p4q6
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~~[Relates to Fourier series.]~~

(a)

$$a_r = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos rx \, dx \qquad r = 0,1,2,\ldots$$

$$b_r = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin rx \, dx \qquad r = 1,2,\ldots$$

(b) Euler's equation is for θ real

$$e^{i\theta} = \cos\theta + i\sin\theta$$

So, $\cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2}$, $\sin\theta = \dfrac{e^{i\theta}-e^{-i\theta}}{2i}$

Hence,

$$\frac{a_0}{2} + \sum_{r=1}^{\infty}(a_r\cos rx + b_r\sin rx)$$

$$= \frac{a_0}{2} + \sum_{r=1}^{\infty} a_r\left(\frac{e^{irx}+e^{-irx}}{2}\right) + b_r\left(\frac{e^{irx}-e^{-irx}}{2i}\right)$$

$$= \frac{a_0}{2} + \sum_{r=1}^{\infty} a_r\left(\frac{e^{irx}+e^{-irx}}{2}\right) - ib_r\left(\frac{e^{irx}-e^{-irx}}{2}\right)$$

$$= \frac{a_0}{2} + \sum_{r=1}^{\infty}\left(\frac{1}{2}(a_r-ib_r)e^{irx} + \frac{1}{2}(a_r+ib_r)e^{-irx}\right)$$

$$= c_0 + \sum_{r=1}^{\infty}\left(c_r e^{irx} + c_{-r}e^{-irx}\right)$$

$$= \sum_{r=-\infty}^{\infty} c_r e^{irx}$$ where $c_0 = a_0/2$

$(r=1,2,\ldots)$ $c_r = \frac{1}{2}(a_r - ib_r)$

$c_{-r} = 1/2(a_r + ib_r)$

(c)

$$c_0 = \frac{a_0}{2} = \frac{1}{2}\,\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos 0x \, dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx$$

For $r \geq 1$,

$$c_r = \frac{1}{2}\left(a_r - i b_r\right)$$

$$= \frac{1}{2}\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos rx \;-\; i\,\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin rx\, dx\right)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\left(\cos rx - i\sin rx\right) dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-irx}\, dx$$ by Euler's equation

For $r \geq 1$

$$c_{-r} = \frac{1}{2}\left(a_r + i b_r\right)$$

$$= \frac{1}{2}\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos rx\, dx + i\,\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin rx\, dx\right)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\left(\cos rx + i\sin rx\right) dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-i(-r)x}\, dx$$

So in all cases we have

$$c_r = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-irx}\, dx \qquad (r = 0, \pm 1, \pm 2, \ldots)$$

(d) To find the complex Fourier coefficients of $f(x-\alpha)$ we use the expression from part (c)

$$\cancel{c_r =} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-\alpha)\, e^{-irx}\, dx$$

Set, $x' = x - \alpha$ then $\frac{dx'}{dx} = 1$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-\alpha)\, e^{-irx}\, dx = \frac{1}{2\pi}\int_{-\pi-\alpha}^{\pi-\alpha} f(x')\, e^{-ir(x'+\alpha)}\, dx'$$

$$= e^{-ir\alpha}\,\frac{1}{2\pi}\int_{-\pi-\alpha}^{-\pi-\alpha} f(x')\, e^{-irx'}\, dx'$$

$$= e^{-ir\alpha}\, c_r$$

x' is dummy variable

(e)

$$h(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)\,g(y)\,dy$$

To show that $h(x)$ is periodic of period 2π we must show that $h(x+k2\pi) = h(x)$ for all integers, k.

$$h(x+k2\pi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+k2\pi-y)\,g(y)\,dy$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)\,g(y)\,dy$$

$$= h(x)$$ since f is periodic of period 2π.

The complex Fourier coefficients of $h(x)$ are given by

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} h(x)\,e^{-irx}\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)\,g(y)\,dy\right)e^{-irx}\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)\,e^{-irx}\,dx\right)g(y)\,dy$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)\, e^{-ir(x-y)}\,dx\right) e^{-iry}\, g(y)\,dy$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} c_r\, g(y)\, e^{-iry}\,dy$$

$$= c_r \frac{1}{2\pi}\int_{-\pi}^{\pi} g(y)\, e^{-iry}\,dy$$

$$= c_r d_r \qquad r = 0, \pm 1, \pm 2, \ldots$$