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Complexity Theory

1. (a) The class NP is the class of all languages that can be decided by a non-deterministic Turing machine in polynomial time.

   (b) The class co-NP contains all problems whose complement is in NP.

   (c) A polynomial time reduction from a language A to a language B is a function f that can be computed in polynomial time such that, for any x, $f(x) \in B$ if, and only if, $x \in A$.

   (d) A language A is NP-complete, if it is in NP and for every language L in NP, there is a polynomial time reduction from L to A.

2. For any graph $G = (V, E)$, define its dual graph $D(G)$ to be the graph whose set of vertices is E, and such that there is an edge from vertex (u, v) to vertex (u', v') if, and only if, $\{u, v\} \cap \{u', v'\}$ is non-empty. Note that the function D is computable in time $O(n^2)$.

   To show that D is the required reduction, we need to show that $D(G)$ is 3-node-colourable if, and only if, G is 3-edge-colourable.

   First, assume G is 3-edge-colourable, and let χ be the colouring. Since the nodes of $D(G)$ are exactly the edges of G, χ assigns a colour to each node of $D(G)$. To see that this is a valid colouring, note that if there is an edge in $D(G)$ between two vertices of the same colour, the corresponding two edges in G would have a vertex in common, contradicting the assumption that χ is a valid colouring of G.

   In the other direction, suppose $D(G)$ is 3-node-colourable. An argument as above shows that the colouring of $D(G)$ yields a valid colouring of the nodes of G.

3. (a) True. Since 3-edge-colourability is NP-complete, there is a polynomial time reduction to it from any problem in NP. So, we only need to show that 3-node-colourability is in NP. But, t is trivial to verify (in time $O(n^2)$) that a given colouring of a graph is valid. The colouring can be generated non-deterministically in $O(n)$ steps.

(b) True. We saw above that there is a reduction from 3-edge-colourability to 3-node-coourability, from which it follows by compostion of reductions that every problemm in NP is reduible to the latter. Since we also just saw that 3-node-colourbility is in NP, it is NP-complete.

(c) True. Every problem in NP can be decided with polynomial space.