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DS2. P1
p 9 q 15
GW

Denotational Semantics Qu 2. 2004

(a) Let $h: D \to D$ be cts for domain D.

Define $fix(h) = \bigsqcup_n h^n(\bot)$.

Clearly if $h(x) \sqsubseteq x$, ie. x is a prefixed pt., then $\bot \sqsubseteq x$.

Assuming, inductively, that $h^n(\bot) \sqsubseteq x$

$h^{n+1}(\bot) \sqsubseteq h(x) \sqsubseteq x$

by monotonicity of h., so $h^{n+1}(\bot) \sqsubseteq x$.

By induction, $h^n(\bot) \sqsubseteq x$ so $fix(h) \sqsubseteq x$.

(b) Let $f: D \times E \to D$ and $g: D \times E \to E$ be cts. for domains D, E. Then

$\langle f, g \rangle : D \times E \to D \times E$ is cts. and has a least prefixed pt. (d_0, e_0)

$\hat{d} =_{def} fix(\lambda d. f(d, \hat{e}))$ where

$\hat{e} =_{def} fix(\lambda e. g(fix(\lambda d. f(d, e)), e))$.

(i) $\langle f, g\rangle(\hat{d}, \hat{e}) = \cancel{(f(\hat{d}), g(\hat{e}))}$
$(f(\hat{d},\hat{e}), g(\hat{d},\hat{e}))$

But $\hat{d} = f(\hat{d}, \hat{e})$
directly from the defn of $\hat{d}$ as a least fix pt.
of $\lambda d.\ f(d, \hat{e})$.

And $\hat{e} = g(\mathrm{fix}(\lambda d.\ f(d,\hat{e})), \hat{e})$ directly
from its defn.

$= g(\hat{d}, \hat{e})$.

Hence $\langle f, g\rangle(\hat{d}, \hat{e}) = (\hat{d}, \hat{e})$ and

$(d_0, e_0) \sqsubseteq (\hat{d}, \hat{e})$.

(ii) We require $(\hat{d}, \hat{e}) \sqsubseteq (d_0, e_0)$

By defn. $\hat{e}$ is least element s.t.

$\hat{e} = g(\mathrm{fix}(\lambda d. f(d,\hat{e})), \hat{e})$.

We show ~~Consider~~ $g(\mathrm{fix}(\lambda d.\ f(d, e_0)), e_0) \sqsubseteq e_0$ so $\hat{e} \sqsubseteq e_0$.

Let $d_1 = \mathrm{fix}(\lambda d.\ f(d, e_0))$. Then

as $f(d_0, e_0) \sqsubseteq d_0$, $\mathrm{fix}(\lambda d. f(d, e_0)) \sqsubseteq d_0$.
being the least prefixed pt of $d \mapsto f(d, e_0)$.

Hence
$$g(\mathrm{fix}(\lambda d. f(d,\hat{e}_0)), \hat{e}_0) \sqsubseteq g(d_0, e_0) = e_0$$

so $\hat{e} \sqsubseteq e_0$ because $\hat{e}$ is the least prefixed pt of
$e \mapsto g(\mathrm{fix}(\lambda d. f(d,e)), e)$.

Now $f(d_0, \hat{e}) \sqsubseteq f(d_0, e_0) = d_0$ so

d_0 is a prefixed pt of $d \mapsto f(d, \hat{e})$ and hence

$\hat{d} \sqsubseteq d_0$.

$\boxtimes$