1999 CST Part II Denotational Semantics, paper 7, q 9

AMP

fix(f) satisfies

(1) $f(fix(f)) \sqsubseteq fix(f)$

2 (2) $\forall d \in D\ (f(d) \sqsubseteq d \Rightarrow fix(f) \sqsubseteq d)$

Existence of fix(f) (Tarski's fixed point theorem):

Define $f^n(\bot) \in D$ ($n \in \mathbb{N}$) by

$$\begin{cases} f^0(\bot) \stackrel{\text{def}}{=} \bot & \text{(least element of } D\text{)} \\ f^{n+1}(\bot) \stackrel{\text{def}}{=} f(f^n(\bot)). \end{cases}$$

Claim: $f^n(\bot)$ forms an ascending chain,

[because $\forall n \in \mathbb{N}\ (f^n(\bot) \sqsubseteq f^{n+1}(\bot))$ holds by induction on n:

- base case: $f^0(\bot) = \bot \sqsubseteq f^1(\bot)$ 'cos $\bot$ is least
- induction step: if $f^n(\bot) \sqsubseteq f^{n+1}(\bot)$, then by monotonicity of f $f(f^n(\bot)) \sqsubseteq f(f^{n+1}(\bot))$, i.e. $f^{n+1}(\bot) \sqsubseteq f^{n+2}(\bot)$, as required.]

Thus we can form

$$fix(f) \stackrel{\text{def}}{=} \bigsqcup_{n \geq 0} f^n(\bot)$$

Have to check it satisfies (1) & (2).

(1):
$$\begin{aligned} f(fix(f)) &= f\Big(\bigsqcup_{n \geq 0} f^n(\bot)\Big) \\ &= \bigsqcup_{n \geq 0} f(f^n(\bot)) && \text{'cos } f \text{ is cts} \\ &= \bigsqcup_{n \geq 0} f^{n+1}(\bot) \\ &= \bigsqcup_{m \geq 0} f^m(\bot) && \text{'cos } f^n(\bot) \text{ an ascending chain} \end{aligned}$$

Note that this proves not only (1), but also that $fix(f)$ is a fixed point of f, i.e. $f(fix(f)) = fix(f)$.

(2) Suppose $f(d) \sqsubseteq d$.

<u>Claim</u>: $\forall n \geq 0 \, (f^n(\perp) \sqsubseteq d)$

[by induction on n:

- base case: $f^0(\perp) = \perp \sqsubseteq d$, cos $\perp$ least
- induction step: if $f^n(\perp) \sqsubseteq d$, then
  $f^{n+1}(\perp) = f(f^n(\perp)) \sqsubseteq f(d)$ by monotonicity
  $\sqsubseteq d$ by assumption.]

Hence $fix(f) \sqsubseteq d$, since it is the least upper bound of $(f^n(\perp) \mid n \geq 0)$ and we've just proved d is an upper bound.

Note that not only have we proved $fix(f)$ is a fixed point, but by (2) it's the <u>least</u> fixed
12 point, since if $f(d) = d$, then $fix(f) \sqsubseteq d$ by (2).

(d,e) is defined to be the least fixed point of the continuous function

$$h : D \times E \to D \times E$$
$$(x,y) \mapsto (f(x), g(x,y))$$

From above $fix(h)$ exists and equals (d,e).

So (d,e) satisfies

$\forall (x,y) \in D\times E \ ((f(x), g(x,y)) \sqsubseteq (x,y) \Rightarrow (d,e) \sqsubseteq (x,y))$

i.e.

(3) $\forall x \in D, y \in E \ (f(x) \sqsubseteq x \ \& \ g(x,y) \sqsubseteq y \Rightarrow d \sqsubseteq x \ \& \ e \sqsubseteq y)$.

Since $d = f(d)$, by (2) we have

(4) $\quad fix(f) \sqsubseteq d$

Then taking $x = fix(f)$ & $y = e$ in (3),
Since

$$\begin{cases} f(x) = f(fix(f)) = fix(f) = x \\ g(x,y) = g(fix(f), e) \sqsubseteq g(d,e) \quad \text{by (4)} \\ \qquad\qquad\qquad\qquad\qquad\quad = e \end{cases}$$

(3) implies $d \sqsubseteq x = fix(f)$ ($\&$ $e \sqsubseteq y = e$)
so combined with (4) we get $fix(f) = d$, as required.

6

20