Probability 2000

Model Answer

p2 q5
FHK

Let w_n = number of patterns of n bits in which there are no instances of adjacent 0s.

There are two possible cases:

1. Leading digit is a 1.
   In this case there are w_{n-1} possibilities – the 1 may be followed by any valid pattern of $n-1$ bits.

2. Leading digit is a 0.
   In this case the second digit must be a 1 and the 01 may be followed by any valid pattern of $n-2$ bits. There are w_{n-2} such patterns.

Hence $\quad w_n = w_{n-1} + w_{n-2} \quad$ --- (1)

If u_n = Probability of no instances of adjacent 0s

then $u_n = \dfrac{w_n}{2^n}$

From (1) $\quad 2^n u_n = 2^{n-1} u_{n-1} + 2^{n-2} u_{n-2}$

Hence $\quad 4u_n - 2u_{n-1} - u_{n-2} = 0 \quad$ [8 marks]

With registers of zero or one bits it is clearly impossible to have two adjacent 0s so

$u_0 = u_1 = 1 \quad$ [2 marks]

Solving the equation:

$$4u_n - 2u_{n-1} - u_{n-2} = 0$$

Auxiliary equation is $4w^2 - 2w - 1 = 0$ so $w = \dfrac{2 \pm \sqrt{4+16}}{8}$

General solution is $u_n = A\left(\dfrac{1+\sqrt{5}}{4}\right)^n + B\left(\dfrac{1-\sqrt{5}}{4}\right)^n$

$u_0 = 1$ gives $A + B = 1$ --- ①

$u_1 = 1$ gives $\dfrac{A+B}{4} + \dfrac{A-B}{4}\cdot\sqrt{5} = 1$

so $\dfrac{1}{4} + \dfrac{A-B}{4}\cdot\sqrt{5} = 1$

$$\left.\begin{aligned} \therefore\ A - B &= \frac{3}{\sqrt{5}} \\ \text{From ①} \rightarrow\ A + B &= 1 \end{aligned}\right\} \quad \begin{aligned} 2A &= 1 + \frac{3}{\sqrt{5}} \quad \therefore A = \frac{3+\sqrt{5}}{2\sqrt{5}} \\ 2B &= 1 - \frac{3}{\sqrt{5}} \quad \therefore B = -\frac{3-\sqrt{5}}{2\sqrt{5}} \end{aligned}$$

From general solution:

$$u_n = \frac{1}{2\sqrt{5}}\left[(3+\sqrt{5})\left(\frac{1+\sqrt{5}}{4}\right)^n - (3-\sqrt{5})\left(\frac{1-\sqrt{5}}{4}\right)^n\right]$$

[10 marks]