Quantum Computing 2004 – Paper 7 Question 9 (AD)

(a) You are given one of two quantum states of a single qubit: either $|\phi\rangle = |0\rangle$ or $|\psi\rangle = \cos\theta|0\rangle + \sin\theta|1\rangle$.

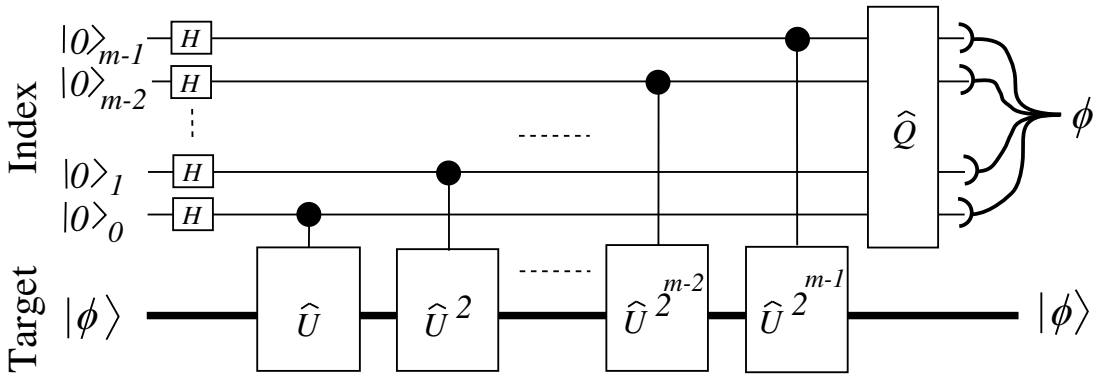
The probability of correctly identifying the state is:

$$\begin{aligned} P(\text{success}) &= P(|1\rangle) \times P(|\psi\rangle||1\rangle) + P(|0\rangle) \times P(|\phi\rangle||0\rangle) \\ &= \frac{\sin^2\theta}{2} \times 1 + \frac{1+\cos^2\theta}{2} \times \frac{1}{1+\cos^2\theta} \\ &= \frac{\sin^2\theta}{2} + \frac{1}{2} \end{aligned}$$

[3 marks]

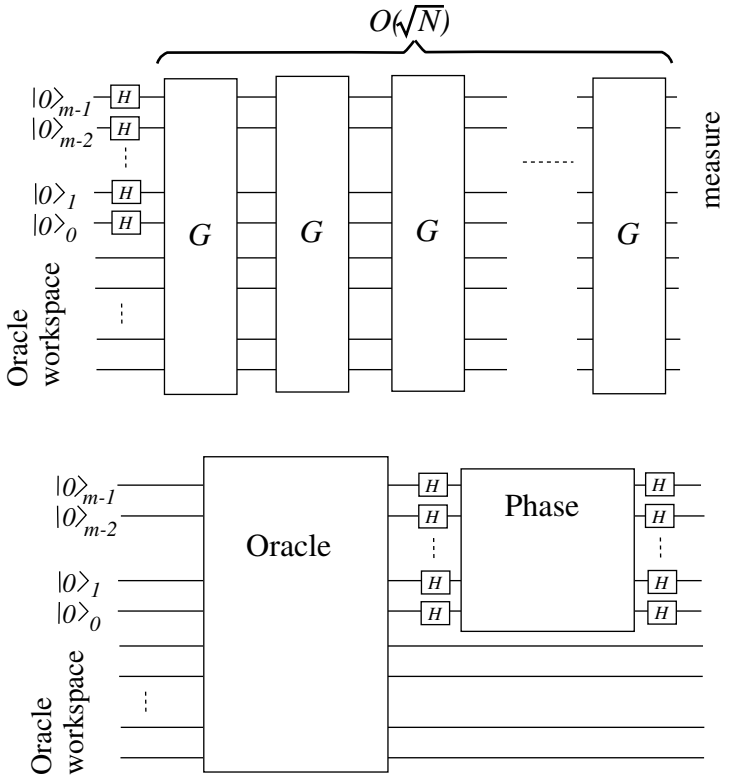
(*b*) Draw a labelled schematic circuit diagram for:

(*i*) the phase estimation algorithm.



[4 marks]

(*ii*) Grover's algorithm.



[4 marks]

(*c*) Suppose a search problem has an unknown number M of solutions. Show how phase estimation and Grover's algorithm can be combined to estimate M to a high accuracy using $O(\sqrt{N})$ oracle calls.
(Hint: The Grover iterate, G, has eigenvalues $\exp(\pm i\theta)$ where $\sin^2(\theta/2) = M/N$.)

[5 marks]

Reproduce the diagram for the phase estimation algorithm with the Us replaced by Grover iterates. Use upto 2^k of these to get a k-bit approximation to θ where $\exp(\pm i\theta)$ is an eigenvalue of G. Note that, as there are only two eigenvalues and they are of the form $\exp(\pm i\theta)$, it doesn't matter what initial state you use. Then, obtain M as $N \sin^2(\theta/2)$.

(*d*) Suppose there is an algorithm which can determine the number M (as in part (c) above) state in an unsorted search space of size N using only $O(\log(N))$ oracle calls. Explain why this would allow us to solve NP-complete problems in polynomial time.

An NP-complete problem on an input of size n has a solution search space of size $O(2^n)$. We need to determine whether this search space has any solutions at all. Thus, estimating the number M will allow us to determine whether or not it is 0. Taking $N = 2^n$, if this can be done in in $O(\log N)$, i.e., $O(n)$ steps, it offers a polynomial time solution.

[4 marks]

Part (a) is based on Lecture 1. Parts (b) and (c) combine elements of Lecture 5 and 6. The final part is based on Lecture 8.