Solution notes 2002 p3q5

Paper 2002 Paper 3 Continuous Maths RJG

[RELATES to Fourier series]

(a) Fourier coefficients given by [4 marks]

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_r = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos rx\,dx \qquad r = 1,2,3,\ldots$$

$$b_r = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin rx\,dx \qquad r = 1,2,3,\ldots$$

(b) Dirichlet conditions on $f(x)$ periodic with period 2π are: [2 marks]

(i) $f(x)$ continuous at every point in interval $-\pi < x \leq \pi$ except for a finite number of finite discontinuities

(ii) $f(x)$ has a finite number of maxima or minima in the interval $-\pi < x \leq \pi$.

(c) [3 marks]

$f(x)$ even $\Rightarrow f(x) = f(-x) \ \forall x$

So, $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx$$

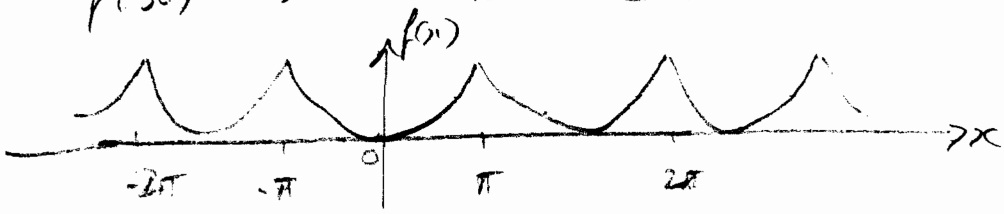
$$a_r = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos rx\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos rx\,dx \qquad r=1,2,\ldots$$

(since both f and cos are even so $f(x)\cos rx$ is even)

$$b_r = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin rx\,dx = 0$$

(since f even, sin odd so $f(x)\sin rx$ is odd)

(d) $f(x) = x^2 \quad -\pi < x \leq \pi$ [2 marks]



Within $-\pi < x \leq \pi$
(i) $f(x)$ is continuous every where
(ii) has finite number of maxima & minima
Hence Dirichlet conditions are satisfied.

(e) $f(x)$ is even function so (using part (c))

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2\,dx$$

$$= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2}{\pi}\cdot\frac{\pi^3}{3} = \frac{2\pi^2}{3}$$

$$a_r = \frac{2}{\pi}\int_0^{\pi} x^2 \cos rx\,dx \qquad r = 1, 2, \ldots$$

$$= \frac{2}{\pi}\left\{\left[\frac{x^2 \sin rx}{r}\right]_0^{\pi} - \frac{2}{r}\int_0^{\pi} x \sin rx\,dx\right\} \quad \text{(by parts)}$$

$$= -\frac{4}{\pi r}\int_0^{\pi} x \sin rx\,dx \qquad \left(\begin{array}{l}\text{as}\\ \sin r\pi = 0\\ \sin 0 = 0\end{array}\right)$$

$$= -\frac{4}{\pi r}\left\{\left[-x\frac{\cos rx}{r}\right]_0^{\pi} - \int_0^{\pi} -\frac{\cos rx}{r}\,dx\right\} \quad \text{(by parts)}$$

$$= \frac{4\cos r\pi}{r^2} - \frac{4}{\pi r^2}\int_0^{\pi} \cos rx\,dx$$

$$= \frac{4\cos r\pi}{r^2} - \frac{4}{\pi r^2}\left[\frac{\sin rx}{r}\right]_0^{\pi} \qquad \left(\begin{array}{l}\text{as } \sin r\pi = 0\\ \sin 0 = 0\end{array}\right)$$

$$= \frac{4(-1)^r}{r^2} \qquad (\text{as } \cos r\pi = (-1)^r)$$

So Fourier series is

$$f(x) = x^2 = \frac{\pi^2}{3} + 4\sum_{r=1}^{\infty} (-1)^r \frac{\cos rx}{r^2} \qquad -\pi < x \leq \pi$$

[3 marks]

f) Substitute $x = 0$ to give

$$0 = \frac{\pi^2}{3} + 4\sum_{r=1}^{\infty} \frac{(-1)^r}{r^2}$$

(as $\cos 0 = 1$)

$$\text{So} \quad \frac{\pi^2}{12} = \sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r^2}$$