p9 q13 ✓.
AMP

CST II 2000. Paper 9, q P3

Types

PLC typing judgements take the form

$$\Gamma \vdash M : \tau$$

where $\Gamma = (\Gamma_{tv}, \Gamma_{ta})$

Γ_{tv} = finite set of type variables

Γ_{ta} = finite function from variables to types

M = PLC expression

τ = PLC type.

Write "Γ ok" to mean free type variables of types in Γ_{ta} are contained in Γ_{tv}.

The valid typing judgements are inductively generated by the following axiom & rules

1 (var) $\Gamma \vdash x : \tau$ if Γ ok & $(x:\tau) \in \Gamma_{ta}$

1 (fn) $$\frac{\Gamma, x:\tau_1 \vdash M : \tau_2}{\Gamma \vdash \lambda x:\tau_1 (M) : \tau_1 \to \tau_2} \quad x \notin dom(\Gamma_{ta})$$

2 (gen) $$\frac{\alpha, \Gamma \vdash M : \tau}{\Gamma \vdash \Lambda \alpha (M) : \forall \alpha (\tau)} \quad \alpha \notin \Gamma_{tv} \ \& \ \Gamma_{ok}$$

1 (spec) $$\frac{\Gamma \vdash M : \forall \alpha (\tau)}{\Gamma \vdash M\tau' : \tau[\tau'/\alpha]} \quad ftv(\tau') \subseteq \Gamma_{tv}.$$

1 (app) $$\frac{\Gamma \vdash M : \tau_1 \to \tau_2 \quad \Gamma \vdash M' : \tau_1}{\Gamma \vdash M M' : \tau_2}$$

6

1 $\rho(\forall\alpha(\alpha\to\alpha)) = \rho[\alpha\mapsto true](\alpha\to\alpha) \,\&\, \rho[\alpha\mapsto false](\alpha\to\alpha)$

$= (true \Rightarrow true) \,\&\, (false \Rightarrow false)$

$= true \,\&\, true$

$= true$

1 $\rho(\forall\alpha(\alpha)) = \rho[\alpha\mapsto true](\alpha) \,\&\, \rho[\alpha\mapsto false](\alpha)$

$= true \,\&\, false$

(2) $= false$

Writing $\Phi(\Gamma, M, \tau)$ to mean

$\forall\rho.\ \rho(\tau) = false \Rightarrow \exists\,(x':\tau') \in \Gamma_{ta}.\ \rho(\tau') = false$,

we show Φ is closed under (var)—(spec) and hence that if $\Gamma \vdash M : \tau$ holds then so does $\Phi(\Gamma, M, \tau)$.

1 Case (var): — trivial.

Case (fn): Assume

(1) $\Phi(\Gamma, x:\tau_1, M, \tau_2)$ with $x \notin dom(\Gamma_{ta})$.

If $\rho(\tau_1\to\tau_2) = false$, ie. if $\rho(\tau_1) \Rightarrow \rho(\tau_2) = false$, then can only have

(2) $\rho(\tau_1) = true$

(3) $\rho(\tau_2) = false$ $\qquad \Gamma_{ta} \overset{+}{\cup} \{(x:\tau_1)\}$

Now (1) + (3) imply $\exists\,(x':\tau') \in (\Gamma, x:\tau_1)_{ta}$ with $\rho(\tau') = false$. By (2), can't have $(x':\tau') = (x:\tau_1)$, so $(x':\tau') \in \Gamma_{ta}$.

2 So $\Phi(\Gamma, \lambda x:\tau_1(M), \tau_1\to\tau_2)$ holds. ✓

Case (gen): Assume

(4) $\Phi(\alpha, \Gamma, M, \tau)$ with $\alpha \notin \Gamma_{tv}$ & Γ ok.

If $\rho(\forall\alpha(\tau)) = false$,

i.e. $\rho[\alpha \mapsto true](\tau)$ & $\rho[\alpha \mapsto false](\tau) = false$,

then

either $\rho[\alpha \mapsto true](\tau) = false$

or $\rho[\alpha \mapsto false](\tau) = false$

In either case, by (4) we have

$\exists\ (x' : \tau') \in (\alpha, \Gamma)_{ta} = \Gamma_{ta}$ with

either $\rho[\alpha \mapsto true](\tau') = false$

or $\rho[\alpha \mapsto false](\tau') = false$.

But since Γ ok, $ftv(\tau') \subseteq \Gamma_{tv} \not\ni \alpha$.

So $\rho[\alpha \mapsto \underline{b}](\tau') = \rho(\tau')$.

Hence in either case $\rho(\tau') = false$.

3 Thus $\Phi(\Gamma, \Lambda\alpha(M), \forall\alpha(\tau))$ holds. ✓

Case (spec): Assume

(5) $\Phi(\Gamma, M, \forall\alpha(\tau))$ & $ftv(\tau') \subseteq \Gamma_{tv}$.

If $\rho(\tau[\tau'/\alpha]) = false$, then

$\rho[\alpha \mapsto \rho(\tau')](\tau) = false$.

so $\rho[\alpha \mapsto true](\tau)$ & $\rho[\alpha \mapsto false](\tau) = false$

(Since $\rho(\tau') \in \{true, false\}$)

i.e. $\rho(\forall\alpha(\tau)) = false$. So by (5), $\exists\ (x' : \tau'') \in \Gamma_{ta}$

2 with $\rho(\tau'') = false$. Thus $\Phi(\Gamma, M\tau', \tau[\tau'/\alpha])$ holds. ✓

Case (app): Assume

(6) $\Phi(\Gamma, M, \tau_1 \to \tau_2)$

(7) $\Phi(\Gamma, M', \tau_1)$

If $\rho(\tau_2) = false$, then

either $\rho(\tau_1) = false$, in which case by (7)

$\exists (x' : \tau') \in \Gamma_{ta}$ with $\rho(\tau') = false$

or $\rho(\tau_1) = true$, in which case

$\rho(\tau_1 \to \tau_2) = \rho(\tau_1) \Rightarrow \rho(\tau_2) = (true \Rightarrow false) = false$

so by (6) $\exists (x' : \tau') \in \Gamma_{ta}$ with $\rho(\tau') = false$.

2 So in either case $\Phi(\Gamma, MM', \tau_2)$ holds. ✓.

(10)

If there was closed $M : \forall\alpha(\alpha)$, then

1 $\emptyset, \emptyset \vdash M : \forall\alpha(\alpha)$

would hold. We saw that, choosing any ρ,
$\rho(\forall\alpha(\alpha)) = false$. So by the result above

1 $\Gamma = (\emptyset, \emptyset)$ has to satisfy $\exists (x' : \tau') \in \Gamma_{ta} \cdot \rho(\tau') = false$

which is impossible.

(2)
