Paper 4/11 (Solution)

i) f is a TR function of a single variable iff a) f is computable (μR)
and b) f is everywhere defined.

ii) $\{f_n : \mathbb{N} \to \mathbb{N}\}$ is recursively enumerable if their values can be determined within a single computational framework — equivalently:

a) the set of Gödel numbers for $\{f_n\}$ can be recursively enumerated (? define this!);

b) there is a TR fn $u(n, x)$ of arity 2 s.t. $f_n(x) = u(n, x)$ $\forall n, x$.

Suppose as above that $u(n, x) = f_n(x)$ enumerates some sequence $\{f_n\}$ of TR functions.

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Define a function $g: \mathbb{N} \to \mathbb{N}$ as follows:

$$g(x) = S\{u(x,x)\}.$$

Then $g \neq f_n$, since $g(n) = f_n(n) + 1$.

But g is everywhere defined and computable, hence a TR function distinct from each f_n

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To establish a sequence $\{g_n\}$ of TR functions with the required properties we proceed exactly as above, starting with the definition of $g_0 = g$. We include the new function g_0 with the enumeration of the $\{f_n\}$ by a new function $u_1(n,x)$:

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$$u_1(0,x) = g_0(x) = S\{u(x,x)\}$$

$$u_1(n+1,x) = f_n(x) = u(x,x).$$

We can now define a new TR function g_1 distinct from the functions enumerated by $u_1(n,x)$. Include g_1 in an enumeration $u_2(n,x)$, derive g_2, and so on.

The essence of the construction is to ensure that a newly defined function g_n differs from g_x for $x < n$ at argument x, and from each function f_m in the original enumeration at arguments $x \geqslant n$.

We now write $N\{h(x)\}$ to indicate a computed value $\neq h(x)$.

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The required scheme is defined as follows:

$$\begin{cases} v(n,x) = N\{v(x,x)\} & \forall\, x < n \\ v(n,x) = N\{u(x-n,x)\} & \forall\, x \geqslant n \end{cases}$$

Eliminating the recursion, we may write:

$$\begin{cases} v(n,x) = N\{N\{u(0,x)\}\} & \forall\, x < n \\ v(n,x) = N\{u(x-n,x)\} & \forall\, x \geqslant n \end{cases}$$

A neat way of realising this is:

$$\begin{cases} v(n,x) = u(0,x) & \forall\, x < n \\ v(n,x) = S\{u(x-n,x)\} & \forall\, x \geqslant n \end{cases}$$