p8q1
GW

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Denotational semantics (1) [Uses fixed point induction section in Den. Sem notes]

(3) (a) Fixed point induction: Let $\varphi : D \to D$ be a continuous function on a ~~cpo~~ domain D. Let Q be an admissible property of D (i.e. ~~a prop~~ s.t. $Q(\bot)$ and $Q(d_n)$ for all $d_0 \sqsubseteq d_1 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \cdots$ implies $Q(\bigsqcup_n d_n)$. Then

$$Q(fix\,\varphi) \quad \text{if} \quad \forall d \in D. \quad Q(x) \Rightarrow Q(\varphi(x)).$$

(4) (b) $h(f^*(x)) = h(if(h(x), x, f^*(h(x))))$

$$= \begin{cases} h(x) & \text{if } h(x) = true \\ hf^*(h(x)) & \text{if } h(x) = false \\ h(\bot) & \text{if } h(x) = \bot \end{cases}$$

$$= \begin{cases} h(x) & \text{if } h(x) = true \\ hf^*(h(x)) & \text{if } h(x) = false \\ \bot & \text{if } h(x) = \bot \end{cases}$$

by strictness of h

$$= if(h(x),\ h(x),\ hf^*h(x)).$$

(5) (c) For Q to be admissible we need

$Q(\bot)$: But $\forall x \in D.\ h(\bot(x)) = \bot \sqsubseteq true$

by strictness of h.

and that Q is chain closed: let $d_0 \sqsubseteq \cdots \sqsubseteq d_n \sqsubseteq \cdots$

be a chain in D for which $Q(d_n)$, all n.

ie. $h(f(d_n)) \sqsubseteq true$.

Then $h(f(\bigsqcup_n d_n)) = \bigsqcup_n h f(d_n)$

by cty of f & h.

$\sqsubseteq true$.

(8) (d) To show $Q(f^*)$ by fixed point induction we now need $Q(f) \Rightarrow Q(\varphi(f))$ where

$$\varphi(f) = \lambda x \in D.\ if(h(x), x, f\,k(x)).$$

(Then $f^* = fix\,\varphi$).

Assume $Q(f)$. ie. $h(f(x)) \sqsubseteq true$, all $x \in D$.

$$h(\varphi f(x)) = h(if(h(x), x, f\,k(x)))$$

$$= if(h(x), h(x), h\,f\,k(x)) \text{ by strictness of } h \text{ (cf. (b))}$$

$$= \begin{cases} h(x) & \text{if } h(x) = true \\ h\,f\,k(x) & \text{if } h(x) = false \\ \bot & \text{if } h(x) = \bot \end{cases}$$

$\sqsubseteq true$ (considering each case).

$\boxtimes$