SOLUTION NOTES

Complexity Theory 2001 Paper 5 Question 12 (AD)

(*a*) The class NP contains all problems that can be solved by a non-deterministic machine in polynomial time. Or equivalently, it contains all problems for which a solution can be verified in polynomial time.

A problem is NP-complete if it is in NP and every problem in NP is polynomial time reducible to it.

[2 marks each]

(*b*) (*i*) A solution (or certificate) for the problem consists of a map $\chi : V \to \{c_1, \ldots, c_k\}$ satisfying the condition if $(u, v) \in E$, then $\chi(u) \neq \chi(v)$. This certificate can be verified by examining each edge only once. This is therefore a polynomial time algorithm.

[2 marks]

(*ii*) If $k = 1$, the only graphs that are k-colourable are the ones with no edges. This is in PTIME. If $k = 2$, a graph is k-colourable if, and only if, it contains no odd cycles. This can be checked in polynomial time. [2 marks]

(*iii*) For any value of $k \geq 3$, the problem is NP-complete. This can be proved first for $k = 3$ by a reduction from propositional satisfiability, and for larger k by reduction from 3-colourability.

[2 marks]

(*c*) (*i*) The problem is to check whether a particular graph is 20-colourable. The problem could be solved by a general purpose graph colouring algorithm – essentially exhaustive search, or by using heuristics that are specific to the problem domain.

[2 marks]

(*ii*) The problem that the company is trying to solve is a single instance. Thus, it is not sensible to speak of its asymptotic complexity. However, a solution is likely to employ a general purpose algorithm. Since the problem 20-colourability is NP-complete, the only such general purpose algorithms are likely to be of exponential complexity, and thus highly inefficient on a graph with 2000 nodes.

[3 marks]

(*d*) For the short note – observe that the complexity of the exhaustive search algorithm is exponential. Thus, if 2000 masts required a week, double the number of nodes is likely to be completely impractical. Two weeks would certainly not suffice. As alternatives – use heuristics specific to the problem to narrow down the search. For instance, use the fact that only masts within 50 miles of each other are connected by an edge to put a limit on the valency of the graph. Another alternative would be to reanalyse the problem as an optimization problem and use an approximation algorithm.

[5 marks]