Solution notes

Information Theory and Coding 2005 – Paper 7 Question 8

(a) [*This question relates to entropy definitions and binary communication channels.*]

(i) The uncertainty about the input X given the observed output Y from the channel is the conditional entropy $H(X|Y)$, which is defined as:

$$H(X|Y) = -\sum_{x,y} p(x,y) \log p(x|y)$$

So, we need to calculate both the joint probability distribution $p(X,Y)$ and the conditional probability distribution $p(X|Y)$, and then combine their terms according to the above summation.

The joint probability distribution $p(X,Y)$ is

$$\begin{pmatrix} 0.5(1-\epsilon) & 0.5\epsilon \\ 0.5\epsilon & 0.5(1-\epsilon) \end{pmatrix}$$

and the conditional probability distribution $p(X|Y)$ is

$$\begin{pmatrix} 1-\epsilon & \epsilon \\ \epsilon & 1-\epsilon \end{pmatrix}$$

Combining these matrix elements accordingly gives us the conditional entropy:

$$H(X|Y) = -\left[0.5(1-\epsilon)\log(1-\epsilon) + 0.5\epsilon\log(\epsilon) + 0.5\epsilon\log(\epsilon) + 0.5(1-\epsilon)\log(1-\epsilon)\right]$$

$= \underline{-(1-\epsilon)\log(1-\epsilon) - \epsilon\log(\epsilon)}$ [5 marks]

(ii) One definition of mutual information is $I(X;Y) = H(X) - H(X|Y)$. Since the two input symbols are equi-probable, clearly $H(X) = 1$ bit. We know from (i) above that $H(X|Y) = -(1-\epsilon)\log(1-\epsilon) - \epsilon\log(\epsilon)$, and so therefore, the mutual information of this channel is:

$$I(X;Y) = 1 + (1-\epsilon)\log(1-\epsilon) + \epsilon\log(\epsilon)$$

[2 marks]

(iii) The uncertainty $H(X|Y)$ about the input, given the output, is maximised when $\underline{\epsilon = 0.5}$, in which case it is 1 bit. [1 mark]

(*b*) The analysis and synthesis (or forward and inverse) continuous Fourier transforms are, respectively:

(*i*) $G(k) = \int_{-\infty}^{+\infty} g(x) e^{-ikx} dx$ [2 marks]

(*ii*) $g(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} G(k) e^{ikx} dk$ [2 marks]

(*c*) The Fourier representation becomes simplified as follows:

(*i*) If the function is real-valued rather than complex-valued, then its Fourier transform has Hermitian symmetry: the real-part of the Fourier transform has even symmetry, and the imaginary part has odd-symmetry. [1 mark]

(*ii*) If the function has even symmetry, then its Fourier transform is purely real-valued. [1 mark]

(*iii*) If the function has odd symmetry, then its Fourier transform is purely imaginary-valued. [1 mark]

(*d*) (*i*) $11111111111110 = 1^{13}0$
The unary code word for 13 is simply 13 ones, followed by a final zero. [1 mark]

(*ii*) $1111010 = 1^{4}0\ 10$
We first divide $n = 13$ by $b = 3$ and obtain the representation $n = q \times b + r = 4 \times 3 + 1$ with remainder $r = 1$. We then encode $q = 4$ as the unary code word "11110". To this we need to attach an encoding of $r = 1$. Since r could have a value in the range $\{0, \ldots, b-1\} = \{0, 1, 2\}$, we first use all $\lfloor \log_2 b \rfloor = 1$-bit words that have a leading zero (here only "0" for $r = 0$), before encoding the remaining possible values of r using $\lceil \log_2 b \rceil = 2$-bit values that have a leading one (here "10" for $r = 1$ and "11" for $r = 2$). [2 marks]

(*iii*) $1110101 = 1^{3}0\ 101$
We first determine the length indicator $m = \lfloor \log_2 13 \rfloor = 3$ (because $2^3 \leq 13 < 2^4$) and encode it using the unary code word "1110", followed by the binary representation of 13 (1101_2) with the leading one removed: "101". [2 marks]

[*This question relates to variable-length codes.*]