Discrete mathematics 2001

Short question

Prove the fundamental theorem of arithmetic, that any natural number can be expressed as a product of powers of primes and that such an expression is unique up to the order of the primes. [4 marks]

Given a natural number n, let $d(n)$ be the number of divisors of n (including 1 and n).

If $p_1, p_2, \ldots, p_k$ are distinct primes prove that $d(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}) = \prod_{i=1}^{k} (\alpha_i + 1)$. [3 marks]

What is the smallest number with 36 factors? [3 marks]

Answer

Existence: Use contradiction. Pick a minimal counter-example. Either it is prime and we are done or it can be factored into two smaller numbers which consequently have expressions.

Uniqueness. Use contradiction. Pick a minimal counter-example and express it as two different products of powers of primes. Pick a prime in the first expression. It must appear in the second so divide by it to give two expressions for a smaller number, which must be the same.

Any factor of n must consist of a product of lower powers of the same primes.

$36 = 2^2 3^2$, so $\alpha_1=1$, $\alpha_2=1$, $\alpha_3=2$ and $\alpha_4=2$, and the smallest number will be $2^2 3^2 5^1 7^1 = 1260$.