Types 2005 - Paper 9 Question 10 (AMP)

A PLC type for represting lists:

② $\mathrm{list}_\alpha \triangleq \forall \alpha' (\alpha' \to (\alpha \to \alpha' \to \alpha') \to \alpha')$

Constructors:

2 $\mathrm{Nil} \triangleq \Lambda \alpha, \alpha' (\lambda x' : \alpha', f : \alpha \to \alpha' \to \alpha' (x'))$

2 $\mathrm{Cons} \triangleq \Lambda \alpha (\lambda x : \alpha, l : \mathrm{list}_\alpha (\Lambda \alpha' (\lambda x' : \alpha', f : \alpha \to \alpha' \to \alpha' (f\, x\, (l\, \alpha'\, x'\, f)))))$

Iteration function:

2 $\mathrm{iter} \triangleq \Lambda \alpha, \alpha' (\lambda x' : \alpha', f : \alpha \to \alpha' \to \alpha' (\lambda l : \mathrm{list}_\alpha\ (l\, \alpha'\, x'\, f)))$

Types of these expressions:

1 (1) $\vdash \mathrm{Nil} : \forall \alpha (\mathrm{list}_\alpha)$

1 (2) $\vdash \mathrm{Cons} : \forall \alpha (\alpha \to \mathrm{list}_\alpha \to \mathrm{list}_\alpha)$

1 (3) $\vdash \mathrm{iter} : \forall \alpha, \alpha' (\alpha' \to (\alpha \to \alpha' \to \alpha') \to \mathrm{list}_\alpha \to \alpha')$

<u>Proof of (1)</u>:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\{x' : \alpha', f : \alpha \to \alpha' \to \alpha'\} \vdash x' : \alpha'}\,(\mathrm{var})}{\{x' : \alpha'\} \vdash \lambda f : \alpha \to \alpha' \to \alpha' (x') : (\alpha \to \alpha' \to \alpha') \to \alpha}\,(\mathrm{fn})}{\vdash \lambda x' : \alpha', f : \alpha \to \alpha' \to \alpha' (x') : \alpha' \to (\alpha \to \alpha' \to \alpha') \to \alpha}\,(\mathrm{fn})}{\vdash \Lambda \alpha (\lambda x' : \alpha', f : \alpha \to \alpha' \to \alpha' (x')) : \mathrm{list}_\alpha}\,(\mathrm{gen})}{\vdash \mathrm{Nil} : \forall \alpha (\mathrm{list}_\alpha)}\,(\mathrm{gen})$$

1

Proof of (2):

Writing $\Gamma = \{x:\alpha,\ l:list_\alpha,\ x':\alpha',\ f:\alpha\to\alpha'\to\alpha'\}$, we have

$$\dfrac{\dfrac{\dfrac{}{\Gamma\vdash l:list_\alpha}\,(var)}{\Gamma\vdash l\alpha':\alpha'\to(\alpha\to\alpha'\to\alpha')\to\alpha'}\,(spec)\qquad \dfrac{}{\Gamma\vdash x':\alpha'}\,(var)}{\dfrac{\Gamma\vdash l\alpha'x':(\alpha\to\alpha'\to\alpha')\to\alpha'\qquad \dfrac{}{\Gamma\vdash f:\alpha\to\alpha'\to\alpha'}\,(var)}{ⓐ\ \Gamma\vdash l\alpha'x'f:\alpha'}\,(app)}$$

and

$$\dfrac{\dfrac{}{\Gamma\vdash f:\alpha\to\alpha'\to\alpha'}\,(var)\qquad \dfrac{}{\Gamma\vdash x:\alpha}\,(var)}{ⓑ\ \Gamma\vdash fx:\alpha'\to\alpha'}$$

so

$$\dfrac{\dfrac{\dfrac{ⓑ\qquad ⓐ}{\Gamma\vdash fx(l\alpha'x'f):\alpha'}\,(app)}{\{x:\alpha,\ l:list_\alpha\}\vdash\Lambda\alpha'(\lambda x';f(fx(l\alpha'x'f))):list_\alpha}\,(fn)^2(gen)}{\vdash Cons:\forall\alpha(\alpha\to list_\alpha\to list_\alpha)\,.}\,(fn)^2(gen)$$

2

Proof of (3):

Writing $\Gamma'=\{x':\alpha',\ f:\alpha\to\alpha'\to\alpha',\ l:list_\alpha\}$, as in the proof of ⓐ above, we have

$$\Gamma'\vdash l\alpha'x'f:\alpha'$$

and then applying $(fn)^3(gen)^2$ we get

$$\vdash iter:\forall\alpha,\alpha'(\alpha'\to(\alpha\to\alpha'\to\alpha')\to list_\alpha\to\alpha')\,.$$

(13)

β-conversion properties of iter:

1 (4) $\quad iter\,\alpha\alpha' x' f\,(Nil\,\alpha) =_\beta x'$

1 (5) $\quad iter\,\alpha\alpha' x' f\,(Cons\,\alpha\, x\, l) =_\beta f x\,(iter\,\alpha\alpha' x' f\, l)$

Proof of (4):

$iter\,\alpha\alpha' x f\,(Nil\,\alpha)$

$\xrightarrow{(5)} (Nil\,\alpha)\,\alpha' x' f$ by def^n of iter

1 $\xrightarrow{(4)} x'$ " " " Nil

Proof of (5):

$iter\,\alpha\alpha' x f\,(Cons\,\alpha\, x\, l)$

$\xrightarrow{(5)} (Cons\,\alpha\, x\, l)\,\alpha' x' f$ by def^n of iter

1 $\xrightarrow{(6)} f x\,(l\,\alpha' x' f)$ " " " Cons

and

$f x\,(iter\,\alpha\alpha' x' f\, l)$

5 1 $\xrightarrow{(5)} f x\,(l\,\alpha' x' f)$ by def^n of iter

Commentary

This material is book work, covered in lecture 7.