MODEL ANSWER

Information Theory and Coding 2002 Paper 7 Question 12 (JGD)

(Subject areas: Signal encoding; variable-length prefix codes; discrete FT.)

A.

1.

Nyquist's Sampling Theorem: If a signal $f(x)$ is strictly bandlimited so that it contains no frequency components higher than W, i.e. its Fourier Transform $F(k)$ satisfies the condition

$$F(k) = 0 \text{ for } |k| > W$$

then $f(x)$ is completely determined just by sampling its values at a rate of at least $2W$. The signal $f(x)$ can be exactly recovered by using each sampled value to fix the amplitude of a sinc(x) function,

$$\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

whose width is scaled by the bandwidth parameter W and whose location corresponds to each of the sample points. The continuous signal $f(x)$ can be perfectly recovered from its discrete samples $f_n(\frac{n\pi}{W})$ just by adding all of those displaced sinc(x) functions together, with their amplitudes equal to the samples taken:

$$f(x) = \sum_n f_n\left(\frac{n\pi}{W}\right)\frac{\sin(Wx - n\pi)}{(Wx - n\pi)}$$

Thus we see that any signal that is limited in its bandwidth to W, during some duration T has at most $2WT$ degrees-of-freedom. It can be completely specified by just $2WT$ real numbers.

2.

Logan's Theorem: If a signal $f(x)$ is strictly bandlimited to one octave or less, so that the highest frequency component it contains is no greater than twice the lowest frequency component it contains

$$k_{max} \leq 2k_{min}$$

i.e. $F(k)$ the Fourier Transform of $f(x)$ obeys

$$F(|k| > k_{max} = 2k_{min}) = 0$$

and

$$F(|k| < k_{min}) = 0$$

and if it is also true that the signal $f(x)$ contains no complex zeroes in common with its Hilbert Transform, then the original signal $f(x)$ can be perfectly recovered (up to an amplitude scale constant) merely from knowledge of the set $\{x_i\}$ of zero-crossings of $f(x)$ alone.

$$\{x_i\} \text{ such that } f(x_i) = 0$$

Obviously there is only a finite and countable number of zero-crossings in any given length of the bandlimited signal, and yet these "quanta" suffice to recover the original continuous signal completely (up to a scale constant).

[10 marks]

B.

The N binary code word lengths $n_1 \leq n_2 \leq n_3 \cdots \leq n_N$ must satisfy the *Kraft-McMillan Inequality* if they are to constitute a uniquely decodable prefix code:

$$\sum_{i=1}^{N} \frac{1}{2^{n_i}} \leq 1$$

[4 marks]

C.

The Discrete Fourier Transform G_k of the regular sequence $\{g_n\} = \{g_0,\ g_1,\ ...,\ g_{N-1}\}$ is:

$$\{G_k\} = \sum_{n=0}^{N-1} g_n \exp\left(-\frac{2\pi i}{N}kn\right), \quad (k = 0, 1, ..., N-1)$$

The Inverse Transform (or synthesis equation) which recovers $\{g_n\}$ from $\{G_k\}$ is:

$$\{g_n\} = \frac{1}{N}\sum_{k=0}^{N-1} G_k \exp\left(\frac{2\pi i}{N}kn\right), \quad (n = 0, 1, ..., N-1)$$

[6 marks]