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PR

Discrete mathematics – long question A

Recall the Fibonacci numbers defined by:

- $f_0 = 0$
- $f_1 = 1$
- $f_n = f_{n-1} + f_{n-2}$ for $n > 1$

Using induction, or otherwise, show that $f_{m+n} = f_{m\,1} f_n + f_m f_{n+1}$. [4 marks]

Deduce that $\forall\ m, n > 0\ .\ m \mid n \Rightarrow f_m \mid f_n$. [4 marks]

Deduce further that $\forall\ n > 4\ .\ f_n$ prime $\Rightarrow n$ prime. [2 marks]

Given $n \in \mathbb{N}$, let $g_i = f_i \bmod n$, and consider the pairs (g_1, g_2), (g_2, g_3), …, (g_i, g_{i+1}), …. Show that there must be a repetition in the first n^2+1 pairs. Let $r < s$ be the least values with $(g_r, g_{r+1}) = (g_s, g_{s+1})$. Show that $g_{r-1} = g_{s-1}$, and deduce that $r = 1$. Calculate g_1 and g_2, and deduce that $g_{s-1} = 0$. Hence show that one of the first n^2 Fibonacci numbers is divisible by n.
[10 marks]

Solution

Induction on n: statement	[1]
base case	[1]
inductive step	[2]
Use induction on i to show that f_{mi} is divisible by f_m	[1+1+2]
$n > 4$ composite $\Rightarrow n = rs$ with $s > 2$. Now $f_s > 1$ and $f_s \mid f_n$, so f_n is composite	[1+1]
Follow the instructions: only n values for g_i	[1]
so only n^2 values for pair	[1]
so repetition by counting	[1]
Suppose $r > 1$	[1]
use inductive definition of Fibonacci numbers	[1]
observe r not minimal	[1]
$f_1 = f_2 = 1$, so $g_1 = g_2 = 1$	[1]
use inductive definition of Fibonacci numbers again	[1]
$s > r = 1$ and $g_{s-1} = 0$, so f_{s-1} is divisible by n	[2]