Numerical Analysis II 2002

p9 q11
MRO'D

Solution Notes – Question B.

Question concerns: Differential equations – Euler's method, multistep methods, stability theory, predictor-corrector methods.

(a) Local error is the error over a single step.
A method is of order k if its local error is $O(h^{k+1})$.
So method (i) has order 1, and method (ii) has order 2. [2 marks]

(b) Method (i): $y_{n+1} = y_n - 5hy_n = 0.5y_n$.
Method (ii): $y_{n+1} = y_{n-1} - 10hy_n = y_{n-1} - y_n$.

x_n	(i)	(ii)	exact
0.0	1.0	1.0	1.0
0.1	0.5	0.61	0.61
0.2	0.25	0.39	0.37
0.3	0.12	0.22	0.22
0.4	0.062	0.17	0.14
0.5	0.031	0.050	0.082
0.6	0.016	0.12	0.050
0.7	0.0078	−0.070	0.030
0.8	0.0039	0.19	0.018
0.9	0.0020	−0.26	0.011
1.0	0.00098	0.45	0.0067

[7 marks]

(c) Method (ii) is more accurate initially because it is of higher order. However it becomes numerically unstable, losing accuracy considerably for $x > 0.5$.
Method (i) is never very accurate because h is too large, but remains stable, mimicking the shape of the solution. [3 marks]

(d) $\frac{dy}{dx} = -5y$ $\quad\therefore \int \frac{dy}{y} = -5\int dx$

$\therefore \ln y = -5x + c$

$\therefore y = e^{-5x+c}$

Since $y(0)=1$, $c=0$ so the solution of the ODE is
$y = e^{-5x}$.

In method (i) $y_{n+1} = \frac{1}{2} y_n$, $y_0 = 1$.

So

$$y_n = 2^{-n} = 2^{-10x}.$$

Absolute error in method (i)

$$= \left| e^{-5x} - 2^{-10x} \right| \to 0 \quad \text{as } x \to \infty.$$

The absolute error in method (ii) will $\to \infty$ as $x \to \infty$, oscillating in sign at each iteration. [5 marks]

(e) Method (i) is a suitable predictor for a corrector of order 1. Method (ii) is a suitable predictor for a corrector of order 2. This is for reasons of efficiency. The stability of each method is irrelevant as a predictor is only used over a single step between applications of the corrector.

[3 marks]