A$.
and surjective : $\forall b \in B\, \exists a \in A.\ f(a) = b$.

The bijection
$\theta : (A \times B \to C) \to (A \to (B \to C))$ is defined by

$$\theta(f) = \lambda a \in A\ \lambda b \in B.\ f(a,b)$$

Its inverse θ^{-1} is defined by

$$\theta^{-1}(g) = \lambda(a,b).\ (g(a))(b).$$

(b) From the hint it suffices to give injections into $\mathbb{N}$ to establish countability.

(i) As A, B are countable there are injections $f : A \to \mathbb{N}$ and $g : B \to \mathbb{N}$. Define the injection $h : A \times B \to \mathbb{N}$ by

$$h(a,b) = 2^{f(a)} 3^{g(b)}.$$

(ii) As A, B are countable they are associated with injections f, g as above. Define the injection $h : A \cup B \to \mathbb{N}$ by

$$h(x) = \begin{cases} 2^{f(x)} & \text{if } x \in A \\ 3^{g(x)} & \text{if } x \in B \setminus A. \end{cases}$$

(iii) Suppose $\mathcal{P}(\mathbb{N})$ were countable, i.e. that were an injection $\theta : \mathcal{P}(\mathbb{N}) \to \mathbb{N}$.

Define $Y = \{ n \in \mathbb{N} \mid \exists X.\ n = \theta(X) \ \& \ n \notin X \}$.

Then $\theta(Y) = m$. Either $m \in Y$ or $m \notin Y$. But if $m \in Y$ then $m \notin Y$, and if $m \notin Y$ then $m \in Y$ — a contradiction.

(iv) $f : \mathcal{P}_{fin}(\mathbb{N}) \to \mathbb{N}$ is an injection where

$$f(X) = \prod_{x \in X} p_x$$ where p_x is the xth prime.

(c) If $A = \{ x \mid x \text{ is a set} \}$ were a set, then by comprehension $\{ y \in A \mid y \notin y \}$ i.e. Russell's ~~set~~ collection would ~~be~~ be a set. — contradiction.

(d) Eg. define $\prec = \{ (r, b) \mid r \in \mathbb{R} \}$ where $b \notin \mathbb{R}$. Then $\prec$ is a well-founded relation on $\mathbb{R} \cup \{b\}$.