p8 q 4
JGD

Model Answer, Information Theory and Coding, Question 2. 2000

A.
If a continuous signal is discretely sampled by multiplying it with a sequence of uniformly-spaced Dirac delta functions, having frequency f_s, then its Fourier spectrum becomes reproduced in endless repeated copies along the frequency axis at all integer multiples of f_s. This is because multiplication in the signal domain is equivalent to convolution in the frequency domain, and convolution with a string of delta functions at frequency positions f_s, $2f_s$, $3f_s$, ... , (which is the comb-like Fourier Transform of the comb-like sampling function), has the consequence of reproducing the entire original spectrum at every one of those positions.

[3 marks]

B.

$$p(x|y) = \frac{p(y|x)p(x)}{p(y)}$$

[3 marks]

C.
1. Entropy of the source, $H(X)$, is 1 bit.

[1 mark]

2. Output probabilities are $p(y = 0) = (0.5)(1 - \epsilon) + (0.5)\epsilon = 0.5$ and $p(y = 1) = (0.5)(1 - \epsilon) + (0.5)\epsilon = 0.5$. Entropy of this distribution is $H(Y) = 1$ bit, just as for the entropy $H(X)$ of the input distribution.

[3 marks]

3. Joint probability distribution $p(X, Y)$ is

$$\begin{pmatrix} 0.5(1-\epsilon) & 0.5\epsilon \\ 0.5\epsilon & 0.5(1-\epsilon) \end{pmatrix}$$

and the entropy of this joint distribution is $H(X,Y) = -\sum_{x,y} p(x,y)\log_2 p(x,y)$

$= -(1-\epsilon)\log(0.5(1-\epsilon)) - \epsilon\log(0.5\epsilon) = (1-\epsilon) - (1-\epsilon)\log(1-\epsilon) + \epsilon - \epsilon\log(\epsilon)$

$= 1 - \epsilon\log(\epsilon) - (1-\epsilon)\log(1-\epsilon)$

[3 marks]

4. The mutual information is $I(X;Y) = H(X) + H(Y) - H(X,Y)$, which we can evaluate from the quantities above as: $1 + \epsilon\log(\epsilon) + (1-\epsilon)\log(1-\epsilon)$.

[2 marks]

5. In the two cases of $\epsilon = 0$ and $\epsilon = 1$ (perfect transmission, and perfectly erroneous transmission), the mutual information reaches its maximum of 1 bit and this is also then the channel capacity.

[3 marks]

6. If $\epsilon = 0.5$, the channel capacity is minimal and equal to 0.

[2 marks]