**SOLUTION NOTES**

## Logic and Proof 2001 Paper 5 Question 11 (LCP)

Bookwork, see Sections 2.2, 11.1 and 11.2 of the notes. For S4 we consider model frames where accessibility is reflexive and transitive.

This claimed equivalence fails:

$$(P \land (Q \to R)) \to S \quad \simeq \quad (\neg P \lor \neg Q \lor S) \land (\neg P \lor \neg R \lor S)$$

Put $P = Q =$ **true** and $R = S =$ **false**. Then the left-hand side evaluates to **true** while the right-hand side evaluates to **false**. Here is one way of finding this countermodel. Converting the left-hand side to CNF yields $(\neg P \lor Q \lor S) \land (\neg P \lor \neg R \lor S)$, which differs from the right-hand side in the sign of $Q$.

This equivalence holds:

$$(P \to Q) \to (Q \to P) \quad \simeq \quad (Q \to P)$$

We can prove this by converting both sides to CNF. The transformation preserves the meaning, so if the two results are identical then the equivalence is valid. The right-hand side is obviously $\neg Q \lor P$. For the left-hand side, we first convert to NNF and get $(P \land \neg Q) \lor \neg Q \lor P$. Applying the distributive law yields $(P \lor \neg Q \lor P) \land (\neg Q \lor \neg Q \lor P)$. This simplifies to $(P \lor \neg Q) \land (\neg Q \lor P)$, that is, $P \lor \neg Q$.

This equivalence holds:

$$\forall xy\,(P(x) \lor \neg P(y)) \quad \simeq \quad \forall xy\,(P(x) \leftrightarrow P(y))$$

By the definition of $\leftrightarrow$, the right-hand side is equivalent to

$$\forall xy\,((P(x) \to P(y)) \land (P(y) \to P(x))).$$

Because $\forall$ distributes over $\land$, this is equivalent to

$$\forall xy\,(P(x) \to P(y)) \land \forall xy\,(P(y) \to P(x)).$$

Renaming the bound variables in the first conjunct, this becomes

$$\forall yx\,(P(y) \to P(x)) \land \forall xy\,(P(y) \to P(x)).$$

We can re-order the quantified variables in the first conjunct, obtaining

$$\forall xy\,(P(y) \to P(x)) \land \forall xy\,(P(y) \to P(x)),$$

which collapses to $\forall xy\,(P(y) \to P(x))$. By the definition of $\to$, this is equivalent to the left-hand

Proofs using the sequent calculus, resolution or (in the propositional cases) OBDDs are equally acceptable.