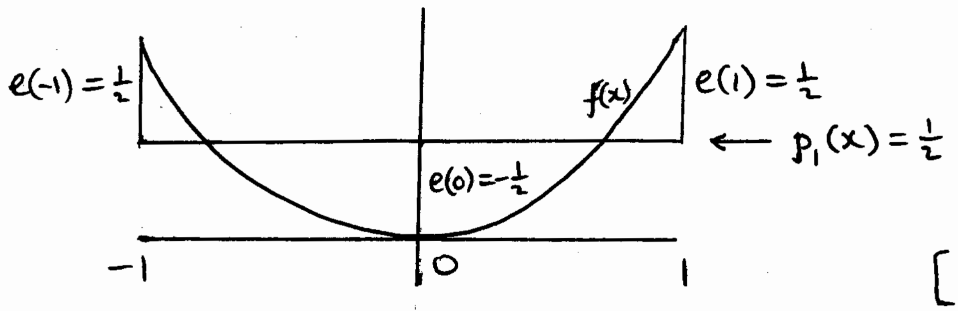
Numerical Analysis II Paper 8 Question 12

Solution Notes

Context: Chebyshev characterisation theorem.

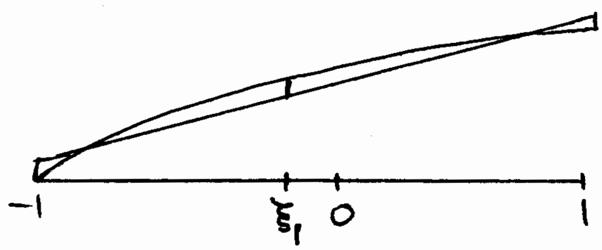
(a)



[3 marks]

(b) $f(x) = \dfrac{x+1}{x+\frac{5}{3}}$. $f(-1)=0$, $f(0)=0.6$, $f(1)=0.75$

$$f'(x) = \frac{\cancel{x}+\frac{5}{3}-(\cancel{x}+1)}{(x+\frac{5}{3})^2} = \frac{\frac{2}{3}}{(x+\frac{5}{3})^2} > 0 \text{ for all } x$$



Clearly ± 1 are extrema of $e(x)$. The third extremum in $[-1,1]$ is at ξ_1 where $e'(\xi_1)=0$, and $\xi_1<0$.

Write $p_1(x) = ax+b$.

$$e(x) = \frac{x+1}{x+\frac{5}{3}} - (ax+b)$$

$$e(-1) = a-b \quad , \quad e(+1) = \tfrac{3}{4}-a-b$$

From the theorem, $e(-1) = -e(\xi_1) = e(+1)$

so $a-b = \frac{3}{4}-a-b \quad \therefore a = \frac{3}{8}$.

$$e'(x) = \frac{\frac{2}{3}}{(x+\frac{5}{3})^2} - a$$

so $e'(\xi_1)=0 \Rightarrow (\xi_1+\frac{5}{3})^2 = \frac{2}{3a} = \frac{16}{9}$

$$\therefore \xi_1 + \tfrac{5}{3} = \tfrac{4}{3}$$

$$\therefore \xi_1 = -\tfrac{1}{3}.$$

over

Now $e(-1) = -e(-\frac{1}{3})$

so $\frac{5}{8} - b = -\frac{3}{8} + b \quad \therefore b = \frac{1}{2}.$

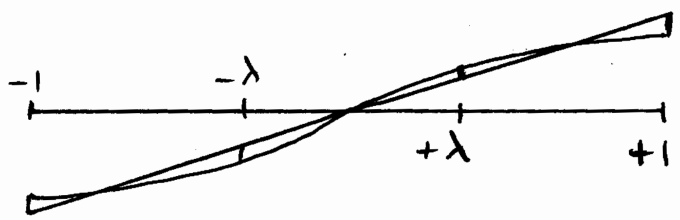
so $p_1(x) = \frac{3}{8}x + \frac{1}{2}.$ [9 marks]

(c) If $f(x) = x/(9x^2+16)$ then $f(-x) = -f(x)$, i.e. $f(x)$ is an odd function, so the best approximation only contains odd powers of x, by symmetry, and must be of the form $p_2(x) = ax$.

$f(0) = 0, \ f(1) = 1/25.$

$$f'(x) = \frac{9x^2+16-18x^2}{(9x^2+16)^2} = \frac{16-9x^2}{(9x^2+16)^2} > 0 \text{ for } x \in [-1,1]$$

$f'(0) = 1/16, \ f'(1) = 7/625$



$e'(x) = f'(x) - a$

$$e'(\lambda) = 0 \Rightarrow a = f'(\lambda) = \frac{16-9\lambda^2}{(9\lambda^2+16)^2} \quad (1)$$

Also $e(1) = -e(\lambda)$ so

$$\frac{1}{25} - a = a\lambda - \frac{\lambda}{9\lambda^2+16}$$

$$\therefore a(1+\lambda) = \frac{9\lambda^2+25\lambda+16}{25(9\lambda^2+16)} = \frac{(9\lambda+16)(1+\lambda)}{25(9\lambda^2+16)}$$

$$\therefore a = \frac{9\lambda+16}{25(9\lambda^2+16)} \quad (2)$$

If $\lambda = 4/9$ then (1) and (2) should yield the same value of a.

(1) gives $a = \dfrac{16 - 16/9}{(16/9+16)^2} = \dfrac{9}{16} \cdot \dfrac{8}{100} = \dfrac{9}{200}.$

(2) gives $a = \dfrac{4+16}{25(\frac{16}{9}+16)} = \dfrac{20.9}{25.160} = \dfrac{9}{200}.$

[8 marks]