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PR

Long question B

Let $(A, \leq_A)$ and $(B, \leq_B)$ be partially ordered sets. Define the ***product order*** on $A \times B$ and prove that it is a partial order. [4 marks]

The ***upper bound*** of a set $S \subseteq A$ is an element $u \in A$ (but not necessarily in S) such that $\forall\, s \in S \,.\, s \leq u$. The ***least upper bound*** of S is an upper bound of S that is less than every other upper bound of S. The ***greatest lower bound*** is defined similarly.

A *lattice* is a partially ordered set in which every pair of elements has both a least upper bound and a greatest lower bound.

Prove that $(\mathbb{N}, |)$, the natural numbers under the divisibility order, form a lattice. [4 marks]

Given a set X, prove that $(\mathcal{P}(X), \subseteq)$, the power set of X under set inclusion, forms a lattice. [4 marks]

Does every subset of $(\mathbb{N}, |)$ have a least upper bound and a greatest lower bound? Justify your answer. What about $(\mathbb{N}_0, |)$ and $(\mathcal{P}(X), \subseteq)$? [4 marks]

If $(A, \leq_A)$ and $(B, \leq_B)$ are lattices, show that $A \times B$ is a lattice under the product order. [4 marks]

Answer

$(a_1, b_1) \leq (a_2, b_2) \Leftrightarrow (a_1 \leq_A a_2) \wedge (b_1 \leq_A b_2)$. Reflexive, anti-symmetric and transitive.

Lowest common multiple and greatest common divisor.

Union and intersection.

$\mathbb{N}$ itself has no least upper bound. Otherwise yes. 0 is LUB for $\mathbb{N}_0$.

LUB of pair is pair of LUBs and so on.