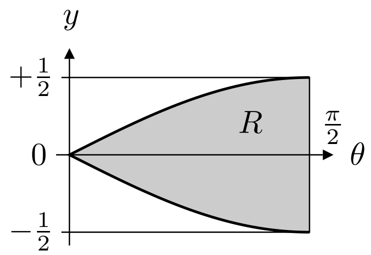
Probability Solution Notes 2005 – Paper 2 Question 5 (FHK)

The knitting-needle problem. This exploits bivariate continuous distributions (Lecture 10) and the Binomial distribution (Lecture 5).

Part (*a*)

The portion of the θ–y plane for which the value of the relevant uniform bivariate distribution function is non-zero is bounded by $0 \leqslant \theta \leqslant \frac{\pi}{2}$ and $-\frac{1}{2} \leqslant y \leqslant +\frac{1}{2}$. Region R is shown shaded. The upper and lower bounds of R are given by $y = \pm\frac{1}{2}\sin\theta$.



The area of R is given by:

$$R = 2\int_0^{\frac{\pi}{2}} \frac{1}{2}\sin\theta\, d\theta = [-\cos\theta]_0^{\frac{\pi}{2}} = 1$$

The total area of interest is $1 \times \frac{\pi}{2}$ so:

$$\mathrm{P}(hit) = \frac{1}{\pi/2} = \frac{2}{\pi}$$

An experimental estimate of this probability is simply *hits*/*drops* which may be equated to $2/\pi$:

$$\frac{hits}{drops} = \frac{2}{\pi} \qquad \text{so} \qquad \pi \approx \frac{2.drops}{hits}$$

[8 marks]

Part (*b*)

Given that X is distributed Binomial$(n, 2/\pi)$, the standard expressions np and npq can be used for:

$$\mu = \mathrm{E}(X) = \frac{2n}{\pi} \qquad \text{and} \qquad \sigma^2 = \mathrm{V}(X) = \frac{2n}{\pi}\left(1 - \frac{2}{\pi}\right)$$

[2 marks]

P.T.O.

Part (c)

The estimated value of π when the number of hits is two standard deviations below the expected number is:

$$\pi_e = \frac{2n}{\frac{2n}{\pi} - 2\sqrt{\frac{2n}{\pi}(1-\frac{2}{\pi})}} = \frac{\pi}{1 - \frac{\pi}{n}\sqrt{\frac{2n}{\pi}(1-\frac{2}{\pi})}} = \frac{\pi}{1 - \sqrt{\frac{2\pi}{n}(1-\frac{2}{\pi})}} = \frac{\pi}{1 - \sqrt{\frac{2(\pi-2)}{n}}}$$

Now consider the inequality:

$$\pi_e < \pi + 0.1$$

This is satisfied when the estimate π_e does not exceed the true value of π by more than 0.1 and, using the value of π_e just derived, this requirement is:

$$\frac{\pi}{1 - \sqrt{\frac{2(\pi-2)}{n}}} < \pi + 0.1$$

So:

$$\pi < \pi - \pi\sqrt{\frac{2(\pi-2)}{n}} + 0.1 - 0.1\sqrt{\frac{2(\pi-2)}{n}}$$

and:

$$0 < 0.1 - \sqrt{\frac{2(\pi-2)}{n}}\,(\pi + 0.1)$$

and:

$$\sqrt{\frac{2(\pi-2)}{n}}\,(\pi + 0.1) < 0.1$$

Squaring both sides:

$$\frac{2(\pi-2)}{n}\,(\pi+0.1)^2 < 0.01$$

Rearranging:

$$n > \frac{2(\pi-2)\,(\pi+0.1)^2}{0.01}$$

[10 marks]