SOLUTION NOTES

Complexity Theory 2003 Paper 5 Question 12 (AD)

1. $A \leq_P B$ if there is a function $f : \Sigma_1^* \rightarrow \Sigma_2^*$ computable in polynomial time such that $f(x) \in B$ if, and only if, $x \in A$.

   [2 marks]

2. (a) It is reflexive. To see that $A \leq_P A$, we can take f in the above definition to be the identity function.

   [2 marks]

   (b) It is not symmetric. For instance, let A be a language decidable in polynomial time and B a language that is not (for instance, and EXPTIME-complete language). To see that $A \leq_P B$, choose two strings $s_1 \in B$ and $s_2 \notin B$. Let $f : \Sigma_1^* \rightarrow \Sigma_2^*$ be the function such that $f(x) = s_1$ if $x \in A$ and $f(x) = s_2$ if $x \notin A$. Then f is computable in polynomial time, using the procedure for deciding A. To see that $B \not\leq_P A$, note that if this were the case then B would be decidable in polynomial time.

   [4 marks]

   (c) It is transitive. If f is a reduction from A to B and g is a reduction from B to C, then $g \circ f$ is a reduction from A to C.

   [3 marks]

3. Suppose L is any language other than $\emptyset$ and Σ^*. Then we can find two strings $s_1 \in L$ and $s_2 \notin L$. By the construction for 2(b) above, for any language A that is in P, we can construct a reduction f of A to B. Now, suppose that P=NP and L is a language in NP. It then follows that any language in P, and therefore in NP, is reducible to L, which is therefore NP-complete. $\emptyset$ cannot be NP-complete because the only language reducible to it is itself. If A contains any string x and $A \leq_P \emptyset$ by a reduction f, then $f(x) \in \emptyset$, which is a contradiction. Similarly, the only language reducible to Σ^* is itself.

   [9 marks]