Solution notes

Information Theory and Coding 2005 – Paper 8 Question 10

(*a*) [*This question relates to the extension of Information Theory to continuous random variables.*]

(*i*) The *differential entropy* $h(X)$ is defined as:

$$h(X) = \int_{-\infty}^{+\infty} p(x) \log \left(\frac{1}{p(x)} \right) dx$$

[1 mark]

(*ii*) The *joint entropy* $h(X, Y)$ of random variables X and Y is:

$$h(X, Y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} p(x, y) \log \left(\frac{1}{p(x, y)} \right) dxdy$$

[1 mark]

(*iii*) The *conditional entropy* $h(X|Y)$ of X, given Y, is:

$$h(X|Y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} p(x, y) \log \left(\frac{p(y)}{p(x, y)} \right) dxdy$$

[1 mark]

(*iv*) The *mutual information* $i(X;Y)$ between continuous random variables X and Y is:

$$i(X;Y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} p(x, y) \log \left(\frac{p(x|y)}{p(x)} \right) dxdy$$

[1 mark]

(*v*) The *capacity* of a continuous communication channel is computed by finding the maximum of the above expression for mutual information $i(X;Y)$ over all possible input distributions for X. [1 mark]

(*b*) The continuous signal $g(t)$ is *modulated* into a selected part of the frequency spectrum, defined by a transmitter carrier frequency. The signal is just multiplied by that carrier frequency (in complex form, i.e. as a complex exponential of frequency ω). The modulation theorem asserts that then the Fourier transform of the original signal is merely shifted by an amount equal to that carrier frequency ω:

$$g(t)e^{i\omega t} \rightleftharpoons G(k - \omega)$$

Many different signals can each be thus modulated into their own frequency bands and transmitted together over the electromagnetic spectrum using a common antenna. Upon reception, the reverse operation is performed by a tuner, i.e. multiplication of the received signal by the complex conjugate complex exponential $e^{-i\omega t}$ [and filtering

away any other transmitted frequencies], thus restoring the original signal $g(t)$.
[4 marks]

(c) [*These questions relate to coding theory and transforms.*]

(i) The Fourier transform of the n^{th} derivative of $g(x)$ is:
$(ik)^n G(k)$ [2 marks]

(ii) The entropy of the new alphabet of symbol blocks is simply n times the entropy of the original alphabet:

$$H(\mathcal{S}^n) = nH(\mathcal{S})$$

[2 marks]

(iii) The *efficiency* of the coding is defined as

$$\eta = \frac{H}{R}$$

[2 marks]

(d) (i) 10 V = 10^7 μV = (20×7) dBμV = 140 dBμV [1 mark]

(ii) Human colour vision splits the red/green/blue input signal into separate luminosity and colour channels. Compression algorithms can achieve a simple approximation of this by taking a linear combination of about 30% red, 60% green, and 10% blue as the luminance signal $Y = 0.3R + 0.6G + 0.1B$ (the exact coefficients differ between standards and do not matter here). The remaining colour information can be preserved, without adding redundancy, in the form of the difference signals $R - Y$ and $B - Y$. These are usually encoded scaled as $Cb = (B - Y)/2 + 0.5$ and $Cr = (R - Y)/1.6 + 0.5$, such that the colour cube remains, after this "rotation", entirely within the encoded unit cube, assuming that the original RGB values were all in the interval $[0, 1]$. [4 marks]

[*This question relates to (i) the section on perceptual scales, (ii) the section on colour coordinates.*]