2003
p9q14
GW

Den Sem Qu 2

(1) • fixed point induction

• fix(f) is the least prefixed point of f

• math. induction on the approximations $f^n(\perp)$ of fix(f)

• well-founded induction.

(2) $fix(f \circ f) = \bigsqcup_n f^{2n}(\perp)$

$fix(f) = \bigsqcup_n f^n(\perp)$

The two chains

$\perp \quad f(\perp) \quad f^2(\perp) \quad f^3(\perp) \quad f^4(\perp) \ldots f^{2n}(\perp)$

and

$\perp \quad f^2(\perp) \quad f^4 \ldots f^{2n}(\perp)$.

have the same least upper bounds because clearly they have the same upper bounds.

(3) $f\circ g(f(\mathrm{fix}(g\circ f))) = f(g\circ f\,(\mathrm{fix}(g\circ f)))$,

$= f(\mathrm{fix}(g\circ f))$

$\therefore$ $f(\mathrm{fix}(g\circ f))$ is a (pre-) fixed point of $f\circ g$

~~gof~~ $\therefore$ $\mathrm{fix}(f\circ g) \sqsubseteq f(\mathrm{fix}(g\circ f))$.

(4) $f(\mathrm{fix}(g\circ f)) = f(\bigsqcup_n (g\circ f)^n(\bot))$

$= \bigsqcup_n f\circ(g\circ f)^n(\bot)$ (f cts)

$= \bigsqcup_n (f\circ g)^n(f(\bot))$ by rearranging.

The result will follow from

$\forall n.\ (f\circ g)^n(f(\bot)) \sqsubseteq \mathrm{fix}(f\circ g)$.

We show this by mathematical induction:

<u>Basis</u> $n=0$ $f(\bot) \sqsubseteq f(g(\bot)) \sqsubseteq \bigsqcup_n f\circ g(\bot)$

$= \mathrm{fix}(f\circ g)$.

<u>Step</u> Assume $(f\circ g)^n(f(\bot)) \sqsubseteq \mathrm{fix}(f\circ g)$.

$(f\circ g)^{n+1}(f(\bot)) \sqsubseteq f\circ g(\mathrm{fix}(f\circ g)) = \mathrm{fix}(f\circ g)$

$\boxtimes$