AMP 1.

CST II Types

2002, Paper 7, question 13

(a)

2 $$(\text{let})\ \frac{\begin{cases} A \cup A', \Gamma \vdash M : \tau \\ A', \Gamma[x \mapsto \forall A(\tau)] \vdash M' : \tau' \end{cases}}{A', \Gamma \vdash \text{let } x = M \text{ in } M' : \tau'}$$

1 provided $A \cap A' = \emptyset$
$x \notin dom(\Gamma)$

(where $\Gamma[x \mapsto \forall A(\tau)]$ maps x to $\forall A(\tau)$

3 and otherwise acts like Γ.).

(b) Consider the case when

$M = \lambda y\ (y)$

$M' = x\ x$

Claim that

1 $\{\alpha\}, \emptyset \vdash \text{let } x = \lambda y\ (y) \text{ in } x\ x : \alpha \to \alpha$

3 Proof uses typing rules for λ-abstraction, application, and variables:

$$(\text{fn})\ \frac{A, \Gamma[x \mapsto \tau] \vdash M : \tau'}{A, \Gamma \vdash \lambda x(M) : \tau \to \tau'} \text{ if } x \notin dom(\Gamma)$$

$$(\text{app})\ \frac{\begin{cases} A, \Gamma \vdash M_1 : \tau \to \tau' \\ A, \Gamma \vdash M_2 : \tau \end{cases}}{A, \Gamma \vdash M_1 M_2 : \tau'}$$

$(\text{var}\succ)$ $A, \Gamma \vdash x : \tau$ if $ftv(\Gamma) \subseteq A$
$ftv(\tau) \subseteq A$
& for some $x \in dom(\Gamma)$,
with $\Gamma(x) = \sigma$ say,
$\sigma \succ \tau$.

Then we have

(1) $\{\alpha, \beta\}, [y \mapsto \beta] \vdash y : \beta$ by $(\text{var}\succ)$
(2) $\{\alpha, \beta\}, \emptyset \vdash \lambda y(y) : \beta \to \beta$ by (fn) on (1)
(3) $\{\alpha\}, [x \mapsto \forall\{\beta\}(\beta \to \beta)] \vdash x : (\alpha \to \alpha) \to (\alpha \to \alpha)$ by $(\text{var}\succ)$
(4) $\{\alpha\}, [x \mapsto \forall\{\beta\}(\beta \to \beta)] \vdash x : \alpha \to \alpha$ by $(\text{var}\succ)$
(5) $\{\alpha\}, [x \mapsto \forall\{\beta\}(\beta \to \beta)] \vdash x x : \alpha \to \alpha$ by (app) on (3) & (4)
(6) $\{\alpha\}, \emptyset \vdash \text{let } x = \lambda y(y) \text{ in } x\,x : \alpha \to \alpha$ by (let) on (2) & (5)

1 and note that this occurrence of x has implicit type $(\alpha \to \alpha) \to (\alpha \to \alpha)$ (cf. line (3)), whereas this one has implicit type $\alpha \to \alpha$ (cf. line (4)).
5 So the let-bound variable x occurs polymorphically.

(c)

1 (unit) $A, \Gamma \vdash () : unit$

provided $ftv(\Gamma) \subseteq A$

1 (ref) $$\frac{A, \Gamma \vdash M : \tau}{A, \Gamma \vdash ref\, M : \tau\, ref}$$

1 (get) $$\frac{A, \Gamma \vdash M : \tau\, ref}{A, \Gamma \vdash\, !M : \tau}$$

1 (set) $$\frac{A, \Gamma \vdash M : \tau\, ref \quad A, \Gamma \vdash M' : \tau}{A, \Gamma \vdash M := M' : unit}$$

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(d) Consider the expression

1 $M = $ let $r = ref\, \lambda x(x)$ in
let $u = (r := \lambda y(ref(!y)))$ in
$(!r)()$

If we try to evaluate this expression, we first create a fresh storage location (r) containing $\lambda x(x)$, then update it to contain $\lambda y(ref(!y))$, then apply this contents to $()$; this application simplifies to evaluating $ref(!())$, which results in trying to evaluate $!()$, which fails because

2 $()$ is not a storage location.

However, even though evaluation of M goes wrong, it is typeable:

1 $\quad \emptyset, \emptyset \vdash M : unit$

2 follows from (let) on

(7) $\{\alpha\}, \emptyset \vdash ref\ \lambda x(x) : (\alpha \to \alpha)\, ref$

& (8) $\emptyset, [r \mapsto \forall\{\alpha\}((\alpha \to \alpha)\, ref)] \vdash (let\ u = (r := \lambda y(ref(!y)))\ in\ (!r)()) : unit$

(7) holds by (ref) on

$$\{\alpha\}, \emptyset \vdash \lambda x(x) : \alpha \to \alpha$$

which is proved by (var>) & (fn); and

(8) holds by (let) on

(9) $\emptyset, \Gamma \vdash r := \lambda y(ref(!y)) : unit$

(10) $\emptyset, \Gamma[u \mapsto unit] : (!r)() : unit$

where we write Γ for $[r \mapsto \forall\{\alpha\}((\alpha \to \alpha)ref)]$.

The proof of (9) is by (set) on

$\quad \emptyset, \Gamma \vdash r : (\alpha\, ref \to \alpha\, ref)\, ref$ by (var>)

& $\quad \emptyset, \Gamma \vdash \lambda y(ref(!y)) : \alpha\, ref \to \alpha\, ref$, by (var>), (get) & (ref).

The proof of (10) is by (app) on

$\quad \emptyset, \Gamma[u \mapsto unit] \vdash\ !r : unit \to unit$ by (var>) & (get)

$\quad \emptyset, \Gamma[u \mapsto unit] \vdash () : unit$ by (unit).

Revised ML modifies the rule (let) by imposing the side condition

2 "$A = \emptyset$ or M is a <u>value</u>" $(::= x \mid \lambda x(M) \mid () \mid \ldots)$

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To which parts of the lecture course does this question refer?

(a) A definition given in lecture 2.

(b) Examples of this phenomenon were given in lectures 2 & 3.

(c) & (d) are bookwork from lecture 5.