//

pb q 11
LCP

Logic & Proof 2 2000

For each of the given pairs of terms, give a most general unifier or indicate why none exists. (Here x, y, z are variables while a, b are constant symbols.)

$$\begin{array}{lcl} h(x,y,x) & \text{and} & h(y,z,u) \\ h(x,y,z) & \text{and} & h(f(y),z,x) \\ h(x,y,b) & \text{and} & h(a,x,y) \\ h(x,y,z) & \text{and} & h(g(y,y),g(z,z),g(u,u)) \end{array}$$

[4 marks]

A standard unification algorithm takes a pair of terms t_1 and t_2 and returns a substitution θ such that $t_1\theta = t_2\theta$. Show how this algorithm can be used to find the unifier of several ($n > 2$) terms t_1, t_2, ..., t_n: a substitution θ such that $t_1\theta = t_2\theta = \cdots = t_n\theta$. Indicate how the unifier is constructed from the unifiers of $n-1$ pairs of terms. (Assume that all required unifiers exist and ignore the question of whether the unifiers are most general.)

[6 marks]

Prove using resolution the formula

$$(\forall x\,[P(x) \leftrightarrow (Q(x) \land \neg Q(f(x)))]) \to \exists y\, \neg P(y)$$

or indicate why this formula is not a theorem. [10 marks]

Solution notes

For $h(x,y,x)$ and $h(y,z,u)$ we get $x = y = z = u$. A most general unifier is $[u/x, u/y, u/z]$

For $h(x,y,z)$ and $h(f(y),z,x)$ we get $x = f(y) = f(z) = f(x)$ and there is no unifier (occurs check).

For $h(x,y,b)$ and $h(a,x,y)$ we get $a = x = y = b$ and there is no unifier because a and b are distinct constant symbols.

For $h(x,y,z)$ and $h(g(y,y),g(z,z),g(u,u))$ we get the identities $x = g(y,y)$ and $y = g(z,z)$ and $z = g(u,u)$. A most general unifier is

$$[g(g(g(u,u),g(u,u)),g(g(u,u),g(u,u)))/x,\ g(g(u,u),g(u,u))/y,\ g(u,u)/z].$$

To unify the terms t_1, t_2, ..., t_n we merely need to call the standard unification algorithm as follows:

Unify t_1 with t_2, yielding θ_1 such that $t_1\theta_1 = t_2\theta_1$.

Unify $t_2\theta_1$ with $t_3\theta_1$, yielding θ_2 such that $t_2\theta_1\theta_2 = t_3\theta_1\theta_2$. Continue like this until the final step:

Unify $t_{n-1}\theta_1 \dots \theta_{n-2}$ with $t_n\theta_1 \dots \theta_{n-2}$, yielding θ_{n-1} such that

$$t_{n-1}\theta_1 \dots \theta_{n-2} = t_n\theta_1 \dots \theta_{n-1}.$$

The required result is $\theta_1 \circ \theta_2 \cdots \circ \theta_{n-1}$.

The steps shown above will be carried out if we apply the pairwise unification algorithm to the pair of terms

$$f(x, \dots, x) \quad \text{and} \quad f(t_1, \dots, t_n)$$

where f is an n-place function symbol.

(One can show that if all the pairwise unifiers are most general then so is the final computed unifier, and that if any of the pairwise unifications fail then the n terms have no unifier.)

To convert the problem to clause form, first we negate the formula, getting

$$\forall x\,[P(x) \leftrightarrow (Q(x) \land \neg Q(f(x)))] \land \forall y\,P(y).$$

Obviously one clause will be $\{P(y)\}$, so we focus on the first conjunct and drop the quantifier. We expand the biconditional, getting

$$(P(x) \to (Q(x) \land \neg Q(f(x)))) \land ((Q(x) \land \neg Q(f(x))) \to P(x)).$$

Obviously the second conjunct gives rise to the clause $\{\neg Q(x), Q(f(x)), P(x)\}$, so again we focus on the first conjunct. We expand the implications, getting

$$\neg P(x) \lor (Q(x) \land \neg Q(f(x)))$$

and distribute the $\land$ over the $\lor$ to get

$$(\neg P(x) \lor Q(x)) \land (\neg P(x) \lor \neg Q(f(x)))$$

The clauses are therefore

$$\{\neg P(x), Q(x)\} \quad \{\neg P(x), \neg Q(f(x))\} \quad \{\neg Q(x), Q(f(x)), P(x)\} \quad \{P(y)\}.$$

Now from $\{P(y)\}$ and $\{\neg P(x), Q(x)\}$ we derive the clause $\{Q(x)\}$.

From $\{P(y)\}$ and $\{\neg P(x), \neg Q(f(x))\}$ we derive the clause $\{\neg Q(f(x))\}$.

From $\{Q(x)\}$ we derive $\{Q(x')\}$ by renaming; from this and $\{\neg Q(f(x))\}$ we derive the empty clause. Therefore the formula is indeed a theorem.