Probability Solution Notes 2005 – Paper 2 Question 4 (FHK)

The two-bit random number problem. This exploits the Geometric distribution (Lectures 4 and 6) and generating functions (Lecture 6).

Part (*a*) [Entirely book work]

A random variable X distributed Geometric(p) has:

$$\mathrm{P}(X = r) = q^r p \quad \text{where} \quad p + q = 1$$

The associated generating function is:

$$\begin{aligned} G(\eta) &= q^0 p\eta^0 + q^1 p\eta^1 + q^2 p\eta^2 + q^3 p\eta^3 + \cdots \\ &= p\big((q\eta)^0 + (q\eta)^1 + (q\eta)^2 + (q\eta)^3 + \cdots\big) \end{aligned}$$

The item in brackets is easily recognised as the expansion of $(1 - q\eta)^{-1}$, so the generating function and its first two derivatives are:

$$\begin{aligned} G(\eta) &= p(1 - q\eta)^{-1} \\ G'(\eta) &= p(1 - q\eta)^{-2} q \\ G''(\eta) &= 2p(1 - q\eta)^{-3} q^2 \end{aligned}$$

Accordingly:

$$\begin{aligned} G(1) &= p(1 - q)^{-1} = p.p^{-1} = 1 \\ G'(1) &= p(1 - q)^{-2} q = \frac{p}{p^2} q = \frac{q}{p} \\ G''(1) &= 2p(1 - q)^{-3} q^2 = 2\frac{p}{p^3} q^2 = 2\frac{q^2}{p^2} \end{aligned}$$

The expectation $\mathrm{E}(X)$ is given by $G'(1) = \dfrac{q}{p}$.

The variance $\mathrm{V}(X)$ is given by:

$$\begin{aligned} \mathrm{V}(X) &= G''(1) + G'(1) - \big(G'(1)\big)^2 \\ &= 2\frac{q^2}{p^2} + \frac{q}{p} - \frac{q^2}{p^2} \\ &= \frac{q}{p}\left(\frac{q}{p} + 1\right) \\ &= \frac{q}{p}\left(\frac{q + p}{p}\right) \\ &= \frac{q}{p^2} \end{aligned}$$

[6 marks]

P.T.O.

Part (*b*)

Regard the two-bit random number generator as yielding a *success* if the bits are 01 or 10 and a *failure* if the bits are 00 or 11. Given that the probability of each bit being 1 is p and being 0 is q:

The probability of *success* $= 2pq$
The probability of *failure* $= p^2 + q^2$

The value r of the random variable X is the length of the game measured in steps. If the game is to end at step r, there must be $r-1$ *failures* before the first *success*. Accordingly:

$$\mathrm{P}(X = r) = \begin{cases} 0, & \text{if } r = 0 \\ (p^2+q^2)^{r-1}\, 2pq, & \text{if } r > 0 \end{cases}$$

[5 marks]

$$\begin{aligned}\sum_{r=0}^{\infty} \mathrm{P}(X = r) &= 0 + (p^2+q^2)^0\, 2pq + (p^2+q^2)^1\, 2pq + (p^2+q^2)^2\, 2pq + \cdots \\ &= \frac{2pq}{1-(p^2+q^2)} = \frac{2pq}{(p+q)^2-(p^2+q^2)} = \frac{2pq}{2pq} = 1\end{aligned}$$

[4 marks]

Let Y be a random number whose value s is the number of *failures* before the first *success*. Y is therefore distributed Geometric($2pq$).
The result for the expectation derived in part (a) was q/p for a random variable distributed Geometric(p). So for a random variable distributed Geometric($2pq$):

$$\text{Expected } \textit{failures} \text{ before first } \textit{success} \ = \frac{p^2+q^2}{2pq}$$

Accordingly:

$$\text{Expected length of a game } = 1 + \frac{p^2+q^2}{2pq} = \frac{2pq+p^2+q^2}{2pq} = \frac{1}{2pq}$$

[5 marks]