Solution Notes

Numerical Analysis II 2001

p9q11 p13q10

MROD

Question B

Differentiate with respect to a:

$$T_N'(a) = f'(a) - f'(a) + (x-a)f''(a) - (x-a)f''(a) + \dots$$

$$\dots + \frac{(x-a)^{N-1}}{(N-1)!} f^{(N)}(a) - \frac{(x-a)^{N-1}}{(N-1)!} f^{(N)}(a) + \frac{(x-a)^N}{N!} f^{(N+1)}(a)$$

$$= \frac{(x-a)^N}{N!} f^{(N+1)}(a).$$

For any $g \in C^1[a,b]$,

$$g(x) = g(a) + \int_a^x g'(t)\,dt$$

so

$$T_N(x) = T_N(a) + \int_a^x T_N'(t)\,dt.$$ [6 marks]

If Qf represents the quadrature rule in Peano's theorem then

$$E(f) = \int_a^b f(x)\,dx - Qf.$$

E_x is the same as E but, to resolve ambiguity, indicates that integration is with respect to x.

$$(x-t)_+^N = \begin{cases} (x-t)^N, & x > t \\ 0, & x \le t. \end{cases}$$ [4 marks]

Since $f \in C^{N+1}[a,b]$ and $E(P_r) = 0$ for $r \le N$ it follows that

$$E_x[T_N(a)] = 0$$

so we get directly from Taylor's theorem

$$E(f) = \frac{1}{N!} E_x\left\{ \int_a^x f^{(N+1)}(t)(x-t)^N\,dt \right\}.$$

It makes no difference to write

$$E(f) = \frac{1}{N!} E_x\left\{ \int_a^x f^{(N+1)}(t)(x-t)_+^N\,dt \right\}$$

but we can then replace the upper limit of the integral by b:

$$E(f) = \frac{1}{N!} E_x\left\{ \int_a^b f^{(N+1)}(t)(x-t)_+^N\,dt \right\}.$$

Now E_x can be taken inside the integral and Peano's theorem is proved.

[8 marks]

The simplified formula may be used if $K(t)$ does not change sign in $[a,b]$. [2 marks]