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Notes on the Model Answer to Paper 2 Question 4

There are various approaches to both parts of this question. A candidate who wishes to play safe will simply enumerate all 32 arrangements of the switches and note those which provide a connection. The solutions below use the Inclusion–Exclusion Theorem...

Label the switches from left to right, top to bottom as a, b, c, d and e and let A be the event 'switch a is closed'. Let the events B, C, D and E be likewise defined for the other switches.

The solution to the first part is derived as follows:

$$\begin{aligned}
& \mathrm{P}\big((A\cap B)\cup C\cup(D\cap E)\big) \\
&= \mathrm{P}(A\cap B)+\mathrm{P}(C)+\mathrm{P}(D\cap E) \\
&- \mathrm{P}(A\cap B\cap C)-\mathrm{P}(A\cap B\cap D\cap E)-\mathrm{P}(C\cap D\cap E) \\
&+ \mathrm{P}(A\cap B\cap C\cap D\cap E) \\
&= p^2+p+p^2 \\
&- p^3-p^4-p^3 \\
&+ p^5 \\
&= p+2p^2-2p^3-p^4+p^5
\end{aligned}$$

The solution to the second part is derived as follows:

$$\begin{aligned}
& \mathrm{P}\big((A\cap B)\cup(A\cap C\cap E)\cup(B\cap C\cap D)\cup(D\cap E)\big) \\
&= \mathrm{P}(A\cap B)+\mathrm{P}(A\cap C\cap E)+\mathrm{P}(B\cap C\cap D)+\mathrm{P}(D\cap E) \\
&- \mathrm{P}(A\cap B\cap C\cap E)-\mathrm{P}(A\cap B\cap C\cap D)-\mathrm{P}(A\cap B\cap D\cap E) \\
&\qquad -\mathrm{P}(A\cap B\cap C\cap D\cap E)-\mathrm{P}(A\cap C\cap D\cap E)-\mathrm{P}(B\cap C\cap D\cap E) \\
&+ \mathrm{P}(A\cap B\cap C\cap D\cap E)+\mathrm{P}(A\cap B\cap C\cap D\cap E) \\
&\qquad +\mathrm{P}(A\cap B\cap C\cap D\cap E)+\mathrm{P}(A\cap B\cap C\cap D\cap E) \\
&- \mathrm{P}(A\cap B\cap C\cap D\cap E) \\
&= p^2+p^3+p^3+p^2 \\
&- p^4-p^4-p^4-p^5-p^4-p^4 \\
&+ p^5+p^5+p^5+p^5 \\
&- p^5 \\
&= 2p^2+2p^3-5p^4+2p^5
\end{aligned}$$