SOLUTION NOTES – Semantics of Programming Languages 2005 – Paper 5 Question 11 (PMS)

Part (a) is bookwork, from Chapter 4 of the notes but with the minor difference that this is an implicitly-typed syntax.

Parts (b) and (c) are inductive proofs, as in Chapter 3 of the notes. The first is most easily proved by a structural induction; the second by a rule induction using a substitution lemma.

2005 Paper 5 question 11 Semantics PMS

a) Typing:

$$\Gamma, x:T \vdash x:T \qquad \frac{\Gamma \vdash e_1 : T_1 \to T_2 \quad \Gamma \vdash e_2 : T_1}{\Gamma \vdash e_1\ e_2 : T_2}$$

$$\frac{\Gamma, x:T_1 \vdash e : T_2}{\Gamma \vdash \mathrm{fn}\ x \Rightarrow e : T_1 \to T_2}$$

$T ::= \ldots$
$\quad | \ T \to T$

~~b)~~ Operational semantics:

$$\frac{e_1 \longrightarrow e_1'}{e_1\ e_2 \longrightarrow e_1'\ e_2} \qquad (\mathrm{fn}\ x \Rightarrow e_1)\,e_2 \to \{e_2/x\}e_1$$

b) Determinacy: If $e \to e_1$ and $e \to e_2$ then $e_1 = e_2$.

Proof: Let $\Phi(e) = \forall e_1, e_2.\ e \to e_1 \ \&\ e \to e_2 \Rightarrow e_1 = e_2$

Prove $\forall e.\ \Phi(e)$ by structural induction.

Case x. There is no e_1 such that $x \to e_1$, so $\Phi(x)$ holds trivially.

case $\mathrm{fn}\, x \Rightarrow e$. There is no e' such that $(\mathrm{fn}\, x \Rightarrow e) \to e'$ (as no rule has a conclusion of that form) so $\Phi(\mathrm{fn}\, x \Rightarrow e)$ holds trivially.

case $e_1\ e_2$. suppose $e_1\ e_2 \to \hat{e}_1$ (*)

& $e_1\ e_2 \to \hat{e}_2$ (**)

consider the last rule used in the derivation of (*). There are two cases:

(a) e_1 is of the form $(\mathrm{fn}\, x \Rightarrow e)$ and $\hat{e}_1 = \{e_2/x\} e$.

(b) for some e_1' we have $e_1 \to e_1'$ and $\hat{e}_1 = e_1'\ e_2$

These are disjoint, as $\neg \exists e''. (\mathrm{fn}\, x \Rightarrow e) \to e''$, so the last ~~dem~~ rule used in the derivation of (**) must correspond.

If the cases ~~rules~~ were (a), (a) then $\hat{e}_1 = \{e_2/x\} e = \hat{e}_2$ ✓

If the cases ~~rules~~ were (b), (b) then we have e_1' and e_1'' such that $e_1 \to e_1'$ $\hat{e}_1 = e_1'\ e_2$

$e_1 \to e_1''$ $\hat{e}_2 = e_1''\ e_2$

then by $\Phi(e_1)$ we know $e_1' = e_1''$, so $\hat{e}_1 = \hat{e}_2$ ✓

c) Type Preservation   If $\Gamma \vdash e : T$ and $e \to e'$ then $\Gamma \vdash e' : T$.

Let $\Phi(e, e') : \forall \Gamma, T.\quad \Gamma \vdash e : T \Rightarrow \Gamma \vdash e' : T$

Prove $\forall e, e'.\ e \to e' \Rightarrow \Phi(e, e')$ by rule induction.

Case $e_1 e_2 \to e_1' e_2$ and $e_1 \to e_1'$.

Assume $\Gamma \vdash e_1 e_2 : T$. The derivation must be of the form ~~Inverting the defn~~

$$\frac{\Gamma \vdash e_1 : T_2 \to T \qquad \Gamma \vdash e_2 : T_2}{\Gamma \vdash e_1\ e_2 : T}$$

By the induction hypothesis $\Phi(e_1, e_1')$ we have $\Gamma \vdash e_1' : T_2 \to T$

Hence using the type rule again $\Gamma \vdash e_1' e_2 : T$.

Case $(\mathrm{fn}\, x \Rightarrow e_1)\, e_2 \to \{e_2/x\}\, e_1$

Assume $\Gamma \vdash (\mathrm{fn}\, x \Rightarrow e_1)\, e_2 : T$. The derivation must be of the form

$$\frac{\dfrac{\overline{\Gamma, x : T_2 \vdash e_1 : T}}{\Gamma \vdash (\mathrm{fn}\, x \Rightarrow e_1) : T_2 \to T} \qquad \Gamma \vdash e_2 : T_2}{\Gamma \vdash (\mathrm{fn}\, x \Rightarrow e_1)\ e_2 : T}$$

Use Lemma [substitution] If $\Gamma \vdash e_2 : T_2$ & $\Gamma, x : T_2 \vdash e_1 : T$ then $\Gamma \vdash \{e_2/x\}\, e_1 : T$

to conclude $\Gamma \vdash \{e_2/x\} e_1 : T$.