p7q5 ~~p8q16~~
RJG

Computer Systems Modelling 2004/5

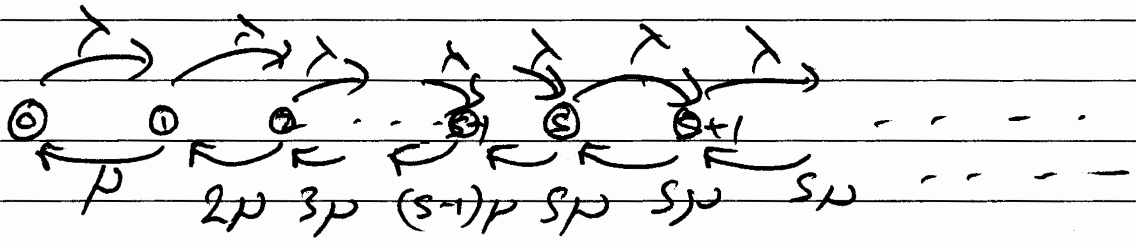
[Relates to birth-death processes and their application to modelling queues.]

(a) Arrival rate λ,
Service rate μ
s servers.

State space
i = # customers present
$i \in \{0, 1, 2, \ldots\}$

Birth rates $\lambda_i = \lambda \quad (i \geq 0)$

Death rates $\mu_i = \mu \min(i, s) \quad (i \geq 1)$



(b) Detailed balance equations for equilibrium distribution $\pi_i \quad (i = 0, 1, \ldots)$

$$\pi_i \lambda_i = \pi_{i+1} \mu_{i+1}$$

ie
$$\lambda \pi_i = \begin{cases} \mu (i+1) \pi_{i+1} & 0 \leq i < s \\ \mu s \, \pi_{i+1} & i \geq s \end{cases}$$

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(c) Fix π_0 then

$$\pi_1 = \frac{\lambda}{\mu(0+1)}\pi_0 = \frac{\lambda}{\mu}\pi_0$$

$$\pi_2 = \frac{\lambda}{\mu.2}\pi_1 = \left(\frac{\lambda}{\mu}\right)^2\frac{\pi_0}{2}$$

$$\pi_3 = \frac{\lambda}{\mu.3}\pi_2 = \left(\frac{\lambda}{\mu}\right)^3\frac{\pi_0}{3!}$$

$$\vdots$$

$$\pi_s = \left(\frac{\lambda}{\mu}\right)^s\frac{\pi_0}{s!}$$

$$\pi_{s+1} = \frac{\lambda}{\mu s}\pi_s = \left(\frac{\lambda}{\mu}\right)^{s+1}\frac{\pi_0}{s.s!}$$

$$\pi_{s+2} = \frac{\lambda}{\mu.s}\pi_{s+1} = \left(\frac{\lambda}{\mu}\right)^{s+2}\frac{1}{s^2}\frac{\pi_0}{s!}$$

$$\vdots$$

$$\pi_i = \left(\frac{\lambda}{\mu}\right)^i\frac{\pi_0}{s^{i-s}\,s!} \qquad i \geq s$$

$$\vdots$$

So,

$$\pi_i = \begin{cases} \left(\frac{\lambda}{\mu}\right)^i\frac{\pi_0}{i!} & 0 \leq i < s \\ \left(\frac{\lambda}{s\mu}\right)^i\frac{s^s\pi_0}{s!} & i \geq s \end{cases}$$

(d)

To find the normalising value of π_0 we must solve $\sum_{i=0}^{\infty} \pi_i = 1$ That is

$$\sum_{i=0}^{\infty} \pi_0 \left(\sum_{i=0}^{s-1} \left(\frac{\lambda}{\mu}\right)^i \Big/ i! + \frac{s^s}{s!} \underbrace{\sum_{i=s}^{\infty} \left(\frac{\lambda}{\mu s}\right)^i}_{\substack{\text{converges} \\ \text{iff } \frac{\lambda}{\mu s} < 1 \\ \text{iff } \lambda < \mu s}} \right) = 1$$

Stationary distribution exists iff $\lambda < \mu s$ and

$$\pi_0^{-1} = \sum_{i=0}^{s-1} \left(\frac{\lambda}{\mu}\right)^i \Big/ i! + \frac{s^s}{s!} \sum_{i=s}^{\infty} \left(\frac{\lambda}{\mu s}\right)^i$$

gives stationary distribution using

$$\pi_i = \begin{cases} \left(\frac{\lambda}{\mu}\right)^i \frac{\pi_0}{i!} & 0 \le i < s \\ \left(\frac{\lambda}{s\mu}\right)^i \frac{s^s \pi_0}{s!} & i \ge s \end{cases}$$