2005
MROD

Numerical Analysis II Paper 9 Question 12

Solution Notes

Context: singular value decomposition.

(a) A is positive semi-definite if $\underline{x}^T A \underline{x} \geqslant 0$ for any vector $\underline{x}$.

Now

$$(A^TA)^T = A^T(A^T)^T = A^TA$$

so A^TA is symmetric. Also

$$\underline{x}^T(A^TA)\underline{x} = (A\underline{x})^T(A\underline{x}) \geqslant 0$$

so A^TA is semi-definite. [3 marks]

(b) $\|A\|_2 = \sqrt{\text{maximum eigenvalue of } A^TA}$

$= \text{maximum singular value of } A.$

Schwarz's inequality is

$$\|A\underline{x}\|_V \leqslant \|A\|_M \cdot \|\underline{x}\|_V$$

where $\|.\|_V$ is a vector norm and $\|\cdot\|_M$ is the compatible matrix norm. [2 marks]

(c) Let $\lambda_1, \lambda_2, \ldots \lambda_n$ be the eigenvalues of A^TA in non-increasing order. If $\sigma_j = \sqrt{\lambda_j}$ then $\sigma_1, \sigma_2, \ldots \sigma_n$ are the singular values of A. W is the diagonal matrix of singular values, i.e. $\text{diag}(\sigma_1, \sigma_2, \ldots \sigma_n)$.
V is the square matrix of eigenvectors corresponding to $\{\lambda_j\}$ so that $V^TV = I$.
If A is an $m \times n$ matrix then U is also $m \times n$ and is such that $U^TU = I_n$. [3 marks]

(d) $A\underline{e} = A\underline{x} - A\hat{\underline{x}} = b - A\hat{\underline{x}} = \underline{r}$

so $\underline{e} = A^{-1}\underline{r}$

$\|\underline{e}\| = \|A^{-1}\underline{r}\|$

$\leqslant \|A^{-1}\|.\|\underline{r}\|$ by Schwarz

over

Dividing by $\|\underline{x}\|$ we get

$$\frac{\|\underline{e}\|}{\|\underline{x}\|} \leqslant \|A^{-1}\| . \frac{\|\underline{r}\|}{\|\underline{x}\|}$$

but we cannot compute the right-hand side as we do not know $\|\underline{x}\|$. But

$$\|\underline{b}\| = \|A\underline{x}\| \leqslant \|A\| . \|\underline{x}\| \quad \text{by Schwarz}$$

so

$$\frac{\|\underline{e}\|}{\|\underline{x}\|} \leqslant \|A\| . \|A^{-1}\| . \frac{\|\underline{r}\|}{\|\underline{b}\|}$$

and the right-hand side is computable.

If the ℓ_2 norm is used, the condition number

$$K_n = \|A\|_2 . \|A^{-1}\|_2 = \frac{\sigma_1}{\sigma_n} .$$

[8 marks]

(e) $K_3 = \dfrac{10^2}{10^{-10}} = 10^{12}$, $K_4 = \dfrac{10^2}{10^{-16}} = 10^{18}$, $K_5 = \dfrac{10^2}{10^{-22}} = 10^{24}$, $K_6 = \dfrac{10^2}{10^{-29}} = 10^{31}$.

(i) $\epsilon_m = 10^{-15}$ and $K_4 > \dfrac{1}{\epsilon_m}$ so choose rank 3

$$w^+ = \operatorname{diag}(10^{-2}, 10^4, 10^{10}, 0, 0, 0, 0).$$

(ii) $\epsilon_m = 10^{-30}$ and $K_6 > \dfrac{1}{\epsilon_m}$ so choose rank 5

$$w^+ = \operatorname{diag}(10^{-2}, 10^4, 10^{10}, 10^{16}, 10^{22}, 0, 0).$$

[4 marks]

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