Numerical Analysis I 2000

p4 q9
p11 q10
MROD

Acceptable Answer B

Let x^* denote the floating point representation of x.

Absolute error is defined by $x^* = x + \epsilon_x$.

Relative error is defined by $x^* = x(1+\delta_x) = x + x\delta_x$.

Comparing these definitions,

$$\epsilon_x = x\delta_x,$$

$$\text{so} \quad \delta_x = \frac{\epsilon_x}{x} \quad \text{if } x \neq 0.$$

Loss of significance occurs when relative error grows much larger during the course of a calculation.

Machine epsilon is the smallest $\epsilon_m > 0$ such that

$$(1+\epsilon_m)^* > 1.$$

[4 marks]

$$|\delta_{xy}| \leq |\delta_x| + |\delta_y|$$

$$|\epsilon_{xy}| \leq |xy|\left(|\delta_x| + |\delta_y|\right)$$

$$|\epsilon_{xy+z}| \leq |\epsilon_{xy}| + |\epsilon_z|$$

$$\leq |xy|\left(|\delta_x| + |\delta_y|\right) + |z\delta_z|$$

$$|\delta_{xy+z}| \leq \frac{|xy|\left(|\delta_x| + |\delta_y|\right) + |z\delta_z|}{|xy+z|} ; \quad xy+z \neq 0$$

[6 marks]

If $|\delta_x| = |\delta_y| = |\delta_z| = \epsilon_m$

$$\frac{|\delta_{xy+z}|}{\epsilon_m} \leq \frac{2|xy| + |z|}{|xy+z|}$$

We expect loss of significance if $|xy| \simeq |z|$ and xy and z have opposite signs.

[3 marks]

PTO.

(a) $h = 10^{-3}$ gives a poor estimate because the discretization error $\frac{h}{2}|f''(x)|$ is too large.

(b) $h = 10^{-8}$ gives a poor estimate because the rounding error $\frac{\epsilon_m}{h}|f(x)|$ is too large. [4 marks]

As $f(0.2) = O(1)$ then $h = \sqrt{\epsilon_m} = 10^{-5}$ should be a more suitable value. In this case

$$\text{discretization error} \simeq \text{rounding error} \simeq h$$

so roughly 5 significant decimal digits would be expected in the answer. [3 marks]