Numerical Analysis I 2002

p4q7
MRO'D

Solution Notes – Question B

Question concerns: Errors in numerical methods.

(a) Let x^* be the floating-point representation of a number x.
Absolute error ϵ_x is defined by $x^* = x + \epsilon_x$
Relative error δ_x is defined by $x^* = x(1+\delta_x) = x + x\delta_x$

so $\epsilon_x = x\delta_x$.

$x^* + y^* = x + y + \epsilon_x + \epsilon_y$
$\Rightarrow$ worst-case absolute error is $|\epsilon_x| + |\epsilon_y|$

$x^* y^* = xy(1+\delta_x)(1+\delta_y) = xy(1+\delta_x+\delta_y) + O(\delta^2)$
$\Rightarrow$ worst-case relative error is $|\delta_x| + |\delta_y|$. [5 marks]

(b) Loss of significance occurs when relative error grows large when absolute error does not. [1 mark]

(c) Worst-case relative error in evaluating x^2 is $2|\delta_x|$.
Using $\epsilon = x\delta$, worst-case absolute error is $2x^2|\delta_x|$.
Worst-case absolute error in evaluating $x^2 - y^2$ is

$$2(x^2|\delta_x| + y^2|\delta_y|).$$ [3 marks]

(d) Assume 1,2 are represented exactly.
Worst-case relative error in evaluating $\sin 2\theta$ is $|\delta_s| + |\delta_c|$.
Worst-case absolute error in evaluating $\cos 2\theta$ is $2\cos^2\theta|\delta_c|$,
so worst-case relative error is

$$\frac{2\cos^2\theta|\delta_c|}{|2\cos^2\theta - 1|}, \quad 2\cos^2\theta - 1 \neq 0.$$

The relative error will grow large, hence loss of significance, when $|2\cos^2\theta - 1|$ is small, i.e. when

$$2\cos^2\theta \simeq 1$$
$$\text{i.e. } \cos\theta \simeq \pm\frac{1}{\sqrt{2}}.$$ [6 marks]

(e) (i) If $|x|$ is very small then $|y| \simeq |z|$ implies that loss of significance can occur because $|x^2+y^2-z^2|$ will be small.

(ii) If $|x| \simeq |y| \simeq |z|$ then $x^2+y^2-z^2 \simeq x^2$ so loss of significance cannot occur as $|x|$ is not small. [4 marks]

(f) Given accurate data, $x^2+y^2-z^2$ can be computed more accurately by factorising as

$$x^2 + (y+z)(y-z)$$

provided at least one guard digit is in use. [1 mark]