Numerical Analysis II Question A 1999

p 8 q 14
~~p 12 q 13~~
MROD

Model Answer

$$T_{n+1} = 2x\,T_n - T_{n-1}$$

$$T_0 = 1$$

$$T_1 = x$$

$$T_2 = 2x^2 - 1$$

$$T_3 = 4x^3 - 3x$$

$$T_4 = 8x^4 - 8x^2 + 1$$

$$T_5 = 16x^5 - 20x^3 + 5x$$

[4 marks]

Define $\overline{T}_n(x) = T_n(x)/2^{n-1}$.

The best polynomial approximation to x^n of lower degree is $x^n - \overline{T}_n(x)$ which is of degree $n-2$.

Let $P_n(x) = a_0 + a_1 x + \dots + a_n x^n$ be a truncated Taylor series. The best approximation to $P_n(x)$ by a polynomial of lower degree is therefore

$$L_{n-1}(x) = P_n(x) - a_n \overline{T}_n(x).$$

The process of forming $L_{n-1}(x)$ is called <u>economization</u>.

Since $|T_n(x)| \leq 1$ over $[-1,1]$, $|\overline{T}_n(x)| \leq \frac{1}{2^{n-1}}$ over $[-1,1]$.

It follows that

$$|L_{n-1}(x) - P_n(x)| \leq \frac{a_n}{2^{n-1}}.$$

[7 marks]

The best abscissae $\{x_j\}$ for interpolation are the zeros of the Chebyshev polynomial $T_n(x)$ since these minimize the term $\prod_{j=1}^{n}(x - x_j)$ in the Lagrange error formula.

[2 marks]

Let $E_n(x) = \sin x - P_n(x)$.

Then $|E_3(x)| \leq |E_3(1)| \simeq \frac{1}{5!} = \frac{1}{120} \simeq 8 \times 10^{-3}$,

$|E_5(x)| \leq |E_5(1)| \simeq \frac{1}{\ } = \frac{1}{\ } \simeq 2 \times 10^{-4}$.

- Performing economization,

$$\sin x \simeq P_5(x) - \frac{1}{5!}\bar{T}_5(x) \; ; \; \bar{T}_5(x) = x^5 - \frac{5}{4}x^3 + \frac{5}{16}x$$

$$= x - \frac{x^3}{3!} - \frac{1}{5!}\left[-\frac{5}{4}x^3 + \frac{5}{16}x\right]$$

$$= \frac{383}{384}x - \frac{15}{16}\frac{x^3}{3!} .$$

Maximum absolute error $\simeq |E_5(1)| + \frac{a_5}{2^4}$

$$= \frac{1}{5040} + \frac{1}{1920}$$

$$\simeq 7 \times 10^{-4}$$

$$< |E_3(1)|$$

[7 marks]