1.

CST IB/II(G)/Dip. Computation Theory AMP
2002, Paper 4(11), question 6 (7)

(a) A partial recursive function (of arity n, say) is a partial function (from n-tuples of numbers to numbers) that can be built up from the
1 basic functions by repeated use of the operations of composition, primitive recursion and minimization.

• Basic functions:

1 projections $Proj^n_i \in Fun(\mathbb{N}^n, \mathbb{N})$

$Proj^n_i(x_1, \ldots x_n) \triangleq x_i$

1 zero $zero^n \in Fun(\mathbb{N}^n, \mathbb{N})$

$zero^n(x_1, \ldots, x_n) \triangleq 0$

1 successor $Suc \in Fun(\mathbb{N}, \mathbb{N})$

$Suc(x) \triangleq x+1$

1 • Composition: given $f \in PFn(\mathbb{N}^n, \mathbb{N})$ and $g_1, \ldots, g_n \in PFn(\mathbb{N}^m, \mathbb{N})$, then $f \circ (g_1, \ldots, g_n) \in PFn(\mathbb{N}^m, \mathbb{N})$ is the unique partial function such that

$$(f \circ (g_1, \ldots, g_n))(x_1, \ldots, x_m) \equiv f(g_1(x_1, \ldots, x_m), \ldots, g_n(x_1, \ldots, x_m))$$

one side defined iff the other is & then they're =.

for all $(x_1, \ldots, x_m) \in \mathbb{N}^m$.

2 • Primitive recursion: given $f \in PFn(\mathbb{N}^n, \mathbb{N})$ and $g \in PFn(\mathbb{N}^{n+2}, \mathbb{N})$, $\rho^n(f, g) \in PFn(\mathbb{N}^{n+1}, \mathbb{N})$

is the unique partial function such that

$$\begin{cases} \rho^n(f,g)(x_1,\dots,x_n,0) \equiv f(x_1,\dots,x_n) \\ \rho^n(f,g)(x_1,\dots,x_n,x+1) \equiv g(x_1,\dots,x_n,x,\rho^n(f,g)(x_1,\dots,x_n,x)) \end{cases}$$

all $(x_1,\dots x_n) \in \mathbb{N}^n$ & $x \in \mathbb{N}$.

2 • Minimization: given $f \in \mathrm{PFn}(\mathbb{N}^{n+1}, \mathbb{N})$, $\mu(f) \in \mathrm{PFn}(\mathbb{N}^n, \mathbb{N})$ is the unique partial function satisfying

$$\mu(f)(x_1,\dots,x_n) \equiv \begin{cases} \text{least } x \text{ such that} \\ f(x_1,\dots,x_n,x) > 0 \text{ and} \\ f(x_1,\dots,x_n,y) = 0 \text{, all } 0 \leq y < x \end{cases}$$

(so is undefined if no such x exists)

all $(x_1,\dots,x_n) \in \mathbb{N}^n$.

1 $f \in \mathrm{PFn}(\mathbb{N}^n, \mathbb{N})$ is (total) <u>recursive</u> if it is partial recursive and totally defined, i.e. $f(x_1,\dots,x_n)$ defined for all $(x_1,\dots,x_n) \in \mathbb{N}^n$.

2 Since each partial recursive function has a finite desciption (formal expression built up from proj^n_i, zero^n, suc using $\circ$, ρ & μ), we can enumerate them all. So there are only countably many partial recursive functions from $\mathbb{N}$ to $\mathbb{N}$, but uncountably many arbitrary such functions: hence not all functions can be partial recursive.

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① (b) (i) S is decidable if its characteristic function $\begin{cases} \chi_S \in Fun(\mathbb{N}, \mathbb{N}) \\ \chi_S(x) = \begin{cases} 0 & \text{if } x \notin S \\ 1 & \text{if } x \in S \end{cases} \end{cases}$
is (register machine) computable.

② (ii) S is r.e. if either $S = \emptyset$ or $S = \{ f(n) \mid n \in \mathbb{N} \}$ for some recursive $f \in Fun(\mathbb{N}, \mathbb{N})$.

(c) Suppose S is both r.e. and co-r.e.
If $S = \emptyset$ or $S = \mathbb{N}$, clearly it is decidable.
1 So suppose $S \neq \emptyset, \mathbb{N}$ and hence $\mathbb{N} - S \neq \emptyset, \mathbb{N}$.
Then by hypothesis there are recursive functions f, g so ~~that~~

$$S = \{ f(n) \mid n \in \mathbb{N} \}$$
$$\mathbb{N} - S = \{ g(n) \mid n \in \mathbb{N} \}$$

1 We use the fact that partial recursive functions are precisely register machine computable functions.
Hence we can find register machines computing f & g. From these we can construct a register machine such that given x in R1 (& all other registers zeroed), the machine computes successive elements of the list

1 $f(0), g(0), f(1), g(1), \dots$

stopping the first time one of these elements is equal to x and halting with

$R0 = 0$ if the element was in odd position (i.e. was a g-value)

$\quad = 1$ if the element was in even position (i.e. was an f-value)

2 Since each $x \in \mathbb{N}$ is either in S (ie is some f-value) or in $\mathbb{N} - S$ (i.e. is some g-value), but not both, <u>the machine always halts</u> giving

5 the correct value of $\chi_S(x)$. Hence <u>S is decidable</u>.

To which parts of the course does this question refer?

(a) Mostly bookwork from lectures 8–10. The last sentence is made easier by the statement in the first sentence.

(b) (i) – definition from lecture 5;
(ii) – definition from lecture 11.

(c) This result was covered in lecture 12.