Solution Notes

(a) A is positive semi-definite if $\underline{x}^T A \underline{x} \geq 0$ for any vector $\underline{x}$.

[1 mark]

(b) Write $\|.\|_M$, $\|.\|_V$ for compatible matrix and vector norms.
Schwarz's inequality for AB is

$$\|AB\|_M \leq \|A\|_M \|B\|_M .$$

For $A\underline{x}$ we have

$$\|A\underline{x}\|_V \leq \|A\|_M \|\underline{x}\|_V .$$

If $\lambda_1, \lambda_2, \ldots \lambda_n$ are the real, non-negative eigenvalues of A^TA in non-increasing order of magnitude the singular values $\sigma_1, \sigma_2, \ldots \sigma_n$ are given by $\sigma_j = \sqrt{\lambda_j}$. Also

$$\|A\|_2 = \max_j \sqrt{\lambda_j} = \sigma_1 .$$

[4 marks]

(c) The singular value decomposition $A = UWV^T$ is such that $W = \mathrm{diag}\{\sigma_1, \sigma_2, \ldots \sigma_n\}$, V is an orthogonal matrix of orthonormalised eigenvectors of A^TA, and U is another orthogonal matrix.
Given the singular value decomposition of A, to solve $A\underline{x} = \underline{b}$ when A has full rank,

$$UWV^T\underline{x} = \underline{b}$$

so

$$\underline{x} = VW^{-1}U^T\underline{b} .$$

If A has rank $r < n$ then W^{-1} is replaced by

$$W^+ = \mathrm{diag}\{\sigma_1^{-1}, \sigma_2^{-1}, \ldots \sigma_r^{-1}, 0, \ldots 0\}.$$

[4 marks]

PTO.

(d) $A\underline{e} = A\underline{x} - A\hat{\underline{x}} = \underline{b} - A\hat{\underline{x}} = \underline{r}$

So $\underline{e} = A^{-1}\underline{r}$

$$\|\underline{e}\| = \|A^{-1}\underline{r}\|$$

$$\leq \|A^{-1}\|.\|\underline{r}\| \quad \text{by Schwarz}$$

So
$$\frac{\|\underline{e}\|}{\|\underline{x}\|} \leq \|A^{-1}\|.\frac{\|\underline{r}\|}{\|\underline{x}\|}.$$

But $\|\underline{b}\| = \|A\underline{x}\| \leq \|A\|.\|\underline{x}\|$ by Schwarz

so
$$\frac{\|\underline{e}\|}{\|\underline{x}\|} \leq \|A\|\,\|A^{-1}\|\,\frac{\|\underline{r}\|}{\|\underline{b}\|}$$

and the right-hand side is computable.

If the l_2 norm is used, the condition number

$$K_n = \|A\|_2\|A^{-1}\|_2 = \frac{\sigma_1}{\sigma_n}.$$

[8 marks]

(e) The condition number of an approximate solution with rank r is K_r, and r should be chosen so that

$$K_r < \frac{1}{\text{machine epsilon}} = 10^{15}.$$

Now $K_2 = \dfrac{10^3}{1}$ but $K_3 = \dfrac{10^3}{10^{-14}} = 10^{17}$

so choose rank 2. Then

$$W^+ = \text{diag}\{10^{-3}, 1, 0, 0, 0\}.$$

[3 marks]