Numerical Analysis I Question 6 2003 p4q7 MROD

Solution Notes

(a) To solve a system of the shape

$$\begin{bmatrix} \times & \times & \times & \times & \times \\ & \times & \times & \times & \times \\ & & \times & \times & \times \\ & & & a_{n-1,n-1} & a_{n-1,n} \\ & & & & a_{nn} \end{bmatrix} \begin{bmatrix} \cdots \\ \cdots \\ \cdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} \cdots \\ \cdots \\ \cdots \\ b_{n-1} \\ b_n \end{bmatrix}$$

we can first solve the nth equation

$$a_{nn} x_n = b_n$$

$$x_n = b_n / a_{nn}$$

and hence substitute in the $(n-1)$th equation to find x_{n-1}, etc. This is back substitution and is important because it is the last step in Gaussian elimination, and is also used in the Choleski factorisation method. Forward substitution applies to a lower triangular matrix

$$\begin{bmatrix} \times & & & & \\ \times & \times & & & \\ \times & \times & \times & & \\ \times & \times & \times & \times & \\ \times & \times & \times & \times & \times \end{bmatrix}$$

in which the first equation is solved first, the the second, etc.

[5 marks]

(b) The matrix A is symmetric if $A = A^T$, and positive definite if $\underline{x}^T A \underline{x} > 0$ for any $\underline{x} \neq \underline{0}$.

[2 marks]

(c) This is an example of a Choleski factorisation $A = LDL^T$ but, in this case $D = I$, so $A = LL^T$. We need to solve

$$LL^T \underline{x} = \underline{b} .$$

We write

$$L^T \underline{x} = \underline{y}$$

so

$$L \underline{y} = \underline{b} .$$

PTO.

Now $L\underline{y} = \underline{b}$ is a lower triangular system, i.e.

$$\begin{bmatrix} 1 & \\ 2 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

Using forward substitution, $y_1 = 3$, $y_2 = -2$.

Then $L^T\underline{x} = \underline{y}$ is an upper triangular system, i.e.

$$\begin{bmatrix} 1 & 2 \\ & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}.$$

Using back substitution, $x_2 = -2$, $x_1 = 7$.

So the solution is $\underline{x} = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$.

[6 marks]

(d) Using a compact notation

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & -1 & -4 \\ 1 & 0 & -1 \end{bmatrix}\left[\begin{array}{ccc|c} 3 & 4 & 1 & 16 \\ 0 & 8 & 2 & 14 \\ 3 & 2 & 5 & 8 \end{array}\right] = \left[\begin{array}{ccc|c} 3 & 4 & 1 & 16 \\ 0 & 0 & -18 & 18 \\ 0 & 2 & -4 & 8 \end{array}\right].$$

Observe that the rows can now be re-ordered to upper triangular form

$$\begin{bmatrix} 3 & 4 & 1 \\ & 2 & -4 \\ & & -18 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 16 \\ 8 \\ 18 \end{bmatrix}.$$

Solve by back substitution

$$x_3 = -1,$$
$$x_2 = 2,$$
$$x_1 = 3.$$

So the solution is $\underline{x} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$.

[7 marks]