Solution Notes

Numerical Analysis I – Question 8 2004

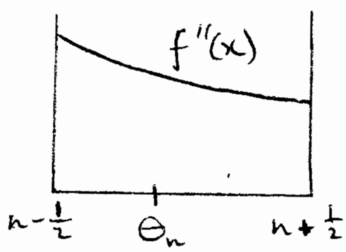
Context: quadrature, summation of series.

(a)
$$S_{1,\infty} = S_{1,N} + S_{N+1,\infty}$$
$$= S_{1,N} + \sum_{n=N+1}^{\infty} (I_n - e_n)$$
$$= S_{1,N} + \int_{N+\frac{1}{2}}^{\infty} f(x)\,dx - \frac{1}{24}\sum_{n=N+1}^{\infty} f''(\theta_n)$$

so the integral remainder is $\int_{N+\frac{1}{2}}^{\infty} f(x)\,dx$. [5 marks]

(b) Let
$$E_N = -\frac{1}{24}\sum_{n=N+1}^{\infty} f''(\theta_n).$$

Consider some interval $[n-\frac{1}{2}, n+\frac{1}{2}]$



If $f''(x)$ is positive decreasing
$$f''(\theta_n) \leqslant f''(n-\tfrac{1}{2})$$
so if $f''(x)$ is positive decreasing for $x > N+\frac{1}{2}$
$$|E_N| \leqslant \frac{1}{24}\sum_{n=N+1}^{\infty} f''(n-\tfrac{1}{2})$$
$$\simeq \frac{1}{24}\int_N^{\infty} f''(x)\,dx.$$
$$\therefore \quad |E_N| \leqslant -\frac{f'(N)}{24} \quad \text{approximately.}$$ [5 marks]

(c) If
$$f(x) = \frac{1}{(1+x)\sqrt{x}}$$
$$f'(x) = -\frac{\frac{1}{2}\frac{1+x}{\sqrt{x}} + \sqrt{x}}{(1+x)^2 x} = -\frac{1+3x}{2x^{3/2}(1+x)^2}$$

over.

$$f''(x) = \frac{-6x^{3/2}(1+x)^2 + 2(1+3x)\left[\frac{3}{2}\sqrt{x}(1+x)^2 + 2x^{\frac{3}{2}}(1+x)\right]}{4x^3(1+x)^4}$$

$$= \frac{-6x(1+x) + (1+3x)\left[3(1+x)+4x\right]}{4x^{5/2}(1+x)^3}$$

$$= \frac{15x^2+10x+3}{4x^{5/2}(1+x)^3} > 0 \quad \text{for } x>0.$$

As $x \to \infty$

$$f''(x) \to \frac{15}{4x^{7/2}} \quad \text{which is decreasing.}$$

The integral remainder is

$$\int_{N+\frac{1}{2}}^{\infty} \frac{dx}{(1+x)\sqrt{x}} = 2\tan^{-1}\sqrt{x}\Big]_{N+\frac{1}{2}}^{\infty}$$

$$= 2\left[\frac{\pi}{2} - \tan^{-1}\sqrt{N+\tfrac{1}{2}}\right]$$

$$= \pi - 2\tan^{-1}\sqrt{N+\tfrac{1}{2}}.$$

[6 marks]

(d) Let

$$\epsilon = -\frac{f'(N)}{24} = \frac{1+3N}{48N^{3/2}(1+N)^2}$$

$$\simeq \frac{3N}{48N^{3/2}N^2} \quad \text{since } N \gg 1$$

$$\simeq \frac{1}{16N^{5/2}}.$$

So

$$N \simeq \frac{1}{(16\epsilon)^{2/5}}$$

$$\simeq \frac{1}{(32\times10^{-15})^{2/5}}$$

$$\simeq 0.25\times10^6$$

$$\simeq 250\,000.$$

[4 marks]