2004 DS 1 P. 1

p 8 q 15

GW

Denotational semantics Qu. 1

(a) (i) ~~Proof~~:

Induction hypothesis: $\lambda f.\, f^n(\bot)$ is cts.

Base case $n = 0$. $\lambda f.\, f^0(\bot) = \lambda f.\, \bot$ a constant function so clearly cts.

Induction step. Assume inductively that $\lambda f.\, f^n(\bot)$ is cts. The function $\lambda f.\, f^{n+1}(\bot)$ factorises as

$$f \overset{\text{diagonal}}{\longmapsto} (f, f) \overset{id \times \lambda f. f^n(\bot)}{\longmapsto} (f, f^n(\bot)) \overset{\text{evaln.}}{\longmapsto} f(f^n(\bot)) = f^{n+1}(\bot)$$

the composition of cts. functions — it is therefore cts.

(ii) $$\left(\bigsqcup_n \left(\lambda f.\, f^n(\bot)\right)\right)(g)$$

$$= \bigsqcup_n g^n(\bot) \quad \text{as lubs of fns are got ptwise.}$$

$$= fix(g).$$

Hence as g is arbitrary

$$fix = \bigsqcup_n \left(\lambda f.\, f^n(\bot)\right).$$

(b) (i)

$$\frac{\Gamma, x : \tau \vdash t : \tau}{\Gamma \vdash \mathbf{rec}\, x : \tau.\, t : \tau}$$

(ii) $$\frac{t[\text{rec}x:\tau/t/x] \Downarrow V}{\text{rec}x:\tau.\ t \Downarrow V}$$

(iii) $[\![\Gamma \vdash \text{rec}x:\tau.\ t]\!]\rho =$

$$fix\left(\lambda d \in [\![\tau]\!].[\![\Gamma, x:\tau \vdash t]\!]\rho[d/x]\right)$$

(iv) $$fix_\tau(f) = f(fix_\tau(f))$$

where $f: ([\![\tau]\!] \to [\![\tau]\!]) \to [\![\tau]\!]$.

i.e. $$fix_\tau = \lambda f.\ (fix_\tau(f))$$

Thus we can define fix_τ as

$$\text{rec}\,F.\ \lambda f.\ (F(f)).$$