Probability 2003 Paper 2 Question 5 (FHK)

Transformation Functions Problem — Solution Notes

(a) The principal constraints on any transformation function $y(x)$ are:

- Throughout the useful range of x, both $y(x)$ and its inverse $x(y)$ must be defined and must be single-valued.
- Throughout this range, $\dfrac{dx}{dy}$ must be defined and *either* $\dfrac{dx}{dy} \geqslant 0$ *or* $\dfrac{dx}{dy} \leqslant 0$.

The use of $|\frac{dx}{dy}|$ is a consequence of the second constraint and this use is significant because, when setting up a differential equation to determine a transformation function, one may choose the sign as in:

$$\frac{dx}{dy} = \frac{g(y)}{f(x(y))} \qquad \text{or} \qquad \frac{dx}{dy} = -\frac{g(y)}{f(x(y))}$$

(b) In the case of the Uniform Distribution $f(x) = 1$ over the range of interest and the differential equations become:

$$\frac{dx}{dy} = g(y) \qquad \text{or} \qquad \frac{dx}{dy} = -g(y)$$

In all four cases the second (minus sign) option will be used.

(i) Here $\frac{dx}{dy} = -\lambda.e^{-\lambda y}$ so $x = e^{-\lambda y} + c$ where c is some constant. In this example, as x runs upwards from 0 to 1, y runs downwards from ∞ to 0 so the constant is 0. Re-arranging the result in terms of y gives the required transformation function:

$$y = -\frac{1}{\lambda}.\ln x$$

(ii) Here $\frac{dx}{dy} = -\sin y$ so $x = \cos y + c$ where c is some constant. In this example, as x runs upwards from 0 to 1, y runs downwards from $\frac{\pi}{2}$ to 0 so the constant is 0. Re-arranging the result in terms of y gives the required transformation function:

$$y = \cos^{-1} x$$

(*iii*) Here $\frac{dx}{dy} = -\frac{1}{2}(2-y)$ so $x = \frac{1}{4}(2-y)^2+c$ where c is some constant. In this example, as x runs upwards from 0 to 1, y runs downwards from 2 to 0 so the constant is 0. Re-arranging the result in terms of y gives the required transformation function:

$$y = 2(1 - \sqrt{x})$$

(*iv*) Here $\frac{dx}{dy} = -\frac{3}{8}(2-y)^2$ so $x = \frac{1}{8}(2-y)^3 + c$ where c is some constant. In this example, as x runs upwards from 0 to 1, y runs downwards from 2 to 0 so the constant is 0. Re-arranging the result in terms of y gives the required transformation function:

$$y = 2(1 - \sqrt[3]{x})$$

In cases (*iii*) and (*iv*) it is possible to produce alternative acceptable solutions starting with the monadic minus sign omitted: $y = 2(1 - \sqrt{1-x})$ and $y = 2(1 - \sqrt[3]{1-x})$ respectively and as x runs 0 to 1, y runs 0 to 2 in both cases.