~~Maths~~ for Computation Theory LECTURES 10, 11 | 3

Paper 11 (solution) 2004 p11 q 9 JKMM

$$X \geq B, \quad X \geq AX.$$

Suppose $X \geq A^n . B$, $n \geq 0$.

Then $X \geq A(A^n . B) = A^{n+1} . B$.

BUT $X \geq A^0 . B$. Hence by induction

$X \geq A^n . B \quad \forall n \geq 0$.

$$\therefore X \geq \sum_{n=0}^{\infty} A^n . B = \left(\sum_{n=0}^{\infty} A^n\right) B$$

$$= A^* B$$

BUT $A^* B = (I + AA^*) B$

$$= B + A(A^* B)$$

satisfies the inequality, hence it is the least solution.

Paper 11 (solution) ctd)

$$MY = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} (A+BD^*C)^* & A^*B(D+CA^*B)^* \\ D^*C(A+BD^*C)^* & (D+CA^*B)^* \end{pmatrix}$$

$$= \begin{pmatrix} (A+BD^*C)(A+BD^*C)^* & (AA^*B+B)(D+CA^*B)^* \\ ? & ? \end{pmatrix}$$

Now $I + (A+BD^*C)(A+BD^*C)^* = (A+BD^*C)^*$

$(AA^*B + B) = (AA^* + I)B = A^*B$

The bottom row entries may be verified similarly.

Note that combining the first two parts gives the formula for $Y = M^*$ when $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Transition matrix

Maths for Computation Theory

Paper 11 (solution) ctd)

$$
M = \begin{array}{c} \alpha \\ \beta \\ \gamma \\ \\ \end{array}
\begin{array}{c}
\left(\begin{array}{c|cc}
0 & b & a \\
\hline
b & 0 & a \\
0 & b & a
\end{array}\right) \\
\begin{array}{ccc} \alpha & \beta & \gamma \end{array}
\end{array}
$$

The accepting entry is in the top left corner of M^*, so we must compute –

$$
D^* = \begin{pmatrix} (aa^*b)^* & a(a+ba)^* \\ a^*b(aa^*b)^* & (a+ba)^* \end{pmatrix}
$$

$$
D^*C = \begin{pmatrix} (aa^*b)^*b & 0 \\ a^*b(aa^*b)^*b & 0 \end{pmatrix}
$$

Paper 11 (solution) ctd)

$$BD^*C = b(aa^*b)^*b + aa^*b(aa^*b)^*b = a^*b(aa^*b)^*b$$

Since $A = 0$, the required event is

$$[a^*b(aa^*b)^*b]^*$$

There's one piece of simplification given above, but otherwise this is straightforward. I've checked the other block subdivision of M, and it comes out just as easily, once again with a single simplification.