Computation Theory 2005 - Paper 4 Question 9 (AMP)

$S \subseteq \mathbb{N}$ is register machine decidable iff there is a register machine M so that: for all $x \in \mathbb{N}$, M started with $R1 = x$ and all other registers zeroed, always halts, with $R0 = 1$ if
3 $x \in S$ and with $R0 = 0$ if $x \notin S$

Since $S_f \neq \mathbb{N}$, there is some $e_1 \in \mathbb{N} - S_f$ i.e. with $f(e_1) \neq f(e_0)$.
Given $e, n \in \mathbb{N}$ consider the following
2 register machine program $M_{e,n}$:
- input x, then compute $\varphi_e(n)$ and if that halts, compute $\varphi_{e_1}(x)$.

Now let M' be the following register machine
4 built from M:
- input e and n,
  Compute the index $i(e,n)$ of the machine $M_{e,n}$ and apply M to $i(e,n)$

Thus for any $e, n \in \mathbb{N}$:

- if $\varphi_e(n)\downarrow$, then $M_{e,n}$ just computes φ_{e_1},
4 so <u>by extensionality</u> $f(\ulcorner M_{e,n} \urcorner) = f(e_1) \neq f(e_0)$, so $\ulcorner M_{e,n} \urcorner \notin S_f$, so M on $\ulcorner M_{e,n} \urcorner$ gives 0, so M' on e,n gives 0.

4 • if $\varphi_e(n)\uparrow$, then $M_{e,n}$ just computes φ_{e_0}
so by extensionality $f(\ulcorner M_{e,n}\urcorner) = f(e_0)$,
so $\ulcorner M_{e,n}\urcorner \in S_f$, so M on $\ulcorner M_{e,n}\urcorner$ gives 1,
4 so M' on e,n gives 1.

We appeal to the undecidability of the
2 Halting Problem, which tells us ~~that~~ no
register machine M' with properties as in
part (a) can exist. Therefore there can
be no register machine M deciding S_f $(\neq \mathbb{N})$.
3 1 Thus if $S_f \neq \mathbb{N}$, S_f is not decidable.

Commentary

This question concerns material from Lectures 11 & 12. The main part of the question is Rice's Theorem in disguise. That theorem is not covered in full generality in the course, although instances of it are covered. In particular the (tricky!) construction of M′ from M in part (a) is similar to constructions that are dealt with in lectures.