Probability 2001

p 2 q 4

FHK

Model Answer

$u_n - 2u_{n-1} = 6$

Let $u_n = A w^n \rightarrow w - 2 = 0$ so $u_n = A.2^n$ [homogeneous case]

Try $u_n = A.2^n + k$ where $k - 2k = 6$ so $k = -6$

Hence $u_n = A.2^n - 6$ but $u_1 = A.2 - 6 = 0$ so $A = 6/2 = 3$

solution $\underline{u_n = 3(2^n - 2)}$ [5 marks]

[1 mark]

$P(\text{first } k \text{ digits all the same}) = 3/3^k$

Let u_k = no. ways first k digits consist of just 2 of the 3 available.

$u_k = 2(u_{k-1} + 3)$ and $u_1 = 0$

2 ways of getting all three next time

first $k-1$ contain two only

first $k-1$ are all the same

[5 marks]

Hence $u_k = 3(2^k - 2)$

$\text{Prob}(\text{first } k \text{ contain two only}) = \dfrac{3(2^k-2)}{3^k}$ [1 mark]

$\text{Prob}(\text{stream is length } r) = \dfrac{1}{\cancel{3}} \cdot \underbrace{\dfrac{\cancel{3}(2^{r-1}-2)}{3^{r-1}}}_{\text{Prob(first } r-1 \text{ contain 2 only)}}$ $r \geq 2$ [3 marks]

[P(new digit completes the set) = 1/3]

Expected length $= \sum r P_r = \sum_{r=2}^{\infty} r\left(\frac{2}{3}\right)^{r-1} - 2\sum_{r=2}^{\infty} r\left(\frac{1}{3}\right)^{r-1}$

$= \sum_{r=1}^{\infty} r\left(\frac{2}{3}\right)^{r-1} - \underbrace{[1]}_{r=1 \text{ case}} - 2\sum_{r=1}^{\infty} r\left(\frac{1}{3}\right)^{r-1} + 2\underbrace{[1]}_{r=1 \text{ case}}$

$= \dfrac{1}{(1-2/3)^2} - \dfrac{2}{(1-1/3)^2} + 1 = \dfrac{1}{(1/3)^2} - \dfrac{2}{(2/3)^2} + 1$

$= 9 - \dfrac{9}{2} + 1$

$= 5\frac{1}{2}$ [5 marks]