2001

Continuous Mathematics - paper 4 - SOLUTION NOTES

Let $c_k = v_k + i w_k$
$c_{-k} = v_k - i w_k$

Consider the case $k \geq 1$ and extract the positive & negative k^{th} terms from the sum:

$$(v_k + i w_k) e^{i 2\pi k x} + (v_k - i w_k) e^{-i 2\pi k x}$$
$$= 2 v_k \cos 2\pi k x + i v_k (\sin 2\pi k x - \sin 2\pi k x)$$
$$+ i w_k (\cos 2\pi k x - \cos 2\pi k x) + i^2 w_k . 2 \sin 2\pi k x$$
$$= 2 v_k \cos 2\pi k x - 2 w_k \sin 2\pi k x$$

Now, let $2 v_k = A_k \cos \phi_k$
and $-2 w_k = A_k \sin \phi_k$

then this equation becomes:

$$A_k \cos \phi_k \cos 2\pi k x + A_k \sin \phi_k \sin 2\pi k x$$
$$= A_k \cos (2\pi k x - \phi_k)$$

by the identity $\cos(\theta - \psi) = \cos\psi \cos\theta + \sin\psi \sin\theta$

So: $\phi_k = \tan^{-1}\left(\frac{-w_k}{v_k}\right)$ and $A_k = 2\sqrt{v_k^2 + w_k^2}$

Which is how $c_k = v_k + i w_k$ encodes A_k and ϕ_k for $k \geq 1$

For $k = 0$

$c_0 = v_0 + i w_0 = v_0 - i w_0$

$\therefore c_0 = v_0$

$\therefore v_0 e^0 = A_0 \cos(-\phi_0)$

$\Rightarrow v_0 = A_0 \cos(-\phi_0)$

we assert that $v_0 = A_0$, $\phi_0 = 0$

i.e. we cannot determine A_0 and ϕ_0 from the analysis, although if we treat $k=0$ as ~~we~~ we treated $k \geq 1$ we would have to say $\phi_0 = 0$, but there is no mathematical justification

It is clear that:

$$f(x) = f_L(x) * \text{comb}(x)$$

where $\text{comb}(x) = \sum_{j=-\infty}^{\infty} \delta(x - Tj)$

and $\delta(x)$ is the Dirac delta function.

Now, convolution in one domain is equivalent to multiplication in the other, so:

$$F(\nu) = F_L(\nu) \times \text{Comb}(\nu)$$

where $\text{Comb}(\nu) = \sum_{l=-\infty}^{\infty} \delta\left(\nu - \frac{l}{T}\right)$

Thus: $F(\nu) = \sum_{l=-\infty}^{\infty} F_L\left(\frac{l}{T}\right) . \delta\left(\nu - \frac{l}{T}\right)$

We know that the inverse F.T. of $\delta(\nu - \alpha)$ is $e^{i2\pi\alpha x}$, therefore:

$$f(x) = \sum_{l=-\infty}^{\infty} F_L\left(\frac{l}{T}\right) e^{i2\pi\frac{l}{T}x}$$

and $c_k = F_L\left(\frac{k}{T}\right)$, as required.