P5 q 11

LCP

Logic & Proof 1 2000

Given a propositional formula, we wish to test whether it is a tautology and, if it is not, to compute an interpretation that makes it false. Two techniques for doing this are the sequent calculus and ordered-binary decision diagrams. Give a brief outline of these techniques, applying both of them to the formulas

$$(A \to B) \to (B \to A) \quad \text{and} \quad (A \vee B) \to (\neg B \to A).$$

[7+7 marks].

It is proposed to replace the usual sequent calculus rule for disjunction on the left by this rule:

$$\frac{\Gamma, A \Rightarrow \Delta \quad \Gamma, B \Rightarrow \Delta, A}{\Gamma, A \vee B \Rightarrow \Delta}$$

Is this rule sound? Justify your answer. [3 marks]

Give an example to show that using this rule instead of the usual one makes some proofs shorter. [3 marks]

Solution notes

This material comes from the notes. The sequent calculus operates on sequents of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are sets of formulae: Γ is interpreted as a conjunction, Δ as a disjunction. There are two rules for each connective, one introducing it on the left side of the $\Rightarrow$ symbol and one introducing it on the right. E.g. for implication we have

$$\frac{\Gamma, B \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, A}{\Gamma, A \to B \Rightarrow \Delta}(\to l) \qquad \frac{\Gamma, A \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B}(\to r)$$

Proof are constructed upwards from the desired theorem until basic sequents are reached: sequents with a common formula on both sides of the arrow. The first example is

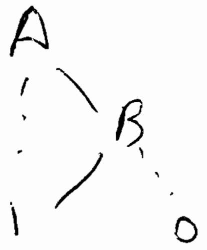
$$\cfrac{\cfrac{\cfrac{B \Rightarrow A \quad B \Rightarrow A}{A \to B, B \Rightarrow A}}{A \to B \Rightarrow B \to A}}{\Rightarrow (A \to B) \to (B \to A)}$$

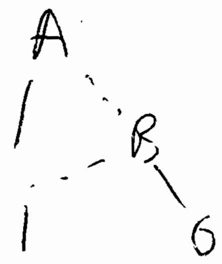
The proof fails because $B \Rightarrow A$ cannot be reduced further and is not basic. We see that B can be true and A false, which is a countermodel. The other proof succeeds:

$$\cfrac{\cfrac{\cfrac{\cfrac{\overline{A \Rightarrow A} \quad \overline{B \Rightarrow B}}{A \vee B \Rightarrow A, B}}{A \vee B, \neg B \Rightarrow A}}{A \vee B \Rightarrow \neg B \to A}}{\Rightarrow (A \vee B) \to (\neg B \to A)}$$

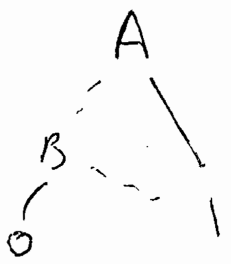
OBDDs are directed graphs representing if-then-else decisions on Boolean values. The variables are tested in a fixed order, e.g. alphabetically. Equivalent subgraphs are identified. Tests whose outcome is the same in either case are omitted. Hashing can be used to speed up the translation of large formulae by caching the outcomes of previous translations. Hashing is essential anyway in order to identify equivalent subgraphs, though this requires a separate hash table.

In the first example we get for $A \to B$ and for $B \to A$



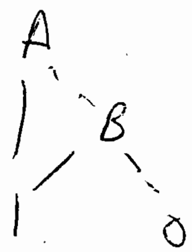


Their implication is



and is not a tautology: it yields 0 (false) if $B = 1$ and $A = 0$.

In the other example, we convert $A \lor B$ and $\neg B \to A$ like this:



These are the same OBDD, so their implication is 1: a tautology.

The rule is sound, for note that $A \lor B \simeq A \land (B \land \neg A)$ and that applying $(\lor l)$, $\land l)$ and $(\neg l)$ to the latter formula gives the effect of the new rule on any sequent..

Here is an example where the rule shortens a proof:

$$\frac{A \Rightarrow \qquad \overline{A \Rightarrow A}}{A \lor A \Rightarrow}$$

The formula A is inconsistent but the proof is long. With the ordinary rule, the proof would have to appear twice, but with the new rule, the second premise is just a basic sequent.