Types 2003, p 9, q 10 AMP

(a) $M_1 =_\beta M_2$ holds if there is a finite
1 chain of reductions between M_1 & M_2:

$$M_1 - \bullet - \bullet \cdots \bullet - \bullet - M_2$$

where each $-$ is either $\rightarrow$ or $\leftarrow$ and:

- $M_1 \rightarrow M_2$ holds if M_2 is obtained from M_1 by replacing a subterm of the form

1 $(\lambda x : \tau(M))N$ by $M[N/x]$

1 or $(\Lambda\alpha(M))\,\tau$ by $N[\tau/\alpha]$

- $M_1 \leftarrow M_2$ holds if $M_1 \rightarrow M_2$ does.

To decide $M_1 =_\beta M_2$:

1 - reduce M_1 & M_2 to β-normal forms N_1 & N_2 (because M_1 & M_2 are typeable, such n.f.'s always exist (by Strong Normalisation result) and
1 are unique (by Church-Rosser result)).
1 - compare N_1 & N_2 up to α-equivalence (which is decidable).

1 This doesn't work for untyped terms, because
1 they may have no β-normal form (e.g.
8 $(\lambda x : \tau(x\,x))\,\lambda x : \tau(x\,x)$ reduces to itself).

(b) We can take

2 $I \triangleq \Lambda\alpha(\lambda x : \alpha(\Lambda\alpha_1(\lambda y : \forall\alpha_2(\alpha_2 \to \alpha_1)(y\,\alpha\,x))))$

(i) Putting $\Gamma = [x:\alpha,\ y:\forall\alpha_2(\alpha_2\to\alpha_1)]$, we have

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma\vdash y:\forall\alpha_2(\alpha_2\to\alpha_1)}\,(var)}{\Gamma\vdash y\alpha:\alpha\to\alpha_1}\,(spec)\qquad \dfrac{}{\Gamma\vdash x:\alpha}\,(var)}{\Gamma\vdash y\alpha x:\alpha_1}\,(app)}{x:\alpha\vdash\lambda y:\forall\alpha_2(\alpha_2\to\alpha_1)(y\alpha x):(\forall\alpha_2(\alpha_2\to\alpha_1))\to\alpha_1}\,(fn)}{x:\alpha\vdash\Lambda\alpha_1(\cdots\ \cdots\ \cdots\ \cdots):\omega}\,(gen)}{\Phi\vdash\lambda x:\alpha(\cdots\ \cdots\ \cdots\ \cdots):\alpha\to\omega}\,(fn)}{\Phi\vdash I:\forall\alpha(\alpha\to\omega)}\,(gen)$$

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(ii) Suppose $\Phi\vdash M_i:\tau \quad (i=1,2)$.
As mentioned in part (a), we therefore know that
1 $M_i\to^* N_i$ for (unique) β-normal forms N_i $(i=1,2)$
Then from the definition of I we get

1 $I\tau M_i\alpha x \to^* x\tau M_i \to^* x\tau N_i$

1 and $x\tau N_i$ is in normal form, because N_i is.
1 So if $M_1 =_\beta M_2$, then $I\tau M_1\alpha x =_\beta I M_2\alpha x$
so the latter terms reduce to the same normal form, so from above $x\tau N_1 = x\tau N_2$ and
1 hence $N_1 = N_2$. Therefore $M_1 =_\beta N_1 = N_2 =_\beta M_2$,
(12) 1 i.e. $M_1 =_\beta M_2$, as required.

Commentary

This question concerns material covered in lectures 6 & 7 of the course.

Part (a) is bookwork.

Part (b) is a new problem. Given the definition of ω, the candidate has to discover a term I satisfying (i): there is an obvious candidate once one uses techniques illustrated by example in the lectures. Part (ii) then follows if one takes the hint; perhaps the hardest part is to realise that N_1 and N_2 are α-equivalent if $x\tau N_1$ and $x\tau N_2$ are.