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/ 2002 Discrete Mathematics [20 marks question]

(1) $$\Big(\bigcap_{B\in\mathcal{B}} B\Big) \cup \Big(\bigcap_{C\in\mathcal{C}} C\Big) = \bigcap_{(B,C)\in\mathcal{B}\times\mathcal{C}} (B\cup C)$$

"$\subseteq$" Let $x \in (\bigcap_B B)\cup(\bigcap_C C)$. Then
① ($x\in B$, all $B\in\mathcal{B}$) or ② ($x\in C$, all $C\in\mathcal{C}$)

If ①, then $x \in B\cup C$ for all $B\in\mathcal{B}$ and $C\in\mathcal{C}$, so
$$x\in\bigcap_{(B,C)\in\mathcal{B}\times\mathcal{C}}(B\cup C).$$
And similarly, if ②. Hence "$\subseteq$".

"$\supseteq$" Let $x\in\bigcap_{(B,C)} B\cup C$. Then,

$x\in B\cup C$ for all $B\in\mathcal{B}$, $C\in\mathcal{C}$.

~~Suppose~~
Suppose $x\notin\bigcap_{B\in\mathcal{B}}B$ i.e. $x\notin B_0$ for some $B_0\in\mathcal{B}$.

Then $x\in B_0\cup C$ for all $C\in\mathcal{C}$.
As $x\notin B_0$, we must have $x\in C$ for all $C\in\mathcal{C}$.
i.e. $x\in\bigcap_{C\in\mathcal{C}} C$. Hence "$\supseteq$".

To prove:

(2) $C \in \mathcal{A} \iff C$ is R-closed.

(a) "$\Rightarrow$" Suppose $C \in \mathcal{A}$ and $(X, y) \in R_{\mathcal{A}}$. Then

(B3) means $X \subseteq C \Rightarrow y \in C$, by the defn. of $R_{\mathcal{A}}$. Hence

C is $R_{\mathcal{A}}$-closed.

(b) "$\Leftarrow$" Suppose C is R-closed. Define $\mathcal{B}$ by

$\mathcal{B} = \{A \in \mathcal{A} \mid C \subseteq A\}$

Then $\bigcap_{B \in \mathcal{B}} B \in \mathcal{A}$, as $\mathcal{A}$ is intersection closed.

Clearly $C \subseteq \bigcap_{B \in \mathcal{B}} B$.

Suppose $y \in \bigcap_{B \in \mathcal{B}} B$, i.e. $\forall A \in \mathcal{A}.\ C \subseteq A \Rightarrow y \in A$.

Recall

$(X, y) \in R_{\mathcal{A}}$ iff $\forall A \in \mathcal{A}.\ X \subseteq A \Rightarrow y \in A$.

It follows that $(C, y) \in R_{\mathcal{A}}$.

But C is R-closed. Hence $y \in C$ and we

have $\bigcap_{B \in \mathcal{B}} B \subseteq C$. $\therefore$ $C = \bigcap_{B \in \mathcal{B}} B \in \mathcal{A}$.

[Hint: Consider

$\mathcal{B} = \{A \in \mathcal{A} \mid C \subseteq A\}$]