2005
MROD

Numerical Analysis I Paper 4 Question 7

Solution Notes

Context: errors; loss of significance; quadratic equations.

(a) Let x^* denote the floating-point representation of x.

Absolute error ϵ is defined by $x^* = x + \epsilon$.

Relative error δ is defined by $x^* = x(1+\delta) = x + x\delta$.

Comparing these definitions,

$$\epsilon = x\delta \quad \text{so} \quad \delta = \frac{\epsilon}{x} \quad \text{if } x \neq 0.$$

Loss of significance occurs when relative error grows much larger during the course of a calculation. [3 marks]

(b) The formula

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \qquad (1)$$

is unsuitable if $b^2 \gg |4ac|$ because of loss of significance in the numerator for the positive square root. For this case consider

$$x = \frac{-b+\sqrt{b^2-4ac}}{2a} \cdot \frac{-b-\sqrt{b^2-4ac}}{-b-\sqrt{b^2-4ac}}$$

$$= \frac{\cancel{b^2} - (\cancel{b^2} - 4ac)}{-2a\left(b+\sqrt{b^2-4ac}\right)} = \frac{-2c}{b+\sqrt{b^2-4ac}}. \qquad (2)$$

There is no loss of significance in formula (2) when $b>0$. Consider $a=30$, $b=3000$, $c=1$ with 5 sig. figs. of accuracy

$$\left(\sqrt{b^2-4ac}\right)^* = (3000 - 60/3000)^* = (3000 - 0.02)^*$$
$$= (2999.98)^* = 3000.0.$$

For the positive square root

(1) gives $x = \left(\frac{-3000+3000.0}{60}\right)^* = 0$

(2) gives $x = \left(\frac{-2}{3000+3000.0}\right)^* = -3.3333 \times 10^{-4}$

[10 marks] over

(c) Write $\cos x = a_0 - a_2 + a_4 - \dots$

$$a_6 = \frac{x^6}{6!} = a_4 \cdot \frac{x^2}{5.6}$$

$$a_8 = \frac{x^8}{8!} = a_6 \cdot \frac{x^2}{7.8}$$

For $x = 6$, $a_4 < a_6 > a_8$, i.e. a_6 is the largest.

$$a_6 = \frac{6^6}{6!} = \frac{6^4}{4.5} = 64 \cdot 8$$

If relative error in $a_6 \simeq 0.5 \times 10^{-p}$

then absolute error in $a_6 \simeq 0.5 \times 10^{-p}\, a_6$

$\simeq 32.4 \times 10^{-p}$

Estimate relative error in $\cos 6$

$$\simeq \frac{|\text{absolute error in largest term}|}{|\cos 6|}$$

$\simeq 0.3 \times 10^{-(p-2)}$

so about 2 significant digits are lost. [7 marks]