CST II 2001, paper 9, question 6

Types AMP

PLC typing judgements take the form $\Gamma \vdash M : \tau$

where $\Gamma = (\Gamma_{tv}, \Gamma_{ta})$

Γ_{tv} = finite set of type variables

Γ_{ta} = finite map from variables to types

and where M is a PLC term & τ a PLC type.

Write Γ ok to mean free type vars of Γ_{ta} are in Γ_{tv}. Valid typing judgements are inductively generated by following axiom & rules:

(var) $\Gamma \vdash x : \tau$ if Γ ok & $(x : \tau) \in \Gamma_{ta}$

(fn) $$\frac{\Gamma, x : \tau_1 \vdash M : \tau_2}{\Gamma \vdash \lambda x : \tau_1 (M) : \tau_1 \to \tau_2}$$ if $x \notin dom(\Gamma_{ta})$

where $\begin{cases} (\Gamma, x:\tau_1)_{tv} = \Gamma_{tv} \\ (\Gamma, x:\tau_1)_{ta} = \Gamma_{ta}[x \mapsto \tau_1] \end{cases}$

(gen) $$\frac{\alpha, \Gamma \vdash M : \tau}{\Gamma \vdash \Lambda\alpha(M) : \forall\alpha(\tau)}$$ if $\alpha \notin \Gamma_{tv}$ & Γ ok

where $\begin{cases} (\alpha, \Gamma)_{tv} = \{\alpha\} \cup \Gamma_{tv} \\ (\alpha, \Gamma)_{ta} = \Gamma_{ta} \end{cases}$

(spec) $$\frac{\Gamma \vdash M : \forall\alpha(\tau)}{\Gamma \vdash M\tau' : \tau[\tau'/\alpha]}$$ if $ftv(\tau') \subseteq \Gamma_{tv}$

result of substituting τ' for free occurrences of α in τ

(app) $$\frac{\Gamma \vdash M : \tau_1 \to \tau_2 \quad \Gamma \vdash M' : \tau_1}{\Gamma \vdash M M' : \tau_2}$$

(5)

1 (a) Yes: $\tau = \forall\alpha(\alpha) \to \forall\alpha(\alpha)$

<u>Proof</u>:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\{\beta\},\{x:\forall\alpha(\alpha)\} \vdash x : \forall\alpha(\alpha)}\,(\text{var})}{\{\beta\},\{x:\forall\alpha(\alpha)\} \vdash x\beta : \beta}\,(\text{spec})}{\phi,\{x:\forall\alpha(\alpha)\} \vdash \Lambda\beta(x\beta) : \forall\beta(\beta)}\,(\text{gen})}{\phi,\phi \vdash \lambda x:\forall\alpha(\alpha)(\Lambda\beta(x\beta)) : \underbrace{\forall\alpha(\alpha)\to\forall\beta(\beta)}_{=_\alpha\ \forall\alpha(\alpha)\to\forall\alpha(\alpha).}}\,(\text{fn})$$

2

(3)

1 (b) No such τ exists.

If $\phi,\phi \vdash \Lambda\alpha(\lambda x:\alpha(\Lambda\beta(x\beta))) : \tau$ were provable, it would have had to be proved by an application of rule (gen) to

$$\{\alpha\},\phi \vdash \lambda x:\alpha(\Lambda\beta(x\beta)) : \tau_1$$

where $\tau = \forall\alpha(\tau_1)$. This in turn would have to have been proved by applying (fn) to

$$\{\alpha\},\{x:\alpha\} \vdash \Lambda\beta(x\beta) : \tau_2$$

where $\tau_1 = \alpha \to \tau_2$. This in turn must have come from (gen) on

$$\{\alpha,\beta\},\{x:\alpha\} \vdash x\beta : \tau_3$$

where $\tau_2 = \forall\beta(\tau_3)$. And this must have come from (spec) on

$$\{\alpha,\beta\},\{x:\alpha\} \vdash x : \forall\gamma(\tau_4)$$

2 where $\tau_3 = \tau_4[\beta/\gamma]$. But the latter is not provable,

(3) because $\alpha \neq \forall\gamma(\tau_4)$ for any γ, τ_4.

1 (c) Yes : $\tau = \forall\alpha(\alpha)$.

Proof

2

$$\dfrac{\dfrac{\dfrac{}{\{\alpha\},\{x:\tau\}\vdash x:\forall\alpha(\alpha)}\,(\text{var})}{\{\alpha\},\{x:\tau\}\vdash x(\alpha\to\alpha):\alpha\to\alpha}\,(\text{spec}) \qquad \dfrac{\dfrac{}{\{\alpha\},\{x:\tau\}\vdash x:\forall\alpha(\alpha)}\,(\text{var})}{\{\alpha\},\{x:\tau\}\vdash x\alpha:\alpha}\,(\text{spec})}{\{\alpha\},\{x:\tau\}\vdash x(\alpha\to\alpha)(x\alpha):\alpha}\,(\text{app})$$

$$\dfrac{\{\alpha\},\{x:\tau\}\vdash x(\alpha\to\alpha)(x\alpha):\alpha}{\emptyset,\{x:\tau\}\vdash \Lambda\alpha(x(\alpha\to\alpha)(x\alpha)):\forall\alpha(\alpha) =_\alpha \forall\beta(\beta)}\,(\text{gen})$$

3

$$\dfrac{\emptyset,\{x:\tau\}\vdash \Lambda\alpha(x(\alpha\to\alpha)(x\alpha)):\forall\beta(\beta)}{\emptyset,\emptyset\vdash \lambda x:\tau(\Lambda\alpha(x(\alpha\to\alpha)(x\alpha))):\tau\to\forall\beta(\beta).}\,(\text{fn})$$

2 (d) Yes : $\tau = \forall\alpha(\alpha\to\alpha)$

Proof Writing $\Gamma \triangleq \{\alpha\},\{x:\tau\}$,

1

$$\dfrac{\dfrac{\dfrac{}{\Gamma\vdash x:\forall\alpha(\alpha\to\alpha)}\,(\text{var})}{\Gamma\vdash x(\alpha\to\alpha):(\alpha\to\alpha)\to(\alpha\to\alpha)}\,(\text{spec}) \qquad \dfrac{\dfrac{}{\Gamma\vdash x:\forall\alpha(\alpha\to\alpha)}\,(\text{var})}{\Gamma\vdash x\alpha:\alpha\to\alpha}\,(\text{spec})}{\Gamma\vdash x(\alpha\to\alpha)(x\alpha):\alpha\to\alpha}\,(\text{app})$$

$$\dfrac{\Gamma\vdash x(\alpha\to\alpha)(x\alpha):\alpha\to\alpha}{\emptyset,\{x:\tau\}\vdash \Lambda\alpha(x(\alpha\to\alpha)(x\alpha)):\forall\alpha(\alpha\to\alpha)}\,(\text{gen})$$

3

$$\dfrac{\emptyset,\{x:\tau\}\vdash \Lambda\alpha(x(\alpha\to\alpha)(x\alpha)):\forall\alpha(\alpha\to\alpha)}{\emptyset,\emptyset\vdash \lambda x:\tau(\Lambda\alpha(x(\alpha\to\alpha)(x\alpha))):\tau\to\forall\alpha(\alpha\to\alpha).}\,(\text{fn})$$

1 (e) No.

Proof. If $\emptyset,\emptyset\vdash \Lambda\alpha(\lambda x:\tau(x(\alpha\to\alpha)(x\alpha))):\forall\alpha(\alpha\to\alpha)$ were provable, it must have been proved by applying

2 (gen) to $\{\alpha\},\emptyset\vdash \lambda x:\tau(x(\alpha\to\alpha)(x\alpha)):\alpha\to\alpha$

and for this to be provable, it must be the case that $\tau=\alpha$ & $\{\alpha\},\{x:\alpha\}\vdash x(\alpha\to\alpha)(x\alpha):\alpha$ is provable : but this requires $x\alpha$ to be typeable in context $\{\alpha\},\{x:\alpha\}$, which is impossible because

3 α is not of the form $\forall\gamma(\tau)$ for any γ & τ (cf. (a)).