CST IB 2001, paper 5, question 9

Semantics of Programming Languages AMP

1.

Structural induction: to prove $\forall E\ \Phi(E)$ it suffices to show

1 (1) $\forall n\ \Phi(n)$

1 (2) $\forall X\ \Phi(X)$

1 (3) $\forall E\,(\Phi(E) \Rightarrow \Phi(-E))$

2 (4) $\forall E_1, E_2\,(\Phi(E_1)\ \&\ \Phi(E_2) \Rightarrow \Phi(E_1 + E_2))$

(5)

Suppose (1)–(4) hold. Note that

$$\#(E) \triangleq \text{number of symbols in } E$$

1 is always finite; so to prove $\forall E\ \Phi(E)$ it suffices to prove $\forall k\ \Psi(k)$, where

1 $$\Psi(k) \overset{\Delta}{\Leftrightarrow} \forall E\,(\#(E) \leq k \Rightarrow \Phi(E)).$$

We do this by mathematical induction on k:

1 Base case: $\Psi(0)$ holds trivially, because $\#(E) \leq 0$ for no E.

2 Induction step: If $\Psi(k)$ holds, given any E with $\#(E) \leq k+1$

- either $E = n$ – in which case $\Phi(E)$ by (1)
- or $E = X$ – in which case $\Phi(E)$ by (2)
- or $E = -E'$ – in which case $\#(E') = \#(E) - 1 \leq k$ so by $\Psi(k)$ we have $\Phi(E')$ & hence by (3), $\Phi(E)$.
- or $E = E_1 + E_2$ – in which case $\#(E_i) \leq k$ so by $\Psi(k)$ we have $\Phi(E_i)$ for $i = 1, 2$; hence $\Phi(E)$ by (4).

(5) So in all cases $\Phi(k)$ implies $\Phi(k+1)$.

1 $E, s \Downarrow n$ is defined inductively: it's the least relation closed under the following axioms & rules:

1 (5) $n, s \Downarrow n$

2 (6) $X, s \Downarrow n$ if $X \in \mathrm{dom}(S)$ & $S(X) = n$

1 (7) $\dfrac{E, s \Downarrow n'}{-E, s \Downarrow n}$ if $n = -n'$

2 (8) $\dfrac{E_1, s \Downarrow n_1 \quad E_2, s \Downarrow n_2}{E_1 + E_2, s \Downarrow n}$ if $n = n_1 + n_2$

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Let $\Phi(E)$ be $\forall s, n_1, n_2\ (E, s \Downarrow n_1 \ \&\ E, s \Downarrow n_2 \Rightarrow n_1 = n_2)$ and check (1)–(4) hold.

1 (1) If $n, s \Downarrow n_1$ & $n, s \Downarrow n_2$, then these could only have been deduced using (5), so $n_1 = n = n_2$.

1 (2) If $X, s \Downarrow n_1$ & $X, s \Downarrow n_2$, then these could only have been deduced using (6), so $X \in \mathrm{dom}(S)$ and $n_1 = S(X) = n_2$.

2 (3) Assume $\Phi(E)$. If $-E, s \Downarrow n_1$ & $-E, s \Downarrow n_2$, these can only have been deduced by applying rule (7) to $E, s \Downarrow -n_1$ & $E, s \Downarrow -n_2$. By $\Phi(E)$ we have $-n_1 = -n_2$, so $n_1 = n_2$. Thus $\Phi(-E)$ holds.

3 (4) Assume $\Phi(E_1)$ & $\Phi(E_2)$. If $E_1 + E_2, s \Downarrow n_1$ and $E_1 + E_2, s \Downarrow n_2$, these can only have been deduced by applying rule (8) to
$E_1, s \Downarrow n_{11}$ & $E_2, s \Downarrow n_{12}$ where $n_1 = n_{11} + n_{12}$
$E_1, s \Downarrow n_{21}$ & $E_2, s \Downarrow n_{22}$ where $n_2 = n_{21} + n_{22}$.

By $\Phi(E_1)$ we have $n_{11} = n_{21}$
and by $\Phi(E_2)$ we have $n_{12} = n_{22}$.
So $n_1 = n_{11} + n_{12} = n_{21} + n_{22} = n_2$.
(7) Thus $\Phi(E_1 + E_2)$ holds.

If the domain of definition of s contains all the identifiers occurring in E, then there exists n with $E, s \Downarrow n$.
(Proof by structural induction on E, but not required.)
(1)