Solution notes

Logic and Proof 2005 – Paper 5 Question 9 (LCP)

Context: lectures 7–8.

(a) As pointed out in the course notes, the separate parts of the formula can be converted into clauses separately. We begin by negating the conclusion, namely $\forall x\, S(x)$, to obtain $\exists x\, \neg S(x)$. After Skolemization, we obtain the clause

$$\{\neg S(a)\}.$$

From $\forall x\, [P(x) \vee Q \rightarrow \neg R(x)]$, we obtain $\forall x\, [\neg(P(x) \vee Q) \vee \neg R(x)]$ and thus (distributing) $\forall x\, [(\neg P(x) \vee \neg R(x)) \wedge (\neg Q \vee \neg R(x)]$. We obtain two additional clauses:

$$\{\neg P(x), \neg R(x)\} \quad \{\neg Q, \neg R(x)\}.$$

From $\forall x\, [(Q \rightarrow \neg S(x)) \rightarrow (P(x) \wedge R(x))]$, we obtain $\forall x\, [\neg(\neg Q \vee \neg S(x)) \vee (P(x) \wedge R(x))]$ and then $\forall x\, [(Q \wedge S(x)) \vee (P(x) \wedge R(x))]$. Distributing yields four additional clauses:

$$\{Q, P(x)\} \quad \{Q, R(x)\} \quad \{S(x), P(x)\} \quad \{S(x), R(x)\}$$

Context: lectures 9–10.

(b) It is impossible because the clauses are satisfiable. Let $P(x, y)$ denote $x < y$ and let $f(x)$ denote $x+1$, with variables ranging over the integers. Then the clauses say $x \not< x$, $x < x + 1$ and $x < y \wedge y < z \rightarrow x < z$, which are all true.

(c) We derive the empty clause as follows.

Resolve $\{\neg S(x), \neg R(x), Q(x)\}$ and $\{S(b)\}$ to obtain $\{\neg R(b), Q(b)\}$.

Resolve $\{\neg R(b), Q(b)\}$ and {R(b)} to obtain $\{Q(b)\}$.

Resolve $\{\neg Q(x), P(x), \neg R(y), \neg Q(y)\}$ and $\{R(b)\}$ to obtain $\{\neg Q(x), P(x), \neg Q(b)\}$

Resolve $\{\neg Q(x), P(x), \neg Q(b)\}$ and $\{Q(b)\}$ to obtain $\{\neg Q(x), P(x)\}$

Resolve $\{\neg Q(x), P(x)\}$ and $\{Q(a)\}$ to obtain $\{P(a)\}$.

Resolve $\{P(a)\}$ and $\{\neg P(a)\}$ to obtain $\Box$.